Rigorous asymptotics for the Lamé, Mathieu and spheroidal wave equations with a large parameter

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Doctor of Philosophy

The University of Edinburgh

2016

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise. This work has not been submitted for any other degree or professional qualification.

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September 26, 2016

Publications

[1] K. Ogilvie and A. B. Olde Daalhuis, Rigorous asymptotics for the Lamé and Mathieu functions and their respective eigenvalues with a large parameter, SIGMA Symmetry Integrability Geom. Methods Appl., vol. 11, pp. Paper 095, 11, 2015.

Lay Summary

Real world phenomena are frequently described in science by differential equations. Since their introduction by Newton and Leibniz in the latter half of the 17th century, the theory of such equations has been extensively studied by generations of mathematicians and physicists.

Many differential equations have a free parameter. When this parameter takes on special values, termed *eigenvalues*, the differential equation admits special solutions called *eigenfunctions*. These are the solutions which are typically of interest in physical applications.

Often differential equations cannot be solved explicitly, and thus some approximation theory is needed to describe solutions, and in turn the corresponding eigenvalues, if they exist. If the case of interest concerns some variable or parameter which tends to some limit, we do this using asymptotic methods.

This thesis concerns three differential equations, the so-called Lamé, Mathieu and spheroidal wave equations. The case of interest in physical applications is a parameter in these equations becomes large. We concentrate on approximating the eigenfunctions and eigenvalues in each of these three cases, using uniform asymptotic methods.

Abstract

We are interested in rigorous asymptotic results pertaining to three different differential equations which lie in the Heun class (see [1] §31). The Heun class contains those ordinary linear second-order differential equations with four regular singularities.

We first investigate the Lamé equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(h - \nu(\nu+1)k^2 \operatorname{sn}^2(z,k)\right) w = 0, \quad z \in [-K, K],$$

where $0 < k < 1$, $\operatorname{sn}(z,k)$ is a Jacobi elliptic function, and

$$K = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

is the complete elliptic integral of the first kind. We obtain rigorous uniform asymptotic approximations complete with error bounds for the Lamé functions $Ec_\nu^m\left(z,k^2\right)$ and $Es_\nu^{m+1}\left(z,k^2\right)$ for $z \in [0,K]$ and $m \in \mathbb{N}_0$, and rigorous approximations for their respective eigenvalues a_ν^m and b_ν^{m+1}, as $\nu \to \infty$. Then we obtain asymptotic expansions for the Lamé functions complete with error bounds, which hold only in a shrinking neighbourhood of the origin as $\nu \to \infty$. We also find corresponding expansions for the eigenvalues complete with order estimates for the errors. Then finally we give rigorous result for the exponentially small difference between the eigenvalues b_ν^{m+1} and a_ν^m as $\nu \to \infty$.

Second we investigate Mathieu's equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\lambda - 2h^2 \cos 2z\right) w = 0, \quad z \in [0, \pi],$$

and obtain analogous results for the Mathieu functions $ce_m(h,z)$ and $se_{m+1}(h,z)$ and their corresponding eigenvalues a_m and b_{m+1} for $m \in \mathbb{N}_0$ as $h \to \infty$, which are derived from those of Lamé's equation by considering a limiting case.

Lastly we investigate the spheroidal wave equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)\frac{\mathrm{d}w}{\mathrm{d}z}\right) + \left(\lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right) w = 0, \quad z \in [-1, 1],$$

and consider separately the cases where $\gamma^2 > 0$ and $\gamma^2 < 0$. In the first case we give similar results to those previously for the prolate spheroidal wave functions $\mathrm{Ps}(z, \gamma^2)$ and their corresponding eigenvalues λ_n^m for $m, n \in \mathbb{N}_0$ and $n \geq m$ as $\gamma^2 \to \infty$, and in the second we discuss the gap in theory which makes it difficult to obtain rigorous results as $\gamma^2 \to -\infty$, and how one would bridge this gap to obtain these.

Acknowledgements

For the last many years, I give my greatest thanks to my supervisor Adri. His support, encouragement, and mathematical guidance has led me to this point.

Contents

1

Introduction

This thesis is a study of second-order linear ordinary differential equations which lie in the so-called Heun class of differential equations, which are those ordinary linear second-order differential equations with four regular singularities. This class appears to be a simple step up from the more widely known hypergeometric class, which are those ordinary linear second-order differential equations with three regular singularities, yet far fewer results are known for the Heun class. This is due to the analytical complexity which is introduced. As such, rigorous results for special eigenfunctions and corresponding eigenvalues are lacking, a problem which we aim to resolve for several differential equations. Rigorous and explicit error bounds for approximations to solutions of differential equations developed by Frank W. J. Olver [2] provide the analytical tools to tackle these problems.

This thesis is structured as follows. First in the introduction we give some background theory for differential equations, some basic asymptotic results, and discuss uniform asymptotic approximations for solutions of certain differential equations. Chapter 2 will discuss the Lamé differential equation, and we provide rigorous results for the Lamé functions and their corresponding eigenvalues for a parameter in the differential equation becoming large. In chapter 3 we will realise analagous results for the Mathieu functions and their corresponding eigenvalues as a limiting case of Lamé's equation. In chapter 4, the problems are twofold. First we consider the case where a parameter becomes large and positive, and provide rigorous results for the prolate spheroidal wave functions and their corresponding eigenvalues, which are similar to those given in the previous chapters. Secondly we give a discussion on an unfinished piece of work. In the case where a parameter becomes large and negative, solutions are called oblate spheroidal wave functions. The analysis to provide rigorous and explicit error bounds for approximations to these functions is yet to be provided, and as such we only provide an outline of how one would tackle these problems.

Now we discuss some introductory theory.

1.1 Theory of ordinary differential equations

Many special functions are known to satisfy linear ordinary differential equations of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + f(x)\frac{\mathrm{d}w}{\mathrm{d}x} + g(x)w = 0. \tag{1.1}$$

If $f(x)$ and $g(x)$ are continuous on a finite or infinite interval (a, b) then (1.1) has infinitely many solutions which are twice continuously differentiable on (a, b). If the values of w and $\mathrm{d}w/\mathrm{d}x$ are prescribed at any point then the solution is unique.

Linearly independent solutions

The differential equation (1.1) has two linearly independent solutions w_1 and w_2 in (a, b), which satisfy the Wronskian relation

$$\mathcal{W}(w_1(x), w_2(x)) = w_1(x)w_2'(x) - w_1'(x)w_2(x) \neq 0 \tag{1.2}$$

for all $x \in (a, b)$. Every solution w of (1.1) on (a, b) can be expressed in the form

$$w(x) = Aw_1(x) + Bw_2(x), \tag{1.3}$$

where A and B are constants, and such a pair of solutions is referred to as a fundamental pair.

Classification of singularities

An important topic of study is the behaviour of solutions to equations of type (1.1) in the neighbourhood of singularities. If

(i) f and g are both analytic at $x = x_0$, then x_0 is called an *ordinary point*;

(ii) x_0 is not an ordinary point but both $(x - x_0)f(x)$ and $(x - x_0)^2 g(x)$ are both analytic there, then x_0 is called a *regular singularity*;

(iii) x_0 is neither an ordinary point nor a regular singularity, then it is called an *irregular singularity.*

Irregular singularities can be further classed into ranks. Corresponding to each singularity is an exponent pair which corresponds to the behaviour of solutions in a neighbourhood of the singularity. The theory concerning solutions in the neighbourhood of singular points is rich and interesting, but is not relevant to this thesis so we will not discuss the details here. The reader is directed to [2] for a comprehensive discussion.

Numerically satisfactory solutions

Two solutions of a differential equation of the form (1.1) are *numerically satisfactory* if in a neighbourhood, or sectorial neighbourhood, of a singularity x_0, one solution is recessive, r(x) say, and one is dominant, d(x) say, meaning that

$$\frac{r(x)}{d(x)} \to 0 \quad (x \to x_0). \tag{1.4}$$

Any solution which is linearly independent of the recessive solution at a singularity is referred to as a dominant solution. A solution which is recessive at one singularity is not usually recessive at another, hence if there is more than one singularity then several standard solutions usually need to be chosen to have numerically satisfactory representations everywhere.

Equations with a real parameter

Consider the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + f(u, x)\frac{\mathrm{d}w}{\mathrm{d}x} + g(u, x)w = 0, \tag{1.5}$$

where $u \in [u_0, u_1]$, $x \in [a, b]$, and f and g are functions which are continuous in both variables. If x_0 is a fixed point such that the values of w and $\partial w/\partial x$ at x_0 are prescribed as continuous functions of u, then w, $\partial w/\partial x$ and $\partial^2 w/\partial x^2$ are continuous in both variables.

The hypergeometric differential equation

In this section we look at the Heun differential equation, and discuss the complexities associated with it. It is natural to first introduce a simpler, widely known differential equation called the *hypergeometric equation*, typically written in the form

$$x(1-x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + (c - (a+b+1)x)\frac{\mathrm{d}w}{\mathrm{d}x} - abw = 0, \tag{1.6}$$

where a, b and c are real or complex parameters. This equation has regular singularities at $0, 1$ and ∞ with exponent pairs $\{0, 1-c\}$, $\{0, c-a-b\}$, and $\{a, b\}$ respectively. One way to see the regularity of the singularity at ∞ is to map it to the origin. Any homogeneous linear second-order differential equation with at most three regular singularities can be transformed into the hypergeometric equation by a change of variables.

If one replaces x by x/b and lets $b \to \infty$, and subsequently replaces c by b, we get the limiting form of the hypergeometric equation called the *confluent hypergeometric equation* in the form

$$x\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + (b-x)\frac{\mathrm{d}w}{\mathrm{d}x} - aw = 0. \tag{1.7}$$

This equation has a regular singularity at the origin with exponent $\{0, 1-b\}$, and an irregular singularity at ∞. The regular singularities of the hypergeometric equation at b and ∞ are coalesced into an irregular singularity at ∞. This special case of the hypergeometric equation is of great interest in physical applications, and special cases of the confluent hypergeometric equation include the Airy, Bessel, Hermite, Laguerre, and parabolic cylinder differential equations (see [1] §13.6).

1.2 Heun's differential equation

Since many of the special functions of interest that arise in physical applications are special cases of the hypergeometric differential equation, a unified theory of this class of equations was of much interest. It might then seem a simple step to consider a general linear second-order differential equation with four regular singularities instead of three, with the property that any linear second-order differential equation with at most four regular singularities can be transformed into it. For this reason Heun in 1889 introduced the following equation known as Heun's differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\alpha\beta x - q}{x(x-1)(x-a)} w = 0, \tag{1.8}$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon,\ a$ and q are parameters, and where $a \neq 0,\ 1$. Any linear second-order differential equation with at most four regular singularities can be transformed into the form (1.8). This equation has regular singularities at $0,\ 1,\ a,\ \infty$, with exponent pairs $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$, and $\{\alpha, \beta\}$ respectively. The sum of these eight exponents must equal two (theory of Fuchsian differential equations (see [3]) state that if a differential equation of second-order has ρ finite singularities and a singularity at ∞, then the sum of its exponents must equal $\rho - 1$) hence there is a condition on the parameters that

$$\alpha + \beta + 1 = \gamma + \delta + \epsilon. \tag{1.9}$$

The parameters $\alpha,\ \beta,\ \gamma,\ \delta$, and ϵ are regarded as exponent parameters, a as a singularity parameter, and q an accessory parameter which plays the role of an eigenvalue parameter in some applications.

This class of equations is one step up from the hypergeometric class of differential equations, but the complexity introduced in studying this equation is significant. We give two important examples of the difficulties faced. When one tries to obtain power series expansions for solutions, the recurrence relations which determine the coefficients are now more complicated than for the hypergeometric case and it is generally impossible to write down explicit series representations. Another analytic drawback is the general lacking of integral representations for solutions in terms of simpler functions (in general "simpler" means that the differential equations they satisfy are less complicated), which are available to those in the hypergeometric class.

Just as the hypergeometric equation has the confluent hypergeometric equation as a significant special case of interest, the Heun equation has four special cases of interest which occur when regular singularities merge to form irregular singularities: the confluent Heun equation, the doubly-confluent Heun equation, the biconfluent Heun equation, and the triconfluent Heun equation. These result by the confluence of a finite singularity with the singularity at ∞, the confluence of two separate pairs of singularities, the confluence of two finite singularities with the singularity at ∞ and when the three finite singularities confluence with the singularity at ∞ respectively.

The special cases of Heun's equation discussed in this thesis are as follows.

Lamé's equation

In (1.8) setting $\gamma = \delta = \epsilon = \frac{1}{2}$ means necessarily from (1.9) that $\alpha + \beta = \frac{1}{2}$, and writing

$$\alpha = -\frac{1}{2}\nu, \ \beta = \frac{1}{2}(\nu + 1), \ q = -\frac{1}{4}ah, \tag{1.10}$$

(1.8) becomes Lamé's equation of order ν

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-a}\right)\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{ah - \nu(\nu+1)x}{4x(x-1)(x-a)}w = 0. \tag{1.11}$$

Without loss of generality it is assumed that $\nu \geq -\frac{1}{2}$ since (1.11) is unchanged when ν is replaced by $-\nu - 1$. Setting

$$a = k^{-2}, \quad x = \operatorname{sn}^2(z, k^2), \tag{1.12}$$

where $\operatorname{sn}(z, k^2)$ is the Jacobi elliptic sine function with modulus k $(0 < k < 1)$, (1.11) becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(h - \nu(\nu+1)k^2 \operatorname{sn}^2(z, k^2)\right) w = 0. \tag{1.13}$$

We consider in this thesis solutions in the interval $[-K, K]$, and where h, k and ν are real parameters such that $0 < k < 1$ and $\nu \geq -\frac{1}{2}$, and $K = K(k)$ is Legendre's complete elliptic integral of the first kind (see [1] §19.2(ii)). When h assumes the special values a_ν^m or b_ν^{m+1} for $m \in \mathbb{N}_0$, Lamé's equation admits even or odd periodic solutions denoted $Ec_\nu^m\left(z, k^2\right)$ or $Es_\nu^{m+1}\left(z, k^2\right)$ respectively.

Lamé's equation first appeared in a paper by Gabriel Lamé in 1837 [4]. It appears in the method of separation of variables applied to the Laplace equation in elliptic coordinates. Lamé functions have applications in antenna research, occur when studying bifurcations in chaotic Hamiltonian systems, and in the theory of Bose-Einstein condensates, to name but a few (see [1] §29.19).

Mathieu's equation

Mathieu's equation is a special case of the confluenced Heun equation. It can be recognised as a limiting form of Lamé's equation, where $k \to 0^+$. Mathieu's equation in its most recognisable form is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\lambda - 2h^2 \cos 2z\right) w = 0. \tag{1.14}$$

In this thesis we consider solutions in the interval $[0, \pi]$, and take λ and h to be real parameters. When λ assumes the special values a_m or b_{m+1} for $m \in \mathbb{N}_0$, Mathieu's equation admits even or odd periodic solutions denoted $ce_m(h, z)$ or $se_{m+1}(h, z)$ respectively.

These functions first arose from physical applications in 1868 in Émile Mathieu's study of vibrations in an elliptic drum [5]. Since they have appeared in problems pertaining to vibrational systems, electrical and thermal diffusion, electromagnetic wave guides, elliptical cylinders in viscous fluids, and diffraction of sound and electromagnetic waves, to name but a few. In general, they appear when studying solutions of differential equations that are separable in elliptic cylindrical coordinates. For an insight as to how Mathieu functions appear in physical applications see [6].

The spheroidal wave equation

The spheroidal wave equation is another special case of the confluenced Heun equation. It is most commonly given as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1 - z^2)\frac{\mathrm{d}w}{\mathrm{d}z}\right) + \left(\lambda + \gamma^2(1 - z^2) - \frac{\mu^2}{1 - z^2}\right) w = 0, \tag{1.15}$$

and in this thesis we consider solutions in the interval $[-1, 1]$, where λ, γ^2 and μ are real parameters. If λ assumes the special values λ_n^m for $m, n \in \mathbb{N}_0$ and $n \geq m$, and

$\mu = m$, then the corresponding special solutions are split into two classes; if $\gamma^2 > 0$ then solutions are called the *prolate* spheroidal wave functions, and if $\gamma^2 < 0$ then solutions are called the *oblate* spheroidal wave functions. These are the non-trivial solutions which are bounded at the singularities of the differential equation at $z = \pm 1$ and are denoted by $\mathrm{Ps}_n^m(z, \gamma^2)$.

This equation comes from separation of a special partial differential equation in spheroidal coordinates where γ^2 is real, but negative or positive depending on whether the spheroids are oblate or prolate (see [1] §30.13 & §13.14). It is a generalisation of Mathieu's equation, and appears most prominently in applications related to signal analysis.

1.3 Introduction to asymptotics

In physical applications, situations of interest often include when some parameter in the system becomes very small or large, and to study these one typically uses asymptotic analysis. In particular in this thesis, we will study the behaviour of a function when some parameter in the function becomes large.

Symbols

To describe the behaviour of a function as its variable tends to ∞ we use the following notation:

(i) If $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$ then in this limit, $f(x) \sim g(x)$,

(ii) If $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$ then in this limit, $f(x) = o(g(x))$,

(iii) If $\frac{f(x)}{g(x)}$ is bounded as $x \to \infty$ then in this limit, $f(x) = \mathcal{O}(g(x))$.

Two notable cases include $f(x) = \mathcal{O}(1)$ as $x \to \infty$ which means $f(x)$ is bounded in this limit, and $f(x) = o(1)$ as $x \to \infty$ which means $f(x)$ vanishes in this limit. This notation can also be used when the limit is some other point.

Asymptotic expansions

The following results about asymptotic expansions can be found in §1 of [2].

Theorem 1.1. *If* $\sum_{s=0}^{\infty} a_s z^s$ *converges for* $|z| < r$*, then for each non-negative integer* n

$$\sum_{s=n}^{\infty} a_s z^s = \mathcal{O}(z^n). \tag{1.16}$$

However many expansions are often not convergent and are called *divergent* series. Suppose we have the function $f(z)$, where z is real or complex, and the formal power series expansion (not necessarily convergent) $\sum_s a_s z^{-s}$ such that

$$f(z) = \sum_{s=0}^{n-1} a_s z^{-s} + R_n(z). \tag{1.17}$$

If $R_n(z)$ is $\mathcal{O}(z^{-n})$ as $z \to \infty$ in some unbounded region $\mathbf{Z}$ in $\mathbb{R}$ or $\mathbb{C}$ for each fixed n then we say that $\sum_s a_s z^{-s}$ is an *asymptotic (Poincaré) expansion* of $f(z)$ as $z \to \infty$ and that

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + ... \quad (z \to \infty \text{ in } \mathbf{Z}). \tag{1.18}$$

This definition can be altered for some other finite limit point in $\mathbf{Z}$. All convergent series are asymptotic series by definition, but the use of the term usually refers to divergent series. Divergent series are, in a sense, often more powerful than convergent series, as typically when divergent series are truncated at an *optimal point* the errors are beyond all orders in the expansion parameter.

1.4 Uniform asymptotics for differential equations with a large parameter

Many special functions satisfy an equation of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(u^2 f(u,z) + g(u,z)\right) w, \tag{1.19}$$

where u is a real or complex parameter, and solutions of interest involve $|u|$ becoming large. In this case, asymptotic solutions which are uniform with respect to u are needed for z in some interval or region $\mathbf{Z}$ in $\mathbb{R}$ or $\mathbb{C}$. Here uniform means that the error bound corresponding to the asymptotic approximations holds uniformly in **Z**.

The form of the asymptotic approximations for solutions depends on the nature of the *transition points* of the differential equation. Transition points are zeros and poles of $f(u,z)$, with the zeros of $f(u,z)$ being more commonly referred to as *turning points.* This leads to a discussion of three main cases. We first discuss the methodology that applies to all three cases.

Introducing new variables $\{\zeta, W\}$, related to $\{z, w\}$ respectively by

$$W = \dot{z}^{-1/2} w, \quad \text{where } \dot{z} = \frac{\mathrm{d}z}{\mathrm{d}\zeta}, \tag{1.20}$$

(1.19) takes the form

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\zeta^2} = \left(u^2 \dot{z}^2 f(u,z) + \psi(u,\zeta)\right) W, \tag{1.21}$$

where

$$\psi(u,\zeta) = \dot{z}^2 g(u,z) + \dot{z}^{1/2} \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left(\dot{z}^{-1/2}\right). \tag{1.22}$$

The transformation is specialised by specifying the relationship between z and ζ. This is done by defining $\tilde{f}$ such that

$$\dot{z}^2 f(u,z) = \tilde{f}(u,\zeta), \tag{1.23}$$

where z and ζ are analytic functions of one another, and such that

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\zeta^2} = u^2 \tilde{f}(u,z) W \tag{1.24}$$

has solutions which are functions of a single variable. In order for this transformation to make sense, $\widetilde{f}(u,z)$ must have the same number and order of poles and zeros as $f(u,z)$ in the region considered, and $\psi(u,\zeta)$ should be negligible as $u \to \infty$. This transformation is commonly referred to as the *Liouville transformation.* One chooses $\tilde{f}(u,\zeta)$ conveniently such that it satisfies the above conditions and so that approximants are in the most natural form. These approximants exhibit similar behaviour as the solutions you want to approximate. We now discuss the simplest case, where there are no transition points.

(I) If $f(u,z)$ has no zeros or singularities, specifying that

$$\dot{z}^2 f(u,z) = 1, \text{ giving } \zeta = \int f^{1/2}(z)\mathrm{d}z, \tag{1.25}$$

gives approximations for solutions in terms of exponential functions.

In this case, formal solutions of (1.19) can be given in the form

$$\begin{aligned} w_1(z) &\sim f^{-1/4}(u,z)\exp\left(\int f^{1/2}(u,z)\mathrm{d}z\right), \\ w_2(z) &\sim f^{-1/4}(u,z)\exp\left(-\int f^{1/2}(u,z)\mathrm{d}z\right). \end{aligned} \tag{1.26}$$

These approximations are widely known as the WKB approximations, named so after Wentzel (1926), Kramers (1926) and Brillouin (1926) for their independent contributions to the theory. Their contributions however weren't the form of the approximations themselves, but rather determining the arbitrary constants in the formulae which connect the approximate solutions on either side of a turning point. Even this work was unknowingly predated by Jeffreys (1924) who solved this connection problem, and as such the term WKB is sometimes amended to WKBJ, or some permutation thereof. Jeffreys also later noted that he himself had overlooked an even earlier derivation by Gans. The approximations themselves date back to Liouville (1837) and Green (1837) who published them independently, with similar techniques dating back as far back to the work of Carlini (1812). As such, the solutions are also sometimes referred to, especially in the work of Olver, as the Liouville-Green approximations. The proof that these approximations are asymptotically equivalent to particular solutions of the differential equation can be shown by constructing a so-called Volterra integral equation for the difference between the solution and approximation.

This method is now a part of a larger general theory which dates back to the work of Langer, and is not how these results were first derived. Langer, in a series of papers starting from 1931, first developed a transformation for this simple case in a similar manner to the above. The singularity of the functions in the approximations at a turning point, i.e. at a zero of $f(u,z)$, is introduced by the transformations given to solve the problem. Thus he developed a similar method for the case where there is a zero of $f(u,z)$, and provided uniform asymptotic approximations for solutions in this case, which we will call case (II). Previously only local approximations in the neighbourhood of a turning point were available.

(II) If $f(u,z)$ contains one zero point, specifying that

$$\dot{z}^2 f(u,z) = \zeta, \text{ giving } \frac{2}{3}\zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t)\mathrm{d}t, \tag{1.27}$$

where z_0 is the zero of $f(u,z)$ in the z-plane, gives approximations for solutions in terms of Airy functions.

In this manner, the zero of $f(u,z)$ in the z-plane is mapped to $\zeta = 0$ in the ζ-plane. Langer provided the modified version of Liouville's transformation for this case, and gave the leading term of the asymptotic approximation together with a Liouville-Neumann series for the corresponding error. This theory was further developed later by Cherry (1950) and Olver [2], the latter who derived explicit bounds for the error terms.

The earliest investigation of a transition point being a simple pole appears to be in Langer's 1935 paper. We call this case (III).

(III) If $f(u,z)$ has a simple pole, specifying that

$$\dot{z}^2 f(u,z) = \frac{1}{\zeta}, \text{ giving } 2\zeta^{1/2} = \int_{z_0}^{z} f^{1/2}(t)\mathrm{d}t, \tag{1.28}$$

where z_0 is the simple pole of $f(u,z)$ in the z-plane, gives approximations for solutions in terms of Bessel functions.

In this manner, the simple pole of $f(u,z)$ in the z-plane is mapped to the simple pole at $\zeta = 0$ in the ζ-plane. For large values of u, Langer derived asymptotic approximations for solutions in terms of Bessel functions, which were valid in a shrinking neighbourhood of the transition point, which vanished as $|u| \to \infty$.

Further situations are treated in a similar manner. To summarise this method using the Liouville transformation, the differential equation is essentially transformed into a form such that approximants for the transformed equation behave similarly and in a sense "simpler", again meaning generally that the differential equation corresponding to the approximant is simpler. The theory of such transformations and explicit error bounds corresponding to the approximations can be found in [2]. It is these developments by Olver in constructing explicit error bounds corresponding to uniform asymptotic approximations for these types of equations which allow us to compose the work contained in this thesis.

Summarising Olver's technique for the error analysis, once the differential equation is in the form

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\zeta^2} = \Big(u^2 \tilde{f}(u,z) + \psi(u,\zeta)\Big) W, \tag{1.29}$$

a pair of solutions can be written down as

$$\begin{aligned} W_1(u,\zeta) &= V_1(u,\zeta) + \epsilon_1(u,\zeta), \\ W_2(u,\zeta) &= V_2(u,\zeta) + \epsilon_2(u,\zeta), \end{aligned} \tag{1.30}$$

where V_1 and V_2 are solutions of (1.24). To obtain rigorous bounds for ϵ_1, substituting the first of (1.30) into (1.29) gives the differential equation for ϵ_1,

$$\frac{\mathrm{d}^2 \epsilon_1(u,\zeta)}{\mathrm{d}\zeta^2} = \Big(u^2 \widetilde{f}(u,z) + \psi(u,\zeta)\Big) \epsilon_1(u,\zeta) + \psi(u,\zeta) V_1(u,\zeta). \tag{1.31}$$

Then using variation of parameters, we get the Volterra-type integral equation

$$\epsilon_1(u,\zeta) = \int^{\zeta} K(\zeta,t)\psi(u,t)\Big(\epsilon_1(u,t) + V_1(u,t)\Big)\mathrm{d}t, \tag{1.32}$$

where

$$K(\zeta,t) = V_1(u,\zeta)V_2(u,t) - V_2(u,\zeta)V_1(u,\zeta), \tag{1.33}$$

and the limits of the integral are chosen to identify the approximation with the correct solutions. The same type of integral equation can be written down for $\epsilon_2(u,\zeta)$.

In the rest of this section we give the general theory for how to bound errors of this type, which can be found in §6 of [2]. Take the standard form of the integral equation to be

$$h(\xi) = \int_\alpha^\xi K(\xi,\nu)\Big(\phi(\nu)J(\nu) + \psi_0(\nu)h(\nu) + \psi_1(\nu)h'(\nu)\Big)\mathrm{d}\nu. \tag{1.34}$$

Assumptions are as follows:

(i) The path of integration lies along a given path $\mathcal{P}$ comprising a finite chain of R_2 arcs in the complex plane. Either, or both, of the endpoints α and β, say, may be at infinity.

(ii) The real or complex functions $J(\nu)$, $\phi(\nu)$, $\psi_0(\nu)$ and $\psi_1(\nu)$ are continuous when $\nu \in (\alpha,\beta)_{\mathcal{P}}$, save for a finite number of discontinuities and infinities.

(iii) The real or complex kernel $K(\xi,\nu)$ and its first two partial ξ derivatives are continuous functions of both variables when $\xi,\nu \in (\alpha,\beta)_{\mathcal{P}}$, including the arc junctions. Here, and in what follows, all differentiations with respect to ξ are performed along $\mathcal{P}$.

(iv) $K(\xi,\xi) = 0$.

(v) When $\xi \in (\alpha,\beta)_{\mathcal{P}}$ and $\nu \in (\alpha,\xi]_{\mathcal{P}}$

$$|K(\xi,\nu)| \le P_0(\xi)Q(\nu), \quad \left|\frac{\partial K(\xi,\nu)}{\partial \xi}\right| \le P_1(\xi)Q(\nu), \quad \left|\frac{\partial^2 K(\xi,\nu)}{\partial \xi^2}\right| \le P_2(\xi)Q(\nu), \tag{1.35}$$

where the $P_j(\xi)$ and $Q(\nu)$ are continuous real functions, the $P_j(\xi)$ being positive.

(vi) When $\xi \in (\alpha,\beta)_{\mathcal{P}}$, the following integrals converge

$$\Phi(\xi) = \int_\alpha^\xi |\phi(\nu)\mathrm{d}\nu|, \quad \Psi_0(\xi) = \int_\alpha^\xi |\psi_0(\nu)\mathrm{d}\nu|, \quad \Psi_1(\xi) = \int_\alpha^\xi |\psi_1(\nu)\mathrm{d}\nu|, \tag{1.36}$$

and the following suprema are finite

$$\kappa \equiv \sup\left(Q(\xi)|J(\xi)|\right), \quad \kappa_0 \equiv \sup\left(P_0(\xi)Q(\xi)\right), \quad \kappa_1 \equiv \sup\left(P_1(\xi)Q(\xi)\right), \tag{1.37}$$

except that κ_1 need not exist when $\psi_1(\nu) = 0$.

Theorem 1.2. *With the foregoing conditions, equation (1.34) has a unique solution* $h(\xi)$ *which is continuously differential in* $(\alpha, \beta)_{\mathcal{P}}$ *and satisfies*

$$h(\xi)/P_0(\xi) \to 0, \quad h'(\xi)/P_1(\xi) \to 0 \quad (\xi \to \alpha \textit{ along } \mathcal{P}). \tag{1.38}$$

Furthermore,

$$\frac{|h(\xi)|}{P_0(\xi)}, \frac{|h'(\xi)|}{P_1(\xi)} \leq \kappa \Phi(\xi) \exp\left(\kappa_0 \Psi_0(\xi) + \kappa_1 \Psi_1(\xi)\right), \tag{1.39}$$

and $h''(\xi)$ *is continuous except at the discontinuities (if any) of* $\phi(\xi) J(\xi)$, $\psi_0(\xi)$, *and* $\psi_1(\xi)$.

The bounds for $h(\xi)$ and $h'(\xi)$ can be sharpened in the following common case:

Theorem 1.3. *Assume the conditions above, and also that* $\phi(\nu) = \psi_0(\nu)$, $\psi_1(\nu) = 0$. *Then the solution* $h(\xi)$ *satisfies*

$$\frac{|h(\xi)|}{P_0(\xi)}, \frac{|h'(\xi)|}{P_1(\xi)} \leq \frac{\kappa}{\kappa_0}\Big(\exp\left(\kappa_0 \Phi(\xi)\right) - 1\Big). \tag{1.40}$$

Olver derived these bounds by constructing uniformly convergent series for the errors, with bounds derived by majorizing these expansions (see [2] §6.10 for details). These bounds are only meaningful if $\psi(u, \xi)$ is in some sense small compared to $u^2 \tilde{f}(u, z)$ as $|u|$ becomes large. This technique of bounding the errors is prevalent in this thesis.

In the next three chapters of this thesis, we consider differential equations on the real line which have two turning points which coalesce as a parameter in the equation becomes large. In this case, specifying that

$$\dot{z}^2 f(u, z) = \zeta^2 - \alpha^2, \tag{1.41}$$

gives approximations for solutions in terms of parabolic cylinder functions. In this manner, the coalescing zeros of $f(u, z)$ we have in our equations in the z-plane are mapped to $\zeta = \pm\alpha$ in the ζ-plane, and these will coalesce into the origin there in this case. The theory involving this case and explicit error bounds are provided by Olver in [7], which is an important paper in relation to this thesis. We summarise the main result as follows.

Consider the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} = \left(u^2(\zeta^2 - \alpha^2) + \psi(u,\alpha,\zeta)\right) w. \tag{1.42}$$

Denote ζ_1 and ζ_2 to be the endpoints of an interval containing ζ, with ζ_1 negative and ζ_2 positive, where both may depend on α or be infinite. Additionally, $\Omega(z)$ denotes a conveniently chosen continuous function of the real variable z that is positive, except possibly when $z = 0$, and satisfies

$$\Omega(z) = \mathcal{O}(z) \quad (z \to \pm\infty). \tag{1.43}$$

With this, the error-control function is introduced as

$$F(u,\alpha,\zeta) = \int \frac{|\psi(u,\alpha,\zeta)|}{\Omega(\sqrt{2u}\zeta)}\mathrm{d}\zeta, \tag{1.44}$$

the choice of integration constant being immaterial.

Theorem 1.4. *Assume that for each value of u, the function $\psi(u,\alpha,\zeta)$ is continuous in the region $\alpha \in [0,A]$, $\zeta \in [0,\zeta_2)$, and*

$$\mathcal{V}_{0,\zeta_2}(F) \equiv \int_0^{\zeta_2} \frac{|\psi(u,\alpha,t)|}{\Omega(\sqrt{2u}t)}\mathrm{d}t \tag{1.45}$$

converges uniformly with respect to α. Then in this region equation (1.42) has solutions $w_1(u,\alpha,\zeta)$ and $w_2(u,\alpha,\zeta)$ which are continuous, have continuous first and second partial ζ-derivatives, and are given by

$$\begin{aligned} w_1(u,\alpha,\zeta) &= U\left(-\frac{1}{2}u\alpha^2, \sqrt{2u}\zeta\right) + \epsilon_1(u,\alpha,\zeta), \\ w_2(u,\alpha,\zeta) &= \overline{U}\left(-\frac{1}{2}u\alpha^2, \sqrt{2u}\zeta\right) + \epsilon_2(u,\alpha,\zeta), \end{aligned} \tag{1.46}$$

where

$$\begin{aligned} &\frac{|\epsilon_1(u,\alpha,\zeta)|}{\mathbf{M}(-\frac{1}{2}u\alpha^2, \sqrt{2u}\zeta)}, \frac{|\partial\epsilon_1(u,\alpha,\zeta)/\partial\zeta|}{\sqrt{2u}\mathbf{N}(-\frac{1}{2}u\alpha^2, \sqrt{2u}\zeta)} \\ &\qquad \le \mathbf{E}^{-1}\left(-\frac{1}{2}u\alpha^2, \sqrt{2u}\zeta\right)\left(\exp\left(\frac{1}{2}\sqrt{\frac{\pi}{u}}l_1\left(-\frac{1}{2}u\alpha^2\right)\mathcal{V}_{\zeta,\zeta_2}(F)\right) - 1\right), \end{aligned} \tag{1.47}$$

$$\frac{|\epsilon_2(u,\alpha,\zeta)|}{\mathbf{M}(-\frac{1}{2}u\alpha^2,\sqrt{2u}\zeta)},\ \frac{|\partial\epsilon_2(u,\alpha,\zeta)/\partial\zeta|}{\sqrt{2u}\mathbf{N}(-\frac{1}{2}u\alpha^2,\sqrt{2u}\zeta)}$$
$$\leq \mathbf{E}\left(-\frac{1}{2}u\alpha^2,\sqrt{2u}\zeta\right)\left(\exp\left(\frac{1}{2}\sqrt{\frac{\pi}{u}}l_1\left(-\frac{1}{2}u\alpha^2\right)\mathcal{V}_{0,\zeta}(F)\right)-1\right), \qquad (1.48)$$

$$\textit{and}\ \ l_1(b)=\sup_{z\in(0,\infty)}\left(\Omega(z)\mathbf{M}^2(b,z)/\Gamma\left(\frac{1}{2}-b\right)\right)\quad (b\leq 0). \qquad (1.49)$$

The functions U and $\overline{U}$ are parabolic cylinder functions, and the functions **E**, **M** and **N** functions are defined in terms of parabolic cylinder functions, and will be defined explicitly in the following section.

At the end of chapter 4, we will discuss briefly another case, where a turning point coalesces with a simple pole when a parameter becomes large. If $f(u,z)$ has a simple pole and a turning point, if we specify that

$$\dot{z}^2 f(u,z)=\frac{\zeta-\alpha}{\zeta}, \qquad (1.50)$$

then approximations for solutions are given in terms of Whittaker functions. In this manner, the simple pole of $f(u,z)$ is mapped to the origin of the ζ-plane, and the turning point will be mapped to $\zeta=\alpha$. In our equation the tuning point will coalesce with the simple pole, thus we will have $\alpha\to 0$ as $u\to\infty$. The theory for this is as of yet incomplete, and as such we only provide a short discussion of the problem.

For all of these cases one might consider whether asymptotic expansions can be derived for solutions of the differential equations, instead of just a one term approximations. Indeed for case (I), one could try for solutions of the form

$$W_1(u,\zeta)\sim e^{u\zeta}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{u^s},\ \ W_2(u,\zeta)\sim e^{-u\zeta}\sum_{s=0}^{\infty}(-1)^s\frac{A_s(\zeta)}{u^s}, \qquad (1.51)$$

as $u\to\infty$. In case (II), one would need to add a second term in the solutions involving the derivative of the Airy functions,

$$W_1(u,\zeta)\sim \mathrm{Ai}(u^{2/3}\zeta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{u^{2s}}+\frac{\mathrm{Ai}'(u^{2/3}\zeta)}{u^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{u^{2s}}, \qquad (1.52)$$

$$W_2(u,\zeta) \sim \mathrm{Bi}(u^{2/3}\zeta)\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{u^{2s}} + \frac{\mathrm{Bi}'(u^{2/3}\zeta)}{u^{4/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{u^{2s}}, \tag{1.53}$$

since, if this second term wasn't there, the recurrence relations the coefficients A_s must satisfy can only hold when $\psi(u,\zeta) = 0$. The other cases are treated in a similar manner, which we shall discuss in this thesis. The error analysis for these expansions follows similarly from the theory detailed above.

1.5 Preliminaries: Parabolic cylinder functions

The parabolic cylinder functions satisfy the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\tfrac{1}{4}z^2 + b\right) w. \tag{1.54}$$

When $b < 0$ this equation has real turning points at $z = \pm 2\sqrt{-b}$, and solutions oscillate in the interval defined by these endpoints. When $b > 0$ there are no turning points and no oscillations.

The two standard solutions of (1.54) we shall use are $U(b,z)$ and $\overline{U}(b,z)$, and the case $b < 0$ will be of interest in this thesis.

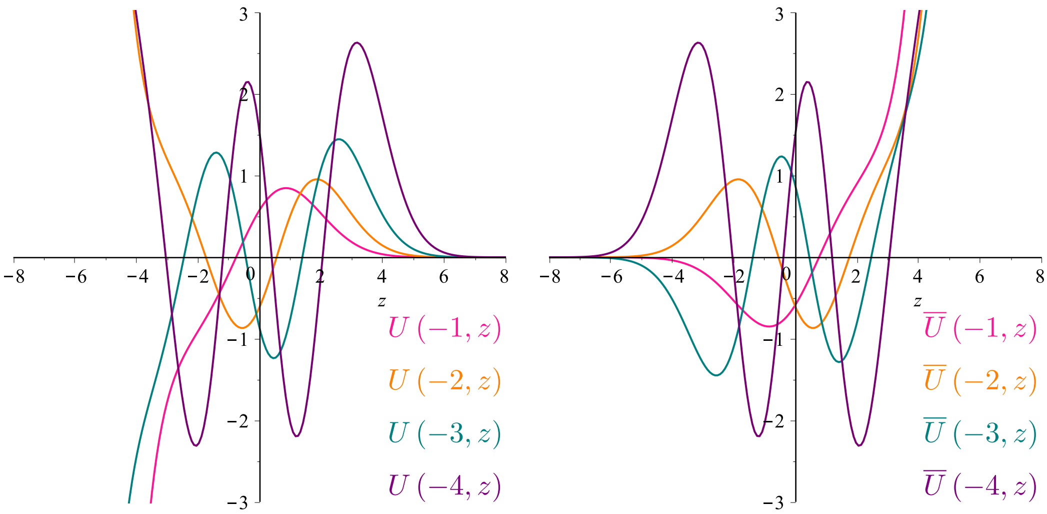

Figure 1.1: Plots for assorted parabolic cylinder U and $\overline{U}$ functions.

Properties of U and $\overline{U}$

Here we will summarise the relevant properties of the parabolic cylinder functions, which can be found in (§12 [1]) and [7].

These functions satisfy the Wronskian relation

$$\mathcal{W}\left(U(b,z),\overline{U}(b,z)\right)=\sqrt{\frac{2}{\pi}}\Gamma\left(\tfrac{1}{2}-b\right), \tag{1.55}$$

take the following values at $z=0$

$$\begin{aligned} U(b,0)&=\frac{\sqrt{\pi}}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma\left(\frac{3}{4}+\frac{1}{2}b\right)}, \quad U'(b,0)=\frac{\sqrt{\pi}}{2^{\frac{1}{2}a-\frac{1}{4}}\Gamma\left(\frac{1}{4}+\frac{1}{2}b\right)},\\ \overline{U}(b,0)&=\frac{\sqrt{\pi}2^{\frac{1}{2}a+\frac{1}{4}}\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{3}{4}-\frac{1}{2}a\right)^2\Gamma\left(\frac{1}{4}+\frac{1}{2}b\right)}, \quad \overline{U}'(b,0)=\frac{\sqrt{\pi}2^{\frac{1}{2}a+\frac{3}{4}}\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{4}-\frac{1}{2}a\right)^2\Gamma\left(\frac{3}{4}+\frac{1}{2}b\right)}, \end{aligned} \tag{1.56}$$

and satisfy the reflection formulae

$$\begin{aligned} U(b,-z)&=-\sin(\pi b)U(b,z)+\pi\frac{\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{2}+b\right)}\overline{U}(b,z),\\ \overline{U}(b,-z)&=-\frac{\cos(\pi b)}{\Gamma\left(\frac{1}{2}-b\right)^2}U(b,z)+\sin(\pi a)\overline{U}(b,z). \end{aligned} \tag{1.57}$$

One can then see that when $b=-m-\frac{1}{2}$ for $m\in\mathbb{N}_0$ we have

$$\begin{aligned} U(-m-\tfrac{1}{2},-z)&=(-1)^m U(-m-\tfrac{1}{2},z),\\ \overline{U}(-m-\tfrac{1}{2},-z)&=(-1)^{m+1}\overline{U}(-m-\tfrac{1}{2},z). \end{aligned} \tag{1.58}$$

This parameter case is of special importance and we use the given notation

$$U(-m-\tfrac{1}{2},z)=D_m(z), \tag{1.59}$$

and define

$$\overline{U}(-m-\tfrac{1}{2},z)=\overline{D}_m(z). \tag{1.60}$$

The requirement $m\in\mathbb{N}_0$ is not necessary for this notation. When $m\in\mathbb{N}_0$ the parabolic cylinder function $D_m(z)$ can be written in terms of Hermite polynomials by the relation

$$D_m(z) = e^{-\frac{1}{4}z^2} He_m(z) = 2^{-n/2} e^{-\frac{1}{4}z^2} H_m(z/\sqrt{2}). \tag{1.61}$$

These functions satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} D_m(z) D_n(z) \mathrm{d}z = \delta_{mn} m! \sqrt{2\pi}, \tag{1.62}$$

and the following recurrence relations are satisfied by both $U(b,z)$ and $\overline{U}(b,z)$

$$\begin{aligned} zW(b,z) - W(b-1,z) + (b+\tfrac{1}{2})W(a+1,z) &= 0, \\ W'(b,z) - \tfrac{1}{2}zW(b,z) + W(b-1,z) &= 0. \end{aligned} \tag{1.63}$$

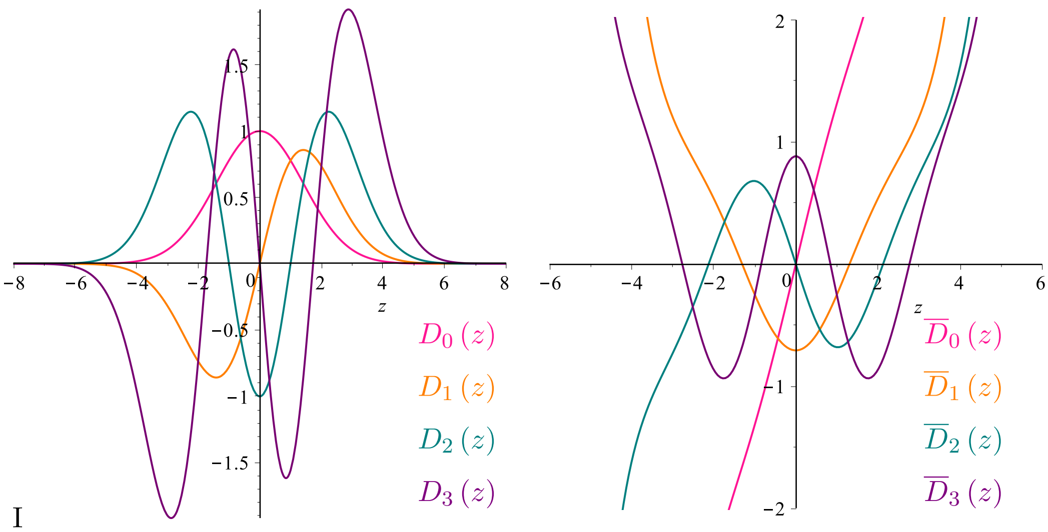

Figure 1.2: Plots for assorted parabolic cylinder D and $\overline{D}$ functions.

These functions exhibit the following behaviour as $z \to \infty$

$$\begin{aligned} U(b,z) &\sim z^{-b-\frac{1}{2}} e^{-\frac{1}{4}z^2}, \\ U'(b,z) &\sim -\tfrac{1}{2} z^{\frac{1}{2}-b} e^{-\frac{1}{4}z^2}, \\ \overline{U}(b,z) &\sim \sqrt{\tfrac{2}{\pi}} \Gamma\left(\tfrac{1}{2}-b\right) z^{b-\frac{1}{2}} e^{\frac{1}{4}z^2}, \\ \overline{U}'(b,z) &\sim \sqrt{2\pi}^{-1} \Gamma\left(\tfrac{1}{2}-b\right) z^{b+\frac{1}{2}} e^{\frac{1}{4}z^2}. \end{aligned} \tag{1.64}$$

When

$$
\begin{array}{rl}
b \geq -\frac{1}{2} & U(b,z) \text{ has no real zeros,} \\
-\frac{3}{2} < b < -\frac{1}{2} & U(b,z) \text{ has no positive real zeros,} \\
-2n - \frac{3}{2} < b < -2n + \frac{1}{2},\ n = 1, 2, .. & U(b,z) \text{ has } n \text{ positive real zeros,} \\
b = -n - \frac{1}{2},\ n = 1, 2, .. & U(b,z) \text{ has } n \text{ real zeros} \\
 & \text{in the interval } [-2\sqrt{-b}, 2\sqrt{-b}].
\end{array}
\tag{1.65}
$$

Auxiliary functions

The following is a summary of definitions given in [7], which are relevant in this part of the thesis. Denote $\rho(b)$ to be the largest real root of the equation

$$\overline{U}(b,z) = U(b,z). \tag{1.66}$$

For $b \leq 0$,

$$\mathbf{E}(b,z) = 1 \quad (0 \leq x \leq \rho(b)), \quad \mathbf{E}(b,z) = \sqrt{\frac{\overline{U}(b,z)}{U(b,z)}} \quad (x \geq \rho(b)). \tag{1.67}$$

When b is fixed, $\mathbf{E}(b,x)$ is non-decreasing on the interval $[0,\infty)$. For $b \leq 0$ and $x \geq 0$, we define

$$
\begin{aligned}
U(b,z) &= \mathbf{E}^{-1}(b,z)\mathbf{M}(b,z)\sin\theta(b,z), \quad \overline{U}(b,z) = \mathbf{E}(b,z)\mathbf{M}(b,z)\cos\theta(b,z), \\
U'(b,z) &= \mathbf{E}^{-1}(b,z)\mathbf{N}(b,z)\sin\omega(b,z), \quad \overline{U}'(b,z) = \mathbf{E}(b,z)\mathbf{N}(b,z)\cos\omega(b,z),
\end{aligned}
\tag{1.68}
$$

where $\mathbf{E}^{-1}(b,z) = 1/\mathbf{E}(b,z)$. Thus for $0 \leq x \leq \rho(b)$,

$$
\begin{aligned}
\mathbf{M}(b,z) &= \sqrt{U^2(b,z) + \overline{U}^2(b,z)}, \quad \theta(b,z) = \arctan\{U(b,z)/\overline{U}(b,z)\}, \\
\mathbf{N}(b,z) &= \sqrt{U'^2(b,z) + \overline{U}'^2(b,z)}, \quad \omega(b,z) = \arctan\{U'(b,z)/\overline{U}'(b,z)\},
\end{aligned}
\tag{1.69}
$$

whereas for $x \geq \rho(b)$

$$\begin{aligned}
\mathbf{M}(b,z) &= \sqrt{2U(b,z)\overline{U}(b,z)}, \quad \theta(b,z) = \frac{\pi}{4}, \\
\mathbf{N}(b,z) &= \sqrt{\frac{U'^2(b,z)\overline{U}^2(b,z) + \overline{U}'^2(b,z)U^2(b,z)}{U(b,z)\overline{U}(b,z)}}, \\
\omega(b,z) &= \arctan\left\{\frac{U'(b,z)\overline{U}(b,z)}{\overline{U}'(b,z)U(b,z)}\right\}.
\end{aligned} \tag{1.70}$$

For large x,

$$\mathbf{M}(b,z) \sim \left(\frac{8}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\Gamma(\frac{1}{2}-b)}{x}}, \quad \mathbf{N}(b,z) \sim \frac{\sqrt{\Gamma(\frac{1}{2}-b)x}}{(2\pi)^{1/4}}. \tag{1.71}$$

These hold for fixed b and also when b ranges over the compact interval $(-\infty, 0]$.

2

Rigorous asymptotics for the Lamé equation with a large parameter

This chapter and the next concerns linear ordinary differential equations with periodic coefficients. These equations mainly arise from the study of partial differential equations where new coordinates are introduced and then the equations are "separated" into several ordinary differential equations in the new coordinate system. Solutions of interest in applications have to satisfy certain boundary conditions at special surfaces. For a satisfactory discussion of such equations see [8].

In this chapter we discuss the Lamé equation and in particular we present rigorous results for the Lamé functions, and their corresponding eigenvalues as a parameter in its differential equation becomes large. In preparation for the analysis we will discuss the properties of these functions and give a concise literature review related to the relevant problems.

2.1 Properties

We will summarise the important properties here, for a fuller treatment see ([1] §29). Lamé's equation is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(h - \nu(\nu+1)k^2 \operatorname{sn}^2(z, k^2)\right) w = 0, \tag{2.1}$$

and we consider h, k and ν to be real parameters such that $0 < k < 1$ and $\nu \geq -\frac{1}{2}$, and $\operatorname{sn}(z, k)$ is the Jacobian elliptic sine function (see [1] §22.2). We consider always the interval $z \in [-K, K]$ unless stated otherwise,

$$K = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}} \tag{2.2}$$

is Legendre's complete elliptic integral of the first kind (see [1] §19.2(ii)). When h assumes the special values a_ν^m or b_ν^{m+1} for $m \in \mathbb{N}_0$, Lamé's equation admits even or odd periodic solutions denoted $Ec_\nu^m\left(z, k^2\right)$ or $Es_\nu^{m+1}\left(z, k^2\right)$ respectively. These functions are either $2K$-periodic or $2K$-antiperiodic, depending on the parity of m. For a summary of these properties, including the boundary conditions which the functions satisfy, see Table 2.1.

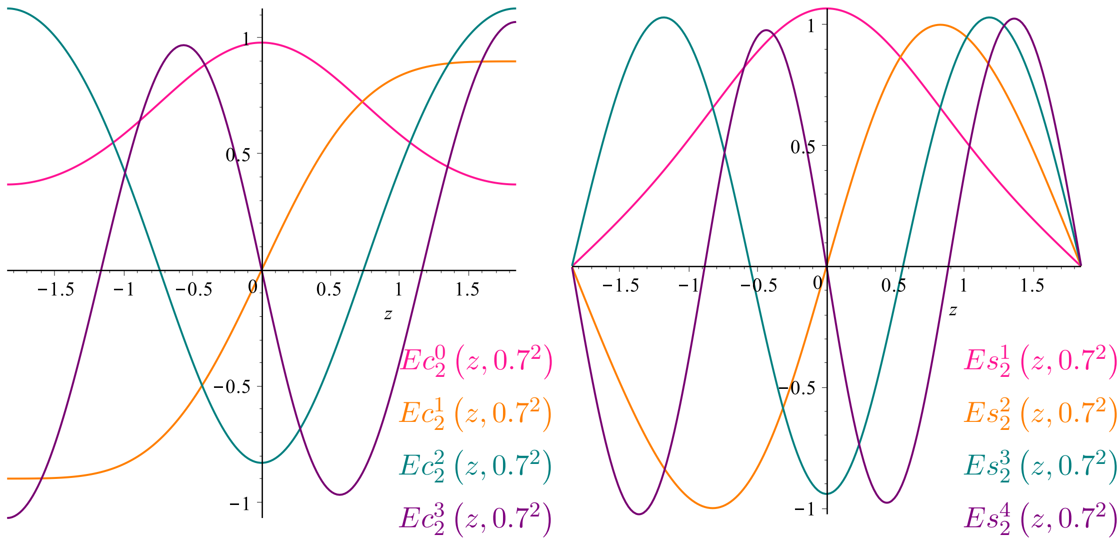

Figure 2.1: Plots for assorted Lamé functions in $[-K, K]$.

These functions have exactly m zeros in the interval $(-K, K)$ and their eigenvalues are ordered such that

$$a_\nu^0 < a_\nu^1 < a_\nu^2 < \ldots , \qquad b_\nu^1 < b_\nu^2 < b_\nu^3 < \ldots , \tag{2.3}$$

interlace such that

$$a_\nu^m < b_\nu^{m+1}, \qquad b_\nu^m < a_\nu^{m+1}, \tag{2.4}$$

and coalesce such that

$$a_\nu^m = b_\nu^m, \text{ when } \nu = 0, 1, .., m-1. \tag{2.5}$$

Since the Jacobian elliptic function $\mathrm{dn}(z,k)$ (see [1] §22) is even, we can rewrite the normalisations given in ([1] §29.3) as

$$\int_{-K}^{K} \mathrm{dn}(z,k) \left\{ Ec_\nu^m \left(z,k^2\right)\right\}^2 \mathrm{d}z = \int_{-K}^{K} \mathrm{dn}(z,k) \left\{ Es_\nu^{m+1} \left(z,k^2\right)\right\}^2 \mathrm{d}z = \frac{\pi}{2}. \tag{2.6}$$

To complete their definitions we have

$$Ec_\nu^m \left(K,k^2\right) > 0, \qquad \text{and} \qquad \left.\frac{\mathrm{d}Es_\nu^m \left(z,k^2\right)}{\mathrm{d}z}\right|_{z=K} < 0. \tag{2.7}$$

They satisfy the orthogonality conditions for $m \neq n$, $(n \in \mathbb{N}_0)$

$$\int_{-K}^{K} Ec_\nu^m \left(z,k^2\right) Ec_\nu^n \left(z,k^2\right) \mathrm{d}z = 0, \tag{2.8}$$

$$\text{and} \quad \int_{-K}^{K} Es_\nu^{m+1} \left(z,k^2\right) Es_\nu^{n+1} \left(z,k^2\right) \mathrm{d}z = 0. \tag{2.9}$$

We summarise their properties and give boundary conditions in Table 2.1.

Table 2.1: properties and boundary conditions for Lamé functions

Eigenfunctions	Eigenvalues	Periodicity	Parity at $z = 0, K$	Boundary conditions
$Ec_\nu^{2m}\left(z,k^2\right)$	a_ν^{2m}	Period $2K$	even, even	$w'(0) = w'(K) = 0$
$Ec_\nu^{2m+1}\left(z,k^2\right)$	a_ν^{2m+1}	Antiperiod $2K$	even, odd	$w(0) = w'(K) = 0$
$Es_\nu^{2m+1}\left(z,k^2\right)$	b_ν^{2m+1}	Antiperiod $2K$	odd, even	$w'(0) = w(K) = 0$
$Es_\nu^{2m+2}\left(z,k^2\right)$	b_ν^{2m+2}	Period $2K$	odd, odd	$w(0) = w(K) = 0$

When ν becomes large the oscillatory region of the Lamé functions in $[-K, K]$ shrinks into a neighbourhood of the origin, and otherwise the functions are exponentially small approaching the endpoints at $z = \pm K$. Formal results given in the literature (see [1] §29.7) indicate that $Es_\nu^{m+1}\left(z,k^2\right)$ and $Ec_\nu^m\left(z,k^2\right)$ behave asymptotically the same as $\nu \to \infty$.

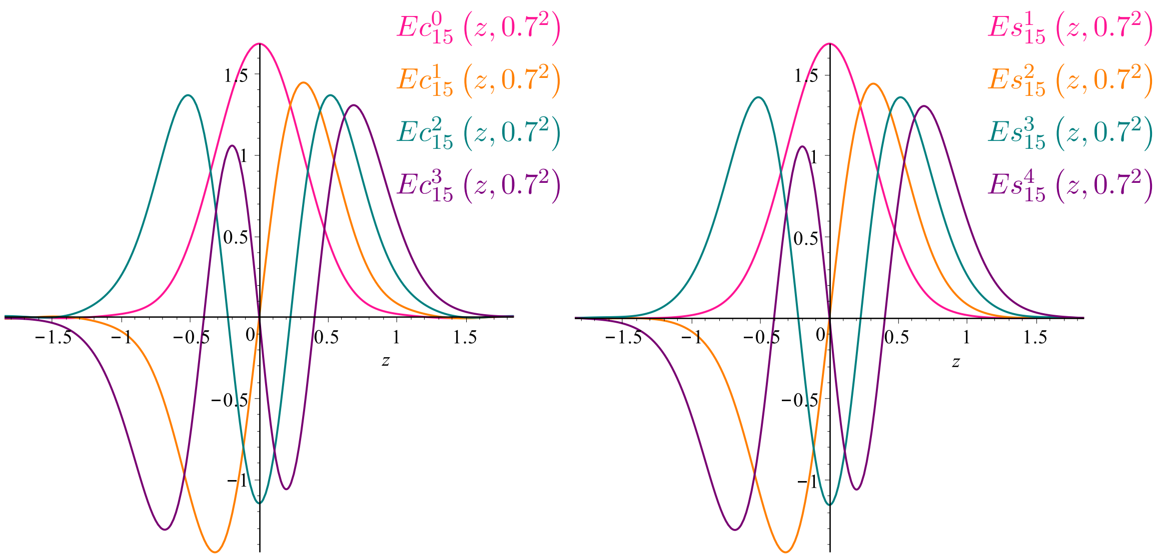

Figure 2.2: Plots for assorted Lamé functions in $[-K, K]$.

Figure 2.2 shows that even for ν as large as 15, $Es_{\nu}^{m+1}\left(z,k^2\right)$ and $Ec_{\nu}^{m}\left(z,k^2\right)$ already look very asymptotically similar.

Formal results given in the literature (see [1] §29.7) also indicate that the difference between the corresponding eigenvalues b_{ν}^{m+1} and a_{ν}^{m} is exponentially small as $\nu \to \infty$.

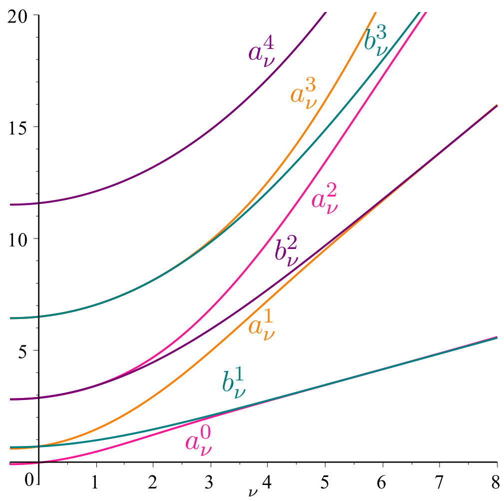

Figure 2.3: The first few eigenvalues as a function of ν for $k = 0.7$.

Figure 2.3 shows that as ν becomes large, b_{ν}^{m+1} and a_{ν}^{m} tend to each other asymptotically. It also shows that as m becomes larger, ν must become larger to see that they are asymptotically the same in this limit.

2.2 Previous Results

For a general overview of the theory concerning Lamé functions and their corresponding eigenvalues see [8], [9] and [10]. In this part of the thesis we will be investigating two types of results. First the asymptotic expansions of the Lamé functions and their respective eigenvalues for ν large, and secondly the exponentially small difference between the eigenvalues b_ν^{m+1} and a_ν^m in the same limit.

The results for asymptotic expansions of the Lamé functions and their respective eigenvalues for parameter ν large is not so abundant. The main results can be found in [11], [12], [13], [14], and [15], and are all formal, meaning they are not accompanied with any error analysis. None of the results about the functions have been published in ([1] §29.7), and none are in a particularly satisfactory form. In ([1] §29.7 (ii)) it is stated that one could derive from the results of [16] asymptotic approximations for the Lamé functions. However in that paper the results are given without much justification and the error bounds given for the approximations do not make sense in the intervals where the approximant is exponentially small. In ([1] §29.7) only limited formal results about the corresponding eigenvalues can be found, and are given as follows;

As $\nu \to \infty$,

$$a_\nu^m \sim p\kappa - \tau_0 - \tau_1\kappa^{-1} - \tau_2\kappa^{-2} - \dots \tag{2.10}$$

where

$$\begin{aligned}
&\kappa = \sqrt{\nu(\nu+1)}k,\\
&p = 2m+1,\\
&\tau_0 = \frac{1}{2^3}(1+k^2)(1+p^2), \quad \tau_1 = \frac{p}{2^6}((1+k^2)^2(p^2+3) - 4k^2(p^2+5)).
\end{aligned} \tag{2.11}$$

The same Poincaré expansion holds for b_ν^{m+1}, since the difference between b_ν^{m+1} and a_ν^m is exponentially small, which has been given in [17] as

$$b_\nu^{m+1} - a_\nu^m = \frac{(1-k^2)^{-m-1/2}}{m!\sqrt{2\pi}}(8k\nu)^{m+3/2}\left(\frac{1-k}{1+k}\right)^{\nu+1/2}(1+\mathcal{O}(\nu^{-1/2})) \quad (\nu\to\infty). \tag{2.12}$$

This result is reasonable but due to issues in the paper it was derived from we will be able to improve it, and give another term in the expansion.

We wish to obtain rigorous results for these types of problems concerning the Lamé equation.

2.3 Chapter outline

We now present the analysis concerning the aforementioned problems. We obtain uniform approximations for the Lamé functions $Ec_\nu^m\left(z,k^2\right)$ and $Es_\nu^{m+1}\left(z,k^2\right)$ as $\nu\to\infty$ on the interval $[0,K]$, in terms of parabolic cylinder U functions, using the theory developed by Olver in [7]. Using this method we are also able to give rigorous approximations for the eigenvalues a_ν^m and b_ν^{m+1} in this limit. By a judicious renaming of parameters we then give uniform asymptotic approximations for the Lamé functions in terms of the more natural parabolic cylinder D functions. Then we give formally a second term in the asymptotic approximation for the Lamé functions.

We then derive uniform asymptotic expansions for the Lamé functions, which hold only in a shrinking neighbourhood for the origin as ν becomes large, but encapsulate all the interesting oscillatory behaviour of the functions. The coefficients in the expansions are polynomials and we can compute as many as we like. Simultaneously we give asymptotic expansions for the eigenvalues, where again, we can compute as many terms as we like, and provide order estimates for the error when the expansions are truncated.

Finally we give an expression for the exponentially small difference between a_ν^m and b_ν^{m+1} as $\nu\to\infty$, and are able to say more, and give stronger results, than has been previously given.

Results given in the next two sections are published in the authors paper [18].

2.4 Uniform asymptotic approximations for the Lamé functions

We first give the main result of this section, and later give the proof after building up some machinery.

Theorem 2.1. *Let* $\kappa = \sqrt{\nu(\nu+1)}k$*,* $m \in \mathbb{N}_0$ *and* $0 < k < 1$*. Then for* $z \in [0, K]$ *as* $\kappa \to \infty$

$$\begin{aligned} Ec_\nu^m\left(z,k^2\right) &= C_\nu^m \left(\frac{\zeta^2-(\sigma_\nu^m)^2}{x^2-(s_\nu^m)^2}\right)^{1/4} \Big(D_m\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,1}^m(\zeta,k^2) \\ &\qquad +\eta_{\nu,c}^m\left(\overline{D}_m\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,2}^m(\zeta,k^2)\right)\Big), \\ Es_\nu^{m+1}\left(z,k^2\right) &= S_\nu^{m+1} \left(\frac{\zeta^2-(\sigma_\nu^m)^2}{x^2-(s_\nu^m)^2}\right)^{1/4} \Big(D_m\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,1}^m(\zeta,k^2) \\ &\qquad +\eta_{\nu,s}^{m+1}\left(\overline{D}_m\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,2}^m(\zeta,k^2)\right)\Big), \end{aligned} \tag{2.13}$$

where

$$\epsilon_{\nu,1}^m(\zeta) = \frac{\mathbf{M}_m\left(\sqrt{2\kappa}\zeta\right)}{\mathbf{E}_m\left(\sqrt{2\kappa}\zeta\right)}\mathcal{O}\left(\kappa^{-1}\right), \quad \epsilon_{\nu,2}^m(\zeta) = \mathbf{E}_m\left(\sqrt{2\kappa}\zeta\right)\mathbf{M}_m\left(\sqrt{2\kappa}\zeta\right)\mathcal{O}\left(\kappa^{-1}\right), \tag{2.14}$$

and

$$\frac{\partial\epsilon_{\nu,1}^m\left(\zeta,k^2\right)}{\partial\zeta} = \frac{\mathbf{N}_m\left(\sqrt{2\kappa}\zeta\right)}{\mathbf{E}_m\left(\sqrt{2\kappa}\zeta\right)}\mathcal{O}\left(\kappa^{-1/2}\right), \quad \frac{\partial\epsilon_{\nu,2}^m\left(\zeta,k^2\right)/\partial\zeta}{\mathbf{E}_m\left(\sqrt{2\kappa}\zeta\right)} = \mathbf{N}_m\left(\sqrt{2\kappa}\zeta\right)\mathcal{O}\left(\kappa^{-1/2}\right), \tag{2.15}$$

with the order terms in the errors differing depending on whether we consider Ec_ν^m *or* Es_ν^{m+1}*, the functions* U*,* $\overline{U}$*,* $\mathbf{E}$*,* $\mathbf{M}$ *and* $\mathbf{N}$ *are all defined in section 1.5, both* $\eta_{\nu,c}^m$ *and* $\eta_{\nu,s}^{m+1}$ *are* $\mathcal{O}\left(e^{-\kappa\zeta_*^2}\kappa^{m+1/2}\right)$*, with* $\zeta_* \sim \sqrt{\frac{2\operatorname{arctanh}(k)}{k}}$*, as* $\kappa \to \infty$*, the relationship between* z *and* x *is defined by*

$$x = \operatorname{sn}(z,k), \tag{2.16}$$

and x and ζ by

$$\begin{aligned}
\int_{x}^{-s_\nu^m} \sqrt{\frac{t^2-(s_\nu^m)^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t &= \int_{\zeta}^{-\sigma_\nu^m} \sqrt{\tau^2-(\sigma_\nu^m)^2}\mathrm{d}\tau \quad (-1<x\le -s_\nu^m)\,,\\
\int_{-s_\nu^m}^{x} \sqrt{\frac{(s_\nu^m)^2-t^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t &= \int_{-\sigma_\nu^m}^{\zeta} \sqrt{(\sigma_\nu^m)^2-\tau^2}\mathrm{d}\tau \quad (-s_\nu^m\le x\le s_\nu^m)\,,\\
\int_{s_\nu^m}^{x} \sqrt{\frac{t^2-(s_\nu^m)^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t &= \int_{\sigma_\nu^m}^{\zeta} \sqrt{\tau^2-(\sigma_\nu^m)^2}\mathrm{d}\tau \quad (s_\nu^m\le x<1)\,,
\end{aligned} \tag{2.17}$$

where

$$(s_\nu^m)^2 = \frac{h_\nu^m}{\kappa^2}, \quad (\sigma_\nu^m)^2 = \frac{2}{\pi}\int_{-s_\nu^m}^{s_\nu^m}\sqrt{\frac{(s_\nu^m)^2-t^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t, \tag{2.18}$$

where h_ν^m corresponds to either a_ν^m or b_ν^{m+1} depending on whether we are considering the even or odd solution solution, and

$$\left.\begin{matrix} C_\nu^m \\ S_\nu^{m+1}\end{matrix}\right\} \sim \frac{(\pi\kappa)^{1/4}}{\sqrt{2m!^2}}\left(1-\frac{2m+1}{8\kappa}\right). \tag{2.19}$$

Again as $\kappa\to\infty$

$$a_\nu^m = (2m+1)\kappa + \mathcal{O}(1), \qquad b_\nu^{m+1} = (2m+1)\kappa + \mathcal{O}(1). \tag{2.20}$$

2.4.1 Approximations in terms of parabolic cylinder U functions

In (2.1), we denote the solution w_ν^m such that

$$\frac{\mathrm{d}^2 w_\nu^m(z,k^2)}{\mathrm{d}z^2} + \left(h_\nu^m - \nu(\nu+1)k^2\,\mathrm{sn}^2\,(z,k^2)\right) w_\nu^m(z,k^2) = 0, \tag{2.21}$$

where h_ν^m corresponds to either a_ν^m or b_ν^{m+1}. The periodic coefficient in (2.21) is troublesome, thus to obtain uniform asymptotic approximations for w_ν^m we transform the independent variable to obtain an algebraic equation, and then transform the dependent variable to remove the subsequent term in the first derivative. This is done by letting

$$x = \mathrm{sn}(z,k), \quad w_\nu^m(z,k^2) = \left((1-x^2)(1-k^2x^2)\right)^{-1/4}\widetilde{w}_\nu^m(x,k^2), \tag{2.22}$$

and denoting $\kappa = \sqrt{\nu(\nu+1)}k$, we obtain the differential equation corresponding to the Lamé's functions in the form

$$\frac{\mathrm{d}^2 \widetilde{w}_\nu^m(x,k^2)}{\mathrm{d}x^2} = \left(\kappa^2 \frac{x^2-(s_\nu^m)^2}{(1-x^2)(1-k^2x^2)} + \phi_\nu^m\left(x,k^2\right)\right)\widetilde{w}_\nu^m(x,k^2), \tag{2.23}$$

where s_ν^m is defined by the first of (2.18) and

$$\phi_\nu^m\left(x,k^2\right) = -\frac{2k^2(k^2+1)x^4+(k^4-10k^2+1)x^2+2(1+k^2)}{4\left(1-x^2\right)^2\left(1-k^2x^2\right)^2}. \tag{2.24}$$

We note that the formal asymptotic expansions given in ([1] §29.7) indicate that

$$h_\nu^m = \mathcal{O}(\kappa) \text{ as } \kappa \to \infty. \tag{2.25}$$

Corresponding to the transformation we consider the interval $x \in [-1,1]$, where $x = -1, 0, 1$ corresponds to $z = -K, 0, K$.

We deduce from (2.25) and the first of (2.18) that $s_\nu^m \to 0$ as $\kappa \to \infty$, hence in this limit (2.23) has two turning points which coalesce into the origin. The turning points of our equation lie at $x = \pm s_\nu^m$ and

$$\frac{x^2-(s_\nu^m)^2}{(1-x^2)(1-k^2x^2)} < 0 \quad (-s_\nu^m < x < s_\nu^m), \tag{2.26}$$

thus we apply the theory of Case I in [7], which is detailed in section 1.4. In this case uniform asymptotic approximations are in terms of the parabolic cylinder functions $U(-\frac{1}{2}\kappa\left(\sigma_\nu^m\right)^2, \sqrt{2\kappa}\zeta)$ and $\overline{U}(-\frac{1}{2}\kappa\left(\sigma_\nu^m\right)^2, \sqrt{2\kappa}\zeta)$, where σ_ν^m is defined in terms of s_ν^m.

Following Olver, new variables relating $\{x, \widetilde{w}_\nu^m\}$ to $\{\zeta, W_\nu^m\}$ are introduced by the appropriate Liouville transformation given by

$$W_\nu^m(\zeta,k^2) = \dot{x}^{-\frac{1}{2}}\widetilde{w}_\nu^m(x,k^2), \quad \dot{x}^2\frac{x^2-(s_\nu^m)^2}{(1-x^2)(1-k^2x^2)} = \zeta^2 - \left(\sigma_\nu^m\right)^2, \tag{2.27}$$

the dot signifying differentiation with respect to ζ. It then follows that σ_ν^m is defined as in the second of (2.18). From this we denote that

$$0 < s_\nu^m < 1 \text{ corresponds to } 0 < \sigma_\nu^m < \sigma_{\nu,*}^m, \text{ where } \sigma_{\nu,*}^m = 2\sqrt{\frac{\arcsin(k)}{\pi k}}. \tag{2.28}$$

Since $\zeta = \pm\sigma_\nu^m$ corresponds to $x = \pm s_\nu^m$, integration of the second of (2.27) yields (2.17). These equations define ζ as a real analytic function of x. There is a one-to-one correspondence between the variables x and ζ, where ζ is an increasing function of x, and we denote $\zeta = -\zeta_*, 0, \zeta_*$ to correspond to $x = -1, 0, 1$. It follows that $x(\zeta, \sigma_\nu^m)$ is analytic both in ζ and σ_ν^m for $\zeta \in [-\zeta_*, \zeta_*]$ and $\sigma_\nu^m \in (-\sigma_{\nu,*}^m, \sigma_{\nu,*}^m)$. Also $\dot{x}$ is non-zero in these intervals.

Performing the substitution $t = s_\nu^m \tau$ in the second of (2.18), we expand the integral and obtain

$$(\sigma_\nu^m)^2 = (s_\nu^m)^2 + \frac{1+k^2}{8}(s_\nu^m)^4 + \frac{3+2k^2+3k^4}{64}(s_\nu^m)^6 + \mathcal{O}((s_\nu^m)^8) \quad (s_\nu^m \to 0), \quad (2.29)$$

and by reversion

$$(s_\nu^m)^2 = (\sigma_\nu^m)^2 - \frac{1+k^2}{8}(\sigma_\nu^m)^4 - \frac{(1-k^2)^2}{64}(\sigma_\nu^m)^6 + \mathcal{O}\left((\sigma_\nu^m)^8\right) \quad (\sigma_\nu^m \to 0). \quad (2.30)$$

In the critical case $s_\nu^m = \sigma_\nu^m = 0$ we have from the third of (2.17)

$$\frac{\operatorname{arctanh}(k)}{k} = \int_0^1 \frac{t}{\sqrt{(1-t^2)(1-k^2t^2)}}\mathrm{d}t = \int_0^{\zeta_*} \tau \mathrm{d}\tau = \tfrac{1}{2}\zeta_*^2, \quad (2.31)$$

which gives

$$\zeta_* = \sqrt{\frac{2\operatorname{arctanh}(k)}{k}}. \quad (2.32)$$

Thus we deduce that as $s_\nu^m, \sigma_\nu^m \to 0$

$$\zeta_* \to \sqrt{\frac{2\operatorname{arctanh}(k)}{k}}. \quad (2.33)$$

The transformed differential equation is now of the form

$$\frac{\mathrm{d}^2 W_\nu^m(\zeta,k^2)}{\mathrm{d}\zeta^2} = \left(\kappa^2\left(\zeta^2 - (\sigma_\nu^m)^2\right) + \psi_\nu^m\left(\zeta,k^2\right)\right) W_\nu^m(\zeta,k^2), \quad (2.34)$$

where

$$\psi_\nu^m\left(\zeta,k^2\right) = \dot{x}^2\phi\left(x,k^2\right) + \dot{x}^{\frac{1}{2}}\frac{d^2}{d\zeta^2}\left(\dot{x}^{-\frac{1}{2}}\right)$$

$$= \frac{2\left(\sigma_\nu^m\right)^2 + 3\zeta^2}{4\left(\left(\sigma_\nu^m\right)^2 - \zeta^2\right)^2} + \frac{\left(\sigma_\nu^m\right)^2 - \zeta^2}{4}\left(-k^2 + \frac{1 + k^2(1 - 3\left(s_\nu^m\right)^2)}{\left(s_\nu^m\right)^2 - x^2}\right.$$
$$\left. + \frac{3(1 - 2(1+k^2)\left(s_\nu^m\right)^2 + 3k^2\left(s_\nu^m\right)^4)}{\left(\left(s_\nu^m\right)^2 - x^2\right)^2} + \frac{5\left(s_\nu^m\right)^2\left(\left(s_\nu^m\right)^2 - 1\right)(1 - k^2\left(s_\nu^m\right)^2)}{\left(\left(s_\nu^m\right)^2 - x^2\right)^3}\right). \tag{2.35}$$

By construction, the apparent singularities in the above function at $\zeta = \pm\sigma_\nu^m$, corresponding to $x = \pm s_\nu^m$, cancel each other out so that $\psi_\nu^m\left(\zeta, k^2\right)$ is well-behaved there. To check this one could expand $\psi_\nu^m\left(\zeta, k^2\right)$ around this point. One could also note that in (2.23), $\phi_\nu^m\left(x, k^2\right)$ has singularities at $x = \pm 1$, where as $\psi_\nu^m\left(\zeta, k^2\right)$ does not blow up there. However ζ and therefore $\psi_\nu^m\left(x, k^2\right)$ will have branch point singularities there.

Proposition 2.1. *Let $\kappa = \sqrt{\nu(\nu+1)}k$, $m \in \mathbb{N}_0$ and $0 < k < 1$. Then for $z \in [0, K]$ as $\kappa \to \infty$, (2.34) has two solutions of the form*

$$\begin{aligned} W_{\nu,1}^m\left(z, k^2\right) &= U\left(-\tfrac{1}{2}\kappa\widetilde{\sigma}^2, \sqrt{2\kappa}\zeta\right) + \epsilon_{\nu,1}^m\left(\zeta, k^2\right), \\ W_{\nu,2}^m\left(z, k^2\right) &= \overline{U}\left(-\tfrac{1}{2}\kappa\widetilde{\sigma}^2, \sqrt{2\kappa}\zeta\right) + \epsilon_{\nu,2}^m\left(\zeta, k^2\right), \end{aligned} \tag{2.36}$$

with

$$\begin{aligned} \epsilon_{\nu,1}^m\left(\zeta, k^2\right) &= \frac{\mathbf{M}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)}{\mathbf{E}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)}\mathcal{O}\left(\kappa^{-1}\right), \\ \epsilon_{\nu,2}^m\left(\zeta, k^2\right) &= \mathbf{E}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)\mathbf{M}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)\mathcal{O}\left(\kappa^{-1}\right), \end{aligned} \tag{2.37}$$

and

$$\begin{aligned} \frac{\partial\epsilon_{\nu,1}^m\left(\zeta, k^2\right)}{\partial\zeta} &= \frac{\mathbf{N}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)}{\mathbf{E}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)}\mathcal{O}\left(\kappa^{-\frac{1}{2}}\right), \\ \frac{\partial\epsilon_{\nu,2}^m\left(\zeta, k^2\right)}{\partial\zeta} &= \mathbf{E}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)\mathbf{N}\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)\mathcal{O}\left(\kappa^{-\frac{1}{2}}\right), \end{aligned} \tag{2.38}$$

where

$$\tilde{\sigma}_\nu^m = \left(\sigma_\nu^m\right)^2 + \frac{\left(\sigma_\nu^m\right)^4 - \left(s_\nu^m\right)^4}{2\kappa^2\left(\sigma_\nu^m\right)^2\left(s_\nu^m\right)^4}. \tag{2.39}$$

and s_ν^m and σ_ν^m are defined as in (2.18).

Proof. For our advantage in the error analysis, we write the differential equation (2.34) in the form

$$\frac{\mathrm{d}^2 W_\nu^m(\zeta,k^2)}{\mathrm{d}\zeta^2} = \left(\kappa^2\left(\zeta^2 - (\widetilde{\sigma}_\nu^m)^2\right) + \widetilde{\psi}_\nu^m\left(\zeta,k^2\right)\right) W_\nu^m(\zeta,k^2), \tag{2.40}$$

where

$$\widetilde{\psi}_\nu^m\left(\zeta,k^2\right) = \psi_\nu^m\left(\zeta,k^2\right) - \psi_\nu^m\left(0,k^2\right) = \psi_\nu^m\left(\zeta,k^2\right) + \frac{(\sigma_\nu^m)^4 - (s_\nu^m)^4}{2\,(\sigma_\nu^m)^2\,(s_\nu^m)^4}, \tag{2.41}$$

and thus correspondingly we define (2.39) where

$$\frac{(\sigma_\nu^m)^4 - (s_\nu^m)^4}{2\,(\sigma_\nu^m)^2\,(s_\nu^m)^4} = \frac{1+k^2}{8} + \mathcal{O}((s_\nu^m)^2) \quad (s_\nu^m \to 0). \tag{2.42}$$

This gives

$$\widetilde{\psi}_\nu^m\left(0,k^2\right) = 0. \tag{2.43}$$

On inspection it follows that since s_ν^m, σ_ν^m and the variables x and ζ are all bounded as $\kappa \to \infty$, and the apparent singularities at $x = \pm s_\nu^m$ and $\zeta = \pm\sigma_\nu^m$ cancel each other, we have for $\zeta \in [0,\zeta_*]$

$$\widetilde{\psi}_\nu^m\left(\zeta,k^2\right) = \mathcal{O}(1) \tag{2.44}$$

uniformly in this limit. On applying Theorem I of ([7] §6) with $u = \kappa$, $\alpha = \widetilde{\sigma}_\nu^m$ and $\zeta_2 = \zeta_*$, for $\zeta \in [0,\zeta_*]$ we obtain the solutions (2.36) of (2.34).

To obtain bounds for the errors, Olver defines the variational operator

$$\mathcal{V}_{a,b}\left(\widetilde{\psi}_\nu^m\right) = \int_a^b \frac{|\widetilde{\psi}_\nu^m\left(t,k^2\right)|}{\Omega(\sqrt{2\kappa}t)}\mathrm{d}t, \tag{2.45}$$

where $\Omega(z)$ is to be chosen with the condition that

$$\Omega(z) \sim z, \quad \text{as} \quad z \to \infty. \tag{2.46}$$

Here, choosing as a consequence of (2.43)

$$\Omega(z) = z, \tag{2.47}$$

and thus defining the variational operator as

$$\mathcal{V}_{a,b}\left(\widetilde{\psi}_\nu^m\right) = \int_a^b \frac{|\widetilde{\psi}_\nu^m\left(t,k^2\right)|}{\sqrt{2\kappa t}}\mathrm{d}t, \tag{2.48}$$

the bounds obtained are

$$|\epsilon_{\nu,1}^m\left(\zeta,k^2\right)| \leq \frac{\mathbf{M}\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)}{\mathbf{E}\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)} \times \left[\exp\left(\frac{1}{2}\sqrt{\pi/\kappa}\, l_1\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2\right)\mathcal{V}_{\zeta,\zeta_*}\left(\widetilde{\psi}_\nu^m\right)\right) - 1\right], \tag{2.49}$$

$$\frac{|\epsilon_{\nu,2}^m\left(\zeta,k^2\right)|}{\mathbf{E}\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right)} \leq \mathbf{M}\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, \sqrt{2\kappa}\zeta\right) \times \left[\exp\left(\frac{1}{2}\sqrt{\pi/\kappa}\, l_1\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2\right)\mathcal{V}_{0,\zeta}\left(\widetilde{\psi}_\nu^m\right)\right) - 1\right], \tag{2.50}$$

where

$$l_1\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2\right) = \sup_{z\in(0,\infty)}\left\{\frac{\Omega(z)\mathbf{M}^2\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, z\right)}{\Gamma\left(\frac{1}{2}\left(1+\kappa\left(\widetilde{\sigma}_\nu^m\right)^2\right)\right)}\right\}. \tag{2.51}$$

It follows from (2.43) and the evenness of $\widetilde{\psi}\left(\zeta,k^2\right)$ that

$$\mathcal{V}_{0,\zeta}\left(\widetilde{\psi}_\nu^m\right) = \mathcal{O}\left(\kappa^{-1/2}\right) \quad \text{and} \quad \mathcal{V}_{\zeta,\zeta_*}\left(\widetilde{\psi}_\nu^m\right) = \mathcal{O}\left(\kappa^{-1/2}\right) \quad (\kappa\to\infty). \tag{2.52}$$

From the first of (1.71) we have

$$\mathbf{M}^2\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2, z\right) = \mathcal{O}\left(z^{-1}\right) \quad (z\to\infty), \tag{2.53}$$

thus clearly $l_1\left(-\frac{1}{2}\kappa\left(\widetilde{\sigma}_\nu^m\right)^2\right)$ is bounded as $\kappa\to\infty$. Hence we obtain (2.37). The bounds given in (2.38) follow similarly from the same theorem. □

Corollary 2.1. *Let* $\kappa = \sqrt{\nu(\nu+1)}k$, $m \in \mathbb{N}_0$ *and* $0 < k < 1$. *Then for* $z \in [0, K]$ *as* $\kappa \to \infty$, *the Lamé functions can be written in the form*

$$\begin{aligned} Ec_\nu^m\left(z,k^2\right) &= C_\nu^m \left(\frac{\zeta^2-(\sigma_\nu^m)^2}{x^2-(s_\nu^m)^2}\right)^{1/4} \left(W_{\nu,1}^m(\zeta,k^2)+\eta_{\nu,c}^m W_{\nu,2}^m(\zeta,k^2)\right), \\ Es_\nu^{m+1}\left(z,k^2\right) &= S_\nu^{m+1} \left(\frac{\zeta^2-(\sigma_\nu^m)^2}{x^2-(s_\nu^m)^2}\right)^{1/4} \left(W_{\nu,1}^m(\zeta,k^2)+\eta_{\nu,s}^{m+1} W_{\nu,2}^m(\zeta,k^2)\right), \end{aligned} \tag{2.54}$$

where C_ν^m *and* S_ν^{m+1} *are normalisation constants,* $\eta_{\nu,c}^m$ *and* $\eta_{\nu,s}^{m+1}$ *are connection formulae constants, and* $W_{\nu,1}^m(\zeta,k^2)$ *and* $W_{\nu,2}^m(\zeta,k^2)$ *are as in Propositiom 2.1.*

Proof. These results can be easily found by transforming the solutions found in Proposition 2.1 back into the z-plane using (2.27) and (2.22), and the fact that any solution of a second-order differential equation can be written as a linear combination of two standard solutions of the equation. □

Lemma 2.1. *Letting* $m \in \mathbb{N}_0$, *as* $\kappa \to \infty$

$$(\sigma_\nu^m)^2 = \frac{2m+1}{\kappa} + \mathcal{O}(\kappa^{-2}). \tag{2.55}$$

Proof. Let's first consider m odd in (2.54). In correspondence with the boundary conditions we require

$$Ec_\nu^m\left(0,k^2\right) = \left.\frac{\mathrm{d}\ Ec_\nu^m\left(z,k^2\right)}{\mathrm{d}z}\right|_{z=K} = 0, \tag{2.56}$$

$$Es_\nu^{m+1}\left(0,k^2\right) = Es_\nu^{m+1}\left(K,k^2\right) = 0. \tag{2.57}$$

The requirement at $z = K$ in (2.56) gives

$$\begin{aligned} &\tfrac{1}{2}\zeta_* W_{\nu,1}^m(\zeta_*,k^2) + \left(\zeta_*^2-(\sigma_\nu^m)^2\right)\left.\frac{\mathrm{d}\ W_{\nu,1}^m(\zeta,k^2)}{\mathrm{d}\zeta}\right|_{\zeta=\zeta_*} \\ &\qquad + \eta_{\nu,c}^m\left(\tfrac{1}{2}\zeta_* W_{\nu,2}^m(\zeta_*,k^2) + \left(\zeta_*^2-(\sigma_\nu^m)^2\right)\left.\frac{\mathrm{d}\ W_{\nu,2}^m(\zeta,k^2)}{\mathrm{d}\zeta}\right|_{\zeta=\zeta_*}\right) = 0, \end{aligned} \tag{2.58}$$

thus from (2.36), (2.37), (2.38), and the large variable behaviour of the parabolic cylinder functions given in (1.64), necessarily $\eta_{\nu,c}^{m}$ is exponentially small in this limit. Note that here we have assumed that $\kappa\widetilde{\sigma}^2 = \mathcal{O}(1)$ as $\kappa \to \infty$, and we show later, in (2.61) and (2.64), that this assumption is justified.

Similarly the requirement at $z = K$ in (2.57) gives

$$W_{\nu,1}^{m}(\zeta_*, k^2) + \eta_{\nu,s}^{m+1} W_{\nu,2}^{m}(\zeta_*, k^2) = 0, \tag{2.59}$$

then again it can be noted that necessarily $\eta_{\nu,s}^{m+1}$ is exponentially small as $\kappa \to \infty$.

Hence for both of these cases, by considering the requirements at $z = 0$ we have the condition

$$U\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_{\nu}^{m}\right)^2, 0\right) + \mathcal{O}(\kappa^{-1}) = \frac{2^{\left(\kappa(\widetilde{\sigma}_{\nu}^{m})^2 - 1\right)/4}\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\left(3 - \kappa\left(\widetilde{\sigma}_{\nu}^{m}\right)^2\right)\right)} + \mathcal{O}(\kappa^{-1}) = 0, \tag{2.60}$$

and to satisfy this we require

$$\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_{\nu}^{m}\right)^2 = j + \tfrac{1}{2} + \mathcal{O}(\kappa^{-1}), \tag{2.61}$$

where j is an odd integer.

Let's now consider m even. In correspondence with the boundary conditions we require

$$\begin{aligned} \left.\frac{\mathrm{d}\, Ec_{\nu}^{m}\left(z, k^2\right)}{\mathrm{d}z}\right|_{z=0} &= \left.\frac{\mathrm{d}\, Ec_{\nu}^{m}\left(z, k^2\right)}{\mathrm{d}z}\right|_{z=K} = 0, \\ \left.\frac{\mathrm{d}\, Es_{\nu}^{m+1}\left(z, k^2\right)}{\mathrm{d}z}\right|_{z=0} &= Es_{\nu}^{m+1}\left(K, k^2\right) = 0. \end{aligned} \tag{2.62}$$

By similar reason to the m odd case, when considering the boundary condition at $z = K$ both $\eta_{\nu,c}^{m}$ and $\eta_{\nu,s}^{m+1}$ are necessarily exponentially small as $\kappa \to \infty$. Hence by using the boundary condition at $z = 0$ we have the condition

$$U'\left(-\tfrac{1}{2}\kappa\left(\widetilde{\sigma}_{\nu}^{m}\right)^2, 0\right) + \mathcal{O}(\kappa^{-1}) = -\frac{2^{\left(\kappa(\widetilde{\sigma}_{\nu}^{m})^2 + 1\right)/4}\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\left(1 - \kappa\left(\widetilde{\sigma}_{\nu}^{m}\right)^2\right)\right)} + \mathcal{O}(\kappa^{-1}) = 0. \tag{2.63}$$

To satisfy this we require that

$$\tfrac{1}{2}\kappa\,(\widetilde{\sigma}_\nu^m)^2 = j + \tfrac{1}{2} + \mathcal{O}(\kappa^{-1}), \tag{2.64}$$

where j is an even integer.

As $\kappa \to \infty$, the zeros of $U(-\frac{1}{2}\kappa\,(\widetilde{\sigma}_\nu^m)^2\,, \sqrt{2\kappa}\zeta)$ tend to the zeros of $D_j(\sqrt{2\kappa}\zeta)$, the parabolic cylinder functions with exactly j zeros in it's oscillatory interval (see 1.65). Thus in correspondence with the number of zeros of the Lamé functions and those of $D_j(\sqrt{2\kappa}\zeta)$, we deduce that $j = m$ and as such from (2.64) we have the result stated in (2.55). □

2.4.2 Approximations in terms of parabolic cylinder D functions

The Lamé functions decay exponentially on either side of the oscillatory interval in $[-K, K]$. Our approximant in the previous subsection, $U\left(-\frac{1}{2}\kappa\widetilde{\sigma}^2, \sqrt{2\kappa}\zeta\right)$, displays the appropriate exponentially decreasing behaviour when ζ is large and positive, but when the variable is large and negative it becomes exponentially large. If our argument $-\frac{1}{2}\kappa\widetilde{\sigma}^2$ had been exactly a negative half-integer, which it is not, then the approximant would have exhibited this wanted exponentially decaying behaviour when the variable is both large and positive and large and negative. In this subsection we will redefine the parameters and functions in the differential equation (2.34) so that we obtain approximants with this favourable behaviour. This leads us finally to the proof of the Theorem 2.1.

Proof of Theorem 2.1. With respect to (2.55) we define

$$(\omega_\nu^m)^2 = (\sigma_\nu^m)^2 - \frac{2m+1}{\kappa} = \mathcal{O}(\kappa^{-2}) \quad (\kappa \to \infty), \tag{2.65}$$

thus it makes sense to split up (2.34) so that

$$\frac{\mathrm{d}^2 W_\nu^m(\zeta,k^2)}{\mathrm{d}\zeta^2} = \left(\kappa^2\left(\zeta^2 - \frac{2m+1}{\kappa}\right) + \widehat{\psi}_\nu^m\left(\zeta,k^2\right)\right) W_\nu^m(\zeta,k^2), \tag{2.66}$$

where

$$\widehat{\psi}_{\nu}^{m}\left(\zeta,k^{2}\right)=\psi_{\nu}^{m}\left(\zeta,k^{2}\right)-\kappa^{2}\left(\omega_{\nu}^{m}\right)^{2}. \tag{2.67}$$

From (2.65) and since s_{ν}^{m}, σ_{ν}^{m} and the variables x and ζ are all bounded as $\kappa\to\infty$, we have for $\zeta\in[-\zeta_{*},\zeta_{*}]$

$$\widehat{\psi}_{\nu}^{m}\left(\zeta,k^{2}\right)=\mathcal{O}(1), \tag{2.68}$$

in this limit uniformly. Now our approximant will have the desired property of decaying exponentially on either side of the oscillatory interval for large positive and large negative ζ. We obtain the solutions for (2.40)

$$\begin{aligned}W_{\nu,1}^{m}\left(\zeta,k^{2}\right)&=D_{m}\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,1}^{m}\left(\zeta,k^{2}\right),\\ W_{\nu,2}^{m}\left(\zeta,k^{2}\right)&=\overline{D}_{m}\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,2}^{m}\left(\zeta,k^{2}\right),\end{aligned} \tag{2.69}$$

valid when $\zeta\in[0,\zeta_{*}]$. We have that

$$\widehat{\psi}_{\nu}^{m}\left(0,k^{2}\right)=-\kappa^{2}\left(\omega_{\nu}^{m}\right)^{2}+\psi_{\nu}^{m}\left(0,k^{2}\right)=-\kappa^{2}\left(\sigma_{\nu}^{m}\right)^{2}+(2m+1)\kappa+\frac{\left(s_{\nu}^{m}\right)^{4}-\left(\sigma_{\nu}^{m}\right)^{4}}{2\left(\sigma_{\nu}^{m}\right)^{2}\left(s_{\nu}^{m}\right)^{4}}. \tag{2.70}$$

From Theorem 2.2, we will prove that

$$\left(\sigma_{\nu}^{m}\right)^{2}=\frac{2m+1}{\kappa}-\frac{1+k^{2}}{8\kappa^{2}}+\mathcal{O}(\kappa^{-3}). \tag{2.71}$$

We combine this with (2.30) and obtain

$$\widehat{\psi}_{\nu}^{m}\left(0,k^{2}\right)=\mathcal{O}(\kappa^{-1})\quad(\kappa\to\infty). \tag{2.72}$$

This time in the error analysis we choose it so that

$$\Omega(z)=1+z, \tag{2.73}$$

and, hence, take as the variational operator

$$\mathcal{V}_{a,b}\left(\widehat{\psi}_{\nu}^{m}\right) = \int_{a}^{b} \frac{|\widetilde{\psi}_{\nu}^{m}\left(t,k^{2}\right)|}{1+\sqrt{2\kappa t}}\mathrm{d}t. \tag{2.74}$$

The bounds for the errors are

$$|\epsilon_{\nu,1}^{m}\left(\zeta,k^{2}\right)| \leq \frac{\mathbf{M}_{m}\left(\sqrt{2\kappa}\zeta\right)}{\mathbf{E}_{m}\left(\sqrt{2\kappa}\zeta\right)}\left[\exp\left(\tfrac{1}{2}\sqrt{\pi/\kappa}\, l_{m,1}\mathcal{V}_{\zeta,\zeta_{*}}\left(\widehat{\psi}_{\nu}^{m}\right)\right)-1\right], \tag{2.75}$$

$$|\epsilon_{\nu,2}^{m}\left(\zeta,k^{2}\right)| \leq \mathbf{E}_{m}\left(\sqrt{2\kappa}\zeta\right)\mathbf{M}_{m}\left(\sqrt{2\kappa}\zeta\right)\left[\exp\left(\tfrac{1}{2}\sqrt{\pi/\kappa}\, l_{m,1}\mathcal{V}_{0,\zeta}\left(\widehat{\psi}_{\nu}^{m}\right)\right)-1\right], \tag{2.76}$$

where

$$l_{m,1} = \sup_{z\in(0,\infty)}\left\{\frac{\Omega(z)\mathbf{M}_{m}^{2}\left(z\right)}{\Gamma\left(m+1\right)}\right\}. \tag{2.77}$$

We have the bounds

$$\mathcal{V}_{a,b}\left(\widehat{\psi}_{\nu}^{m}\right) = \int_{a}^{b}\frac{|\widehat{\psi}_{\nu}^{m}\left(t,k^{2}\right)|}{1+\sqrt{2\kappa t}}\mathrm{d}t \leq \int_{a}^{b}\frac{|\widehat{\psi}_{\nu}^{m}\left(0,k^{2}\right)|}{1+\sqrt{2\kappa t}}\mathrm{d}t + \int_{a}^{b}\frac{|\widehat{\psi}_{\nu}^{m}\left(t,k^{2}\right)-\widehat{\psi}_{\nu}^{m}\left(0,k^{2}\right)|}{\sqrt{2\kappa t}}\mathrm{d}t, \tag{2.78}$$

thus from (2.72) we have

$$\mathcal{V}_{0,\zeta}\left(\widehat{\psi}_{\nu}^{m}\right) = \mathcal{O}\left(\kappa^{-1/2}\right) \quad \text{and} \quad \mathcal{V}_{\zeta,\zeta_{*}}\left(\widehat{\psi}_{\nu}^{m}\right) = \mathcal{O}\left(\kappa^{-1/2}\right) \quad (\kappa\to\infty). \tag{2.79}$$

From (2.53) clearly $l_{m,1}$ is a bounded constant as $\kappa\to\infty$. Hence we obtain the results given in (2.14). Applying similar analysis to the above we obtain (2.15). Thus from (2.69), (2.27) and (2.22) we obtain the solutions given in (2.13). Since we have the boundary conditions

$$\left.\frac{\mathrm{d}Ec_{\nu}^{m}\left(z,k^{2}\right)}{\mathrm{d}z}\right|_{z=K} = 0, \quad Es_{\nu}^{m+1}\left(K,k^{2}\right) = 0, \tag{2.80}$$

we obtain that necessarily both $\eta_{\nu,c}^{m}$ and $\eta_{\nu,s}^{m+1}$ are $\mathcal{O}\left(e^{-\kappa\zeta_{*}^{2}}\kappa^{m+1/2}\right)$ as $\kappa\to\infty$. Only when ζ nears the endpoint the contribution from the $\overline{D}_{m}\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,2}^{m}(\zeta,k^{2})$ term is comparable to $D_{m}\left(\sqrt{2\kappa}\zeta\right)+\epsilon_{\nu,1}^{m}(\zeta,k^{2})$. These functions will be made unique by their normalisation.

Due to the complicated nature of the mapping between x and ζ, it is not simple to express one in terms of the other. With respect to (2.6) we consider the integral

$$(C_\nu^m)^2 \int_{-K}^{K} \mathrm{dn}(z,k)\sqrt{\frac{\zeta^2-(\sigma_\nu^m)^2}{x^2-(s_\nu^m)^2}}D_m^2\left(\sqrt{2\kappa}\zeta\right)\mathrm{d}z. \tag{2.81}$$

Remember that x and ζ are defined in terms of z by (2.22) and (2.17). Using only this first term in an expansion for the solution, we can only obtain coefficients for up to $1/\kappa$ terms. Any further terms in an expansion for the solution would contribute to further terms in the normalisation constant expansion, which we will not consider here. Since the first terms in their respective function uniform approximations are the same for large κ, it will be the same for both C_ν^m and S_ν^{m+1}.

First we perform the transformation $x = \mathrm{sn}(z,k)$ to obtain

$$(C_\nu^m)^2 \int_{-1}^{1} \sqrt{\frac{\zeta^2-(\sigma_\nu^m)^2}{(x^2-(s_\nu^m)^2)(1-x^2)}}D_m^2\left(\sqrt{2\kappa}\zeta\right)\mathrm{d}x \tag{2.82}$$

then another transformation to the ζ variable to obtain

$$(C_\nu^m)^2 \int_{-\zeta_*}^{\zeta^*} \frac{\zeta^2-(\sigma_\nu^m)^2}{x^2-(s_\nu^m)^2}\sqrt{1-k^2x^2}D_m^2\left(\sqrt{2\kappa}\zeta\right)\mathrm{d}\zeta. \tag{2.83}$$

As the oscillatory behaviour of $D_m(\sqrt{2\kappa}\zeta)$ happens in a shrinking region of the origin as $\kappa \to \infty$, we seek to approximate the integral around this point to get an approximation for C_ν^m. Thus we consider an expansion of the form

$$x = \sum_{k=1}^{\infty} c_k \zeta^k, \tag{2.84}$$

as $\zeta \to 0$, and substituting this into the relation

$$\frac{\mathrm{d}x}{\mathrm{d}\zeta} = \sqrt{\frac{(\zeta^2-(\sigma_\nu^m)^2)(1-x^2)(1-k^2x^2)}{x^2-(s_\nu^m)^2}}. \tag{2.85}$$

From this we can determine, by matching (2.84) and (2.85), the c_k terms, and obtain

$$x = \frac{\sigma_\nu^m}{s_\nu^m}\zeta + \left(\frac{(\sigma_\nu^m)^4-(s_\nu^m)^4}{6(s_\nu^m)^5\sigma_\nu^m} - \frac{(1+k^2)(\sigma_\nu^m)^3}{6(s_\nu^m)^3}\right)\zeta^3 + \mathcal{O}(\zeta^5), \tag{2.86}$$

as $\zeta \to 0$. From this we obtain

$$\frac{\zeta^2 - (\sigma_\nu^m)^2}{x^2 - (s_\nu^m)^2}\sqrt{1-k^2x^2} = \frac{(\sigma_\nu^m)^2}{(s_\nu^m)^2} + \frac{2\,(\sigma_\nu^m)^4 - 2\,(s_\nu^m)^4 - k^2\,(s_\nu^m)^2\,(\sigma_\nu^m)^4}{2\,(s_\nu^m)^6}\zeta^2 + \mathcal{O}(\zeta^4), \tag{2.87}$$

as $\zeta \to 0$. With respect to this, we consider as a first approximation for C_ν^m the integral

$$(C_\nu^m)^2 \int_{-\zeta_*}^{\zeta_*} \left(\frac{(\sigma_\nu^m)^2}{(s_\nu^m)^2} + \frac{(2\sigma_\nu^m)^4 - 2\,(s_\nu^m)^4 - k^2\,(s_\nu^m)^2\,(\sigma_\nu^m)^4}{2\,(s_\nu^m)^6}\zeta^2 \right) D_m^2(\sqrt{2\kappa}\zeta)\mathrm{d}\zeta. \tag{2.88}$$

and letting $t = \sqrt{2\kappa}\zeta$, we have the approximation

$$\frac{(C_\nu^m)^2}{\sqrt{2\kappa}} \int_{-\infty}^{\infty} \left(1 + \frac{(1+k^2)(2m+1) + t^2 - k^2t^2}{8\kappa}\right) D_m^2(t)\mathrm{d}t \sim \frac{(C_\nu^m)^2\, m!}{\sqrt{\kappa/\pi}} \left(1 + \frac{2m+1}{4\kappa}\right). \tag{2.89}$$

This approximation is computed using integral formulas for the parabolic cylinder functions. We do this using the representation of D_m in terms of Hermite polynomials. In correspondence with (2.7) and that for large z,

$$D_m(z) > 0, \quad D_m'(z) < 0 \ \ \text{(see [1] §12.9 (i))}, \tag{2.90}$$

we deduce that as $\kappa \to \infty$, from (2.6) we obtain (2.19). We will not consider any error analysis for this normalisation constant expansion.

The last statement in the Theorem (2.20) is derived easily from Proposition 2.1, (2.30) and the first of (2.18). □

2.4.3 A second term in the approximation in terms of parabolic cylinder D functions

In §11 of [7] uniform asymptotic expansions are included. The two-term solutions of (2.66) are of the form

$$\begin{aligned} W_{\nu,1}^m\left(\zeta, k^2\right) &\sim A_0 D_m\left(\sqrt{2\kappa}\zeta\right) + \frac{B_0(\zeta,\kappa)}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}D_m\left(\sqrt{2\kappa}\zeta\right), \\ W_{\nu,2}^m\left(\zeta, k^2\right) &\sim A_0 \overline{D}_m\left(\sqrt{2\kappa}\zeta\right) + \frac{B_0(\zeta,\kappa)}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\overline{D}_m\left(\sqrt{2\kappa}\zeta\right), \end{aligned} \tag{2.91}$$

in which we have taken $A_0 = 1$ and for $B_0(\zeta, \kappa)$ we have

$$B_0(\zeta,\kappa) = \int_{\tilde{\sigma}}^{\zeta} \frac{\widehat{\psi}_{\nu}^{m}\left(t,k^2\right)}{2\sqrt{(\zeta^2-\tilde{\sigma}^2)(t^2-\tilde{\sigma}^2)}}\,\mathrm{d}t, \tag{2.92}$$

where $\tilde{\sigma} = \sqrt{(2m+1)/\kappa}$. This coefficient depends on κ and its dominant part is

$$B_0(\zeta) = \lim_{\kappa\to\infty} B_0(\zeta,\kappa) = \frac{1}{2\zeta}\int_0^{\zeta} \frac{\lim_{\kappa\to\infty}\widehat{\psi}_{\nu}^{m}\left(t,k^2\right)}{t}\,\mathrm{d}t. \tag{2.93}$$

To simplify the integrand we take the final equation in (2.17) and let $\kappa \to \infty$. The result is

$$\tfrac{1}{2}\zeta^2 = \int_0^x \frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{1}{k}\ln\left(\frac{1+k}{\sqrt{1-k^2x^2}+k\sqrt{1-x^2}}\right). \tag{2.94}$$

This relation can be inverted to obtain

$$x^2 = k^{-1}\sinh\left(k\zeta^2\right) - \left(1+k^{-2}\right)\sinh^2\left(k\zeta^2/2\right). \tag{2.95}$$

We use this in

$$\lim_{\kappa\to\infty}\widehat{\psi}_{\nu}^{m}\left(\zeta,k^2\right) = \frac{3}{4\zeta^2} + \frac{\zeta^2}{4}\left(\frac{k^2(x^2+x^4)+x^2-3}{x^4}\right) + \frac{1+k^2}{8}, \tag{2.96}$$

and are able to evaluate the integral in (2.93) and obtain

$$32\zeta B_0(\zeta) = (k^2+1)\ln\left(\tfrac{1}{4}\zeta^2 C(\zeta,k)\right) - \frac{3\left(k^2-1\right)^2}{2C(\zeta,k)} + 3k\coth\left(k\zeta^2/2\right) + 2k^2\zeta^2 - \frac{6}{\zeta^2}, \tag{2.97}$$

where $C(\zeta,k) = 2k\coth\left(k\zeta^2/2\right) - k^2 - 1$. Note that to obtain a bounded solution for $B_0(\zeta)$, an additional term in the eigenvalue expansion needed to be determined, which we derive in following section and give in Theorem 2.2.

At this time we have not yet finished the analysis to obtain rigorous bounds for the two-term approximations, but it should be performed similarly to the error analysis used previously for the one-term uniform approximation. Thus we give the following formal two-term uniform approximations for the Lamé functions as $\kappa \to \infty$

$$\begin{aligned} Ec_{\nu}^{m}\left(z,k^{2}\right) &\sim C_{\nu}^{m}\left(\frac{\zeta^{2}-(\sigma_{\nu}^{m})^{2}}{x^{2}-(s_{\nu}^{m})^{2}}\right)^{1/4}\left(D_{m}\left(\sqrt{2\kappa}\zeta\right)+\frac{B_{0}(\zeta)}{\kappa^{2}}\frac{d}{d\zeta}D_{m}\left(\sqrt{2\kappa}\zeta\right)\right), \\ Es_{\nu}^{m+1}\left(z,k^{2}\right) &\sim S_{\nu}^{m+1}\left(\frac{\zeta^{2}-(\sigma_{\nu}^{m})^{2}}{x^{2}-(s_{\nu}^{m})^{2}}\right)^{1/4}\left(D_{m}\left(\sqrt{2\kappa}\zeta\right)+\frac{B_{0}(\zeta)}{\kappa^{2}}\frac{d}{d\zeta}D_{m}\left(\sqrt{2\kappa}\zeta\right)\right), \end{aligned} \tag{2.98}$$

where C_{ν}^{m} and S_{ν}^{m+1} are normalisation constants given in (2.19), and $B_0(\zeta)$ is as in (2.97).

2.4.4 Numerics

We will now provide some numerical clarification we have performed. In the figures below we give a plot of a Lamé function, and next to it a plot of the absolute error of our one-term uniform approximation.

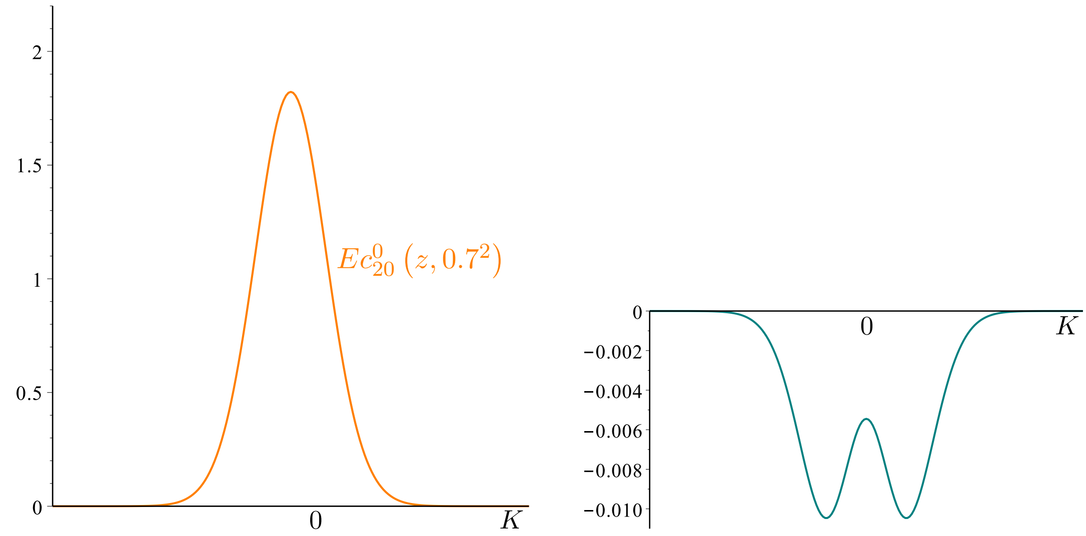

Figure 2.4: (left) Plots for the even Lamé function $Ec_{20}^{0}\left(z, 0.7^2\right)$ in $[-K, K]$, and (right) the absolute error corresponding to our uniform approximation.

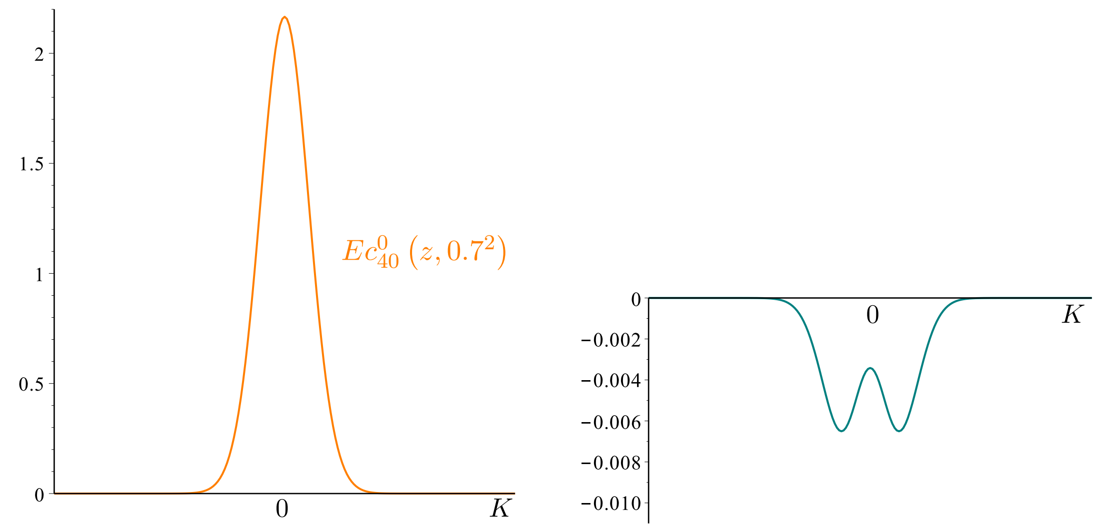

Figure 2.5: (left) Plot for the even Lamé function $Ec_{40}^{0}\left(z, 0.7^2\right)$ in $[-K, K]$, and (right) the absolute error corresponding to our uniform approximation.

We note that as ν becomes larger, the absolute error corresponding to our approximations becomes smaller.

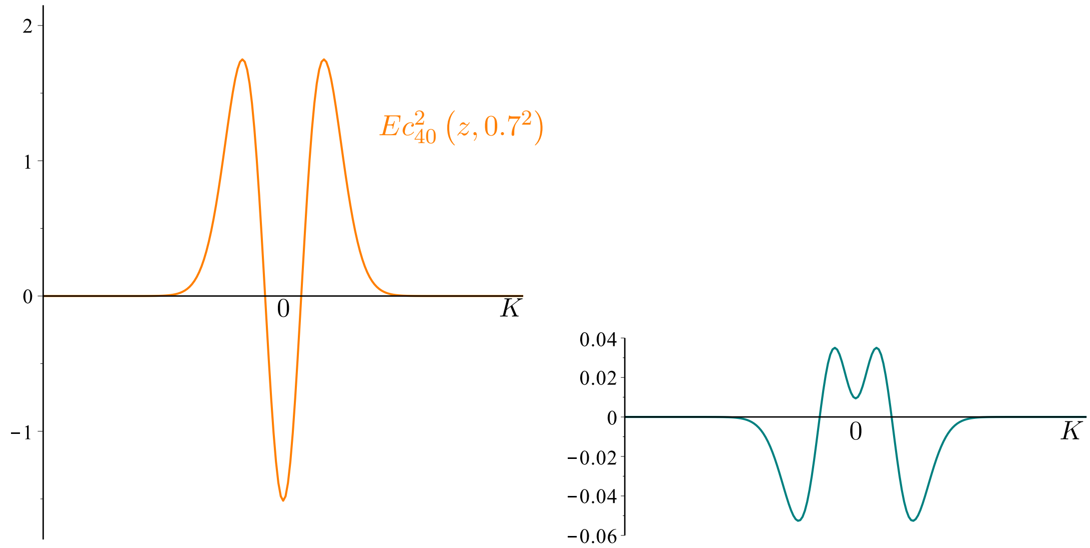

Figure 2.6: (left) Plot for the even Lamé function $Ec_{40}^{2}\left(z, 0.7^2\right)$ in $[-K, K]$, and (right) the absolute error corresponding to our uniform approximation.

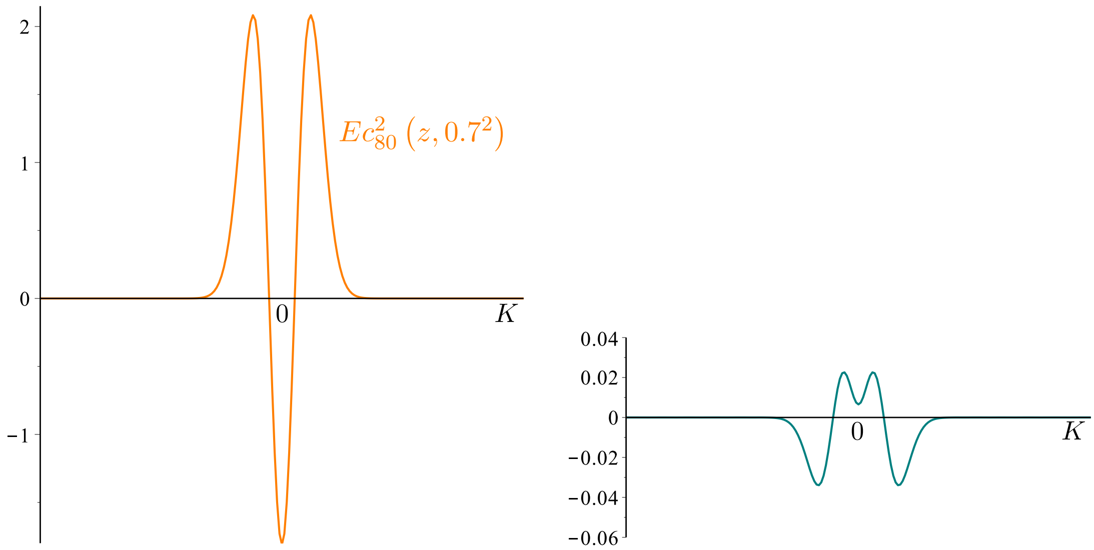

Figure 2.7: (left) Plot for the even Lamé function $Ec_{80}^{2}\left(z, 0.7^2\right)$ in $[-K, K]$, and (right) the absolute error corresponding to our uniform approximation.

We note that as m becomes larger, ν must become even larger to see a significantly small absolute error corresponding to our approximations. Investigating numerical errors for other approximations gives similarly good results. Numerically the difference isn't notable for the two-term approximation so we don't provide that here.

2.5 Asymptotic expansions for the Lamé functions and their eigenvalues

We have given one and two-term uniform asymptotic approximations for the Lamé functions when ν became large. A third term in an asymptotic expansion could not be computed due to the complicated nature of the transformation of the independent variable. In this section we employ a simpler transformation than in the previous subsection such that we can construct formal asymptotic expansions for the Lamé functions and their corresponding eigenvalues.

Theorem 2.2. *Let $\kappa = \sqrt{\nu(\nu+1)}k$, $m \in \mathbb{N}_0$ and $0 < k < 1$. Then as $\kappa \to \infty$*

$$\left.\begin{matrix} a_\nu^m \\ b_\nu^{m+1} \end{matrix}\right\} = (2m+1)\kappa + 2\sum_{s=0}^{n-1} \frac{\mu_{s+1}}{\kappa^s} + \mathcal{O}(\kappa^{-n}), \tag{2.99}$$

valid for integral $n \geq 1$, where the order term will be different in both cases. The μ_s terms are constant coefficients which depend on k and m, found recursively with the function expansions. We give here the first two terms:

$$\mu_1 = -\frac{k^2+1}{8}\left(1+2m+2m^2\right), \qquad \mu_2 = -\frac{2m+1}{32}\left(\left(k^2-1\right)^2\left(1+m+m^2\right) - 2k^2\right).$$

Letting $t = \sqrt{2\kappa}\,\mathrm{sn}\,(z,k)$, for $z = \mathcal{O}(\kappa^{-1/2})$ as $\kappa \to \infty$

$$\begin{aligned} Ec_\nu^m(z,k^2) &= C_\nu^m \left(D_m(t) \sum_{s=0}^{n} \frac{A_s(t)}{\kappa^s} + D_m'(t) \sum_{s=0}^{n} \frac{B_s(t)}{\kappa^s} + \mathcal{O}(\kappa^{-n-1}) \right), \\ \frac{Es_\nu^{m+1}(z,k^2)}{\mathrm{cn}(z,k)} &= S_\nu^{m+1} \left(D_m(t) \sum_{s=0}^{n} \frac{P_s(t)}{\kappa^s} + D_m'(t) \sum_{s=0}^{n} \frac{Q_s(t)}{\kappa^s} + \mathcal{O}(\kappa^{-n-1}) \right), \end{aligned} \tag{2.100}$$

valid for $n \in \mathbb{N}_0$, where

$$\left.\begin{matrix} C_\nu^m \\ S_\nu^{m+1} \end{matrix}\right\} \sim \frac{(\pi\kappa)^{1/4}}{\sqrt{2m!}} \left(1 + \frac{\eta_1}{\kappa} + \frac{\eta_2}{\kappa^2} \ldots\right)^{-1/2}. \tag{2.101}$$

Both $A_s(t)$ and $P_s(t)$ are even polynomials, and both $B_s(t)$ and $Q_s(t)$ are odd polynomials. These polynomials are found recursively and we give here the first two terms of *each, and the first term in the normalisation constant expansion:*

$$\begin{aligned} &A_0 = 1, \quad A_1 = \frac{k^2+1}{32}t^2, \quad B_0 = 0, \quad B_1 = \frac{k^2+1}{16}\left(t^3 - (2m+1)t\right), \\ &P_0 = 1, \quad P_1 = \frac{k^2+9}{32}t^2, \quad Q_0 = 0, \quad Q_1 = B_1, \quad \eta_1 = \frac{3-k^2}{16}(2m+1). \end{aligned} \tag{2.102}$$

2.5.1 Formal uniform asymptotic expansions of the Lamé functions and eigenvalues

The oscillatory behaviour of the Lamé functions happens in a shrinking neighbourhood of the origin as $\kappa \to \infty$, and it can be shown that around the origin the ζ variable used in the uniform approximations given in the last section behaves approximately like $\mathrm{sn}(z,k)$. Thus the variable in the parabolic cylinder function used in the approximation around this point behaves approximately like $\sqrt{2\kappa}\,\mathrm{sn}(z,k)$. This motivates the next simpler transformation.

Letting $t = \sqrt{2\kappa}\,\mathrm{sn}(z,k)$ in (2.21) we have

$$\begin{aligned} &\frac{\mathrm{d}^2 w_\nu^m(z,k^2)}{\mathrm{d}t^2} + \frac{2h - t^2\kappa}{4\kappa} w_\nu^m(z,k^2) \\ &\qquad - \frac{1}{2\kappa}\left(\left((1+k^2)t^2 - \frac{k^2t^4}{2\kappa}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + t\left(1 + k^2 - \frac{k^2t^2}{\kappa}\right)\frac{\mathrm{d}}{\mathrm{d}t}\right) w_\nu^m(z,k^2) = 0, \end{aligned} \tag{2.103}$$

where $z \in [-K, K]$ corresponds to $t \in [-\sqrt{2\kappa}, \sqrt{2\kappa}]$. We suppose in accordance with (2.20) that

$$\frac{h_\nu^m}{2\kappa} = m + \tfrac{1}{2} + \sum_{s=1}^{n} \frac{\mu_s}{\kappa^s} + \frac{\widetilde{\mu}_{n+1}}{\kappa^{n+1}}, \tag{2.104}$$

for both $h_\nu^m = a_\nu^m$ and $h_\nu^m = b_\nu^{m+1}$, where integral $n \geq 1$ and $\widetilde{\mu}_n$ can be re-expanded in a sensible manner. We can then write Lamé's equation in the form

$$\begin{aligned} &\frac{\mathrm{d}^2 w_\nu^m(z,k^2)}{\mathrm{d}t^2} + \left(m + \frac{1}{2} - \frac{t^2}{4}\right) w_\nu^m(z,k^2) + \frac{1}{\kappa}\left(\left(-\frac{1}{2}(1+k^2)t^2 + \frac{k^2t^4}{4\kappa}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right. \\ &\qquad \left. + \frac{t}{2}\left(-1 - k^2 + \frac{k^2t^2}{\kappa}\right)\frac{\mathrm{d}}{\mathrm{d}t} + \sum_{s=0}^{n} \frac{\mu_{s+1}}{\kappa^s} + \frac{\widetilde{\mu}_{n+1}}{\kappa^{n+1}}\right) w_\nu^m(z,k^2) = 0. \end{aligned} \tag{2.105}$$

This equation is split in such a way that constructing a formal asymptotic expansion in terms of parabolic cylinder functions $D_m(t)$ in the form

$$w_\nu^m(t,k^2) = D_m\left(t\right)\sum_{s=0}^{\infty}\frac{A_s\left(t\right)}{\kappa^s} + D'_m\left(t\right)\sum_{s=0}^{\infty}\frac{B_s\left(t\right)}{\kappa^s} \tag{2.106}$$

seems sensible. However one should observe that this splitting only makes sense when $t = \mathcal{O}(1)$ as $\kappa \to \infty$. We denote this range as $[-t_*, t_*]$. One should note that whilst this appears to be a new ansatz, there are similar expansions given in the literature for the Mathieu functions which we discussed in section 2.2. These expansions are given in terms of $D_{m-2j}(t)$ for $j \in \mathbb{Z}$ instead of in terms of $D_m(t)$ and $D'_m(t)$, but using the recurrence relations for the parabolic cylinder functions it is obvious these expansions are equivalent up to normalisation. An expansion in the form we have given allows us to differentiate our expansions easily and hence seems the most natural in this case, also allowing us to perform rigorous error analysis.

We seek solutions which are either even or odd respective to the parity of m. Since $D_m(t)$ is either even or odd respective to when m is either even or odd, we deduce that $A_s(t)$ and $B_s(t)$ must be even and odd respectively.

Substituting (2.106) into (2.105) and equating powers of κ, we have the recurrence relations for $A_s(t)$ and $B_s(t)$

$$\begin{aligned}2A'_s(t) + B''_s(t) &- \frac{t+tk^2}{2}\left(2tA'_{s-1}(t) + A_{s-1}(t) + tB''_{s-1}(t) + B'_{s-1}(t) + \tilde{t}(m)tB_{s-1}(t)\right)\\ &+ \frac{k^2t^3}{4}\left(2tA'_{s-2}(t) + 2A_{s-2}(t) + tB''_{s-2}(t) + 2B'_{s-2}(t) + \tilde{t}(m)tB_{s-2}(t)\right)\\ &+ \sum_{j=1}^{s}\mu_j B_{s-j}(t) = 0,\end{aligned} \tag{2.107}$$

$$\begin{aligned}&A''_s(t) + 2\tilde{t}(m)B'_s(t) + \frac{t}{2}B_s(t) + \sum_{j=1}^{s}\mu_j A_{s-j}(t)\\ &- \frac{t+tk^2}{2}\left(tA''_{s-1}(t) + A'_{s-1}(t) + \frac{t^2}{2}B_{s-1}(t) + \tilde{t}(m)\left(tA_{s-1}(t) + 2tB'_{s-1}(t) + B_{s-1}(t)\right)\right)\\ &+ \frac{k^2t^3}{4}\left(tA''_{s-2}(t) + 2A'_{s-2}(t) + \frac{t^2}{2}B_{s-2}(t) + \tilde{t}(m)\left(tA_{s-2}(t) + 2tB'_{s-2}(t) + 2B_{s-2}(t)\right)\right) = 0,\end{aligned} \tag{2.108}$$

where $\tilde{t}(m) = \frac{1}{4}t^2 - m - \frac{1}{2}$. Neither of these relations separately determine solutions for $A_s(t)$ or $B_s(t)$ from previous coefficients, thus we differentiate (2.107) to obtain an expression for $A_s''(t)$ and substitute it into (2.108); this gives the third order inhomogeneous differential equation for $B_s(t)$

$$B_s'''(t) - \left(t^2 - 4m - 2\right) B_s'(t) - tB_s(t) = b_s(t), \tag{2.109}$$

where

$$\begin{aligned} b_s(t) = &\sum_{j=1}^{s} \mu_j \left(2A_{s-j}(t) - B_{s-j}'(t)\right) \\ &+ \frac{1+k^2}{2}\Big[3tA_{s-1}'(t) + A_{s-1}(t) + t^2 B_{s-1}'''(t) + 3tB_{s-1}''(t) + B_{s-1}'(t) \\ &\qquad -\tfrac{1}{2}t^3 B_{s-1}(t) - \tilde{t}(m)t^2\left(2A_{s-1}(t) + 3B_{s-1}'(t)\right)\Big] \\ &- \frac{k^2t^2}{2}\Big[3tA_{s-2}'(t) + 3A_{s-2}(t) + \tfrac{1}{2}t^2 B_{s-2}'''(t) + 3tB_{s-2}''(t) + 3B_{s-2}'(t) \\ &\qquad -\tfrac{1}{4}t^3 B_{s-2}(t) - \tfrac{1}{2}t^2\tilde{t}(m)\left(2A_{s-2}(t) + 3B_{s-2}'(t)\right)\Big]. \end{aligned} \tag{2.110}$$

Once $B_s(t)$ is determined, we can use (2.107) to determine $A_s(t)$. There will be freedom in choosing the integration constants in the $A_s(t)$ terms, with identification of our solutions made unique by their normalisation.

2.5.2 General coefficients $B_s(t)$ and $A_s(t)$

Using variation of parameters we obtain the general solution for $B_s(t)$

$$B_s(t) = b_s^1(t)\, D_m^2(t) + b_s^2(t)\, \overline{D}_m^2(t) + b_s^3(t)\, D_m(t)\, \overline{D}_m(t), \tag{2.111}$$

where

$$\begin{aligned} b_s^1(t) &= \int \frac{b_s(t)\,\mathcal{W}\{\overline{D}_m^2, D_m\overline{D}_m\}}{\mathcal{W}\{D_m^2, \overline{D}_m^2, D_m\overline{D}_m\}}\mathrm{d}t + c_1, \qquad b_s^2(t) = -\int \frac{b_s(t)\,\mathcal{W}\{D_m^2, D_m\overline{D}_m\}}{\mathcal{W}\{D_m^2, \overline{D}_m^2, D_m\overline{D}_m\}}\mathrm{d}t + c_2, \\ b_s^3(t) &= \int \frac{b_s(t)\,\mathcal{W}\{D_m^2, \overline{D}_m^2\}}{\mathcal{W}\{D_m^2, \overline{D}_m^2, D_m\overline{D}_m\}}\mathrm{d}t + c_3. \end{aligned} \tag{2.112}$$

By expanding the Wronskians we derive the relations

$$\mathcal{W}\{D_m^2, D_m\overline{D}_m\} = \mathcal{W}\{D_m, \overline{D}_m\}D_m^2(t)\,,\quad \mathcal{W}\{\overline{D}_m^2, D_m\overline{D}_m\} = -\mathcal{W}\{D_m, \overline{D}_m\}\overline{D}_m^2(t)\,,$$
$$\mathcal{W}\{D_m^2, \overline{D}_m^2\} = 2\mathcal{W}\{D_m, \overline{D}_m\}D_m(t)\,\overline{D}_m(t)\,,\quad \mathcal{W}\{D_m^2, \overline{D}_m^2, D_m\overline{D}_m\} = -2\mathcal{W}\{D_m, \overline{D}_m\}^3. \tag{2.113}$$

Then without loss of generality, since we have the constants c_1, c_2 and c_3, we can rewrite the indefinite integrals in $\left\{b_s^i(t)\right\}_{i=1}^3$ as definite integrals from 0 to t, and since $B_s(t)$ is supposed to be an odd function we have to take $c_1 = c_2 = 0$. Thus

$$B_s(t) = \frac{\pi}{4m!^2}\int_0^t b_s(\tau)\left(\overline{D}_m(\tau)D_m(t) - D_m(\tau)\overline{D}_m(t)\right)^2 \mathrm{d}\tau + c_3 D_m(t)\,\overline{D}_m(t)\,. \tag{2.114}$$

Although we consider t to be $\mathcal{O}(1)$ as $\kappa \to \infty$, we still want an expansion which exhibits the correct behaviour at the endpoints of the interval. Expanding the squared term in (2.114) and splitting it into three separate integrals, large variable asymptotics for the parabolic cylinder functions given in (1.64) tells us that the terms involving $D_m^2(t)$ or $D_m(t)\overline{D}_m(t)$ grow no faster than polynomials when t becomes large. Since

$$\overline{D}_m(t)^2 \sim \frac{2m!^2}{\pi}e^{t^2/2}t^{-2m-2} \qquad (t \to \infty), \tag{2.115}$$

to ensure the boundedness of our formal expansion as t becomes large, μ_s is determined uniquely by the condition that

$$\int_0^\infty b_s(\tau)\, D_m^2(\tau)\mathrm{d}\tau = 0. \tag{2.116}$$

This condition will also ensure boundedness as $t \to -\infty$. Note that in the next subsection we will derive an alternative method to compute the μ_s terms which we use to compute the $A_s(t)$ and $B_s(t)$ coefficients as it is simpler.

Consider first $s = 0$. From (2.110) we see that $b_0(t) = 0$, thus ensuring that $B_0(t)$ is odd we obtain the general solution

$$B_0(t) = c_3 D_m(t)\overline{D}_m(t) \tag{2.117}$$

and then

$$A_0(t) = -\tfrac{1}{2}c_3\left(D'_m(t)\overline{D}_m(t) + D_m(t)\overline{D}'_m(t)\right) + c. \tag{2.118}$$

Rearranging, on the κ^0 level of the asymptotic expansion (2.106) we have

$$\left(\tfrac{1}{2}c_3\mathcal{W}(D_m,\overline{D}_m) + c\right) D_m\left(t\right). \tag{2.119}$$

Thus having this term in $B_0(t)$ equates to $B_0(t) = 0$ and an extra constant term in $A_0(t)$, and since we have freedom in the arbitrary constant terms in $A_s(t)$, we can take $c_3 = 0$ without loss of generality. For simplicity we take $A_0(t) = 1$ and adopt the convention $A_s(t) = 0$ for $s \geq 1$. With these choices we have

$$b_1(t) = 2\mu_1 + \frac{1+k^2}{4}\left(2 - t^2\left(t^2 - 4m - 2\right)\right), \tag{2.120}$$

and expressing the $D_m(t)$ in (2.116) via ([1] 12.7.2) in terms of Hermite polynomials we can evaluate the integral in (2.116) and obtain

$$\mu_1 = -\tfrac{1}{8}\left(1+k^2\right)\left(1 + 2m + 2m^2\right). \tag{2.121}$$

In the case $s = 1$, it can be shown using integration by parts that for μ_1 which satisfies (2.116)

$$\begin{aligned}\frac{\pi}{4m!^2}\int_0^t b_1\left(\tau\right)\left(\overline{D}_m(\tau)D_m\left(t\right) - D_m(\tau)\overline{D}_m\left(t\right)\right)^2 \mathrm{d}\tau \\ = \frac{1+k^2}{16}\left(t^3 - (2m+1)t + (-1)^m\,\frac{2m+1}{m!}\sqrt{\frac{\pi}{2}}D_m(t)\overline{D}_m\left(t\right)\right).\end{aligned} \tag{2.122}$$

Thus clearly in this case, if the correct c_3 is chosen in (2.114) then $B_1(t)$ is exactly an odd polynomial. Using similar observations as in the $s = 0$ case, we also note that if this multiple of $D_m\overline{D}_m$ is included in the $B_1(t)$ term, the expansion can be rearranged so that this term is instead represented in the constant term of $A_1(t)$. Taking $B_1(t)$ to be exactly an odd polynomial, we get that $A_1(t)$ is an even polynomial. If one would go through the details to compute the representation for $B_1(t)$ as a polynomial plus this multiple of $D_m\overline{D}_m$, one would see that it would appear for all $s \geq 1$ that if the previous $A_s(t)$ and $B_s(t)$ terms are all polynomials, then the $A_s(t)$ and $B_s(t)$ terms can be represented as polynomials.

2.5.3 Polynomial coefficients $B_s(t)$ and $A_s(t)$

To obtain explicit expressions for $A_s(t)$ and $B_s(t)$ we try substituting in polynomial expansions with undetermined coefficients. Take $A_0(t) = 1$ and $B_0(t) = 0$, then for $s \geq 1$ we consider $A_s(t)$ and $B_s(t)$ in the form

$$A_s(t) = \sum_{i=1}^{\infty} a_{s,i} t^{2i} \quad \text{and} \quad B_s(t) = \sum_{i=0}^{\infty} b_{s,i} t^{2i+1}. \tag{2.123}$$

Substituting these into (2.109) and (2.107) we obtain the recurrence relations for coefficients

$$\begin{aligned}
&(2i+3)(2i+2)(2i+1)\, b_{s,i+1} + (4m+2)(2i+1)\, b_{s,i} - 2i b_{s,i-1} + 2\sum_{j=0}^{s-1} \mu_{s-j} \left(\left(i+\tfrac{1}{2}\right) b_{j,i} - a_{j,i}\right) \\
&\quad - \left(1+k^2\right)\Big[\left(3i+\tfrac{1}{2}\right) a_{s-1,i} + \tfrac{1}{2}(2i+1)^3 b_{s-1,i} + \left(m+\tfrac{1}{2}\right)\left(a_{s-1,i-1} + \left(3i-\tfrac{3}{2}\right) b_{s-1,i-1}\right) \\
&\quad -\tfrac{1}{4} a_{s-1,i-2} - \left(\tfrac{3}{4}i - \tfrac{7}{8}\right) b_{s-1,i-2}\Big] + k^2\Big[\left(3i-\tfrac{3}{2}\right) a_{s-2,i-1} + 2i\left(i^2-\tfrac{1}{4}\right) b_{s-2,i-1} \\
&\quad + (2m+1)\left(\tfrac{1}{4} a_{s-2,i-2} + \left(\tfrac{3}{4}i - \tfrac{9}{8}\right) b_{s-2,i-2}\right) - \tfrac{1}{8} a_{s-2,i-3} - \left(\tfrac{3}{8}i - \tfrac{13}{16}\right) b_{s-2,i-3}\Big] = 0,
\end{aligned} \tag{2.124}$$

$$\begin{aligned}
&2(2i+2)\, a_{s,i+1} + (2i+3)(2i+2)\, b_{s,i+1} + \sum_{j=0}^{s-1} \mu_{s-j} b_{j,i} \\
&\quad - \tfrac{1}{2}\left(1+k^2\right)\left((4i+1)\, a_{s-1,i} + (2i+1)^2 b_{s-1,i} - \left(m+\tfrac{1}{2}\right) b_{s-1,i-1} + \tfrac{1}{4} b_{s-1,i-2}\right) \\
&\quad + k^2\left(i\left(i-\tfrac{1}{2}\right) b_{s-2,i-1} + \left(i-\tfrac{1}{2}\right) a_{s-2,i-1} - \tfrac{1}{8}(2m+1)\, b_{s-2,i-2} + \tfrac{1}{16} b_{s-2,i-3}\right) = 0.
\end{aligned} \tag{2.125}$$

From these recurrence relations it is observed that only a finite number of the $a_{s,i}$ and $b_{s,i}$ are non-zero. The orders are displayed in Table 2.2. In this manner the coefficients

Table 2.2: The orders of the polynomials.

s	$A_s(t)$	$B_s(t)$
even	$4s$	$4s-3$
odd	$4s-2$	$4s-1$

are determined recursively. We deduce by considering the difference of (2.124) and (2.125) in the case $i = 0$ that

$$\mu_s = (2m+1)b_{s,0} - 2a_{s,1}. \tag{2.126}$$

In the case that s is even then we start with $i = 2s-1$ in (2.124) to determine $b_{s,2s-2}$. Then we take $i = 2s-2$ and determine $b_{s,2s-3}$, and so on. Once every power of t is eliminated, we are left with a constant equation which we must make zero by specifying μ_s; in this manner the eigenvalue terms are determined uniquely. These μ_s terms are the same as those specified by the condition (2.116), since it is required for solutions $B_s(t)$ which grow no faster than polynomials. Once the coefficients in the polynomial expression for $B_s(t)$ are determined, and the corresponding μ_s, then the coefficients in the polynomial expression for $A_s(t)$ can be determined from (2.125). In the case that s is odd then we have to start with $i = 2s$ in (2.124) to determine $b_{s,2s-1}$.

2.5.4 Returning to the z-plane: odd solutions

In (§28.8 [1]) expansions for the odd Mathieu functions are given which exhibit the correct odd behaviour. Once we transform our formal expansion back into the z-plane, over the whole real line they behave like the even Lamé functions $Ec_\nu^m(z,k^2)$. We want to construct similar expansions which when transformed back into the z-plane behave like the odd Lamé functions $Es_\nu^{m+1}(z,k^2)$. We need our expansion to have the property

$$w_\nu^m(-K,k^2) = w_\nu^m(K,k^2) = 0. \tag{2.127}$$

The Jacobi elliptic function $\mathrm{cn}(z,k)$ is even around the origin and odd around $z = -K$ and $z = K$. Hence again letting $t = \sqrt{2\kappa}\,\mathrm{sn}(z,k)$ we consider a formal expansion for a solution of (2.105) of the form

$$w_\nu^m(z,k^2) = \mathrm{cn}(z,k)\left(D_m(t)\sum_{s=0}^{\infty}\frac{P_s(t)}{\kappa^s} + D'_m(t)\sum_{s=1}^{\infty}\frac{Q_s(t)}{\kappa^s}\right). \tag{2.128}$$

This has the correct behaviour at $z = -K$ and $z = K$. Again we will require that $P_s(t)$ is even and $Q_s(t)$ is odd. By writing

$$\operatorname{cn}\left(\operatorname{arcsn}\left(\frac{t}{\sqrt{2\kappa}}, k\right), k\right) = \sqrt{1 - \frac{t^2}{2\kappa}}, \tag{2.129}$$

we can express the formal solution in the form

$$w_\nu^m(z, k^2) = \sqrt{1 - \frac{t^2}{2\kappa}} \left(D_m(t) \sum_{s=0}^{\infty} \frac{P_s(t)}{\kappa^s} + D'_m(t) \sum_{s=1}^{\infty} \frac{Q_s(t)}{\kappa^s} \right). \tag{2.130}$$

Expanding this square root, we can rewrite this formal expansion as

$$w_\nu^m(z, k^2) = D_m(t) \sum_{s=0}^{\infty} \frac{A_s(t)}{\kappa^s} + D'_m(t) \sum_{s=1}^{\infty} \frac{B_s(t)}{\kappa^s}, \tag{2.131}$$

where the connection between $P_s(t)$ and $A_s(t)$, and $Q_s(t)$ and $B_s(t)$ is given by

$$A_s(t) = \sum_{j=0}^{s} \binom{\frac{1}{2}}{j} \left(-\frac{t^2}{2}\right)^j P_{s-j}(t), \quad B_s(t) = \sum_{j=0}^{s} \binom{\frac{1}{2}}{j} \left(-\frac{t^2}{2}\right)^j Q_{s-j}(t), \tag{2.132}$$

and where $\binom{a}{b}$ is the generalised binomial coefficient. Thus we determine the $P_s(t)$ and $Q_s(t)$ terms by connection with the $A_s(t)$ and $B_s(t)$ derived previously. Correspondingly we get the same eigenvalue expansion to all orders for both a_ν^m and b_ν^{m+1}, as in the previous case.

To prove Theorem 2.2, we need to use the following theorem, which is Theorem 1 in ([19] §1.52). First we set up the machinery.

We want to match our eigenvalue expansions with the eigenvalues a_ν^m and b_ν^{m+1}, and give order estimates for the expansions on truncation. The theory used for this proof is included in [19], where it is employed for Mathieu's equation. For this we consider Sturm-Liouville theory; for a fuller treatment on Sturm-Liouville theory see [20]. We have the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + (\lambda w(x) - q(x))\, y = 0 \tag{2.133}$$

and we consider $p(x), w(x) > 0$, and $p(x), p'(x)$, $q(x)$ and $w(x)$ to be continuous functions over a finite real interval $[a, b]$. When (a, b) is bounded and $p(x)$ does not vanish on $[a, b]$, this is a regular SL problem, otherwise it is singular. Consider the regular problem. We want to find special values of λ called eigenvalues for which there exists a non-trivial solution satisfying the separated boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad (\alpha_1^2 + \alpha_2^2 > 0), \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \quad (\beta_1^2 + \beta_2^2 > 0). \end{aligned} \tag{2.134}$$

The regular SL theory states that these eigenvalues are real and can be ordered such that

$$\lambda_0 < \lambda_1 < ... < \lambda_\nu < ... \to \infty$$

and these eigenvalues λ_ν correspond to unique eigenfunctions $y_\nu(x)$ with exactly ν zeros in (a, b). Normalised eigenfunctions form an orthonormal basis such that

$$(y_n, y_m) = \int_a^b y_n(x) y_m(x) w(x) \mathrm{d}x = \delta_{mn} \tag{2.135}$$

in the Hilbert space $L^2 \left[[a, b], w(x)\mathrm{d}x\right]$.

Theorem 2.3. *Consider SL to be a self-adjoint differential operator defined on a subspace A of $L^2 \left[[a, b], w(x)\mathrm{d}x\right]$ containing all the twice differentiable functions which satisfy the boundary conditions (2.134) and corresponding λ_ν such that*

$$(SL + \lambda_\nu) y_\nu(x) = 0. \tag{2.136}$$

Let $\widetilde{y}(x) \in A$ be such that $(\widetilde{y}, \widetilde{y}) = 1$. Now define a remainder function $R(x)$ such that

$$(SL + \lambda)\widetilde{y}(x) = R(x) \tag{2.137}$$

for some constant λ. If $R(x) \in L^2 \left[[a, b], w(x)\mathrm{d}x\right]$ then

$$\min_k |\lambda - \lambda_k| < \sqrt{(R, R)}. \tag{2.138}$$

Proof. We have that

$$(R, y_k) = ((SL + \lambda)\widetilde{y}, y_k) = (\widetilde{y}, SL y_k) + \lambda(\widetilde{y}, y_k) = (\lambda - \lambda_k)(\widetilde{y}, y_k), \tag{2.139}$$

and using this in

$$1 = (\widetilde{y}, \widetilde{y}) = \sum_k |(\widetilde{y}, y_k)|^2 = \sum_k \frac{|(R, y_k)|^2}{|\lambda - \lambda_k|^2} < \frac{(R, R)}{\min_k |\lambda - \lambda_k|^2}, \tag{2.140}$$

we have that

$$\min_k |\lambda - \lambda_k| < \sqrt{(R, R)}. \tag{2.141}$$

□

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. We now apply Theorem 2.3 to Lamé's equation to obtain order estimates upon truncation of asymptotic expansions of the eigenvalues a_ν^m and b_ν^{m+1}. Special eigenvalues a_ν^m correspond to non-trivial solutions satisfying the separated boundary conditions

$$\left.\frac{\mathrm{d}}{\mathrm{d}z} Ec_\nu^m(z, k^2)\right|_{z=-K} = \left.\frac{\mathrm{d}}{\mathrm{d}z} Ec_\nu^m(z, k^2)\right|_{z=K} = 0 \tag{2.142}$$

and these eigenvalues are real and can be ordered such that

$$a_\nu^0 < a_\nu^1 < ... < a_\nu^m < ... \to \infty$$

The functions $Ec_\nu^m(z, k^2)$ have exactly m zeros in $(-K, K)$. The normalised eigenfunctions $w_\nu^m(z, k^2)$ form an orthonormal basis such that

$$(w_\nu^i, w_\nu^j) = \int_{-K}^{K} w_\nu^i(z, k^2) w_\nu^j(z, k^2) \mathrm{d}z = \delta_{ij} \tag{2.143}$$

in the Hilbert space $L^2\left[[-K, K], \mathrm{d}z\right]$.

Let L_ν be the operator

$$L_\nu := \frac{\mathrm{d}^2}{\mathrm{d}z^2} - \kappa^2 \operatorname{sn}^2(z, k) \tag{2.144}$$

so that

$$\left(L_\nu + a_\nu^m\right) w_\nu^m\left(z, k^2\right) = 0. \tag{2.145}$$

Letting $t = \sqrt{2\kappa}\,\mathrm{sn}(z,k)$, we define the truncated expansions corresponding to $w_\nu^m\left(z,k^2\right)$ as

$$w_{\nu,n}^m(z,k^2) = c_{\nu,n}^m\left(D_m\left(t\right)\sum_{s=0}^{n}\frac{A_s\left(t\right)}{\kappa^s} + D_m'\left(t\right)\sum_{s=0}^{n}\frac{B_s\left(t\right)}{\kappa^s}\right), \tag{2.146}$$

where the $A_s(t)$ and $B_s(t)$ terms were derived previously, and $c_{\nu,n}^m$ is defined to be a function of κ so that

$$\int_{-K}^{K}\left(w_{\nu,n}^m(z,k^2)\right)^2 \mathrm{d}z = 1. \tag{2.147}$$

Then we can write the derivative of $w_{\nu,n}^m(z,k^2)$ with respect to z as

$$\begin{aligned}\frac{\mathrm{d}w_{\nu,n}^m(z,k^2)}{\mathrm{d}z} = c_{\nu,n}^m\sqrt{2\kappa}\,\mathrm{cn}(z,k)\,\mathrm{dn}(z,k)\Bigg(D_m\left(t\right)\sum_{s=0}^{n}\frac{A_s'\left(t\right)+\left(\frac{t^2}{4}-m-\frac{1}{2}\right)B_s(t)}{\kappa^s}\\ +D_m'\left(t\right)\sum_{s=0}^{n}\frac{A_s\left(t\right)+B_s'(t)}{\kappa^s}\Bigg)\end{aligned} \tag{2.148}$$

where the dash represents differentiation with respect to t. Thus clearly $w_{\nu,n}^m(z,k^2) \in A$ since $\mathrm{cn}(-K,k) = \mathrm{cn}(K,k) = 0$. We also define the truncated eigenvalue expansions

$$a_{\nu,n}^m = (2m+1)\kappa + 2\sum_{s=0}^{n-1}\frac{\mu_{s+1}}{\kappa^s}, \tag{2.149}$$

where the μ_s were derived previously and define $R_{\nu,n}^m(z,k^2)$ such that

$$\left(L_\nu + a_{\nu,n}^m\right) w_{\nu,n}^m(z,k^2) = R_{\nu,n}^m(z,k^2). \tag{2.150}$$

We consider the operator L_ν acting on $D_m\left(t\right)$ and derive

$$L_\nu\left(D_m\left(t\right)\right) = \left((2m+1)\left(\frac{1+k^2}{2}t^2 - \kappa\right) - \left(k^2(2m+1) + \kappa(1+k^2)\right)\frac{t^4}{4\kappa} + \frac{k^2t^6}{8\kappa}\right)D_m\left(t\right)$$

$$+\left(-(1+k^2)t+\frac{k^2t^3}{\kappa}\right)D'_m(t)\,. \tag{2.151}$$

Using the recurrence relations given in (1.63), we can rewrite $w^m_{\nu,n}(z,k^2)$ given in (2.146) such that for $s \in \{0,...,n\}$, $A_s(t)\,D_m(t)$ and $B_s(t)\,D'_m(t)$ are sums of varying orders of parabolic cylinder functions with constant coefficients dependent only on m and k; it then follows from (2.151) that with the solution rewritten in this form, we have

$$R^m_{\nu,n}(z,k^2) = c^m_{\nu,n}\kappa^{-n}\left[(...)+\kappa^{-1}(...)+...\right], \tag{2.152}$$

where the terms inside the brackets are sums of varying orders of parabolic cylinder functions with constant coefficients dependent only on k and m. Note that this remainder term will be a finite sum and clearly $R^m_{\nu,n}(z,k^2) \in L^2\left[[-K,K],\mathrm{d}z\right]$. As $w^m_{\nu,n}(z,k^2) \in A$ we have that

$$\min_k \left|a^m_{\nu,n}-a^k_\nu\right|^2 < \sqrt{\left(R^m_{\nu,n},R^m_{\nu,n}\right)}. \tag{2.153}$$

In the previous section we proved that

$$a^m_\nu = (2m+1)\kappa+\mathcal{O}(1) \quad \text{as} \quad \kappa\to\infty, \tag{2.154}$$

hence for κ large enough, necessarily

$$\min_k \left|a^m_{\nu,n}-a^k_\nu\right|^2 = \left|a^m_{\nu,n}-a^m_\nu\right|^2 \tag{2.155}$$

and thus

$$\left|a^m_{\nu,n}-a^m_\nu\right| < \sqrt{\left(R^m_{\nu,n},R^m_{\nu,n}\right)}. \tag{2.156}$$

We need an order estimate for $\left(R^m_{\nu,n},R^m_{\nu,n}\right)$. The integrals we must consider then are of the form

$$I = \int_{-K}^{K} D_i\left(\sqrt{2\kappa}\,\mathrm{sn}(z,k)\right)D_j\left(\sqrt{2\kappa}\,\mathrm{sn}(z,k)\right)\mathrm{d}z, \tag{2.157}$$

where $i, j \in \mathbb{N}_0$. Performing the substitution $t = \sqrt{2\kappa}\,\mathrm{sn}(z,k)$ we obtain

$$I = \frac{1}{\sqrt{2\kappa}} \int_{-\sqrt{2\kappa}}^{\sqrt{2\kappa}} \frac{1}{\sqrt{1-\frac{t^2}{2\kappa}}\sqrt{1-\frac{k^2t^2}{2\kappa}}} D_i(t)\, D_j(t)\,\mathrm{d}t. \tag{2.158}$$

Since when t is large D_m is exponentially small it follows that

$$I = \mathcal{O}\left(\kappa^{-\frac{1}{2}}\right), \text{ as } \kappa \to \infty. \tag{2.159}$$

Considering the first term of the expansion for $w^m_{\nu,n}(z,k^2)$ in (2.146) we have

$$\begin{aligned}\left(c^m_{\nu,n}\right)^2 \int_{-K}^{K} D^2_m(\sqrt{2\kappa}\,\mathrm{sn}(z,k))\mathrm{d}z &= \frac{\left(c^m_{\nu,n}\right)^2}{\sqrt{2\kappa}} \int_{-\sqrt{2\kappa}}^{\sqrt{2\kappa}} \frac{1}{\sqrt{1-\frac{t^2}{2\kappa}}\sqrt{1-\frac{k^2t^2}{2\kappa}}} D^2_m(t)\mathrm{d}t \\ &\sim \frac{\left(c^m_{\nu,n}\right)^2}{\sqrt{2\kappa}} \int_{-\infty}^{\infty} D^2_m(t)\mathrm{d}t = \left(c^m_{\nu,n}\right)^2 m!\sqrt{\frac{\pi}{\kappa}},\end{aligned} \tag{2.160}$$

as $\kappa \to \infty$. Hence it follows from (2.147) that $c_{m,n} = \mathcal{O}(\kappa^{1/4})$ as $\kappa \to \infty$. Similar observations would give us that

$$\int_{-K}^{K} \left(R_{m,n}(z)\right)^2 \mathrm{d}z = \mathcal{O}\left(\kappa^{-2n}\right), \tag{2.161}$$

and so

$$a^m_\nu - a^m_{\nu,n} = \mathcal{O}\left(\kappa^{-n}\right), \tag{2.162}$$

as $\kappa \to \infty$. The error analysis for $b^m_{\nu,n}$ would in the same manner give the order estimate

$$b^{m+1}_\nu - b^{m+1}_{\nu,n} = \mathcal{O}\left(\kappa^{-n}\right) \quad \text{as} \quad \kappa \to \infty. \tag{2.163}$$

Hence we have (2.99).

We now use these results to obtain strict and realistic error bounds for the functions expansions derived previously. Define the differential operator

$$\begin{aligned}L^m_\nu :=& \frac{\mathrm{d}^2}{\mathrm{d}t^2} + m + \frac{1}{2} - \frac{t^2}{4} \\ &+ \frac{1}{2\kappa}\left(\left(-t^2(1+k^2) + \frac{k^2t^4}{2\kappa}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2} - t\left(1+k^2-\frac{k^2t^2}{\kappa}\right)\frac{\mathrm{d}}{\mathrm{d}t} + h - \kappa\,(2m+1)\right),\end{aligned} \tag{2.164}$$

and consider $t \in [-t_*, t_*]$. We have the truncated expansion corresponding to an even solution

$$w_{\nu,n}^m(t,k^2) = D_m(t)\sum_{s=0}^{n}\frac{A_s(t)}{\kappa^s} + D_m'(t)\sum_{s=1}^{n}\frac{B_s(t)}{\kappa^s}, \tag{2.165}$$

such that we have the exact solution

$$w_\nu^m(t,k^2) = w_{\nu,n}^m(t,k^2) + \epsilon_{\nu,n}^m(t,k^2). \tag{2.166}$$

We define the remainder term $R_{\nu,n}^m(t,k^2)$ such that

$$L_\nu^m\left(w_{\nu,n}^m(t,k^2)\right) = R_{\nu,n}^m(t,k^2) \tag{2.167}$$

and split the eigenvalue such that

$$\frac{h_\nu^m}{2\kappa} = m + \tfrac{1}{2} + \sum_{s=1}^{n}\frac{\mu_s}{\kappa^s} + \frac{\widetilde{\mu}_{n+1}}{\kappa^{n+1}}, \tag{2.168}$$

and note we just proved that $\widetilde{\mu}_{n+1} = \mathcal{O}(1)$ as $\kappa \to \infty$. Since the coefficients $A_s(t)$ and $B_s(t)$ satisfy (2.107) and (2.108) it follows that

$$R_{\nu,n}^m(t,k^2) = \mathcal{O}(\kappa^{-n-1}), \tag{2.169}$$

as $\kappa \to \infty$. Applying L_ν to (2.166) we obtain

$$\begin{aligned}\left(\epsilon_{\nu,n}^m\right)'' + \left(m + \frac{1}{2} - \frac{t^2}{4}\right)\epsilon_{\nu,n}^m = \frac{1}{1 - \frac{t^2}{2\kappa}\left(1 + k^2 - \frac{k^2t^2}{2\kappa}\right)}\Bigg[\frac{t}{2\kappa}\left(1 + k^2 - \frac{k^2t^2}{\kappa}\right)\left(\epsilon_{\nu,n}^m\right)' \\ + \left(t^2\left(-1 - k^2 + \frac{k^2t^2}{2\kappa}\right)\left(2m + 1 - \frac{t^2}{2}\right) - h_\nu^m + \kappa\left(2m+1\right)\right)\frac{\epsilon_{\nu,n}^m}{2\kappa} - R_{\nu,n}^m\Bigg],\end{aligned} \tag{2.170}$$

and denoting the right hand side of this equation $\Omega_{\nu,n}^m(t,k^2)$, by use of variation of parameters we have

$$\epsilon_{\nu,n}^m(t,k^2) = \frac{\sqrt{\pi/2}}{m!}\int_t^{t_*}\left[D_m(t)\overline{D}_m(\tau) - D_m(\tau)\overline{D}_m(t)\right]\Omega_{\nu,n}^m(\tau,k^2)\mathrm{d}\tau. \tag{2.171}$$

In accordance with the theory outlined in section 1.4 we define $J(\tau) = 1$, $H(\tau) = 1 + k^2 - \frac{k^2\tau^2}{2\kappa}$ and

$$K(t,\tau) = \frac{\sqrt{\pi/2}}{m!}\left(D_m(t)\,\overline{D}_m(\tau) - D_m(\tau)\,\overline{D}_m(t)\right), \qquad \phi(\tau) = \frac{-R^m_{\nu,n}(\tau,k^2)}{1 - \frac{\tau^2}{2\kappa}H(\tau)},$$
$$\psi_0(\tau) = \frac{\kappa(2m+1) - h^m_\nu - \tau^2 H(\tau)\left(2m+1-\frac{\tau^2}{2}\right)}{2\kappa - \tau^2 H(\tau)}, \qquad \psi_1(\tau) = \frac{\tau\left(1+k^2-\frac{k^2\tau^2}{\kappa}\right)}{2\kappa - \tau^2 H(\tau)}, \tag{2.172}$$

$$\Phi(t) = \int_t^{t_*} |\phi(\tau)\,\mathrm{d}\tau|, \qquad \Psi_0(t) = \int_t^{t_*} |\psi_0(\tau)\,\mathrm{d}\tau|, \qquad \Psi_1(t) = \int_t^{t_*} |\psi_1(\tau)\,\mathrm{d}\tau|. \tag{2.173}$$

Since we consider $t \in [-t_*, t_*]$ where $t_* = \mathcal{O}(1)$ as $\kappa \to \infty$, the error analysis is much simpler than the analysis Olver uses in [7] as we have

$$|K(t,\tau)| \le k_0, \text{ and } |\partial K(t,\tau)/\partial t| \le k_1, \tag{2.174}$$

where k_0 and k_1 are $\mathcal{O}(1)$ as $\kappa \to \infty$, whereas Olver's bounds were in terms of parabolic cylinder functions. Thus we define

$$P_0(t) = k_0, \qquad Q(\tau) = 1, \qquad P_1(t) = k_1, \tag{2.175}$$

(we do not define $P_2(t)$ as we do not need to bound $|\partial^2 K(t,\tau)/\partial t^2|$ to carry out our analysis), and finally the constants

$$\widetilde{\kappa} = 1, \quad \widetilde{\kappa}_0 = k_0, \quad \widetilde{\kappa}_1 = k_1. \tag{2.176}$$

Hence it follows from Theorem 10.1 in (§6 [7]) that

$$|\epsilon^m_{\nu,n}(t,k^2)| \le P_0(t)\,\widetilde{\kappa}\;\Phi(t)\exp\left[\widetilde{\kappa}_0\;\Psi_0(t) + \widetilde{\kappa}_1\;\Psi_1(t)\right]. \tag{2.177}$$

Since

$$h^m_\nu - \kappa(2m+1) = \mathcal{O}(1) \tag{2.178}$$

for both $h = a_\nu^m$ and $h = b_\nu^{m+1}$ we obtain

$$\Phi(t) = \mathcal{O}\left(\kappa^{-n-1}\right), \qquad \Psi_0(t) = \mathcal{O}\left(\kappa^{-1}\right), \qquad \Psi_1(t) = \mathcal{O}\left(\kappa^{-1}\right), \tag{2.179}$$

as $\kappa \to \infty$. Then substituting expressions from (2.179), (2.176) and the first of (2.175) into (2.177) we have

$$\epsilon_{\nu,n}^m(t,k^2) = \mathcal{O}\left(\kappa^{-n-1}\right) \quad \text{as } \kappa \to \infty, \quad \text{for } \ t \in [-t_*, t_*]. \tag{2.180}$$

Now we identify the solutions derived with the Lamé functions. We give the identification for $t \in [-t_*, t_*]$

$$Ec_\nu^m(z,k^2) = C_\nu^m \Bigg(D_m(t) \sum_{s=0}^{n} \frac{A_s(t)}{\kappa^s} + D_m'(t) \sum_{s=0}^{n} \frac{B_s(t)}{\kappa^s} + \tfrac{1}{2}\left(\epsilon_{\nu,n}^m(t,k^2) + (-1)^m \epsilon_{\nu,n}^m(-t,k^2)\right) \Bigg), \tag{2.181}$$

$$Es_\nu^{m+1}(z,k^2) = S_\nu^{m+1} \sqrt{1 - \frac{t^2}{2\kappa}} \Bigg(D_m(t) \sum_{s=0}^{n} \frac{P_s(t)}{\kappa^s} + D_m'(t) \sum_{s=0}^{n} \frac{Q_s(t)}{\kappa^s} + \tfrac{1}{2}\left(\epsilon_{\nu,n}^m(t,k^2) + (-1)^m \epsilon_{\nu,n}^m(-t,k^2)\right) \Bigg), \tag{2.182}$$

where the errors are defined according to $h_\nu^m = a_\nu^m$ or $h_\nu^m = b_\nu^{m+1}$ respectively. We consider now just C_ν^m since we will obtain the same asymptotic expansion from S_ν^{m+1} by construction. To obtain an asymptotic expansion for these constants we consider with respect to (2.6) the integral

$$(C_\nu^m)^2 \int_{-K}^{K} \mathrm{dn}(z,k) \left(D_m(t) \sum_{s=0}^{\infty} \frac{A_s(t)}{\kappa^s} + D_m'(t) \sum_{s=0}^{\infty} \frac{B_s(t)}{\kappa^s} \right)^2 \mathrm{d}z. \tag{2.183}$$

In the integral we let $t = \sqrt{2\kappa}\,\mathrm{sn}(z,k)$ and obtain

$$\frac{(C_\nu^m)^2}{\sqrt{2\kappa}} \int_{-\sqrt{2\kappa}}^{\sqrt{2\kappa}} \frac{1}{\sqrt{1 - \frac{t^2}{2\kappa}}} \left(D_m(t) \sum_{s=0}^{\infty} \frac{A_s(t)}{\kappa^s} + D_m'(t) \sum_{s=0}^{\infty} \frac{B_s(t)}{\kappa^s} \right)^2 \mathrm{d}t. \tag{2.184}$$

Since the parabolic cylinder functions are exponentially small when the variable is large, we consider the integral from $-\infty$ to ∞ and express the integral in the form

$$\frac{(C_{\nu}^{m})^2}{\sqrt{2\kappa}} \int_{-\infty}^{\infty} \sum_{s=0}^{\infty} \kappa^{-s} \sum_{j=0}^{s} \binom{-\frac{1}{2}}{j} \left(t^2/2\right)^j \sum_{i=0}^{s-j} \Big(A_i(t) A_{s-j-i}(t) D_m^2(t) \\ + 2A_i(t) B_{s-j-i}(t) D_m(t) D_m'(t) + B_i(t) B_{s-j-i}(t) \left(D_m'(t)\right)^2 \Big) \mathrm{d}t. \tag{2.185}$$

Then from (2.6) we have the formal asymptotic expansions for the normalisation constants given in (2.101). □

2.5.5 Numerics

The coefficients in both the function and eigenvalue expansions match the formal results given in (§29.7 [1]). Below we give tables of relative errors corresponding to some of our eigenvalue expansions. Below, $a_{\nu,0}^{m}$ corresponds to the expansion as in (2.99) with $n = 1$,i.e., the large order behaviour, $a_{\nu,1}^{m}$ with $n = 2$,i.e., with the $\mathcal{O}(\kappa^{-1})$ term included, and so on.

Table 2.3: Relative errors for eigenvalue approximations for $\nu = 50$ and $k = 0.7$

m	$1 - \frac{a_{\nu,0}^{m}}{a_{\nu}^{m}}$	$1 - \frac{a_{\nu,1}^{m}}{a_{\nu}^{m}}$	$1 - \frac{a_{\nu,2}^{m}}{a_{\nu}^{m}}$
0	-0.000036	4.31×10^{-7}	1.49×10^{-8}
5	-0.000452	0.000075	0.000011
10	0.002207	0.000634	0.000180
15	0.006384	0.002671	0.001124
20	0.015941	0.008905	0.005062
25	0.050049	0.037957	0.029773

Table 2.4: Relative errors for eigenvalue approximations for $\nu = 150$ and $k = 0.7$

m	$1 - \frac{a^m_{\nu,0}}{a^m_\nu}$	$1 - \frac{a^m_{\nu,1}}{a^m_\nu}$	$1 - \frac{a^m_{\nu,2}}{a^m_\nu}$
15	0.000427	0.000060	8.27×10^{-6}
20	0.000807	0.000148	0.000027
25	0.001350	0.000307	0.000070

From the tables it is clear that the asymptotic expansions for the eigenvalues are numerically very good. The numerics also indicate that as m becomes larger, ν needs to become much larger to see good numerical results for the relative errors in the approximations.

We will now provide some numerical clarification we have performed for the function expansions. In the figures below we give a plot of the function, and next to it plots of the absolute error corresponding to our one,two and three-term local approximations.

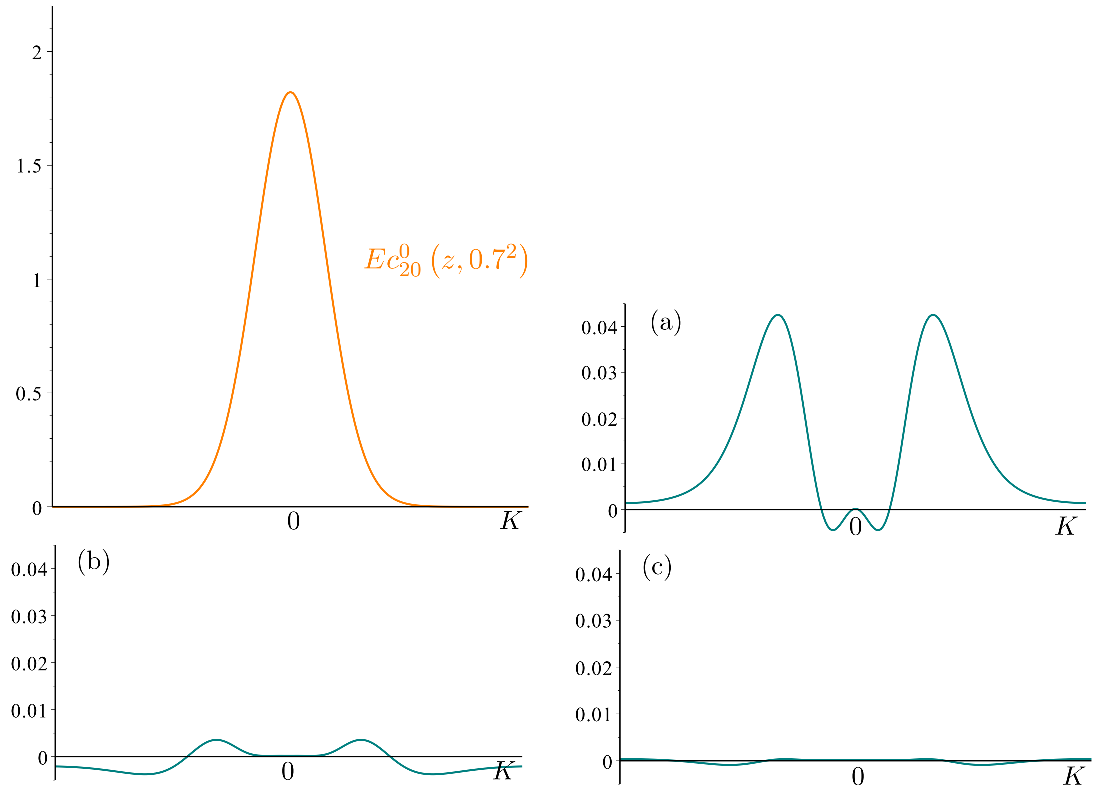

Figure 2.8: (top left) Plot for the Lamé function $Ec_{20}^{0}\left(z, 0.7^2\right)$ in $[-K, K]$. In (a) we have the absolute error corresponding to the one-term approximation, in (b) corresponding to the two-term approximation, and in (c) corresponding to the three-term approximation.

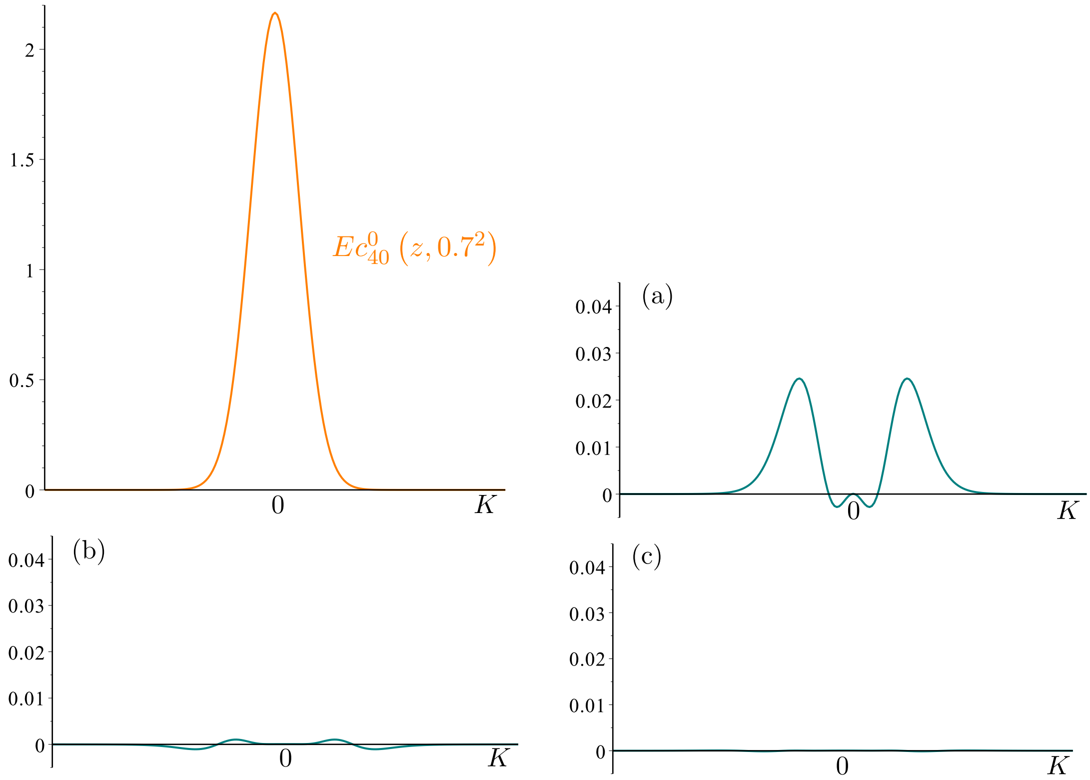

Figure 2.9: (top left) Plot for the Lamé function $Ec_{40}^{0}\left(z, 0.7^2\right)$ in $[-K, K]$. In (a) we have the absolute error corresponding to the one-term approximation, in (b) corresponding to the two-term approximation, and in (c) corresponding to the three-term approximation.

We note that as ν becomes larger, the absolute error corresponding to our approximations becomes smaller, and that for both values of ν, adding further terms in the function approximation gives much smaller errors.

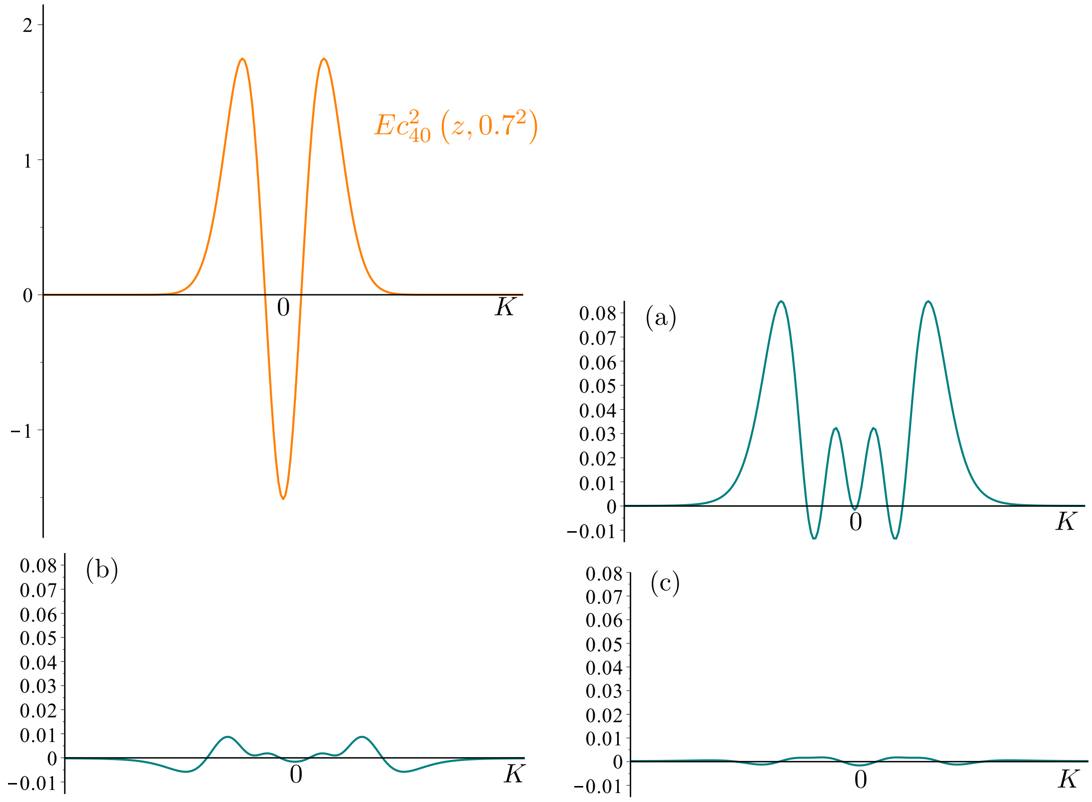

Figure 2.10: (top left) Plot for the Lamé function $Ec_{40}^2\left(z, 0.7^2\right)$ in $[-K, K]$. In (a) we have the absolute error corresponding to the one-term approximation, in (b) corresponding to the two-term approximation, and in (c) corresponding to the three-term approximation.

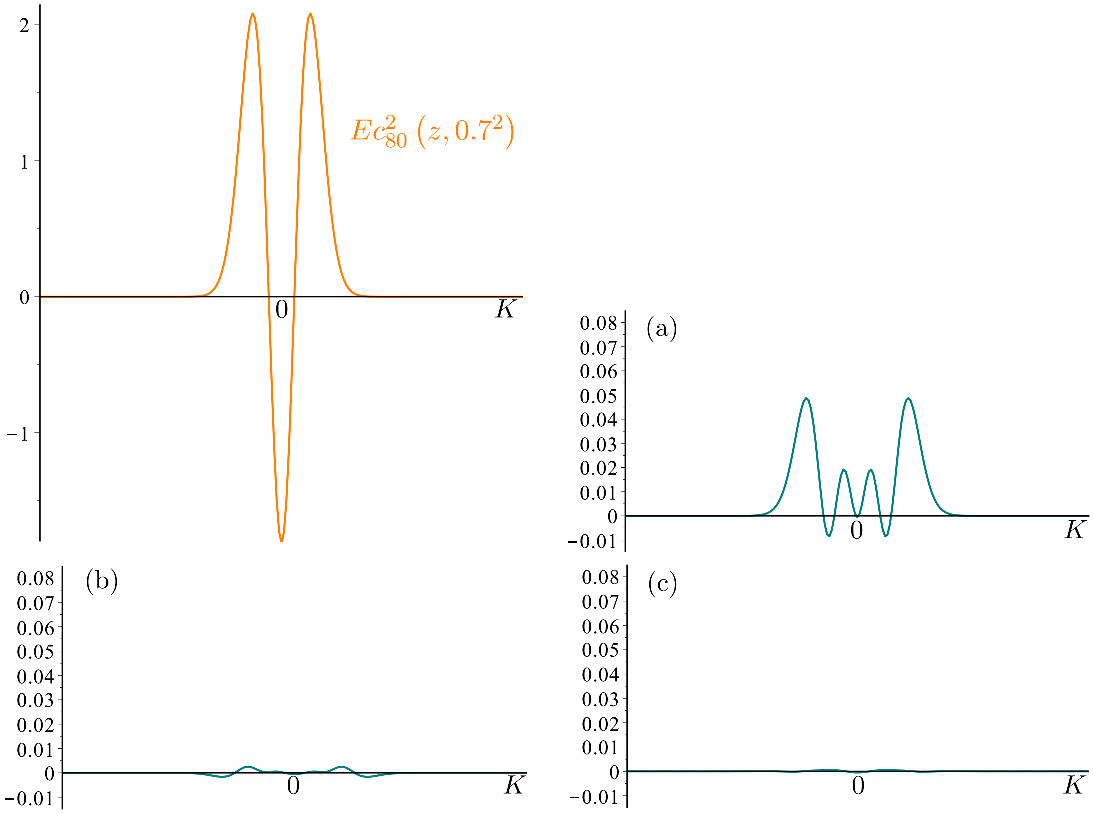

Figure 2.11: (top left) Plot for the Lamé function $Ec_{80}^{2}\left(z, 0.7^2\right)$ in $[-K, K]$. In (a) we have the absolute error corresponding to the one-term approximation, in (b) corresponding to the two-term approximation, and in (c) corresponding to the three-term approximation.

We also note that as m becomes larger, ν must become even larger to see a significantly small absolute error corresponding to our approximations. Investigating numerical errors for other approximations gives similar good results.

Comparing these results with the numerics given in subsection 2.4.4, we see these local expansions are better around the origin once more that one term is taken.

2.6 Exponentially small difference between a_ν^m and b_ν^{m+1}

We want to derive a rigorous expression for the exponentially small difference between the eigenvalues a_ν^m and b_ν^{m+1} of Lamé's equation as $\kappa \to \infty$. We will do this by using the asymptotic expansions derived for the eigenvalues a_ν^m and b_ν^{m+1} in the previous section with respect to the parameter κ in the equation being large, and also the uniform approximations derived in the section before for the corresponding Lamé functions.

Theorem 2.4. *Let $\kappa = \sqrt{\nu(\nu+1)}k$, m a non-negative integer and $0 < k < 1$. Then as $\kappa \to \infty$*

$$b_\nu^{m+1} - a_\nu^m = (1-k^2)^{-m-1/2} \frac{1}{m!} \sqrt{\frac{2}{\pi}} \left(\frac{1-k}{1+k}\right)^{\frac{\kappa}{k}} 2^{3m+7/2} \kappa^{m+3/2} \left(1 + \frac{P(k)}{\kappa} + \mathcal{O}(\kappa^{-2})\right), \tag{2.186}$$

where

$$P(k) = \frac{1}{32R(k)} \Big((2m+1)^2(4 - 3R(k)(1+k^2)) + 2R(k)(1+k^2) \times (Q(k) - 4(2m+1)) + \frac{R^{3/2}(k)}{\sqrt{2}}(2\ln 2 + 3) - 16(1 + m(m+1))\Big), \tag{2.187}$$

with

$$R(k) = \frac{\ln\left(\frac{1+k}{1-k}\right)}{k}, \quad \textit{and} \quad Q(k) = \ln(1-k^2) - 2\ln 2 + \ln R. \tag{2.188}$$

Proposition 2.2. *Defining $w_{\nu,1}^m$ and $w_{\nu,2}^m$ to be two standard solutions of (2.21), then*

$$\frac{w_{\nu,1}^{m\prime}(-K,k^2) w_{\nu,2}^m(K,k^2) - w_{\nu,2}^{m\prime}(-K,k^2) w_{\nu,1}^m(K,k^2)}{w_{\nu,1}^{m\prime}(-K,k^2) w_{\nu,2}^m(K,k^2) - w_{\nu,2}^{m\prime}(-K,k^2) w_{\nu,1}^m(K,k^2)} = \pm 1 \tag{2.189}$$

where the + corresponds to $h_\nu^m = a_\nu^m$, and the − to $h_\nu^m = b_\nu^{m+1}$ when m is even, and vice-versa when m is odd.

Proof. Define the fundamental pair of solutions $\widetilde{w}_1\left(z,k^2\right)$ and $\widetilde{w}_2\left(z,k^2\right)$ of (2.1) as

$$\widetilde{w}_1\left(-K,k^2\right) = 1, \quad \widetilde{w}_1'\left(-K,k^2\right) = 0, \quad \widetilde{w}_2\left(-K,k^2\right) = 0, \quad \widetilde{w}_2'\left(-K,k^2\right) = 1, \tag{2.190}$$

and Floquet solutions $w_F\left(z,k^2\right)$ of (2.1) which satisfy

$$w_F\left(z+2K,k^2\right)=e^{i\pi\eta(h)}w_F\left(z,k^2\right), \tag{2.191}$$

where $\eta(h)$ is the *characteristic exponent* and is dependent on the eigenvalue h. Writing this Floquet solution in terms of $\widetilde{w}_1\left(z,k^2\right)$ and $\widetilde{w}_2\left(z,k^2\right)$,

$$w_F\left(z,k^2\right)=A\widetilde{w}_1\left(z,k^2\right)+B\widetilde{w}_2\left(z,k^2\right), \tag{2.192}$$

one obtains

$$\begin{aligned}w_F\left(K,k^2\right)&=A\widetilde{w}_1\left(K,k^2\right)+B\widetilde{w}_2\left(K,k^2\right)\\&=e^{i\pi\eta(h)}w_F\left(-K,k^2\right)=e^{i\pi\eta(h)}\left(A\widetilde{w}_1(-K)+B\widetilde{w}_2(-K)\right)=e^{i\pi\eta(h)}A,\end{aligned} \tag{2.193}$$

$$\begin{aligned}w_F'\left(K,k^2\right)&=A\widetilde{w}_1'\left(K,k^2\right)+B\widetilde{w}_2'\left(K,k^2\right)\\&=e^{i\pi\eta(h)}w_F'\left(-K,k^2\right)=e^{i\pi\eta(h)}\left(A\widetilde{w}_1'(-K)+B\widetilde{w}_2'(-K)\right)=e^{i\pi\eta(h)}B,\end{aligned} \tag{2.194}$$

which gives the simultaneous equations

$$\begin{aligned}A\left(\widetilde{w}_1(K)-e^{i\pi\eta(h)}\right)+B\widetilde{w}_2(K)&=0,\\A\widetilde{w}_1'(K)+B\left(\widetilde{w}_2'(K)-e^{i\pi\eta(h)}\right)&=0.\end{aligned} \tag{2.195}$$

For non-trivial Floquet solutions to exist, we must have

$$\begin{vmatrix}\widetilde{w}_1(K)-e^{i\pi\eta(h)} & \widetilde{w}_2(K)\\ \widetilde{w}_1'(K) & \widetilde{w}_2'(K)-e^{i\pi\eta(h)}\end{vmatrix}=e^{i2\pi\eta(h)}-e^{i\pi\eta(h)}(\widetilde{w}_1(K)+\widetilde{w}_2'(K))+1=0. \tag{2.196}$$

It can be shown easily using connection formulae that $\widetilde{w}_1(K)=\widetilde{w}_2'(K)$, hence

$$\widetilde{w}_1\left(K,k^2\right)=\cos\left(\pi\eta(h)\right). \tag{2.197}$$

Defining $h_\nu^{m\pm}$ to be the value of h so that

$$\widetilde{w}_1\left(K,k^2\right)=\cos\left(\pi\eta(h_\nu^{m\pm})\right)=\pm 1, \tag{2.198}$$

it then follows from (2.190) and (2.191) that h_ν^{m+} gives rise to $2K$-periodic solutions, and that h_ν^{m-} gives rise to $2K$-anti-periodic solutions. When m is even, $Ec_\nu^m(z,k^2)$ is $2K$-periodic and $Es_\nu^{m+1}(z,k^2)$ is $2K$-antiperiodic, and vice-versa when m is odd, thus it follows that when m is even, h_ν^{m+} corresponds to a_ν^m and h_ν^{m-} corresponds to b_ν^{m+1}, and vise-versa when m is odd.

Writing $\widetilde{w}_1(z,k^2)$ using two standard solutions w_1 and w_2 of (2.1) we have

$$\widetilde{w}_1(z,k^2)=\frac{w_1'(-K,k^2)w_2(z,k^2)-w_2'(-K,k^2)w_1(z,k^2)}{w_1'(-K,k^2)w_2(-K,k^2)-w_2'(-K,k^2)w_1(-K,k^2)}, \tag{2.199}$$

thus we obtain the condition (2.189). □

In section 2.5 we derived asymptotic expansions for the eigenvalues a_ν^m and b_ν^{m+1}. The expansions were the same for both eigenvalues to all algegraic orders of κ, thus the difference between the two eigenvalues must be exponentially small as $\kappa\to\infty$. Condition (2.189) gives for the first time a way to analytically distinguish between a_ν^m and b_ν^{m+1}. With the ultimate goal to determine the exponentially small difference between these eigenvalues, we will perturb the m variable in the eigenvalue expansions as such;

Lemma 2.2. *In the asymptotic expansion of a_ν^m given in (2.99), if m is even let $m=\rho_\nu^{m+}-\varepsilon_\nu^{m+}$, otherwise let $m=\rho_\nu^{m-}-\varepsilon_\nu^{m-}$. Similarly in the asymptotic expansion of b_ν^{m+1} given in (2.99), if m is even let $m=\rho_\nu^{m-}-\varepsilon_\nu^{m-}$, otherwise let $m=\rho_\nu^{m+}-\varepsilon_\nu^{m+}$. Here both ε_ν^{m+} and ε_ν^{m-} are exponentialy small as $\kappa\to\infty$.*

Then we can write the difference between the eigenvalues as

$$b_\nu^{m+1}-a_\nu^m=(-1)^m 2\kappa\left(1-\frac{(1+k^2)(2m+1)}{4\kappa}+\mathcal{O}(\kappa^{-2})\right)\left(\varepsilon_\nu^{m-}-\varepsilon_\nu^{m+}\right). \tag{2.200}$$

Proof. Consider m even. Letting $m=\rho_\nu^{m\pm}-\varepsilon_\nu^{m+}$ in the expansions for a_ν^m given in (2.99), and $m=\rho_\nu^{m\pm}-\varepsilon_\nu^{m-}$ in the expansions for b_ν^{m+1} also given in (2.99), we have the expansion for the eigenvalues as

$$a_\nu^m = (2\left(\rho_\nu^{m\pm} - \varepsilon_\nu^{m+}\right) + 1)\kappa - \frac{k^2+1}{4}(1 + 2\left(\rho_\nu^{m\pm} - \varepsilon_\nu^{m+}\right) + 2\left(m + \varepsilon_\nu^{m+}\right)^2) + 2\sum_{s=1}^{n-1}\frac{\mu_{s+1}}{\kappa^s} + \mathcal{O}(\kappa^{-n}) \quad (\kappa \to \infty), \tag{2.201}$$

$$b_\nu^{m+1} = (2\left(\rho_\nu^{m\pm} - \varepsilon_\nu^{m-}\right) + 1)\kappa - \frac{k^2+1}{4}(1 + 2\left(m + \varepsilon_\nu^{m-}\right) + 2\left(\rho_\nu^{m\pm} - \varepsilon_\nu^{m+}\right)^2) + 2\sum_{s=1}^{n-1}\frac{\mu_{s+1}}{\kappa^s} + \mathcal{O}(\kappa^{-n}) \quad (\kappa \to \infty), \tag{2.202}$$

for $n = 2, 3, \ldots$, since the exponentially small term $\varepsilon_\nu^{m\pm}$ has no impact up to algebraic orders of κ. Taking the difference of (2.202) and (2.201) gives (2.200) in the case m is even. Similar analysis for m odd also clarifies (2.200). □

We now need to pick solutions to use for determining the condition in (2.189). We want to use the two-term uniform asymptotic expansions given in section 2.4, but will need to incorporate the exponentially small perturbation of m, as this function expansion would correspond with the eigenvalue expansion with the same perturbed m.

Conjecture 2.1. *Let $\rho_\nu^{m\pm} = m + \varepsilon_\nu^{m\pm}$, as in Lemma 2.2. Denoting*

$$\Phi(z) = \left(\frac{\zeta^2 - (\sigma_m^\pm)^2}{x^2 - \left(s_m^\pm\right)^2}\right)^{\frac{1}{4}}, \tag{2.203}$$

the following solutions to (2.21) are available for $z \in [0, K]$;

$$w_{\nu,1}^{m\pm}(z,k^2) = \Phi(z)\left(D_{\rho_\nu^{m\pm}}\left(\sqrt{2\kappa}\zeta\right) + \frac{1}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(D_{\rho_\nu^{m\pm}}\left(\sqrt{2\kappa}\zeta\right)\right)B_0(\zeta) + \epsilon_{\nu,1}^{m\pm}(\zeta,k^2)\right),$$
$$w_{\nu,2}^{m\pm}(z,k^2) = \Phi(z)\left(\overline{D}_{\rho_\nu^{m\pm}}\left(\sqrt{2\kappa}\zeta\right) + \frac{1}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\overline{D}_{\rho_\nu^{m\pm}}\left(\sqrt{2\kappa}\zeta\right)\right)B_0(\zeta) + \epsilon_{\nu,2}^{m\pm}(\zeta,k^2)\right), \tag{2.204}$$

with

$$32\zeta B_0(\zeta) = (k^2+1)\ln\left(\tfrac{1}{4}\zeta^2 C(\zeta,k)\right) - \frac{3\left(k^2-1\right)^2}{2C(\zeta,k)} + 3k\coth\left(k\zeta^2/2\right) + 2k^2\zeta^2 - \frac{6}{\zeta^2}, \tag{2.205}$$

where $C(\zeta,k) = 2k\coth\left(k\zeta^2/2\right) - k^2 - 1$,

$$\left(s_\nu^{m\pm}\right)^2 = \frac{h}{\kappa^2}, \quad \left(\sigma_\nu^{m\pm}\right)^2 = \frac{2}{\pi}\int_{-s_\nu^{m\pm}}^{s_\nu^{m\pm}} \sqrt{\frac{\left(s_\nu^{m\pm}\right)^2 - t^2}{(1-t^2)(1-k^2t^2)}}dt \tag{2.206}$$

where x is defined as a function of z by $x = \mathrm{sn}(z,k^2)$, and ζ is defined as a function of x by

$$\begin{aligned}
\int_x^{-s_\nu^{m\pm}} \sqrt{\frac{t^2 - \left(s_\nu^{m\pm}\right)^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t &= \int_\zeta^{-\sigma_\nu^{m\pm}} \sqrt{\tau^2 - \left(\sigma_\nu^{m\pm}\right)^2}\mathrm{d}\tau \quad \left(-1 < x \le -s_\nu^{m\pm}\right), \\
\int_{-s_\nu^{m\pm}}^x \sqrt{\frac{\left(s_\nu^{m\pm}\right)^2 - t^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t &= \int_{-\sigma_\nu^{m\pm}}^\zeta \sqrt{\left(\sigma_\nu^{m\pm}\right)^2 - \tau^2}\mathrm{d}\tau \quad \left(-s_\nu^{m\pm} \le x \le s_\nu^{m\pm}\right), \\
\int_{s_\nu^{m\pm}}^x \sqrt{\frac{t^2 - \left(s_\nu^{m\pm}\right)^2}{(1-t^2)(1-k^2t^2)}}\mathrm{d}t &= \int_{\sigma_\nu^{m\pm}}^\zeta \sqrt{\tau^2 - \left(\sigma_\nu^{m\pm}\right)^2}\mathrm{d}\tau \quad \left(s_\nu^{m\pm} \le x < 1\right),
\end{aligned} \tag{2.207}$$

and

$$\begin{aligned}
\epsilon_{\nu,1}^{m\pm}(\zeta,k^2) &= \mathbf{E}_\nu^{-1}(\sqrt{2\kappa}\zeta)\mathbf{M}_\nu(\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-2}), \\
\epsilon_{\nu,2}^{m\pm}(\zeta,k^2) &= \mathbf{E}_\nu(\sqrt{2\kappa}\zeta)\mathbf{M}_\nu(\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-2}), \\
d\epsilon_{\nu,1}^{m\pm}(\zeta,k^2)/d\zeta &= \mathbf{E}_\nu^{-1}(\sqrt{2\kappa}\zeta)\mathbf{N}_\nu(\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-3/2}), \\
d\epsilon_{\nu,2}^{m\pm}(\zeta,k^2)/d\zeta &= \mathbf{E}_\nu(\sqrt{2\kappa}\zeta)\mathbf{N}_\nu(\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-3/2}),
\end{aligned}$$

The two-term approximation follows easily from letting $\rho_\nu^{m\pm} = m + \varepsilon_\nu^{m\pm}$ in the results given in subsection 2.4.3.

We now need to determine solutions for $\zeta < 0$ by use of connection formulae.

Conjecture 2.2. *For $\zeta \in [-K, 0]$ we have as $\kappa \to \infty$*

$$\begin{aligned} w_{\nu,1}^{m\pm}(\zeta, k^2) &= A_{\nu,1}^{m\pm} w_{\nu,3}^{m\pm}(z, k^2) + B_{\nu,1}^{m\pm} w_{\nu,4}^{m\pm}(z, k^2), \\ w_{\nu,2}^{m\pm}(\zeta, k^2) &= A_{\nu,2}^{m\pm} w_{\nu,3}^{m\pm}(z, k^2) + B_{\nu,2}^{m\pm} w_{\nu,4}^{m\pm}(z, k^2), \end{aligned} \tag{2.208}$$

. where

$$\begin{aligned} w_{\nu,3}^{m\pm}(z,k^2) &= \Phi(z)\left(D_{\rho_\nu^{m\pm}}\left(-\sqrt{2\kappa}\zeta\right) - \frac{1}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(D_{\rho_\nu^{m\pm}}\left(-\sqrt{2\kappa}\zeta\right)\right)B_0(-\zeta) + \epsilon_3(\zeta,k^2)\right), \\ w_{\nu,4}^{m\pm}(z,k^2) &= \Phi(z)\left(\overline{D}_{\rho_\nu^{m\pm}}\left(-\sqrt{2\kappa}\zeta\right) - \frac{1}{\kappa^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\overline{D}_{\rho_\nu^{m\pm}}\left(-\sqrt{2\kappa}\zeta\right)\right)B_0(-\zeta) + \epsilon_4(\zeta,k^2)\right), \end{aligned} \tag{2.209}$$

and

$$\begin{aligned} A_{\nu,1}^{m\pm} &= \sin\left(\pi\left(\rho_\nu^{m\pm} + \tfrac{1}{2}\right)\right) + \mathcal{O}\left(\kappa^{-2}\right), \quad B_{\nu,1}^{m\pm} = \cos\left(\pi\left(\rho_\nu^{m\pm} + \tfrac{1}{2}\right)\right) + \mathcal{O}\left(\kappa^{-2}\right), \\ A_{\nu,2}^{m\pm} &= \cos\left(\pi\left(\rho_\nu^{m\pm} + \tfrac{1}{2}\right)\right) + \mathcal{O}\left(\kappa^{-2}\right), \quad B_{\nu,1}^{m\pm} = -\sin\left(\pi\left(\rho_\nu^{m\pm} + \tfrac{1}{2}\right)\right) + \mathcal{O}\left(\kappa^{-2}\right), \end{aligned} \tag{2.210}$$

and

$$\begin{aligned} \epsilon_{\nu,3}^{m\pm}(\zeta,k^2) &= \mathbf{E}_\nu^{-1}(-\sqrt{2\kappa}\zeta)\mathbf{M}_\nu(-\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-2}), \\ \epsilon_{\nu,4}^{m\pm}(\zeta,k^2) &= \mathbf{E}_\nu(-\sqrt{2\kappa}\zeta)\mathbf{M}_\nu(-\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-2}), \\ d\epsilon_{\nu,3}^{m\pm}(\zeta,k^2)/d\zeta &= \mathbf{E}_\nu^{-1}(-\sqrt{2\kappa}\zeta)\mathbf{N}_\nu(-\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-3/2}), \\ d\epsilon_{\nu,4}^{m\pm}(\zeta,k^2)/d\zeta &= \mathbf{E}_\nu(-\sqrt{2\kappa}\zeta)\mathbf{N}_\nu(-\sqrt{2\kappa}\zeta)\mathcal{O}(\kappa^{-3/2}). \end{aligned} \tag{2.211}$$

This result is outlined in [7], where ζ is replaced with $-\zeta$ to obtain solutions for $z \in [-K, 0]$. The proof would follow easily after the proof of Conjecture 2.1 is complete.

To prove Theorem 2.4 we need to derive an expansion for ζ_* in descending powers of κ.

Lemma 2.3. *Let $\kappa = \sqrt{\nu(\nu+1)}k$, m a non-negative integer and $0 < k < 1$. Then as $\kappa \to \infty$*

$$\zeta_* = \sqrt{R} + \frac{2\rho+1}{4\kappa}\frac{Q(k)}{\sqrt{R}} + \frac{1}{2^6 R^{\frac{3}{2}}\kappa^2}\left((2m+1)^2\left(3R\left(1+k^2\right) - 2Q(k)(Q(k)-4) - 4\right)\right.$$

$$-2RQ(k)\left(1+k^2\right)\Big) + \mathcal{O}(\kappa^{-3}), \tag{2.212}$$

where

$$R(k) = \frac{\ln\left(\frac{1+k}{1-k}\right)}{k}, \quad \textit{and} \quad Q(k) = \ln\left(1-k^2\right) - 2\ln 2 + \ln R. \tag{2.213}$$

For simplicity, in the following two proofs we will suppress notation and use s, σ, ρ, ϵ and w to refer to $s_\nu^{m\pm}$, $\sigma_\nu^{m\pm}$, $\rho_\nu^{m\pm}$, $\varepsilon_\nu^{m\pm}$ and $w_\nu^{m\pm}$ respectively, and will return them later to match them with their appropriate eigenvalue.

Proof. We have the integral relation

$$\int_s^1 \sqrt{\frac{t^2-s^2}{\left(1-t^2\right)\left(1-k^2t^2\right)}}\mathrm{d}t = \int_\sigma^{\zeta_*} \sqrt{\tau^2-\sigma^2}\mathrm{d}\tau. \tag{2.214}$$

To obtain an expansion for ζ_* we need to expand the first integral as $s \to 0$ and the second as $\sigma \to 0$. We call the integral on the left $I_x(s)$ and the integral on the right $I_\zeta(\sigma)$. We consider first $I_x(s)$.

$$\begin{aligned}
&\int_s^1 \sqrt{\frac{t^2-s^2}{\left(1-t^2\right)\left(1-k^2t^2\right)}}\mathrm{d}t = \frac{1}{2}\int_{s^2}^1 \sqrt{\frac{\tau-s^2}{\tau\left(1-\tau\right)\left(1-k^2\tau\right)}}\mathrm{d}\tau \\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!}k^{2n}\int_{s^2}^1 \sqrt{\frac{\tau-s^2}{1-\tau}}\tau^{n-\frac{1}{2}}\mathrm{d}\tau \quad \text{(expanding the } 1-k^2\tau \text{ term)}, \\
&= \frac{1-s^2}{4}\pi\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}k^{2n}\,{}_2F_1\left({\tfrac{1}{2}-n}\,{}_2\,{\tfrac{1}{2}};1-s^2\right) \quad \text{(see [1] §15.5)}, \\
&= \frac{1-s^2}{4}\pi\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}k^{2n}\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)}\sum_{j=0}^{n}\frac{\left(\frac{1}{2}-n\right)_j\left(\frac{1}{2}\right)_j(n-j)!}{j!}\left(-s^2\right)^j\right. \\
&\qquad -\frac{\left(-s^2\right)^{n+1}}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)_j\left(n+\frac{3}{2}\right)_j}{j!\,(n+1+j)!}\left(s^2\right)^j\big(2\ln s-\Psi(1+j)-\Psi(n+2+j) \\
&\qquad\qquad \left.+\Psi\left(\frac{3}{2}+j\right)+\Psi\left(n+\frac{3}{2}+j\right)\big)\right) \\
&\qquad\qquad\qquad \text{(see [1] §15.8(ii))}.
\end{aligned} \tag{2.215}$$

Thus the first few terms in the expansion for the integral as $s \to 0$ are

$$s^0 \text{ term}: \qquad \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \pi}{4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} = \frac{1}{2k} \ln\left(\frac{1+k}{1-k}\right), \tag{2.216}$$

$$\begin{aligned} s^2 \text{ term}: \qquad & -\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \pi}{4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} - \sum_{n=1}^{\infty} \frac{\pi\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-n\right)}{8n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} \\ & + \frac{\pi}{4\Gamma\left(\frac{1}{2}\right)^2}\left(2\ln a - \Psi(1) - \Psi(2) + 2\Psi\left(\frac{3}{2}\right)\right) \\ & = \frac{1}{4}\ln\left(1-k^2\right) + \frac{1}{2}\ln s - \ln 2 - \frac{1}{4}, \end{aligned} \tag{2.217}$$

$$\begin{aligned} s^4 \text{ term}: \quad & \sum_{n=1}^{\infty} \frac{\pi\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-n\right)}{8n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} - \frac{\pi}{4\Gamma\left(\frac{1}{2}\right)^2}\left(2\ln a - \Psi(1) - \Psi(2) + 2\Psi\left(\frac{3}{2}\right)\right) \\ & + \sum_{n=2}^{\infty} \frac{3\pi\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-n\right)_2}{32(n-1)n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} \\ & - \frac{\pi k^2}{16\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(2\ln s - \Psi(1) - \Psi(3) + \Psi\left(\frac{3}{2}\right) + \Psi\left(\frac{5}{2}\right)\right) \\ & + \frac{9\pi}{32\Gamma\left(\frac{1}{2}\right)^2}\left(2\ln s - \Psi(2) - \Psi(3) + 2\Psi\left(\frac{5}{2}\right)\right) \\ = & \left(1+k^2\right)\left(\frac{1}{32}\ln(1-k^2) + \frac{1}{16}\ln s - \frac{1}{8}\ln 2 + \frac{3}{64}\right), \end{aligned} \tag{2.218}$$

which gives

$$\begin{aligned} I_x(s) = & \frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right) + \left(\frac{1}{4}\ln\left(1-k^2\right) + \frac{1}{2}\ln s - \ln 2 - \frac{1}{4}\right)s^2 \\ & + (1+k^2)\left(\frac{1}{32}\ln(1-k^2) + \frac{1}{16}\ln s - \frac{1}{8}\ln 2 + \frac{3}{64}\right)s^4 + \mathcal{O}(s^6 \ln s). \end{aligned} \tag{2.219}$$

$I_\zeta(\sigma)$ simply yields

$$I_\zeta(\sigma) = \frac{\zeta_*^2}{2} + \left(\frac{1}{2}\ln\sigma - \frac{1}{4} - \frac{1}{2}\ln 2\zeta_*\right)\sigma^2 + \frac{1}{16\zeta_*^2}\sigma^4 + \mathcal{O}(\sigma^6). \tag{2.220}$$

Combining these give

$$\begin{aligned}\zeta_*^2 =& \frac{1}{k}\ln\frac{1+k}{1-k} + 2\left(\frac{1}{4}\ln(1-k^2) + \frac{1}{2}\ln s - \ln 2 - \frac{1}{4}\right)s^2 \\ &+ 2(1+k^2)\left(\frac{1}{32}\ln(1-k^2) + \frac{1}{16}\ln s - \frac{1}{8}\ln 2 + \frac{3}{64}\right)s^4 \\ &- 2\left(\frac{1}{2}\ln\sigma - \frac{1}{4} - \frac{1}{2}\ln 2\zeta_*\right)\sigma^2 - \frac{1}{8\zeta_*^2}\sigma^4 + ..., \end{aligned} \tag{2.221}$$

and substituting in the expansions for s and σ as $\kappa \to \infty$, we have the expansion for ζ_* in κ as (2.212). □

Now we are ready to provide the proof the main result in Theorem 2.4. This proof relies on Conjectures 2.1 and 2.2 being true. Note that we could rigorously prove the result in Theorem 2.4 up to $\mathcal{O}(\kappa^{-1})$ by using the error analysis given for the one-term approximations given in section (2.4).

Proof of Theorem 2.4, conditional on Conjectures 2.1 and 2.2. We want to say something about ε, which is determined by ensuring condition (2.189) is satisfied. Now we consider first the denominator in (2.189). Denoting

$$1 + \mathcal{O}(\kappa^{-2}) = \mathcal{O}_\kappa, \tag{2.222}$$

we derive an expression for the Wronskian

$$w_1'(-K,k^2)w_2(-K,k^2) - w_2'(-K,k^2)w_1(-K,k^2) \tag{2.223}$$

by instead considering

$$w_1'(0,k^2)w_2(0,k^2) - w_2'(0,k^2)w_1(0,k^2), \tag{2.224}$$

since the differential equation has no first derivative term and thus the Wronskian is constant. By using the relations

$$\Phi'(0) = 0, \quad B_0(0) = 0, \quad B_0'(0) = -\frac{5}{256}k^2 + \frac{11}{128}k^2 - \frac{5}{256}, \tag{2.225}$$

it is easy to obtain that

$$w_1(0,k^2) = \left(\frac{\sigma^2}{s^2}\right)^{\frac{1}{4}} D_\rho(0)\,\mathcal{O}_\kappa, \quad w_1'(0,k^2) = \sqrt{2\kappa}\left(\frac{s^2}{\sigma^2}\right)^{\frac{1}{4}} D_\rho'(0)\,\mathcal{O}_\kappa,$$
$$w_1(0,k^2) = \left(\frac{\sigma^2}{s^2}\right)^{\frac{1}{4}} \overline{D}_\rho(0)\,\mathcal{O}_\kappa, \quad w_1'(0,k^2) = \sqrt{2\kappa}\left(\frac{s^2}{\sigma^2}\right)^{\frac{1}{4}} \overline{D}_\rho'(0)\,\mathcal{O}_\kappa. \tag{2.226}$$

Substituting this into (2.224) we obtain

$$w_1'(-K,k^2)w_2(-K,k^2) - w_2'(-K,k^2)w_1(-K,k^2) = -2\sqrt{\frac{\kappa}{\pi}}\Gamma(\rho+1)\mathcal{O}_\kappa. \tag{2.227}$$

Now we consider the numerator

$$w_1'(-K,k^2)w_2(K,k^2) - w_2'(-K,k^2)w_1(K,k^2). \tag{2.228}$$

Letting $\beta = \pi(\rho+1/2)$, we have the relations

$$w_1(K,k^2) \quad = \Phi(K)\left(D_\rho(\sqrt{2\kappa}\zeta_*) + \frac{\sqrt{2\kappa}}{\kappa^2}D_\rho'(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\mathcal{O}_\kappa, \tag{2.229}$$

$$w_2(K,k^2) \quad = \Phi(K)\left(\overline{D}_\rho(\sqrt{2\kappa}\zeta_*) + \frac{\sqrt{2\kappa}}{\kappa^2}\overline{D}_\rho'(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\mathcal{O}_\kappa, \tag{2.230}$$

$$\begin{aligned}
w_1'(-K,k^2) \quad &= \Phi'(-K)\left(\sin\beta\left(D_\rho(\sqrt{2\kappa}\zeta_*) + \frac{\sqrt{2\kappa}}{\kappa^2}D_\rho'(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\right.\\
&\left.+\cos\beta\left(\overline{D}_\rho(\sqrt{2\kappa}\zeta_*) + \frac{\sqrt{2\kappa}}{\kappa^2}\overline{D}_\rho'(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\right)\mathcal{O}_\kappa\\
&+\frac{\mathrm{d}x}{\mathrm{d}z}\Big|_{z=-K}\frac{\mathrm{d}\zeta}{\mathrm{d}x}\Big|_{x=-1}\Phi(-K)\left(\sin\beta\left(\left(\frac{2\rho+1}{\kappa}-\zeta_*^2\right)D_\rho(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right.\right.\\
&\qquad\left.-\sqrt{2\kappa}\left(1+\frac{B_0'(\zeta_*)}{\kappa^2}\right)D_\rho'(\sqrt{2\kappa}\zeta_*)\right)\mathcal{O}_\kappa\\
&+\cos\beta\left(\left(\frac{2\rho+1}{\kappa}-\zeta_*^2\right)\overline{D}_\rho(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right.\\
&\qquad\left.\left.-\sqrt{2\kappa}\left(1+\frac{B_0'(\zeta_*)}{\kappa^2}\right)\overline{D}_\rho'(\sqrt{2\kappa}\zeta_*)\right)\mathcal{O}_\kappa\right),
\end{aligned} \tag{2.231}$$

$$\begin{aligned}
w_2'(-K,k^2) \quad &= \Phi'(-K)\left(\cos\beta\left(D_\rho(\sqrt{2\kappa}\zeta_*) + \frac{\sqrt{2\kappa}}{\kappa^2}D_\rho'(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\right.\\
&\left.-\sin\beta\left(\overline{D}_\rho(\sqrt{2\kappa}\zeta_*) + \frac{\sqrt{2\kappa}}{\kappa^2}\overline{D}_\rho'(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\right)\mathcal{O}_\kappa
\end{aligned}$$

$$+\frac{\mathrm{d}x}{\mathrm{d}z}\Big|_{z=-K}\frac{\mathrm{d}\zeta}{\mathrm{d}x}\Big|_{x=-1}\Phi(-K)\left(\cos\beta\left(\left(\frac{2\rho+1}{\kappa}-\zeta_*^2\right)D_\rho(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right.\right.$$
$$\left.-\sqrt{2\kappa}\left(1+\frac{B_0'(\zeta_*)}{\kappa^2}\right)D_\rho'(\sqrt{2\kappa}\zeta_*)\right)\mathcal{O}_\kappa$$
$$-\sin\beta\left(\left(\frac{2\rho+1}{\kappa}-\zeta_*^2\right)\overline{D}_\rho(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right.$$
$$\left.\left.-\sqrt{2\kappa}\left(1+\frac{B_0'(\zeta_*)}{\kappa^2}\right)\overline{D}_\rho'(\sqrt{2\kappa}\zeta_*)\right)\mathcal{O}_\kappa\right). \tag{2.232}$$

Since

$$\Phi(z)=\left(\frac{\zeta^2-\sigma^2}{x^2-s^2}\right)^{\frac{1}{4}},\quad \frac{\mathrm{d}x}{\mathrm{d}z}=\sqrt{1-x^2}\sqrt{1-k^2x^2},$$
$$\frac{\mathrm{d}\zeta}{\mathrm{d}x}=\sqrt{\frac{x^2-s^2}{(\zeta^2-\sigma^2)(1-x^2)(1-k^2x^2)}}, \tag{2.233}$$

we have

$$\Phi(K)\Phi(-K)\frac{\mathrm{d}x}{\mathrm{d}z}\Big|_{z=-K}\frac{\mathrm{d}\zeta}{\mathrm{d}x}\Big|_{x=-1}=1. \tag{2.234}$$

Considering the large variable behaviour of the parabolic cylinder functions given in (1.64), we obtain

$$w_1'(-K,k^2)w_2(K,k^2)-w_2'(-K,k^2)w_1(K,k^2)$$
$$=\cos\beta\left(\left(\Phi(K)\Phi'(-K)+\left(\frac{2\rho+1}{\kappa}-\zeta_*^2\right)B_0(\zeta_*)\right)\overline{D}_\rho^2(\sqrt{2\kappa}\zeta_*)\right.$$
$$\left.-\sqrt{2\kappa}\overline{D}_\rho(\sqrt{2\kappa}\zeta_*)\overline{D}_\rho'(\sqrt{2\kappa}\zeta_*)-\frac{2}{\kappa}\overline{D}_\rho'^2(\sqrt{2\kappa}\zeta_*)B_0(\zeta_*)\right)\mathcal{O}_\kappa \tag{2.235}$$

The large variable behaviour given in (1.64) then gives

$$w_1'(-K,k^2)w_2(K,k^2)-w_2'(-K,k^2)w_1(K,k^2)$$
$$=-\sqrt{2\kappa}\cos\beta\frac{\Gamma(\rho+1)^2}{\pi}e^{\kappa\zeta_*^2}(\sqrt{2\kappa}\zeta_*)^{-2\rho-1}$$
$$\times\left(1-\frac{2\zeta_*\left(\Phi(K)\Phi'(-K)-2\zeta_*^2B_0(\zeta_*)\right)-\rho(\rho+1)}{2\kappa\zeta_*^2}\right)\mathcal{O}_\kappa \tag{2.236}$$

We now need to derive an expansion for ζ_* in descending powers of κ. We have the integral relation

$$\int_s^1 \sqrt{\frac{t^2-s^2}{\left(1-t^2\right)\left(1-k^2t^2\right)}}\mathrm{d}t = \int_\sigma^{\zeta_*} \sqrt{\tau^2-\sigma^2}\mathrm{d}\tau. \tag{2.237}$$

To obtain an expansion for ζ_* we need to expand the first integral as $s \to 0$ and the second as $\sigma \to 0$. We call the integral on the left $I_x(s)$ and the integral on the right $I_\zeta(\sigma)$. We consider first $I_x(s)$.

$$\begin{aligned}
&\int_s^1 \sqrt{\frac{t^2-s^2}{\left(1-t^2\right)\left(1-k^2t^2\right)}}\mathrm{d}t = \frac{1}{2}\int_{s^2}^1 \sqrt{\frac{\tau-s^2}{\tau\left(1-\tau\right)\left(1-k^2\tau\right)}}\mathrm{d}\tau \\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!}k^{2n}\int_{s^2}^1 \sqrt{\frac{\tau-s^2}{1-\tau}}\tau^{n-\frac{1}{2}}\mathrm{d}\tau \quad \text{(expanding the } 1-k^2\tau \text{ term)}, \\
&= \frac{1-s^2}{4}\pi\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}k^{2n}\,{}_2F_1\left({\tfrac{1}{2}-n \atop 2}\,{\tfrac{1}{2}};1-s^2\right) \quad \text{(see [1] §15.5)}, \\
&= \frac{1-s^2}{4}\pi\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}k^{2n}\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)}\sum_{j=0}^{n}\frac{\left(\frac{1}{2}-n\right)_j\left(\frac{1}{2}\right)_j(n-j)!}{j!}\left(-s^2\right)^j\right. \\
&\qquad -\frac{\left(-s^2\right)^{n+1}}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)_j\left(n+\frac{3}{2}\right)_j}{j!\,(n+1+j)!}\left(s^2\right)^j\left(2\ln s-\Psi\left(1+j\right)-\Psi\left(n+2+j\right)\right. \\
&\qquad\qquad \left.\left.+\Psi\left(\frac{3}{2}+j\right)+\Psi\left(n+\frac{3}{2}+j\right)\right)\right) \\
&\qquad\qquad \text{(see [1] §15.8(ii))}.
\end{aligned} \tag{2.238}$$

Thus the first few terms in the expansion for the integral as $s \to 0$ are

$$s^0 \text{ term}: \qquad \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\pi}{4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)}k^{2n} = \frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right), \tag{2.239}$$

$$\begin{aligned}
s^2 \text{ term}: \qquad & -\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\pi}{4\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)}k^{2n} - \sum_{n=1}^{\infty}\frac{\pi\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-n\right)}{8n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)}k^{2n} \\
&\qquad + \frac{\pi}{4\Gamma\left(\frac{1}{2}\right)^2}\left(2\ln a-\Psi\left(1\right)-\Psi\left(2\right)+2\Psi\left(\frac{3}{2}\right)\right) \\
&= \frac{1}{4}\ln\left(1-k^2\right)+\frac{1}{2}\ln s-\ln 2-\frac{1}{4},
\end{aligned} \tag{2.240}$$

$$
\begin{aligned}
s^4 \text{ term}: \quad & \sum_{n=1}^{\infty} \frac{\pi \left(\frac{1}{2}\right)_n \left(\frac{1}{2}-n\right)}{8n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} - \frac{\pi}{4\Gamma\left(\frac{1}{2}\right)^2}\left(2\ln a - \Psi(1) - \Psi(2) + 2\Psi\left(\frac{3}{2}\right)\right) \\
& \quad + \sum_{n=2}^{\infty} \frac{3\pi \left(\frac{1}{2}\right)_n \left(\frac{1}{2}-n\right)_2}{32(n-1)n\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} k^{2n} \\
& \quad - \frac{\pi k^2}{16\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(2\ln s - \Psi(1) - \Psi(3) + \Psi\left(\frac{3}{2}\right) + \Psi\left(\frac{5}{2}\right)\right) \\
& \quad + \frac{9\pi}{32\Gamma\left(\frac{1}{2}\right)^2}\left(2\ln s - \Psi(2) - \Psi(3) + 2\Psi\left(\frac{5}{2}\right)\right) \\
& = (1+k^2)\left(\frac{1}{32}\ln(1-k^2) + \frac{1}{16}\ln s - \frac{1}{8}\ln 2 + \frac{3}{64}\right),
\end{aligned} \tag{2.241}
$$

which gives

$$
\begin{aligned}
I_x(s) =& \frac{1}{2k}\ln\left(\frac{1+k}{1-k}\right) + \left(\frac{1}{4}\ln\left(1-k^2\right) + \frac{1}{2}\ln s - \ln 2 - \frac{1}{4}\right)s^2 \\
& + (1+k^2)\left(\frac{1}{32}\ln(1-k^2) + \frac{1}{16}\ln s - \frac{1}{8}\ln 2 + \frac{3}{64}\right)s^4 + \mathcal{O}(s^6 \ln s).
\end{aligned} \tag{2.242}
$$

$I_\zeta(\sigma)$ simply yields

$$
I_\zeta(\sigma) = \frac{\zeta_*^2}{2} + \left(\frac{1}{2}\ln\sigma - \frac{1}{4} - \frac{1}{2}\ln 2\zeta_*\right)\sigma^2 + \frac{1}{16\zeta_*^2}\sigma^4 + \mathcal{O}(\sigma^6). \tag{2.243}
$$

Combining these give

$$
\begin{aligned}
\zeta_*^2 =& \frac{1}{k}\ln\frac{1+k}{1-k} + 2\left(\frac{1}{4}\ln(1-k^2) + \frac{1}{2}\ln s - \ln 2 - \frac{1}{4}\right)s^2 \\
& + 2(1+k^2)\left(\frac{1}{32}\ln(1-k^2) + \frac{1}{16}\ln s - \frac{1}{8}\ln 2 + \frac{3}{64}\right)s^4 \\
& - 2\left(\frac{1}{2}\ln\sigma - \frac{1}{4} - \frac{1}{2}\ln 2\zeta_*\right)\sigma^2 - \frac{1}{8\zeta_*^2}\sigma^4 + \ldots,
\end{aligned} \tag{2.244}
$$

and substituting in the expansions for s and σ as $\kappa \to \infty$, we have the expansion for ζ_* in κ as

$$
\begin{aligned}
\zeta_* =& \sqrt{R} + \frac{2\rho+1}{4\kappa}\frac{Q(k)}{\sqrt{R}} + \frac{1}{2^6 R^{\frac{3}{2}}\kappa^2}\left((2m+1)^2\left(3R\left(1+k^2\right) - 2Q(k)(Q(k)-4) - 4\right)\right. \\
& \left. -2RQ(k)\left(1+k^2\right)\right) + \mathcal{O}(\kappa^{-3}),
\end{aligned} \tag{2.245}
$$

where

$$R(k)=\frac{\ln\left(\frac{1+k}{1-k}\right)}{k},\quad \text{and}\quad Q(k)=\ln\left(1-k^2\right)-2\ln 2+\ln R. \tag{2.246}$$

Expanding the functions in (2.235) as $\kappa\to\infty$, we have

$$\begin{aligned}
\cos\beta &= (-1)^{m+1}\pi\varepsilon+\mathcal{O}(\varepsilon^2),\\
e^{-\kappa\zeta_*^2} &= \left(\frac{1-k}{1+k}\right)^{\frac{\kappa}{k}} e^{-(m+1/2)Q(k)}\left(1+\frac{1}{2^5R(k)\kappa}\left((2m+1)^2(4-8Q(k)-3R(k)(1+k^2))\right.\right.\\
&\qquad \left.\left. +2R(k)Q(k)(1+k^2)\right)+\mathcal{O}(\kappa^{-2})\right),\\
\zeta_*^{2\rho+1} &= R^{m+1/2}(k)\left(1+\frac{(2m+1)^2}{4R(k)\kappa}Q(k)+\mathcal{O}(\kappa^{-2})\right),\\
B_0(\zeta_*) &= \frac{1}{2^5}\sqrt{\frac{1}{R(k)}}\left((1+k^2)\ln\left(\frac{1}{4}R(k)\left(2k\coth\left(\frac{1}{2}kR(k)\right)-(1+k^2)\right)\right)\right.\\
&\quad \left. -\frac{3}{2}\frac{(1-k^2)^2}{2k\coth\left(\frac{1}{2}kR(k)\right)-(1+k^2)}+3k\coth\left(\frac{1}{2}kR(k)\right)+2k^2R(k)-\frac{6}{R(k)}\right),\\
\Phi(K)\Phi'(-K) &= -\frac{1}{2\sqrt{R(k)}}+\mathcal{O}(\kappa^{-1}),
\end{aligned} \tag{2.247}$$

and using these we obtain

$$\begin{aligned}
&w_1'(-K,k^2)w_2(K,k^2)-w_2'(-K,k^2)w_1(K,k^2)\\
&=(-1)^m\varepsilon\Gamma(m+1)^2\left(\frac{1+k}{1-k}\right)^{\frac{\kappa}{k}} e^{(m+1/2)Q(k)}(2\kappa)^{-m}R^{-m-1/2}(k)\\
&\qquad\times\left(1-\frac{2\zeta_*\left(\Phi(K)\Phi'(-K)-2\zeta_*^2B_0(\zeta_*)\right)-m(m+1)}{2\kappa\zeta_*^2}\right)\mathcal{O}_\kappa.
\end{aligned} \tag{2.248}$$

Substituting (2.248) and (2.227) into (2.189) we obtain

$$\begin{aligned}
\varepsilon_\nu^{m\pm} &= \mp(-1)^m(1-k^2)^{-m-1/2}\frac{1}{m!}\sqrt{\frac{2}{\pi}}\left(\frac{1-k}{1+k}\right)^{\frac{\kappa}{k}} 2^{3m+3/2}\kappa^{m+1/2}\\
&\quad\times\left(1+\frac{1}{32R(k)\kappa}\left((2m+1)^2(4-3R(k)(1+k^2))+2R(k)Q(k)(1+k^2)\right.\right.\\
&\qquad\left.\left.+\frac{R^{3/2}(k)}{\sqrt{2}}(2\ln 2+3)-16(1+m(m+1))\right)\right)\mathcal{O}_\kappa.
\end{aligned} \tag{2.249}$$

Substituting $\varepsilon_\nu^{m\pm}$ into (2.200), we obtain (2.186). □

The first term here matches Volker's result (2.12), when replacing $\kappa = \sqrt{\nu(\nu+1)}k$, where he only gives the first term.

In this work me made conjectures about the errors of the solutions to obtain this result. If instead we had just wanted a one-term approximation, we would have used one-term approximations for the solutions for which he have already obtained in section 2.4 rigorous error bounds for, thus we would have gotten without any conjecture rigorously the result above with $\mathcal{O}(\kappa^{-1})$ instead of the κ^{-2} term, which is still stronger than Volkmer's result. We do not provide numerical justification for this result as the numerics we were using to calculate the eigenvalues were not stable enough to deal with such exponentially small numbers. We will however provide numerics for the analogous problem concerning the eigenvalues of Mathieu's equation, in section 3.5.

3

Rigorous asymptotics for the Mathieu equation with a large parameter

In this chapter we discuss the Mathieu equation and in particular we present rigorous results for the Mathieu functions, and their corresponding eigenvalues as a parameter in its differential equation becomes large. In preparation for the analysis we will discuss the properties of these functions and give a concise literature review related to the relevant problems.

3.1 Properties of the Mathieu functions

We will summarise their important properties here, for a fuller treatment see ([1] §28). Mathieu's equation is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(\lambda - 2h^2 \cos 2z\right) w = 0, \tag{3.1}$$

where λ and h are real parameters. We consider always the interval $z \in [0, \pi]$ unless stated otherwise. When λ assumes the special values a_m or b_{m+1} for $m = 0, 1, 2, ...,$ Mathieu's equation admits even or odd periodic solutions denoted $ce_m(h, z)$ or $se_{m+1}(h, z)$ respectively. These are either π-periodic or π-antiperiodic, depending on the parity of m. For a summary of these properties, including the boundary conditions which the functions satisfy, see Table 3.1.

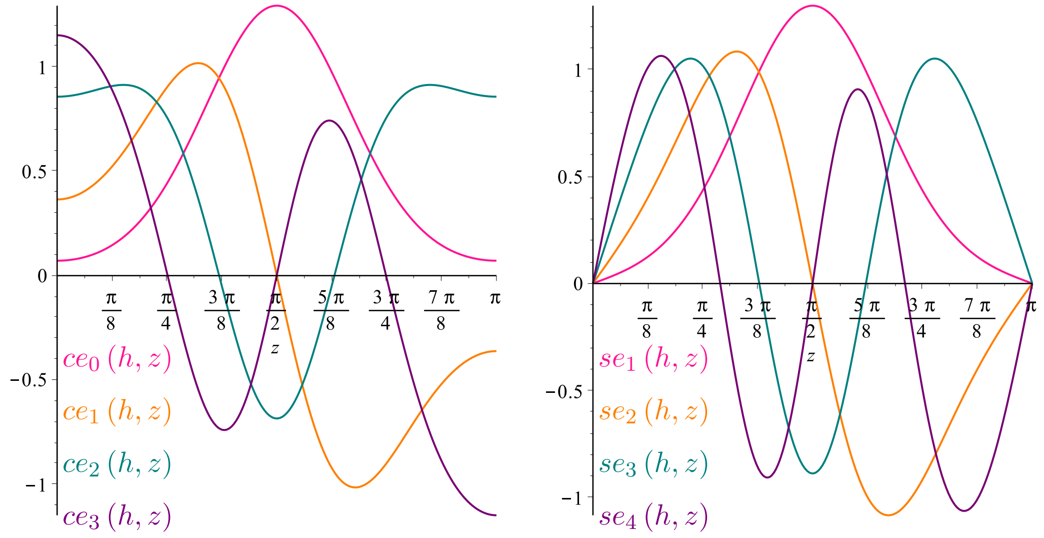

Figure 3.1: Plots for assorted Mathieu functions in $[0,\pi]$ for $h=2$.

Both functions have m zeros in the interval $(0,\pi)$ and their eigenvalues are ordered such that

$$a_0 < a_1 < \dots \to \infty, \qquad b_1 < b_2 < \dots \to \infty, \tag{3.2}$$

and interlace such that

$$a_0 < b_1 < a_1 < b_2 < a_2 < \dots . \tag{3.3}$$

The normalisations in ([1] §28.2) can be rewritten as

$$\int_0^\pi \{ce_m(h,z)\}^2\,\mathrm{d}z = \int_0^\pi \{se_{m+1}(h,z)\}^2\,\mathrm{d}z = \frac{\pi}{2}, \tag{3.4}$$

and to complete their definitions, the signs are determined by continuity from

$$ce_0(0,z) = \frac{1}{\sqrt{2}}, \quad ce_m(0,z) = \cos mz, \quad se_m(0,z) = \sin mz.$$

We summarise their properties and give boundary conditions in Table 3.1.

Table 3.1: properties and boundary conditions for Mathieu functions

Eigenfunctions	Eigenvalues	Periodicity	Parity at $z = 0, \frac{\pi}{2}$	Boundary conditions
$ce_{2m}(h, z)$	a_{2m}	Period π	even, even	$w'(0) = w'(\pi/2) = 0$
$ce_{2m+1}(h, z)$	a_{2m+1}	Antiperiod π	even, odd	$w'(0) = w(\pi/2) = 0$
$se_{2m+1}(h, z)$	b_{2m+1}	Antiperiod π	odd, even	$w(0) = w'(\pi/2) = 0$
$se_{2m+2}(h, z)$	b_{2m+2}	Period π	odd, odd	$w(0) = w(\pi/2) = 0$

For $z \in [0, \pi]$, when h becomes large the oscillatory regions of the Mathieu functions lie in a neighbourhood of $\frac{\pi}{2}$ which shrinks in this limit, and otherwise they are exponentially small outside of this region and approaching the endpoints 0 and π. Formal results given in the literature indicate that $se_{m+1}(h, z)$ and $ce_m(h, z)$ behave asymptotically the same as $h \to \infty$.

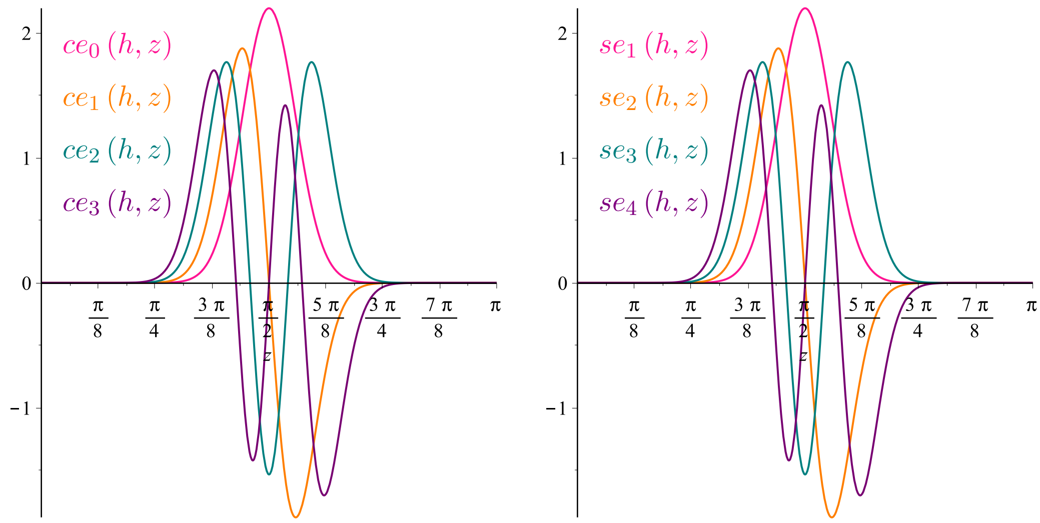

Figure 3.2: Plots for assorted Mathieu functions in $[0, \pi]$ for $h = 15$

Figure 3.2 shows that even for h as large as 15, $se_{m+1}(h, z)$ and $ce_m(h, z)$ already look very asymptotically similar.

Formal results given in the literature also indicate that the difference between the corresponding eigenvalues b_{m+1} and a_m is exponentially small as $h \to \infty$.

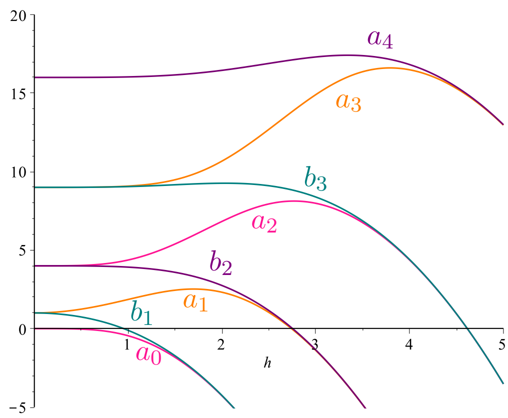

Figure 3.3: The first few eigenvalues as a function of h

Figure 3.3 shows that as h becomes large, b_{m+1} and a_m tend to each other asymptotically. It also shows that as m becomes larger, h must become larger to see that they are asymptotically the same in this limit.

3.2 Previous Results

For a general overview of the theory concerning Mathieu's equation, see [9], [10], [8] and for a more detailed study see [19]. The main formal results concerning the Mathieu function can be found in (in chronological order) [21], [22], [23], [24], [25], [26], [19], [27], [28], [29] and [30]. They are all equivalent, thus we detail those given in [29]:

As $h \to \infty$, letting $s = 2m+1$ and $z = \frac{\pi}{2} + \lambda h^{-\frac{1}{4}}$, where λ is a real constant such that $|\lambda| < 2^{\frac{1}{4}}$. Also let $\zeta = 2\sqrt{h}\cos z$. Then as $h \to +\infty$

$$\begin{aligned} ce_m(h,z) &= \hat{C}_m(U_m(\zeta) + V_m(\zeta)), \\ \frac{se_{m+1}(h,z)}{\sin x} &= \hat{S}_m(U_m(\zeta) - V_m(\zeta)), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} U_m(\zeta) \sim & D_m(\zeta) - \frac{1}{2^6 h}\left(D_{m+4}(\zeta) - 4!\binom{m}{4}D_{m-4}(\zeta)\right) \\ & + \frac{1}{2^{13}h^2}\left(D_{m+8}(\zeta) - 2^5(m+2)D_{m+4}(\zeta) + 4!2^5(m-1)\binom{m}{4}D_{m-4}(\zeta)\right. \\ & \left. +8!\binom{m}{8}D_{m-8}(\zeta)\right) + \ldots, \end{aligned} \tag{3.6}$$

$$\begin{aligned} V_m(\zeta) \sim & \frac{1}{2^4 h}\left(-D_{m+2}(\zeta) - m(m-1)D_{m-2}(\zeta)\right) \\ & + \frac{1}{2^{10}h^2}\Big(D_{m+6}(\zeta) + (m^2-25m-36)D_{m+2}(\zeta) \\ & -m(m-1)(m^2+27m-10)D_{m-2}(\zeta) + 6!\binom{m}{6}D_{m-6}(\zeta)\Big) \\ & + ..., \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \hat{C}_m &\sim \left(\frac{\pi h}{2(m!)^2}\right)^{1/4}\left(1 + \frac{2m+1}{8h} + \frac{m^4+2m^3+263m^2+262m+108}{2048h^2} + ...\right)^{-1/2}, \\ \hat{S}_m &\sim \left(\frac{\pi h}{2(m!)^2}\right)^{1/4}\left(1 - \frac{2m+1}{8h} + \frac{m^4+2m^3-121m^2-122m-84}{2048h^2} + ...\right)^{-1/2}. \end{aligned} \tag{3.8}$$

No error analysis was given for these function expansions. In [19], only a first term was given in these expansions, and came provided with on order estimate for the error, however this estimate does not make sense in the regions the functions are exponentially small. Kurz extended this to an expansion in [30], however similar issues occur for the error estimates related to the expansions. In [16] a one-term uniform approximation was given, but again the order estimate for the error provided does not make sense in the regions the functions are exponentially small.

The most satisfactory work thus far is contained in [31]. Here Dunster derives uniform asymptotic approximations for all complex values z when $-2h^2 \le \lambda \le (2-d)h^2$, $(d>0)$, with error bounds either included or available for all approximations. These approximations involve both elementary functions and Whittaker functions. He also includes rigorous statements related to the eigenvalues a_m and b_{m+1}. Here we consider only the real interval $z \in [0,\pi]$ as many physical applications are restricted to real variables thus we want different results which will be given in the most natural and strong form for the case we consider.

Similarly formal results concerning the eigenvalues relating to the Mathieu function can be found in (in chronological order) [21], [22], [23], [24], [25], [26], [19], [27], [28], [29] and [30]. All of these except the last are all formal expansions and no error analysis was given. The expansions in all the papers are equivalent and are as follows:

As $h \to \infty$, letting $s = 2m+1$,

$$\left.\begin{matrix} a_\nu^m \\ b_\nu^{m+1} \end{matrix}\right\} \sim -2h^2 + 2sh - \frac{1}{8}(s^2+1) - \frac{1}{2^7 h}(s^3+3s) - \frac{1}{2^{12}h^2}(5s^4+34s^2+9)$$
$$- \frac{1}{2^{12}h^3}(33s^5+410s^3+405s) - \frac{1}{2^{20}h^4}(63s^6+1260s^4+2943s^2+486)$$
$$- \frac{1}{2^{25}h^5}(527s^7+15617s^5+69001s^3+41607s) + \dots \,. \tag{3.9}$$

In [30], an attempt was made to provide error estimates for these expansions, and she gives that when the expansion is truncated on the h^{-n-1} term, the error is $\mathcal{O}(h^{-n})$. This result is reasonable, but methods of obtaining terms in the eigenvalue expansions seem cumbersome.

The result for the exponentially small difference between the eigenvalues of Mathieu's equation is stated without proof in [19] as

$$b_{m+1} - a_m \sim \frac{2^{4m+5}}{m!}\left(\frac{2}{\pi}\right)^{1/2} h^{m+3/2} e^{-4h}\left(1 - \frac{6m^2+14m+7}{32h}\right) \quad (h \to \infty). \tag{3.10}$$

Later in [16] the result (with $\lambda = \sqrt{2}h$)

$$b_{m+1} - a_m = \frac{2^{4m+5}}{m!}\left(\frac{2}{\pi}\right)^{1/2} h^{m+3/2} e^{-4h}\left(1 + \mathcal{O}(h^{-1/2})\right) \quad (h \to \infty) \tag{3.11}$$

is given. However the error analysis used in this paper is not satisfactory and at times the steps are non-rigorous, thus a stronger order estimate was not obtained. Here nothing was said either of the second term given in [19].

We now give results for Mathieu's equation which are analgous to those we gave for Lamé's equation, and for which the proof relies heavily on.

3.3 Uniform asymptotic approximations for the Mathieu functions

Theorem 3.1. *For $m \in \mathbb{N}_0$ and $z \in \left[0, \frac{\pi}{2}\right]$, as $h \to \infty$*

$$ce_m(h,z) = C_m \left(\frac{\zeta^2 - \sigma_m^2}{x^2 - s_m^2}\right)^{1/4} \left(D_m\left(2\sqrt{h}\zeta\right) + \epsilon_{m,1}(\zeta) + \eta_m^c \left(\overline{D}_m\left(\sqrt{2}h\zeta\right) + \epsilon_{m,2}(\zeta)\right)\right), \quad (3.12)$$

$$se_{m+1}(h,z) = S_{m+1} \left(\frac{\zeta^2 - (\sigma_\nu^m)^2}{x^2 - (s_\nu^m)^2}\right)^{1/4} \left(D_m\left(2\sqrt{h}\zeta\right) + \epsilon_{m,1}(\zeta) + \eta_{m+1}^s \left(\overline{D}_m\left(\sqrt{2}h\zeta\right) + \epsilon_{m,2}(\zeta)\right)\right), \quad (3.13)$$

where

$$\epsilon_{m,1}(\zeta) = \frac{\mathbf{M}_m\left(\sqrt{2}h\zeta\right)}{\mathbf{E}_m\left(\sqrt{2}h\zeta\right)} \mathcal{O}\left(h^{-1}\right), \quad \epsilon_{m,2}(\zeta) = \mathbf{E}_m\left(\sqrt{2}h\zeta\right)\mathbf{M}_m\left(\sqrt{2}h\zeta\right)\mathcal{O}\left(h^{-1}\right) \quad (3.14)$$

both η_m^c and η_{m+1}^s are $\mathcal{O}\left(e^{-2h\zeta_*^2} h^{m+1/2}\right)$, with $\zeta_* \sim \sqrt{2}$, as $h \to \infty$, the relationship between z and x is defined by $x = \cos z$, and between x and ζ by

$$\begin{aligned} \int_x^{-s_m} \sqrt{\frac{t^2 - s_m^2}{1 - t^2}} \mathrm{d}t &= \int_\zeta^{-\sigma_m} \sqrt{\tau^2 - \sigma_m^2} \mathrm{d}\tau \quad (-1 < x \le -s_m), \\ \int_{-s_m}^x \sqrt{\frac{s_m^2 - t^2}{1 - t^2}} \mathrm{d}t &= \int_{-\sigma_m}^\zeta \sqrt{\sigma_m^2 - \tau^2} \mathrm{d}\tau \quad (-s_m \le x \le s_m), \\ \int_{s_m}^x \sqrt{\frac{t^2 - s_m^2}{1 - t^2}} \mathrm{d}t &= \int_{\sigma_m}^\zeta \sqrt{\tau^2 - \sigma_m^2} \mathrm{d}\tau \quad (s_m \le x < 1), \end{aligned} \quad (3.15)$$

where

$$s_m^2 = \frac{\lambda_m + 2h^2}{4h^2}, \quad \sigma_m^2 = s_m^2 \, {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 2\end{matrix}; s_m^2\right), \quad (3.16)$$

and λ_m corresponds to either $\lambda = a_m$ or $\lambda = b_{m+1}$ depending on the solution we are considering, and

$$\left.\begin{matrix} C_m \\ S_{m+1} \end{matrix}\right\} \sim \left(\frac{\pi h}{2m!^2}\right)^{1/4}\left(1 - \frac{2m+1}{16h}\right). \tag{3.17}$$

Again as $h \to \infty$

$$a_m = -2h^2 + (4m+2)h + \mathcal{O}(1), \qquad b_{m+1} = -2h^2 + (4m+2)h + \mathcal{O}(1), \tag{3.18}$$

Proof. We denote for the moment the parameter h in (2.1) to be h_L, to avoid confusion with the parameter h in Mathieu's equation. We have in the limit $k \to 0_+$ from ([1] §22.5 (ii)) that

$$\lim_{k \to 0_+} \operatorname{sn}(z,k) = \sin z, \quad \text{and} \quad \lim_{k \to 0_+} K(k) = \frac{\pi}{2}, \tag{3.19}$$

thus if $\nu \to \infty$ in such a way that as $k \to 0_+$, $\sqrt{\nu(\nu+1)}k = 2h$ for some constant h, then we can rewrite the limit of (2.1) in the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \left(h_L - 2h^2 - 2h^2 \cos 2\left(\frac{\pi}{2} - z\right)\right) w = 0. \tag{3.20}$$

Thus for $\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} + \left(\frac{2h}{k}\right)^2}$ we have

$$\lim_{k \to 0_+} Ec_\nu^m(z,k^2) = ce_m\left(h, \frac{\pi}{2} - z\right), \qquad \lim_{k \to 0_+} Es_\nu^{m+1}(z,k^2) = se_{m+1}\left(h, \frac{\pi}{2} - z\right), \tag{3.21}$$

and

$$a_m = \lim_{k \to 0_+} a_\nu^m - 2h^2, \qquad b_{m+1} = \lim_{k \to 0_+} b_\nu^{m+1} - 2h^2. \tag{3.22}$$

Thus we obtain all the results in Theorem 3.1 as the limiting cases of those results proved in Theorem 2.1. □

two-term approximations can be deduced from subsection 2.4.3, and thus are given by

$$\begin{aligned} ce_m(h,z) &\sim C_m\left(D_m\left(2\sqrt{h}\zeta\right)+\frac{B_0(\zeta)}{h^2}\frac{d}{d\zeta}D_m\left(2\sqrt{h}\zeta\right)\right), \\ se_{m+1}(h,z) &\sim S_{m+1}\left(D_m\left(2\sqrt{h}\zeta\right)+\frac{B_0(\zeta)}{h^2}\frac{d}{d\zeta}D_m\left(2\sqrt{h}\zeta\right)\right), \end{aligned} \tag{3.23}$$

where for $B_0(\zeta)$ we have the relation

$$256\ B_0(\zeta)=\frac{3\zeta}{4-\zeta^2}-\frac{2}{\zeta}\ln\left(1-\frac{\zeta^2}{4}\right). \tag{3.24}$$

3.4 Asymptotic expansions for the Lamé functions and their eigenvalues

Similarly to the Lamé case, if one considers the two-term uniform approximation for the Mathieu functions you would observe that the oscillatory behaviour of the Mathieu functions happen in a shrinking neighbourhood of the $z=\frac{\pi}{2}$ as $h\to\infty$. It can be shown that in a shrinking neighbourhood of this point, the ζ variable used in the uniform approximations behaves approximately like $\cos(z)$. Thus the variable in the parabolic cylinder function around this point behaves approximately like $\sqrt{2}h\cos(z)$. This would motivate a much simpler transformation.

Letting $t=2\sqrt{h}\cos z$ in (3.1) we present a solution of Mathieu's equation in the algebraic form

$$\frac{\mathrm{d}^2w_m(h,z)}{\mathrm{d}t^2}+\left(\frac{\lambda_m+2h^2}{4h}-\frac{t^2}{4}\right)w_m(h,z)-\frac{1}{4h}\left(t^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}+t\frac{\mathrm{d}}{\mathrm{d}t}\right)w_m(h,z)=0, \tag{3.25}$$

where λ_m corresponds to a_m or b_{m+1}, and since (3.18) gives

$$\frac{\lambda_m+2h^2}{4}-h\left(m+\tfrac{1}{2}\right)=\mathcal{O}(1),\quad (h\to\infty) \tag{3.26}$$

for both $\lambda_m=a_m$ and $\lambda_m=b_{m+1}$, we have

$$\frac{\mathrm{d}^2w_m(h,z)}{\mathrm{d}t^2}+\left(m+\frac{1}{2}-\frac{t^2}{4}\right)w_m(h,z)$$

$$+\frac{1}{h}\left(-\frac{t^2}{4}\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\frac{t}{4}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\lambda+2h^2}{4}-h\left(m+\frac{1}{2}\right)\right)w_m(h,z)=0. \tag{3.27}$$

In a similar manner to the previous subsections coefficients in the expansions for the functions and eigenvalues can be computed using the same ansatz, although they would only make sense asymptotically in a shrinking neighbourhood of $z=\frac{\pi}{2}$. However these results again are also realised by considering the Mathieu functions as a special case of the Lamé functions.

Theorem 3.2. *For $m \in \mathbb{N}_0$, as $h \to \infty$*

$$\left.\begin{array}{c} a_m \\ b_{m+1} \end{array}\right\} = -2h^2 + 4h\sum_{s=0}^{n}\frac{\mu_s}{h^s} + \mathcal{O}(h^{-n}), \tag{3.28}$$

where the order term will be different in both cases. The μ_s terms are constant coefficients which depend on m, found recursively with the eigenfunction expansions. We give here the first few terms:

$$\mu_0 = m+\tfrac{1}{2},\quad \mu_1 = -\tfrac{1}{16}\left(1+2m+2m^2\right),\quad \mu_2 = -\tfrac{1}{128}\left(1+3m+3m^2+2m^3\right).$$

Letting $t = 2\sqrt{h}\cos z$, for $z=\frac{\pi}{2}+\mathcal{O}(h^{-1/2})$ as $h\to\infty$

$$\begin{aligned} ce_m(h,a_m,z) &= C_m\left(D_m(t)\sum_{s=0}^{n}\frac{A_s(t)}{h^s}+D_m'(t)\sum_{s=0}^{n}\frac{B_s(t)}{h^s}+\mathcal{O}(h^{-n-1})\right),\\ \frac{se_{m+1}(h,b_{m+1},z)}{\sin z} &= S_{m+1}\left(D_m(t)\sum_{s=0}^{n}\frac{P_s(t)}{h^s}+D_m'(t)\sum_{s=0}^{n}\frac{Q_s(t)}{h^s}+\mathcal{O}(h^{-n-1})\right), \end{aligned} \tag{3.29}$$

where

$$\left.\begin{array}{c} C_m \\ S_{m+1} \end{array}\right\} \sim \left(\frac{\pi h}{2\,(m!)^2}\right)^{1/4}\left(1+\frac{\eta_1}{h}+\frac{\eta_2}{h^2}+...\right)^{-1/2}. \tag{3.30}$$

Both $A_s(t)$ and $P_s(t)$ are even polynomials, and both $B_s(t)$ and $Q_s(t)$ are odd polyno-

mials. These polynomials are found recursively and we give here the first few terms:

$$
\begin{aligned}
&A_0 = 1, \quad A_1 = \frac{t^2}{2^6}, \quad B_0 = Q_0 = 0, \quad B_1 = Q_1 = \frac{t^3}{2^5} - (1+2m)\,\frac{t}{2^5}, \quad P_0 = 1, \quad P_1 = \frac{9t^2}{2^6},\\
&A_2 = \frac{t^8}{2^{13}} - (1+2m)\,\frac{t^6}{2^{11}} + \left(9+10m+10m^2\right)\frac{t^4}{2^{12}} + \left(5+6m-12m^2-8m^3\right)\frac{t^2}{2^{12}},\\
&B_2 = \frac{t^5}{2^8} - (1+2m)\,\frac{5t^3}{2^{11}} - \left(11+20m+20m^2\right)\frac{t}{2^{11}},\\
&P_2 = \frac{t^8}{2^{13}} - (1+2m)\,\frac{t^6}{2^{11}} + \left(113+10m+10m^2\right)\frac{t^4}{2^{12}} + \left(5+6m-12m^2-8m^3\right)\frac{t^2}{2^{12}},\\
&Q_2 = \frac{t^5}{2^7} - (1+2m)\,\frac{13t^3}{2^{11}} - \left(11-20m-20m^2\right)\frac{t}{2^{11}}, \qquad \eta_1 = \frac{6m+3}{32}.
\end{aligned}
\tag{3.31}
$$

Proof. The proof follows from Proof 3.3 and Theorem 2.2. □

These function and eigenvalue coefficients match the formal results given in (§28.8 [1]), (§2 [19]), and various other papers discussed at the start of this chapter. Note that similar results to (3.29) and (3.28) were given in [30], but there the function expansions did not make sense in the interval they were stated to hold in, and the method used there to obtain the coefficients in the expansions for the functions and eigenvalue was very cumbersome. Our methods were simple and enabled us to perform rigorous error analysis on these expansions.

3.5 Exponentially small difference between a_m and b_{m+1}

Theorem 3.3. *We have the eigenvalue difference*

$$b_\nu^{m+1} - a_\nu^m = \frac{2^{4m+5}}{m!}\sqrt{\frac{2}{\pi}}h^{m+3/2}e^{-4h}(1+\mathcal{O}(h^{-1})), \tag{3.32}$$

and obtain the expression with a second term

$$b_\nu^{m+1} - a_\nu^m \sim \frac{2^{4m+5}}{m!}\sqrt{\frac{2}{\pi}}h^{m+3/2}e^{-4h}\left(1 - \frac{6m^2+14m+7}{32h}\right), \tag{3.33}$$

which matches that given in [19], where it was given without any kind of derivation.

Proof. The proof follows from Proof 3.3 and Theorem 2.4. □

We give some numerics for this exponentially small difference, for both the one-term and two-term approximations.

Table 3.2: Relative errors for the exponentially small difference for $h = 50$.

m	one-term	two-term
2	-0.0367	-0.0123
3	-0.0634	-0.0214
4	-0.0966	-0.0328
5	-0.1255	-0.0466

Table 3.3: Relative errors for the exponentially small difference for $h = 150$.

m	one-term	two-term
2	0.000209	0.000023
3	0.001029	0.000110
4	0.003096	0.000323
5	0.007372	0.000747

We note that as m becomes larger, h must become much larger to see a significantly small relative error, and that the errors are much smaller when the second term is incorporated, especially as h becomes larger.

4

Rigorous asymptotics for the spheroidal wave equation for a parameter being large and positive, and being large and negative

The differential equation discussed in this chapter is the spheroidal wave equation, which arises in a similar way to those discussed in chapters 2 and 3. We are particularly interested in obtaining rigorous results for the prolate spheroidal wave functions and the oblate spheroidal wave functions, and their corresponding eigenvalues as a parameter in their equations becomes large. These two types of solutions occur by taking a parameter to be either positive or negative respectively. The analysis concerning the prolate case is very similar to that in chapter 2. The discussion concerning the oblate case is unfinished work, but has a different theme than those given before it.

In preparation for the analysis we first discuss the properties of these functions.

4.1 Properties of the spheroidal wave functions

We will summarise their important properties here, for a fuller treatment see ([1] §30). The spheroidal wave equation is

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)\frac{\mathrm{d}w}{\mathrm{d}z}\right)+\left(\lambda+\gamma^2(1-z^2)-\frac{\mu^2}{1-z^2}\right)w=0, \tag{4.1}$$

and we consider λ, γ^2 and μ to be real. We will consider the interval $[-1,1]$ unless stated otherwise. When $\mu = m \in \mathbb{N}_0$, and λ assumes the special values λ_n^m, where $n = m, m+1, ..$, the spheroidal wave equation admits solutions denoted by $\mathrm{Ps}_n^m(z,\gamma^2)$.

These special solutions are bounded at the singularities $z = \pm 1$ of the differential equation. When $\gamma^2 > 0$, the functions are called prolate spheroidal wave functions, and when $\gamma^2 < 0$ they are called oblate spheroidal wave functions. When $\gamma = 0$, $\mathrm{Ps}_n^m(z, 0)$ reduces to the Ferrers function $\mathrm{P}_n^m(z)$.

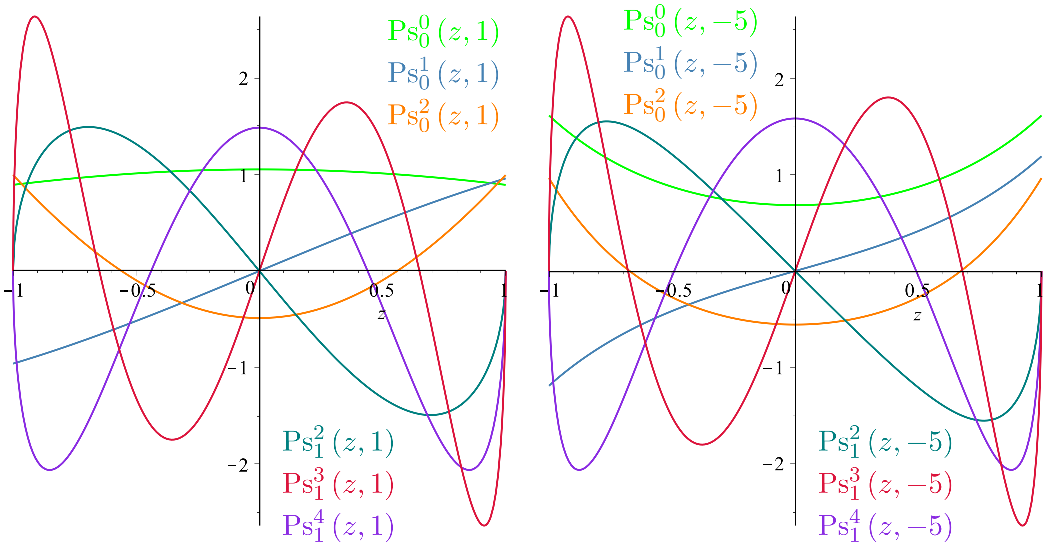

Figure 4.1: Plots for assorted (left) prolate spheroidal wave functions and (right) oblate spheroidal wave functions on $[-1, 1]$.

The spheroidal wave functions can be written in the form

$$\mathrm{Ps}_n^m(z, \gamma^2) = (1 - z^2)^{\frac{m}{2}} g(z), \tag{4.2}$$

where $g(z)$ is an entire function. They have exactly $(n-m)$ zeros in the interval $(-1, 1)$ and satisfy the reflection formula

$$\mathrm{Ps}_n^m(-z, \gamma^2) = (-1)^{n-m} \mathrm{Ps}_n^m(z, \gamma^2). \tag{4.3}$$

The eigenvalues are ordered such that

$$\lambda_m^m < \lambda_{m+1}^m < \lambda_{m+2}^m < \ldots \tag{4.4}$$

The functions satisfy the orthogonality relation

$$\int_{-1}^{1} \mathrm{Ps}_k^m(z, \gamma^2) \mathrm{Ps}_n^m(z, \gamma^2) \mathrm{d}z = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{kn}, \tag{4.5}$$

and to complete their definition we have the signs

$$\begin{aligned} \frac{\mathrm{Ps}_n^m(\gamma^2,0)}{|\mathrm{Ps}_n^m(\gamma^2,0)|} &= (-1)^{(n+m)/2} \quad \text{when } (n-m) \text{ is even,} \\ \frac{\mathrm{dPs}_n^m(z,\gamma^2)/\mathrm{d}z\big|_{z=0}}{\big|\mathrm{dPs}_n^m(z,\gamma^2)/dz\big|_{z=0}\big|} &= (-1)^{(n+m-1)/2} \text{ when } (n-m) \text{ is odd.} \end{aligned} \tag{4.6}$$

When γ^2 becomes large and positive, the functions decay when approaching the endpoints at $z = \pm 1$, and oscillate in between.

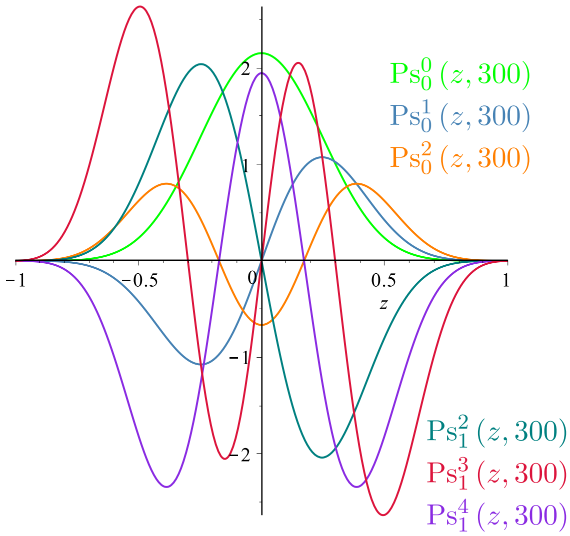

Figure 4.2: Plots for assorted prolate spheroidal wave functions on $[-1, 1]$.

When γ^2 becomes large and negative, the functions decay around the origin and oscillate when approaching the endpoints at $z = \pm 1$.

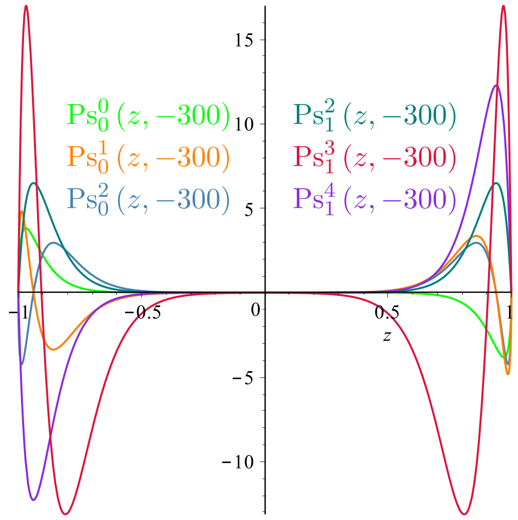

Figure 4.3: Plots for assorted oblate spheroidal wave functions on $[-1, 1]$.

The eigenvalues become large and negative when γ^2 is large and positive, and large and positive when γ^2 is large and negative.

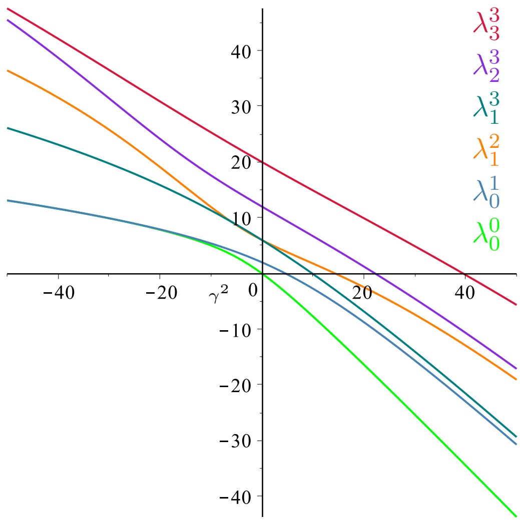

Figure 4.4: Plots for assorted eigenvalues λ_n^m.

4.2 Previous Results

For some general theory concerning the prolate spheroidal wave equation, see [8], [9] and [10], and for a more detailed study see [19]. The results for the spheroidal wave functions are not so abundant as those for the Mathieu equation, but there are more so than those for Lamé's equation.

In [19], one-term asymptotic approximations are written down for the prolate spheroidal wave functions in terms of parabolic cylinder functions, and one-term asymptotic approximations for the oblate spheroidal wave functions in terms of associated Laguerre functions. These are given as follows:

For $z \in [-1, 1]$, as $\gamma \to \infty$

$$\mathrm{Ps}_n^m\left(z, \gamma^2\right) = (-1)^m \left(\frac{4\gamma}{\pi}\right)^{\frac{1}{4}} \frac{1}{(n-m)!} \sqrt{\frac{(n+m)!}{2n+1}} \left(1 - z^2\right)^{\frac{m}{2}} D_{n-m}\left(\sqrt{2\gamma} z\right) + \mathcal{O}\left(\gamma^{-1}\right), \tag{4.7}$$

and letting $\gamma^2 = -\gamma^{*2}$, with $q = 2p + m + 1$, where $2p = n - m$ when $n - m$ is even, or $2p + 1 = n - m$ when $n - m$ is odd, as $\gamma^* \to \infty$

$$\begin{aligned} \mathrm{Ps}_n^m\left(z, \gamma^2\right) = &(-1)^m \gamma^{*\frac{m+1}{2}} \sqrt{\frac{p!}{(m+p)!} \frac{(n+m)!}{(n-m)!} \frac{1}{2n+1}} \left(1 - z^2\right)^{\frac{m}{2}} \\ &\times \left(e^{-\gamma^*(1-z)} L_p^m\left(2\gamma^*(1-z)\right) \pm e^{-\gamma^*(1+z)} L_p^m\left(2\gamma^*(1+z)\right)\right) + \mathcal{O}\left(\gamma^{-\frac{1}{2}}\right). \end{aligned} \tag{4.8}$$

The errors here are given without justification and clearly don't make sense when the functions are exponentially small.

Results for expansions of the eigenvalues in both cases are also provided in [19], and are given as follows:

Let $q = 2(n - m) + 1$, then as $\gamma \to \infty$

$$\lambda_n^m = -\gamma^2 + \gamma q + m^2 - \frac{1}{8}(q^2 + 5) + ... + \frac{1}{\gamma^5}(...) + \mathcal{O}(\gamma^{-6}), \tag{4.9}$$

and letting $\gamma^2 = -\gamma^{*2}$, with $q = n + 1$ if $n - m$ is even, or $q = n$ if $n - m$ is odd, as $\gamma^* \to \infty$

$$\lambda_n^m = 2q\gamma^* - \frac{1}{2}(q^2 - m^2 + 1) + ... + \frac{1}{\gamma^{*5}}(...) + \mathcal{O}(\gamma^{*-6}). \tag{4.10}$$

Three pairs of formal expansions for solutions of the prolate spheroidal wave equations are shown to exist in [32], one pair in terms of trigonometric functions and two in terms of Hermite functions. Formal expansions for the corresponding eigenvalues are also given. Three pairs of formal expansions are also shown to exist for solutions of the oblate spheroidal wave equations in [33], one pair in terms of trigonometric functions and two in terms of associated Laguerre functions. Formal expansions for the corresponding eigenvalues are also given.

In [34] Dunster gives uniform asymptotic approximations for solutions of the prolate spheroidal wave equation, which consider a parameter case which is not relevant here, and not used much in applications. In [35], Dunster gives what he calls simplified asymptotic solutions to solutions of differential equations having two turning points, in comparison to Olver's 1975 paper [7] on two turning points. He then, which was several years ago, gave a talk in Hong Kong where he applied this paper to present results on uniform asymptotic approximations for the prolate spheroidal wave functions, this time in the parameter case which of interest in applications, and of which we study here. He has not yet at this time published these results.

In [36] and [37], Dunster gives uniform asymptotic expansions for solutions of solutions of the oblate spheroidal wave equation in terms of elementary, Airy and Bessel functions, these being valid in certain subdomains of the complex plane.

4.3 Chapter outline

We now present the analysis concerning the aforementioned problems. We give uniform approximations for the prolate spheroidal wave functions $\mathrm{Ps}_n^m\left(z,\gamma^2\right)$ as $\gamma\to\infty$ on the interval $[0,1-\epsilon]$, for some small $\epsilon>0$, in terms of parabolic cylinder U functions, using the theory developed by Olver in [7], which is outlined in section 1.4. Using this method we are also able to give rigorous approximations for the eigenvalues λ_n^m in this limit. By a judicious renaming of parameters we further give uniform asymptotic approximations for the prolate spheroidal wave functions again now in terms of the more natural parabolic cylinder D functions.

We then give uniform asymptotic expansions for the functions, which hold only in a shrinking neighbourhood for the origin as γ becomes large, but encapsulate all the interesting oscillatory behaviour. The coefficients in the expansions are polynomials and we can compute as many as we like. Simultaneously we give asymptotic expansions for the eigenvalues where again we can compute as many terms as we like, and provide order estimates for the error when the expansions are truncated.

Finally we give a discussion on the analogous problems concerning the oblate spheroidal wave functions, and discuss how the problems could be solved by bridging the gap in the related theory.

4.4 Uniform asymptotic approximations for the prolate spheroidal wave functions

We first give the main result of this section, and later give the proof after building up some machinery.

Theorem 4.1. *Let $\epsilon > 0$ and $m, n \in \mathbb{N}_0$ with $n \geq m$. Them for $z \in [0, 1-\epsilon]$ as $\gamma \to \infty$*

$$\mathrm{Ps}_n^m\left(z,\gamma^2\right) = P_n^m \left(\frac{\zeta^2 - (\alpha_n^m))^2}{(z^2-(a_n^m)^2)(1-z^2)}\right)^{1/4} \times \left(D_{n-m}\left(\sqrt{2\gamma}\zeta\right) + \epsilon_{n,1}^m(\zeta,\gamma^2) + \eta_n^m\left(\overline{D}_{n-m}\left(\sqrt{2\gamma}\zeta\right) + \epsilon_{n,2}^m(\zeta,\gamma^2)\right)\right), \tag{4.11}$$

where

$$\begin{aligned} \epsilon_{n,1}^m\left(\zeta,\gamma^2\right) &= \frac{\mathbf{M}_{n-m}\left(\sqrt{2\gamma}\zeta\right)}{\mathbf{E}_{n-m}\left(\sqrt{2\gamma}\zeta\right)}\mathcal{O}\left(\gamma^{-1}\right), \\ \epsilon_{n,2}^m\left(\zeta,\gamma^2\right) &= \mathbf{E}_{n-m}\left(\sqrt{2\gamma}\zeta\right)\mathbf{M}_{n-m}\left(\sqrt{2\gamma}\zeta\right)\mathcal{O}\left(\gamma^{-1}\right), \end{aligned} \tag{4.12}$$

with η_n^m exponentially small as $\gamma \to \infty$, the relationship between z and ζ defined by

$$\begin{aligned} \int_z^{-a_n^m}\sqrt{\frac{t^2-(a_n^m)^2}{1-t^2}}\mathrm{d}t &= \int_\zeta^{-\alpha_n^m}\sqrt{\tau^2-(\alpha_n^m)^2}\mathrm{d}\tau \quad (-1 < z \leq -a_n^m), \\ \int_{-a_n^m}^{z}\sqrt{\frac{(a_n^m)^2-t^2}{1-t^2}}\mathrm{d}t &= \int_{-\alpha_n^m}^{\zeta}\sqrt{(\alpha_n^m)^2-\tau^2}\mathrm{d}\tau \quad (-a_n^m \leq z \leq a_n^m), \\ \int_{a_n^m}^{z}\sqrt{\frac{t^2-(a_n^m)^2}{1-t^2}}\mathrm{d}t &= \int_{\alpha_n^m}^{\zeta}\sqrt{\tau^2-(\alpha_n^m)^2}\mathrm{d}\tau \quad (a_n^m \leq z < 1), \end{aligned} \tag{4.13}$$

where

$$(a_n^m)^2 = 1 + \frac{\lambda_n^m + 1}{\gamma^2}, \quad (\alpha_n^m)^2 = \frac{2}{\pi}\int_{-a_n^m}^{a_n^m}\sqrt{\frac{(a_n^m)^2-t^2}{1-t^2}}\mathrm{d}t, \tag{4.14}$$

with λ_n^m *being the eigenvalue* λ *corresponding to the prolate spheroidal wave functions, and*

$$P_n^m \sim (-1)^m \frac{a_n^m}{\alpha_n^m} \left(\frac{4\gamma}{\pi}\right)^{1/4} \sqrt{\frac{(n+m)!}{(2n+1)(n-m)!^2}}. \tag{4.15}$$

Again as $\gamma \to \infty$

$$\lambda_n^m = -\gamma^2 + (2(n-m)+1)\gamma + \mathcal{O}(1). \tag{4.16}$$

4.4.1 Approximations in terms of parabolic cylinder U functions

In (4.1), we set $\mu = m$ and denote the solution w_n^m such that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)\frac{\mathrm{d}w_n^m(z,\gamma^2)}{\mathrm{d}z}\right) + \left(\lambda_n^m + \gamma^2(1-z^2) - \frac{m^2}{1-z^2}\right) w_n^m(z,\gamma^2) = 0, \tag{4.17}$$

where λ_n^m is the eigenvalue corresponding to the prolate spheroidal wave functions. To remove the first derivative term in (4.17) we transform the dependent variable by

$$w_n^m(z,\gamma^2) = \frac{1}{\sqrt{1-z^2}}\widetilde{w}_n^m(z,\gamma^2) \tag{4.18}$$

we obtain the equation in the amenable form

$$\frac{\mathrm{d}^2\widetilde{w}_n^m(z,\gamma^2)}{\mathrm{d}z^2} = \left(\gamma^2\frac{z^2-(a_n^m)^2}{1-z^2} + \frac{m^2-z^2}{(1-z^2)^2}\right)\widetilde{w}_n^m(z,\gamma^2), \tag{4.19}$$

where

$$(a_n^m)^2 = 1 + \frac{\lambda_n^m + 1}{\gamma^2}. \tag{4.20}$$

Note that formal asymptotic expansions given in ([1] §30.9) indicate that

$$\lambda_n^m = -\gamma^2 + \mathcal{O}(\gamma^{-1}). \tag{4.21}$$

The turning points of our equation are at $z = \pm a_n^m$ and

$$\frac{z^2-(a_n^m)^2}{1-z^2} < 0 \quad (-a_n^m < z < a_n^m). \tag{4.22}$$

We deduce from (4.21) and (4.20) that $a_n^m \to 0$ as $\gamma \to \infty$, hence in this limit (4.19) has two coalescing turning points and we apply the theory of Case I in [7]. In this case uniform asymptotic approximations are in terms of the parabolic cylinder functions $U(-\frac{1}{2}\gamma\,(\alpha_n^m)^2\,, \sqrt{2\gamma}\zeta)$ and $\overline{U}(-\frac{1}{2}\gamma\,(\alpha_n^m)^2\,, \sqrt{2\gamma}\zeta)$, where α_n^m is defined in terms of a_n^m.

Following Olver, new variables relating $\{z, \widetilde{w}_n^m\}$ to $\{\zeta, W_n^m\}$ are introduced by the appropriate Liouville transformation given by

$$W_n^m(\zeta,\gamma^2) = \dot{z}^{-\frac{1}{2}}\,\widetilde{w}_n^m(z,\gamma^2), \quad \dot{z}^2\frac{z^2-(a_n^m)^2}{1-z^2} = \zeta^2 - (\alpha_n^m)^2\,, \tag{4.23}$$

the dot signifying differentiation with respect to ζ, where α_n^m is defined by the second of (4.20). From this we denote that

$$0 < a_n^m < 1 \text{ corresponds to } 0 < \alpha_n^m < \alpha_{n,*}^m, \text{ where } \alpha_{n,*}^m = \frac{2}{\sqrt{\pi}}. \tag{4.24}$$

Since $\zeta = \pm\sigma_n^m$ corresponds to $z = \pm a_n^m$, integration of the second of (4.23) yields (4.13). These equations define ζ as a real analytic function of z. There is a one-to-one correspondence between the variables z and ζ, where ζ is an increasing function of z, and we denote $\zeta = -\zeta_*, 0, \zeta_*$ to correspond to $z = -1, 0, 1$. It follows that $z(\zeta, \alpha_n^m)$ is analytic both in ζ and α_n^m for $\zeta \in [-\zeta_*, \zeta_*]$ and $\alpha_n^m \in (-\alpha_{n,*}^m, \alpha_{n,*}^m)$. Also $\dot{z}$ is non-zero in these intervals.

Performing the substitution $t = a_n^m\tau$ in the second of (4.20), we expand the integral and obtain

$$\begin{aligned}(\alpha_n^m)^2 &= \frac{4}{\pi}\,(a_n^m)^2\int_0^1\sqrt{\frac{1-\tau^2}{1-(a_n^m)^2\,\tau^2}}\mathrm{d}\tau = (a_n^m)^2\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 2\end{matrix};(a_n^m)^2\right)\\ &= (a_n^m)^2 + \frac{1}{8}\,(a_n^m)^4 + \frac{3}{64}\,(a_n^m)^6 + \mathcal{O}\left((a_n^m)^8\right) \quad (a_n^m \to 0)\end{aligned} \tag{4.25}$$

and by reversion

$$(a_n^m)^2 = (\alpha_n^m)^2 - \frac{1}{8}\,(\alpha_n^m)^4 - \frac{1}{64}\,(\alpha_n^m)^6 + \mathcal{O}\left((\alpha_n^m)^8\right) \quad (\alpha_n^m \to 0). \tag{4.26}$$

In the critical case $a_n^m = \alpha_n^m = 0$ we have from the third of (4.13)

$$\int_0^1 \frac{t}{\sqrt{1-t^2}}\mathrm{d}t = \int_0^{\zeta_*}\tau\mathrm{d}\tau, \tag{4.27}$$

which gives

$$\zeta_* = \sqrt{2}. \tag{4.28}$$

Thus we deduce that as $a_n^m, \alpha_n^m \to 0$,

$$\zeta_* \to \sqrt{2}. \tag{4.29}$$

The transformed differential equation is now of the form

$$\frac{\mathrm{d}^2 W_n^m(\zeta,\gamma^2)}{\mathrm{d}\zeta^2} = \left(\gamma^2\left(\zeta^2 - (\alpha_n^m)^2\right) + \psi_n^m\left(\zeta,\gamma^2\right)\right) W_n^m(\zeta,\gamma^2), \tag{4.30}$$

where

$$\begin{aligned}\psi_n^m\left(\zeta,\gamma^2\right) =& \frac{2\left(\alpha_n^m\right)^2 + 3\zeta^2}{4\left(\left(\alpha_n^m\right)^2 - \zeta^2\right)^2} \\ &+ \frac{\left(\alpha_n^m\right)^2 - \zeta^2}{4((a_n^m)^2 - z^2)}\left(4 + \frac{3 - 6\left(a_n^m\right)^2}{\left(a_n^m\right)^2 - z^2} + \frac{5\left(a_n^m\right)^2\left(\left(a_n^m\right)^2 - 1\right)}{(\left(a_n^m\right)^2 - z^2)^2} + \frac{4m^2 - 1}{1 - z^2}\right).\end{aligned} \tag{4.31}$$

By construction, the apparent singularities in the above function at $\zeta = \pm\alpha_n^m$, corresponding to $z = \pm a_n^m$, cancel each other out so that $\psi_n^m\left(\zeta,\gamma^2\right)$ is well-behaved there. To check this one could expand $\psi_n^m\left(\zeta,\gamma^2\right)$ around this point.

Proposition 4.1. *Let $\tilde{\epsilon} > 0$ and $m, n \in \mathbb{N}_0$ with $n \geq m$. Them for $z \in [0, 1-\tilde{\epsilon}]$ as $\gamma \to \infty$*

$$\begin{aligned} W_{n,1}^m\left(\zeta,\gamma^2\right) &= U\left(-\tfrac{1}{2}\gamma\widetilde{\alpha}^2, \sqrt{2\gamma}\zeta\right) + \epsilon_{n,1}^m\left(\zeta,\gamma^2\right), \\ W_{n,2}^m\left(\zeta,\gamma^2\right) &= \overline{U}\left(-\tfrac{1}{2}\gamma\widetilde{\alpha}^2, \sqrt{2\gamma}\zeta\right) + \epsilon_{n,2}^m\left(\zeta,\gamma^2\right), \end{aligned} \tag{4.32}$$

with

$$\begin{aligned} \epsilon_{n,1}^m\left(\zeta,\gamma^2\right) &= \frac{\mathbf{M}\left(-\frac{1}{2}\gamma\widetilde{\alpha}^2, \sqrt{2\gamma}\zeta\right)}{\mathbf{E}\left(-\frac{1}{2}\gamma\widetilde{\alpha}^2, \sqrt{2\gamma}\zeta\right)}\mathcal{O}\left(\gamma^{-1}\right), \\ \epsilon_{n,2}^m\left(\zeta,\gamma^2\right) &= \mathbf{E}\left(-\tfrac{1}{2}\gamma\widetilde{\alpha}^2, \sqrt{2\gamma}\zeta\right)\mathbf{M}\left(-\tfrac{1}{2}\gamma\widetilde{\alpha}^2, \sqrt{2\gamma}\zeta\right)\mathcal{O}\left(\gamma^{-1}\right), \end{aligned} \tag{4.33}$$

where

$$(\widetilde{\alpha}_n^m)^2 = (\alpha_n^m)^2 + \frac{(\alpha_n^m)^4 - (2m^2+1)\,(a_n^m)^2\,(\alpha_n^m)^4 - (a_n^m)^4}{2\gamma^2\,(a_n^m)^4\,(\alpha_n^m)^2}, \tag{4.34}$$

and a_n^m *and* α_n^m *are defined as in (4.20).*

Proof. For our advantage in the error analysis, we write our differential equation in the form

$$\frac{\mathrm{d}^2 W_n^m(\zeta,\gamma^2)}{\mathrm{d}\zeta^2} = \left(\gamma^2\left(\zeta^2 - (\widetilde{\sigma}_n^m)^2\right) + \widetilde{\psi}_n^m\left(\zeta,\gamma^2\right)\right) W_n^m(\zeta,\gamma^2), \tag{4.35}$$

where

$$\widetilde{\psi}_n^m\left(\zeta,\gamma^2\right) = \psi_n^m\left(\zeta,\gamma^2\right) - \psi_n^m\left(0,\gamma^2\right) = \psi_n^m\left(\zeta,\gamma^2\right) + \frac{(\alpha_n^m)^4 - (2m^2+1)\,(a_n^m)^2\,(\alpha_n^m)^4 - (a_n^m)^4}{2\,(a_n^m)^4\,(\alpha_n^m)^2}, \tag{4.36}$$

and correspondingly

$$(\widetilde{\alpha}_n^m)^2 = (\alpha_n^m)^2 + \frac{(\alpha_n^m)^4 - (2m^2+1)\,(a_n^m)^2\,(\alpha_n^m)^4 - (a_n^m)^4}{2\gamma^2\,(a_n^m)^4\,(\alpha_n^m)^2}, \tag{4.37}$$

where

$$\frac{(\alpha_n^m)^4 - 2\,(a_n^m)^2\,(\alpha_n^m)^4\,m^2 - (a_n^m)^2\,(\alpha_n^m)^4 - (a_n^m)^4}{2\,(a_n^m)^4\,(\alpha_n^m)^2} = -\left(m^2 + \frac{3}{8}\right) + \mathcal{O}((a_n^m)^2), \quad (a_n^m \to 0). \tag{4.38}$$

This gives

$$\widetilde{\psi}_n^m\left(0,\gamma^2\right) = 0. \tag{4.39}$$

These is a singularity at ζ_* in $\widetilde{\psi}_n^m\left(\zeta,\gamma^2\right)$ at $\zeta = \zeta_*$ corresponding to $z = 1$, which behaves roughly like $\mathcal{O}\left((\zeta-\zeta_*)^{-2}\right)$ as $\zeta \to \zeta_*$. Thus we omit this point from our analysis. On inspection it follows that since a_n^m, α_n^m and the variables z and ζ are all bounded as $\gamma \to \infty$, and the apparent singularities at $z = \pm a_n^m$ and $\zeta = \pm\alpha_n^m$ cancel each other out, we have for $\zeta \in [0, \zeta_* - \tilde{\epsilon}]$

$$\widetilde{\psi}_n^m \left(\zeta, \gamma^2\right) = \mathcal{O}(1), \tag{4.40}$$

uniformly in this limit. On applying Theorem I of ([7] §6), which is outlined in section 1.4, with $x = z$, $u = \gamma$, and $\zeta_2 = \zeta_* - \tilde{\epsilon}$, we obtain the solutions (4.32). In a very similar manner to the proof of Proposition 2.1 in subsection 2.4.1. We obtain the errors given in (4.33). □

Corollary 4.1. *Let $\tilde{\epsilon} > 0$ and $m, n \in \mathbb{N}_0$ with $n \geq m$. Them for $z \in [0, 1 - \widetilde{\epsilon}]$ as $\gamma \to \infty$*

$$\mathrm{Ps}_n^m \left(z, \gamma^2\right) = P_n^m \left(\frac{\zeta^2 - (\alpha_n^m))^2}{(z^2 - (a_n^m)^2)(1 - z^2)} \right)^{1/4} \left(W_{n,1}^m(\zeta, \gamma^2) + \eta_n^m W_{n,2}^m(\zeta, \gamma^2)\right), \tag{4.41}$$

where P_n^m is the normalisation constant, η_n^m is the connection folmulae constant, and $W_{n,1}^m(\zeta, \gamma^2)$ and $W_{n,2}^m(\zeta, \gamma^2)$ are as in Proposition 4.1.

Proof. These results can easily be found by transforming the solutions found in Proposition 4.1 back into the z-plane using (4.23), and the fact that any solution of a second-order differential equation can be written as a linear combination of two standard solutions of the equation. □

Lemma 4.1. *Let $m, n \in \mathbb{N}_0$ with $n \geq m$, then as $\gamma \to \infty$*

$$(\alpha_n^m)^2 = \frac{2(n - m) + 1}{\gamma} + \mathcal{O}(\gamma^{-2}). \tag{4.42}$$

The proof of this follows in an identical manner to Lemma 2.1, this we omit it here.

4.4.2 Approximations in terms of parabolic cylinder D functions

The prolate spheroidal wave functions decay exponentially on either side of the oscillatory interval in $[-1, 1]$. Again if our argument $-\frac{1}{2}\gamma\widetilde{\alpha}^2$ had been exactly a negative half-integer, which it is not, then the approximant would have exhibited this wanted exponentially decaying behaviour when the variable is both large and positive and large and negative. In this subsection we will redefine the parameters and functions in the differential equation (4.30) so that we obtain approximants with this favourable behaviour. This leads us finally to the proof of Theorem 4.1.

Proof of Theorem 4.1. With respect to (4.42) we define

$$(\omega_n^m)^2 = (\alpha_n^m)^2 - \frac{2(n-m)+1}{\gamma} = \mathcal{O}(\gamma^{-2}) \;\; (\gamma \to \infty), \tag{4.43}$$

thus it makes sense to split up (4.35) so that

$$\frac{d^2 W_n^m(\zeta,\gamma^2)}{d\zeta^2} = \left(\gamma^2\left(\zeta^2 - \frac{2(n-m)+1}{\gamma}\right) + \widehat{\psi}_n^m\left(\zeta,\gamma^2\right)\right) W_n^m(\zeta,\gamma^2), \tag{4.44}$$

where

$$\widehat{\psi}_n^m\left(\zeta,\gamma^2\right) = \psi_n^m\left(\zeta,\gamma^2\right) - \gamma^2\left(\omega_n^m\right)^2. \tag{4.45}$$

From (4.43) and since a_n^m, α_n^m and the variables z and ζ are all bounded as $\gamma \to \infty$, we have for $\zeta \in [0, \zeta_* - \tilde{\epsilon}]$

$$\widehat{\psi}_n^m\left(\zeta,\gamma^2\right) = \mathcal{O}(1), \tag{4.46}$$

in this limit uniformly. Now our approximant will have the desired property of decaying exponentially on either side of the oscillatory interval for large positive and large negative ζ. We obtain the solutions for (4.44)

$$W_{n,1}^m\left(\zeta,\gamma^2\right) = D_{n-m}\left(\sqrt{2\gamma}\zeta\right) + \epsilon_{n,1}^m\left(\zeta,\gamma^2\right), \quad W_{n,2}^m\left(\zeta,\gamma^2\right) = \overline{D}_{n-m}\left(\sqrt{2\gamma}\zeta\right) + \epsilon_{n,2}^m\left(\zeta,\gamma^2\right), \tag{4.47}$$

valid when $\zeta \in [0, \widetilde{\zeta}_* - \tilde{\epsilon}]$. We have that

$$\begin{aligned}\widehat{\psi}_n^m\left(0,\gamma^2\right) &= -\gamma^2\left(\omega_n^m\right)^2 + \psi_n^m\left(0,\gamma^2\right) \\ &= -\gamma^2\left(\alpha_n^m\right)^2 + (2(n-m)+1)\gamma + \frac{\left(a_n^m\right)^4 + \left(2m^2+1\right)\left(a_n^m\right)^2\left(\alpha_n^m\right)^4 - \left(\alpha_n^m\right)^4}{2\left(\alpha_n^m\right)^2\left(a_n^m\right)^4}.\end{aligned} \tag{4.48}$$

In Theorem 4.2 we will show that

$$(\alpha_n^m)^2 = \frac{2(n-m)+1}{\gamma} + \frac{8m^2+3}{8\gamma^2} + \mathcal{O}(\gamma^{-3}), \tag{4.49}$$

and combining this with (4.26) we obtain

$$\widehat{\psi}_n^m\left(0,\gamma^2\right) = \mathcal{O}(\gamma^{-1}) \quad (\gamma \to \infty). \tag{4.50}$$

Thus performing the error analysis similarly as in section 2.4 of chapter 2 we get expressions for the errors as in (4.12).

Thus from (4.47), (4.23) and (4.18) we obtain (4.11).

Similarly as in the proof of Theorem 2.1, considering that

$$\mathrm{Ps}_n^m\left(z,\gamma^2\right) \sim P_n^m \left(\frac{\zeta^2 - (\alpha_n^m))^2}{(z^2-(a_n^m)^2)(1-z^2)}\right)^{1/4} D_{n-m}\left(\sqrt{2\gamma}\zeta\right) \quad (\gamma \to \infty) \tag{4.51}$$

with respect to (4.5) we consider the integral

$$(P_n^m)^2 \int_{-1}^{1} \sqrt{\frac{\zeta^2-(\alpha_n^m)^2}{(z^2-(a_n^m)^2)(1-z^2)}} D_{n-m}^2\left(\sqrt{2\gamma}\zeta\right) \mathrm{d}z. \tag{4.52}$$

and perform the transformation from z to ζ to obtain

$$(P_n^m)^2 \int_{-\zeta_*}^{\zeta^*} \frac{\zeta^2-(\alpha_n^m)^2}{z^2-(a_n^m)^2} D_{n-m}^2\left(\sqrt{2\gamma}\zeta\right) \mathrm{d}\zeta. \tag{4.53}$$

Approximating the integral about the origin where the oscillatory behaviour occurs, we deduce from (4.5) that as $\gamma \to \infty$ we have (4.15). □

4.5 Uniform asymptotic expansions of the prolate spheroidal wave functions

We have given one-term uniform asymptotic approximations for the prolate spheroidal wave functions when γ became large. A second term in an asymptotic expansion was not computed due to the complicated nature of the transformation between the variables z and ζ. In this section we employ a simpler transformation than in the previous subsection, just like in the previous chapters, so that we can construct formal asymptotic expansions for the prolate spheroidal wave functions and their corresponding eigenvalues.

Theorem 4.2. *Let $m, n \in \mathbb{N}_0$, with $n \geq m$. Then as $\gamma \to \infty$*

$$\lambda_n^m = -\gamma^2 + (2(n-m)+1)\gamma + 2\sum_{s=0}^{r-1} \frac{\mu_{s+1}}{\gamma^s} + \mathcal{O}(\gamma^{-r}), \ \text{as } \gamma \to \infty. \tag{4.54}$$

The μ_s terms are constant coefficients which depend on m and n, found recursively with the eigenfunction expansions. We give here the first two terms:

$$\begin{aligned} \mu_1 &= -\frac{1}{2^4}\left(8m^2 - (2(n-m)+1)^2 - 5\right), \\ \mu_2 &= -\frac{1}{2^7}\left((2(n-m)+1)^3 + 11(2(n-m)+1) - 32m^2(2(n-m)+1)\right). \end{aligned} \tag{4.55}$$

Letting $t = \sqrt{2\gamma}z$, for $z = \mathcal{O}(\gamma^{-1/2})$ as $\gamma \to \infty$

$$\mathrm{Ps}_n^m(z,\gamma^2) = P_n^m \left(D_{n-m}(t) \sum_{s=0}^{r} \frac{A_s(t)}{\gamma^s} + D'_{n-m}(t) \sum_{s=0}^{r} \frac{B_s(t)}{\gamma^s} + \mathcal{O}(\gamma^{-r-1}) \right), \tag{4.56}$$

valid for $r \in \mathbb{N}_0$, where

$$P_n^m \sim (-1)^m \frac{a_n^m}{\alpha_n^m} \left(\frac{4\gamma}{\pi}\right)^{1/4} \sqrt{\frac{(n+m)!}{(2n+1)(n-m)!^2}} \left(1 + \frac{\eta_1}{\gamma} + \frac{\eta_2}{\gamma^2} + \ldots\right)^{-\frac{1}{2}}. \tag{4.57}$$

The $A_s(t)$ terms and are even polynomials, and the $B_s(t)$ terms are odd polynomials. These polynomials are found recursively and we give here the first two terms:

$$A_0 = 1, \quad A_1 = \frac{5}{32}t^2, \quad B_0 = 0, \quad B_1 = -\frac{1}{8}\left((n-m) + \frac{1}{2}\right)t + \frac{1}{2^4}t^3 \tag{4.58}$$

4.5.1 Formal uniform asymptotic expansions of the prolate spheroidal wave functions and eigenvalues

The oscillatory behaviour of the prolate spheroidal wave functions happens in a shrinking neighbourhood of the origin as $\gamma \to \infty$, and it can be shown that around the origin the ζ variable used in the uniform approximation given in the last section behaves approximately z. Thus the variable in the parabolic cylinder functions behaves approximately like $\sqrt{2\gamma}z$. This motivates the next simpler transformation.

Letting $t = \sqrt{2\gamma}z$ and $\mu = m$ in (4.17) we have

$$\frac{\mathrm{d}^2 w_n^m(t,\gamma^2)}{\mathrm{d}t^2} + \left(\frac{\lambda+\gamma^2}{2\gamma} - \frac{t^2}{4}\right) w_n^m(t,\gamma^2) - \frac{1}{\gamma}\left(\frac{t^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + t\frac{\mathrm{d}}{\mathrm{d}t} + \frac{m^2}{2\left(1-\frac{t^2}{2\gamma}\right)}\right) w_n^m(t,\gamma^2) = 0, \tag{4.59}$$

where $z \in [-1,1]$ corresponds to $t \in [-\sqrt{2\gamma}, \sqrt{2\gamma}]$. We suppose in accordance with (4.16) that

$$\frac{\lambda_n^m + \gamma^2}{2\gamma} = n - m + \tfrac{1}{2} + \sum_{s=1}^{r} \frac{\mu_s}{\gamma^s} + \frac{\widetilde{\mu}_{r+1}}{\gamma^{r+1}}, \tag{4.60}$$

where r is a positive integer and $\widetilde{\mu}_r$ can be re-expanded in a sensible manner. We can then write the differential equation in the form

$$\frac{\mathrm{d}^2 w_n^m(t,\gamma^2)}{\mathrm{d}t^2} + \left(n - m + \frac{1}{2} - \frac{t^2}{4}\right) w_n^m(t,\gamma^2) - \frac{1}{\gamma}\left(\frac{t^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + t\frac{\mathrm{d}}{\mathrm{d}t} + \frac{m^2}{2\left(1-\frac{t^2}{2\gamma}\right)} - \sum_{s=0}^{r-1}\frac{\mu_{s+1}}{\gamma^s} - \frac{\widetilde{\mu}_{r+1}}{\gamma^r}\right) w_n^m(t,\gamma^2) = 0. \tag{4.61}$$

This equation is split in such a way that constructing a formal asymptotic expansion in terms of parabolic cylinder functions $D_{n-m}(t)$ in the form

$$w_n^m(t,\gamma^2) = D_{n-m}(t)\sum_{s=0}^{\infty}\frac{A_s(t)}{\gamma^s} + D'_{n-m}(t)\sum_{s=0}^{\infty}\frac{B_s(t)}{\gamma^s} \tag{4.62}$$

seems sensible. However one should observe that this splitting only makes sense when $t = \mathcal{O}(1)$ as $\gamma \to \infty$, hence this formal expansion is only sensible for this range of t. We denote this range as $[-t_*, t_*]$. We seek solutions which are either even or odd respective to the parity of $n-m$. Since $D_{n-m}(t)$ is either even or odd respective to when $n-m$ is either even or odd, we deduce that $A_s(t)$ and $B_s(t)$ must be even and odd respectively.

Substituting (4.62) into (4.61) and equating powers of γ, we have the recurrence relations for $A_s(t)$ and $B_s(t)$

$$2A_s'(t) + B_s''(t) - \frac{t^2}{2}\left(2A_{s-1}'(t) + B_{s-1}''(t) + \tilde{t}(m)B_{s-1}(t)\right) - t\left(A_{s-1}(t) + B_{s-1}'(t)\right) - \frac{m^2}{2}\sum_{j=0}^{s}\frac{t^{2j}}{2^j}B_{s-j-1}(t) + \sum_{j=1}^{s}\mu_j B_{s-j}(t) = 0, \tag{4.63}$$

$$A_s''(t) + 2\tilde{t}(m)B_s'(t) + \frac{t}{2}B_s(t) - \frac{t^2}{2}\left(A_{s-1}''(t) + \frac{t}{2}B_{s-1}(t) + \tilde{t}(m)\left(A_{s-1}(t) + 2B_{s-1}'(t)\right)\right) - t\left(A_{s-1}'(t) + \tilde{t}(m)B_{s-1}(t)\right) - \frac{m^2}{2}\sum_{j=0}^{s}\frac{t^{2j}}{2^j}A_{s-j-1}(t) + \sum_{j=1}^{s}\mu_j A_{s-j}(t) = 0, \tag{4.64}$$

where $\tilde{t}(m) = \frac{1}{4}t^2 - (n-m) - \frac{1}{2}$. Neither of these relations separately determine solutions for $A_s(t)$ or $B_s(t)$ from previous coefficients, thus we differentiate the first to obtain an expression for $A_s''(t)$ and substitute it into the second; this gives the third order inhomogeneous differential equation for $B_s(t)$

$$B_s'''(t) - \left(t^2 - 4(n-m) - 2\right)B_s'(t) - tB_s(t) - b_s(t) = 0, \tag{4.65}$$

where

$$b_s(t) = \sum_{j=1}^{s}\mu_j\left(2A_{s-j}(t) - B_{s-j}'(t)\right) + \frac{m^2}{2}\sum_{j=0}^{s} j\frac{t^{2j-1}}{2^{j-1}}B_{s-j-1}(t) + \frac{t^{2j}}{2^j}\left(B_{s-j-1}'(t) - 2A_{s-j-1}(t)\right) + A_{s-1}(t) + B_{s-1}'(t) + t\left(A_{s-1}'(t) + 2B_{s-1}'(t) - \tilde{t}(m)B_{s-1}(t)\right) + t^2\left(\frac{1}{2}B_{s-1}'''(t) - \frac{3}{2}\tilde{t}(m)B_{s-1}'(t) - \frac{1}{4}tB_{s-1}(t) - \tilde{t}(m)A_{s-1}(t)\right). \tag{4.66}$$

Once $B_s(t)$ is determined, we can use (4.63) to determine $A_s(t)$. There will be freedom in choosing the integration constants in the $A_s(t)$ terms, with identification of our solutions made unique by their normalisation.

General coefficients $B_s(t)$ and $A_s(t)$

As in chapter 2, using variation of parameters we obtain the general solution for $B_s(t)$

$$B_s(t) = \frac{\pi}{4(n-m)!^2} \int_0^t b_s(\tau) \left(\overline{D}_{n-m}(\tau) D_{n-m}(t) - D_{n-m}(\tau) \overline{D}_{n-m}(t)\right)^2 \mathrm{d}\tau + c D_{n-m}(t) \overline{D}_{n-m}(t), \tag{4.67}$$

where c is some constant. In an identical manner to the Lamé case by checking the $s = 1$ case, one would see that it would appear for all $s \geq 1$ that if the previous $A_s(t)$ and $B_s(t)$ terms are all polynomials, then the $A_s(t)$ and $B_s(t)$ terms can be represented as polynomials, by the correct choice of the constant c.

Polynomial coefficients $B_s(t)$ and $A_s(t)$

To obtain explicit expressions for $A_s(t)$ and $B_s(t)$ we try substituting in polynomial expansions with undetermined coefficients. Take $A_0(t) = 1$ and $B_0(t) = 0$, then for $s \geq 1$ we consider $A_s(t)$ and $B_s(t)$ in the form

$$A_s(t) = \sum_{i=1}^{\infty} a_{s,i} t^{2i} \quad \text{and} \quad B_s(t) = \sum_{i=0}^{\infty} b_{s,i} t^{2i+1}, \tag{4.68}$$

then one can easily write down recurrence relations for these $a_{s,i}$ and $b_{s,i}$ terms, and obtain more detailed information about the polynomials such as their orders corresponding to s and their relation to the μ_s terms in the eigenvalue expansion.

Proof of Theorem 4.2. We now apply the theory used in the analagous problem relating to the Lamé functions to the prolate spheroidal wave equation to obtain order estimates upon truncation of asymptotic expansions of the eigenvalues λ_n^m.

The normalised eigenfunctions $w_n^m(z, \gamma^2)$ corresponding to eigenvalues λ_n^m form an orthonormal basis such that

$$(w_i^m, w_j^m) = \int_{-1}^{1} w_i^m(z, \gamma^2) w_j^m(z, \gamma^2) \mathrm{d}z = \delta_{ij} \tag{4.69}$$

in the Hilbert space $L^2\left[[-1,1], \mathrm{d}z\right]$. Consider Pr_n^m to be the self-adjoint differential operator

$$Pr_n^m := (1-z^2)\frac{d^2}{dz^2} - 2z\frac{d}{dz} + \gamma^2(1-z^2) - \frac{m^2}{1-z^2} \tag{4.70}$$

defined on a subspace A of $L^2\left[[-1,1],dz\right]$ containing all the twice differentiable functions which are bounded at the endpoints $z=\pm 1$. Then

$$\left(Pr_n^m + \lambda_n^m\right) w_n^m\left(z,\gamma^2\right) = 0. \tag{4.71}$$

The Theorem 2.3 gives that if there exists some $\widetilde{w}(z) \in A$ such that $(\widetilde{w},\widetilde{w}) = 1$, define a remainder function $R(z)$ such that

$$(\mathrm{Ps}_n^m + \lambda)\widetilde{w}(z) = R(z) \tag{4.72}$$

for some constant λ. If $R(z) \in L^2\left[[-1,1],dz\right]$ then

$$\min_k |\lambda - \lambda_k| < \sqrt{(R,R)}. \tag{4.73}$$

Letting $t = \sqrt{2\gamma}z$, we define the truncated expansions corresponding to $w_n^m\left(z,\gamma^2\right)$ as

$$w_{n,r}^m(z,\gamma^2) = c_{n,r}^m\left(D_{n-m}(t)\sum_{s=0}^{r}\frac{A_s(t)}{\gamma^s} + D'_{n-m}(t)\sum_{s=0}^{r}\frac{B_s(t)}{\gamma^s}\right), \tag{4.74}$$

where the $A_s(t)$ and $B_s(t)$ terms were derived previously, and $c_{n,r}^m$ is defined to be a function of γ so that

$$\int_{-1}^{1}\left(w_{n,r}^m(z,\gamma^2)\right)^2 \mathrm{d}z = 1. \tag{4.75}$$

Then we can write the derivative of $w_{n,r}^m(z,\gamma^2)$ with respect to z as

$$\frac{\mathrm{d}w_{n,r}^m(z,\gamma^2)}{\mathrm{d}z} = c_{n,r}^m\sqrt{2\gamma}\left(D_{n-m}(t)\sum_{s=0}^{r}\frac{A'_s(t) + \left(\frac{t^2}{4} - (n-m) - \frac{1}{2}\right)B_s(t)}{\gamma^s}\right.$$
$$\left. + D'_m(t)\sum_{s=0}^{r}\frac{A_s(t) + B'_s(t)}{\gamma^s}\right) \tag{4.76}$$

where the dash represents differentiation with respect to t. Thus clearly $w^m_{n,r}(z,\gamma^2) \in A$ since it is bounded at both $z=-1$ and $z=1$. We also define the truncated eigenvalue expansions

$$\lambda^m_{n,r} = -\gamma^2 + (2(n-m)+1)\gamma + 2\gamma \sum_{s=1}^{r} \frac{\mu_s}{\gamma^s}, \tag{4.77}$$

where the μ_s are those derived previously, and define $R^m_{n,r}(z,\gamma^2)$ such that

$$\left(Pr^m_n + \lambda^m_{n,r}\right) w^m_{n,r}(z,\gamma^2) = R^m_{n,r}(z,\gamma^2). \tag{4.78}$$

We consider the operator Pr^m_n acting on $D_{n-m}(t)$ and derive

$$Pr^m_n\left(D_{n-m}(t)\right) = \left(\gamma^2 - \left(2(n-m)+\frac{1}{2}\right)\gamma - \frac{m^2}{1-\frac{t^2}{2\gamma}}\right) D_{n-m}(t) - 2tD'_{n-m}(t)\,. \tag{4.79}$$

Using the recurrence relations given in (1.63) we can rewrite $w^m_{n,r}(z,\gamma^2)$ given in (4.74) such that for $s \in \{0,...,r\}$, $A_s(t)\,D_{n-m}(t)$ and $B_s(t)\,D'_{n-m}(t)$ are sums of varying orders of parabolic cylinder functions with constant coefficients dependent only on m and n; it then follows from (4.79) that with the solution rewritten in this form, we have

$$R^m_{n,r}(z,\gamma^2) = c^m_{n,r}\gamma^{-n}\left[(...) + \gamma^{-1}(...) + ...\right], \tag{4.80}$$

where the terms inside the brackets are sums of varying orders of parabolic cylinder functions with constant coefficients dependent only on n and m. Note that this remainder term will be a finite sum and clearly $R^m_{n,r}(z,\gamma^2) \in L^2\left[[-1,1],dz\right]$. As $w^m_{n,r}(z,\gamma^2) \in A$ we have that

$$\min_k \left|\lambda^m_{n,r} - \lambda^m_k\right|^2 < \sqrt{\left(R^m_{n,r}, R^m_{n,r}\right)}. \tag{4.81}$$

In section 4.4 we proved that

$$\lambda^m_n = -\gamma^2 + (2(n-m)+1)\gamma + \mathcal{O}(1) \quad \text{as} \quad \gamma \to \infty, \tag{4.82}$$

hence for γ large enough, necessarily

$$\min_k \left|\lambda_{n,r}^m - \lambda_k^m\right|^2 = \left|\lambda_{n,r}^m - \lambda_n^m\right|^2 \tag{4.83}$$

and thus

$$|\lambda_{n,r}^m - \lambda_n^m| < \sqrt{\left(R_{n,r}^m, R_{n,r}^m\right)}. \tag{4.84}$$

It follows in an identical manner to the Lamé case that

$$\int_{-1}^{1} \left(R_{n,r}^m(z)\right)^2 \mathrm{d}z = \mathcal{O}\left(\gamma^{-2r}\right), \tag{4.85}$$

and so

$$\lambda_n^m - \lambda_{n,r}^m = \mathcal{O}\left(\gamma^{-r}\right), \tag{4.86}$$

as $\gamma \to \infty$. Hence from (4.60) in Theorem 4.2 we have (4.54).

We now obtain strict and realistic error bounds for the functions expansions. Define the differential operator

$$Pr_n^m := \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\lambda_n^m + \gamma^2}{2\gamma} - \frac{t^2}{4} - \frac{1}{\gamma}\left(\frac{t^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + t\frac{\mathrm{d}}{\mathrm{d}t} + \frac{m^2}{2(1-\frac{t^2}{2\gamma})}\right), \tag{4.87}$$

and consider the truncated expansion

$$w_{n,r}^m(t,\gamma^2) = D_{n-m}\left(t\right)\sum_{s=0}^{r}\frac{A_s\left(t\right)}{\gamma^s} + D'_{n-m}\left(t\right)\sum_{s=1}^{r}\frac{B_s\left(t\right)}{\gamma^s}, \tag{4.88}$$

such that we have the exact solution

$$w_n^m(t,\gamma^2) = w_{n,r}^m(t,\gamma^2) + \epsilon_{n,r}^m(t,\gamma^2). \tag{4.89}$$

We define the remainder term $R_{n,r}^m(t,\gamma^2)$ such that

$$Pr_n^m\left(w_{n,r}^m(t,\gamma^2)\right) = R_{n,r}^m(t,\gamma^2) \tag{4.90}$$

and split the eigenvalue such that

$$\frac{\lambda_n^m+\gamma^2}{2\gamma}=n-m+\tfrac{1}{2}+\sum_{s=1}^{r}\frac{\mu_s}{\gamma^s}+\frac{\widetilde{\mu}_{r+1}}{\gamma^{r+1}}, \tag{4.91}$$

and note we just proved that $\widetilde{\mu}_{n+1}=\mathcal{O}(1)$ as $\gamma\to\infty$. By construction of the coefficients $A_s(t)$ and $B_s(t)$ it follows that

$$R_{n,r}^m(t,\gamma^2)=\mathcal{O}(\gamma^{-r-1}), \tag{4.92}$$

as $\gamma\to\infty$. Applying Pr_n^m to (4.89) we obtain

$$\begin{aligned}\left(\epsilon_{n,r}^m\right)''+\left(n-m+\frac{1}{2}-\frac{t^2}{4}\right)\epsilon_{n,r}^m=\\ \frac{1}{(1-\frac{t^2}{2\gamma})}\left(\frac{t}{\gamma}(\epsilon_{n,t}^m)'+\left(\frac{t^2}{2\gamma}(n-m+\frac{1}{2}-\frac{t^2}{4})+\frac{m^2}{2\gamma(1-\frac{t^2}{2\gamma})}-\sum_{s=1}^{r}\frac{\mu_s}{\gamma^s}-\frac{\widetilde{\mu}_{r+1}}{\gamma^{r+1}}\right)\epsilon_{n,r}^m\right.\\ \left.-R_{n,r}^m(t,\gamma^2)\right),\end{aligned} \tag{4.93}$$

and denoting the right hand side of this equation as $\Omega_{n,r}^m(t,\gamma^2)$, by use of variation of parameters we have

$$\epsilon_{n,r}^m(t,\gamma^2)=\frac{\sqrt{\pi/2}}{(n-m)!}\int_t^{t_*}\left[D_{n-m}(t)\overline{D}_{n-m}(\tau)-D_{n-m}(\tau)\overline{D}_{n-m}(t)\right]\Omega_{n,r}^m(\tau,\gamma^2)\mathrm{d}\tau. \tag{4.94}$$

We define $J(\tau)=1$ and

$$\begin{aligned}K(t,\tau)&=\frac{\sqrt{\pi/2}}{(n-m)!}\left(D_{n-m}(t)\,\overline{D}_{n-m}(\tau)-D_{n-m}(\tau)\,\overline{D}_{n-m}(t)\right),\\ \phi(\tau)&=\frac{-R_{n,r}^m(\tau,\gamma^2)}{1-\frac{\tau^2}{2\gamma}},\\ \psi_0(\tau)&=\frac{1}{(1-\frac{\tau^2}{2\gamma})}\left(\frac{\tau^2}{2\gamma}(n-m+\frac{1}{2}-\frac{\tau^2}{4})+\frac{m^2}{2\gamma(1-\frac{\tau^2}{2\gamma})}-\sum_{s=1}^{r}\frac{\mu_s}{\gamma^s}-\frac{\widetilde{\mu}_{r+1}}{\gamma^{r+1}}\right),\\ \psi_1(\tau)&=\frac{\tau}{\gamma(1-\frac{\tau^2}{2\gamma})},\end{aligned} \tag{4.95}$$

$$\Phi\left(t\right) = \int_{t}^{t_*} \left|\phi(\tau)\,\mathrm{d}\tau\right|, \qquad \Psi_0\left(t\right) = \int_{t}^{t_*} \left|\psi_0(\tau)\,\mathrm{d}\tau\right|, \qquad \Psi_1\left(t\right) = \int_{t}^{t_*} \left|\psi_1(\tau)\,\mathrm{d}\tau\right|. \tag{4.96}$$

Again since we consider $t \in [-t_*, t_*]$ where $t_* = \mathcal{O}(1)$ as $\gamma \to \infty$, the error analysis is much simpler than the analysis Olver uses in [7] as we have

$$\left|K\left(t,\tau\right)\right| \le k_0, \text{ and } \left|\partial K\left(t,\tau\right)/\partial t\right| \le k_1, \tag{4.97}$$

where k_0 and k_1 are $\mathcal{O}(1)$ as $\gamma \to \infty$, whereas Olver's bounds are in terms of parabolic cylinder functions. Then we define

$$P_0\left(t\right) = k_0, \qquad Q\left(\tau\right) = 1, \qquad P_1\left(t\right) = k_1, \tag{4.98}$$

(we do not define $P_2(t)$ as we do not need to bound $\left|\partial^2 K\left(t,\tau\right)/\partial t^2\right|$ to carry out our analysis), and finally the constants

$$\widetilde{\kappa} = 1, \quad \widetilde{\kappa}_0 = k_0, \quad \widetilde{\kappa}_1 = k_1. \tag{4.99}$$

It follows from Theorem 10.1 in ([2] §6.10.2) detailed in section 1.4 that we have the bound

$$\left|\epsilon_{n,r}^{m}(t,\gamma^2)\right| \le P_0\left(t\right)\widetilde{\kappa}\;\Phi\left(t\right)\exp\left[\widetilde{\gamma}_0\;\Psi_0\left(t\right) + \widetilde{\gamma}_1\;\Psi_1\left(t\right)\right]. \tag{4.100}$$

Substituting in expressions from (4.99) and the first of (4.98) we have

$$\epsilon_{n,r}^{m}(t,\gamma^2) = \mathcal{O}\left(\gamma^{-r-1}\right) \quad \text{as } \gamma \to \infty, \quad \text{for } \; t \in [-t_*, t_*]. \tag{4.101}$$

Now we identify the derived expansions with the prolate spheroidal wave functions. We give the identification

$$\begin{aligned}\mathrm{Ps}_n^m(z,\gamma^2) = P_n^m \Bigg(& D_{n-m}\left(t\right)\sum_{s=0}^{n}\frac{A_s\left(t\right)}{\gamma^s} + D'_{n-m}\left(t\right)\sum_{s=0}^{n}\frac{B_s\left(t\right)}{\gamma^s} \\ & + \tfrac{1}{2}\left(\epsilon_{n,r}^{m}(t,\gamma^2) + (-1)^{n-m}\epsilon_{n,r}^{m}(-t,\gamma^2)\right)\Bigg),\end{aligned} \tag{4.102}$$

where P_n^m is the normalisation constant. To obtain an asymptotic expansion for this constant we substitute (4.102) into the first of (4.5) and consider the integral

$$(P_n^m)^2 \int_{-1}^{1} \left(D_{n-m}(t) \sum_{s=0}^{\infty} \frac{A_s(t)}{\gamma^s} + D'_{n-m}(t) \sum_{s=0}^{\infty} \frac{B_s(t)}{\gamma^s} \right)^2 \mathrm{d}z. \tag{4.103}$$

In the integral we let $t = \sqrt{2\gamma}z$ and obtain

$$\frac{(P_n^m)^2}{\sqrt{2\gamma}} \int_{-\sqrt{2\gamma}}^{\sqrt{2\gamma}} \left(D_m(t) \sum_{s=0}^{\infty} \frac{A_s(t)}{\gamma^s} + D'_m(t) \sum_{s=0}^{\infty} \frac{B_s(t)}{\gamma^s} \right)^2 \mathrm{d}t. \tag{4.104}$$

Since the parabolic cylinder functions are exponentially small with their variable is large, we consider the integral from $-\infty$ to ∞ and express the integral in the form

$$\frac{(C_n^m)^2}{\sqrt{2\gamma}} \int_{-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{\sum_{j=0}^{s} A_j A_{s-j} D_{n-m}^2(t) + A_j B_{s-j} D_{n-m}(t) D'_{n-m}(t) + B_j B_{s-j} \left(D'_{n-m}(t)\right)^2}{\gamma^s} \mathrm{d}t \tag{4.105}$$

Then from (4.5) we have the formal asymptotic expansions for the normalisation constant

$$P_n^m \sim (-1)^m \frac{a_n^m}{\alpha_n^m} \left(\frac{4\gamma}{\pi}\right)^{1/4} \sqrt{\frac{(n+m)!}{(2n+1)(n-m)!^2}} \left(1 + \frac{\eta_1}{\gamma} + \frac{\eta_2}{\gamma^2} + ...\right)^{-\frac{1}{2}}. \tag{4.106}$$

□

4.6 Discussion outline on further work

We now present a discussion on the aforementioned problems in the introduction concerning the oblate spheroidal wave functions. We set up the differential equation in such a way as to obtain uniform approximations for the oblate spheroidal wave functions $\mathrm{Ps}_n^m\left(z,\gamma^2\right)$ as $\gamma^2 \to -\infty$ in terms of Whittaker functions. To obtain the same kind of results that are contained in chapter previous chapters and the previous section, one would need to develop rigorous and realistic error bounds for asymptotic approximations in this case which are similar to those given in [2]. As this has not yet been developed satisfactorily, we only discuss the possible results that could be obtained for the functions and the eigenvalues, and how one would obtain approximations for the functions in terms of the more natural associated Laguerre polynomials.

We then give formal uniform asymptotic expansions for the functions, which would hold only in a shrinking neighbourhood containing the oscillatory behaviour of the functions. The coefficients in the formal expansions are polynomials and we can compute as many as we like. Simultaneously we give asymptotic expansions for the eigenvalues where again we can compute as many terms as we like. If one could overcome the issue or providing the error analysis discussed above, these results would likely become rigorous.

4.6.1 Uniform asymptotic approximations of the oblate spheroidal wave functions

In (4.1) we let $\gamma = \mathrm{i}\Gamma$, and define w_n^m so that we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\left((1-z^2)\frac{\mathrm{d}w_n^m(z,\gamma^2)}{\mathrm{d}z}\right)+\left(\lambda_n^m-\Gamma^2(1-z^2)-\frac{\mu^2}{1-z^2}\right)w_n^m(z,\gamma^2)=0, \tag{4.107}$$

where λ_n^m is the special eigenvalue corresponding to the oblate spheroidal wave functions. Now $\gamma^2 \to -\infty$ corresponds to $\Gamma \to \infty$. To eliminate the first derivative term, and transform (4.107) into a suitable form, we let

$$w_n^m(z,\gamma^2)=\frac{1}{\sqrt{1-z^2}}\tilde{w}_n^m(z,\gamma^2), \quad x=1-z, \tag{4.108}$$

so that we have

$$\frac{\mathrm{d}^2\tilde{w}_n^m(z,\gamma^2)}{\mathrm{d}x^2} = \left(\Gamma^2\frac{x(2-x)-a}{x(2-x)} + \frac{m^2-(1-x)^2}{(x(2-x))^2}\right)\tilde{w}_n^m(z,\gamma^2), \tag{4.109}$$

where

$$a_n^m = \frac{\lambda_n^m+1}{\Gamma^2}, \tag{4.110}$$

and $z \in [-1,1]$ corresponds to $x \in [0,2]$. Turning points of this differential equation lie at $x = 1 \pm \sqrt{1-a_n^m}$, and formal analysis in §30.9 [1] indicates that $a_n^m \to 0$ as $\Gamma \to \infty$, hence these turning points will coalesce into the origin and into the point at $x = 2$ in this limit. Since there are singularities at these points, we are looking at the special behaviour of a singularity and a turning point coalescing. For the following sections we restrict x to the interval $[0,1]$, since we only want to consider the turning point coalescing with the singularity at $x = 0$.

Note: To otherwise consider the z interval $[-1,0]$, in (4.107) one would have instead let $x = 1+z$, and similar analysis would follow for again $x \in [0,1]$.

New variables relating $\{x, \tilde{w}_n^m\}$ to $\{\zeta, W_n^m\}$ are introduced by the appropriate Liouville transformation given by

$$W_n^m(\zeta,\gamma^2) = \dot{x}^{-\frac{1}{2}}\tilde{w}_n^m(z,\gamma^2), \quad \dot{x}^2\frac{x(2-x)-_n^m a}{x(2-x)} = \frac{\zeta-\alpha_n^m}{\zeta}, \tag{4.111}$$

the dot signifying differentiation with respect to ζ, where letting $b_n^m = 1-\sqrt{1-a_n^m}$, α is defined as

$$\alpha_n^m = \frac{2}{\pi}\int_0^{b_n^m}\left\{\frac{a_n^m-\xi(2-\xi)}{\xi(2-\xi)}\right\}^{\frac{1}{2}}\mathrm{d}\xi. \tag{4.112}$$

From this we note that

$$0 < b_n^m < 1 \ \text{ corresponds to } \ 0 < \alpha_n^m < \alpha_{n,*}^m, \quad \text{where} \quad \alpha_{n,*}^m = \frac{2}{\pi}. \tag{4.113}$$

Since b_n^m corresponds to α_n^m, integration of the second of (4.111) yields

$$\begin{aligned} &(0 \leq x \leq b_n^m) \quad \int_x^{b_n^m} \left\{ \frac{a_n^m - \xi(2-\xi)}{\xi(2-\xi)} \right\}^{\frac{1}{2}} \mathrm{d}\xi = \int_\zeta^{\alpha_n^m} \left\{ \frac{\alpha_n^m - \tau}{\tau} \right\}^{\frac{1}{2}} \mathrm{d}\tau, \\ &(b_n^m \leq x \leq 1) \quad \int_{b_n^m}^{x} \left\{ \frac{\xi(2-\xi) - a_n^m}{\xi(2-\xi)} \right\}^{\frac{1}{2}} \mathrm{d}\xi = \int_{\alpha_n^m}^{\zeta} \left\{ \frac{\tau - \alpha_n^m}{\tau} \right\}^{\frac{1}{2}} \mathrm{d}\tau. \end{aligned} \tag{4.114}$$

These equations define ζ as a real analytic function of x. There is a one-to-one correspondence between the variables x and ζ, where ζ is an increasing function of x, and we denote $\zeta = 0, \zeta_*$ to correspond to $x = 0, 1$. Also $\dot{x}$ is non-zero in these intervals.

In the critical case $a = \alpha = 0$ we have from (4.114)

$$1 = \int_0^1 d\xi = \int_0^{\zeta_*} d\tau = \zeta_*. \tag{4.115}$$

Thus we deduce that as $a,\ \alpha \to 0$,

$$\zeta_* \to 1. \tag{4.116}$$

The transformed differential equation is now of the form

$$\frac{\mathrm{d}^2 W_n^m(\zeta,\gamma^2)}{\mathrm{d}z^2} = \left(\Gamma^2 \frac{\zeta - \alpha}{\zeta} + \frac{1}{4}\frac{m^2-1}{\zeta^2} + \frac{\psi_n^m(\zeta,\gamma^2)}{\zeta} \right) W_n^m(\zeta,\gamma^2), \tag{4.117}$$

where

$$\psi_n^m(\zeta,\gamma^2) = \zeta \dot{x}^2 \frac{m^2 - (1-x)^2}{(x(2-x))^2} - \frac{1}{4}\frac{m^2-1}{\zeta} + \zeta \dot{x}^{\frac{1}{2}} \frac{d^2(\dot{x}^{-\frac{1}{2}})}{d\zeta^2}. \tag{4.118}$$

Then solutions will be of the form

$$\begin{aligned} W_1(\zeta,\gamma^2) &= M_{\frac{\alpha\Gamma}{2},\frac{m}{2}}(2\Gamma\zeta) + \epsilon_{n,1}^m(\zeta,\gamma^2), \\ W_2(\zeta,\gamma^2) &= W_{\frac{\alpha\Gamma}{2},\frac{m}{2}}(2\Gamma\zeta) + \epsilon_{n,2}^m(\zeta,\gamma^2). \end{aligned} \tag{4.119}$$

In [38] error analysis is provided for equations of this type, however our case corresponds to their case (I), which has a restriction on the parameters such that (translated into our case) the error bounds are only meaningful when

$$\frac{m^2-1}{4} \leq 0, \tag{4.120}$$

which is unsuitable for our analysis since we consider $m \in \mathbb{N}_0$. In [39] Dunster gives error analysis in the whole complex plane for equations of this type however they are too complicated for just considering the real line. Thus getting meaningful and realistic error analysis for real variables in this parameter case would be of interest, to then provide uniform asymptotic approximations for the oblate spheroidal wave equations. The following is an indication of how the results would be expected to turn out.

Via connection formulae it is very likely one would be able to make rigorous statements about the eigenvalues λ_n^m, which would hopefully would be of the form

$$\lambda_m^n = 4\Gamma\left(k + \frac{1}{2}(m+1)\right) + \mathcal{O}(1), \tag{4.121}$$

where $k = \frac{1}{2}(n-m)$ if $(n-m)$ is even, and otherwise $k = \frac{1}{2}(n-m-1)$. Using this, by expanding the corresponding a_n^m and α_n^m one would obtain that

$$\frac{\alpha_n^m \Gamma}{2} \sim \frac{m}{2} + \frac{1}{2} + k, \quad \Gamma \to \infty. \tag{4.122}$$

Hence in a similar manner to that of chapter 2, writing the differential equation as

$$\frac{\mathrm{d}^2 W_n^m(\zeta,\gamma^2)}{\mathrm{d}z^2} = \left(\Gamma^2 \frac{\zeta - \frac{m+1+2k}{\Gamma}}{\zeta} + \frac{1}{4}\frac{m^2-1}{\zeta^2} + \frac{\widetilde{\psi}(\zeta,\gamma^2)}{\zeta}\right) W_n^m(\zeta,\gamma^2), \tag{4.123}$$

where

$$\widetilde{\psi}_n^m(\zeta,\gamma^2) = \psi_n^m(\zeta,\gamma^2) + (m+1+2k)\Gamma - \alpha_n^m \Gamma^2, \tag{4.124}$$

then a suitable approximation can be given more naturally in terms of the solutions

$$\begin{aligned} W_{n,1}^m(\zeta,\gamma^2) &= M_{\frac{m}{2}+\frac{1}{2}+k,\frac{m}{2}}(2\Gamma\zeta) + \epsilon_{n,1}^m(\zeta,\gamma^2), \\ &= (-1)^k k! e^{-\Gamma\zeta}(2\Gamma\zeta)^{\frac{1}{2}(m+1)} L_k^m(2\Gamma\zeta) + \epsilon_{n,1}^m(\zeta,\gamma^2), \end{aligned} \tag{4.125}$$

$$W_{n,2}^m(\zeta,\gamma^2) = M_{-\frac{m}{2}-\frac{1}{2}-k,\frac{m}{2}}(e^{\pi \mathrm{i}} 2\Gamma\zeta) + \epsilon_{n,2}^m(\zeta,\gamma^2). \tag{4.126}$$

where L_k^m is the associated Laguerre polynomial (see §18.3 [1]).

Formally for $\zeta \in [0, \zeta_*]$,

$$\mathrm{Ps}_n^m(z,\gamma^2) \sim P_n^m \left(\frac{\zeta - \alpha_n^m}{\zeta x(2-x)(x(2-x) - a_n^m)} \right)^{\frac{1}{4}} e^{-\Gamma\zeta} \zeta^{\frac{1}{2}(m+1)} L_k^m(2\Gamma\zeta), \tag{4.127}$$

where P_n^m is the normalisation constant. If this case, using the orthogonality relation for the associated Laguerre polynomials

$$\int_0^\infty e^{-x} x^m L_i^m(x) L_j^m(x) \mathrm{d}x = \frac{(i+m)!}{i!}\delta_{ij}, \tag{4.128}$$

and matching with the signs given in (4.6), with some work one would obtain

$$P_n^m \sim (-1)^m \Gamma^{\frac{m+1}{2}} \sqrt{2^{m+1} \frac{a_n^m}{\alpha_n^m} \frac{k!}{(m+k)!} \frac{(n+m)!}{(n-m)!} \frac{1}{2n+1}}. \tag{4.129}$$

4.6.2 Asymptotic expansions of the oblate spheroidal wave functions and their eigenvalues

The prevoius uniform asymptotic approximations for the oblate spheroidal wave functions when $\gamma^2 \to -\infty$ would likely only lead to a one-term approximation, with more terms in an asymptotic expansion not being computed due to the complicated nature of the transformation between z and ζ. The oscillatory behaviour of the oblate spheroidal wave functions happens in shrinking neighbourhoods of $z = -1$ and $z = 1$ as $\gamma^2 \to -\infty$. It can be shown in the previous analysis that around $z = 1$, ζ behaves approximately like $1 - z$. If the uniform approximation had been done in the z interval $[-1, 0]$, around $z = -1$, ζ would have behaved there like $1 + z$. Hence again letting $\Gamma = \mathrm{i}\gamma$ so that $\gamma^2 \to -\infty$ corresponds to $\Gamma \to \infty$, we can employ simpler transformations again to obtain asymptotic expansions for the functions and the eigenvalues.

Considering again (4.107), first letting

$$t_1 = 2\Gamma\left(1 - z\right), \quad \text{and} \quad w_n^m = e^{-\frac{t_1}{2}} \left(1 - z^2\right)^{\frac{m}{2}} W_n^m, \tag{4.130}$$

then W_n^m satisfies

$$t_1 \frac{\mathrm{d}^2 W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1^2} + (m+1-t_1)\frac{\mathrm{d}W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1} + \left(\frac{\lambda}{4\Gamma} - \frac{1}{2}(m+1)\right) W_n^m(t_1,\gamma^2)$$
$$- \frac{1}{4\Gamma}\left(t_1^2 \frac{\mathrm{d}^2 W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1^2} + \left(2t_1(m+1) - t_1^2\right)\frac{\mathrm{d}W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1}\right.$$
$$\left. + \left(m(m+1) - (m+1)t_1\right) W_n^m(t_1,\gamma^2)\right) = 0. \tag{4.131}$$

Let $k = \frac{1}{2}(n-m)$ if the parity of $(n-m)$ is even, and otherwise $k = \frac{1}{2}(n-m-1)$ and suppose

$$\lambda_m^n = 4\Gamma\left(k + \frac{1}{2}(m+1)\right) + \sum_{s=0}^{\infty}\frac{\mu_s}{\Gamma^s}, \tag{4.132}$$

which would make sense from the previous analysis if one have derived the desired rigorous statements about the eigenvalues, so that the differential equation can be written in the form

$$t_1 \frac{\mathrm{d}^2 W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1^2} + (m+1-t_1)\frac{\mathrm{d}W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1} + (k)\, W_n^m(t_1,\gamma^2)$$
$$- \frac{1}{4\Gamma}\left(t_1^2 \frac{\mathrm{d}^2 W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1^2} + \left(2t_1(m+1) - t_1^2\right)\frac{\mathrm{d}W_n^m(t_1,\gamma^2)}{\mathrm{d}t_1}\right.$$
$$\left. + \left(m(m+1) - (m+1)t_1 - \sum_{s=0}^{\rho-1}\frac{\mu_{s+1}}{\Gamma^s} + \frac{\widetilde{\mu}_{\rho+1}}{\Gamma^\rho}\right) W_n^m(t_1,\gamma^2)\right) = 0, \tag{4.133}$$

where $\widetilde{\mu}_{\rho+1}$ can be re-expanded appropriately. Then supposing there is a solution of the form

$$W_n^m(t_1,\gamma^2) \sim L_k^m(t_1)\sum_{s=0}^{\infty}\frac{A_s(t_1)}{\Gamma^s} + \frac{\mathrm{d}}{\mathrm{d}t_1}L_k^m(t_1)\sum_{s=0}^{\infty}\frac{B_s(t_1)}{\Gamma^s}, \tag{4.134}$$

by substituting this into (4.131) one can obtain recurrence relations for these coefficients. Again substituting in polynomials with undetermined coefficients for A_s and B_s as before (but this time without the restriction of even and oddness of the polynomials as it is unnecessary here), one can obtain explicit expressions for these coefficients in terms of polynomials whose order match the index s.

Instead in (4.107) letting

$$t_2 = 2\Gamma\,(1+z)\,, \quad \text{and} \quad w_n^m = e^{-\frac{t_2}{2}}\left(1-z^2\right)^{\frac{m}{2}} W_n^m, \tag{4.135}$$

$W_n^m(t_2,\gamma^2)$ satisfies

$$\begin{aligned} t_2\frac{\mathrm{d}^2 W_n^m(t_2,\gamma^2)}{\mathrm{d}t_1^2} + (m+1-t_2)\,\frac{\mathrm{d}W_n^m(t_2,\gamma^2)}{\mathrm{d}t_1} + \left(\frac{\lambda}{4\Gamma} - \frac{1}{2}\,(m+1)\right) W_n^m(t_2,\gamma^2) \\ - \frac{1}{4\Gamma}\left(t_2^2\frac{\mathrm{d}^2 W_n^m(t_2,\gamma^2)}{\mathrm{d}t_1^2} + \left(2t_2\,(m+1) - t_2^2\right)\frac{\mathrm{d}W_n^m(t_2,\gamma^2)}{\mathrm{d}t_1}\right. \\ \left. + \left(m(m+1) - (m+1)t_2\right)\right) W_n^m(t_2,\gamma^2) = 0, \end{aligned} \tag{4.136}$$

which is the same differential equation as before with the new independent variable. Thus using a similar method one would obtain the same eigenvalue and functions expansions as before, with t_2 in place of t_1.

The error and eigenvalue error analysis for these discussed expansions is very similar to the work in the previous chapters, but since we haven't proved that $\lambda_m^n = 4\Gamma\left(k + \frac{1}{2}(m+1)\right) + \mathcal{O}(1)$ as $\Gamma \to \infty$, then we cannot write it down rigorously at this time. Once the error analysis for the eigenvalues would be completed, the error analysis for the functions would follow.

Formally we give

$$\begin{aligned} \mathrm{Ps}_n^m(z,\gamma^2) \sim P_n^m\left(1-z^2\right)^{\frac{m}{2}} \Bigg(e^{-\frac{t_1}{2}} \left(L_k^m(t_1)\sum_{s=0}^{\infty}\frac{A_s(t_1)}{\Gamma^s} + \frac{\mathrm{d}}{\mathrm{d}t_1}L_k^m(t_1)\sum_{s=0}^{\infty}\frac{B_s(t_1)}{\Gamma^s}\right) \\ + (-1)^{n-m} e^{-\frac{t_2}{2}} \left(L_k^m(t_2)\sum_{s=0}^{\infty}\frac{A_s(t_2)}{\Gamma^s} + \frac{\mathrm{d}}{\mathrm{d}t_2}L_k^m(t_2)\sum_{s=0}^{\infty}\frac{B_s(t_2)}{\Gamma^s}\right)\Bigg), \end{aligned} \tag{4.137}$$

where

$$P_n^m \sim (-1)^m \Gamma^{\frac{m+1}{2}} \sqrt{\frac{k!}{(m+k)!}\frac{(n+m)!}{(n-m)!}\frac{1}{2n+1}}\left(1 + \frac{\eta_1}{\Gamma} + \frac{\eta_2}{\Gamma^2} + \ldots\right), \tag{4.138}$$

where the η_s terms can be computed using integral relations for the associated Laguerre polynomials. Since outside their oscillatory regions the two expressions in t_1 and t_2 are exponentially small, meaningful order estimates for $z \in [-1, -1 + \mathcal{O}(\Gamma^{-1})]$, and

$z \in [1 - \mathcal{O}(\Gamma^{-1})), 1]$ would be attainable using the theory of chapter 2.

We give the first few terms of both the function and eigenvalue formal expansions:

$$\begin{aligned}
&A_0(t) = 1, \quad A_1(t) = \frac{1}{4}\left(m + k + 1\right), \\
&B_0(t) = 0, \quad B_1(t) = -\frac{1}{4}\left(m + 2k + 1\right), \\
&\mu_1 = -2k^2 - (2k+1)(m+1), \quad \mu_2 = -\frac{1}{4}(2k+m+1)(2k^2 + (2k+1)(m+1)).
\end{aligned} \tag{4.139}$$

The results for the eigenvalues correspond with the formal results given in the literature.

5

Conclusion

5.1 Results obtained in this thesis

By application of the theory for second-order linear differential equations with two turning points developed in [7] and briefly described in section 1.4, we obtained uniform asymptotic approximations for the Lamé, Mathieu and prolate spheroidal wave functions with a large real parameter. These approximations are expressed in terms of parabolic cylinder functions, and are uniformly valid in their respective real open intervals. In all cases explicit bounds are supplied for the error terms associated with the approximations. We also obtained approximations for the large order behaviour for the respective eigenvalues. We restricted ourselves to a two term uniform approximation. Theoretically more terms in these approximations could be computed, but the coefficients would be very complicated.

Using a simplified method we obtained uniform asymptotic expansions for these functions where the coefficients are just polynomials and satisfy simple recurrence relations. The price to pay for this simplification was that these asymptotic expansions hold only in a shrinking interval as their respective parameters become large; this interval however does encapsulate all the interesting oscillatory behaviour of the functions. This simplified method also gives many terms in asymptotic expansions for these eigenvalues, derived simultaneously with the coefficients in the function expansions. We provided rigorous realistic error bounds for the function expansions when truncated and order estimates for the error when the eigenvalue expansions are truncated. With our work we confirmed that many of the formal results in the literature are correct.

We also derived an expression for the exponentially small difference between the eigenvalues a_ν^m and b_ν^{m+1} of the Lamé equations, and a_m and b_{m+1} of the Mathieu equation. We did this using a result from Floquet theory, and were able to derive a second term in the expansion.

5.2 Potential future work in the field

Exponentially small difference between the eigenvalues a_ν^m and b_ν^{m+1} of the Lamé equations, and a_m and b_{m+1} of the Mathieu equation. In sections 2.5 and 3.4, for both a_ν^m and b_ν^{m+1}, and a_m and b_{m+1}, the asymptotic expansions were the same to all orders, thus there must be some difference between the two which is exponentially small. We derived an expression in section 2.6 for the exponentially small term including a second term, which was stated in [19] without proof and is displayed in [1], but the proof of the error term lacks the proofs of Conjectures 2.1 and 2.2. The proofs of these Conjectures should be attainable with some work, and something I will hopefully complete in the near future, after submitting this thesis.

Oblate spheroidal wave functions. More potential future work has already been outlined in section 4.6 regarding obtaining rigorous uniform asymptotic for the oblate spheroidal wave functions and their respective eigenvalues with a large parameter. For these desired results the key piece of missing machinery is satisfactory error analysis for approximating solutions of differential equations where a turning point coalesces with a simple pole. In [38] some work has been done towards this, however the parameter case which relates to the oblate spheroidal wave functions was specifically omitted due to the complicated nature of the analysis. For this problem one could write a paper giving satisfactory analysis for this parameter case, then write a paper which applies this theory to the oblate spheroidal wave functions and their eigenvalues.

Zeros of Lamé and Mathieu functions. In this thesis we gave asymptotic expansions for the Lamé and Mathieu functions which held in the oscillatory regions of both functions, for which we can compute as many coefficients as we like. From these one should be able to give rigorous results for the zeros of the Lamé and Mathieu functions.

Rigorous asymptotics for the Lamé and Mathieu functions when both a parameter and the order of the functions becomes large. So far the work I have done in this thesis has only considered rigorous asymptotics for a parameter in the equation becoming large. In physical applications it is also a case of interest to consider the parameter which corresponds to the order of the functions becoming large simultaneously to the large parameter. Some formal results already available in the literature relating to spheroidal wave functions and with advances in rigorous asymptotic for differential equation it could be possible to do this rigorously.

Theory for Heun functions. Despite their numerous applications, the theory of functions which lie in the Heun class remains far from complete. There are some analytical works on the Heun functions (see [40]), but as a whole there are many gaps in our knowledge of these functions. Particularly, the connection problem for the Heun functions is not solved, i.e., one cannot connect two local solutions at different singular points using known constant coefficients. Another example of a serious gap in the general theory of the Heun functions in general is the absence of integral representations analogous to the one for hypergeometric functions. However, it is likely that these problems cannot be solved.

5.3 Physical and mathematical applications

Lamé functions have been shown to occur at bifurcations in chaotic Hamiltonian systems [41], and in the theory of Bose-Einstein condensates [42]. Transforming the wave equation

$$\nabla^2 u + \omega^2 u = 0 \tag{5.1}$$

to sphero-conal coordinates, solutions can be written in terms of spherical Bessel functions and solutions of Lamé's equation. Alternatively transforming (5.1) using ellipsoidal coordinates, solutions can be written entirely in terms of solutions of Lamé's equation. Solutions of Lamé's equation also have applications in spherical and ellipsoidal harmonics [9], and to the theory of conformal mappings [43]. They also appear in all sorts of problems relating to solutions of differential equations, for example see [44].

Mathieu functions occur in physical applications in two main categories, in boundary-value problems and initial value problems. Topics these occur in include elliptic membranes and electromagnetic waves [19], vibrating systems [45] and ring antennas [46]. Transforming the two-dimensional wave equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0 \tag{5.2}$$

to elliptical coordinates, solutions can be written in terms of solutions of Mathieu's equation. Elliptical coordinates were chosen as boundary conditions in physical problems often relate to the perimeter of an ellipse.

The spheroidal wave functions appear in a mathematical context in a very similar way to solutions of the Lamé and Mathieu equations. The spheroidal wave functions have applications in signal analysis, see [47] and [48].

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