

QUARK MODELS AND THEIR APPLICATIONS TO BARYON SPECTROSCOPY

Thesis

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BARRIE FENWICK TYNEMOUTH

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To

Theima and Raymond

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ABSTRACT

In this thesis we aim to study the phenomenology of the low-lying baryons using a non-relativistic harmonic oscillator quark model with "QCD-inspired" relativistic corrections. We give in chapter 2 derivations of the short-range effects and long-range effects, coming from one-gluon-exchange and confinement, respectively. In chapter 3 we firstly outline the model of Isgur and Karl and show that their judicious choice of hyperfine interactions ensures their results reproduce many of the features of the presently observed spectrum. In the second part of chapter 3 we show how the inclusion of the spin-orbit interactions, from one-gluon-exchange and confinement, in a manner which is consistent with the original philosophy, produces problems in the P-wave baryons. This leads us to study the non-strange baryons belonging to the $N=2$ level to see if the inclusion of these spin-orbit interactions causes the same problems. Our results are in good agreement with the experimental spectrum. We briefly study the decay of the ΔF_{35} , which seems to imply that the spin-orbit interactions are not present with the correct relative strength to the spin-spin interactions, as Isgur and Karl suspect. In chapter 4 we introduce $Sp(12, R)$, a spectrum generating algebra for the harmonic-oscillator quark model and show how it may be used to label the oscillator eigenstates. Using $Sp(12, R)$ we show how Bowler et al produce an algebraic mass formula which reproduces the splitting pattern of the five $N=2$ $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ supermultiplets under the addition of a two-body anharmonic perturbation. We extend this work on two-body perturbations to a more general type of symmetry-breaking potential as considered by Gromes and Stamatescu in the context of explicit wavefunction calculations. A new algebraic mass formula for the splitting of the five $N=2$ supermultiplets is derived. In chapter 5 we introduce lattice gauge theories and consider the construction of operators which represent hadrons in their ground state and first excited state.

This is one small step for mankind,
but a giant leap for a man.

Barrie Tynemouth,
Edinburgh, August 1983.

(adapted from the original quote by
Neil Armstrong, Sea of Tranquility,
July 1969).

CHAPTER 1

INTRODUCTION

Throughout the history of modern physical science, the search for a greater understanding of the basic nature of matter has repeatedly been aided by the idea that a complex system is evidence for a substructure of constituents, which are small on the scale of the more complex system. This ansatz has led from materials to molecules, from molecules to atoms, from atoms to nuclei and electrons, from nuclei to nucleons, and from nucleons to quarks as a means to aid in the elucidation of the properties of matter. For the purposes of this thesis this concatenation of divisibility will stop at quarks, although there are at present theories in which quarks are themselves composite particles. Recent decades have seen the construction of particle accelerators of ever increasing energy, which allow, as a consequence of quantum mechanics, the probing of systems on a decreasing length scale. This has led to remarkable progress in the understanding of these basic building blocks of nature, called quarks (q) and antiquarks (\bar{q}), their fundamental interactions and the many ways in which they manifest themselves in the guise of hadrons.

This progress in the understanding of the physics of these elementary particles has been made on two levels:

The first is the major advances in quantum field theory. These include a better understanding of the problems of renormalizability, the spontaneous breaking of the symmetry of the

Lagrangian into solutions of a lower symmetry, and a coupling strength which is small at short distances and very big at large distances. This final problem which leads to the breakdown in perturbation theory due to these infra-red (or low energy) singularities, has led to the inception of lattice gauge theories. In such theories space and time are discretised in order to aid computation; this will be discussed in more detail in Chapter 5 of this thesis.

The second is the achievement of major quantitative advances in correlating detailed experimental information with predictions made by taking naive quark models of strong interactions seriously and then applying to these models the more familiar tools of symmetry, non-relativistic quantum mechanics, relativistic corrections and so forth. The majority of this thesis is dedicated to such models.

Hadrons, or strongly interacting particles, come in two types of large families; the first is baryons which are fermions (i.e. have half integral spins); the second is mesons which are bosons (i.e. have integral spins). Applying this criterion from the beginning implies that both baryons and mesons are composite particles. Assuming quarks are spin $\frac{1}{2}$ fermions, then baryons would be built from qqq and mesons from $q\bar{q}$. In the original quark model proposed by Gell-Mann and Zweig (for a review of the origins of the quark model, see Gell-Mann and Ne'eman, 1964), quarks were assigned to the fundamental representation of a broken, $SU(3)$ symmetry, and the hadrons could be classified in higher irreducible representations of the group. At that time and up until the mid-1970's,

all the experimentally observed hadrons could be accounted for with the above composition and allowing for internal excitations of the quarks, using only three quark 'flavours' (or types). These three flavours are 'up' (u), 'down' (d) and 'strange' (s). In this thesis we shall only be considering baryons which are made from these three flavours. In 1974 with the discovery of the ψ , this list was lengthened to include a fourth flavour called 'charm' (c). Another flavour called 'bottom' (b) was added in 1976 when the T , another vector meson, was discovered. The recent discovery of a heavy lepton, tau (τ), and the appealing idea of quark-lepton symmetry, strongly favour the existence of a sixth flavour, called 'top' (t). However there is no experimental evidence for its existence at the present, only a strong theoretical prejudice.

The quark model, in the above form, encountered one major difficulty: the Pauli Exclusion Principle, which states that: The total wavefunction of a collection of fermions should be antisymmetric under the interchange of any two identical fermions. The state that is usually quoted in connection with this problem, is the $\Delta^{++}(1232)^{3/2^+}$. In the quark model this state's composition is $u\uparrow u\uparrow u\uparrow$, (the \uparrow denotes the quark's spin) which gives the correct total angular momentum, J , and isospin I . This part of the wavefunction is symmetric by inspection. This resonance is assumed to be a ground state baryon, so all the quarks are presumably in relative s-waves; then the spacial wavefunction is also permutation symmetric. Thus the total wavefunction is permutationally symmetric, flagrantly contravening the Pauli principle. To circumnavigate this problem it was suggested that quarks could possess a new, triple valued quantum number, called colour. So a quark, of a given flavour,

can exist in three different colour states, red, blue and green. It is now possible to construct a colour wavefunction which is totally antisymmetric under the interchange of any two quarks. Thus for the $\Delta^{++}(1232)^{3/2^+}$ the wavefunction is $\epsilon_{\alpha\beta\gamma} u^\alpha \uparrow u^\beta \uparrow u^\gamma \uparrow$, where $\epsilon_{\alpha\beta\gamma}$ is totally antisymmetric under the interchange of any two indices and α, β, γ are colour indices.

The introduction of colour would appear to increase the number of baryons twenty-sevenfold and the number of mesons ninefold, because of these new 'colourful' states. The absence of experimental evidence for all of these states suggests that all observable states should be 'colourless'. Then all baryonic states would be of the form: $B_{ijk} \sim \epsilon_{\alpha\beta\gamma} q_i^\alpha q_j^\beta q_k^\gamma$, and mesonic states would be of the form $M_i^j \sim \delta_\alpha^\beta q_i^\alpha \bar{q}^j_\beta$, where i, j, k are flavour indices and α, β, γ are colour indices.

Although there is no direct experimental evidence for the existence of this new quantum number, colour, there is strong indirect evidence to support the theory:-

- a) The calculated decay rate for the process; $\pi^0 \rightarrow 2\gamma$, using the triangle anomaly and PCAC, is a factor of nine too small. Allowing the quarks to carry colour introduces a factor of $n_c = 3$ (where n_c is the number of colours), giving good agreement with experimental data.
- b) The experimental value of the ratio,

$$R = \frac{\sigma(e^+e^- \longrightarrow \text{hadrons})}{\sigma(e^+e^- \longrightarrow \mu^+\mu^-)},$$

is well behaved away from resonance behaviour. If the process in the numerator proceeds via $e^+e^- \rightarrow \gamma \rightarrow q\bar{q} \rightarrow \chi$, then the predicted ratio

is given by the sum of the squares of charges on the relevant quarks. This prediction is approximately a third of the experimental value. The inclusion of colour into the calculation introduces a factor $n_c = 3$, which means there is now good agreement between the experimental and theoretical values.

c) Another test is the ratio

$$R' = \frac{\Gamma(\tau \rightarrow \text{hadrons})}{\Gamma(\tau \rightarrow \ell + \nu_\ell + \nu_\tau)}$$

where the process in the numerator is $\tau^- \rightarrow d + \bar{u} + \nu_\tau$. This theoretical prediction is a factor of three too small, compared to the experimentally observed value, without colour. With colour the width in the numerator is multiplied by a factor of $n_c = 3$, which then gives the correct value.

There are further examples of evidence for colour, for these and for a wider review of the quark model see Greenberg (1978), Gasiorowicz and Rosner (1981) and references therein.

Colour has a more important role to play in strong interactions, other than the one outlined above. This is in Quantum Chromodynamics (QCD), which is the presently accepted theory of strong interactions, based on a local $SU(3)_{\text{colour}}$ gauge group. In this theory the 'colour' force is transmitted between 'coloured' particles by eight intermediate vector bosons, called gluons. These gluons are the analogues of the photon in Quantum Electrodynamics (QED). In QED the photon transmits the electric force between two electrically charged particles, but it is electrically neutral itself. However, the gluons in QCD carry colour themselves; this property allows them to interact with other gluons, as well as with quarks, unlike their sister

the photon. It is this property of QCD, that gluons carry colour, which gives QCD its idiosyncratic properties of asymptotic freedom (which means that at small interquark distances, or high momentum transfer, the quarks act like free particles) and infra-red slavery (which means that at large interquark separation, or small momentum transfer, the coupling strength is large). This property of asymptotic freedom allows us to use perturbation theory to calculate hadron masses; this is because the quarks are in a bound state (and therefore their separation is small) which, from above, implies that they are weakly coupled. Infra-red slavery implies that all coloured objects such as quarks, antiquarks, gluons, diquarks etc., should be permanently confined to the interior of colourless objects, such as hadrons.

Confinement of colour has not been proved rigorously yet, but it is strongly supported by experimental evidence, or rather, by the lack of evidence for the existence of free quarks, and also by various results from lattice gauge theory calculations. The search for free quarks has been conducted in high-energy particle accelerators, in cosmic-rays, and in stable matter, all of which has not found any evidence for free quarks. However, there is one experiment by Fairbank et al. (1977), a derivative of Millikan's oil drop experiment, which claims to have observed fractional charges. Other experiments of a similar nature have failed to confirm these results (Greenberg, 1978).

In the next chapter we give a more formal introduction to QCD, the presently accepted theory of strong interactions, and how non-relativistic potentials and their incumbent relativistic corrections may be derived from the theory. These potentials will then be used

in Chapter 3 to calculate baryon masses using a non-relativistic harmonic oscillator quark model. In Chapter 4 we explore the group theoretical aspects of this harmonic oscillator model and anharmonic symmetry breaking. Finally, in Chapter 5, look at the calculation of baryon masses using lattice gauge theories.

CHAPTER 2

QUANTUM CHROMODYNAMICS AND NON-RELATIVISTIC POTENTIALS

2.1 An Introduction to QCD

Quantum Chromodynamics is the quantum field theory of colour. It is hoped that one day QCD will provide theorists with a complete description of strong interactions, just as Quantum Electrodynamics (QED), which describes the interaction of electrically charged particles, has provided the most accurate theory known to mankind. In this section we shall draw on the similarities and the differences between the more familiar theory of QED and the, possibly, not so familiar QCD.

In the introduction it was hypothesised that, although quarks and gluons are coloured objects, only colourless (or white) objects are allowed to exist in nature. This implies that quarks and gluons, in QCD, are permanently confined to the interiors of bound states, called hadrons. Bound states also exist in QED, e.g. positronium (which is the bound state of an electron and a positron), but in this case the constituent particles can exist on their own outside the bound state. Positronium (e^+e^-), in QED, may be thought of as the analogue of a meson ($q\bar{q}$), in QCD, as they are both bound states of a particle and an antiparticle. But there is no analogue in QED of a baryon (qqq) in QCD, i.e. a bound state of three electrons has not been observed experimentally. It is worth noting that the existence of baryons implies that any pair of quarks (or diquark, qq) in the baryon is attracting the remaining quark in a similar fashion to an antiquark attracting the quark in a meson.

QED is a quantum field theory, in which the Lagrangian is invariant under local gauge transformations belonging to the abelian group $U(1)$. As $U(1)$ is a one-parameter group, there is one gauge boson, the photon, which transmits the electromagnetic force, even though it is electrically neutral itself. QCD is also a quantum field theory, in this case the Lagrangian is invariant under local gauge transformations belonging to the group $SU(3)_{\text{colour}}$ (the subscript is to denote the difference between the colour and flavour $SU(3)$ symmetry). $SU(3)_{\text{colour}}$ is an eight parameter group, hence there are eight independent massless gauge bosons, the gluons. The gluons carry the colour force between colour charges, because of the non-abelian nature of the theory, which will be expounded below.

The Lagrangian in QCD is,

$$\begin{aligned} \mathcal{L}_{\text{QCD}} = & \bar{q}(x) \{ \gamma^\mu (\partial_\mu + ig F_\mu^a G_\mu^a(x)) - m \} q(x) \\ & - \frac{1}{4} G_{\mu\nu}^a(x) G^{\mu\nu a}(x), \end{aligned} \quad (2.1)$$

where

$$G_{\mu\nu}^a(x) = \partial_\mu G_\nu^a(x) - \partial_\nu G_\mu^a(x) - g f^{abc} G_\mu^b(x) G_\nu^c(x), \quad (2.2)$$

μ, ν are Lorentz indices ($= 0, 1, 2, 3$), a, b, c are group indices ($= 1, 2, \dots, 8$) and repeated indices are summed over. The quarks, $q(x)$, transform as the fundamental representation of $SU(3)_{\text{colour}}$, which is a triplet, i.e. $\underline{3}$ (giving the three colours), the anti-quarks transform as the second fundamental representation of $SU(3)_{\text{colour}}$, which is a $\bar{\underline{3}}$ (giving the three anticolours) and the gluons, $G^a(x)$, transform as an $\underline{8}$ of $SU(3)_{\text{colour}}$. In equation

(2.1) the F^a 's are the $SU(3)_{\text{colour}}$ matrices which obey the commutation relation,

$$[F^a, F^b] = if^{abc} F^c, \quad (2.3)$$

where $a, b, c = 1, 2, \dots, 8$ and the f^{abc} are the structure constants of $SU(3)_{\text{colour}}$ (see Lichtenberg, 1978). The F^a 's are related to the original Gell-Mann λ^a matrices by the simple relation; $F^a = \frac{\lambda^a}{2}$. A small amount of algebra will show that the QCD Lagrangian (2.1) is indeed invariant under transformations of the type:

$$q(x) \rightarrow q'(x) = U(x) q(x), \quad (2.4a)$$

where

$$U(x) = \exp(i \theta^a(x) F^a) \quad (2.4b)$$

and the $\theta^a(x)$'s contain the space-time parts of the transformation, providing the gluon fields transform like

$$G_{\mu}^a(x) \rightarrow G_{\mu}^{a'}(x) = U(x) G_{\mu}^a(x) U^{-1}(x) + \frac{i}{g} U(x) \partial_{\mu} U^{-1}(x). \quad (2.4c)$$

It is the term in (2.2) proportional to f^{abc} which allows the gluons to interact with themselves in the Lagrangian (2.1). However, in abelian groups, like $U(1)$, the structure constants f^{abc} , as in (2.3), are all zero because all the elements commute. In equations (2.1), (2.2) and (2.4c) g is the coupling constant which determines the strength of the interaction between gluons and objects which have a colour charge. This is just the analogue of the electric charge e , which determines the strength of electromagnetic interactions.

The QCD Lagrangian, (2.1), contains the colour charge g , just

as the QED Lagrangian contains the electric charge, e , these are both characteristic of these respective theories. However, it is useful to introduce quantities which depend on the four-momentum transfer, Q , of the process under consideration. Consider, firstly, a calculation of the scattering amplitude for a simple process, in QED, such as electron-electron scattering. The lowest order Feynman diagram for this process is shown in Figure 2.1. The Feynman rules tell us that associated with each electron-photon vertex is a factor

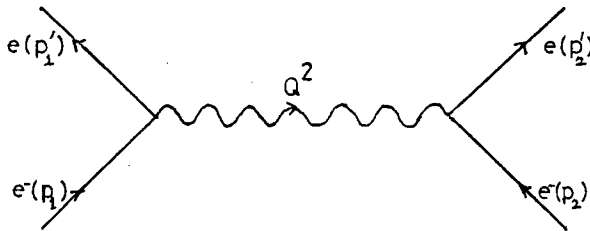


Figure 2.1. Lowest-order contribution to electron-electron scattering.

$-ie\gamma_\mu$ (for further details see Bjorken and Drell, 1964). Thus the scattering amplitude for the process shown in Figure 2.1 is proportional to e^2 . It is thus customary to work with the quantity $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$, called the fine structure constant. There are other Feynman diagrams which can contribute to the scattering amplitude for electron-electron scattering, but these are of higher order in the fine structure constant, α . Because α is such a small number the contribution from such graphs, in most cases, may be neglected. However, there is a certain class of Feynman diagram which has a non-negligible contribution, when the four-momentum transfer (usually given as Q^2) is high enough, i.e. when the separation is small enough. Such a graph is shown in Figure 2.2, and is called a vacuum polarization graph. There are other such graphs coming from vertex

corrections and self-energy diagrams, but it is these vacuum polarization graphs which have the dominant contribution. The contribution from the graph in Figure 2.2 is proportional to $\alpha^2 \log\left(\frac{Q^2}{m_e^2}\right)$, where

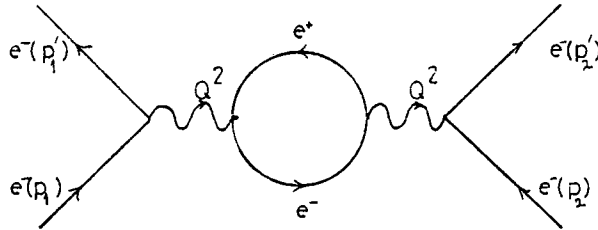


Figure 2.2. Vacuum polarization diagram in electron-electron scattering.

where m_e is the electron mass. Therefore as Q^2 increases, the largeness of the log eventually nullifies the effect of the smallness of α . The Feynman diagram in Figure 2.2 has only one electron-positron loop. However there are Feynman diagrams with any number of electron-positron loops which all have the same functional form as for the first diagram in Figure 2.2.

As all the vacuum polarization diagrams have the same functional form, it is possible to re-define α as a function of Q^2 , so that

$$\alpha(Q^2) = \alpha \left\{ 1 + \frac{\alpha}{3\pi} \log\left(\frac{Q^2}{m_e^2}\right) + \dots \right\} \quad (2.5)$$

At high energies, $Q^2 \gg m_e^2$, it is possible to explicitly sum these dominant, or leading, terms to give

$$\alpha(Q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \log\left(\frac{Q^2}{m_e^2}\right)} \quad (2.6)$$

Thus as Q^2 increases $\alpha(Q^2)$ also increases, which means that at small separation the electron's effective charge increases. As $Q^2 \rightarrow 0$, equation (2.6) is not correct, because the summation is only valid for large Q^2 , but the correct expression reduces to α , the fine structure constant.

The physical interpretation of equation (2.6) is as follows: At small Q^2 (corresponding to large interparticle distances), the charge on an electron is shielded by these virtual electron-positron pairs in the vacuum, thus reducing the effective charge, whereas at large Q^2 (corresponding to small interparticle distances) there are fewer of these virtual electron-positron pairs to shield the electron's charge, hence the effective charge and $\alpha(Q^2)$ is bigger.

Equation (2.6) has a singularity when $Q^2 = m_e^2 \exp(\frac{3\pi}{\alpha})$. Long before such a value of Q^2 is reached gravitational effects will play an important rôle. This breakdown in perturbation theory in QED at high Q^2 is referred to as an ultraviolet breakdown (Lichtenberg, 1981).

We shall now study the analogous scattering process in QCD, which is quark-quark scattering. The lowest order Feynman diagram for this process is shown in Figure 2.3. In this case the exchanged particle is a gluon, not a photon. The QCD Lagrangian and the Feynman Rules tell us that with each quark-gluon vertex is associated a factor $-ig\gamma_\mu F^a$, where g is the colour charge on the quark, F^a is one of the eight traceless 3×3 matrices forming the adjoint representation of $SU(3)_{\text{colour}}$, $a = 1, 2, \dots, 8$ and $\mu = 0, 1, 2, 3$. Following the earlier example we define a coupling constant $\alpha_s(\mu^2) = \frac{g^2}{4\pi}$, where μ is the normalization point, or the mass scale of QCD (in the case of QED $\mu \sim 0$), as the lowest order

scattering amplitude is proportional to $\alpha_s(\mu^2)$. Again most of the

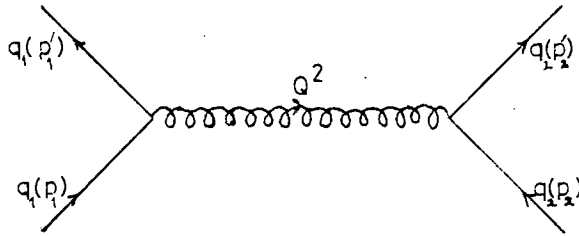


Figure 2.3. Lowest order contribution to quark-quark scattering.

higher graphs can be neglected, as they are of higher order in $\alpha_s(\mu^2)$. That is apart from the ubiquitous vacuum polarization diagrams, which at high enough Q^2 give a non-negligible contribution. Such a diagram is given in Figure 2.4. The internal fermion loop in this case is a quark and an antiquark; there is no internal electron loop as electrons do not carry colour so they cannot couple to gluons. The contribution to the scattering amplitude from this diagram in Figure 2.4, is proportional to $N_f \alpha_s^2(\mu^2) \log\left(\frac{Q^2}{\mu^2}\right)$, where N_f is the number

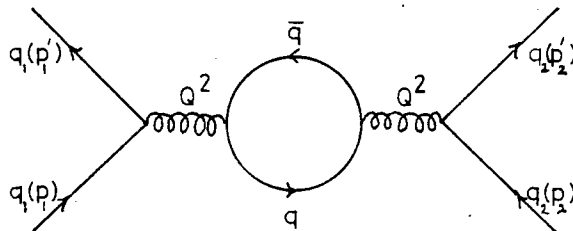


Figure 2.4. Vacuum polarization graph in quark-quark scattering.

of quark flavours which are allowed to circulate in the loop. So far the calculation of the scattering amplitude in QCD has been very similar to the previous calculation in QED. But the non-abelian nature of QCD

allows another graph to contribute to the scattering amplitude at high enough Q^2 . This is another vacuum polarization graph, but with an internal gluon loop, as in Figure 2.5, and its contribution to the scattering amplitude is proportional to $\alpha_s^2(\mu^2) \log\left(\frac{Q^2}{\mu^2}\right)$. It is the term proportional to f_{abc} in equation (2.2) which is responsible

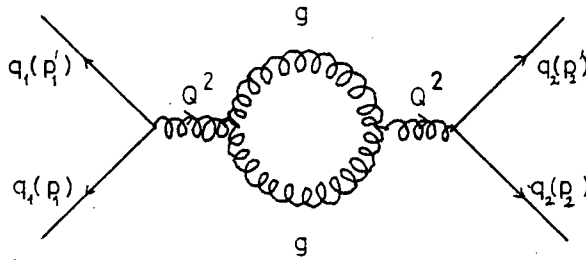


Figure 2.5. Gluon vacuum polarization diagram in quark-quark scattering.

for this graph; in QED all the f_{abc} 's are zero.

It is once again convenient to define a strong coupling function $\alpha_s(Q^2)$, which contains these momentum-dependent contributions of the vacuum polarization diagrams, with internal quark and gluon loops;

$$\alpha_s(Q^2) = \alpha_s(\mu^2) \left\{ 1 + \frac{N_f \alpha_s(\mu^2)}{6\pi} \log\left(\frac{Q^2}{\mu^2}\right) - \frac{11\alpha_s(\mu^2)}{4\pi} \log\left(\frac{Q^2}{\mu^2}\right) + \dots \right\} . \quad (2.7)$$

In equation (2.7) the second term in the parenthesis is from internal quark loops, Figure 2.4, and the third term is from internal gluon loops, Figure 2.5. For $Q^2 \gg \mu^2$ the series in (2.7) can be summed in the leading logarithm approximation, giving

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{33-2N_f}{12\pi} \alpha_s(\mu^2) \log\left(\frac{Q^2}{\mu^2}\right)} \quad (2.8)$$

In QCD the virtual quark-antiquark pairs in the vacuum shield the colour charge on the quark, just as the electron-positron pairs shield the electron charge in QED. However, the gluon loops around the quark spread out its colour charge, hence increasing the effective colour charge of the quark; this effect is called antishielding. Thus, as long as the number of quark flavours, N_f , is less than seventeen then the effective coupling strength, $\alpha_s(Q^2)$, will decrease as the four-momentum transfer, Q^2 , increases (i.e. as the separation decreases), and vice versa. This is the opposite behaviour to that of the coupling strength $\alpha(Q^2)$ in QED. This property of a decreasing effective coupling strength with decreasing separation is called asymptotic freedom, as we mentioned earlier. As Q^2 decreases, or the separation increases, the effective coupling strength increases until perturbation theory breaks down, and equation (2.8) is no longer valid. Unfortunately, the behaviour of QCD at small Q^2 is not known at present. But it is thought that this increasing effective coupling strength will permanently confine colourful objects, such as quarks and gluons, to the interior of colourless composite objects, such as hadrons; this is called infra-red slavery.

QCD, like QED, should only depend on one parameter. However, equation (2.8) seems to depend on two parameters $\alpha_s(\mu^2)$ and μ^2 ; these two parameters are not independent, as may be seen by eliminating these two parameters in favour of a single new parameter Λ ,

defined by:

$$\log \Lambda^2 = \log \mu^2 - \frac{12\pi}{(33-2N_f)\alpha_s(\mu^2)} \quad (2.9)$$

Then equation (2.8) reduces to

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2N_f) \log \left(\frac{Q^2}{\Lambda^2}\right)} \quad \cdot Q^2 \gg \Lambda^2, \quad (2.10)$$

which obviously depends only on one parameter. In QED the fine structure constant, α , is dimensionless; in QCD the parameter Λ has the dimension of energy (working with $\hbar = c = 1$). Empirically the value of Λ is not very well known, but it is thought to lie somewhere in the range 200-500 MeV; the reason being that in the region where perturbation theory works best, $\alpha_s(Q^2)$ is relatively insensitive to the value of Λ . To illustrate this, take $\Lambda \sim 400$ MeV, then $\alpha_s[(1 \text{ GeV})^2] \sim 0.76$ and $\alpha_s[(10 \text{ GeV})^2] \sim 0.25$. In the first case $N_f = 3$ in equation (2.10) (only u , d and s quarks are relevant) and in the second case, $N_f = 5$ (now u , d , s , c and b quarks are relevant). In QED, by comparison, $\alpha[(\sim 0 \text{ GeV})^2] = 0.073 (= \frac{1}{137})$ and $\alpha[(10 \text{ GeV})^2] = 0.0074$. Thus the relative change in $\alpha_s(Q^2)$ is much smaller than the relative change in $\alpha(Q^2)$, over the same energy range; at even higher energies than 10 GeV, $\alpha_s(Q^2)$ decreases very slowly, because of the argument in the logarithm (Lichtenberg, 1981). Thus, despite asymptotic freedom QCD perturbation theory converges more slowly than QED perturbation theory for all presently accessible energies.

As we have just seen, coloured objects such as quarks and gluons are permanently confined to the interior of hadrons in the form of baryons or mesons. The original theory assumed that baryons were made

from three quarks, qqq , and mesons from a quark and an antiquark, $q\bar{q}$. Are there no other combinations of quarks and antiquarks which will give a colour singlet state?

Using Young tableaux we can calculate the higher-dimensional representations of $SU(3)_{\text{colour}}$, obtained by combining quarks, which transform as a $\square = \underline{3}$ of $SU(3)_{\text{colour}}$, and antiquarks which transform as a $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = \bar{\underline{3}}$ of $SU(3)_{\text{colour}}$. (For an excellent introduction to Young tableaux and rules and regulations for combining them, see Lichtenberg (1978) or Wybourne (1970).) The results for various combinations are given in Table 2.1, from which it can be seen that the only place colourless objects (colour singlets) occur is in the combinations qqq (and hence \overline{qqq}) and $q\bar{q}$, which have already been associated with baryons (antibaryons) and mesons. Note also the diquark (qq) combination has a part which transforms like a $\bar{\underline{3}}$, which in a baryon will act like an antiquark and attract the remaining quark.

It is also possible to make other colourless objects by combining two states which are colourless, such as $qq\overline{qq}$ and $qqqq\bar{q}\bar{q}$. There are other ways of constructing colour singlets which contain gluons or are totally composed of gluons; such states are called hermaphrodites and glueballs and are outwith the scope of this thesis. (For further details, see Close (1979)).

Combination	Young Tableaux Decomposition	$SU(3)_{\text{colour}}$ Representations
q		$\underline{3}$
\bar{q}		$\bar{\underline{3}}$
qq	\otimes = \oplus	$\underline{3} \otimes \underline{3} = \underline{6} \oplus \bar{\underline{3}}$
$q\bar{q}$	\otimes = \oplus \cdot	$\underline{3} \otimes \bar{\underline{3}} = \underline{8} \oplus \underline{1}$
$q\bar{q}\bar{q}$	\otimes \otimes = \oplus \oplus \oplus \oplus \cdot	$\underline{3} \otimes \underline{3} \otimes \bar{\underline{3}} = \bar{\underline{15}} \oplus \bar{\underline{6}} \oplus \underline{3} \oplus \underline{3}$
qqq	\otimes \otimes = \oplus \oplus \oplus \cdot	$\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{10} \oplus \underline{8} \oplus \underline{8} \oplus \underline{1}$
.	.	.
.	.	.
.	.	.

Table 2.1. $SU(3)_{\text{colour}}$ Representations for q, \bar{q} combinations.

2.2 Non-Relativistic Potentials in QCD

The objective of this section is to show how to extract from QCD, the relativistic quantum field theory of strong interactions, a non-relativistic potential with relativistic corrections, which may then be used in a Schrödinger equation to calculate baryon masses. To do this we must restrict ourselves to considering only two-body interactions between the three quarks within a baryon. In such a scenario a colour flux tube would join each pair of quarks inside a baryon (or a quark and an antiquark in a meson), outside of which the gluon (or colour) field is not allowed to propagate. The form of the QCD Lagrangian does allow three-body forces to be present within a baryon, due to the three-gluon vertex, see Figure 2.6.

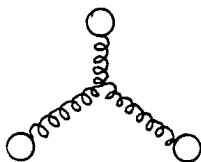


Figure 2.6. Three body force in a baryon.

However, using techniques from the study of lattice gauge theories, Dosch and Müller (1976) showed that any such three-body force present in a baryon can be effectively expressed as a linear combination of two-body forces.

In relativistic quantum mechanics the appropriate equation to use for the bound state of two-fermions is

$$\{(i\rlap{/}\partial - m)_1 (i\rlap{/}\partial - m)_2 - I_{12}\}\Psi(x_1, x_2) = 0 \quad (2.9)$$

where $\rlap{/}\partial = \gamma^\mu \partial_\mu = \gamma^0 \partial_0 - \underline{\gamma} \cdot \nabla$, $\Psi(x_1, x_2)$ is the wavefunction of

the bound state and I_{12} is the interaction operator which is given by an infinite series, i.e. the appropriate Feynman graph expansion. Equation (2.9) is called the Bethe-Salpeter equation (Bethe and Salpeter, 1957) and was first used in QED to describe relativistic, hydrogen-like bound states, such as positronium. It was found that, in an instantaneous approximation with only the first-order interaction term coming from one-photon exchange, the non-relativistic reduction of equation (2.9) to order v^2/c^2 , produces the Fermi-Breit Hamiltonian (Bethe and Salpeter, 1957).

This is what we wish to do in QCD, produce a non-relativistic Hamiltonian with relativistic corrections to order v^2/c^2 for three quarks within a baryon. A full reduction of the Bethe-Salpeter equation for any process is a somewhat tedious and not very enlightening procedure, so here we shall use a different method. The interested, or merely masochistic reader is directed to either Gromes (1977) or Schwinger (1973), both of which give the gory details of such reduction. Instead, we use the method expounded by Berstetskii et al. (1971), which involves equating scattering amplitudes calculated by using relativistic and non-relativistic quantum mechanics. The relativistic scattering amplitude is calculated from the Feynman graph expansion for the process using the usual Feynman rules (see Bjorken and Drell, 1964). The non-relativistic scattering amplitude is obtained using the Born approximation (see, e.g. Schiff, 1968), in which a particle interaction operator $U_{op}(r_{12})$, is sandwiched between initial and final plane-wave states:-

$$\begin{aligned}
 S_{fi} &= i(2\pi)^4 \delta^4(p_\mu^f - p_\mu^i) T_{fi} \\
 &= i(2\pi) \delta(E^f - E^i) \iint d^3x_1 \cdot d^3x_2 \exp\{-i(\underline{p}_1' \cdot \underline{x}_1 + \underline{p}_2' \cdot \underline{x}_2)\} U_{op}(r_{12}) \\
 &\quad \times \exp\{i(\underline{p}_1 \cdot \underline{x}_1 + \underline{p}_2 \cdot \underline{x}_2)\} \tag{2.10}
 \end{aligned}$$

By performing a non-relativistic decomposition of the T-matrix element in powers of v^2/c^2 to the required order, then taking the momentum Fourier transform and using (2.10), we obtain the relevant non-relativistic interaction operator in orders v^2/c^2 .

In the original work by Berstetskii et al. (1971), they considered one-photon exchange between an electron and a positron; however in their calculation they made two small errors, as we will see later. Here we will hopefully give a correct derivation for the analogous process in QCD. It is convenient to use a small amount of foresight; in order to calculate the effective non-relativistic potential between quarks in a baryon it is necessary to perform two of these non-relativistic decompositions: one for short-range effects, the other for long-range effects. This is because the potential obtained in the non-relativistic reduction of one-gluon-exchange, which is thought to be responsible for short-range effects, is not confining and the non-relativistic reduction of confinement, which gives rise to the long-range effects, does not provide the spin dependent hyperfine interactions required to produce the observed mass splittings, as we shall now show:-

(i) One-gluon Exchange

Following De Rujula, Georgi and Glashow (1975), the relevant process to consider when calculating short-range effects in a baryon is

one-gluon-exchange; see Figure 2.3. Using the Feynman rules for QCD (see, e.g., Ross, 1981) the T-matrix element for one-gluon-exchange between quarks i and j is

$$T_{fi} = \frac{m_i m_j}{\sqrt{E_i' E_i E_j' E_j}} g^2 \bar{\psi}^\alpha(p_i') (F_i^a)_{\alpha\beta} \gamma_\mu \psi^\beta(p_i) D_{ab}^{\mu\nu}(q^2) \bar{\psi}^\delta(p_j') (F_j^b)_{\delta\epsilon} \gamma_\nu \psi^\epsilon(p_j) \quad (2.11)$$

where $\mu, \nu = 0, 1, 2, 3$ are Lorentz indices, $a, b = 1, 2, \dots, 8$ label the matrices of the adjoint representation of $SU(3)_{\text{colour}}$, $\alpha, \beta, \gamma, \delta = 1, 2, 3$ are $SU(3)$ colour indices and $q^\phi = p_i^\phi - p_i'^\phi = p_j'^\phi - p_j^\phi$ is the four momentum transfer at which the process is being carried out. The quark spinors obey the Dirac equation,

$$\{i\gamma^\mu(\partial_\mu + iF^a G_\mu^a) - mc\}\psi(x) = 0, \quad (2.12)$$

and we use the standard representation for the γ matrices (Bjorken and Drell, 1964), i.e.

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (2.13)$$

where the σ 's are the Pauli spin matrices which obey the commutation relation

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = i2\epsilon_{ijk} \sigma_k. \quad (2.14)$$

The T-matrix element must be invariant under $SU(3)_{\text{colour}}$ transformations, therefore the propagator must have the form

$$D_{ab}^{\mu\nu}(q^2) = \delta_{ab} D^{\mu\nu}(q^2), \quad (2.15)$$

then the $SU(3)_{\text{colour}}$ part of the matrix element is proportional to

$\underline{F}_i \cdot \underline{F}_j$, where \underline{F}_i acts on quark i . The colour wavefunctions for a baryon is

$$|B_{ijk}\rangle = \frac{1}{\sqrt{6}} \epsilon_{\alpha\beta\gamma} |q_i^\alpha q_j^\beta q_k^\gamma\rangle ; \quad (2.16)$$

using this it is possible to evaluate the above scalar product of colour matrices, knowing the traces of the matrices F^a (e.g. Lichtenberg, 1978).

$$\langle B | \underline{F}_i \cdot \underline{F}_j | B \rangle_{SU(3)_{\text{colour}}} = -2/3 . \quad (2.17)$$

Similarly in a meson the colour part of the matrix element is proportional to $\underline{F}_i \cdot \underline{F}_j^*$ and the meson colour wavefunction is

$$|M_i^j\rangle = \frac{1}{\sqrt{3}} \delta_\alpha^\beta |q_i^\alpha q_j^\beta\rangle , \quad (2.18)$$

hence

$$\langle M | \underline{F}_i \cdot \underline{F}_j^* | M \rangle_{SU(3)_{\text{colour}}} = -4/3 . \quad (2.19)$$

Thus in the calculation of the T-matrix element, (2.11), the colour part only contributes a numerical factor, which is the same for all baryons (and for all mesons), so for convenience we subsume this factor and the factor of g^2 into the propagator, i.e.

$$\tilde{D}^{\mu\nu}(q^2) = \frac{2}{3} g^2 D^{\mu\nu}(q^2). \quad (2.20)$$

To calculate the non-relativistic limit of the T-matrix element, (2.11), we need to know how various individual terms behave in this non-relativistic limit. To do this it is convenient to work in a Coulomb, or transverse gauge, i.e.

$$\underline{\nabla} \cdot \underline{G}^a = 0 ; \quad (2.21)$$

in this gauge the gluon propagator has the form

$$\begin{aligned} \tilde{D}^{00}(q^2) &= D_I(q^2), & \tilde{D}^{0i}(q^2) &= 0, \\ \tilde{D}^{ij}(q^2) &= -D_I(q^2) \left[\delta^{ij} - \frac{q^i q^j}{q^2} \right]. \end{aligned} \quad (2.22)$$

(In QED calculations, it was found that the Coulomb gauge was convenient to use in non-relativistic problems) where $D_I(q^2)$ is the instantaneous part of the propagator. It is possible to find the non-relativistic limit of the quark spinors, $\psi(p_i)$ in equation (2.11), by taking the non-relativistic limit of the Dirac equation (2.14). In the case of weak coupling we find:

$$\psi(p_i) \xrightarrow{v^2/c^2 \rightarrow 0} \left[\begin{array}{c} (1 - \frac{p_i^2}{8m_i^2 c^2}) \omega \\ (\frac{\sigma_i \cdot p_i}{2m_i c}) \omega \end{array} \right] + O(v^3/c^3), \quad (2.23)$$

where ω is a 2-component Pauli spinor. (Beretetskii et al., 1971). To find the limit in equation (2.12) we take $G^a \approx 0$; the quarks are close together and are thus weakly bound due to asymptotic freedom.

It is now possible to study the non-relativistic limit of the T-matrix element (2.11), but first we shall look at the static limit. In this limit only the upper two components of the quark spinor, (2.23), survive and the 00-component of the gluon propagator, (2.22), then in the limit $v^2 \rightarrow 0$ from (2.11), $T_{fi} \rightarrow D_I(q^2)$, (remembering also to take the limit of the external line normalizations, $\sqrt{m_i/E_i}$). Taking the inverse Fourier transform of (2.10) we obtain

$$V_{ij}(\underline{r}_{ij}) = \int d^3q \exp\{i \underline{q} \cdot \underline{r}_{ij}\} D_I(q^2) \quad (2.24)$$

where $V_{\ell_j}(\underline{r}_{ij})$ is the 'static' quark-quark short-range potential. (In QED the static electron-electron potential is just the Coulomb potential $-\alpha/r$).

To include the relativistic corrections we calculate T_{fi} in equation (2.11) to order v^2/c^2 , so that, in this case both the upper and lower components of the quark spinor (2.23) and the spacial part of the gluon propagator (2.22) can contribute. Then, taking the inverse Fourier transform of T_{fi} , after a small amount of labour (!), this leads to

$$\begin{aligned}
 U_{op}(\underline{r}_{ij}) &= V_{\ell_j}(\underline{r}_{ij}) + \underbrace{\frac{1}{8} \left(\frac{1}{m_i^2 c^2} - \frac{1}{m_j^2 c^2} \right) \nabla^2 V_{\ell_j}(\underline{r}_{ij})}_{\text{Darwin Interaction}} \\
 &+ \underbrace{\frac{1}{2m_i m_j c^2} \{ (p_i V_{\ell_j}(\underline{r}_{ij})) p_j - p_i \cdot r_{ij} \frac{1}{r_{ij}} \frac{dV_{\ell_j}(\underline{r}_{ij})}{dr_{ij}} r_{ij} \cdot p_j \}}_{\text{Orbit-orbit Interaction}} \\
 &+ \underbrace{\frac{1}{6m_i m_j c^2} \sigma_i \cdot \sigma_j \nabla^2 V_{\ell_j}(\underline{r}_{ij}) - \frac{1}{4m_i m_j c^2} \left[\frac{d^2 V_{\ell_j}(\underline{r}_{ij})}{dr_{ij}^2} - \frac{1}{r_{ij}} \frac{dV_{\ell_j}(\underline{r}_{ij})}{dr_{ij}} \right] \left(\frac{\sigma_i \cdot r_{ij} \sigma_j \cdot r_{ij}}{r_{ij}^2} - \frac{1}{3} \sigma_i \cdot \sigma_j \right)}_{\substack{\text{Contact term} & \text{Tensor term}}} \\
 &+ \underbrace{\left[\frac{\sigma_i \cdot r_{ij} \times p_i}{4m_i^2 c^2} - \frac{\sigma_j \cdot r_{ij} \times p_j}{4m_j^2 c^2} - \frac{\sigma_i \cdot r_{ij} \times p_j - \sigma_j \cdot r_{ij} \times p_i}{2m_i m_j c^2} \right] \frac{1}{r_{ij}} \frac{dV_{\ell_j}(\underline{r}_{ij})}{dr_{ij}}}_{\text{Spin-spin Interactions}} \\
 &\underbrace{\hspace{10em}}_{\text{Spin-orbit Interaction}}
 \end{aligned}$$

where $\vec{r}_j = \vec{r}_1 - \vec{r}_j$ is the separation of the quarks in the laboratory frame and the notation $\{ \{ P_i P_j \} \} = \frac{4}{3} \{ P_i P_j + P_j P_i + P_i P_j + P_j P_i \}$ is due to Gromes (1977). In their derivation of the Fermi-Breit Hamiltonian Berstetski et al. (1971) miss the external line normalization and have a wrong sign in their equation (83.7) which conspire to give the correct answer (Barnes and Gaudour, 1982). Equation (2.25) agrees with the results of Gromes (1977) and Barnes and Gaudour (1983).

(11) Confinement

Here we shall adopt the assumption of Hendriques et al. (1976), who originally applied it to the charmonium spectrum, that the long-range confining potential is a multi-gluon effect which gives rise to an effective scalar coupling. The vertex in this case is proportional to the unit matrix, as opposed to one-gluon-exchange, which is a vector process where the vertex is proportional to γ_n , which gives a different non-relativistic decomposition.

Again, we shall use the method of Berstetski et al. (1971) to calculate the relevant interaction operator $U_{op}(\vec{r}_{1j})$ from the equating of the scattering amplitudes, see equation (2.10). For scalar coupling the T-matrix element is

$$T_{ij}^{T} = \frac{m_{1j}}{\sqrt{E_i E_j E_i E_j}} \int \psi(\vec{p}_i) \int \psi(\vec{p}_j) D_S(q_2) \psi(\vec{p}_i) \int \psi(\vec{p}_j) \quad (2.26)$$

where $D_S(q_2)$ is the instantaneous part of the scalar propagator, into which has been subsumed any colour factor and coupling strength. To find the non-relativistic limit we assume again that the quark spinor's non-relativistic limit is given by equation (2.23). This

limit is derived for the case of weak coupling - hence it is also assumed, as the confining mechanism is unknown, that this is also the limit in this strong coupling case.

In the static limit, i.e. the zeroth order in v^2/c^2 , $T_{fi}' \rightarrow D^S(q^2)$ and taking the inverse Fourier transform gives

$$V_C(\underline{r}_{ij}) = \int d^3q \exp\{iq \cdot \underline{r}_{ij}\} D^S(q^2) \quad (2.27)$$

where $V_C(\underline{r}_{ij})$ is the static quark confining potential. Again, to include the relativistic corrections to this potential we must expand T_{fi}' , (2.26), to first order in v^2/c^2 and then take the inverse Fourier transform, giving

$$U'_{Op}(\underline{r}_{ij}) = V_C(\underline{r}_{ij}) - \frac{1}{2m_i^2 c^2} \{ \{ \underline{p}_i V_C(\underline{r}_{ij}) \underline{p}_i \} \} - \frac{1}{2m_j^2 c^2} \{ \{ \underline{p}_j V_C(\underline{r}_{ij}) \underline{p}_j \} \}$$

Orbit-orbit interaction

$$- \left[\frac{\sigma_i \cdot \underline{r}_{ij} \times \underline{p}_i}{4m_i^2 c^2} + \frac{\sigma_j \cdot \underline{r}_{ij} \times \underline{p}_j}{4m_j^2 c^2} \right] \frac{1}{r_{ij}} \frac{dV_C(\underline{r}_{ij})}{dr_{ij}}$$

Spin-orbit interaction

(2.28)

where the notation is as in equation (2.25). Our result agrees with Barnes and Gandour (1982), but the spin-independent part disagrees with Gromes (1977), where the spin-independent interaction is,

$$\begin{aligned}
 U_{op}^{S-I}(\underline{r}_{ij}) &= V_C(\underline{r}_{ij}) - \frac{1}{8} \left(\frac{1}{m_i^2 c^2} + \frac{1}{m_j^2 c^2} \right) \nabla^2 V_C(\underline{r}_{ij}) \\
 &\quad - \frac{1}{4m_i^2 c^2} \{ \{ \underline{p}_i \cdot \nabla V_C(\underline{r}_{ij}) \} \underline{p}_i - \underline{p}_i \cdot \underline{r}_{ij} \frac{dV_C(\underline{r}_{ij})}{dr_{ij}} \underline{p}_i \cdot \underline{r}_{ij} \} \\
 &\quad - \frac{1}{4m_j^2 c^2} \{ \{ \underline{p}_j \cdot \nabla V_C(\underline{r}_{ij}) \} \underline{p}_j - \underline{p}_j \cdot \underline{r}_{ij} \frac{1}{r_{ij}} \frac{dV_C(\underline{r}_{ij})}{dr_{ij}} \underline{p}_j \cdot \underline{r}_{ij} \}.
 \end{aligned}
 \tag{2.29}$$

Unfortunately, we shall only be using the spin-dependent terms in (2.28), so our results will not be affected by this difference in spin-independent terms. However, this difference will be, hopefully, tested in calculations of the charmonium spectrum by other authors in the future.

It is now possible to compare the two non-relativistic potentials and their relativistic corrections obtained from a vector coupling, (equation (2.25), and from a scalar coupling, equation (2.28). Note that in the scalar case there are no spin-spin interactions and no spin-other orbit interactions. Experimentally there is no evidence of large spin splittings of resonances with high angular momentum, which is evidence in favour of a scalar confining potential. Another point is the three other possible couplings, pseudoscalar, axial vector and tensor; all have leading order terms which are proportional to $\underline{\sigma}_i \cdot \underline{\sigma}_j$. This would imply, if any of these couplings were responsible for confinement, that only one of the two possible spin-orientations would be confined, the other orientation producing a repulsive potential. This situation is definitely not observed experimentally, so a scalar confining potential seems to be the best candidate.

In the next chapter we shall be making use of both of these non-relativistic potential operators (2.25) and (2.28), or at least

the spin-dependent parts of them, to calculate their effect on the baryon spectrum within the framework of a specific non-relativistic quark model.

2.3 Hadron Masses in a Gauge Theory

The above heading is the title of an article by De Rujula, Georgi and Glashow (1975), in which they explore the implications for hadron spectroscopy of the strong, electromagnetic and weak gauge theories. It was these authors who first suggested that one-gluon-exchange was the process responsible for the short-range forces in a hadron. In their work it is assumed that this short-range potential, coming from one-gluon-exchange, is Coulombic, i.e.

$$V_g(\underline{r}_{ij}) = k \frac{\alpha_s}{r_{ij}}, \quad (2.30)$$

in complete analogy with QED; k is the colour factor, the baryons ($k = -2/3$) and mesons ($k = -4/3$), see §2.2. Substituting the above potential into equation (2.25) and taking $k = -2/3$ gives, for baryons:

$$\begin{aligned} U(r_{ij}) = & -\frac{2}{3} \frac{\alpha_s}{r_{ij}} + \frac{\alpha_s \pi}{3} \delta^3(r_{ij}) \left(\frac{1}{m_i} + \frac{1}{m_j} \right) + \frac{\alpha_s}{3m_i m_j} \left\{ \frac{\underline{p}_i \cdot \underline{p}_j}{r_{ij}} + \frac{\underline{r}_{ij} \cdot (\underline{r}_{ij} \cdot \underline{p}_i) \underline{p}_j}{r_{ij}^3} \right\} \\ & + \frac{16\alpha_s}{9m_i m_j} \delta^3(r_{ij}) \underline{s}_i \cdot \underline{s}_j + \frac{\alpha_s}{m_i m_j} \cdot \frac{1}{r_{ij}^3} \left(\frac{3\underline{s}_i \cdot \underline{r}_{ij} \underline{s}_j \cdot \underline{r}_{ij}}{r_{ij}^2} - \underline{s}_i \cdot \underline{s}_j \right) \\ & + \frac{2\alpha_s}{3r_{ij}^3} \left[\frac{\underline{s}_i \cdot \underline{r}_{ij} \times \underline{p}_i}{2m_i^2} - \frac{\underline{s}_j \cdot \underline{r}_{ij} \times \underline{p}_j}{2m_j^2} - \frac{\underline{s}_i \cdot \underline{r}_{ij} \times \underline{p}_j - \underline{s}_j \cdot \underline{r}_{ij} \times \underline{p}_i}{m_i m_j} \right] \end{aligned} \quad (2.31)$$

which is just the Fermi-Breit interaction, with a colour factor and

α_s instead of α (N.B. we have returned to natural units, $\hbar = c = 1$).

Their full Hamiltonian is

$$H = L(\underline{r}_1, \underline{r}_2, \underline{r}_3) + \sum_{i=1}^3 \left(m_i + \frac{p_i^2}{2m_i} \right) + \sum_{i<j} U(r_{ij}), \quad (2.32)$$

where $U(r_{ij})$ is given by equation (2.31) and $L(\underline{r}_1, \underline{r}_2, \underline{r}_3)$ is the potential which confines the quarks to the interior of a baryon. Allowing $SU(3)_{\text{flavour}}$ breaking to occur through the explicit quark masses, their zeroth order Hamiltonian is

$$H_0 = L(\underline{r}_1, \underline{r}_2, \underline{r}_3) + \sum_{i=1}^3 \left(m_u + \frac{p_i^2}{2m_u} \right) \quad (2.33a)$$

where

$$H = H_0 + V, \quad (2.33b)$$

m_u is the mass of the "up" quark and V is everything else from equation (2.32), which is the first-order perturbation. As the authors do not assume any specific form for the quark confining potential, $L(\underline{r}_1, \underline{r}_2, \underline{r}_3)$, they cannot construct the zeroth-order eigenstates of the Hamiltonian, (2.33a). Instead they parameterise the expectation value of V , and then fit these parameters to the observed spectrum. There are more masses than parameters, which allows the authors to produce several mass formulas.

For the ground state baryons De Rujula et al., using the Hamiltonian (2.32), reproduced the Gell-Mann - Okubo mass formula for the octet:

$$2M_N + 2M_{\Xi} = 3M_{\Lambda} + 3M_{\Sigma}, \quad (2.34)$$

and the equal-spacing rule for the decuplet:

$$M_{\Delta} - M_{\Sigma^*} = M_{\Sigma^*} - M_{\Xi^*} = M_{\Xi^*} - M_{\Omega}. \quad (2.35)$$

They also predicted the SU(6) flavour \otimes spin relation:

$$M_{\Sigma^*} - M_{\Sigma} = M_{\Xi^*} - M_{\Xi} , \quad (2.36)$$

and found a unified mechanism for the $(\Delta - N)$ and $(\Sigma - \Lambda)$ mass splittings, i.e.

$$M_{\Sigma} - M_{\Lambda} = \frac{2}{3} \left(1 - \frac{m_u}{m_s}\right) (M_{\Delta} - M_N) . \quad (2.37)$$

Using the above relation they predicted the ratio m_u/m_s to be ~ 0.6 . In the original work electromagnetic splittings were also calculated by including in (2.32) the Fermi-Breit Hamiltonian for one-photon-exchange. Assuming the quarks' anomalous magnetic moments are negligible, they obtain

$$m_u = \frac{M_p}{\mu(p)} \quad (2.38)$$

where $\mu(p)$ is the proton magnetic moment, which gives $m_u \sim 330$ MeV and hence $m_s \sim 550$ MeV.

In the ground state baryons, because all the quarks are in relative S-waves, there is no contribution from the spin-spin tensor interaction and the spin-orbit interactions, as these are rank 2 and rank 1 tensors respectively. However, both of these interactions will be present in the calculation of the masses of the baryons in the first excited level. De Rujula et al. found it very difficult to parameterise the hyperfine interactions in $U(r_{ij})$, equation (2.31), for these P-wave baryons. So the phenomenology of the baryons in the [70, 1⁻] was left in a somewhat confused state. This question of excited states was taken up by many authors (see Hey, 1980). The method Isgur and Karl (1977, 1978a, 1978b, 1979a) and Gromes and Stamatescu (1976, 1979) chose was to assume a specific form for the quark confining potential.

Gromes and Stamatescu (1976) chose a two-body linear confining potential, from a vector process, using harmonic oscillator wavefunctions and the variational principle produce a fit to the P-wave baryons. However, it was the model of Isgur and Karl (Isgur, 1980) which proved the most phenomenologically successful. It is this model we shall discuss and also use later as a basis for our calculations, in the next chapter.

CHAPTER 3

A STUDY OF THE BARYONS ASSIGNED TO THE N = 1 AND N = 2
LEVELS OF THE NON RELATIVISTIC HARMONIC OSCILLATOR QUARK MODEL

3.1 The Harmonic Oscillator Quark Model

In this chapter we will review the harmonic-oscillator quark model of Isgur and Karl (Isgur, 1980), as it has probably been the most successful in accounting for many features of the presently observed baryon spectrum; it will also provide a basis for our calculations in the later part of the chapter. The foundations of the harmonic-oscillator quark model were laid in the mid-1960's by Dalitz, Greenberg and collaborators (Greenberg 1964, Dalitz 1965). The model has two basic assumptions:

(i) The dynamics of quarks within a baryon is essentially non-relativistic. This allows the centre-of-mass motion of the baryon to be extracted by making a canonical change of co-ordinates from \underline{r}_i , the individual quark co-ordinates in some arbitrary frame, to the internal co-ordinates

$$\underline{\rho} = \frac{1}{\sqrt{2}} (\underline{r}_1 - \underline{r}_2) \quad (3.1a)$$

$$\underline{\lambda} = \frac{1}{\sqrt{6}} (\underline{r}_1 + \underline{r}_2 - 2\underline{r}_3) , \quad (3.1b)$$

in the centre-of-mass frame (for equal mass quarks) and to

$$\underline{R} = \frac{1}{3}(\underline{r}_1 + \underline{r}_2 + \underline{r}_3) , \quad (3.1c)$$

the position vector of the centre-of-mass.

(ii) The forces between the quarks in a baryon can be approximated by two-body harmonic-oscillator forces. It is this one special potential

which allows the three-body problem with two-body forces to be solved exactly.

Therefore, using the above two assumptions the Hamiltonian for a baryon is

$$H_0 = \sum_{i=1}^3 \frac{p_i^2}{2m} + \frac{K}{2} \sum_{i<j} (\underline{r}_i - \underline{r}_j)^2, \quad (3.2)$$

then performing the canonical change of co-ordinates to the internal co-ordinates as defined in equations (3.1a, b, and c), the above Hamiltonian becomes

$$H_0 = \frac{p_R^2}{2(3m)} + \frac{p_\rho^2}{2m} + \frac{p_\lambda^2}{2m} + \frac{3K}{2} (\rho^2 + \lambda^2) \quad (3.3)$$

where K is the spring constant, $p_R = (3m \frac{dR}{dt})$, $p_\rho = m \frac{d\rho}{dt}$ and $p_\lambda = m \frac{d\lambda}{dt}$. We shall be using this Hamiltonian to calculate the rest mass of baryons, so the centre-of-mass term in equation (3.3) can be dropped, which leaves two decoupled degenerate oscillators, one in ρ -space the other in λ -space. It is now possible to calculate the zeroth-order eigenstates for the Hamiltonian (3.3), which will be used later to do perturbation theory.

The harmonic-oscillator potential is not expected to be the confining potential in baryons in the real world, but only an approximation. From lattice gauge theories a more suitable candidate for a confining potential would be a linear potential. Gromes and Stamatescu (1976) tested this approximation by considering a simpler two-particle system with a linear interaction between the particles, then the Airy functions are exact solutions to this problem. Repeating the calculation using harmonic-oscillator wavefunctions, treating the linear potential as a perturbation, and using the variational

principle to calculate the ground state energy and comparing the two sets of energy levels, the authors found that the approximation was good to within 10% up to the fourth excited level. Hence the original approximation seems to be a reasonable one for our purposes, for further details see the original text.

We saw in the introduction that the $SU(3)_{\text{colour}}$ wavefunction is totally antisymmetric under the exchange of any two quarks, and that the quarks are spin- $\frac{1}{2}$ fermions, so the total wavefunction is constructed in the following fashion

$$\underbrace{|\text{Baryon}\rangle}_{\text{Antisymmetric}} = \underbrace{|\text{Flavour}\rangle}_{\text{Symmetric}} \underbrace{|\text{Spin}\rangle}_{\text{Symmetric}} \underbrace{|\text{Space}\rangle}_{\text{Symmetric}} \underbrace{|\text{Colour}\rangle}_{\text{Antisymmetric}} ; \quad (3.4)$$

the harmonic-oscillator forces in the Hamiltonian (3.3) do not depend on the quark's flavour or spin, so the Hamiltonian is a singlet under $SU(6)_{\text{flavour} \otimes \text{spin}}$ transformations. From equations (3.4) our problem is to construct $SU(6)_{\text{flavour} \otimes \text{spin}} \otimes O(3)$ wavefunctions which are permutationally symmetric under the exchange of any two quarks. The $O(3)$ group is associated with the orbital angular momentum (L) and parity (P) quantum numbers. The internal co-ordinates $\underline{\rho}$ and $\underline{\lambda}$ as defined by equations (3.1a and b) form a basis for the 2-dimensional mixed symmetric representation of the permutation group on three objects, S_3 . As the eigenstates of the Hamiltonian are constructed from the internal co-ordinates, $\underline{\rho}$ and $\underline{\lambda}$, it is possible to construct these eigenstates so that they transform as a given representation of the permutation group S_3 . The permutation symmetry of the $SU(6)_{\text{flavour} \otimes \text{spin}}$ wavefunctions can be calculated using the usual Young Tableaux decomposition for a baryon (in $SU(6)_{\text{flavour} \otimes \text{spin}}$

each quark transforms as a $\square = \underline{6}$,

$$\square \otimes \square \otimes \square = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad (3.5)$$

$$\text{SU}(6) \text{ Representations: } \underline{6} \otimes \underline{6} \otimes \underline{6} = \underline{56} \oplus \underline{70} \oplus \underline{70} \oplus \underline{20}$$

$$S_3 \text{ Representations: } M \otimes M \otimes M = S \oplus M_\rho \oplus M_\lambda \oplus A$$

where the S_3 representations are symmetric (S), mixed symmetric ($M_{\rho, \lambda}$) and antisymmetric (A). The reason why there are two mixed symmetric representations in S_3 is that the Young tableaux $\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right)$ has two standard arrangements; the corresponding representation is either symmetric (λ -type) or antisymmetric (ρ -type) under the interchange of particles 1 and 2 (Lichtenberg, 1978).

The allowed $\text{SU}(6)_{\text{flavour}} \otimes \text{spin} \otimes \text{O}(3)$ wavefunctions may be found by using the usual rules for combining representations of S_3 (Wybourne, 1970), i.e.

$$\psi_{\text{Total}} = \sum \psi_{\text{Space}} \circ \psi_{\text{SU}(6)_{\text{flavour}} \otimes \text{spin}} \quad (3.6)$$

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \circ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

$$S = S \circ \underline{56} + M_\rho \underline{70} + A \circ \underline{20}$$

where ψ_{Total} is everything but the colour wavefunction on the R.H.S. of equation (3.4) and \circ denotes the inner product. So equation (3.6) tells us a baryon which is assigned to the 56 dimensional representation of $\text{SU}(6)_{\text{flavour}} \otimes \text{spin}$ should have a spacial wavefunction which is permutationally symmetric under the interchange of any two quarks.

distinguishable from the two non-strange quarks, because of its heavier mass.

Notice that in the second oscillator level there are five degenerate $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ supermultiplets and in the third excited level there are eight degenerate $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ supermultiplets (and in higher oscillator levels there are higher degeneracies). These degeneracies are a consequence of a 'dynamical symmetry' of the harmonic oscillator model. This will be discussed in Chapter 4.

In the next section of this chapter we will be concentrating on Isgur and Karl's version of the harmonic oscillator model. There are, however, other harmonic oscillator quark models of baryons which we have not time to discuss here. The interested reader is directed to the review by Hey (1980) and references therein.

3.2 The Harmonic Oscillator Quark Model of Isgur and Karl

Isgur and Karl (1977, 1978a, 1978b, 1979a, 1979b) in their calculation of the baryon mass spectrum, assume the following form for their Hamiltonian

$$H = \sum_{i=1}^3 m_i + H_0 + \sum_{i<j} H_{\text{hyp}}(ij) \quad (3.7)$$

where

$$H_0 = \sum_{i=1}^3 \frac{p_i^2}{2m_i} + \sum_{i<j} V_c(\underline{r}_{ij}) \quad (3.8)$$

$V_c(\underline{r}_{ij})$ is the confining potential and $H_{\text{hyp}}(ij)$ is the hyperfine interaction between quarks i and j . It is the choice of this hyperfine interaction Hamiltonian which is central to Isgur and Karl's model. The authors assume that it has the form

$$H_{\text{hyp}}(ij) = \frac{2\alpha_s}{3m_i m_j} \left\{ \frac{8\pi}{3} \delta^3(\mathbf{r}_{ij}) \mathbf{s}_i \cdot \mathbf{s}_j + \frac{1}{r_{ij}^3} \left(\frac{3\mathbf{s}_i \cdot \mathbf{r}_{ij} \mathbf{s}_j \cdot \mathbf{r}_{ij}}{r_{ij}^2} - \mathbf{s}_i \cdot \mathbf{s}_j \right) \right\} \quad (3.9)$$

where α_s is some measure of the strong coupling strength, see §2.1. From §2.2, or from equation (2.31) specifically, it can be seen that this hyperfine interaction Hamiltonian is just the spin-spin, contact and tensor terms from the non-relativistic reduction of one-gluon-exchange with a $1/r_{ij}$, i.e. Coulomb potential. The two terms in equation (3.9) have their origin in the same physical interaction, the static interaction of two intrinsic colour-magnetic dipoles. The first term, the contact interaction, comes from the $\underline{\mu}_i \cdot \underline{B}_j^{\text{internal}}$ interaction of the colour-magnet i with the internal colour-magnet j . The second term comes from the $\underline{\mu}_i \cdot \underline{B}_j^{\text{external}}$ interaction of the colour-magnet i with the external colour magnetic-dipole field of the colour-magnet j and is the colour analogue of the macroscopic force between two magnets (Isgur, 1980). This situation is illustrated in Figure 3.2. Isgur and Karl stress that both of these

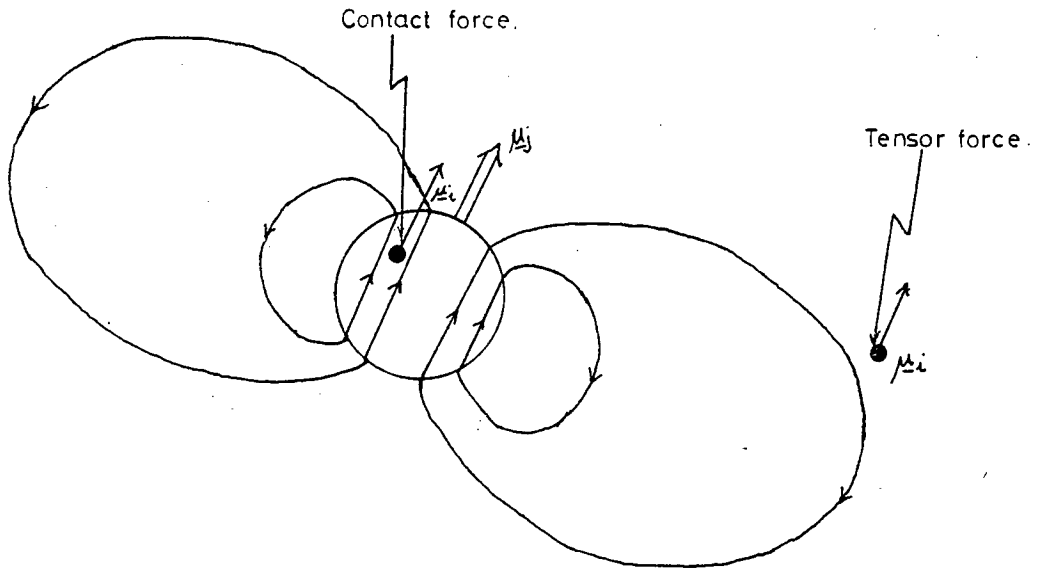


Figure 3.2. A representation of the origin of the contact and tensor hyperfine interaction.

interactions have to be present in the spectrum with the correct relative strength of $(\frac{8\pi}{3})$. The contact term in QED is responsible for the famous 21 cm line in hydrogen, corresponding to the difference in energy of the spin-triplet and the spin-singlet state.

In order to do perturbation theory we need a Hamiltonian which can be solved exactly to give the required eigenfunctions. These may then be used to calculate the matrix elements of the perturbing interactions. In writing down the Hamiltonian (3.8) we have already used the first assumption of a harmonic oscillator model in that the motion is essentially non-relativistic. We now use the second assumption of the model by writing the confining potential as

$$V_c(r_{ij}) = \frac{1}{2} K r_{ij}^2 + U(r_{ij}) \quad , \quad (3.10)$$

where $U(r_{ij})$ is some two-body, anharmonic perturbation. By substituting the above form for the confining potential into equation (3.8), and by neglecting the two-body anharmonic perturbation $U(r_{ij})$, gives a zeroth-order Hamiltonian which can be solved exactly, as we saw in the last section, and the zeroth-order eigenfunctions calculated.

This anharmonic perturbing potential, $U(r_{ij})$ hides a multitude of sins, metaphorically speaking, because as well as containing the corrections to the oscillator potential to give the true confining potential, it also contains the short-range potential coming from one-gluon-exchange and all the spin-independent interactions coming from the non-relativistic decomposition of the aforementioned one-gluon-exchange, see equation (2.31). A point to note here is that all of the terms from the non-relativistic reduction of one-gluon-exchange have been accounted for in the model, except for the spin-orbit interactions. We shall return to this point in section 4 of

this chapter.

It is convenient to split our calculations into two cases, the first is where all three quarks in a baryon have the same mass (e.g. nucleonic resonances, Δ -resonances or the triply strange Ω -resonances); the second is where two quarks have the same mass which is different to the mass of the third quark (i.e. singly strange Σ and Λ resonances and doubly strange Ξ resonances). In this thesis we do not consider baryons which contain charmed or bottom quarks, which would have given the third possible case where all three quarks have different masses. Following the original author's footsteps, we will only consider the non-strange and the singly-strange baryons, as the other two cases follow on trivially, even though the majority of this thesis is concerned with non-strange baryons.

I. The Low-lying Non-strange Baryons

These non-strange baryons only contain "up" and "down" quarks, which in the limit of exact $SU(2)_{\text{isospin}}$ symmetry, have the same mass, i.e. electromagnetic mass differences are being neglected:

$M_N = M_n = M_p$ and $M_{\Delta^+} = M_{\Delta^{++}} = M_{\Delta^0} = M_{\Delta^-}$. So, in equations (3.7) and (3.8), $m_1 = m_2 = m_3 = m$, and the zeroth-order oscillator.

Hamiltonian is

$$H_{HO} = \frac{p_\rho^2}{2m} + \frac{p_\lambda^2}{2m} + \frac{3K}{2} (\rho^2 + \lambda^2) \quad (3.11)$$

where the internal co-ordinates ρ and λ are defined in equations (3.1a and b). $p_\rho = m \frac{d\rho}{dt}$ and $p_\lambda = m \frac{d\lambda}{dt}$ are the momenta associated with these co-ordinates. Comparing the oscillator Hamiltonian in equation (3.11) with the earlier oscillator Hamiltonian, (3.3), it can be seen that the centre-of-mass term has been dropped. This is

because we will be working in the centre-of-mass frame, or the rest frame, of the baryon to calculate its mass. Note also that the fundamental frequency of the two decoupled oscillators, ρ - and λ -type is the same

$$\omega = \omega_{\rho} = \omega_{\lambda} = \left(\frac{3K}{m}\right)^{\frac{1}{2}} . \quad (3.12)$$

The ground state baryons have spacial wavefunctions where the total orbital angular momentum, $L = 0$, both the ρ - and λ -type oscillators are in their ground state, giving $L = L_{\rho} + L_{\lambda} = 0$. With all the quarks in relative S-waves the total wavefunction is permutationally symmetric, thus there is only one allowed $SU(6)_{\text{flavour} \otimes \text{spin}}$ supermultiplet, the $[56, 0^+]$.

The baryons in the first excited level have spacial wavefunctions with one unit of orbital angular momentum. This unit of orbital angular momentum may reside in the ρ - or λ -type oscillator. In one case the relative motion of quarks 1 and 2 has been excited, $L_{\rho} = 1$, $L_{\lambda} = 0$ and $L = L_{\rho} + L_{\lambda} = 1$, and the spacial wavefunction is antisymmetric under the interchange of particles 1 and 2. In the second case the relative motion of quark 3 about the centre-of-mass of quarks 1 and 2 is excited, $L_{\rho} = 0$, $L_{\lambda} = 1$, $L = 1$ and the spacial wavefunction is symmetric under the interchange of quarks 1 and 2. These two wavefunctions are degenerate in energy as they both have the same fundamental frequency, equation (3.12), and when combined with a mixed symmetric representation of $SU(6)_{\text{flavour} \otimes \text{spin}}$ to obtain the totally permutation symmetric wavefunction gives one allowed supermultiplet, the $[70, 1^-]$.

In the second excited level there are two units of orbital angular momentum to be distributed between the oscillators, Because

these two units of angular momentum add vectorially, the total orbital angular momentum can take three values, $L = 0, 1, 2$. It is useful to construct these spacial eigenfunctions of the Hamiltonian (3.11) so that they have definite symmetry properties under S_3 . So we get seven spacial wavefunctions, which when combined with the $SU(6)$ flavour \otimes spin wavefunctions give five allowed degenerate supermultiplets: $[56, 0^+]$, $[70, 0^+]$, $[56, 2^+]$, $[70, 2^+]$, $[20, 1^+]$.

The spacial wavefunctions for these three levels are tabulated in Table 3.1 (Isgur, 1980). The notation is $\psi_{LL_Z}^\pi$, where π is the permutation symmetry of the state [this may be symmetric (S), mixed symmetric of type ρ or λ (M_ρ or M_λ) or antisymmetric (A)], L is the total orbital angular momentum of the state and L_Z is its projection along the z-axis, and $\alpha = (3Km)^{\frac{1}{4}}$. We have only displayed the states with $L = L_Z$.

II. Low-lying $S = -1$ baryons

In this case there are two non-strange quarks, $m_1 = m_2 = m$ and a strange quark, $m_3 = m_s$, so the zeroth-order oscillator Hamiltonian is

$$\tilde{H}_{HO} = \frac{p_\rho^2}{2m} + \frac{p_\lambda^2}{2m_\lambda} + \frac{3K}{2} (\underline{\rho}^2 + \underline{\lambda}^2) \quad (3.13)$$

where $\underline{\rho}$ and $\underline{\lambda}$ are defined by equations (3.1a and b), p_ρ is defined as in equation (3.11) and

$$p_\lambda = m_\lambda \frac{d\underline{\lambda}}{dt} \quad (3.14a)$$

where

$$m = \frac{3mm_s}{2m + m_s} \quad (3.14b)$$

N	States
0	$\psi_{00}^S = \frac{\alpha^3}{\pi^{3/2}} \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
1	$\psi_{11}^{M_\rho} = \frac{\alpha^4}{\pi^{3/2}} \rho_+ \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
1	$\psi_{11}^{M_\lambda} = \frac{\alpha^4}{\pi^{3/2}} \lambda_+ \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{00}^{S'} = \frac{1}{\sqrt{3}} \frac{\alpha^5}{\pi^{3/2}} (\rho^2 + \lambda^2 - 3\alpha^{-2}) \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{00}^{M_\rho} = \frac{2}{\sqrt{3}} \frac{\alpha^5}{\pi^{3/2}} \underline{\rho} \cdot \underline{\lambda} \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{00}^{M_\lambda} = \frac{1}{\sqrt{3}} \frac{\alpha^5}{\pi^{3/2}} (\rho^2 - \lambda^2) \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{22}^S = \frac{1}{2} \frac{\alpha^5}{\pi^{3/2}} (\rho_+^2 + \lambda_+^2) \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{22}^{M_\rho} = \frac{\alpha^5}{\pi^{3/2}} \rho_+ \lambda_+ \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{22}^{M_\lambda} = \frac{1}{2} \cdot \frac{\alpha^5}{\pi^{3/2}} (\rho_+^2 - \lambda_+^2) \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$
2	$\psi_{11}^A = \frac{\alpha^5}{\pi^{3/2}} (\rho_+ \lambda_z - \rho_z \lambda_+) \exp \left\{ -\frac{1}{2} \alpha^2 (\rho^2 + \lambda^2) \right\}$

Table 3.1: Low-lying eigenstates for the Non-strange Harmonic Oscillator Hamiltonian.

Because the strange quark mass is different to that of the other two quarks in a baryon the Hamiltonian, (3.13), is no longer symmetric under the interchange of any two particles, i.e. it is not invariant under S_3 transformations, which the non-strange baryon Hamiltonian (3.11) is. In this case the ρ -mode of oscillation and the λ -mode of oscillation have different fundamental frequencies

$$\omega_\rho = \omega = \left(\frac{3K}{m}\right)^{\frac{1}{2}}, \quad (3.15a)$$

as in the last case, and

$$\omega_\lambda = \left(\frac{3K}{m_\lambda}\right)^{\frac{1}{2}} = \omega\left(\frac{2x+1}{3}\right)^{\frac{1}{2}} \quad (3.15b)$$

where $x = \frac{m_u}{m_s} \sim 0.6$, see §2.3, hence $\omega_\rho > \omega_\lambda$. Hence, the states in the first, second and higher excited levels are no longer degenerate, as it takes less energy to excite the ρ -mode of oscillation than it does to excite the λ mode of oscillation. A pictorial one-dimensional representation of the ρ - and λ -modes of oscillation is shown in Figure 3.3 (Hey, 1980).

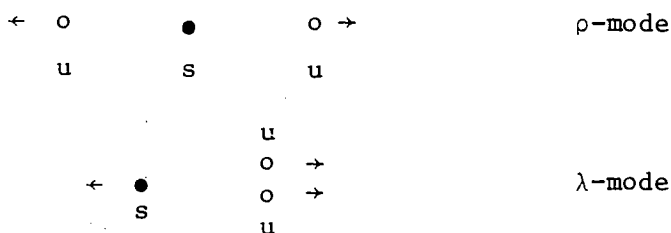


Figure 3.3: A one-dimension representation of the ρ - and λ -modes of oscillation.

Because the S_3 symmetry of the Hamiltonian, (3.13), is lost due to the different mass of the strange quark, it is no longer useful

N	States
0	$\psi_{00} = \frac{\alpha_\rho^{3/2} \alpha_\lambda^{3/2}}{\pi^{3/2}} \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
1	$\psi_{11}^\rho = \frac{\alpha_\rho^{5/2} \alpha_\lambda^{3/2}}{\pi^{3/2}} \rho_+ \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
1	$\psi_{11}^\lambda = \frac{\alpha_\rho^{3/2} \alpha_\lambda^{5/2}}{\pi^{3/2}} \lambda_+ \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{00}^{\lambda\lambda} = \sqrt{\frac{2}{3}} \frac{\alpha_\rho^{3/2} \alpha_\lambda^{7/2}}{\pi^{3/2}} (\lambda^2 - \frac{3}{2} \alpha_\lambda^{-2}) \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{00}^{\rho\lambda} = \frac{2}{\sqrt{3}} \frac{\alpha_\rho^{5/2} \alpha_\lambda^{5/2}}{\pi^{3/2}} \rho \cdot \lambda \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{00}^{\rho\rho} = \sqrt{\frac{2}{3}} \frac{\alpha_\rho^{7/2} \alpha_\lambda^{3/2}}{\pi^{3/2}} (\rho^2 - \frac{3}{2} \alpha_\rho^{-2}) \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{22}^{\lambda\lambda} = \frac{1}{\sqrt{2}} \frac{\alpha_\rho^{3/2} \alpha_\lambda^{7/2}}{\pi^{3/2}} \lambda_+^2 \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{22}^{\rho\lambda} = \frac{\alpha_\rho^{5/2} \alpha_\lambda^{5/2}}{\pi^{3/2}} \rho_+ \lambda_+ \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{22}^{\rho\rho} = \frac{1}{\sqrt{2}} \frac{\alpha_\rho^{7/2} \alpha_\lambda^{3/2}}{\pi^{3/2}} \rho_+^2 \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$
2	$\psi_{11}^{\rho\lambda} = \frac{\alpha_\rho^{5/2} \alpha_\lambda^{5/2}}{\pi^{3/2}} (\rho_+ \lambda_z - \rho_z \lambda_+) \exp\left\{-\frac{1}{2}(\alpha_\rho^2 \rho^2 + \alpha_\lambda^2 \lambda^2)\right\}$

Table 3.2: Low-lying eigenstates for the $S = -1$ Harmonic Oscillator Hamiltonian.

to construct eigenstates which have definite transformation properties under S_3 . The eigenstates of the Hamiltonian for the $S = -1$ baryons, equation (3.13), up to the second excited level, are tabulated in Table 3.2, in a form in which it is more convenient for calculations, (Isgur, 1980). The notation is as for Table 3.1, except:

$$\alpha_\rho = \alpha = (3K_m)^{\frac{1}{4}} \quad (3.16a)$$

$$\alpha_\lambda = (3K_{m_s})^{\frac{1}{4}}, \quad (3.16b)$$

and the suffix on the wavefunction (ψ) now denotes which oscillator mode(s) is(are) excited.

Because the strange quark mass is heavier than the other two non-strange quarks, it spends more time closer to the centre-of-mass of the baryon. Hence, it is possible to treat this strange quark as distinguishable and label it quark number 3. As the S_3 symmetry has been lost, it is now only the symmetry under the interchange of quarks 1 and 2, the two non-strange quarks, which is important. In view of this, Isgur and Karl (Isgur, 1980), introduce the following isospin wavefunctions:

$$\phi_\Sigma = \frac{1}{\sqrt{2}} (ud + du)s \quad (3.17a)$$

and

$$\phi_\Lambda = \frac{1}{\sqrt{2}} (ud - du)s \quad (3.17b)$$

to describe the Σ^0 and Λ states. (Electromagnetic mass differences are being neglected, i.e. we are working in the limit of exact $SU(2)_{\text{isospin}}$ symmetry, so that the two above isospin wavefunctions are sufficient to describe all the $S = -1$ baryons). The above isospin wavefunctions, equations (3.17 a and b), by inspection possess a

definite symmetry under the interchange of quarks 1 and 2, as do the spacial wavefunctions in Table 3.2. These can then be combined with spin wavefunctions whose symmetry under $1 \leftrightarrow 2$ interchange is known to produce a total wavefunction which is symmetric under the interchange of quarks 1 and 2. Again, the antisymmetry required by Fermi-Dirac statistics comes from the totally antisymmetric $SU(3)_{\text{colour}}$ wavefunction. In their work Isgur and Karl (Isgur, 1980) refer to the states in equations (3.17a and b) as the uds-basis, to differentiate it from the $SU(6)_{\text{flavour}} \otimes \text{spin}$ basis used in the non-strange sector. However, it is sometimes more convenient to use the $SU(6)_{\text{flavour}} \otimes \text{spin}$ basis in the $S = -1$ sector when comparing results with the non-strange sector.

Two-body Anharmonic Symmetry Breaking

In Figure 3.1, we saw that five $[SU(6), L^P]$ multiplets in the second excited level and the eight $[SU(6), L^P]$ supermultiplets in the third excited level are degenerate in mass (for the case of exact $SU(3)_{\text{flavour}}$ symmetry). A quick look at the experimental situation (Particle Data Group, 1982) will show that this massive degeneracy is not observed. The addition of the anharmonic perturbation, $U(r_{ij})$, removes this degeneracy of the $[SU(6), L^P]$ supermultiplets within a given oscillator level. For the present we shall restrict ourselves to considering the case of non-strange baryons, where the complete permutational symmetry of their $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ wavefunction allows the following simplification to be made

$$\sum_{i < j} \langle U(r_{ij}) \rangle = 3 \langle U(r_{12}) \rangle, \quad (3.18a)$$

or using (3.1a)

$$\sum_{i<j} \langle U(\underline{r}_{ij}) \rangle = 3 \langle U(\sqrt{2} \underline{\rho}) \rangle \quad (3.18b)$$

which tends to simplify calculations. $U(\underline{r}_{ij})$ only depends on the separation of quarks i and j , not on their flavour or spin; that is to say, it is a $SU(6)_{\text{flavour}} \otimes \text{spin}$ singlet, so it will not remove the degeneracy of the $SU(3)_{\text{flavour}}$ multiplets within a given $SU(6)_{\text{flavour}} \otimes \text{spin}$ supermultiplet.

To calculate the matrix elements of the Hamiltonian, (3.8) which contains the zeroth-order Hamiltonian (3.11) and the anharmonic perturbation (3.10), the first step is to combine the spacial wavefunctions in Table 3.1, for the first three levels, with the appropriate $SU(6)_{\text{flavour}} \otimes \text{spin}$ wavefunctions and then, using (3.18a and b), we obtain:

$$N = 0 : \quad E[\underline{56}, 0^+] = E_0 \quad (3.19)$$

$$N = 1 : \quad E[\underline{70}, 1^-] = E_0 + \Omega \quad (3.20)$$

$$N = 2 : \quad E[\underline{56}', 0^+] = E_0 + 2\Omega - \Delta \quad (3.21a)$$

$$E[\underline{70}, 0^+] = E_0 + 2\Omega - \frac{1}{2}\Delta \quad (3.21b)$$

$$E[\underline{56}, 2^+] = E_0 + 2\Omega - \frac{2}{5}\Delta \quad (3.21c)$$

$$E[\underline{70}, 2^+] = E_0 + 2\Omega - \frac{1}{5}\Delta \quad (3.21d)$$

$$E[\underline{20}, 1^+] = E_0 + 2\Omega, \quad (3.21e)$$

where

$$E_0 = 3m + 3\omega + a \quad (3.22a)$$

$$\Omega = \omega - \frac{1}{2}a + \frac{1}{3}b \quad (3.22b)$$

$$\Delta = -\frac{5}{4}a + \frac{5}{3}b - \frac{1}{3}c. \quad (3.22c)$$

The constants a, b and c are defined below:

$$a = \frac{3\alpha^3}{\pi^{3/2}} \int_0^\infty d^3\rho U(\sqrt{2} \rho) e^{-\alpha^2 \rho^2}, \quad (3.23a)$$

$$b = \frac{3\alpha^5}{\pi^{3/2}} \int_0^\infty d^3\rho U(\sqrt{2} \rho) \rho^2 e^{-\alpha^2 \rho^2}, \quad (3.23b)$$

$$c = \frac{3\alpha^7}{\pi^{3/2}} \int_0^\infty d^3\rho U(\sqrt{2} \rho) \rho^4 e^{-\alpha^2 \rho^2}, \quad (3.23c)$$

Isgur (1980). Note that apart from removing the degeneracy of the five $[SU(6), L^P]$ multiplets in the second excited level, the anharmonic perturbation $U(r_{ij})$ shifts the ground state energy, equation (3.22a), and changes the effective oscillator frequency,

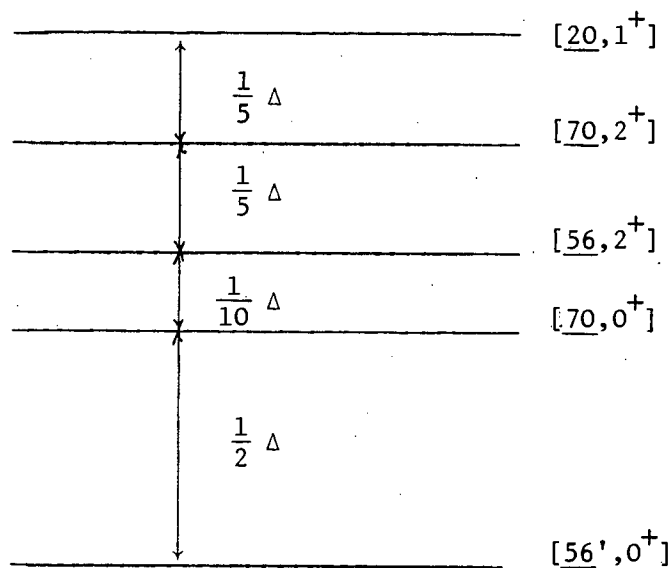


Figure 3.3: The splitting pattern of the five $N = 2$ supermultiplets under the addition of the anharmonic perturbation $U(r_{ij})$.



equation (3.22b). From equations (3.21a - e) the splitting pattern of the five $N = 2$ supermultiplets is shown in Figure (2.3). Isgur and Karl (1980) note that the ratio of the four splittings in this pattern is independent of the form of the two-body anharmonic perturbation, $U(r_{ij})$.

This pattern was first noticed by Gromes and Stamatescu (1976) (and by Horgan (1976)), by assuming that $U(r_{ij})$ has a power law form:

$$U(r_{ij}) = \lambda r_{ij}^k, \quad \lambda, k > 0 \quad (3.24)$$

then the splitting pattern is as in Figure 3.3 if $0 < k < 2$ and inverted for $k > 2$ (for $k = 0, 2$ the levels remain degenerate, as the first case is equivalent to a free particle and the second returns to the harmonic oscillator limit). The invariance of the splitting pattern for the five $N = 2$ multiplets shown in Figure 3.3, suggests that the addition of this two-body anharmonic perturbation, $U(r_{ij})$, breaks a 'dynamical' symmetry of the zeroth-order harmonic oscillator Hamiltonian, (3.11), so it should be possible to reproduce this splitting pattern using group theoretical techniques. The work of Bowler, Corvi, Hey, Jarvis and King (1981), reproduced this splitting pattern using the spectrum generating algebra, $Sp(12, \mathbb{R})$, for the harmonic oscillator quark model of baryons, and its subgroups. This work will be discussed in the next chapter.

In this thesis no specific form will have been assumed for this two-body anharmonic perturbation, $U(r_{ij})$, i.e. the confining potential in a baryon $V_c(r_{ij})$. This is not because there is a shortage of candidates, as will be seen below, but experimental evidence does not favour any specific form for $V_c(r_{ij})$ in a baryon. However, for

mesons, there is experimental evidence from the charmonium spectrum (Henriques et al., 1976) and a theoretical argument (Gromes and Stamatescu, 1979), for a linear confining potential. The argument of Gromes and Stamatescu (1979) is that the long-range force in a meson is due to a colour flux tube connecting the two colour charges. The gluon field is not allowed to propagate outside this tube. This effect makes the dynamics of the meson essentially one-dimensional. The Greens function of Laplace's equation is proportional to the distance

$$\frac{1}{2}\nabla^2|x| = \delta(x) \quad (3.25)$$

so a linear potential is obtained.

Gromes and Stamatescu (1979) considered a scenario where these flux tubes all meet at a point such that the sum of their lengths, S , was minimised, the potential would then be λS , where λ is the slope for the mesonic linear potential. If the circumference defined by the three quarks is C , then elementary geometry gives

$$\frac{1}{2} C \leq S \leq \frac{1}{\sqrt{3}} C \quad (3.26)$$

Therefore the three-body potential in the baryon can be approximated by two-body linear potentials, with a slope of approximately half of that for mesons. This is the argument of Dosch and Müller (1976) introduced in the last chapter. Gromes and Stamatescu (1976) suggested that as the three quarks in a baryon define a plane and the gluon fields would be confined to this plane (just as they were confined to a tube in the mesonic case). This makes the dynamics two-dimensional, and the Greens function of Laplace's equation in two-dimensions is proportional to the logarithm of the separation:

$$\frac{1}{2\pi} \nabla^2 \log r = \delta(r) , \quad (3.27)$$

hence a logarithmic potential is suggested. This 'pancake' of gluon fields is a genuine three-body force, unlike the earlier case. There are other candidates for this interquark confining potential in a baryon which have not been discussed here, hence no specific form will be assumed for $V_c(r_{ij})$.

The above authors also looked at the effect of allowing a potential $V(\underline{r}_1, \underline{r}_2, \underline{r}_3)$, which is rotationally and translationally invariant and symmetric under perturbations - but not necessarily the sum of two-body potentials, to be the anharmonic perturbation which split the five $N = 2$ supermultiplets, discussed earlier. They found that the translational and rotational invariance forced V to have the form $V(\underline{\rho}^2, \underline{\lambda}^2, \underline{\rho} \cdot \underline{\lambda})$, when written in terms of the internal co-ordinates $\underline{\rho}$ and $\underline{\lambda}$, equations (3.1a and b). Because of the permutational symmetry of the non-strange spacial wavefunctions, V can be symmetrised with respect to $\underline{\rho}$ and $\underline{\lambda}$. Then by explicitly calculating the matrix elements of this general anharmonic perturbation, using the oscillator wavefunctions in Table 3.1, they found that the splitting in this case depended on two parameters (in the two-body case the splitting depends on only one parameter, Δ). However, their results show that the four degenerate states $[20, 1^+]$, $[70, 2^+]$, $[56, 2^+]$, $[70, 0^+]$ split in the same way as Figure 3.3, but in this more general case, no statement could be made about the $[56', 0^+]$. This splitting pattern does not depend on the specific form of this more general perturbation, which suggests that the dynamical symmetry of the oscillator is being broken again, as in the two-body case, but by a different breaker. This will be studied in greater detail in the next chapter.

3.3 Hyperfine Interactions à la Isgur and Karl

As we said earlier, the major reason for the success of the Isgur and Karl model (Isgur, 1980), was their judicious choice of the hyperfine interaction Hamiltonian (3.9):

$$H_{\text{hyp}}(ij) = H_{\text{ss}}^{\text{C}}(ij) + H_{\text{ss}}^{\text{T}}(ij) \quad (3.28a)$$

where

$$H_{\text{ss}}^{\text{C}}(ij) = \frac{2\alpha_s}{3m_i m_j} \cdot \frac{8\pi}{3} \cdot \delta^3(r_{ij}) \underline{s}_i \cdot \underline{s}_j, \quad (3.28b)$$

is the spin-spin, contact interaction and

$$H_{\text{ss}}^{\text{T}}(ij) = \frac{2\alpha_s}{3m_i m_j} \cdot \frac{1}{r_{ij}^3} \left[\frac{3\underline{s}_i \cdot \underline{r}_{ij} \underline{s}_j \cdot \underline{r}_{ij} - \underline{s}_i \cdot \underline{s}_j}{r_{ij}^2} \right] \quad (3.28c)$$

is the spin-spin, tensor interaction, between quarks i and j . The complete permutational symmetry of the non-strange sector allows us again to write,

$$\sum_{i < j} \langle H_{\text{hyp}}(ij) \rangle = 3 \langle H_{\text{hyp}}(12) \rangle \quad (3.29)$$

and $m_i = m_j = m$. Then using the internal co-ordinates (3.1a and b), gives

$$H_{\text{ss}}^{\text{C}}(12) = \frac{2\alpha_s}{3m^2} \cdot \frac{8\pi}{3} \delta^3(\sqrt{2} \rho) \underline{s}_1 \cdot \underline{s}_2 \quad (3.28b')$$

and

$$H_{\text{ss}}^{\text{T}}(12) = \frac{2\alpha_s}{m^2 \rho^3} \left(\frac{\underline{s}_1 \cdot \underline{\rho} \underline{s}_2 \cdot \underline{\rho}}{\rho^2} - \frac{1}{3} \underline{s}_1 \cdot \underline{s}_2 \right) \quad (3.28c')$$

for non-strange baryons. In the $S = -1$ sector, Isgur and Karl (1978) make the approximation

$$\sum_{i < j} \langle H_{\text{hyp}}(ij) \rangle = \langle H_{\text{hyp}}(12) \rangle + x [\langle \tilde{H}_{\text{hyp}}(13) \rangle + \langle \tilde{H}_{\text{hyp}}(23) \rangle] \quad (3.29)$$

where $\tilde{H}_{\text{hyp}}(ij)$ is given by equations (3.28a, b and c) with $m_i = m_j = m$ and $x = \frac{m}{m_s}$.

It is these spin-dependent hyperfine interactions which break the $SU(6)_{\text{flavour}} \otimes \text{spin}$ symmetry of the Hamiltonian, i.e. they remove the degeneracy of the $SU(3)_{\text{flavour}}$ multiplets within a given $[SU(6), L^P]$ multiplet. We now present a brief review of Isgur and Karl's results for the first three levels:-

(i) The Ground State Baryons

In Isgur and Karl's model there are two free parameters which describe the baryons in the $[56, 0^+]$ supermultiplet: E_0 , the unperturbed position of the non-strange sector and δ , the overall strength parameter of the hyperfine interaction, defined by

$$\delta = \frac{4\alpha_s \alpha^3}{3\sqrt{2}\pi m^2} \quad (3.30)$$

These two parameters are fixed by the two states, $N(934)P_{11}$ and $\Delta(1232)P_{33}$ [the notation is $R(M_R) L_{2I2J}^{\pi N}$, where R is the resonance, M_R is the mass of the resonance in MeV, I is the isospin and J is the total angular momentum of the resonance and $L^{\pi N}$ is the orbital angular momentum of the π -N partial wave amplitude in which the resonance R is observed].

Isgur and Karl (1979b) take into account second-order effects in the hyperfine interaction by calculating the mixing between the ground-state baryons and the positive-parity excited baryons associated with the second excited level. As a result of this $E_0 > [M_N + M_\Delta]$.

The authors take $E_0 > 1135$ MeV and $\delta \approx 260$ MeV [$\approx M_\Delta - M_N$]; the masses predicted using these values are in good agreement with the experimentally observed spectrum (Isgur and Karl, 1979b).

The mixing of the $N=0$ and $N=2$ states does have important effects in the calculation of the decay widths of the ground state baryons (Koniuk and Isgur, 1980), as well as causing significant mass shifts. One directly observable effect is that the admixture of $[70, 0^+]$ from the $N=2$ level in the nucleon wavefunction gives the neutron and internal charge distribution which explains its observed electric form factor (Isgur, 1980).

(ii) Negative-parity Baryons in the First Excited Level.

As in the ground state, all the baryons in the $N=1$ level are associated with one supermultiplet, in this case the $[70, 1^-]$; this contains a $J^P = 5/2^-$, two $J^P = 3/2^-$ and two $J^P = 1/2^-$ nucleon resonances and a $J^P = 3/2^-$ and two $J^P = 1/2^-$ delta resonances. To calculate the masses of these states the first step is to construct a totally symmetric space \otimes spin \otimes $SU(3)_{\text{flavour}}$ wavefunction. The spacial wavefunctions for the oscillator Hamiltonian (3.11) are tabulated in Table 3.1, the spin wavefunctions are tabulated in Table 3.3 (the superscripts on the kets denote the permutation symmetry) and the $SU(3)_{\text{flavour}}$ wavefunctions are given in many texts (see, e.g. Lichtenberg, 1978). Two example wavefunctions are

$$|4_{\underline{8}}, 5/2^- \rangle = \frac{1}{\sqrt{2}} |3/2, 3/2 \rangle^s (|1, 1 \rangle^0 |8 \rangle^0 + |1, 1 \rangle^\lambda |8 \rangle^\lambda)$$

(3.31a)

State	Composition
$ 3/2, 3/2>^S$	$ \uparrow\uparrow\uparrow\rangle$
$ 3/2, 1/2>^S$	$\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow\rangle + \uparrow\downarrow\uparrow\rangle + \downarrow\uparrow\uparrow\rangle)$
$ 3/2, -1/2>^S$	$\frac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow\rangle + \downarrow\uparrow\downarrow\rangle + \downarrow\downarrow\uparrow\rangle)$
$ 3/2, -3/2>^S$	$ \downarrow\downarrow\downarrow\rangle$
$ 1/2, 1/2>^0$	$\frac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow\rangle - \uparrow\downarrow\uparrow\rangle)$
$ 1/2, -1/2>^0$	$\frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow\rangle - \downarrow\uparrow\downarrow\rangle)$
$ 1/2, 1/2>^\lambda$	$\frac{1}{\sqrt{6}}(2 \uparrow\uparrow\downarrow\rangle - \uparrow\downarrow\uparrow\rangle - \downarrow\uparrow\uparrow\rangle)$
$ 1/2, -1/2>^\lambda$	$\frac{1}{\sqrt{6}}(-2 \downarrow\downarrow\uparrow\rangle + \downarrow\uparrow\downarrow\rangle + \uparrow\downarrow\downarrow\rangle)$

Table 3.3. Quark spin wavefunctions.

and

$$|{}^2\underline{10}, 3/2^-\rangle = \frac{1}{\sqrt{2}}(|1/2, 1/2>^0 |1, 1>^0 + |1/2, 1/2>^\lambda |1, 1>^\lambda) |\underline{10}>^S \quad (3.31b)$$

the notation is $|{}^{2S+1}\text{SU}(3)_{\text{flavour}}, J^P\rangle = |\text{Spin}\rangle |\text{Space}\rangle |\text{SU}(3)_{\text{flavour}}\rangle$.

To aid in the calculation of the spin-spin, contact, hyperfine interaction given by equation (3.28b'), it is useful to use the following identity:

$$\begin{aligned} & \langle R^{2S+1} L_{\pi} J^P | H_{SS}^G(12) | R^{2S'+1} L'_{\pi'} J^P \rangle \\ &= \frac{8\pi\alpha_s}{3\sqrt{2}m^2} \delta_{LL'} \delta_{SS'} \langle SS | \underline{s}_1 \cdot \underline{s}_2 | SS \rangle^{\pi'} \langle \psi_{LL}^{\pi} | \delta^3(\rho) | \psi_{LL}^{\pi'} \rangle, \end{aligned}$$

and in the calculation of the matrix elements of the spin-spin, tensor, hyperfine interaction, given by equation (3.28c'), it is useful to use the identity (Brink and Satchler, 1962)

$$\begin{aligned} & \langle R^{2S+1} L_{\pi} J^P | H_{SS}^T(12) | R^{2S'+1} L'_{\pi'} J^P \rangle \\ &= \frac{\alpha_s}{3\sqrt{2}m^2} (-1)^{J-L-S'} (2L+1)^{\frac{1}{2}} (2S+1)^{\frac{1}{2}} W(LL'SS':2J) \\ & \times \langle S | \frac{1}{2} (s_{1+} s_{2+} + s_{1-} s_{2-} - 4s_{1z} s_{2z}) | S' \rangle^{\pi'} \\ & \times \langle L | \frac{1}{2} (\rho^{-5} (\rho^2 - 3\rho_z^2)) | L' \rangle^{\pi'} \end{aligned} \quad (3.33)$$

where $W(LL'SS': 2J)$ is a Racah coefficient and the last two terms are reduced matrix elements: $|R^{2S+1} L_{\pi} J^P\rangle$, represents a baryon resonance R of total spin S , orbital angular momentum L , permutation symmetry π , total angular momentum J and parity P .

The contact interaction in equation (3.28b'), will elevate the $S = 3/2$ nucleon and $S = 1/2$ delta resonances relative to the $S = 1/2$ nucleon states. This is due to the fact that both the $S = 3/2N$ and $S = 1/2\Delta$ states have wavefunctions where the λ -component of the spacial wavefunction is multiplied by a flavour \otimes spin wavefunction which is symmetric under the interchange of quarks 1 and 2, see equations (3.31a and b). [Remember the part proportional to the ρ -component of the spacial wavefunction vanishes

because of the δ -function in (3.32)]: so where the contact interaction is non-vanishing, quarks 1 and 2 have total spin 1, for the $S = 3/2N$ and $S = 1/2 \Delta$ states: a similar argument shows that when the contact interaction acts for $S = 1/2N$ states - quarks 1 and 2 have total spin 0. Then using the following identity

$$\underline{s}_1 \cdot \underline{s}_2 = \frac{1}{2}(\underline{s}^2 - \underline{s}_1^2 - \underline{s}_2^2) \quad , \quad (3.34)$$

it can be seen that the contact interaction, (3.28b') will raise the $S = 3/2N$ and $S = 1/2 \Delta$ states and lower the $S = 1/2N$ states. This effect is clearly visible in the experimental spectrum for the non-strange members of the $[70, 1^-]$ supermultiplet (Isgur and Karl, 1977).

In the $N=0 [56, 0^+]$ supermultiplet the tensor interaction, (3.28c'), is absent, but here in the $N=1 [70, 1^-]$ supermultiplet it does contribute to the mass splittings and mixings. The matrix elements of the spin-spin tensor interaction are proportional to δ , the value of which was set by the ground state baryons. The only new parameter which enters the calculation is the effective oscillator frequency Ω , equation (3.20), which determines the spacing between the $N=1$ and $N=0$ levels and is taken to be ~ 440 MeV.

In this level it was found that the contact interaction was mainly responsible for the mass shifts and the tensor interaction responsible for mixing states of the same IJ^P . The agreement between the predicted masses and the experimental data for this sector is good (Isgur and Karl, 1977). However, the agreement between the authors' predicted mixing angles and those obtained from the analysis of the experimental data (Cashmore et al., 1975)

is striking; for further details see original texts.

After their successful description of the non-strange baryons in the $N=1$ $[70, 1^-]$ supermultiplet, Isgur and Karl calculate the masses and mixing angles of the $S = -1, -2$ and -3 baryons, commonly assigned to the aforementioned supermultiplet (Isgur and Karl, 1978a and b). Once again, the agreement between the theoretical predictions and the experimentally observed spectrum is remarkably good for such a naive model. In their work they also explain why in the $N=1$ level the $\Lambda(1830)^{5/2^-}$ lies above the $\Sigma^{5/2^-}$ (1765), whereas in the $N=0$ level the $\Sigma(1190)^{1/2^+}$ lies above the $\Lambda(1115)^{1/2^+}$ (Isgur and Karl, 1978b). Their argument shows the usefulness of the uds-basis, see §3.2, which is in essence this: The Σ state and the Λ state in the $N=1$ level, couple to a λ -mode and a ρ -mode oscillator respectively, to preserve the symmetry under the interchange of particles 1 and 2 (see equations (3.17 a and b)); then, as $\omega_\rho > \omega_\lambda$ see equations (3.16a and b), $M_\Lambda > M_\Sigma$ in the $N=1$ level.

However, there were one or two states in their analysis which proved difficult to accommodate in their model, which we shall discuss in §3.4.

(iii) Positive-parity Baryons in the Second Excited Level

In the $N=2$ level there are five $[SU(6), L^P]$ supermultiplets to be considered: $[20, 1^+]$, $[70, 2^+]$, $[56, 2^+]$, $[70, 0^+]$ and $[56, 0^+]$, due to the symmetry properties of the spacial wavefunctions in Table 3.1. In the first section of this chapter we saw how for the zeroth-order Hamiltonian the five $SU(6)$ flavour \otimes spin $\otimes O(3)$ supermultiplets are degenerate, see Figure 3.1. The addition of the anharmonic perturbation $U(r_{ij})$ splits these five supermultiplets into the familiar pattern, characterised by the parameter

Table 3.4 Isgur and Karl's results for the positive parity non-strange excited baryons.

State	Composition	$2S+1$ [SU(6), L ^P]	Mass
N $7/2^+$	1.00	$4[\underline{70}, 2^+]$	1955
Δ $7/2^+$	1.00	$4[\underline{56}, 2^+]$	1915
N $5/2^+$	$\begin{pmatrix} 0.88 & -0.48 & 0.01 \\ 0.48 & 0.84 & 0.27 \\ -0.13 & -0.23 & 0.96 \end{pmatrix}$	$2[\underline{56}, 2^+]$	1715
N $5/2^+$		$2[\underline{70}, 2^+]$	1955
N $5/2^+$		$4[\underline{70}, 2^+]$	2025
Δ $5/2^+$	$\begin{pmatrix} 0.94 & 0.38 \\ -0.38 & 0.94 \end{pmatrix}$	$4[\underline{56}, 2^+]$	1940
Δ $5/2^+$		$2[\underline{70}, 2^+]$	1975
N $3/2^+$	$\begin{pmatrix} -0.17 & 0.84 & -0.52 & 0.03 & 0.00 \\ 0.75 & 0.34 & 0.28 & -0.46 & 0.16 \\ 0.59 & -0.05 & -0.23 & 0.61 & -0.48 \\ -0.23 & 0.41 & 0.77 & 0.28 & -0.34 \\ 0.11 & 0.08 & 0.12 & 0.58 & 0.79 \end{pmatrix}$	$4[\underline{70}, 0^+]$	1710
N $3/2^+$		$2[\underline{56}, 2^+]$	1870
N $3/2^+$		$2[\underline{70}, 2^+]$	1955
N $3/2^+$		$4[\underline{70}, 2^+]$	1980
N $3/2$		$2[\underline{20}, 1^+]$	2060
Δ $3/2^+$	$\begin{pmatrix} 0.98 & 0.18 & -0.10 \\ -0.18 & 0.92 & -0.36 \\ 0.03 & 0.36 & 0.94 \end{pmatrix}$	$4[\underline{56}', 0^+]$	1780
Δ $3/2^+$		$4[\underline{56}, 2^+]$	1925
Δ $3/2^+$		$2[\underline{70}, 2^+]$	1975
N $1/2^+$	$\begin{pmatrix} 0.99 & 0.17 & 0.01 & 0.00 \\ -0.15 & 0.94 & -0.31 & -0.07 \\ -0.06 & 0.30 & 0.83 & 0.46 \\ 0.02 & -0.08 & -0.45 & 0.89 \end{pmatrix}$	$2[\underline{56}', 0^+]$	1405
N $1/2^+$		$2[\underline{70}, 0^+]$	1705
N $1/2^+$		$4[\underline{70}, 2^+]$	1890
N $1/2^+$		$2[\underline{20}, 1^+]$	2055
Δ $1/2^+$	$\begin{pmatrix} 0.64 & 0.77 \\ +0.77 & -0.64 \end{pmatrix}$	$2[\underline{70}, 0^+]$	1875
Δ $1/2^+$		$4[\underline{56}, 2^+]$	1925

Δ and shown in Figure 3.3. It is the addition of the hyperfine interaction Hamiltonian (3.9), which removes the degeneracy of the $SU(3)_{\text{flavour}}$ multiplets from within a given $[SU(6), L^P]$ supermultiplet, just as in the lower two levels. However, assigned to the $N=2$ level of the oscillator model are, for example, five $N^{3/2^+}$ states, seven $\Lambda^{3/2^+}$ states and eight $\Sigma^{3/2^+}$ states, which means a little more labour is required to calculate all the masses and mixing angles for this level than for the two levels discussed earlier.

Isgur and Karl (1979a) set about calculating the masses and mixing angles with the tools which have been honed to a fine edge on the calculations in the lower two levels. The values of the two parameters E_0' ($= E_0 + 2\Omega$, the unsplit $N=2$ Level) and Δ (which characterises the anharmonic symmetry breaking) are chosen empirically by the authors, giving $E_0' \sim 2020$ MeV and $\Delta \sim 420$ MeV. The value of δ which characterises the hyperfine interaction is taken to be ~ 300 MeV. Using these three values for the parameters Isgur and Karl (1978) again produce very good agreement with the experimentally observed spectrum. In Table 3.4 we give Isgur and Karl's results for the non-strange baryons which are commonly assigned to the $N=2$ level. In Figure 3.4 we show their results against current partial-wave analysis data (Cutkosky, 1980), which is obviously not the data the authors used.

Note that from Figure 3.4 the lowest-lying ΔP_{33} state appears to be a problem. This is possibly due to the complicated nature of this channel; the Particle Data Group (1982) give a possible multiple resonance from 1500 - 1900 MeV. Another point to note from the figure is that the theory predicts twenty-one states, yet only

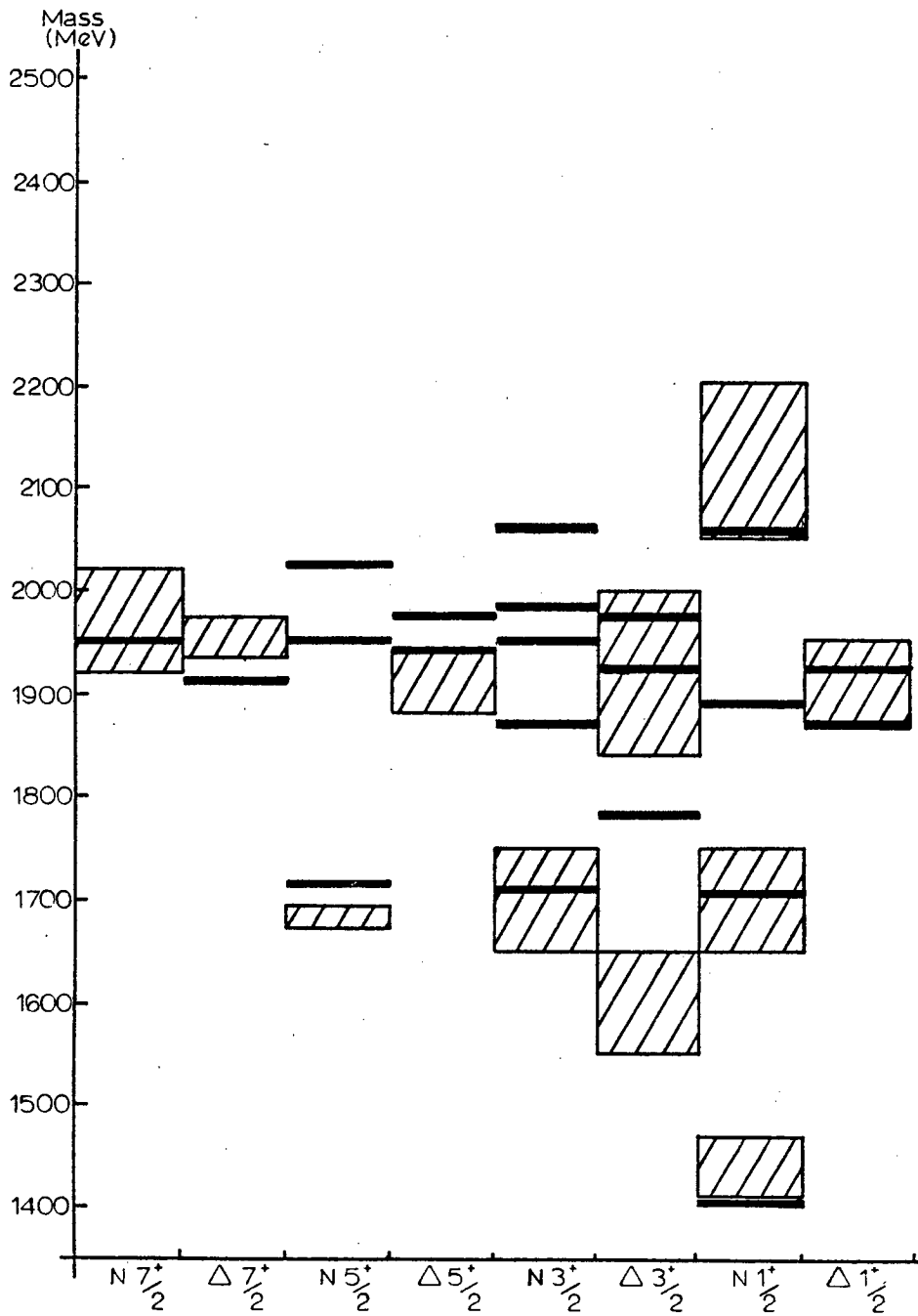


Fig. 3.4 Isgur and Karl's predicted masses for the $N = 2$ non-strange baryon masses (heavy black bars) shown against current experimental data (shaded areas)(Cutkosky 1980).

eleven (or ten) are observed experimentally. Later work, by Koniuk and Isgur (1980) on the decays of baryons, shows that most of the unobserved, or "missing", resonances decouple from the πN channel - thus they would not be observed in the πN partial-wave analyses used to determine the spectrum in Figure 3.4. The authors also found that the states that do not decouple are the states whose predicted masses and decay amplitudes are in good correspondence with the experimentally observed values; for further details, see the original text.

We have seen so far in this chapter how the harmonic oscillator quark model of Isgur and Karl (Isgur, 1980), using the Hamiltonian (3.7) has been remarkably successful in accounting for many of the features of the presently observed spectrum up to and including the second excited level - this success extends to the $N=3$ level, see Corvi (1981) and references therein. For the rest of this chapter we shall explore the implications of the neglect of the spin-orbit interactions which may arise through one-gluon-exchange and confinement, see equations (2.25) and (2.28) respectively.

3.4 Spin-Orbit Interaction in the Baryon Spectrum

It may have been a good idea to put the title of this section in an interrogative form, as we have seen in the previous two sections how the model of Isgur and Karl (Isgur, 1980) has been remarkably successful in accounting for most of the features of the experimentally observed spectrum. However, in their model Isgur and Karl use only the spin-spin hyperfine interactions, coming from one-gluon-exchange, see equation (3.9). To be consistent with the original philosophy of these 'QCD-inspired' interactions, the

spin-orbit interactions, from one-gluon-exchange and from confinement, should be included in the calculation of baryon masses. In this section we investigate the effect of including these spin-orbit interactions on the baryons commonly assigned to the first two excited levels.

From equation (2.25), the spin-orbit interaction obtained from the non-relativistic reduction of one-gluon exchange is:

$$H_{SO}^g(ij) = \frac{1}{r_{ij}} \frac{dV_g(r_{ij})}{dr_{ij}} \times \left[\frac{\underline{s}_i \cdot \underline{r}_{ij} \times \underline{p}_i}{2m_i^2} - \frac{\underline{s}_j \cdot \underline{r}_{ij} \times \underline{p}_j}{2m_j^2} - \frac{\underline{s}_i \cdot \underline{r}_{ij} \times \underline{p}_j - \underline{s}_j \cdot \underline{r}_{ij} \times \underline{p}_i}{m_i m_j} \right] \quad (3.35)$$

and from confinement is:

$$H_{SO}^c(ij) = \frac{1}{r_{ij}} \frac{dV_c(r_{ij})}{dr_{ij}} \left[\frac{\underline{s}_i \cdot \underline{r}_{ij} \times \underline{p}_i}{2m_i^2} - \frac{\underline{s}_j \cdot \underline{r}_{ij} \times \underline{p}_j}{2m_j^2} \right] \quad (3.36)$$

In the equal-mass limit we can write equation (3.35) in the form:

$$H_{SO}^g(ij) = \frac{-1}{4m^2} \frac{1}{r_{ij}} \frac{dV_g(r_{ij})}{dr_{ij}} [3(\underline{s}_i + \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i - \underline{p}_j) - (\underline{s}_i - \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i + \underline{p}_j)] \quad (3.37)$$

and equation (3.26) in the form:

$$H_{SO}^c(ij) = \frac{-1}{4m^2} \frac{1}{r_{ij}} \frac{dV_c(r_{ij})}{dr_{ij}} [(\underline{s}_i + \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i - \underline{p}_j) + (\underline{s}_i - \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i + \underline{p}_j)] \quad (3.38)$$

Note that in equations (3.37) and (3.38) there are terms which are proportional to $(\underline{p}_i + \underline{p}_j)$, which is not invariant under Lorentz transformations. It was originally thought that, because of this non-invariance, these terms had been somehow spuriously generated

by the non-relativistic reductions and thus were neglected. However, Close and Osborn (1970) and others have stressed the need for great care concerning the relationship between the centre-of-mass variables and the laboratory variables when the particles under consideration have intrinsic spin. Conventionally, the calculation and specification of these two-body interactions are carried out in the rest frame of the two particles. Here we are going to calculate the matrix elements of the spin-orbit interactions, equations (3.35) and (3.36), in the rest frame of the baryon which is not the rest frame of the two quarks i and j in which the spin-orbit interactions are derived. The authors show for an N -body system the correct internal degrees of freedom, \underline{r}_{in} , as measured in the rest frame of the two particle system, are given by

$$\underline{r}_{in} = \underline{R} + \sum_{n=1}^N \frac{m_n}{M} \underline{r}_{in} + \frac{1}{2M} \sum_{n=1}^N \left(\frac{s_n}{m_n} - \frac{s_i}{m_i} \right) \times \underline{p}_i \quad ; \quad (3.39)$$

in the limit $\underline{p}_i \rightarrow 0$ the above expression reduces to

$$\underline{r}_i = \underline{R} + \sum_{n=1}^N \frac{m_n}{M} \underline{r}_{in} \quad (3.40)$$

the more familiar Galilean result, where \underline{R} is the centre-of-mass co-ordinate and $M = \sum_{n=1}^N m_n$, is the total mass. For the equal mass case we obtain from equation (3.39),

$$|\underline{r}_i - \underline{r}_j| = |\underline{r}_{ij}| + \frac{1}{2Mm} (s_i - s_j) \cdot \frac{\underline{r}_{ij}}{|\underline{r}_{ij}|} \times \underline{p} + \dots \quad (3.41)$$

where \underline{p} is the momentum of the centre-of-mass, $(\underline{r}_i - \underline{r}_j)$ denotes the relative position of particles i and j in the centre-of-mass frame of the N -body system and \underline{r}_{ij} is the separation in the rest

frame of particles i and j . So care must be taken to differentiate between $\underline{r}_i - \underline{r}_j$ and \underline{r}_{ij} in the leading (spin-independent) potential term $V(\underline{r}_i - \underline{r}_j)$, i.e.

$$V(\underline{r}_i - \underline{r}_j) = V(\underline{r}_{ij}) + \frac{1}{2Mm} (\underline{s}_i - \underline{s}_j) \cdot \underline{r}_{ij} \times \underline{P} \frac{1}{r_{ij}} \frac{dV(r_{ij})}{dr_{ij}}, \quad (3.42)$$

(Reinders, 1980). Taking this into account the correct expressions for the spin-orbit interactions (3.35) and (3.36) are

$$H_{SO}^{\frac{1}{2}}(ij) = \frac{1}{4m^2} \frac{1}{r_{ij}} \frac{dV_{\frac{1}{2}}(r_{ij})}{dr_{ij}} [3(\underline{s}_i + \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i - \underline{p}_j) - (\underline{s}_i - \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i + \underline{p}_j - \frac{2m}{M} \underline{P})] \quad (3.43)$$

and

$$H_{SO}^C(ij) = \frac{1}{4m^2} \frac{1}{r_{ij}} \frac{dV_C(r_{ij})}{dr_{ij}} [(\underline{s}_i + \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i - \underline{p}_j) + (\underline{s}_i - \underline{s}_j) \cdot \underline{r}_{ij} \times (\underline{p}_i + \underline{p}_j - \frac{2m}{M} \underline{P})] \quad (3.44)$$

The spin-orbit interactions given by equations (3.43) and (3.44) are now Lorentz invariant as required. It is also possible to show that the spin-orbit interactions for unequal masses, see equations (3.35) and (3.36), are also Lorentz invariant, when the correct internal co-ordinates are used (Reinders, 1980).

(i) Negative-parity Baryons in the First Excited Level (Revisited)

The effect of including the spin-orbit interactions in the calculation of the masses of the baryons commonly assigned to the $[70, 1^-]$ supermultiplet, has been a subject of discussion in recent years, e.g. Reinders (1980), Gromes (1980) and Close and Dalitz (1981). Isgur and Karl (1978a) also considered the effect on their

results for this sector, of including the spin-orbit interactions from one-gluon-exchange and confinement, equations (3.35) and (3.36) respectively. However, they found the inclusion of these spin-orbit effects perturbed their results too much, so they neglected them in their final analysis. In this sub-section we investigate again the effect of including the spin-orbit interactions in the calculation of the masses of some of the baryons in the $[70, 1^-]$ supermultiplet.

For convenience we only consider the equal mass case, where it is possible to use the simplifying identity given in equation (3.29), thus we write

$$H_{SO}^{\frac{1}{2}}(12) = \frac{\alpha_s}{2\sqrt{2}\rho^3 m^2} ((\underline{s}_1 + \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\rho - \frac{1}{3\sqrt{3}} (\underline{s}_1 - \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\lambda) \quad (3.45)$$

and

$$H_{SO}^c(12) = \frac{1}{4m^2} \frac{1}{\rho} \frac{dV_c(\rho)}{d\rho} ((\underline{s}_1 + \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\rho + \frac{1}{\sqrt{3}} (\underline{s}_1 - \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\lambda), \quad (3.46)$$

where the complete permutational symmetry of the non-strange sector allows us to use the following identity:

$$\sum_{ij} \langle H_{SO}(ij) \rangle = 3 \langle H_{SO}(12) \rangle \quad (3.47)$$

and we have substituted equation (2.30) for $V(r_{ij})$. It is convenient to consider the two spin-orbit terms proportional to $\underline{\rho} \times \underline{p}_\rho = (\underline{L}_\rho)$, from equations (3.45) and (3.46), together so we write:

$$H_{SO}^{2B}(12) = \frac{1}{2m^2} \left(\frac{\alpha_s}{\sqrt{2}\rho^3} - \frac{1}{2\rho} \frac{dV_c(\rho)}{d\rho} \right) (\underline{s}_1 + \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\rho, \quad (3.48)$$

similarly for the two terms proportional to $\underline{\rho} \times \underline{p}_\lambda$ we write:

$$H_{SO}^{3B}(12) = \frac{-1}{2\sqrt{3}m^2} \left(\frac{\alpha_s}{3\sqrt{2}\rho^3} + \frac{1}{2\mathbf{p}} \frac{dV_c(\rho)}{d\rho} \right) (\underline{s}_1 - \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\lambda \quad (3.49)$$

The spin-orbit interactions $H_{SO}^{2B}(12)$ and $H_{SO}^{3B}(12)$, defined in equations (3.48) and (3.49) respectively, are commonly referred to in the literature as two-body and three-body spin-orbit interactions respectively. The terminology "three-body spin-orbit interaction" is somewhat misleading: the name was coined as H_{SO}^{3B} contains the momentum, \underline{p}_λ , associated with the three-body coordinate λ , as defined in equation (3.16), but the spin-orbit interaction in (3.49) called H_{SO}^{3B} is still only an interaction between quarks 1 and 2 (Close and Dalitz, 1981).

Isgur and Karl (1978b) calculate the effect of including in their Hamiltonian spin-orbit interactions from one-gluon-exchange and confinement, equations (3.45) and (3.46) respectively,

$$H_{SO}^T(12) = H_{SO}^{\cancel{E}}(12) + H_{SO}^C(12) = H_{SO}^{2B}(12) + H_{SO}^{3B}(12) \quad (3.50)$$

The authors assume a harmonic confining potential, i.e.

$$V_c(r_{ij}) = \frac{1}{2} K r_{ij}^2, \quad (3.51)$$

then (3.48) becomes

$$H_{SO}^{2B}(12) = \frac{1}{2m^2} \left(\frac{\alpha_s}{\sqrt{2}\rho^3} - K \right) (\underline{s}_1 + \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\rho, \quad (3.52)$$

and similarly (3.49) becomes

$$H_{SO}^{2B}(12) = \frac{1}{6\sqrt{3}m^2} \left(\frac{\alpha_s}{\sqrt{2}\rho^3} + 3K \right) (\underline{s}_1 - \underline{s}_2) \cdot \underline{\rho} \times \underline{p}_\lambda \quad (3.53)$$

The authors found that the matrix elements of the two-body spin-orbit interaction, $H_{SO}^{2B}(12)$, are proportional to $(\delta - \gamma)$ and the

matrix elements of the three-body spin-orbit interactions, $H_{SO}^{3B}(12)$, are proportional to $(\delta + 3\gamma)$, where δ is the spin-spin parameter defined by equation (3.30) and

$$\gamma = \frac{K}{m^2} \quad (3.54)$$

With this choice of confining potential the two parameters which define the spin-orbit interaction, δ and γ , are already determined, so no new free parameters enter the calculation. The value of $\delta \sim 300$ MeV is fixed by the $\Delta - N$ mass difference and $\gamma \sim 215$ MeV which is fixed by the value of the spring constant. Thus $(\delta - \gamma) \sim 95$ MeV and $(\delta + 3\gamma) \sim 945$ MeV, are the values which Isgur and Karl (1978a) use to calculate the spin-orbit effects in the baryons belonging to the $[70, 1^-]$. The authors found that the near cancellation of the two-body parts, coming from $H_{SO}(12)$ and $H_{SO}^C(12)$, made it easy to include H_{SO}^{2B} into their Hamiltonian, (3.7), without perturbing their results too much. However, the large size of these three-body spin-orbit terms, due to the contributions from $H_{SO}^C(12)$ and $H_{SO}(12)$ adding constructively, meant that its inclusion perturbed their results too much.

Close and Dalitz (1981) succinctly expressed the difficulties and quandaries caused by the inclusion of the spin-orbit interaction in the calculation of the masses of the P-wave baryons by considering three mass splittings:

(a) The $\Delta D_{33} - \Delta S_{31}$ Mass Difference

From Table 3.5 (Isgur and Karl, 1978b) it can be seen that if Isgur and Karl's Hamiltonian, (3.7) which only contains spin-spin hyperfine interactions, is employed, these two states are predicted

to be degenerate,

$$M_{\Delta D33} = E_o + \Omega + \frac{1}{4}\delta = M_{\Delta S31} \approx 1685 \text{ MeV.} \quad (3.55)$$

The Particle Data Group (1982) give the masses of the $\Delta D33$ and the $\Delta S31$ to be in the ranges 1630 - 1740 MeV and 1600 - 1650 MeV, respectively, which would seem to confirm, within limits, the above prediction. However, looking at each πN -scattering phase-shift analysis separately shows that they all predict a mass splitting in the range 80 - 100 MeV, but differ in the absolute value of the masses. Thus, it appears that the degeneracy predicted by Isgur and Karl is not observed empirically.

If the spin-orbit interactions, (3.52) and (3.58) are taken into account the mass splitting, from Table 3.5, is

$$M_{\Delta D33} - M_{\Delta S31} = \frac{1}{4}(\delta + 3\gamma) \sim 235 \text{ MeV,} \quad (3.56)$$

which is the correct sign, but the wrong magnitude, i.e. it is almost about three times too big.

(b) The ND15 - $\Delta S31$ Mass Difference

From Table 3.5 it may be seen that the masses of these two states are as follows:

$$M_{ND15} = E_o + \Omega + \frac{1}{4}\delta - \frac{1}{20}\delta \approx 1670 \text{ MeV,} \quad (3.57a)$$

$$M_{\Delta S31} = E_o + \Omega + \frac{1}{4}\delta \approx 1685 \text{ MeV,} \quad (3.57b)$$

using Isgur and Karl's Hamiltonian, equation (3.7), which only contains the spin-spin hyperfine interactions. Hence the predicted splitting

State	$\langle H_{SS}^C \rangle$	$\langle H_{SS}^T \rangle$	$\langle H_{SO}^{2B} \rangle$	$\langle H_{SO}^{3B} \rangle$
$\Delta D33$	$\frac{1}{4} \delta$	0	0	$\frac{1}{4}(\delta+3\gamma)$
$\Delta 331$	$\frac{1}{4} \delta$	0	0	$-\frac{1}{6}(\delta+3\gamma)$
ND15	$\frac{1}{4} \delta$	$-\frac{1}{20} \delta$	$\frac{3}{4}(\delta-\gamma)$	0
$\Lambda D03$	$\frac{1}{4} \delta$	0	$\frac{1}{2}(\delta-\gamma)$	$-\frac{1}{12}(\gamma+3\gamma)$
$\Lambda S01$	$\frac{1}{4} \delta$	0	$-(\delta-\gamma)$	$\frac{1}{6}(\delta+3\gamma)$

Table 3.5: Matrix elements of the spin-spin, contact and tensor, interactions, the two-body and three-body spin-orbit interactions for selected P-wave baryons.

is $\sim \pm 20$ MeV. The Particle Data Group (1982) gives the masses of the ND15 and the $\Delta S31$ to be in the ranges 1660 - 1690 MeV and 1600 - 1650 MeV, respectively. Therefore, Isgur and Karl's splitting is the correct magnitude, but the incorrect sign.

The inclusion of the spin-orbit interactions gives

$$M_{ND15} - M_{\Delta S31} = \frac{1}{20} \delta + \frac{3}{4}(\delta-\gamma) + \frac{1}{6}(\delta+3\gamma) \sim 205 \text{ MeV}, \quad (2.58)$$

see Table 3.5. Now the splitting is of the correct sign, but is an order of magnitude too big.

(c) The Λ_{DO3} - Λ_{SO1} Mass Difference

Using the limit of exact $SU(3)_{\text{flavour}}$ symmetry, Isgur and Kari (1978a) calculate that these two states are degenerate (see Table 3.5),

$$M_{\Lambda_{DO3}} = M_{\Lambda_{SO1}} = E_0 + \Omega + \frac{1}{4}\delta \sim 1490 \text{ MeV} . \quad (3.59)$$

The experimental masses for these two states are ~ 1520 MeV for the Λ_{DO3} and ~ 1405 MeV for the Λ_{SO1} (Particle Data Group, 1982), thus empirically the splitting is ~ 115 MeV. The inclusion of the spin-orbit interactions (which are also calculated in the limit of exact $SU(3)_{\text{flavour}}$ symmetry), giving

$$M_{\Lambda_{DO3}} - M_{\Lambda_{SO1}} = \frac{3}{2}(\delta - \gamma) - \frac{1}{2}(\delta + 3\gamma) \approx -110 \text{ MeV} \quad (3.60)$$

from Table 3.5, which is the correct magnitude, but the incorrect sign. Close and Dalitz (1981) repeated this calculation, taking into account the strange quark mass, and found that in this case the predicted splitting was ~ -190 MeV. Isgur and Karl (1978b) suggest that the low mass of the Λ_{SO1} may be due to its proximity to the $\bar{K}N$ threshold, as it is coupled strongly to this channel.

Thus what conclusions can be drawn about the spin-orbit interactions from these three examples?

It is evident that the spin-orbit interaction has to be present, both two-body and three-body, to lift the degeneracy of certain states, which would be degenerate with only spin-spin hyperfine interactions. But if they are to be present it is at a level which is greatly reduced (10% - 20%) than one would expect if the non-relativistic reductions of one-gluon-exchange and confinement are

to be taken at face value. The $\Delta D33 - \Delta S31$ splitting can only be accounted for by 3-body spin-orbit interactions in such a model, but they would have to be present to only 25-30% of their full value. The $\Delta D03 - \Delta S01$ splitting cannot be accounted for using spin-spin interactions, hence is attributed to spin-orbit effects, even though the splitting produced as the right magnitude but the wrong sign.

The above results were obtained using a harmonic confining potential, but as was seen earlier Isgur and Karl (1978b) do not expect this to be the true confining potential in a baryon and hence the harmonic oscillator wavefunctions may not be an appropriate basis. Corvi (1981b) considered the effect of relaxing the constraints of a harmonic oscillator confining potential and wavefunctions.; the results for the $N=1$ $[70,1^-]$ supermultiplet is shown in Table 3.6. [This method follows the analysis of the P-wave baryons by Reinders (1978), who did not include three-body spin-orbit forces]. The parameters in Table 3.6 are defined below:

$$D_1 = \frac{-\alpha_s}{2m^2} \rho \langle 1 || \rho^{-3} (\underline{\rho} \times \underline{p}_\rho) || 1 \rangle^\rho, \quad (3.61a)$$

$$D_1 = \frac{-\alpha_s}{m^2} \rho \langle 1 || \rho^{-3} (\underline{\rho} \times \underline{p}_\lambda) || 1 \rangle^\lambda \quad (3.61b)$$

$$F_1 = \frac{-1}{2\sqrt{2}m^2} \rho \langle 1 || \frac{1}{\rho} \frac{dV_c(\rho)}{d\rho} (\underline{\rho} \times \underline{p}_\rho) || 1 \rangle^\rho \quad (3.61c)$$

$$F_2 = \frac{-1}{\sqrt{2}m^2} \rho \langle 1 || \frac{1}{\rho} \frac{dV_c(\rho)}{d\rho} (\underline{\rho} \times \underline{p}_\lambda) || 1 \rangle^\lambda. \quad (3.61d)$$

State	H_{SO}^{2B}	H_{SO}^{3B}
${}^4_8, 5/2^-$	$\frac{3}{4}(D_1 - F_1)$	0
${}^2_{10}, 3/2^-$	0	$\frac{1}{12}(D_2 + 3F_2)$
${}^4_8, 3/2^-$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{10}}{8} \\ -\frac{\sqrt{10}}{8} & \frac{1}{4} \end{pmatrix} (D_1 - F_1)$	$\begin{pmatrix} 0 & \frac{\sqrt{10}}{24} \\ \frac{\sqrt{10}}{24} & 0 \end{pmatrix} (D_2 + 3F_2)$
${}^2_8, 3/2^-$		
${}^2_1, 3/2^-$	$\frac{1}{2}(D_1 - F_1)$	$-\frac{1}{12}(D_2 + 3F_2)$
${}^2_{10}, 1/2^-$	0	$-\frac{1}{6}(D_2 + 3F_2)$
${}^4_8, 1/2^-$	$\begin{pmatrix} -\frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} (D_1 - F_1)$	$\begin{pmatrix} 0 & \frac{1}{12} \\ \frac{1}{12} & 0 \end{pmatrix} (D_2 + 3F_2)$
${}^2_8, 1/2^-$		
${}^2_1, 1/2^-$	$-(D_1 - F_1)$	$\frac{1}{6}(D_2 + F_2)$

Table 3.6: Two-body and three-body spin-orbit matrix elements for the P-wave baryons, with arbitrary wavefunctions and confining potential.

If the constraint of using harmonic oscillator wavefunctions is not relaxed, then the four parameters defined above reduce to:

$$D_1 = D_2 = \delta, \quad (3.62a)$$

$$F_1 = F_2. \quad (3.63b)$$

Gromes and Stamatescu (1978) assumed various forms for the confining potential, $V_c(r_{ij})$, such as a power-law potential, i.e.

$$V_c(r_{ij}) = \sum_{i < j} \lambda r_{ij}^k \quad (3.63a)$$

or a logarithmic potential, i.e.

$$V_c(r_{ij}) = \sum_{i < j} \lambda \log\left(\frac{r_{ij}}{r_0}\right), \quad \lambda > 0. \quad (3.63b)$$

However it is still not possible to obtain a value of F_1 ($=F_2$) which gives the correct splitting for the three examples expounded earlier.

By treating the four parameters D_1 , D_2 , F_1 and F_2 , as defined in equations (3.61a - d), as free, then it is possible to assign them values which would account for the troublesome splittings. However, this is somewhat of a Pyrrhic victory (as Corvi points out) as in relaxing the constraint of harmonic oscillator wavefunctions, the naivety and predictive power of the model are lost, which were the major reasons for using the harmonic oscillator model. Also the original philosophy of including these relativistic corrections from one-gluon-exchange and confinement has been somewhat 'swept under the carpet'.

(ii) Positive-parity Non-strange Baryons in the Second Excited Level
(Revisited)

In the last subsection we saw how the inclusion of the spin-orbit interactions, arising from the two non-relativistic decompositions, into Isgur and Karl's Hamiltonian, equation (3.7), causes a few "catch-22" problems in the P-wave baryons. It is possible that some

of these problems are related to the difficulty of experimentally determining the absolute masses of resonances. However, if one is to take the philosophy of the "QCD-inspired" interactions seriously, then the spin-orbit interactions, coming from one-gluon exchange and confinement equations (3.45) and (3.46), should be included as we said earlier. So in this subsection we feel that this dilemma is of sufficient interest to warrant a further investigation of the non-strange baryons in the $N=2$ level, this time including the spin-orbit interactions. Yet again there is little or no experimental evidence in this sector to suggest the presence of these spin-orbit interactions. This can be seen in Table 3.7, the four degenerate Δ -resonances and the two degenerate N -resonances can be accommodated within the framework of Isgur and Karl's model (1979a), which only contains the spin-spin hyperfine interactions. (The experimental data is from Cutkosky (1980) which is not the data the authors originally used.)

Before analysing the effect of including the spin-orbit interactions on the $N=2$ non-strange baryons we will now give the details of the calculation of a matrix element, for the interested reader. The state we choose as an example is the ${}^4\Delta[56,2^+]_{7/2^+}$:

The total spin-orbit Hamiltonian is

$$H_{SO}(ij) = H_{SO}^{\mathcal{G}}(ij) + H_{SO}^C(ij) = H_{SO}^{2B}(ij) + H_{SO}^{3B}(ij) \quad (3.64)$$

where $H_{SO}^{\mathcal{G}}(ij)$ and $H_{SO}^C(ij)$ are, respectively, the spin-orbit interactions from one-gluon-exchange (equation (3.35)), and confinement (equation (3.36)). It is convenient to separate the calculation

State	$\Delta F37$	$\Delta F35$	$\Delta P33$	$\Delta P31$	NF15	NP13
Predicted Mass (MeV)	1955	1940	1925	1925	1715	1710
Experimental Mass (MeV)	1950±15	1910±30	1920±80	1910±10	1680±10	1700±50

Table 3.7: Isgur and Karl's results and experimentally observed masses for some non-strange baryons in the N=2 level.

into two-body and three-body spin-orbit interactions, see equations (3.48) and (3.49). The first step is to construct a $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ wavefunction which is totally symmetric under the interchange of any two quarks, which also possess the correct quantum numbers. For our example

$$|{}^4\Delta, 7/2^+\rangle = |2^S, 3/2^S, 7/2^+\rangle |10\rangle^S \quad (3.65a)$$

the notation is due to Corvi (1981b),

$$|{}^{2S+1}R, J^P\rangle = |L^\alpha, S^\beta, J^P\rangle |SU(3)\rangle^\gamma, \quad (3.65b)$$

R denotes the type of resonance, i.e. N or Δ , J^P the total angular momentum, $\underline{L} = \underline{L}_\rho + \underline{L}_\lambda$, the orbital angular momentum, s the total quark spin and finally α, β, γ give the permutation symmetry of that part of the wavefunction. (Note that we have chosen the easiest example as all part of the wavefunction are permutationally symmetric).

The spin-orbit splitting for this state is

$$E_{SO}({}^4\Delta, 7/2^+) = \sum_{i<j} \langle {}^4\Delta, 7/2^+ | H_{SO}(ij) | {}^4\Delta, 7/2^+ \rangle, \quad (3.66)$$

where $H_{SO}(ij)$ is given by equation (3.64). The complete permutational symmetry of the non-strange sector allows us to write

$$E_{SO}({}^4\Delta, 7/2^+) = 3 \langle {}^4\Delta, 7/2^+ | H_{SO}(12) | {}^4\Delta, 7/2^+ \rangle \quad (3.67a)$$

$$3 \langle {}^4\Delta, 7/2^+ | (H_{SO}^{2B}(12) + H_{SO}^{3B}(12)) | {}^4\Delta, 7/2^+ \rangle. \quad (3.67b)$$

When the state being considered belongs to the 56-dimensional or the 20-dimensional representation of $SU(6)_{\text{flavour}} \otimes \text{spin}$ the matrix

element of the three-body spin-orbit interaction, equation (3.49), automatically vanishes from symmetry arguments. As the state being considered in this example belongs to the $[56, 2^+]$ supermultiplet, the problem reduces to

$$E_{SO}(^4\Delta, 7/2^+) = 3 \langle ^4\Delta, 7/2^+ | H_{SO}^{2B}(12) | ^4\Delta, 7/2^+ \rangle \quad (3.68a)$$

and using equation (3.65a)

$$= 3 \langle 2^S, 3/2^S, 7/2^+ | H_{SO}^{2B}(12) | 2^S, 3/2^S, 7/2^+ \rangle, \quad (3.68b)$$

remembering that the spin-orbit interaction is not flavour dependent, i.e. it is a $SU(3)$ flavour singlet. Using equation (3.45) for $H_{SO}^{2B}(12)$ gives:

$$E_{SO}(^4\Delta, 7/2^+) = \frac{3}{2m^2} \langle 2^S, 3/2^S, 7/2^+ | \left(\frac{\alpha_S}{\sqrt{2}\rho^3} - \tilde{V}_c(\rho) \right) (\underline{s}_1 + \underline{s}_2) \cdot \underline{p} \times \underline{p}_\rho \times | 2^S, 3/2^S, 7/2^+ \rangle \quad (3.69)$$

where $\tilde{V}_c(\rho) = \frac{1}{2\rho} \frac{dV_c(\rho)}{d\rho}$. Throughout the calculation of this matrix element, and for the other states in this section, we shall not assume any specific form for the confining potential, $V_c(\rho)$, leaving discussion fairly general whilst working within an oscillator framework.

To evaluate the matrix element in equation (3.69), we use the general formula for calculating the matrix element of the scalar product of two rank K tensors (Brink and Satchler, 1962):

$$\begin{aligned} & \langle LSJ | R_K S_K | L' S' J' \rangle \\ &= \delta_{JJ'} (-1)^{J-L-S'} (2L+1)^{\frac{1}{2}} (2S+1)^{\frac{1}{2}} W(LL'SS':KJ) \\ & \quad \times \langle L || R_K || L' \rangle \langle S || S_K || S' \rangle, \end{aligned} \quad (3.70)$$

where $\langle L \| R_K \| L' \rangle$ and $\langle S \| S_K \| S' \rangle$ are reduced matrix elements, which may be calculated using the Wigner-Eckart theorem and $W(LL'SS':KJ)$ is a Racah coefficient. (This is the same identity that is used to calculate the tensor spin-spin hyperfine interaction, see equation (3.33)). Using the above identity, equation (3.69) reduces to:

$$E_{SO}(^4\Delta, 7/2^+) = \frac{3}{2m^2} (-1)^{7/2-2-3/2} \sqrt{5} \cdot \sqrt{4} \cdot W(22\frac{3}{2}\frac{3}{2}; 17/2) \\ \times {}^s\langle 3/2 \| (\underline{s}_1 + \underline{s}_2) \| 3/2 \rangle^s {}^s\langle 2 \| \left(\frac{\alpha_s}{\sqrt{2}\rho^3} - \tilde{V}_c(\rho) \right) \underline{p} \times \underline{p}_p \| 2 \rangle^s, \quad (3.71)$$

as we are dealing with vector operators $K = 1$. The Racah coefficient is evaluated using the following identity (Brink and Satchler 1962)

$$W(aa\ bb:1c) = (-1)^{a+b+c-1} \frac{a(a+1) + b(b+1) + c(c+1)}{[4a(a+1)(2a+1)b(b+1)(2b+1)]^{1/2}} \quad (3.72a)$$

which gives

$$W(22\ \frac{3}{2}\ \frac{3}{2}; 17/2) = \frac{1}{5\sqrt{2}} \quad (3.72b)$$

The spin-dependent reduced matrix element in (3.71) is evaluated by using the Wigner-Eckart theorem and the spin wavefunctions in Table 3.3.

$${}^s\langle \frac{3}{2} \| (\underline{s}_1 + \underline{s}_2) \| \frac{3}{2} \rangle^s = \frac{{}^s\langle \frac{3}{2}, \frac{3}{2} | (\underline{s}_1 + \underline{s}_2)_z | \frac{3}{2}, \frac{3}{2} \rangle^s}{{}^s\langle \frac{3}{2}, \frac{3}{2} \ 1, 0; \frac{3}{2}, \frac{3}{2} \rangle} = -\sqrt{\frac{5}{3}} \quad (3.73)$$

Similarly, for the other reduced matrix element:

$${}^s\langle 2 || \left(\frac{\alpha_s}{\sqrt{2}\rho^3} - \tilde{V}_c(\rho) \right) \underline{\rho} \times \underline{p}_\rho || 2 \rangle^s = {}^s\langle 2,2 | \left(\frac{\alpha_s}{\sqrt{2}\rho^3} - \tilde{V}_c(\rho) \right) (\underline{\rho} \times \underline{p}_\rho)_z | 2,2 \rangle^s$$

$$\times \langle 2,2 | 1,0; 2,2 \rangle^{-1} \quad (3.74)$$

from Table 3.1 the required special wavefunction is,

$$\psi_{22}^s = \frac{\alpha^5}{2\pi} (\rho_+^2 + \lambda_+^2) \exp\{-\frac{1}{2}\alpha^2(\rho^2 + \lambda^2)\}$$

or

$$|2,2\rangle^s = |2,2\rangle_\rho + |2,2\rangle_\lambda,$$

identifying $\underline{\rho} \times \underline{p}_\rho = \underline{L}_\rho$ then equation (3.74) reduces to

$$- \frac{3}{2} \cdot 2 \cdot \frac{\alpha^{10}}{4\pi^3} \int d^3\rho d^3\lambda \rho_+^2 \rho_-^2 \left(\frac{\alpha_s}{\sqrt{2}\rho^3} - \tilde{V}_c(\rho) \right) e^{-\alpha^2(\rho^2 + \lambda^2)}$$

$$= - \frac{1}{5} \sqrt{\frac{2}{3}} (3\delta - 4\tau) m^2 \quad (3.75)$$

where δ is Isgur and Karl's spin-spin parameter, defined in equation (3.30), and the mass parameter

$$\tau = \frac{\alpha^7}{m^2\sqrt{\pi}} \int_0^\infty d\rho \tilde{V}_c(\rho) \rho^6 e^{-\alpha^2\rho^2} = \frac{\alpha^7}{m^2\sqrt{\pi}} \int_0^\infty d\rho \left[\frac{dV_c(\rho)}{d\rho} \right] \rho^5 e^{-\alpha^2\rho^2}$$

$$(3.76)$$

Putting together equations (3.75) and (3.72b) and substituting for the required parts in (3.71) leads to (eventually):

$$E_{SO}(^4\Delta, 7/2^+) = \frac{1}{5}(3\delta - 4\tau) = \frac{1}{5}A, \quad (3.77)$$

where the parameter A is defined below.

All the other spin-orbit matrix elements may be calculated in a similar fashion with a large amount of pencil lead. Note that the

form of the identity (3.70) and the fact that the spin-orbit interaction is a $SU(3)_{\text{flavour}}$ singlet allows states of the same I and J to mix. Apart from τ there is only one other new parameter which is required for a complete description of this sector, which is:

$$\epsilon = \frac{\alpha^5}{m^2 \sqrt{\pi}} \int_0^{\infty} d\rho \left[\frac{dV_c(\rho)}{d\rho} \right] \rho^3 e^{-\alpha^2 \rho^2} \quad (3.78)$$

All the matrix elements of the two-body spin-orbit interaction are proportional to

$$A = (3\delta - 4\tau) \quad \text{and} \quad B = (3\delta - 4\epsilon), \quad (3.79a)$$

and all the matrix elements of the three-body spin-orbit interaction are proportional to

$$C = (\delta + 4\tau) \quad \text{and} \quad D = (\delta + 4\epsilon). \quad (3.79b)$$

The matrix elements of the spin-orbit interaction, both two-body and three-body, equations (3.48) and (3.49), from one-gluon-exchange and confinement for the non-strange baryons in the $N=2$ level are given in Table 3.8. Unfortunately, the strange resonances in this level have not been studied due to a lack of stamina in the author.

To re-calculate the masses in this sector we have to collate the matrix elements of the oscillator Hamiltonian, anharmonic perturbation and spin-spin hyperfine interactions, from Isgur and Karl's work (Isgur and Karl, 1979a), with the matrix elements of the spin-orbit interaction in Table 3.8. In their analysis of this sector Isgur and Karl (1979a) use three parameters to calculate the

Table 3.8: Spin-orbit matrix elements for the non-strange baryons in the $N = 2$ level.

${}^4_8[70, 2^+] \frac{7^+}{2}$	$\frac{1}{10}A + \frac{1}{4}B$			
${}^4_{10}[56, 2^+] \frac{7^+}{2}$	$\frac{1}{5}A$			
${}^4_8[56, 2^+] \frac{5^+}{2}$	$\frac{1}{15}A$	$\frac{1}{3\sqrt{2}} \left(-\frac{1}{5}A + \frac{1}{10}C + \frac{1}{4}D \right)$	$\frac{\sqrt{14}}{12} \left(-\frac{1}{5}A + \frac{1}{10}C + \frac{1}{4}D \right)$	}
${}^2_8[70, 2^+] \frac{5^+}{2}$	$\frac{1}{6} \left(\frac{1}{5}A + \frac{1}{2}B \right)$	$\frac{\sqrt{7}}{12} \left(\frac{1}{5}A - \frac{1}{2}B + \frac{1}{5}C - \frac{1}{2}D \right)$		
${}^4_8[70, 2^+] \frac{5^+}{2}$		$-\frac{1}{12} \left(\frac{1}{5}A + \frac{1}{2}B \right)$		
${}^4_{10}[56, 2^+] \frac{5^+}{2}$	$-\frac{1}{30}A$	$\frac{\sqrt{7}}{6} \left(-\frac{1}{5}A + \frac{1}{10}C + \frac{1}{4}D \right)$		
${}^2_{10}[70, 2^+] \frac{5^+}{2}$		$\frac{1}{3} \left(\frac{1}{5}A + \frac{1}{10}C - \frac{1}{4}D \right)$		
${}^4_{10}[56', 0^+] \frac{3^+}{2}$	0	0	0	}
${}^4_{10}[56, 2^+] \frac{3^+}{2}$		$-\frac{1}{5}A$	$\frac{1}{2\sqrt{2}} \left(-\frac{1}{5}A + \frac{1}{10}C + \frac{1}{4}D \right)$	
${}^2_{10}[70, 2^+] \frac{3^+}{2}$			$-\frac{1}{2} \left(\frac{1}{5}A + \frac{1}{10}C - \frac{1}{4}D \right)$	
${}^2_8[56', 0^+] \frac{1^+}{2}$	0	0	0	
${}^2_8[70, 0^+] \frac{1^+}{2}$		0	0	$\left(\frac{1}{6} - B + \frac{1}{2}C - \frac{1}{4}D \right)$
${}^4_8[70, 2^+] \frac{1^+}{2}$		$-\frac{3}{4} \left(\frac{1}{5}A + \frac{1}{2}B \right)$	$\frac{\sqrt{10}}{24} \left(B - \frac{1}{5}C - \frac{1}{2}D \right)$	
${}^2_8[20, 1^+] \frac{1^+}{2}$			$-\frac{1}{6}D$	
${}^2_{10}[70, 0^+] \frac{1^+}{2}$	0	0		
${}^4_{10}[56, 2^+] \frac{1^+}{2}$		$-\frac{3}{10}A$		

$$\begin{array}{l}
 {}^4_8[70, 0^+] \frac{3}{2} \\
 {}^2_8[56, 2^+] \frac{3^+}{2} \\
 {}^2_8[70, 2^+] \frac{3^+}{2} \\
 {}^4_8[70, 2^+] \frac{3^+}{2} \\
 {}^2_8[20, 1^+] \frac{3^+}{2}
 \end{array}
 \left(
 \begin{array}{cccccc}
 0 & 0 & 0 & 0 & 0 & \frac{1}{6\sqrt{2}} \left(-B + \frac{1}{2}C - \frac{1}{4}D \right) \\
 -\frac{1}{10}A & -\frac{1}{2\sqrt{2}} \left(-\frac{1}{5}A + \frac{1}{10}C + \frac{1}{4}D \right) & \frac{1}{4} \left(-\frac{1}{5}A - \frac{1}{2}B + \frac{1}{10}C + \frac{1}{4}D \right) & & & 0 \\
 & -\frac{1}{4} \left(\frac{1}{5}A + \frac{1}{2}B \right) & \frac{1}{4\sqrt{2}} \left(\frac{1}{5}A - \frac{1}{2}B + \frac{1}{10}C + \frac{1}{4}D \right) & & -\frac{1}{12\sqrt{2}} \left(B - \frac{1}{5}C - \frac{1}{2}D \right) & \\
 & & -\frac{1}{2} \left(\frac{1}{5}A + \frac{1}{2}B \right) & & \frac{\sqrt{5}}{24} \left(B - \frac{1}{5}C - \frac{1}{2}D \right) & \\
 & & & & & \frac{1}{12}B
 \end{array}
 \right)$$

spectrum, the first is the unsplit $N=2$ level $E_0' = (E_0 + 2\Omega)$, then Δ which determines the splitting of the five $SU(6)$ flavour \otimes spin supermultiplets and finally δ the spin-spin parameter. Due to the flexibility of their model Isgur and Karl fit E_0 and Δ empirically, which gives the values ~ 2020 MeV and ~ 440 MeV respectively. Their results are shown in Figure 3.4.

In this analysis there are two new parameters which appear, τ and ϵ , as no specific form is assumed for the confining potential $V_c(r_{ij})$. Following in Isgur and Karl's footsteps, we allow a small variation in the spin-spin parameter δ , and re-assign the two parameters E_0' and Δ , while treating τ and ϵ as free parameters. By using current experimental data from πN scattering phase shift analysis (Cutkosky, 1980) and a simple least squares minimisation procedure, we obtain

$$E_0' \sim 2200 \text{ MeV}, \quad \Delta \sim 615 \text{ MeV}, \quad \delta \approx 280 \text{ MeV}$$

$$\tau \sim 145 \text{ MeV} \quad \text{and} \quad \epsilon \sim 315 \text{ MeV} \quad . \quad (3.80)$$

The predicted masses and mixing angles are given in Table 3.9 and are shown against the experimental data in Figure 3.5.

The values obtained for A, B, C and D are 260, -140, 860 and 1540 MeV respectively, which clearly is not a trivial result. Figure 3.5 shows that the predicted spectrum is in good qualitative agreement with the experimentally observed baryon spectrum. Comparing Figures 3.4 and 3.5, shows that for the experimentally observed states the quality of the fit obtained in this analysis is comparable to that obtained by Isgur and Karl (1979a). Note

Table 39: The calculated mass spectrum and composition of the positive-parity excited non-strange baryons.

State	Composition	$2S+1$ $[\text{SU}(6), L^P]$	Mass
N $7/2^+$	1.00	$4[\underline{70}, 2^+]$	2005
Δ $7/2^+$	1.00	$4[\underline{56}, 2^+]$	1935
N $5/2^+$	$\begin{pmatrix} 0.76 & -0.57 & -0.30 \\ 0.58 & 0.81 & 0.06 \\ -0.28 & 0.13 & 0.94 \end{pmatrix}$	$2[\underline{56}, 2^+]$	1695
N $5/2^+$		$2[\underline{70}, 2^+]$	2155
N $5/2^+$		$4[\underline{70}, 2^+]$	2210
Δ $5/2^+$	$\begin{pmatrix} -0.70 & 0.71 \\ 0.71 & 0.70 \end{pmatrix}$	$4[\underline{56}, 2^+]$	1850
Δ $5/2^+$		$2[\underline{70}, 2^+]$	2200
N $3/2^+$	$\begin{pmatrix} 0.19 & 0.36 & 0.64 & -0.36 & -0.53 \\ 0.25 & -0.88 & 0.24 & 0.11 & -0.29 \\ 0.95 & 0.14 & 0.22 & 0.00 & 0.17 \\ -0.04 & -0.25 & -0.11 & -0.91 & 0.30 \\ 0.03 & -0.02 & 0.68 & 0.16 & 0.71 \end{pmatrix}$	$4[\underline{70}, 0^+]$	1650
N $3/2^+$		$2[\underline{56}, 2^+]$	1840
N $3/2^+$		$2[\underline{70}, 2^+]$	1990
N $3/2^+$		$4[\underline{70}, 2^+]$	2260
N $3/2^+$		$2[\underline{20}, 1^+]$	2480
Δ $3/2^+$	$\begin{pmatrix} 0.99 & 0.11 & -0.04 \\ 0.11 & -0.99 & 0.10 \\ 0.03 & 0.10 & 0.99 \end{pmatrix}$	$4[\underline{56}', 0^+]$	1750
Δ $3/2^+$		$4[\underline{56}, 2^+]$	1960
Δ $3/2^+$		$2[\underline{70}, 2^+]$	2230
N $1/2^+$	$\begin{pmatrix} 0.99 & 0.12 & 0.01 & -0.01 \\ 0.09 & -0.65 & 0.42 & 0.63 \\ 0.09 & -0.75 & -0.35 & -0.56 \\ 0.02 & -0.02 & -0.84 & 0.54 \end{pmatrix}$	$2[\underline{56}', 0^+]$	1395
N $1/2^+$		$2[\underline{70}, 0^+]$	1690
N $1/2^+$		$4[\underline{70}, 2^+]$	1885
N $1/2^+$		$2[\underline{20}, 1^+]$	2230
Δ $1/2^+$	$\begin{pmatrix} 0.30 & 0.95 \\ 0.95 & -0.30 \end{pmatrix}$	$2[\underline{70}, 0^+]$	1900
Δ $1/2^+$		$4[\underline{56}, 2^+]$	1980

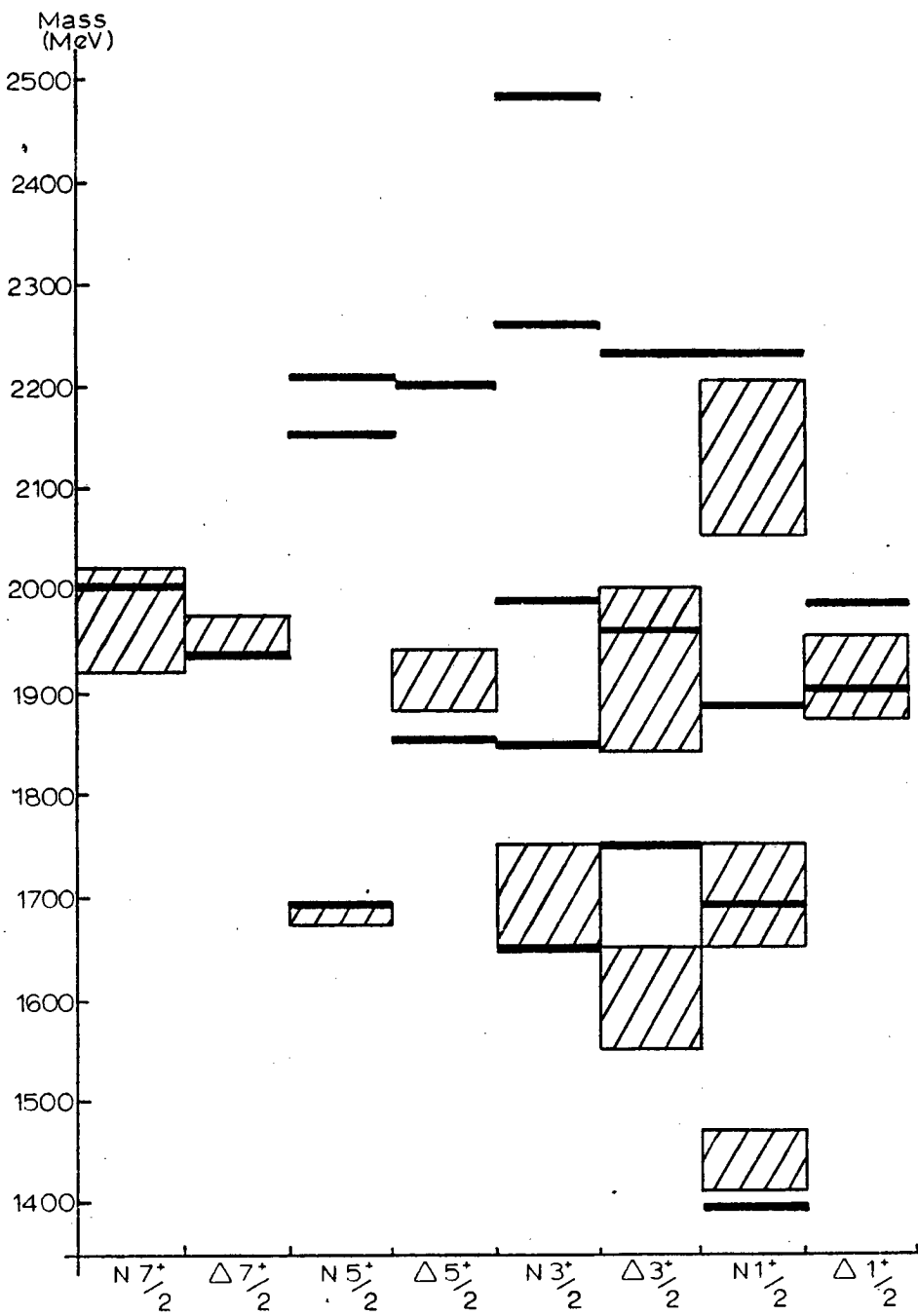


Fig. 3.5 The predicted masses for the $N = 2$ non-strange baryon masses from our analysis (heavy black bars) shown against the same experimental data (shaded areas) as figure 3.4.

in this analysis the agreement between the theoretical and observed mass for the lowest lying $\Delta P33$ resonance is poor, possibly due to the complicated nature of this channel, as we mentioned earlier. Another consequence of incorporating the spin-orbit interactions is that the plethora of states around 2 GeV, from Isgur and Karl's analysis, have been dispersed to higher masses.

3.5 Decay of the $\Delta F35$

So far in this chapter we have been content to study the baryon mass spectrum, making various assumptions about which of the relativistic corrections, from the two non-relativistic reductions, should be included in the calculation. One important area which has been neglected is how the different choices of relativistic corrections affects the internal composition of the baryon. One method of probing the internal structure of the baryon is to study the decay widths of various channels.

We saw in sections 3.3 and 3.4 how a good qualitative fit to the presently observed baryon spectrum, for the non-strange baryons in the $N=2$ level, using Isgur and Karl's Hamiltonian, equation (3.7), or a harmonic oscillator Hamiltonian with spin-spin hyperfine interactions and spin-orbit interactions, Figures 3.4 and 3.5 respectively. Isgur and Koniuk (1980) calculated the decay widths of all the baryons, assigned to the first three levels, into various channels using the compositions predicted for these states by Isgur and Karl (1977, 1978a, 1978b, 1979a, 1979b). The authors found that using a simple model of single quark transitions to calculate various $B \rightarrow \gamma B'$ or $B \rightarrow MB'$ decay widths, the

predicted widths using Isgur and Karl's composition for the baryons were in good agreement with the decay widths for various channels of experimentally observed resonances. Rather than embarking on a complete analysis of the decay widths of all the N=2 non-strange baryons we investigate the effect on such decay widths of including these spin-orbit interactions, from one-gluon exchange and confinement, by considering one particular state, the $\Delta F35$.

By defining a mixing angle θ for the $\Delta F35$ as

$$|\Delta F35\rangle = \sin\theta |^4_{10}[56,2^+]5/2^+\rangle + \cos\theta |^2_{10}[70,2^+]5/2^+\rangle. \quad (3.81)$$

then $\theta \sim 70^\circ$ for Isgur and Karl's analysis and $\theta \sim -45^\circ$ for the analysis which includes the spin-orbit interactions, using the compositions in Tables 3.4 and 3.9 respectively. As the two analyses give such differing results for the mixing angles, it would appear that a study of the decay widths of the $\Delta F35$ would highlight the differences between the two approaches (Isgur, 1983). The two decay channels of the $\Delta F35$ we will be concerned with is the $\Delta_{\frac{5}{2}}^{5+} \rightarrow (\Delta\pi)_{P\text{-wave}}$ and the $\Delta_{\frac{5}{2}}^{5+} \rightarrow (\Delta\pi)_{F\text{-wave}}$, experimentally

$$\frac{\Gamma(\Delta_{\frac{5}{2}}^{5+} \rightarrow (\Delta\pi)_{P\text{-wave}})}{\Gamma(\Delta_{\frac{5}{2}}^{5+} \rightarrow (\Delta\pi)_{F\text{-wave}})} \sim \frac{1}{5}, \quad (3.82)$$

(Metani et al., 1972).

Using the model of Koniuk and Isgur (1980) the predicted ratio of decay widths is

$$\frac{\Gamma(\Delta_{\frac{5}{2}}^{5+} \rightarrow (\Delta\pi)_{P\text{-wave}})}{\Gamma(\Delta_{\frac{5}{2}}^{5+} \rightarrow (\Delta\pi)_{F\text{-wave}})} \sim \frac{1}{3} \quad (3.83a)$$

for Isgur and Karl's mode, where $\theta \sim 70^\circ$, and

$$\frac{\Gamma(\Delta_{\frac{5}{2}}^+ \rightarrow (\Delta\pi)_{\text{P-wave}})}{\Gamma(\Delta_{\frac{5}{2}}^+ \rightarrow (\Delta\pi)_{\text{F-wave}})} \sim 37 \quad (3.83b)$$

for this analysis, where $\theta \sim -45^\circ$. So Isgur and Karl's composition for this state give a ratio which is comparable to the experimentally observed ratio, while this analysis predicts a ratio which is two orders of magnitude wrong.

Looking at Table 3.8 shows that the large mixing of the $\Delta F35$ in this analysis is due to the large off-diagonal three-body spin-orbit terms. So we can now go back and see how much 'spin-orbit' can be present with this new constraint of the empirical decay width, equation (3.82). What we find is that the two-body spin-orbit forces should be present with 20% - 40% of their predicted strength and that the three-body spin-orbit forces should be present with $\sim 0\%$ - 10%, depending on the assumptions made. So with the spin-orbit interactions present in reduced strength it is possible to account for the experimentally observed baryon spectrum, for the $N=2$ non-strange baryons, with this extra constraint of the above decay width.

Yet again we are faced with the same catch-22: To be consistent with the original philosophy of including these "QCD-inspired" relativistic interactions, the spin-orbit interactions should be present with the strength relative to the spin-spin hyperfine interactions as predicted by the non-relativistic decomposition. However, the inclusion of the spin-orbit interactions at the correct strength gives results which are not compatible with the experimental evidence. Possibly a more sophisticated model or a deeper understanding of confinement in QCD is needed to shed some light on this problem.

CHAPTER 4

THE SPECTRUM GENERATING ALGEBRA AND ANHARMONIC SYMMETRY BREAKING

4.1 Classification of the Oscillator States using the Spectrum
Generating Algebra

In the last chapter we made extensive use of explicit harmonic oscillator eigenfunctions to calculate matrix elements of the baryon Hamiltonian. In this chapter we will use the group theoretical properties of the harmonic oscillator as a means of labelling the oscillator wavefunctions and classifying the symmetry breaking. The group theoretical properties of the three-dimensional harmonic oscillator have been discussed widely in the literature (e.g. Wybourne, 1974). The majority of the work in this chapter is based on the work by Bowler, Corvi, Hey, Jarvis and King (1981); sections 4.1 and 4.2 are a synopsis of their work. For further details the interested reader is referred to the original text or to Corvi (1981).

The starting point for the discussion is the zeroth-order harmonic oscillator Hamiltonian, equation (3.11)

$$H_{HO} = \frac{P_{\rho}^2}{2m} + \frac{P_{\lambda}^2}{2m} + \frac{3K}{2} (\underline{\rho}^2 + \underline{\lambda}^2). \quad (4.1)$$

for three quarks of equal mass, where $\underline{\rho}$ and $\underline{\lambda}$ are the usual internal co-ordinates defined by equations (3.1a and b). It is possible to define creation and annihilation operators for the two modes of oscillation:

For the ρ -mode,

$$a_i^+(\rho) = \frac{1}{\sqrt{2}} (\alpha \rho_i - \frac{1}{\alpha} \frac{\partial}{\partial \rho_i}) \quad (4.2a)$$

and

$$a_i(\rho) = \frac{1}{\sqrt{2}} \left(\alpha \rho_i + \frac{1}{\alpha} \frac{\partial}{\partial \rho_i} \right) \quad (4.2b)$$

where $\alpha^4 = 3Km$ ($\hbar = 1$) and $i = x, y, z$, and similarly for the λ -mode.

Both sets of operators obey the following set of commutation relations,

$$[a_i(\rho), a_j(\rho)] = 0, \quad (4.3a)$$

$$[a_i^+(\rho), a_j^+(\rho)] = 0, \quad (4.3b)$$

$$[a_i(\rho), a_j^+(\rho)] = \delta_{ij}. \quad (4.3c)$$

We will require to construct angular momentum eigenstates of the harmonic oscillator Hamiltonian, equation (4.1); using a small amount of foresight it will be convenient to work in a spherical basis rather than a Cartesian basis. So spherical creation operators can be defined:

$$a_{\pm 1}^+(\rho) = \frac{1}{\sqrt{2}} [a_x^+(\rho) \pm ia_y^+(\rho)] \quad (4.4a)$$

$$a_0^+(\rho) = a_z^+(\rho), \quad (4.4b)$$

from the Cartesian operators in equation (4.2a) similarly spherical annihilation operators can be defined using (4.2b) and also for the λ -mode. The commutation relations in equations (4.3a-c) can now be re-written using these spherical operators as:-

$$[a_\mu(\rho), a_\nu(\rho)] = 0, \quad (4.5a)$$

$$[a_\mu^+(\rho), a_\nu^+(\rho)] = 0, \quad (4.5b)$$

$$[a_\mu(\rho), a_\nu^+(\rho)] = \delta_{\mu\nu}. \quad (4.5c)$$

where $\mu, \nu = +1, 0, -1$ and similarly for the λ -mode operators.

Using these spherical creation and annihilation operators, and the fact that ρ and λ form a basis for the two-dimensional mixed symmetric representation of the permutation group on three objects (S_3) it is possible to construct a state of given N , principal quantum number (i.e. oscillator level), L , orbital angular momentum, and P , permutation symmetry. This is achieved by acting on the vacuum state or ground state of the oscillator, defined as

$$|0\rangle = |0\rangle_\rho |0\rangle_\lambda, \quad (4.6)$$

with a monomial of N creation operators; $M_{NLL_Z}^P$, such that

$$|\psi_{N,L,L_Z}^P\rangle = M_{N,L,L_Z}^P |0\rangle. \quad (4.7)$$

Knowledge of the permutational symmetry of the spacial wavefunctions allows us to combine them with the correct $SU(6)_{\text{flavour} \otimes \text{spin}}$ wavefunction, so that the overall wavefunction is totally symmetric under the interchange of any two quarks. Remember that the overall antisymmetry required by Fermi-Dirac statistics comes from the $SU(3)_{\text{colour}}$ wavefunction, which is totally antisymmetric (see chapter 2).

The ρ -type and λ -type creation operators can be written as a six-vector, rather than two three-vectors, i.e.

$$(a_I^+) = (a_{ia}^+) = (\underline{a}^+(\rho), \underline{a}^+(\lambda)), \quad (4.8a)$$

and similarly for the annihilation operators

$$(a_I^-) = (a_{ia}^-) = (\underline{a}^-(\rho), \underline{a}^-(\lambda)), \quad (4.8b)$$

where $I = 1, 2, \dots, 6$, $i = 1, 2, 3$ and $a = 1$ (ρ -type operator), 2 (λ -type operator).

The operators defined in equations (4.8a and b) satisfy the following commutation relations

$$\left[a_I, a_J \right] = 0 \quad (4.9a)$$

$$\left[a_I^+, a_J^+ \right] = 0 \quad (4.9b)$$

$$\left[a_I, a_J^+ \right] = \delta_{IJ} \quad (4.9c)$$

where $I, J = 1, 2, \dots, 6$. It is possible to re-write the harmonic oscillator Hamiltonian, (4.1), using the creation and annihilation operators defined in equations (4.8a and b), as:-

$$H_{HO} = \omega \sum_{I=1}^6 \{ a_I, a_I^+ \}, \quad (4.10)$$

where $\{x, y\} = xy + yx$ and $\omega = \left(\frac{3K}{m} \right)^{\frac{1}{2}}$, is the fundamental frequency of both modes of oscillation. Writing the harmonic oscillator Hamiltonian in the above form, it can be seen, using the commutation relations (4.9a-c), that the 36 bilinear operators;

$$E_{IJ} = \frac{1}{2} \{ a_I, a_J^+ \}, \quad (4.11)$$

all commute with H_{HO} , i.e.

$$\left[H_{HO}, E_{IJ} \right] = 0. \quad (4.12)$$

Using the commutation relations for the creation and annihilation operators, equations (4.9a-c), gives the following commutation relation for these bilinear operators E_{IJ} ;

$$\left[E_{IJ}, E_{KL} \right] = \delta_{JK} E_{IL} - \delta_{IL} E_{JK}, \quad (4.13)$$

where $I, J, K, L = 1, 2, \dots, 6$. The commutation relation (4.13) is the real

Lie algebra of $GL(6, \mathbb{C})$, whose complex form is well known as the Lie algebra of $U(6)$. [$U(6)$ preserves the orthonormality of the oscillator states (4.7), which makes it the more convenient to use]. Note that the operator $\delta_{IJ} E^{IJ}$ is proportional to the harmonic oscillator Hamiltonian, equation (4.1), and it is also the generator of the $U(1)$ subgroup which is associated with N , the principal quantum number, based on the labelling scheme $U(6) \sim SU(6) \otimes U(1)$.

As these bilinear operators, E_{IJ} defined in equation (4.11), commute with the oscillator Hamiltonian then the oscillator states must transform as finite-dimensional irreducible representations of $U(6)$. From the commutation relations, (4.9a-c), it may be seen that the oscillator states of the N^{th} excited level transform as the fully symmetric representation $\{N\}$ of $U(6)$ (for notation see appendix A) the results are shown in table 4.1. Because of this property, $U(6)$ is called the degeneracy group.

Oscillator Level	State Vector	I.R. of $U(6)$	Young tableaux	Dimension of I.R.
Ground State	$ 0\rangle$	$\{0\}$.	<u>1</u>
1 st Excited Level	$a_I^+ 0\rangle$	$\{1\}$	\square	<u>6</u>
2 nd Excited Level	$a_J^+ a_I^+ 0\rangle$	$\{2\}$	$\square \square$	<u>21</u>
3 rd Excited Level	$a_K^+ a_J^+ a_I^+ 0\rangle$	$\{3\}$	$\square \square \square$	<u>56</u>
N^{th} Excited Level	$a_{I_N}^+ \dots a_{I_2}^+ a_{I_1}^+ 0\rangle$	$\{N\}$	$\overbrace{\square \square \dots \square}^N$	$\frac{(N+5)!}{5! N!}$

Table 4.1: Transformation properties of the oscillator wave-functions under the degeneracy group, $U(6)$.

As the name degeneracy group implies, $U(6)$ does not contain any operators which can ladder between different oscillator levels. Such

laddering operators would enable us to generate the oscillator spectrum; to do this a larger group than $U(6)$ must be considered. The first step is to put the creation operators, a_I^+ , and annihilation operators, a_I , as defined in equations (4.8a and b), into a twelve-vector, i.e.

$$(a_A) = (a_{I\alpha}) = (a_{i\alpha}) = (\underline{a}(\rho), \underline{a}(\lambda), \underline{a}^+(\rho), \underline{a}^+(\lambda)) \quad (4.14)$$

where $A = 1, 2, \dots, 12$, $I = 1, 2, \dots, 6$, $\alpha = 1$ (annihilation operator), 2 (creation operator), $a = 1, 2$ and $i = 1, 2, 3$. The commutation relations (4.9a-c) can now be expressed in the following form,

$$[a_A, a_B] = [a_{I\alpha}, a_{J\beta}] = \delta_{IJ} \epsilon_{\alpha\beta} = J_{AB}, \quad (4.15)$$

where $A, B = 1, 2, \dots, 12$,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad J = \delta \otimes \epsilon \quad (4.16)$$

We can now define 78 bilinear operators

$$S_{AB} = \frac{1}{2} \{a_A, a_B\}, \quad (4.17)$$

which obey the following commutation relations

$$[S_{AB}, S_{CD}] = J_{BC} S_{AD} + J_{AD} S_{BC} + J_{BD} S_{AC} + J_{AC} S_{BD}, \quad (4.18)$$

which is the real Lie algebra of $Sp(12, R)$. Note that the bilinear operators S_{AB} , defined in equation (4.17) may contain two creation operators (or two annihilation operators), whereas the bilinear operators E_{IJ} , defined in equation (4.11), always contain one creation and one annihilation operator. Hence when the operators S_{AB} act on an oscillator state, equation (4.7), the principal quantum number (N) may change by $+2, 0, -2$. Therefore, the oscillator spectrum decomposes into two

infinite-dimensional, unitary, irreducible representations of $Sp(12, R)$, i.e. states of even (odd) parity always transform into states of even (odd) parity under the influence of one of the bilinear operators, S_{AB} . It is possible to generate the complete harmonic oscillator spectrum starting from two states of different parity, by acting on these states the appropriate number of times with operators S_{AB} : Hence, $Sp(12, R)$ is called a spectrum generating algebra, for the three quark harmonic oscillator model.

It is the finite-dimensional, non-unitary, irreducible representations of $Sp(12, R)$ which will be of more use to us here. This becomes evident by considering the way these operators a_A , defined by equation (4.14), transform under the action of the operators S_{AB} ,

$$[S_{AB}, a_C] = J_{AC} a_B + J_{BC} a_A \quad (4.19a)$$

similarly

$$[S_{AB}, a_C a_D] = J_{AC} a_B a_D + J_{BC} a_A a_D + J_{AD} a_C a_B + J_{BD} a_C a_A \quad (4.19b)$$

From the above two commutation relations it may be shown that the operators: $1, a_A, a_A a_B + a_B a_A, a_A a_B a_C + a_B a_C a_A + a_C a_A a_B + a_B a_A a_C + a_C a_B a_A + a_A a_C a_B, \dots$ form bases for the fully symmetric representations

$$\langle 0 \rangle = \underline{1}, \langle 1 \rangle = \underline{12}, \langle 2 \rangle = \underline{78}, \langle 3 \rangle = \underline{364}, \dots, \langle N \rangle = \frac{(N+11)!}{11! N!} \quad (\text{for}$$

notation see appendix A) of $Sp(12, R)$. From table 4.1 it can be seen that the monomials which correspond to the physical oscillator states, i.e. $1, a_I^+, a_I^+ a_J^+ (= \frac{1}{2}(a_I^+ a_J^+ + a_J^+ a_I^+))$ etc belong to these basis states. So the harmonic oscillator states may be associated with the fully symmetric, finite-dimensional, non-unitary, irreducible representations of $Sp(12, R)$.

Not all the states in the fully symmetric representation $\langle N \rangle$ of $Sp(12, R)$ are oscillator states belonging to the N^{th} excited level; only those states whose eigenvalue is N for the operator

$$S_{12} = S_{I1J2} \delta^{IJ} = E_{IJ} \delta^{IJ} \quad (4.20)$$

acting as in equations (4.19a and b), are physical oscillator states. The operator defined above is the generator of the group $U(1)$, which was encountered earlier, and is a subgroup of $Sp(2, R)$ whose generators are

$$S_{\alpha\beta} = S_{I\alpha J\beta} \delta^{IJ} \quad (4.21)$$

and which is in turn a subgroup of $Sp(12, R)$. Thus the condition that the physical oscillator states in the N^{th} level must have the eigenvalue N for the generator of the $U(1)$ subgroup, can now be written as: The physical oscillator states of the N^{th} excited level must transform as the $\langle N \rangle = \underline{N+1}$ dimensional representation of $Sp(2, R)$. So states in the $N=0$ level must transform as a $\langle 0 \rangle = \underline{1}$ of $Sp(2, R)$, in the $N=1$ level as a $\langle 1 \rangle = \underline{2}$, in the $N=2$ level a $\langle 2 \rangle = \underline{3}$ of $Sp(2, R)$

The principal quantum number (N) may be associated with the $U(1)$ (or $Sp(2, R)$) subgroup of $Sp(12, R)$. We now wish to find a subgroup with which the orbital angular momentum quantum number (L) may be associated. A sensible starting place would appear to be the orthogonal groups. $Sp(12, R)$ has a subgroup $O(6)$ which occurs in the reduction $Sp(12, R) \supset Sp(2, R) \otimes O(6)$. Using the same method as before, we define the operators

$$O_{IJ} = S_{I\alpha J\beta} \epsilon^{\alpha\beta} = E_{IJ} - E_{JI} \quad (4.22)$$

where $I, J = 1, 2, \dots, 6$ and $\alpha, \beta = 1, 2$, these then satisfy the Lie algebra of $O(6)$,

$$[O_{IJ}, O_{KL}] = \delta_{JK} O_{IL} + \delta_{IL} O_{JK} - \delta_{IK} O_{JL} - \delta_{JL} O_{IK} . \quad (4.23)$$

A further reduction is needed: $O(6) \supset O(3) \otimes O(2)$, where the generators of $O(3)$ and $O(2)$ are

$$O_{ij} = O_{iajb} \delta^{ab} , \quad (4.24)$$

$$O_{ab} = O_{iajb} \delta^{ij} , \quad (4.25)$$

respectively. The usual orbital angular momentum generators, L_i , are related to the $O(3)$ generators, defined above in equation (4.24), by

$$L_i = i \epsilon_{ijk} O^{jk} . \quad (4.26)$$

Thus the orbital angular momentum quantum number (L) may be associated with the transformation properties of the physical oscillator states under the subgroup $O(3)$ of $Sp(12, R)$.

The $O(2)$ subgroup, which appears in the reduction $O(6) \supset O(3) \otimes O(2)$, is responsible for rotations in the two-dimensional space associated with the ρ and λ -modes of oscillation, see equations (4.8a and b). The final label we require to define the physical oscillator states is the permutation symmetry, which gives the states transformation properties under the permutation group on three objects, S_3 , which is a subgroup of $O(2)$. A general rotation in ρ, λ space through an angle θ is given by the matrix

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} . \quad (4.27)$$

The group $O(2)$ also includes the reflection

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.28)$$

This is important as S_3 is a subgroup of $O(2)$, but not of $SO(2)$. This can be seen by noting that the permutations $P(123)$ and $P(12)$ which generate S_3 are given in the 2-dimensional, faithful, irreducible, mixed, representation by;

$$P(123) = R \left(\frac{2\pi}{3} \right) , \quad (4.29a)$$

$$P(12) = \sigma . \quad (4.29b)$$

Thus we now have the subgroup chain which provides all the labels for the physical oscillator states:

$$Sp(12,R) \supset Sp(2,R) \otimes O(6) \quad (4.30a)$$

$$\supset Sp(2,R) \otimes O(3) \otimes O(2) \quad (4.30b)$$

$$\supset U(1) \otimes SO(3) \otimes S_3 \quad (4.30c)$$

where N the principal quantum number is associated with the $U(1)$ subgroup, L the orbital angular momentum is associated with the $SO(3)$ subgroup and P the permutation symmetry is associated with the S_3 subgroup.

To enumerate at each level, specified by N , the $O(3) \otimes S_3$ multiplets and hence the $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ multiplets we require the branching rules for various subgroup embeddings. For continuous groups this is achieved using simple Young diagram techniques which are summarised in appendix A. The fully symmetric tensor representation $\langle N \rangle$ of $Sp(12,R)$, which contains the physical, N^{th} -level, oscillator states, reduces on restriction to $Sp(2,R) \otimes O(6)$ to $\langle \zeta/B \rangle \otimes [\zeta/D]$ summed over all Young diagrams ζ of weight N ; for the definition of the quantities ζ/B and ζ/D see appendix A.

The physical oscillator states in the N^{th} -level must transform as a $\langle N \rangle = \underline{N+1}$ of $Sp(2,R)$ hence the only relevant value of ζ is N . Using the above branching rule, these relevant states belong to the $O(6)$ multiplets $[m]$ with $m = N, N-2, N-4, \dots$ where the sequence will end in a 0 or a 1. This is as expected; the totally symmetric representation $\{N\}$ of $U(6)$ on restriction to its $O(6)$ subgroup gives just these representations $[m]$.

The branching rules for the reduction $O(6) \supset O(3) \otimes O(2)$ are given in appendix A.

The scalar, $[0] = \underline{1}$, and pseudoscalar, $[1^2] = [0]^* = \underline{1}^*$, representations of $O(2)$ are symmetric and antisymmetric under S_3 , respectively, and thus yield on the restriction to S_3 , $(3) = \underline{S}$ and $(1^3) = \underline{A}$. The remaining irreducible representations of $O(2)$ are the doublets $[m] = \underline{2m}$, labelled by a quantum number m (integer or half-integer) such that $R(\theta)$ is mapped onto $R(m\theta)$. In our case only integer values of m occur, thus under the restriction of $O(2)$ to S_3 , (King, 1975).

$$[m] \rightarrow \begin{cases} \underline{S} + \underline{A} & \text{if } m = 0 \pmod{3} \\ \underline{M} & \text{if } m = 1, 2 \pmod{3} \end{cases} \quad (4.31)$$

It is now possible to complete the reduction to determine the $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ supermultiplets at each degenerate level, labelled by N . The results for the first four levels are given in table 4.2.

In this section we have shown how the eigenfunctions of the oscillator Hamiltonian may be labelled using the spectrum generating algebra of $Sp(12,R)$, for further details see the original text (Bowler et al, 1981) and Corvi (1981b), These texts also show how to construct oscillator wavefunctions using group theoretical techniques.

N	O(6)	O(3) \otimes O(2)	[SU(6), L ^P]
0	<u>1</u>	<u>1</u> \otimes <u>1</u>	[<u>56</u> , 0 ⁺]
1	<u>6</u>	<u>3</u> \otimes <u>2</u> ₁	[<u>70</u> , 1 ⁻]
2	20	(<u>5</u> \otimes <u>1</u>) \otimes <u>2</u> ₂	[<u>70</u> , 2 ⁺], [<u>70</u> , 0 ⁺]
		<u>3</u> \otimes <u>1</u> [*]	[<u>20</u> , 1 ⁺]
		<u>5</u> \otimes <u>1</u>	[<u>56</u> , 2 ⁺]
	1	<u>1</u> \otimes <u>1</u>	[<u>56'</u> , 0 ⁺]
3	50	(<u>7</u> \otimes <u>3</u>) \otimes <u>2</u> ₃	[<u>56</u> , 3 ⁻], [<u>20</u> , 3 ⁻], [<u>56</u> , 1 ⁻], [<u>20</u> , 1 ⁻]
		(<u>7</u> \otimes <u>5</u> \otimes <u>3</u>) \otimes <u>2</u> ₁	[<u>70</u> , 3 ⁻], [<u>70</u> , 2 ⁻], [<u>70</u> , 1 ⁻]
	<u>6</u>	<u>3</u> \otimes <u>2</u> ₁	[<u>70</u> , 1 ⁻]
4	<u>105</u>	(<u>9</u> \otimes <u>5</u> \otimes <u>1</u>) \otimes <u>2</u> ₄	[<u>70</u> , 4 ⁺], [<u>70</u> , 2 ⁺], [<u>70</u> , 0 ⁺]
		(<u>9</u> \otimes <u>7</u> \otimes <u>5</u> \otimes <u>3</u>) \otimes <u>2</u> ₂	[<u>70</u> , 4 ⁺], [<u>70</u> , 3 ⁺], [<u>70</u> , 2 ⁺], [<u>70</u> , 1 ⁺]
		(<u>9</u> \otimes <u>5</u> \otimes <u>1</u>) \otimes <u>1</u>	[<u>56</u> , 4 ⁺], [<u>56</u> , 2 ⁺], [<u>56</u> , 0 ⁺]
		(<u>7</u> \otimes <u>5</u>) \otimes <u>1</u> [*]	[<u>20</u> , 3 ⁺], [<u>20</u> , 2 ⁺]
20		(<u>5</u> \otimes <u>1</u>) \otimes <u>2</u> ₂	[<u>70</u> , 2 ⁺], [<u>70</u> , 0 ⁺]
		<u>3</u> \otimes <u>1</u> [*]	[<u>20</u> , 1 ⁺]
		<u>5</u> \otimes <u>1</u>	[<u>56</u> , 1 ⁺]
	1	<u>1</u> \otimes <u>1</u>	[<u>56</u> , 0 ⁺]

Table 4.2. SU(6)_{flavour} \otimes spin \otimes O(3) supermultiplets for the harmonic oscillator quark model up to the fourth excited level.

4.2 Two-body Anharmonic Symmetry Breaking

In chapter 3 we introduced $U(r_{ij})$, the two-body anharmonic perturbation, which is the difference between the true non-relativistic confining potential in a baryon and the harmonic oscillator potential which is used so that the zeroth-order Hamiltonian can be solved exactly, see equation (3.10). We also saw in that chapter when $U(r_{ij})$ is treated in first-order perturbation it removes the degeneracy of the five $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ supermultiplets in the second excited level, see equations (3.21a-e). It was also shown that the splitting pattern produced, figure 3.3., was independent of the specific form of $U(r_{ij})$. In this section our aim is to use the spectrum generating algebra of the harmonic oscillator quark model, $Sp(12, R)$, to classify $U(r_{ij})$, the two-body anharmonic perturbation, and produce an algebraic mass formula of the Gell-Mann-Okubo type for the splitting of these five, $N=2$, supermultiplets.

The total anharmonic symmetry breaking in a baryon is

$$V(\underline{r}_1, \underline{r}_2, \underline{r}_3) = U(\underline{r}_{12}) + U(\underline{r}_{13}) + U(\underline{r}_{23}), \quad (4.32)$$

$U(r_{ij})$ just depends on the magnitude of the separation ($|r_{ij}|$) hence V is obviously permutationally symmetric. Equation (4.32) can be re-written as

$$V(\underline{\rho}, \underline{\lambda}) = U(\sqrt{2} \underline{\rho}) + U\left(-\frac{1}{\sqrt{2}} \underline{\rho} + \sqrt{\frac{3}{2}} \underline{\lambda}\right) + U\left(\frac{1}{\sqrt{2}} \underline{\rho} + \sqrt{\frac{3}{2}} \underline{\lambda}\right) \quad (4.33)$$

where $\underline{\rho}$ and $\underline{\lambda}$ are the usual internal co-ordinates, defined by equations (3.1b and c), where

$$U(\sqrt{2} \underline{\rho}) = \sum_{n=1}^{\infty} \beta_{2n} (\underline{\rho} \cdot \underline{\rho})^n \quad (4.34)$$

and β_{2n} are arbitrary coefficients independent of $\underline{\rho}$. As $v(\underline{\rho}, \underline{\lambda})$ only depends on the internal co-ordinates $\underline{\rho}$ and $\underline{\lambda}$ then it is possible to express it in terms of the creation and annihilation operators, $\underline{a}^+(\underline{\rho})$, $\underline{a}(\underline{\rho})$, $\underline{a}^+(\underline{\lambda})$ and $\underline{a}(\underline{\lambda})$, see equations (4.2a and b), and hence is a function of all twelve components of the vector (a_A) , equation (4.14).

There are two justifications for assuming the above form for $U(\sqrt{2} \underline{\rho})$, see equation (4.34) (Bowler et al, 1981):

Firstly, a large class of potentials may be expected to have an expansion of this form, which is consistent with a perturbation scheme based on the dominant harmonic term having $n=1$.

Secondly, without invoking non-linear realisations, the $Sp(12, R)$ algebra (4.18) of the operators (4.17) is associated with a Fock space in which only multinomials, bilinear in a_A , have a well defined action. We can now write

$$v(\underline{\rho}, \underline{\lambda}) = \sum_{n=1}^{\infty} \beta_{2n} V^{(2n)} \quad , \quad (4.35)$$

where $V^{(2n)}$ is a homogeneous polynomial in $\underline{\rho}$ and $\underline{\lambda}$ and hence a_A of degree $2n$, at each order n . The strengths of each order in the expansion are determined by the coupling constants β_{2n} . The symmetry of equation (4.33) and the construction of the expansion in equation (4.34) ensures that each term $V^{(2n)}$ transforms as the fully symmetric representations $\langle 2n \rangle$ of $Sp(12, R)$.

It is important to know how $V^{(2n)}$ transforms under particular subgroups of $Sp(12, R)$, just as for the oscillator wavefunctions in the last section. By construction each term $V^{(2n)}$ is an $O(3)$ singlet, i.e. the perturbation is spherically symmetric and is an S_3 singlet,

i.e. it is fully symmetric under the interchange of any two quarks, see equation (4.32). Therefore, using equations (4.33), (4.34) and (4.35) gives

$$\begin{aligned}
 v^{(2n)} = & [(\underline{\rho} \cdot \underline{\rho})]^n + [(-\frac{1}{2} \underline{\rho} + \frac{\sqrt{3}}{2} \underline{\lambda}) \cdot (-\frac{1}{2} \underline{\rho} + \frac{\sqrt{3}}{2} \underline{\lambda})]^n \\
 & + [(\frac{1}{2} \underline{\rho} + \frac{\sqrt{3}}{2} \underline{\lambda}) \cdot (\frac{1}{2} \underline{\rho} + \frac{\sqrt{3}}{2} \underline{\lambda})]^n. \quad (4.36)
 \end{aligned}$$

or for small n:

$$n = 0; v^{(0)} = 1 \quad (4.37a)$$

$$n = 1; v^{(2)} = \frac{3}{2} (\underline{\rho} \cdot \underline{\rho} + \underline{\lambda} \cdot \underline{\lambda}) \quad (4.37b)$$

$$\begin{aligned}
 n = 2; v^{(4)} = & \frac{9}{8} (\underline{\rho} \cdot \underline{\rho})^2 + \frac{3}{4} (\underline{\rho} \cdot \underline{\rho}) (\underline{\lambda} \cdot \underline{\lambda}) + \frac{3}{2} (\underline{\rho} \cdot \underline{\lambda})^2 + \frac{9}{8} (\underline{\lambda} \cdot \underline{\lambda})^2 \\
 & (4.37c)
 \end{aligned}$$

$$\begin{aligned}
 n = 3; v^{(6)} = & \frac{33}{32} (\underline{\rho} \cdot \underline{\rho})^3 + \frac{9}{32} (\underline{\rho} \cdot \underline{\rho})^2 (\underline{\lambda} \cdot \underline{\lambda}) + \frac{9}{8} (\underline{\rho} \cdot \underline{\rho}) (\underline{\rho} \cdot \underline{\lambda})^2 \\
 & + \frac{27}{8} (\underline{\lambda} \cdot \underline{\lambda}) (\underline{\rho} \cdot \underline{\lambda})^2 + \frac{27}{32} (\underline{\rho} \cdot \underline{\rho}) (\underline{\lambda} \cdot \underline{\lambda})^2 + \frac{27}{32} (\underline{\lambda} \cdot \underline{\lambda})^3 \\
 & (4.37d)
 \end{aligned}$$

and so on. Note that $v^{(2)}$ is an $O(6)$ singlet, as well as being an $O(3)$ and an $O(2)$ singlet, $v^{(4)}$ is not an $O(6)$ singlet, but is an $O(3)$ and an $O(2)$ singlet and $v^{(6)}$ is not an $O(6)$ singlet, but is an $O(3)$ singlet and transforms like a $[6] = \underline{2}_6$ representation of $O(2)$ which contains an S_3 singlet.

Re-arranging equations (4.2a and b) gives

$$\underline{\rho} = \frac{1}{\sqrt{2\alpha}} (\underline{a}^+(\rho) + \underline{a}(\rho)). \quad (4.38a)$$

$$\underline{\lambda} = \frac{1}{\sqrt{2\alpha}} (\underline{a}^+(\lambda) + \underline{a}(\lambda)), \quad (4.38b)$$

so $\underline{\rho}$ and $\underline{\lambda}$ are symmetric under the interchange of creation and annihilation operators. From equation (4.14), we can write $(a_A) = (a_{i\alpha})$, where α denotes whether the operator is a creation or annihilation operator and is associated with the $Sp(2,R)$ subgroup. Therefore each term $V^{(2n)}$ in the expansion (4.35) transforms as the fully symmetric representation $\langle 2n \rangle = \underline{2n+1}$ of $Sp(2,R)$.

As well as the conditions that require $V^{(2n)}$ to be a singlet under $O(3)$ and S_3 , there is also the condition that the effective part of $V^{(2n)}$ must be a $U(1)$ singlet. If this last condition is not true then the matrix element $\langle N | V^{(2n)} | N \rangle$ vanishes. Expressing $V^{(2n)}$ in terms of creation and annihilation operators, then the above condition implies that the effective part of the monomial must be of the same degree in the operators taken separately.

In the last section we introduced the labelling scheme, equations (4.30a-c), where the first stage is

$$Sp(12,R) \supset Sp(2,R) \otimes O(6),$$

however, in general the terms $V^{(2n)}$ which appear in the expansion do not transform as a single irreducible representation of $O(6)$. It is therefore more convenient to use the following subgroup embedding,

$$Sp(12,R) \supset Sp(6,R) \otimes O(2) \quad (4.39a)$$

$$\supset Sp(2,R) \otimes O(3) \otimes O(2). \quad (4.39b)$$

The conditions which have now been imposed on the effective part of $V^{(2n)}$ are; it must transform as the fully symmetric representation $\langle 2n \rangle = \underline{2n+1}$ of $Sp(2,R)$, it must transform as a singlet of $O(3)$ and

be symmetric under S_3 transformations. These conditions ensure that the effective part of $V^{(2n)}$ transforms as the fully symmetric representation $\langle 2n \rangle = \frac{(2n+5)!}{5!(2n)!}$ of $Sp(6,R)$ (this can be seen from the branching rules in Appendix A). This case of two-body symmetry breaking has a lot in common with the case of a single three-dimensional oscillator with symmetry breaking, due to the permutation symmetry of the three-body problem (Bowler and Corvi, 1981).

As the transformation properties of each term, $V^{(2n)}$, in the expansion of the anharmonic perturbation, equation (4.34), are known it is now possible to calculate the effect of their inclusion into the Hamiltonian (4.1). This involves calculating matrix elements of the form

$$\Delta E^{(2n)} = \langle N | V^{(2n)} | N \rangle , \quad (4.40)$$

where $\Delta E^{(2n)}$ is the splitting caused by $V^{(2n)}$ in the N^{th} excited level. Using the Wigner-Eckart theorem (e.g. Wybourne 1970) the above matrix element reduces to the product of the appropriate Clebsh-Gordon coefficients with a reduced matrix element, $\langle ||2n|| \rangle$. This is the approach used by Gell-Mann and Okubo to produce their algebraic mass formula for the baryon decuplet (see e.g. Lichtenberg 1978).

In the last section we saw how the physical oscillator states in the N^{th} excited level transform as a $\langle N \rangle = \frac{(N+11)!}{11!N!}$ of $Sp(12,R)$. So using the Wigner-Eckart theorem, again, the only symmetry breaking operators which can contribute to the splitting in the N^{th} excited level must belong to the following $Sp(12,R)$ product

$$\langle N \rangle \otimes \langle N \rangle = \langle 2N \rangle \oplus \langle 2N-2 \rangle \oplus \langle 2N-4 \rangle \oplus \dots \oplus \langle 2N-1, 1 \rangle \oplus \dots \oplus \langle N, N \rangle . \quad (4.41)$$

Hence for the ground state, $N = 0$:

$$\begin{aligned}
 \langle 0 \rangle \otimes \langle 0 \rangle &= \langle 0 \rangle \\
 \cdot \otimes \cdot &= \cdot \\
 \underline{1} \otimes \underline{1} &= \underline{1}
 \end{aligned}
 \tag{4.42}$$

this means that only the operator corresponding to $n = 0$ can contribute;

$$\Delta E^{(0)} = 1 \cdot \langle ||0|| \rangle, \tag{4.43}$$

which will shift all the oscillator states by the same amount. For the first excited level, $N = 1$;

$$\begin{aligned}
 \langle 1 \rangle \otimes \langle 1 \rangle &= \langle 2 \rangle \oplus \langle 0 \rangle \oplus \langle 1,1 \rangle \\
 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} &= \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \cdot \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \\
 \underline{12} \otimes \underline{12} &= \underline{78} \oplus \underline{1} \oplus \underline{65}.
 \end{aligned}
 \tag{4.44}$$

Thus an operator transforming as a 78 of $Sp(12, R)$ can now contribute at this level, again this operator does not produce any splitting, as it is just the number operator (or a harmonic term), see equation (4.37b) which is the first order Casimir of $U(1)$, so

$$\Delta E^{(2)} = C_1\{1\} \langle ||2|| \rangle. \tag{4.45}$$

For the second excited level, $N = 2$;

$$\begin{aligned}
 \langle 2 \rangle \otimes \langle 2 \rangle &= \langle 4 \rangle \oplus \langle 2 \rangle \oplus \langle 0 \rangle \oplus \langle 3,1 \rangle \oplus \langle 1,1 \rangle \oplus \langle 2,2 \rangle \\
 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} &= \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \\
 \underline{78} \otimes \underline{78} &= \underline{1365} \oplus \underline{78} \oplus \underline{1} \oplus \underline{2925} \oplus \underline{65} \oplus \underline{1650}.
 \end{aligned}
 \tag{4.46}$$

In section 3.2 we saw how the addition of a two-body anharmonic perturbing potential split the five previously degenerate supermultiplets

in the N=2 level. Equations (4.43) and (4.45) show that the operators transforming as a 1 and 78 of Sp(12,R) cannot be responsible for such a splitting. This implies the operator that is responsible for this splitting is $V^{(4)}$ which transforms as the fully symmetric irreducible representation $\langle 4 \rangle = \underline{1365}$ of Sp(12,R).

In the third excited level, there are two operators which can cause splitting; $V^{(4)}$ and $V^{(6)}$ which transform as a $\langle 4 \rangle = \underline{1365}$ and a $\langle 6 \rangle = \underline{12376}$ of Sp(12,R), respectively.

However, the operator $V^{(4)}$ which splits the N=2 level has to transform as a

$$\begin{array}{cccccccc}
 \langle 4 \rangle & , & \langle 4 \rangle & , & \langle 4 \rangle & , & [0] & , & [0] & , & \{0\} & , & (3) \\
 \underline{1365} & , & \underline{126} & , & \underline{5} & , & \underline{1} & , & \underline{1} & , & \underline{1} & , & \underline{S} & (4.47) \\
 \text{Sp}(12,\mathbb{R}) & , & \text{Sp}(6,\mathbb{R}) & , & \text{Sp}(2,\mathbb{R}) & , & \text{O}(3) & , & \text{O}(2) & , & \text{U}(1) & , & \text{S}_3
 \end{array}$$

to satisfy all the conditions. It is a relatively simple task to write down an arbitrary component of a fully symmetric fourth-order tensor in Sp(12,R);

$$X_{ABCD} = \{S_{AB}, S_{CD}\} + \{S_{AC}, S_{BD}\} + \{S_{AD}, S_{BC}\}, \quad (4.48)$$

where A,B,C,D = 1,2,..., 12 and S_{AB} are the bilinear operators defined in (4.17). The difficult part is to project out of X_{ABCD} the part which has the required transformation properties in equation (4.47):

The first stage is the reduction of Sp(12,R) to Sp(6,R) \otimes O(2); to do this we write A = Pa, where P = (i α) is the Sp(6,R) index and a is the O(2) index, and B = Qb etc. To project out the O(2) singlet piece there are three possible operators which may be used: $\delta^{ab}\delta^{cd}$, $\delta^{ac}\delta^{bd}$, $\delta^{ad}\delta^{bc}$, however there is also the added constraint that the new operator must be fully symmetric on its Sp(6,R) indices, hence

$$Y_{PQRS} \propto X_{PaQbRcSd} \{ \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \} \quad (4.49a)$$

$$\propto \{ X_{PaQbRcSd} + X_{PaRbQcSd} + X_{PaSbQcRd} \} \delta^{ab} \delta^{cd} \quad (4.49b)$$

Y_{PQRS} transforms as a $\underline{126} \otimes \underline{1}$ of $Sp(6, R) \otimes O(2)$.

The next step in the reduction chain, equation (4.39a and b), is the restriction of $Sp(6, R)$ to $Sp(2, R) \otimes O(3)$, this is achieved by writing $P = i\alpha$ where $i = 1, 2, 3$ is the $O(3)$ index and $\alpha = 1, 2$ is the $Sp(2, R)$ index. Once again there are three operators will project out the $O(3)$ singlet part of Y_{PQRS} ; $\delta^{ij} \delta^{kl}$, $\delta^{ik} \delta^{jl}$, $\delta^{il} \delta^{jk}$, there is also the constraint that the new operator has to be symmetric on its $Sp(2, R)$ indices hence

$$Z_{\alpha\beta\gamma\delta} \propto (Y_{i\alpha j\beta k\gamma l\delta} + Y_{i\alpha j\gamma k\beta l\delta} + Y_{i\alpha j\delta k\beta l\gamma}) \delta^{ij} \delta^{kl} \quad (4.50)$$

$Z_{\alpha\beta\gamma\delta}$ transforms as a $\underline{5} \otimes \underline{1}$ of $Sp(2, R) \otimes O(3)$.

The final step is to project out from $Z_{\alpha\beta\delta\gamma}$ the part which transforms as a $U(1)$ singlet, this gives

$$V^{(4)} \propto Z_{1122} \quad (4.51)$$

The above process can now be reversed so that $V^{(4)}$ can be expressed in terms of the quartic operators $\{S_{AB}, S_{CD}\}$, but now with the correct transformation properties, as listed in equation (4.48). This leads to the operator

$$\begin{aligned} V^{(4)} \propto & \frac{1}{2} \{ S_{ialial}, S_{jb2jb2} \} + \{ S_{ialib1}, S_{ja2jb2} \} + \{ S_{ialja1}, S_{ib2jb2} \} \\ & + \{ S_{ialjbl}, S_{ia2jb2} \} + \{ S_{ialjbl}, S_{ib2ja2} \} + \{ S_{ialia2}, S_{jbljlb2} \} \\ & + \{ S_{ialib2}, S_{jaljb2} \} + \{ S_{ialib2}, S_{jblja2} \} + \{ S_{ialja2}, S_{ibljlb2} \} \\ & + \{ S_{ialjlb2}, S_{ialjlb2} \} + \{ S_{ialjlb2}, S_{ibljja2} \} + \{ S_{ialja2}, S_{jbljlb2} \} \\ & + \{ S_{ialjlb2}, S_{jalib2} \} + \{ S_{ialjlb2}, S_{jblia2} \} \end{aligned} \quad (4.52)$$

The spectrum generating algebra of the three-quark harmonic oscillator model has thirteen subgroups, the generators of which are tabulated in table 4.3 (Jarvis, 1979). The full subgroup reduction of $Sp(12, R)$ is shown in figure 4.1. The quadratic Casimir invariants of $Sp(12, R)$ and these thirteen subgroups are given in table 4.4 (Jarvis, 1979). In the following calculation we will also need to include one of the operators

$$\Sigma = \frac{1}{2} \delta^{ij} \delta^{kl} \delta^{ab} \delta^{cd} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \{S_{ia\alpha k c \gamma}, S_{jd\delta l b \beta}\} \quad , \quad (4.53a)$$

$$\Sigma' = \frac{1}{2} \delta^{ij} \delta^{kl} \delta^{ab} \delta^{cd} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} \{S_{ia\alpha k c \gamma}, S_{lb\delta j d \beta}\} \quad , \quad (4.53b)$$

which are invariants of the subgroups $Sp(4, R) \otimes O(3)$ and $Sp(6, R) \otimes O(2)$ of $Sp(12, R)$, respectively, as they commute with all the generators of their respective subgroups. However, they do not belong to the enveloping algebra of the groups and are thus not Casimir invariants. The reason only one of these operators will be required is that the two operators, in equations (4.53a and b) are not independent, they are related by,

$$\Sigma - \Sigma' = C_2[6] + C_2[3] + 4C_2\{3'\} \quad . \quad (4.54)$$

Arbitrarily we choose Σ , which can now be expressed in the same form as the quadratic Casimirs in table 4.4, i.e.:

$$\Sigma = \{S_{ia l j b 1}, S_{ib 2 j a 2}\} - \{S_{ia l j b 2}, S_{ib 1 j a 2}\} \quad . \quad (4.55)$$

It is now possible to express the symmetry breaking operator $V^{(4)}$ in terms of the quadratic Casimir invariants of $Sp(12, R)$, its subgroups and Σ . After a little labour we find

Subgroup	Generators
U(1)	$E_{IJ} \delta^{IJ}$
U(2)	$E_{iajb} \delta^{ij}$
U(3)	$E_{iajb} \delta^{ab}$
U(6)	$E_{IJ} = S_{I2J1}$
Sp(2,R)	$S_{I\alpha J\beta} \delta^{IJ}$
Sp(4,R)	$S_{U_i V_j} \delta^{ij}$
Sp(6,R)	$S_{PaQb} \delta^{ab}$
O(2)	$O_{iajb} \delta^{ij}$
O(3)	$O_{iajb} \delta^{ab}$
O(6)	$O_{IJ} \delta^{IJ}$
U(3)'	$\frac{1}{2}(O_{i1j1} + O_{i2j2} + iO_{i1j2} - iO_{i2j1})$
Sp(6,R)'	$J_{IM}^E M_J + J_{JM}^E M_I$

Table 4.3 Generators of the subgroups of Sp(12,R)

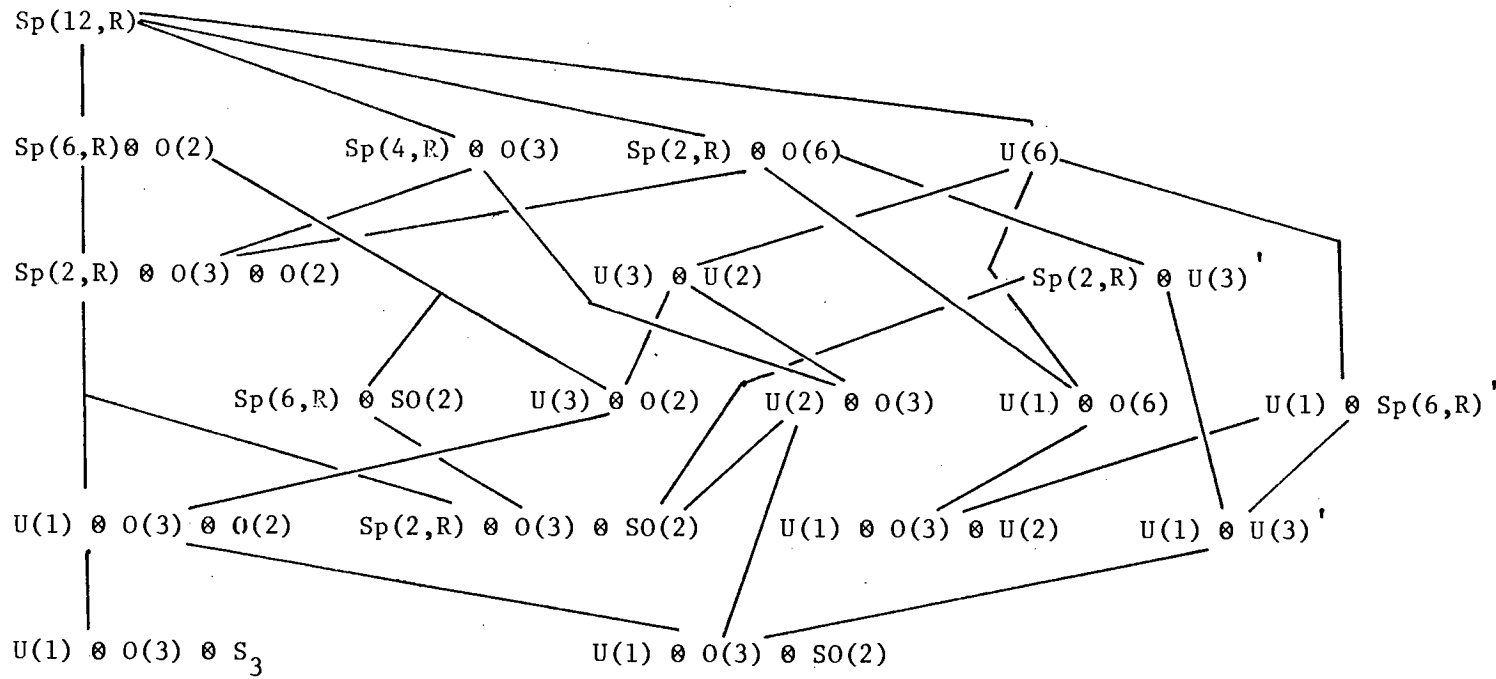


Figure 4.1 Labelling Chains in $Sp(12, R)$

Table 4.4.

Group	Quadratic Casimir Invariant
U(1)	$C_2\{1\} = \frac{1}{2}\{S_{ialia2}, S_{jbljb2}\}$
U(2)	$C_2\{2\} = \frac{1}{2}\{S_{ialib2}, S_{jblja2}\}$
U(3)	$C_2\{3\} = \frac{1}{2}\{S_{ialja2}, S_{jblib2}\}$
U(6)	$C_2\{6\} = \frac{1}{2}\{S_{ialjb2}, S_{jblia2}\}$
Sp(2,R)	$C_2\langle 2 \rangle = \{S_{ialia2}, S_{jbljb2}\} - \{S_{ialial}, S_{jb2jb2}\}$
Sp(4,R)	$C_2\langle 4 \rangle = \{S_{ialib2}, S_{jblja2}\} - \{S_{ialib1}, S_{ja2jb2}\}$
Sp(6,R)	$C_2\langle 6 \rangle = \{S_{ialja2}, S_{jblib2}\} - \{S_{ialja1}, S_{ib2jb2}\}$
Sp(12,R)	$C_2\langle 12 \rangle = \{S_{ialjb2}, S_{ia2jb1}\} - \{S_{ialjb1}, S_{jb2ia2}\}$
O(2)	$C_2[2] = \{S_{ialib2}, S_{jblja2}\} - \{S_{ialib2}, S_{jaljb2}\}$
O(3)	$C_2[3] = \{S_{ialja2}, S_{iblib2}\} - \{S_{ialja2}, S_{ibljb2}\}$
O(6)	$C_2[6] = \{S_{ialjb2}, S_{jblia2}\} - \{S_{ialjb2}, S_{ialjb2}\}$
U(3)	$C_2\{3'\} = \frac{1}{4}[(\{S_{ialjb2}, S_{iblja2}\} + \{S_{ialjb2}, S_{jblia2}\})$ $+ \{S_{ialja2}, S_{jblib2}\}) - (\{S_{ialjb2}, S_{jalib2}\}$ $+ \{S_{ialjb2}, S_{ialjb2}\} + \{S_{ialja2}, S_{ibljb2}\})]$
Sp(6,R)	$C_2\langle 6' \rangle = \{S_{ialjb2}, S_{jblia2}\} + \{S_{ialjb2}, S_{iblja2}\}$ $\{S_{ialja2}, S_{ibljb2}\}$

$$\begin{aligned}
 E^{(4)} = & \langle ||4|| \rangle \{ 3C_2\{1\} + 6C_2\{2\} + 12C_2\{3\} - 4C_2\{3'\} - \frac{1}{2}C_2\langle 2 \rangle \\
 & - C_2\langle 4 \rangle - C_2\langle 6 \rangle + 3C_2\langle 6' \rangle - C_2\langle 12 \rangle - C_2[2] - 3C_2[3] + \Sigma \}
 \end{aligned}
 \tag{4.56}$$

The eigenvalues of these quadratic Casimir operators are given in table 4.5 (Bowler et al, 1981).

The validity of equation (4.56) can be checked by ensuring that the contribution to the splitting vanishes when the states are not physical oscillator states, seven such states transform as

$$\begin{aligned}
 (<1> , <1> , <1>) ; (<1^2> , <2> , <2>) ; (<2> , <2> , <0>) ; \\
 (<1^2> , <1^2> , <2>) ; (<1^2> , <1^2> , <0>) ; (<2> , <1^2> , <2>) ; \\
 (<2> , <1^2> , <0>) ; (<2> , <2> , <0>) ,
 \end{aligned}
 \tag{4.57}$$

of (Sp(12,R), Sp(6,R), Sp(2,R)). In fact the above seven states provide enough conditions to calculate and check the coefficients in equation (4.56), this alternative derivation is due to King (see Corvi, 1981). The splittings for the five physical oscillator states in the N=2 level are given in table 4.5.

Using equations (4.56), (4.45) and (4.43) the total splitting and shift of the five N=2 SU(6)_{flavour} ⊗ spin ⊗ O(3) supermultiplets caused by the addition of the anharmonic perturbation, (4.32) is

$$\Delta E = \Delta E^{(0)} + \Delta E^{(2)} + \Delta E^{(4)},
 \tag{4.58}$$

where $\Delta E^{(0)}$, $\Delta E^{(2)}$, $\Delta E^{(4)}$ are given in (4.43), (4.45) and in table 4.5, respectively. Comparing this with the results of Isgur and Karl (Isgur, 1980) in chapter 3, we can make the following identifications:

$[SU(6), L^P]$	$[20, 1^+]$		$[70, 2^+]$		$[56, 2^+]$		$[70, 0^+]$		$[56, 0^+]$	
Subgroup H	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$	Representation	$C_2(H)$
U(1)	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4	$\{2\} = \underline{1}_2$	4
U(2)	$\{1^2\} = \underline{1}$	2	$\{2\} = \underline{3}$	6	$\{2\} = \underline{3}$	6	$\{2\} = \underline{3}$	6	$\{2\} = \underline{3}$	6
U(3)	$\{1^2\} = \underline{3}$	4	$\{2\} = \underline{6}$	8	$\{2\} = \underline{6}$	8	$\{2\} = \underline{6}$	8	$\{2\} = \underline{66}$	8
U(3)'	$\{2,1\} = \underline{8}$	6	$\{2\} = \underline{6}$	8	$\{2,1\} = \underline{8}$	6	$\{2\} = \underline{6}$	8	$\{0\} = \underline{1}$	0
U(6)	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14	$\{2\} = \underline{21}$	14
Sp(2,R)	$\langle 2 \rangle = \underline{3}$	16	$\langle 2 \rangle = \underline{3}$	16	$\langle 2 \rangle = \underline{3}$	16	$\langle 2 \rangle = \underline{3}$	16	$\langle 2 \rangle = \underline{3}$	16
Sp(4,R)	$\langle 1^2 \rangle = \underline{5}$	16	$\langle 2 \rangle = \underline{10}$	24	$\langle 2 \rangle = \underline{10}$	24	$\langle 2 \rangle = \underline{10}$	24	$\langle 2 \rangle = \underline{10}$	24
Sp(6,R)	$\langle 1^2 \rangle = \underline{14}$	24	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32
Sp(6,R)'	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32	$\langle 2 \rangle = \underline{21}$	32
Sp(12,R)	$\langle 2 \rangle = \underline{78}$	56	$\langle 2 \rangle = \underline{78}$	56	$\langle 2 \rangle = \underline{78}$	56	$\langle 2 \rangle = \underline{78}$	56	$\langle 2 \rangle = \underline{78}$	56
O(2)	$[0]^* = \underline{1}^*$	0	$[2] = \underline{2}_2$	8	$[0] = \underline{1}$	0	$[2] = \underline{2}_2$	8	$[0] = \underline{1}$	0
O(3)	$[1] = \underline{3}$	4	$[2] = \underline{5}$	12	$[2] = \underline{5}$	12	$[0] = \underline{1}$	0	$[0] = \underline{1}$	0
O(6)	$[2] = \underline{20}$	24	$[2] = \underline{20}$	24	$[2] = \underline{20}$	24	$[2] = \underline{20}$	24	$[0] = \underline{1}$	0
Σ		-28		-28		-28		-40		-40
$\frac{\Delta E^{(4)}}{\langle 4 \rangle}$		0		16		32		40		80

Table 4.5 Subgroup representation labels and operator eigenvalues for N = 2.

$$\langle ||0|| \rangle = a \equiv E_0 - (3m + 3\omega) \quad (4.59a)$$

$$\langle ||2|| \rangle = -\frac{1}{2} a + \frac{1}{3} b \equiv \Omega - \omega \quad (4.59b)$$

$$\langle ||4|| \rangle = -\frac{1}{80} \Delta \equiv \frac{1}{16} \left(\frac{1}{4} a - \frac{1}{3} b + \frac{1}{15} c \right) \quad (4.59c)$$

using equations (3.21) and (3.22).

So we have seen in this section how the spectrum generating algebra $Sp(12,R)$ of the three quark harmonic oscillator model and its plethora of subgroups may be used to classify two-body symmetry breaking and reproduce the ubiquitous splitting pattern of the five $N=2$ supermultiplets.

4.3 Three-body Anharmonic Symmetry Breaking

Gromes and Stamatescu (1979) considered the case of a more general perturbing potential, $V(\underline{r}_1, \underline{r}_2, \underline{r}_3)$, one which was translationally and rotationally invariant and symmetric under permutations, but not necessarily the sum of two-body potentials. The authors found, using explicit wavefunction calculations, that the splitting between the four, $N=2$ $[SU(6), 0(3)]$ supermultiplets; $[20, 1^+]$, $[70, 2^+]$, $[56, 2^+]$ and $[70, 0^+]$ is the same as in figure 3.3: However, in this more general case no statement about the $[56', 0^+]$ can be made. In this section we shall show that, using $Sp(12,R)$, the spectrum generating algebra of the three quark harmonic oscillator model, another Gell-Mann-Okubo - type algebraic mass formula can be obtained for this more general splitting pattern.

The Wigner-Eckart theorem tells us that the only symmetry breaking operators which can contribute to the splitting of the $N=2$ level must belong to the $Sp(12,R)$ product,

$$\langle 2 \rangle \otimes \langle 2 \rangle = \langle 4 \rangle \oplus \langle 2 \rangle \oplus \langle 0 \rangle \oplus \langle 3,1 \rangle \oplus \langle 1,1 \rangle \oplus \langle 2,2 \rangle .$$

$$\underline{78} \otimes \underline{78} = \underline{1365} \oplus \underline{78} \oplus \underline{1} \oplus \underline{2925} \oplus \underline{65} \oplus \underline{1650} .$$

Gromes and Stamatescu (1979) note that the conditions imposed on this symmetry breaking potential (V) ensures that it is of the form $V(\underline{\rho}^2, \underline{\lambda}^2, \underline{\rho} \cdot \underline{\lambda})$ and that it can be symmetrised with respect to $\underline{\rho}$ and $\underline{\lambda}$ interchange. Thus, it follows that the splitting at the $N=2$ level is controlled by a single operator which transforms as the fully symmetric representation $\langle 4 \rangle = \underline{1365}$ of $Sp(12, R)$.

For the case of two-body anharmonic perturbations it was possible to argue that under the restriction of $Sp(12, R)$ to $Sp(6, R) \otimes O(2)$ the only $Sp(6, R)$ representation which was allowed to contribute was the totally symmetric tensor representation $\langle 4 \rangle = \underline{126}$. This was because of the permutation symmetry the matrix elements of the perturbation depend only on $U(\sqrt{2}\underline{\rho})$, for the more general case such a simplification cannot be made. Using the methods outlined in appendix A, the full reduction of the fully symmetric $\underline{1365}$ dimensional representation of $Sp(12, R)$ into $Sp(6, R) \otimes O(2)$ is:

$$\begin{array}{c} \square \square \square \square \quad \supset \quad \square \square \square \square \otimes (\square \square \square \square \oplus \square \square \oplus \cdot) \oplus (\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array}) \\ \\ \otimes (\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \cdot) \oplus (\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \square \square \oplus \cdot) \otimes \cdot \end{array}$$

$$\underline{1365} \supset \underline{126} \oplus (\underline{4} \oplus \underline{2} \oplus \underline{1}) \oplus (\underline{189} \oplus \underline{21}) \oplus (\underline{2} \oplus \underline{1}^*) \oplus (\underline{90} \oplus \underline{14} \oplus \underline{1}) \oplus \underline{1}$$

(4.60)

The condition that V must be symmetric under the interchange of any two quarks restricts us to those parts which transforms as a singlet of $O(2)$, see section 4.2. This restricts the discussion to four $Sp(6, R)$ multiplets; $\underline{126}$, $\underline{90}$, $\underline{14}$ and $\underline{1}$. The next step is to look at the

restriction of these four $Sp(6,R)$ multiplets to their $Sp(2,R) \otimes O(3)$ constituents;

$$\begin{aligned}
 \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} & \supset & \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \otimes \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \cdot \right) \oplus \begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \\
 & & \left(\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \end{array} \right) \oplus \cdot \otimes \left(\begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \cdot \right) \\
 \underline{126} & \supset & \underline{5} \otimes (\underline{9} \oplus \underline{5} \oplus \underline{1}) \oplus \underline{3} \otimes (\underline{7} \oplus \underline{5} \oplus \underline{3}) \oplus \underline{1} \otimes (\underline{5} \oplus \underline{1}) \\
 & & \hspace{15em} (4.61a)
 \end{aligned}$$

$$\begin{aligned}
 \begin{array}{|c|c|} \hline & \\ \hline \end{array} & \supset & \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \otimes \left(\begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \cdot \right) \oplus \begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \left(\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \oplus \right. \\
 & & \left. \begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \end{array} \right) \oplus \cdot \otimes \left(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline \end{array} \oplus \cdot \right) \\
 \underline{90} & \supset & \underline{5} \otimes (\underline{5} \oplus \underline{1}) \oplus \underline{3} \otimes (\underline{7} \oplus \underline{5} \oplus \underline{3}) \oplus \underline{1} \otimes (\underline{9} \oplus \underline{5} \oplus \underline{1}) \\
 & & \hspace{15em} (4.61b)
 \end{aligned}$$

$$\begin{aligned}
 \begin{array}{|c|} \hline \\ \hline \end{array} & \supset & \begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} \oplus \cdot \otimes \begin{array}{|c|c|} \hline & \\ \hline \end{array} \\
 \underline{14} & \supset & \underline{3} \otimes \underline{3} \oplus \underline{1} \otimes \underline{5} \\
 & & \hspace{15em} (4.61c)
 \end{aligned}$$

$$\underline{1} \supset \underline{1} \otimes \underline{1} . \hspace{15em} (4.61d)$$

To comply with the condition of rotational invariance $V^{(2n)}$ must transform as a singlet of $O(3)$, and in section 4.2 we saw that $V^{(2n)}$ must transform as the fully symmetric representation $\langle 2n \rangle$ of $Sp(2,R)$. Inspection of equations (4.61 a-d) shows the only $Sp(6,R)$ representations which comply with all the above conditions are the $\langle 4 \rangle = \underline{126}$, which was held to be responsible for the two-body anharmonic symmetry breaking and the $\langle 2,2 \rangle = \underline{90}$, which we shall show corresponds to relaxing the constraints and allowing a more general symmetry breaking.

The operator which corresponds to this new more general type of symmetry breaking operator transforms like,

$$\langle 4 \rangle, \quad \langle 2, 2 \rangle, \quad \langle 4 \rangle, \quad [0], \quad [0], \quad (3)$$

$$\underline{1365}, \quad \underline{90}, \quad \underline{5}, \quad \underline{1}, \quad \underline{1}, \quad \underline{5} \quad (4.62)$$

$$Sp(12, R), \quad Sp(6, R), \quad Sp(2, R), \quad O(3), \quad O(2), \quad S_3$$

An arbitrary component of the fully-symmetric fourth rank tensor in $Sp(12, R)$ is (equation (4.48)),

$$X_{ABCD} = \{S_{AB}, S_{CD}\} + \{S_{AC}, S_{BD}\} + \{S_{AD}, S_{BC}\},$$

where $A, B, C, D = 1, 2, \dots, 12$ and the bilinear operators S_{AB} are defined in equation (4.17). The first step is to project out of X_{ABCD} the piece that transforms like a $\underline{90} \otimes \underline{1}$ of $Sp(6, R) \otimes O(2)$ (this is done as for the $\underline{126} \otimes \underline{1}$ projection in the last section). Writing $A = Pa$, where $P=1, \dots, 6$ is the $Sp(6, R)$ index and $a=1, 2$ is the $O(2)$ index, $B=Qb$ etc., then there are three operators which will project out the $O(2)$ singlet; $\delta^{ab}\delta^{cd}$, $\delta^{ac}\delta^{bd}$, $\delta^{ad}\delta^{bc}$: However, they must be used in such a combination that the part which remains transforms as a $\underline{90}$ of $Sp(6, R)$. In the previous case this was a relatively simple task, as the Young tableau for the fully symmetric representation $\langle 4 \rangle = \underline{126}$ has only one standard arrangement;

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}$$

The Young tableau corresponding to the mixed symmetric representation $\langle 2, 2 \rangle = \underline{90}$ has two standard arrangements;

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

So, the two corresponding tensors are found using the appropriate Young operators (Lichtenberg, 1978), which gives

$$Y^{(1)} \propto X_{PaQbRcSd} (\delta^{ab}_{\delta}{}^{cd} - \delta^{ad}_{\delta}{}^{bc}) \quad (4.63a)$$

and

$$Y^{(2)} = X_{PaQbRcSd} (\delta^{ac}_{\delta}{}^{bd} - \delta^{ad}_{\delta}{}^{bc}). \quad (4.63b)$$

The next step is to write $P = i\alpha$, where $i = 1, 2, 3$ is the $O(3)$ index and $\alpha = 1, 2$ is the $Sp(2, R)$ index, $Q = j\beta$ etc. Then the $O(3)$ singlet part of the above two operators may be projected out, using the operators $\delta^{ij}_{\delta}{}^{kl}$, $\delta^{ik}_{\delta}{}^{jl}$, $\delta^{il}_{\delta}{}^{jk}$, in such a manner that the resulting tensor transforms as the fully symmetric representation $\langle 4 \rangle$ of $Sp(2, R)$. A small amount (!) of labour yields the following result,

$$\begin{aligned} \tilde{Z}_{\alpha\beta\gamma\delta} \propto & (Y^{(1)}_{i\alpha j\beta k\gamma l\delta} - 2Y^{(2)}_{i\alpha j\beta k\gamma l\delta}) \delta^{ij}_{\delta}{}^{kl} \\ & + (Y^{(1)}_{i\alpha j\beta k\gamma l\delta} - 2Y^{(2)}_{i\alpha j\beta k\gamma l\delta}) \delta^{ik}_{\delta}{}^{jl} \\ & + (Y^{(1)}_{i\alpha j\beta k\gamma l\delta} + Y^{(2)}_{i\alpha j\beta k\gamma l\delta}) \delta^{il}_{\delta}{}^{jk}. \end{aligned} \quad (4.64)$$

(The tilde denotes the difference between the above operator and the two body operator in (4.50)). The only part of \tilde{Z} which can contribute to the splitting is a $U(1)$ singlet, i.e.

$$\tilde{V}^{(4)} \propto \tilde{Z}_{1122}. \quad (4.65)$$

As in section 4.2 we can now retrace our footsteps and re-express $\tilde{V}^{(4)}$ in terms of these quartic operators so that it has the correct transformation properties. This eventually leads to the operator

$$\begin{aligned}
 \tilde{V}^{(4)} = & \{S_{ia1ja1}, S_{ib2jb2}\} + \{S_{ia1ib2}, S_{ja1jb2}\} + \{S_{ia1jb2}, S_{ja1ib2}\} \\
 & + \{S_{ia1ja2}, S_{ib1jb2}\} + \{S_{ia1ib1}, S_{ja2jb2}\} + \{S_{ia1jb2}, S_{ja2ib1}\} \\
 & + \{S_{ia1ja2}, S_{jb1ib2}\} + \{S_{ia1jb1}, S_{ja2ib2}\} + \{S_{ia1ib2}, S_{ja2jb1}\} \\
 & - \{S_{ia1ia1}, S_{jb2jb2}\} - 2\{S_{ia1jb2}, S_{ia1jb2}\} - 2\{S_{ia1ia2}, S_{jb1jb2}\} \\
 & - 2\{S_{ia1jb1}, S_{ia2jb2}\} - 2\{S_{ia1jb2}, S_{ia2jb1}\}.
 \end{aligned}
 \tag{4.66}$$

As in the two-body case, it is now possible to write the operator $\tilde{V}^{(4)}$ in terms of the quadratic Casimir invariant operators of $Sp(12,R)$ and its family of subgroups, table 4.4, and the operator Σ which is an invariant of $Sp(4,R) \otimes O(3)$, equation (4.55). The result is

$$\begin{aligned}
 \tilde{V}^{(4)} = & \langle ||\tilde{4}|| \rangle \{-6C_2\{1\} + 6C_2\{2\} + 12C_2\{3\} - 4C_2\{3'\} - 18C_2\{6\} \\
 & + C_2\langle 2 \rangle - C_2\langle 4 \rangle - C_2\langle 6 \rangle + 3C_2\langle 6' \rangle + 2C_2\langle 12 \rangle - C_2[2] \\
 & - 3C_2[3] + 3C_2[6] + \Sigma \}.
 \end{aligned}
 \tag{4.67}$$

where the tilde on the reduced matrix element denotes that in principal it is different to the reduced matrix element in equation (4.56).

The validity of equation (4.67) can be checked by ensuring that it vanishes for the unphysical oscillator states, which transform like:

$$\begin{aligned}
 & (\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle); (\langle 1,1 \rangle, \langle 2 \rangle, \langle 2 \rangle); (\langle 1,1 \rangle, \langle 2 \rangle, \langle 0 \rangle); \\
 & (\langle 1,1 \rangle, \langle 1,1 \rangle, \langle 2 \rangle); (\langle 1,1 \rangle, \langle 1,1 \rangle, \langle 0 \rangle); (\langle 2 \rangle, \langle 1,1 \rangle, \langle 0 \rangle); \\
 & (\langle 2 \rangle, \langle 2 \rangle, \langle 0 \rangle)
 \end{aligned}$$

of $(Sp(12,R), Sp(6,R), Sp(2,R))$ just as for the two-body case.

Comparing the result for $\tilde{V}^{(4)}$, the more general symmetry breaker, in equation (4.66) with the result for $V^{(4)}$, the two-body symmetry

breaker in equation (4.56) it can be seen the two formulae differ only in their dependence on the Casimir operators; $C_2\{1\}$, $C_2\{6\}$, $C_2\{2\}$, $C_2\{12\}$ and finally $C_2[6]$. Table 4.4 shows that the first four Casimir's have the property of not discriminating between the five $N=2$ $SU(6)_{\text{flavour}} \otimes \text{spin} \otimes O(3)$ supermultiplets, whereas the remaining Casimir of $O(6)$, $C_2(6)$, differentiates between the $[56, 0^+]$, which transforms as a singlet of $O(6)$ and the remainder which transforms as a $\underline{20}$ of $O(6)$. Therefore the relative splitting between the four supermultiplets; $[\underline{20}, 1^+]$, $[\underline{70}, 2^+]$, $[\underline{56}, 2^+]$, $[\underline{70}, 0^+]$ remains unaffected when three-body anharmonic perturbations are included, however, in this more general case no statement can be made about the $[56, 0^+]$. Hence, it is the $O(6)$ classification of the $N=2$ supermultiplets which plays a crucial role in reproducing the results of Gromes and Stamatescu (1979) as conjectured by Corvi (1982).

In the first case of two-body anharmonic symmetry breaking, a specific form was chosen and used in the expansion, (4.36), so that the form of each of the terms $V^{(2n)}$ was known, or could be calculated, see equations (4.37a-d). We have not assumed any particular expansion for this more general anharmonic perturbation; however it is possible to re-construct $\tilde{V}^{(4)}$ in terms of ρ and λ starting from equation (4.66). This is achieved by writing the bilinear operators $S_{1c\gamma md\delta}$ in terms of creation and annihilation operators, equation (4.14). The next step is to write the creation and annihilation operators in terms of ρ and λ , using equations (4.3a and b), after a small amount of juggling (a technical term) we find

$$\tilde{V}^{(4)} \propto \underline{\rho}^2 \underline{\lambda}^2 - (\underline{\rho} \cdot \underline{\lambda})^2, \quad (4.68a)$$

$$\propto |\underline{\rho} \times \underline{\lambda}|^2 \quad (4.68b)$$

The last equation shows that $\tilde{V}^{(4)}$ is proportional to the area of the triangle mapped out by the three quarks in a baryon and hence is certainly a three-body force.

CHAPTER 5

HADRON OPERATORS IN A LATTICE GAUGE THEORY5.1 Lattice Quantum Chromodynamics

In chapter 3 of this thesis we used a naive non-relativistic quark model which included some "QCD-inspired" interactions to calculate part of the baryon spectrum. It is impossible to calculate hadron masses directly from QCD as bound states are non-perturbative (due to the infra-red singularities perturbation theory breaks down for QCD, see chapter 2.). Wilson (1975) introduced lattice gauge theories, in which space-time was discretised, as a possible means to calculate hadron masses and to study confinement. In this chapter we aim to write down operators which have the correct quantum numbers to represent various hadrons in their ground state and first excited level. Then, hopefully, by studying the particle correlation length on the lattice, the particle mass can be extracted.

Firstly, we give a skeletal review of how QCD may be written as a lattice gauge theory. For further details and better reviews the interested reader is directed to Drouffe and Itzykson (1978), Kogut (1979, 1983) and Wilson (1975). In continuous Minkowski space in the Feynman-Kac path-integral formalism, the partition function is

$$Z = \sum_{\text{paths}} \exp\{iS[\phi]\} \quad (5.1a)$$

where the action integral depends on the fields $\phi(x)$:

$$S[\phi] = \int \mathcal{L} dt = \int L d^3\underline{x} dt. \quad (5.1b)$$

Performing a Wick rotation, $t \rightarrow -it_E$, then the Euclidean action is defined as $S_E = -iS$, and then for a Euclidean field theory

$$Z = \int D\phi \exp\{-S_E[\phi]\}. \quad (5.2)$$

Thus the mean value $\langle \chi \rangle$ of an observable operator χ is given by

$$\langle \chi \rangle = \frac{1}{Z} \int D\phi \chi \exp\{-S_E[\phi]\}. \quad (5.3)$$

This four dimensional Euclidean space can be discretised to a hypercubical lattice of spacing a . In a lattice gauge theory particle fields are defined on the sites of the lattice and the gauge fields, which mediate the interaction are defined on the links between the lattice sites.

So for QCD, on each link in the lattice there is an $SU(3)$ matrix representing a gauge field,

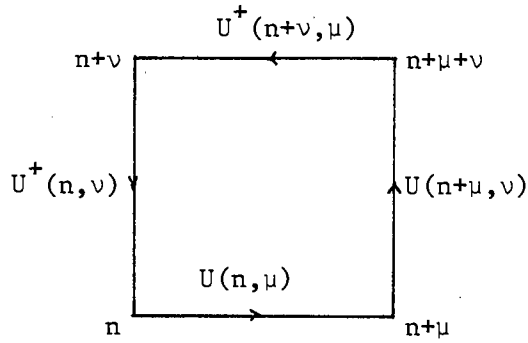
$$U(n,\mu) = \exp\{iB_\mu(n)\}, \quad (5.4)$$

where $B_\mu = iagF \cdot \underline{A}_\mu(n)$, the F 's are the eight 3×3 matrices which form the adjoint representation of $SU(3)$, g is the bare gauge coupling, n labels the lattice site and $\mu = 1,2,3,4$ labels the direction of the link, see figure 5.10. The fermion fields $\psi(n)$ are defined to be at each lattice site n . The action for QCD is taken to be (e.g. Wilson 1975),

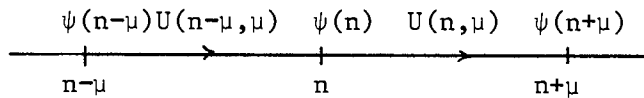
$$\begin{aligned} S = & -\beta \sum_{\square} \text{Re Tr} [U(n,\mu)U(n+\mu,\nu)U^\dagger(n+\nu,\mu)U^\dagger(n,\nu)] \\ & + \sum_{\{n\},\mu} \bar{\psi}(n) \frac{1}{2} \gamma_\mu \{U(n,\mu)\psi(n+\mu) - U^\dagger(n-\mu,\mu)\psi(n-\mu)\} \\ & + \sum_{\{n\}} m \bar{\psi}(n)\psi(n). \end{aligned} \quad (5.5)$$

In the first term the symbol ' \square ' denotes that the sum is over the elementary plaquettes shown in figure 5.1a, in the second term we

have a sum over all lattice sites (n) and directions (μ). In the second term we have used a symmetric form for the discrete difference operator as shown in figure 5.1b.



(a)



(b)

Figures 5.1 (a) Elementary Plaquette.

(b) Interaction picture between fermions.

This action is invariant under local gauge transformations, $\Omega(n)$, if the fermion and gauge fields transform in the following fashion,

$$\psi(n) \rightarrow \Omega(n)\psi(n) \tag{5.6a}$$

$$U(n, \mu) \rightarrow \Omega(n)U(n, \mu)\Omega^\dagger(n+\mu) \tag{5.6b}$$

The lattice action, equation (5.6), reduces to its familiar form in the continuum limit as $a \rightarrow 0$. (see Kogut 1983).

Using this standard lattice action the fermion propagator has extra unwanted poles in the continuum limit, the so-called fermion doubling problem. It is possible to avoid this problem by altering the standard action to project out the unwanted species (Wilson fermions) or by spreading the spinors four components over four lattice sites (Kogut-Susskind fermions). For details see the review articles referenced earlier.

To calculate particle masses we need to be able to calculate expectation values of products of fields, i.e.

$$\langle Q \rangle = \frac{1}{Z} \int DUD\psi D\bar{\psi} Q \exp\{-(S_G + S_F)\} \quad (5.7)$$

where S_G and S_F are the gauge-part and the fermion-part of the standard QCD action (5.6) respectively.

The Mathews-Salam trick (Mathews and Salam, 1954) can be used to integrate out the fermion fields as they are Grassman variables, e.g.

$$\begin{aligned} \int D\psi D\bar{\psi} \psi_i \bar{\psi}_j \exp\{-S_F\} &= \int D\psi D\bar{\psi} \psi_i \bar{\psi}_j \exp\{\Sigma \bar{\psi} (\not{D} + m) \psi\} \\ &= (\not{D} + m)_{ij}^{-1} [\det(\not{D} + m)]^{n_f}. \end{aligned} \quad (5.8a)$$

Thus

$$\langle \psi_i \bar{\psi}_j \rangle = \int DU e^{-S_G} [\det(\not{D} + m)]^{n_f} \underbrace{(\not{D} + m)_{ij}^{-1}}_{G_{ij}} \quad (5.8b)$$

where n_f is the number of quark flavours, \not{D} is the covariant discrete

difference operator and G_{ij} is the single quark propagator which obeys the following equation;

$$\begin{aligned} [\not{D} G](i,j|U) &= \frac{1}{2} \sum_{\mu} \{ (\gamma_{\mu} - r\mathbf{1}) U(i,\mu) G(i+\mu, j|U) \\ &\quad - (\gamma_{\mu} + r\mathbf{1}) U^+(i-\mu, \mu) G(i-\mu, j|U) \} \\ &= \delta_{ij} \quad , \end{aligned} \tag{5.9}$$

where \underline{r} is a 4-vector used in a modified Lagrangian to project out the unwanted fermion species (Wilson, 1975). The quantity $\det[\not{D}+m]$ represents the feedback of closed quark loops on the gluon Green's function. The computation of such a quantity in a Monte Carlo calculation is very time (or memory) consuming because it requires the calculation of a non-local quantity. Hamber and Parisi (1981) suggested making a "quenched" approximation, where $n_f = 0$ (or $\det(\not{D}+m) = 1$). If the number of flavours (N_f) is small compared to the number of colours (N_c), then this is a reasonable approximation: It gives an exact Zweig rule and hadrons are composed of valence quarks and gluons. This result becomes exact when $N_c \rightarrow \infty$ for fixed n_f ($n_f/N_c \rightarrow 0$). It has been noted that including internal quark loops only produces small corrections to physically observable quantities (for $n_f=3$) (Fucito et al, 1982).

Making this "quenched" approximation, it is possible to generate on the lattice an ensemble of gauge field configurations $\{U\}$ using Monte Carlo techniques with only a pure gauge weight, equation (5.8b). For each of these gauge field configurations the quark propagators, $G(n,0)$, can be calculated. This may be achieved using a Gauss-Siedel iterative procedure, i.e.

$$G' = \frac{1}{m} (\not{D}G + \delta) \quad (5.10a)$$

or a relaxation method, by looking at

$$\dot{G} = -(\not{D} + m)G + \delta \quad (5.10b)$$

as a function of a "time" variable. If $\dot{G} \rightarrow 0$ then $G \rightarrow$ required quark Green's function. These single quark Green's functions can then be used to construct the time-slice propagators for various hadrons. These time-slice propagators can then be averaged over the gauge field configurations. One can then attempt to extract the hadron masses by looking at the long distance decay of the Euclidean propagators of the hadron;

$$\sum_{\underline{n}} G_H(n,0) \sim \exp\{-m_H n_4\} \quad (5.10c)$$

see Bowler et al (1983), where m_H is the hadron mass.

5.2 Hadron Operators on a Lattice

The aim of this section is to construct lattice operators which have the correct quantum numbers to represent hadrons, both mesons and baryons. These operators can (theoretically) be used to calculate the mass spectra for the two species using the techniques outlined in the last section. Here we shall restrict ourselves to considering mesons and baryons in their ground state and then in their first excited state.

A general form for a meson operator is

$$M(n)_Q = \bar{\psi}^a(n) \Gamma_Q \psi_b(n) \delta_a^b \quad (5.11)$$

where $a, b = 1, 2, 3$ are $SU(3)_{\text{colour}}$ indices and Γ_Q is one of the 16 4×4 Dirac matrices ($= 1, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5, \sigma_{\mu\nu}$, where $\mu, \nu = 1, 2, 3, 4$). For convenience we shall restrict the discussion to hadrons which are composed of non-strange quarks. To illustrate how particle masses can be calculated in a lattice calculation, we give the outline of the calculation of the pion mass. (For further details see Bowler et al 1983).

An operator which has the correct transformation properties to represent a pion, which is a pseudoscalar meson, is

$$\pi(n) = \bar{\psi}(n)\gamma_5\psi(n). \quad (5.12)$$

The mass can then be calculated from the time slice propagator (Bowler et al, 1983),

$$\begin{aligned} G_\pi(n_4) &= \sum_{\underline{n}} G_\pi(n, 0) \equiv \sum_{\underline{n}} \langle \pi(n)\pi^\dagger(0) \rangle \\ &\propto e^{-m_\pi n_4} \\ & \quad n_4 \rightarrow \infty \end{aligned} \quad (5.13)$$

Therefore using equations (5.7) and (5.12) we find

$$\begin{aligned} G_\pi(n, 0) &\equiv \langle \pi(n)\pi^\dagger(0) \rangle, \\ &= \langle \bar{\psi}(n)\gamma_5\psi(n)\bar{\psi}(0)\gamma_5\psi(0) \rangle \\ &= \frac{1}{Z} \int DU D\psi D\bar{\psi} \{ \bar{\psi}(n)\gamma_5\psi(n)\bar{\psi}(0)\gamma_5\psi(0) \} e^{-(S_F+S_G)} \\ &= \frac{1}{Z} \int DU \left[\prod_{\underline{f}} \det(\not{D}+m) \right] \text{Tr}\{G(0, n)\gamma_5 G(n, 0)\gamma_5\} e^{-S_G} \end{aligned} \quad (5.14)$$

where $G(n, 0)$ is the quark Green's function, equation (5.10), and the

Mathews-Salam trick has been used to integrate out the fermion fields.

Using the following identity

$$\gamma_5 G(0,n) \gamma_5 \equiv G^+(n,0), \quad (5.15)$$

we can now re-write equation (5.14) in the quenched case ($n_f=0$) as

$$G_\pi(n,0) = \int DU \text{Tr}\{G^+(n,0)G(n,0)\} e^{-S_G}. \quad (5.16)$$

The above integral can now be computed using Monte Carlo techniques to calculate the pion mass (from (5.13)).

The lattice operators for some of the low-lying mesons and their transformation properties are given in table 5.1. In general the meson Green's function is

$$G_M(n,0) = \int DU \text{Tr}\{G^+(n,0) \Gamma_A G(n,0) \Gamma_A\} e^{-S_G}. \quad (5.17)$$

For further details about the calculation of these Green's functions see Bowler et al (1983) or Fucito et al (1982) and references therein.

Hasenfratz et al (1982) found that it was not possible (or very difficult) to obtain a mass for the P-wave ($L=1$) mesons using a hopping parameter expansion with Monte Carlo methods. This observation was also noted by Fucito et al (1982). So is there any difference between the two types of local operators, $L=0$ and $L=1$, in table 5.1?

A more familiar world in which to study hadrons is in the non-relativistic limit (at least for the author). In the non-relativistic limit the quark and antiquark spinors behave in the following manner;

$$\psi \rightarrow \begin{pmatrix} \omega \\ \frac{v}{c}\omega' \end{pmatrix} \quad \text{and} \quad \bar{\psi} \rightarrow \left(\frac{v}{c}\omega', \omega \right) \quad (5.18)$$

where ω and ω' are two-component Pauli-spinors (Berestetskii et al, 1971).

Particle	Operator	J^{PC}	L	S	Γ_A
π	$\bar{\psi}\gamma_5\psi$	0^{-+}	0	0	1
δ	$\bar{\psi}\psi$	0^{++}	1	1	γ_5
ρ	$\bar{\psi}\gamma_\mu\psi$	1^{--} $\mu=i$	0	1	$\gamma_\mu\gamma_5$
		0^{++} $\mu=0$	1	1	
A1	$\bar{\psi}\gamma_\mu\gamma_5\psi$	1^{++} $\mu=i$	1	1	γ_μ
		0^{-+} $\mu=0$	0	0	
B	$\bar{\psi}\sigma_{\mu\nu}\gamma_5\psi$	1^{+-} $\mu\nu=ij$	1	0	$\sigma_{\mu\nu}$
		1^{--} $\mu\nu=0i$	0	1	

Table 5.1 Lattice meson operators.

So we can look at the non-relativistic limit of two operators, one with $L=0$ the other with $L=1$.

(a) $\pi : J^{PC} = 0^{-+}$ ($L=0, S=0$),

$$\bar{\psi}\gamma_5\psi \xrightarrow{\frac{v}{c} \rightarrow 0} \begin{pmatrix} \frac{v}{c} \omega_1^t & \omega_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \frac{v}{c} \omega_2^t \end{pmatrix}$$

$$\sim \omega_1\omega_2 + O(v^2/c^2) \quad (5.19a)$$

(b) $\delta : J^{PC} = 0^{++}$ ($L=1, S=1$),

$$\bar{\psi}\psi \xrightarrow{\frac{v}{c} \rightarrow 0} \begin{pmatrix} \frac{v}{c} \omega_1^t, \omega_1 \end{pmatrix} \begin{pmatrix} \omega_2 \\ \frac{v}{c} \omega_2^t \end{pmatrix}$$

$$\sim \frac{v}{c} (\omega_1^t\omega_2 + \omega_1\omega_2^t) \quad (5.19b)$$

Equation (5.19a) shows that the S-wave meson, π , has a non-zero non-relativistic limit, whereas equation (5.19b) shows that the P-wave meson, δ , does not have a well defined limit. Performing the same sort of non-relativistic reduction on all the operators shows that only the S-wave mesons have non-zero, non-relativistic limits. Perhaps there may be a connection between the above observation, that these P-wave meson operators do not have good non-relativistic limits, and the bad pole structure of the propagators for these mesons.

Hasenfratz et al (1982) suggested that for the P-wave mesons extended lattice operators could be defined. In the continuum limit

$$B(x) = \bar{\psi}(x) \gamma_5 D_j \psi(x) , \quad (5.20a)$$

$$A_l(x) = \bar{\psi}(x) \{ \gamma_i D_j - \gamma_j D_i \} \psi(x) , \quad (5.20b)$$

represent operators coupled to the B and A1 P-wave mesons. Thus on a lattice these can be written as

$$B_i(n) = \{ \bar{\psi}(n) \gamma_5 U(n, \hat{i}) \psi(n+\hat{i}) - \bar{\psi}(n+\hat{i}) \gamma_5 U^+(n+\hat{i}) \psi(n) \} , \quad (5.20c)$$

$$A_{l,ij}(n) = \{ \bar{\psi}(n) \gamma_i U(n, \hat{j}) \psi(n+\hat{j}) - \bar{\psi}(n+\hat{j}) \gamma_i U^+(n+\hat{j}) \psi(n) \} - \{ i \leftrightarrow j \} . \quad (5.20d)$$

which have the correct transformation properties. By changing the minus to a plus in the parenthesis in equations (5.20a and b) operators coupled to the π^* - and the ρ -mesons, respectively, are obtained.

The authors found that the results for the masses of these extended operators were too high, but they did have a stable pole structure, whereas no sensible behaviour had been found for the local P-wave meson operators. Hence, it appears that these spacially

extended lattice operators are more appropriate in the study of P-wave mesons.

We now turn to look at baryon operators on a lattice. It is possible to write down a spin- $3/2$ baryon operator,

$$B(n)_{\mu\alpha} = (\psi^a(n)C\gamma_\mu\psi^b(n))\psi_\alpha^c(n)\epsilon_{abc} \quad (5.21a)$$

where $\mu = 1,2,3,4$ is a Lorentz index, α is a spinor index and ϵ_{abc} is the antisymmetric tensor for $SU(3)_{\text{colour}}$, see chapter 2. In (5.21) all the Dirac indices (except the free one) have been contracted out and the charge conjugation matrix (C) is present to ensure the operator has the correct Lorentz transformation properties. Similarly, a spin- $1/2$ baryon operator can be written (Hasenfratz et al, 1982) as

$$B(n)_\alpha = (\psi^a(n)C\gamma_5\psi^b(n))\psi_\alpha^c(n)\epsilon_{abc} \quad (5.21b)$$

where C is the charge conjugation matrix. From these operators we wish to construct the ground state delta resonance and the ground state proton, electromagnetic mass differences are being neglected.

Hence, for the $\Delta(1232)^{3/2^+}$; we could write

$$\Delta_{\mu\alpha}(n) = (U^a(n)C\gamma_\mu U^b(n))U_\alpha^c(n)\epsilon_{abc} \quad (5.22)$$

(e.g. Fucito et al, 1982). Are there any other operators which can be written down to produce an $I=3/2$, $J^P=3/2^+$ current? Thus, we write all possible currents with $I=3/2$, a free Lorentz index

(Ioffe, 1981)

$$\eta_{\mu\alpha}(n) = (u^a(n)C\gamma_\mu O_k u^b(n))O_k u_\alpha^c(n)\epsilon_{abc}, \quad (5.23)$$

where $O_k = (1, \gamma_5, \gamma_\lambda, \gamma_\lambda \gamma_5, \sigma_{\lambda\nu})$. Using the fact that $C^T = -C$, and transposing the spinors u^a and u^b in (5.24) gives,

$$u^a C u^b \epsilon_{abc} = u^a C \gamma_5 u^b \epsilon_{abc} = u^a C \gamma_\lambda \gamma_5 u^b \epsilon_{abc} = 0. \quad (5.24a)$$

Using the above results and method, with a Fierz re-arrangement gives

$$(u^a C \sigma_{\lambda\nu} u^b) \gamma_\mu \sigma_{\lambda\nu} u^c \epsilon_{abc} = 0. \quad (5.24b)$$

Finally, again using the Fierz transformation we find

$$(u^a C \gamma_\lambda u^b) \sigma_{\lambda\mu} u^c \epsilon_{abc} = 4i (u^a C \gamma_\mu u^b) u^c \epsilon_{abc}, \quad (5.24c)$$

in agreement with Ioffe (1981). Thus, the current in (5.23) is $I=3/2$, $J=3/2$ as required, it comes from the product of a vector and a spinor under $SU(2) \otimes (SU(2))$, i.e.

$$(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0) = (1, \frac{1}{2}) \otimes (0, \frac{1}{2}).$$

However, there is another completely symmetric state;

$$\Delta'_{\mu\nu\alpha} = \{ (u^a C \sigma_{\mu\nu} u^b) u^c_{L\alpha} + (u^a C \sigma_{\mu\nu} u^b) u^c_{R\alpha} \} \epsilon_{abc} \quad (5.25)$$

which represents a delta which transforms as a $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$

(Ramond, 1981).

For spin- $3/2$ fermion fields the appropriate equation to use is the Rarita-Schwinger equation,

$$(\not{p}-m)_{\alpha\beta} \eta^\mu_\beta(x) = 0 \quad (5.26a)$$

where the two conditions that

$$\gamma_\mu \eta^\mu_\beta(x) = 0 \quad p_\mu \eta^\mu_\beta(x) = 0 \quad (5.26b)$$

ensure that the operator $\eta^\mu_\beta(x)$ transforms as a spin- $3/2$ field (Rarita

and Schwinger, 1941). Thus, using the above two conditions we can project out of the operator $\Delta_{\mu\alpha}(n)$, defined by equation (5.22), the part which transforms as a pure spin- $3/2$ field.

Once again the non-relativistic limit of this operator can be studied, using equations (5.17a and b) we find,

$$u^a C_{\gamma\mu} u^b \xrightarrow{v/c \rightarrow \infty} -\omega^a \sigma_2 \sigma_i \omega^b + O\left(\frac{v^2}{c^2}\right) \quad \mu = i(=1,2,3) \quad (5.27a)$$

$$\frac{v}{c} (\tilde{\omega}^a \sigma_2 \omega^b + \omega^a \sigma_2 \tilde{\omega}^b) \quad \mu = 4. \quad (5.27b)$$

From above it can be seen that only the spacial part of the operator has a good non-relativistic limit. The r.h.s. of (5.27a) transforms as a vector under SU(2), i.e. has spin 1. Thus the relativistic operator for the Δ has the correct SU(6) flavour \otimes spin wavefunction in the non-relativistic limit, i.e.

$$\Delta_{i\alpha} = (u^a C_{\gamma i} u^b) u^c \epsilon_{abc} \xrightarrow{v/c \rightarrow 0} u_{\uparrow}^a u_{\uparrow}^b u_{\uparrow}^c \epsilon_{abc} \quad (5.28).$$

We now wish to construct an operator which represents the proton; Hasenfratz et al (1982) suggest,

$$p_{\alpha}(n) = \{u^a(n) C_{\gamma 5} d^b(n) - d^a(n) C_{\gamma 5} u^b(n)\} u_{\alpha}^c(n) \epsilon_{abc} \quad (5.29)$$

while Ioffe (1981) suggests

$$p'_{\alpha}(n) = (u^a(n) C_{\gamma} u^b(n)) [\gamma_5 \gamma_{\mu}]_{\alpha\beta} d_{\beta}^c(n) \epsilon_{abc} \quad (5.30a)$$

or

$$p''_{\alpha}(n) = (u^a(n) C \sigma^{\mu\nu} u^b(n)) [\sigma^{\mu\nu} \gamma_5]_{\alpha\beta} d_{\beta}^c(n) \epsilon_{abc}, \quad (5.30b)$$

which all possess the correct quantum numbers. However a Fierz rearrangement of the currents gives

$$p'_\alpha(n) = 2\{(u^a(n)Cd^b(n))[\gamma_5]_{\alpha\beta} u_\beta^c(n) - (u^a(n)C\gamma_5 d^b(n))u_\alpha^c(n)\}\epsilon_{abc} \quad (5.31a)$$

$$p''_\alpha = 2\{(u^a(n)Cd^b(n))[\gamma_5]_{\alpha\beta} u_\beta^c(n) - (u^a(n)C\gamma_5 d^b(n))u_\alpha^c(n)\}\epsilon_{abc} \quad (5.31b)$$

Comparing the above two operators with equation (5.29) shows that there are only two independent spin- $\frac{1}{2}$ currents which may be used to represent the proton. Looking at the non-relativistic limit of these gives

$$u^a C d^b \xrightarrow{\frac{v}{c} \rightarrow 0} \frac{v}{c} (\omega^a \sigma_2 \bar{y}^b + \tilde{\omega}^a \sigma_2 y^b) \quad (5.32a)$$

and

$$u^a C \gamma_5 d^b \xrightarrow{\frac{v}{c} \rightarrow 0} \omega^a \sigma_2 y^b + O\left(\frac{v^2}{c^2}\right), \quad (5.32b)$$

this shows that only one of the two operators has a definite non-relativistic limit.

Even though both terms in Hasenfratz et al's proton operator (5.28) have good non-relativistic limits, it does not reduce to the correct non-relativistic $SU(6)_{\text{flavour}} \otimes \text{spin}$ wavefunction. To satisfy this condition, the following operator should be used

$$p_\alpha(n) = \{u_\delta^a(n)d_\beta^b(n)u_\alpha^c(n) - d_\delta^a(n)u_\beta^b(n)u_\alpha^c(n) + u_\delta^a(n)u_\alpha^b(n)d_\beta^c(n) \\ d_\delta^a(n)u_\alpha^b(n)u_\beta^c(n) + u_\alpha^a(n)u_\beta^b(n)d_\delta^c(n) - u_\alpha^a(n)d_\beta^b(n)u_\delta^c(n)\} \\ \times (C\gamma_5)^{\delta\beta} \epsilon_{abc}. \quad (5.33)$$

What follows in the last of this section is speculation and unfortunately has not been derived rigorously. We intend to write down operators in the continuum limit which have the correct quantum

numbers to represent excited baryon states.

In the first excited level (the $[70,1^-]$) there is a $J^P = 5/2^-$, two $J^P = 3/2^-$ and two $J^P = 1/2^-$ nucleon resonances. The Rarita-Schwinger equations which describes a spin $k + \frac{1}{2}$ field are:

$$(\gamma_\mu \partial^\mu + m)\psi_{\mu_1 \dots \mu_k} = 0, \quad (5.34a)$$

$$\gamma_\alpha \psi_{\alpha \mu_2 \dots \mu_k} = 0, \quad (5.34b)$$

(Rarita and Schwinger, 1941) with the extra conditions that

$$\partial_\alpha \psi_{\alpha \mu_2 \dots \mu_k} = 0 \quad (5.34c)$$

and

$$\psi_{\alpha\alpha \mu_3 \dots \mu_k} = 0. \quad (5.34d)$$

Hence, the continuum operators

$$J^P = 5/2^- : p_{\mu\nu\alpha} = \{ (u^a C D_\mu \gamma_\nu u^b + u^a C D_\nu \gamma_\mu u^b) d^c - \frac{1}{2} \delta_{\mu\nu} (u^a C D_\sigma \gamma_\sigma u^b) d^c \} \epsilon_{abc}, \quad (5.35)$$

$$J^P = 3/2^- : p_{\mu\nu\alpha} = (u^a C D_\mu \gamma_\nu u^b - u^a C D_\nu \gamma_\mu u^b) d^c \epsilon_{abc}, \quad (5.36)$$

$$p_{\mu\alpha} = (u^a C D_\mu \gamma_5 d^b) u^c \epsilon_{abc}, \quad (5.37)$$

$$J^P = 1/2^- : p_\alpha = (u^a C D_\mu \gamma^\mu u^b) d^c \epsilon_{abc}, \quad (5.38)$$

have the correct transformation properties and a definite non-relativistic

limit to represent their respective nucleon resonance. The above set of operators is not a complete set. With a complete set of operators it may be possible to construct operators for the P-wave baryons which reduce to the appropriate $SU(6)_{\text{flavour}} \otimes \text{spin}$ wave-functions in the non-relativistic limit.

Unfortunately, the 'state of the art' is not yet at an advanced enough stage to study the P-wave baryons in these Monte Carlo simulations. Perhaps in the future lattice gauge theories will provide the answers to some of the questions posed by our naive phenomenological model earlier.

APPENDIX A

NOTATION AND BRANCHING RULES

The material in this appendix is a synopsis of the appendix A of the article of Bowler et al (1981).

In chapter 4 and in this appendix, the standard notation for the irreducible representations of unitary, orthogonal and symplectic groups is used, i.e. $\{\lambda_1, \lambda_2, \dots\}$, $[\lambda_1, \lambda_2, \dots]$ and $\langle \lambda_1, \lambda_2, \dots \rangle$, respectively, where $\lambda_1, \lambda_2, \dots$, is the partition specifying a Young diagram with row lengths λ_1, λ_2 etc. For the symmetric group S_n we use the notation $(\lambda_1, \lambda_2, \dots, k)$ to label irreducible representations, where $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$. For further details see Wybourne (1970).

The reduction of tensor representations $\{\underline{\lambda}\}$ of $U(6)$ using Young diagram techniques has been discussed in many texts (e.g. Wybourne, 1974), hence will not be repeated here. Irreducible tensor operators $[\lambda]$ and $\langle \lambda \rangle$ of $O(n)$ and $Sp(n)$, respectively, correspond to Young diagrams with $\leq n/2$ rows (where n is even for $Sp(n)$). (We have dropped the distinction between the various real forms of a complex algebra, as the results for finite-dimensional representations are the same). Tensors of $O(n)$ and $Sp(n)$ are traceless with respect to the appropriate metric tensor, symmetric (δ_{ab}) and antisymmetric (ϵ_{AB}), respectively. For $O(n)$ these are also associated pseudotensor representations $[\lambda]^*$: if the $n/2$ -th row length is non-zero for even n , then $[\lambda]^*$ is equivalent to $[\lambda]$. King (1975) derived the following branching rules for the symmetric and orthogonal groups.

$$Sp(st) \supset Sp(s) \otimes O(t); \langle \lambda \rangle = \sum_{\zeta} \langle \zeta/B \rangle \otimes [((\zeta/A \circ \zeta)/D)] , \quad (A1)$$

$$O(st) \supset O(s) \otimes O(t); [\lambda] = \sum_{\zeta} [\zeta/D] \otimes [((\zeta/C) \circ \zeta)/D] , \quad (A2)$$

$$O(st) \supset Sp(s) \otimes Sp(t); [\lambda] = \sum \langle \zeta/B \rangle \otimes \langle ((\zeta/C) \circ \zeta)/B \rangle . \quad (A3)$$

The notation /A, /B, /C and /D signifies division by all admissible elements of the following infinite collections (in Young tableaux notation);

$$\begin{aligned}
 A &= \{ \cdot, - \begin{array}{|c|} \hline \square \\ \hline \end{array}, + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, - \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \dots, \pm(\text{"legs = arms + 1"}) \} \\
 B &= \{ \cdot, \begin{array}{|c|} \hline \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}, \dots, (\text{even columns}) \} \\
 C &= \{ \cdot, - \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}, + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, - \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \dots, \pm(\text{"arms = legs + 1"}) \} \\
 D &= \{ \cdot, \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \dots, (\text{"even rows"}) \}
 \end{aligned}$$

For a given term μ in $(\lambda/A$ or $\lambda/C)$, ζ runs over all allowed diagrams with the same number of boxes as μ , the produce $\mu \circ \zeta$ is the appropriate Kronecker product of the appropriate permutation group representations.

For S_3 these correspond to the familiar rules,

$$\begin{aligned}
 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} &= \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \\
 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \circ \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}
 \end{aligned}$$

for S_4 (and higher n) the rules are not so familiar, e.g.

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \circ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

In these reductions, non-standard Young tableaux may arise. For these diagrams, there are modification rules whereby a continuous "boundary hook", h, (a line of boxes in contact on the surface of the diagram,

starting in the first box last row) of a certain length is to be removed. If h , beginning in column 1, row p , and ending in column x can be removed to reveal a standard Young diagram, then the contribution of the original diagram is taken to be

$$[\lambda] = (-1)^{x-1} [\lambda - h]^* , \text{ length } |h| = 2p-n \quad (\text{A.4})$$

$$\langle \lambda \rangle = (-1)^x \langle \lambda - h \rangle , \text{ length } |h| = 2p-n-2 . \quad (\text{A.5})$$

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