

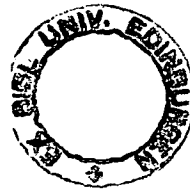
INTERTWINING OPERATORS

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Abstract.

In this thesis we study operators and spaces of operators on a Hilbert space defined by intertwining relations.

The classical Hankel operators are those operators which intertwine the unilateral shift and its adjoint. We consider generalised Hankel operators relative to shifts and relative to families of shifts and give generalisations of the classical theorems of Nehari and Hartman. In contrast to the classical approach our proofs are mainly geometric and rest on the Sz-Nagy Foias lifting theorem.

We show that the closed linear span of the positive Hankel operators is a proper subspace of the Hankel operators and contains all the compact Hankels. Part of this result is also obtained, via Douglas's localisation theory for Toeplitz operators, from the fact that there exist Hankel operators which do not lie in the C^* -algebra generated by the Toeplitz operators.

In chapter 7 we see that certain sums of spaces of intertwining operators are closed and yield C^* -algebras. In fact it is the algebraic properties of these spaces that ensure the automatic closure of their sum. As a consequence we obtain odd/even decompositions for C^* -algebras and von Neumann algebras and related double commutant theorems.

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CHAPTER 1

INTRODUCTION

Let H be a separable complex Hilbert space with complete orthonormal basis $\{e_n ; n = 0, 1, 2, \dots\}$. A Hankel operator on H is one whose representing matrix (a_{ij}) has the special form

$$a_{ij} = \alpha(i + j) \quad (i, j = 0, 1, 2, \dots).$$

The two main classical theorems concerning Hankel operators are due to Nehari [23] and Hartman [16] who found necessary and sufficient conditions for a Hankel operator to be bounded and compact respectively.

Let L^2 be the usual Hilbert space of square integrable functions on the unit circle T , with complete orthonormal basis $\{e_n ; n \in \mathbb{Z}\}$, where e_n denotes the function z^n . For an essentially bounded measurable function ϕ on T the well known multiplication operator M_ϕ is defined by

$$M_\phi f = \phi f \quad (f \in L^2).$$

Let J be the unitary operator on L^2 defined by $Je_n = e_{-n}$ and let P be the orthogonal projection of L^2 onto the Hardy subspace H^2 spanned by $\{e_n ; n \geq 0\}$. Then S_ϕ is defined by

$$S_\phi = PJM_\phi|_{H^2}.$$

Nehari's theorem shows that the Hankel operators on H^2 are precisely the operators S_ϕ and moreover, for each Hankel operator S , there exists a function $\phi \in L^\infty$ such that $S = S_\phi$ and $\|\phi\| = \|S\|$. Hartman's theorem shows that S is compact if and only if the symbol ϕ can be chosen to be continuous. The classical proofs rest heavily on the function theory of the underlying spaces and in particular on the Caratheodory-Fejér interpolation theorem.

The Hankel operators are also characterised by the operator equation

$$SU = U^* S$$

where U is the unilateral shift on H . Page [24] has observed that this characterisation and the elegant lifting theorem of Sz-Nagy and Foias [20] yield a geometric or 'function free' proof of Nehari's theorem which applies to certain generalised Hankel operators intertwining a unilateral shift of arbitrary multiplicity.

Motivated by the above characterisation we make the following definition. Let \underline{U} be a commutative family of unilateral shifts acting on a Hilbert space H . An operator S on H is Hankel relative to \underline{U} if $S \in \text{twi}(\underline{U})$ where

$$\text{twi}(\underline{U}) = \{S \in B(H) ; SU = U^*S, U \in \underline{U}\}$$

These are some of the intertwining operators of the title of this thesis.

In chapters 3 and 4 we shall prove Nehari and Hartman type theorems for some of these generalised Hankel operators. The proofs are essentially geometric and use the Sz-Nagy Foias lifting theorem and an extension of it to continuous semigroups of contractions. These latter theorems and preliminary dilation theory are given in chapter 2.

In chapter 5 we discuss the lifting problem for operators commuting with certain families of contractions. Although we do not always have a Nehari theorem for Hankel operators relative to \underline{U} (as above), we ask whether we do in the case when \underline{U} consists of doubly commuting isometries. In particular do we have Nehari and Hartman theorems for Hankel operators on the Hardy space on the torus?

In chapter 6 we consider certain classical Hankel operators with discontinuous symbol and show that whilst the closed linear span of the positive Hankel operators, \underline{P} say, contains all compact Hankel operators, there are Hankel operators not in \underline{P} .

For $\phi \in L^\infty$ the Toeplitz operator T_ϕ on H^2 is defined by

$$T_\phi = P M_\phi | H^2$$

These operators and their generalisations have been the object of much study in the last twelve years (see for example [9]). Although Hankel operators are formally rather similar to Toeplitz operators they have received much less attention and usually quite different techniques are needed to study them. However we shall use Douglas's localisation theory [9] to extend the results of chapter 6 mentioned above by showing that there exist Hankel operators which do not lie in the algebra generated by the Toeplitz operators.

An immediate corollary of Hartman's theorem is that $H^\infty + C(T)$ is closed. This was first proved by Sarason [33] using a more direct method. Subsequently Rudin [32] has given an elementary abstract theorem applicable to more general situations. We give Rudin's theorem below and use it in chapters 3 and 4 to show that certain analogues of $H^\infty + C(T)$ (not considered in [32]) are also closed.

THEOREM 1.1 Let Y and Z be closed subspaces of a Banach space X and let Ψ be a family of bounded linear maps Δ on X such that

- a) Δ maps X into Y , $\Delta \in \Psi$,
- b) Δ maps Z into Z , $\Delta \in \Psi$,
- c) $\sup\{\|\Delta\|; \Delta \in \Psi\} < \infty$,
- d) For each $y \in Y$ and $\epsilon > 0$ there exists $\Delta \in \Psi$ such that $\|\Delta y - y\| < \epsilon$.

Then $Y + Z$ is closed

Proof. Let M be a constant such that $\|\Delta\| < M$, $\Delta \in \Psi$. Let $y \in Y$, $z \in Z$, $\epsilon > 0$ and choose $\Delta \in \Psi$ such that $\|\Delta y - y\| < \epsilon$. Then

$$\|y - \Delta z\| \leq \|y - \Delta y\| + \|\Delta y - \Delta z\| < \epsilon + M\|y - z\|.$$

Since $\Delta z \in Y \cap Z$ we see that $d(y, Y \cap Z) \leq M d(y, Z)$, $y \in Y$, where $d(,)$ denotes the usual norm distance between sets. It follows that

the canonical embedding of $Y / Y \cap Z$ in Y / Z is bicontinuous. Thus Y / Z is closed in X / Z and its preimage under the canonical mapping of X into X / Z is closed. That is, $Y + Z$ is closed.

In chapter 7 we leave Hankel operators and consider certain spaces of intertwining operators. We will be concerned with the problem of whether the sum of such spaces is closed (using techniques quite different from those of Theorem 1.1). It turns out that the algebraic properties of these spaces are sufficient to ensure the automatic closure of their sum. For example, given a set of operators \underline{F} on a Hilbert space H let

$$\text{com}_{\lambda}(\underline{F}) = \{a \in B(H) ; au = \lambda ua, u \in \underline{F}\} \quad (\lambda \in \mathbb{C}).$$

Then if $\underline{F} = \underline{F}^*$, $\text{com}(\underline{F}) + \text{com}_{-1}(\underline{F})$ is a von Neumann algebra. Also, under natural conditions, we have the following double commutant theorem

$$\text{com}(\text{com}(\underline{F})) = \text{com}(\text{com}_{-1}(\underline{F})) + \text{com}_{-1}(\text{com}_{-1}(\underline{F}))$$

and in this situation we can identify the summands as the closed span in the weak operator topology of even (resp. odd) products of operators in \underline{F} . The key result behind these theorems is the following; Let A be a C^* -algebra of operators and let B be a self-adjoint norm closed linear space of operators such that $BA \subset B$, $AB \subset B$ and $BB \subset A$. Then $A + B$ is a C^* -algebra.

We also obtain generalisations of the above to several spaces of operators and give 'roots of unity' decompositions of C^* -algebras and related double commutant theorems.

We close with a list of some of the problems raised in the thesis.

LIFTING THEOREMS

In this chapter we discuss dilations of contractions on a Hilbert space. We prove the Sz-Nagy Foias lifting theorem [20] and an extension of it to strongly continuous semigroups of contractions.

We first give some preliminary definitions .

Let H be a subspace of a Hilbert space K and let P be the orthogonal projection of K onto H . For $A \in B(H)$ and $B \in B(K)$ we write $A = \text{pr}(B)$ when $A = PB|_H$.

Let $\underline{A} = \{A_\omega : \omega \in \Omega\}$ be a family of operators on H . A family $\underline{B} = \{B_\omega : \omega \in \Omega\}$ of operators on K is called a dilation of \underline{A} if

$$A_{\omega_1}^{n_1} \dots A_{\omega_r}^{n_r} = \text{pr}(B_{\omega_1}^{n_1} \dots B_{\omega_r}^{n_r})$$

for $n_i \geq 0$, $1 \leq i \leq r$, and every finite set of subscripts $\omega_i \in \Omega$, $1 \leq i \leq r$.

\underline{B} is called a minimal dilation of \underline{A} if the smallest reducing subspace for \underline{B} containing H is K .

\underline{B} is called commuting (resp. isometric, unitary etc.) when \underline{B} consists of commuting (resp. isometric, unitary etc.) operators.

Two dilations \underline{B} and \underline{B}' acting on K and K' respectively are isomorphic if there exists a unitary operator Z from K to K' such that

$$(i) \quad Zh = h, \quad h \in H,$$

$$(ii) \quad Z B_\omega' Z^* = B_\omega, \quad \omega \in \Omega.$$

A celebrated theorem of Sz-Nagy [19] states that each contraction on a Hilbert space possesses a minimal unitary dilation and that this dilation is unique up to isomorphism. Inspired by a paper of Sarason [33], Sz-Nagy and Foias [20] have shown that each operator in the

commutant of a contraction can be expressed as the compression of an operator with the same norm in the commutant of the unitary dilation. This result has proved quite useful in the recent work of several authors ([21],[25],[24],[18],[10] and [6]).

We first show that every family of doubly commuting contractions possesses a commuting unitary dilation. This is a well known result which of course extends Sz-Nagy's theorem and is proved in [22]. The proof we give is somewhat more direct.

DEFINITION. Two operators T and S on a Hilbert space doubly commute if $TS = ST$ and $TS^* = S^*T$.

PROPOSITION 2.1. Let $\underline{T} = \{T, T_\alpha, \alpha \in A\}$ be a family of doubly commuting contractions on a Hilbert space H . Then there exists a dilation $\underline{V} = \{V, V_\alpha, \alpha \in A\}$ of doubly commuting contractions on a Hilbert space $K \supset H$ such that the following hold.

- (i) H is invariant for V^* and reduces $V_\alpha, \alpha \in A$.
- (ii) $V^*|_H = T^*$ and $V_\alpha|_H = T_\alpha, \alpha \in A$.
- (iii) V is an isometry and V_α is an isometry for each α such that T_α is an isometry.
- (iv) K is the closed linear span of $\{V^n H; n \in \mathbb{Z}_+\}$.

Proof. Let $R = (I - T^*T)^{\frac{1}{2}}$, $\underline{R} = (RH)^-$ and $K = H \oplus \underline{R} \oplus \underline{R} \oplus \dots$.

Let V be the operator on K defined by the operator matrix

$$\begin{pmatrix} T & & & & \\ R & & & & \\ & I & & & \\ & & I & & \\ & & & \ddots & \ddots \end{pmatrix}$$

Thus for $x = (h, x_1, x_2, \dots) \in K$ we have $Vx = (Th, Rh, x_1, x_2, \dots)$

and so

$$\begin{aligned}
\|Vx\|^2 &= \|Th\|^2 + \|Rh\|^2 + \|x_1\|^2 + \|x_2\|^2 + \dots \\
&= \|h\|^2 + \|x_1\|^2 + \|x_2\|^2 + \dots \\
&= \|x\|^2 .
\end{aligned}$$

Thus V is an isometry. Clearly H is invariant for V^* and $V^*|_H = T^*$. To see that (iv) observe that, identifying H with $H \oplus \{0\} \oplus \{0\} \oplus \dots$, we have

$$\begin{aligned}
Vh - Th &= (0, Rh, 0, 0, \dots) \\
V^2h - VTh &= (0, 0, Rh, 0, \dots)
\end{aligned}$$

etc., for $h \in H$.

Since T_α doubly commutes with T we see that $T_\alpha R = RT_\alpha$ and \underline{R} reduces T_α , $\alpha \in A$. Let $V_\alpha = T_\alpha \oplus T_\alpha|_{\underline{R}} \oplus T_\alpha|_{\underline{R}} \oplus \dots$, $\alpha \in A$, define the operators V_α on K . It is routine to check that $\underline{V} = \{V_\alpha, \alpha \in A\}$ has the properties required of (i), (ii) and (iii) thus completing the proof. (ii) shows that \underline{V} is a dilation of \underline{T} .)

Although the proof of the following theorem is rather involved technically, it is essentially Proposition 2.1 used with Zorn's Lemma.

THEOREM 2.2 Let \underline{T} be a family of doubly commuting contractions on a Hilbert space. Then \underline{T} has a minimal dilation \underline{V} consisting of doubly commuting isometries. Moreover \underline{V} is unique up to isomorphism.

Proof. Let $\underline{T} = \{T_\alpha; \alpha \in A\}$. Let \mathcal{Q} be the set of triples $(\sigma, V^\sigma, R^\sigma)$ where

- (i) $\sigma \subset A$,
- (ii) R^σ is a Hilbert space containing H ,

(iii) $\underline{V}^\sigma = \{V_\alpha^\sigma; \alpha \in A\}$ is a family of doubly commuting contractions on R^σ such that H is invariant for $(V_\alpha^\sigma)^*$, $(V_\alpha^\sigma)^*|_H = T_\alpha^*$ and V_α^σ is an isometry for each $\alpha \in \sigma$.

(iv) The closed linear span of $\{V_1^{n_1} \dots V_r^{n_r} H; V_i \in \underline{V}^\sigma, n_i \in \mathbb{N}\}$ is R^σ .

We define a partial ordering on Q as follows;

$(\sigma, \underline{V}^\sigma, R^\sigma) \leq (\sigma', \underline{V}^{\sigma'}, R^{\sigma'})$ if and only if

(v) $\sigma \subset \sigma'$,

(vi) R^σ is a subspace of $R^{\sigma'}$,

(vii) R^σ is invariant for $(V_\alpha^{\sigma'})^*$ and $(V_\alpha^{\sigma'})^*|_{R^\sigma} = (V_\alpha^\sigma)^*$, $\alpha \in A$,

(viii) If V_α^σ is an isometry, then R^σ reduces $V_\alpha^{\sigma'}$ and $V_\alpha^{\sigma'}$ is an isometry.

For example, note that if we are given $m = (\sigma, \underline{V}^\sigma, R^\sigma)$ in Q and $\alpha \in \sigma$ such that V_α^σ is not an isometry, then Proposition 2.1 shows that there exists $m' = (\sigma \cup \{\alpha\}, \underline{V}^{\sigma \cup \{\alpha\}}, R^{\sigma \cup \{\alpha\}}) \in Q$ such that $m < m'$ but $m \not\leq m'$.

We show that \leq is an inductive ordering on Q . Let Γ be a chain in Q and without loss suppose that Γ has no maximal element (for otherwise we have nothing to prove). Let $\nu \in \bigcup_{\sigma \in \Gamma} \sigma$, where the

notational abuse has its obvious meaning. Let $R_0 = \bigcup_{\sigma \in \Gamma} R^\sigma$ and let R

be the completion of R_0 with respect to the natural induced metric.

Define operators \bar{V}_α^σ on R by

$$\bar{V}_\alpha^\sigma = \begin{cases} V_\alpha^\sigma & \text{on } R^\sigma \\ 0 & \text{on } (R^\sigma)^\perp \end{cases}.$$

Since $\|\bar{V}_\alpha^\sigma\|$ is bounded and R_0 is dense in R , it follows from (vii) that there exists for each α an operator \bar{V}_α on R such that the net \bar{V}_α^σ converges to \bar{V}_α in the weak operator topology.

Set $\bar{\underline{V}} = \{\bar{V}_\alpha; \alpha \in A\}$ and let $q = (\nu, \bar{\underline{V}}, R)$. We claim that

$q \in Q$ and is an upper bound for Γ . Clearly (i) and (ii) hold for q . Since multiplication of operators is continuous on bounded weak operator convergent nets it follows that \bar{V} is a family of doubly commuting contractions. From (iii) and (iv) we see that \bar{V}_α is an isometry for $\alpha \in \mathfrak{A}$ and thus (iii) holds for q . (iv) is easily checked and thus $q \in Q$. From the way we have constructed \bar{V} it follows that $m \leq q$ for all $m \in \Gamma$ and thus that q is an upper bound for Γ .

Since the ordering is inductive, by Zorn's Lemma Q has a maximal element $m = (\mu, \underline{V}, K)$. As we have already noted, if V_α is not an isometry for some $\alpha \in A$ Proposition 2.1 shows that there exists $m' \in Q$ such that $m' > m$ and $m' \neq m$. Since this contradicts the maximality of m , \underline{V} consists of doubly commuting isometries. By (iv) \underline{V} is a minimal dilation of \underline{T} .

It remains to show that such minimal dilations are unique up to isomorphism. We will just sketch here the standard technique for proving this. Let K_0 (resp R_0) be the linear span of $\{V_1^{n_1} \dots V_r^{n_r} h; V_i \in \underline{V}\}$ (resp. $\{W_1^{n_1} \dots W_r^{n_r} h; W_i \in \underline{W}\}$) where $\underline{V} = \{V_\alpha; \alpha \in A\}$ and $\underline{W} = \{W_\alpha; \alpha \in A\}$ are minimal dilations of \underline{T} consisting of doubly commuting isometries acting on K and R respectively. Define ϕ_0 from K_0 to R_0 by

$$\phi_0(V_{\alpha_1}^{n_1} \dots V_{\alpha_r}^{n_r} h) = W_{\alpha_1}^{n_1} \dots W_{\alpha_r}^{n_r} h \quad (h \in H)$$

and by linearity. Using the fact that the V 's and W 's are doubly commuting isometries and that $V_\alpha^*|_H = W_\alpha^*|_H = T_\alpha^*$, $\alpha \in A$, one can show that ϕ_0 is well defined, isometric and, by the minimality conditions, extends to a unitary ϕ from K to R such that $\phi^* W_\alpha \phi = V_\alpha$, $\alpha \in A$. Plainly $\phi h = h$, $h \in H$, and so \underline{V} and \underline{W} are isomorphic dilations.

Remark. If $T_\alpha^{*n} \rightarrow 0$ strongly then V_α is a unilateral shift, $\alpha \in A$.

We will return to this special case in chapter 5.

We now prove the Sz-Nagy Foias lifting theorem, as given in [22], and give a simple extension to operators intertwining continuous semigroups of contractions. These theorems will prove useful in chapters 3 and 4.

PROPOSITION 2.3. Let T be a contraction on H with minimal isometric dilation V on K . Let X be a bounded operator from K to H such that $XV = TX$. Then there exists an operator Y on K such that the following hold.

(i) $X = PY$ where P is the orthogonal projection of K onto H .

(ii) $\|Y\| = \|X\|$.

(iii) $YV = VY$.

Proof. As in the proof of Proposition 2.1 we may assume that

$K = H \oplus \underline{R} \oplus \underline{R} \oplus \dots$, where $\underline{R} = (RH)^\perp$. We may suppose that $\|X\| = 1$.

An operator Y on K satisfying (i) has the form $Y = X + Y_1 + Y_2 + \dots$

where Y_i is an operator from K to the i^{th} \underline{R} summand, \underline{R}_i say.

Since $V\underline{R}_i = \underline{R}_{i+1}$ and V is isometric we may assume that Y has

the form $Y = X + B_0 + VB_1 + V^2B_2 + \dots$ where B_i is an operator

from K to \underline{R}_1 . For such a Y (ii) holds if and only if

$$\|Yg\|^2 = \|Xg\|^2 + \sum_{0 \leq n < \infty} \|B_n g\|^2 \leq \|g\|^2 \quad (g \in K). \quad (1)$$

Also

$$\begin{aligned} VY - YV &= VX - XV + \sum_{0 \leq n < \infty} V^{n+1} B_n - \sum_{0 \leq n < \infty} V^n B_n V \\ &= \sum_{0 \leq n < \infty} V^n (B_{n-1} - B_n V) \end{aligned}$$

where $B_{-1} = VX - XV$.

Thus (iii) holds if and only if

$$B_{-1} = VX - XV \quad \text{and} \quad B_n V = B_{n-1}, \quad n = 0, 1, 2, \dots \quad (2)$$

We find Y satisfying (i) (ii) and (iii) by constructing the B_n by recurrence to satisfy (1) and (2).

Suppose for $N \geq 0$ the B_n ($n < N$) are already determined so that

$$S_N(\xi) = \|X\xi\|^2 + \sum_{0 \leq n < N} \|B_n \xi\|^2 \leq \|\xi\|^2, \quad \xi \in K, \quad (3)_N$$

or equivalently

$$I_K - X^* X - \sum_{0 \leq n < N} B_n^* B_n \geq 0, \quad (4)_N$$

and

$$B_{-1} = VX - XV, \quad B_n V = B_{n-1}, \quad 0 \leq n < N. \quad (5)_N$$

Let D_N be the positive square root of the left hand side of (4)_N.

We first show that

$$\|B_{N-1} \xi\|^2 \leq \|D_N V \xi\|^2, \quad \xi \in K. \quad (6)_N$$

For $N = 0$

$$\begin{aligned} \|B_{-1} \xi\|^2 &= \|(VX - XV)\xi\|^2 = \|(V - T)X\xi\|^2 \\ &= \|X\xi\|^2 - \|TX\xi\|^2 \\ &= \|X\xi\|^2 - \|XV\xi\|^2 \\ &\leq \|V\xi\|^2 - \|XV\xi\|^2 = \|D_0 V \xi\|^2, \quad \xi \in K. \end{aligned} \quad (7)$$

For $N > 0$ (5)_N implies

$$\begin{aligned} S_N(V\xi) &= \|XV\xi\|^2 + \sum_{0 \leq n < N} \|B_n V \xi\|^2 \\ &= \|XV\xi\|^2 + \|B_{-1}\xi\|^2 + \sum_{0 \leq m < N-1} \|B_m V \xi\|^2 \\ &= \|X\xi\|^2 + \sum_{0 \leq m < N-1} \|B_m \xi\|^2 \quad \text{by (7),} \\ &= S_N(\xi) - \|B_{N-1} \xi\|^2, \quad \xi \in K. \end{aligned}$$

Thus (3)_N implies

$\|B_{N-1}g\|^2 \leq \|g\|^2 - S_N(Vg) = \|Vg\|^2 - S_N(Vg) = \|D_N Vg\|^2$,
and so $(6)_N$ holds for $N \geq 0$.

Define C_N on $D_N VK$ by $C_N D_N Vg = B_{N-1}g$, $g \in K$. By $(6)_N$ C_N is well defined and extends to a contraction C_N on $(D_N VK)^{\bar{}}$.

Define C_N to be zero on $(D_N VK)^{\perp}$ to give a contraction on K .

Let $B_N = C_N D_N$. Clearly $B_N V = B_{N-1}$ and $(5)_{N+1}$ holds.

Moreover

$$\|B_N g\|^2 \leq \|C_N D_N g\|^2 \leq \|D_N g\|^2 = \|g\|^2 - \|Xg\|^2 - \sum_{0 \leq n < N} \|B_n g\|^2.$$

Thus $(3)_{N+1}$ holds and we can define operators B_n , $n=-1, 0, 1, 2, \dots$ by recurrence so that (1) and (2) hold, hence giving Y satisfying (i), (ii) and (iii).

2.4

THEOREM (Sz-Nagy and Foias) Let T be a contraction with minimal isometric dilation V . For each operator X commuting with T there exists an operator Y commuting with V such that $X = \text{pr}(Y)$ and $\|Y\| = \|X\|$.

Proof. We may assume as in Proposition 2.1 that T acts on H and V acts on $K = H \oplus \underline{R} \oplus \underline{R} \oplus \dots$. If P is the orthogonal projection of K onto H , then $PV = TP$. Let $X' = XP$ so that

$$X'V = XPV = XTP = TXP = TX'.$$

Proposition 2.3 applies to give $Y \in B(K)$ such that Y commutes with V , $\|Y\| = \|X'\| = \|X\|$ and $X' = PY$. Since $\text{pr}(Y) = PY|_H = X'|_H = X$ the proof is complete.

The following two lifting theorems follow from the above using standard techniques.

THEOREM 2.5. Let T be a contraction with minimal unitary dilation W . Then for each X commuting with T there exists Y commuting with W such that $X = \text{pr}(Y)$ and $\|Y\| = \|X\|$.

Proof. Let T act on the Hilbert space H and have minimal isometric dilation V acting on $K \supset H$. By Theorem 2.5⁴, for $X \in B(H)$ such that $XT = TX$, there exists Y_0 in $B(K)$ such that $Y_0V = VY_0$, $\|Y_0\| = \|X\|$ and $X = \text{pr}(Y_0)$. Now it is a simple consequence of the Wold decomposition for an isometry (see for example [22]) that the minimal isometric dilation of V^* is a unitary operator, we say, acting on $R \supset K$. Thus we can apply Theorem 2.5⁴ to Y_0^* and V^* to show that there exists an operator Y on R such that $YW = WY$ and $\text{pr}(Y) = X$. Since W is the minimal unitary dilation of T the proof is complete.

THEOREM 2.6. Let T (resp. S) be a contraction acting on a Hilbert space H_1 (resp. H_2) with minimal unitary dilation U (resp. V) acting on a Hilbert space K_1 (resp. K_2). For each operator X from H_2 to H_1 such that $TX = XS$ there exists an operator Y from K_2 to K_1 such that $UY = YV$, $\|Y\| = \|X\|$ and $X = PY|_H$ where P is the orthogonal projection of K_1 onto H_1 .

Proof. Let $G = T \oplus S$ acting on $H_1 \oplus H_2$. G is a contraction with minimal unitary dilation $F = U \oplus V$ acting on $K_1 \oplus K_2$. Let X_0 be the operator on $H_1 \oplus H_2$ defined by the operator matrix

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$$

Since X_0 commutes with G , by Theorem 2.5 there exists

$$Y_0 = \begin{pmatrix} Y_1 & Y \\ Y_2 & Y_3 \end{pmatrix}$$

on $K_1 \oplus K_2$, commuting with F such that X_0 is the compression of Y_0 to $H_1 \oplus H_2$ and $\|Y_0\| = \|X_0\| = \|X\|$. It follows that Y has the required properties.

Remarks. 1. Theorems 2.4, 2.5 and 2.6 remain true even when the dilations are not assumed to be minimal. For example, let W , acting on R , be a unitary dilation of the contraction T acting on H . If K is the closed linear span of $\{W^n H; n \in \mathbb{Z}\}$, then K is a reducing subspace for W , and $W|_K$ is a minimal unitary dilation of T . If X commutes with T , then by Theorem 2.5 there exists Y_0 in $B(K)$ commuting with T such that $X = \text{pr}(Y_0)$ and $\|Y_0\| = \|X\|$. If Y is the operator on R which is equal to Y_0 on K and is 0 on $(K)^\perp$ then Y commutes with W , $X = \text{pr}(Y)$ and $\|Y\| = \|X\|$.

2. An alternative proof of the Sz-Nagy Foias theorem has been given by Douglas, Muhly and Pearcy [12].

3. If \underline{T} is a family of commuting contractions which happens to possess a unitary dilation, then we do not necessarily have a lifting theorem for the commutant of \underline{T} . Such counterexamples and related lifting problems are discussed in chapter 5.

The following lifting theorem for operators which intertwine strongly continuous contractive semigroups of operators follows from Theorem 2.6. We shall use this in chapter 4 to prove theorems of a 'Nehari type'.

THEOREM 2.7 Let $\underline{T} = \{T_\alpha : \alpha \in \mathbb{R}_+\}$ (resp. $\underline{S} = \{S_\alpha : \alpha \in \mathbb{R}_+\}$) be a strongly continuous semigroup of contractions on a Hilbert space H_1 (resp. H_2). Let $\underline{U} = \{U_\alpha : \alpha \in \mathbb{R}_+\}$ (resp. $\underline{V} = \{V_\alpha : \alpha \in \mathbb{R}_+\}$) be a strongly continuous semigroup of unitary operators acting on K_1 (resp. K_2) such that $\text{pr}_{K_1}(U_\alpha) = T_\alpha$ (resp. $\text{pr}_{K_2}(V_\alpha) = S_\alpha$), $\alpha \in \mathbb{R}_+$. For each operator X from H_2 to H_1 such that $T_\alpha X = X S_\alpha$, $\alpha \in \mathbb{R}_+$, there exists an operator Y from K_2 to K_1 such that $U_\alpha Y = Y V_\alpha$, $\alpha \in \mathbb{R}_+$, $\|Y\| = \|X\|$ and $X = PY|_{H_2}$ where P is the orthogonal projection of K_1 onto H_1 .

Proof. Let τ_s and τ_w denote the strong and weak operator topologies. Notice that U_α (resp. V_α) is a unitary dilation of T_α (resp. S_α). Thus by Theorem 2.6 there exists Y_α in $B(K_2, K_1)$ such that $U_\alpha Y_\alpha = Y_\alpha V_\alpha$, $PY_\alpha|_{H_2} = X$, $\|Y_\alpha\| = \|X\|$. Since $B(K_2, K_1)_2$ is τ_w compact there exists a subnet Γ of $\{\frac{1}{2}n : n \in \mathbb{N}\}$ and Y in $B(K_2, K_1)$ such that the net Y_σ , $\sigma \in \Gamma$, is τ_w -convergent to Y . Since $U_\sigma Y_\sigma = Y_\sigma V_\sigma$ we have $U_{m(\sigma)} Y_\sigma = Y_\sigma V_{m(\sigma)}$ ($m \in \mathbb{N}$). Given $\alpha \in \mathbb{R}_+$ choose $m(\sigma) \in \mathbb{N}$ such that $m(\sigma)\sigma \rightarrow \alpha$. Since $V_{m(\sigma)\sigma}$, $\sigma \in \Gamma$, is τ_s convergent and $Y_{m(\sigma)}$, $\sigma \in \Gamma$, is a bounded τ_w convergent net, a simple computation shows that $Y_{m(\sigma)} V_{m(\sigma)\sigma}$ is τ_w convergent to $Y V_\alpha$. Similarly, by the ^{bounded} τ_w convergence of $V_{m(\sigma)\sigma}^* Y_{m(\sigma)}^*$ and the bounded strong convergence of $U_{m(\sigma)\sigma}$ we see that $V_{m(\sigma)\sigma}^* Y_\sigma^* U_{m(\sigma)\sigma}$ is τ_w convergent to $V_\alpha^* Y^* U_\alpha$. But $Y_\sigma^* = V_{m(\sigma)\sigma}^* Y_\sigma^* U_{m(\sigma)\sigma}$ and so, since $Y_\sigma^* \rightarrow Y^*$ (τ_w) we see that $U_\alpha Y = Y V_\alpha$. Plainly $PY|_{H_2} = X$ and $\|Y\| = \|X\|$ and the proof is complete.

Remark. Every strongly continuous semigroup of contractions is the

compression of a strongly continuous semigroup of unitaries.

We ask whether Theorem 2.7 remains true when we drop the assumption of strong continuity. An example of such a situation is the following.

Let K be the Hilbert space of complex valued functions on the real line which are square integrable with respect to the counting measure. Let H be the subspace of functions f such that $f(x) = 0$ for $x < 0$ and let $\underline{U} = \{U_\alpha; \alpha \in \mathbb{R}_+\}$ be the semigroup of unilateral shifts on H defined by

$$(U_\alpha f)(x) = f(x - \alpha) \quad , \quad x, \alpha \in \mathbb{R}_+ .$$

The semigroup $\underline{W} = \{W_\alpha; \alpha \in \mathbb{R}_+\}$ of bilateral shifts on K , similarly defined, form a commutative unitary dilation for \underline{U} .

Can we lift $\text{twi}(\underline{U})$ to $\text{twi}(\underline{W})$? This seems plausible and an affirmative answer would yield a Nehari theorem for Hankel operators on the Hardy spaces on the Bohr compactification of the real line. See the conjecture at the end of chapter 4 for an alternative approach to this problem.

HANKEL OPERATORS RELATIVE TO SHIFTS

In this chapter we generalise Nehari's theorem and Hartman's theorem to Hankel operators relative to a shift of arbitrary multiplicity .

Let W be an arbitrary bilateral shift on a Hilbert space K ; that is , W is a unitary operator on K and there exists a subspace L of K such that

$$K = \bigoplus \{W^n L : n \in \mathbb{Z}\} .$$

Let $H = \bigoplus \{W^n L : n \in \mathbb{Z}_+\}$, P the orthogonal projection of K onto H and $U = W|_H$. Then U is a unilateral shift on H with minimal unitary dilation W on K , and every unilateral shift can be derived from a bilateral shift in this way . By analogy with the classical function space representation we denote by K^∞ the commutant of W and take $H^\infty = \{A \in K^\infty : AH \subset H\}$. Let J denote the unitary operator on K such that

$$JW^n f = W^{-n} f \quad (f \in L , n \in \mathbb{Z})$$

and let η denote the mapping given by

$$\eta(A) = \text{pr}(JA) \quad (A \in K^\infty) .$$

Page [24] has observed that the Sz-Nagy Foias lifting theorem yields the following generalisation of Nehari's theorem on classical Hankel operators .

PROPOSITION 3.1. $\text{twi}(U) = \text{pr}(\text{twi}(W)) = \{\eta(A) : A \in K^\infty\}$. Moreover for each $B \in \text{twi}(U)$ there exists $A \in K^\infty$ with $B = \eta(A)$ and $\|B\| = \|A\|$.

Proof. It is elementary to check that $J \in \text{twi}(W)$ and $J \in \text{twi}(W^*)$. For $C \in K^\infty$, $JCW = JWC = W^* JC$ and thus $JK^\infty \subset \text{twi}(W)$. Conversely

let $T \in \text{twi}(W)$. Then $JTW = JW^*T = WJT$. That is, $JT \in K^\infty$.

Since $J^2 = I$, $T \in JK^\infty$ and hence $JK^\infty = \text{twi}(W)$. From Theorem 2.6 it now follows that for each $B \in \text{twi}(U)$ there exists $A \in K^\infty$ with $B = \eta(A)$ and $\|B\| = \|JA\| = \|A\|$. It remains to show that $\text{pr}(\text{twi}(W)) \subset \text{twi}(U)$. Let $C \in \text{twi}(W)$, $D = \text{pr}(C)$, $x, y \in H$. Then

$$\begin{aligned} (DUx, y) &= (CUX, y) = (CWx, y) = (W^*Cx, y) = (Cx, Wy) \\ &= (Dx, Wy) = (Dx, Uy) = (U^*Dx, y). \end{aligned}$$

Thus $D \in \text{twi}(U)$ completing the proof.

A is called the symbol of the Hankel operator $\eta(A)$ and η the symbol map. It is easily verified that the kernel of the mapping η is WH^∞ .

Let \underline{A} denote the norm closure in $B(\overset{K}{H})$ of the subalgebra

$$U\{W^{-n}H^\infty : n \in \mathbb{N}\}$$

and \underline{B} the least norm closed real subalgebra of $B(H)$ containing U and the identity operator I .

DEFINITION. An operator B on a Hilbert space H acts compactly on a space of operators $F \subset B(H)$ if the mapping $S \rightarrow AS$ from F to $B(H)$ is compact.

The following abstract Hartman theorem (Theorem 4 of [6]) characterises those Hankel operators relative to U with symbol in \underline{A} .

THEOREM 3.2 Let $B = \eta(A)$ with $A \in K^\infty$. Then $A \in \underline{A}$ if and only if B acts compactly on \underline{B} .

Proof. Suppose first that B acts compactly on \underline{B} . Since $U^n \in \underline{B}$ ($n = 1, 2, \dots$) there exists a subsequence $\{U^{n_k}\}$ and $T \in B(H)$ such that

$\|BU^{nk} - T\| \rightarrow 0$ as $k \rightarrow \infty$. Since $B \in \text{twi}(U)$ this gives
 $\|U^{*nk}B - T\| \rightarrow 0$ as $k \rightarrow \infty$. But since U is a unilateral shift
the sequence $\{U^{*n}\}$ converges to zero in the strong operator
topology. Therefore $T = 0$, and so $\|BU^{nk}\| \rightarrow 0$ as $k \rightarrow \infty$. Since
this argument applies with the sequence $\{U^{*n}\}$ replaced by each of
its subsequences, it follows that $\|BU^n\| \rightarrow 0$ as $n \rightarrow \infty$.

Since $BU^n \in \text{twi}(U)$, Proposition 3.1 gives the existence of $A_n \in K^\infty$
with $BU^n = \eta(A_n)$ and $\|A_n\| = \|BU^n\|$. We have $\eta(AW^n) = BU^n$ and
so $AW^n - A_n \in \ker \eta = WH^\infty$. Thus there exists $T_n \in H^\infty$ with
 $AW^n - WT_n = A_n$, and, since W is unitary and commutes with T_n ,
this gives

$$\|A - W^{1-n}T_n\| = \|A_n\| = \|BU^n\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore $A \in \underline{A}$.

Suppose next that $C = W^{-n}T$ with $T \in H^\infty$, $n \in \mathbb{N}$. Then
 $CW^{n+1} = WT \in \ker \eta$, and so $\eta(C)U^{n+1} = \eta(CW^{n+1}) = 0$. Therefore

$$\eta(C) = \eta(C)(I - U^{n+1}U^{*n+1}).$$

But $(I - U^{n+1}U^{*n+1})\underline{B}$ has finite dimension. Thus $\eta(C)\underline{B}$ has finite
dimension and $\eta(C)$ acts compactly on \underline{B} .

Finally given $A \in \underline{A}$ and $\epsilon > 0$ there exists C of this form
with $\|A - C\| < \epsilon$. Then

$$\|B - \eta(C)\| = \|\eta(A - C)\| < \epsilon,$$

and so B acts compactly on \underline{B} .

Through the following theorem, due to F.F. Bonsall, we can deduce
a generalisation of the more significant direction of Hartman's Theorem.

The following notation (and also the notation $\text{twi}(\)$) was introduced by Bonsall and Rosenthal in [5].

DEFINITION. Given a set of operators \underline{S} in $B(K)$ let $\text{jor}(\underline{S})$ be the set of operators T such that $ST = T^*S$ for all S in \underline{S} .

THEOREM 3.3. Let B be a compact operator on H . Then B acts compactly on $\text{jor}(B)$.

Proof. Let $F = \text{jor}(B)$, let F_1, H_1 denote the unit balls in F, H respectively, and let E denote the closure of BH_1 in H . Then $\{T^*|E; T \in F_1\}$ is an equicontinuous family of mappings on the compact space E . Also with $T \in F_1$ we have

$$T^*BH_1 = BTH_1 \subset BH_1 \quad ;$$

and so

$$T^*E \subset E \quad (T \in F_1).$$

It follows that, for each $x \in E$, the set $\{T^*x; T \in F_1\}$ is contained in the compact set E , and therefore the Ascoli-Arzelà theorem for metric space valued functions is applicable. Therefore, given $T_n \in F_1$, there exists a subsequence $\{n_k\}$ such that the sequence $T_{n_k}^*|E$ converges uniformly on E . Therefore the sequence $T_{n_k}^*B$ converges uniformly on H_1 ; that is, it converges in operator norm. Finally, $T_{n_k}^*B = BT_{n_k}$.

THEOREM 3.4. Let B be a compact operator on H that is Hankel relative to U . Then $B = \eta(A)$ for some $A \in \underline{A}$ with $\|A\| = \|B\|$.

Proof. Since $B \in \text{twi}(U)$, we have $U \in \text{jor}(B)$. Therefore $B \subset \text{jor}(B)$ and Theorem 3.3 shows that B acts compactly on B . By Proposition 3.1 there exists $A \in K^\infty$ with $\|A\| = \|B\|$ and $\eta(A) = B$. Then

by Theorem 3.2 , $A \in \underline{A}$.

COROLLARY 3.5.(Hartman[16], Page[24]) Let B be a Hankel operator relative to a shift of finite multiplicity. Then B is compact if and only if $B = \eta(A)$ for some $A \in \underline{A}$.

Proof. In view of Theorem 3.4 it suffices to show that $\eta(A)$ is compact if $A \in \underline{A}$. Suppose that $C = W^{-n}T$ with $T \in H^\infty$. Then as before $\eta(C)U^{n+1} = \eta(CW^{n+1}) = \eta(WT) = 0$. Thus

$$\eta(C) = \eta(C)(I - U^{n+1}U^{*n+1}) .$$

Since $I - U^{n+1}U^{*n+1}$ is a finite rank projection when U has finite multiplicity , $\eta(C)$ is compact. The set of such C is dense in \underline{A} and so , since η is linear and contractive , $\eta(A)$ is compact for $A \in \underline{A}$.

Remarks. (1) The equality $\text{twi}(U) = \text{pr}(\text{twi}(W))$ and the existence for each $B \in \text{twi}(U)$ of $A \in \text{twi}(W)$ with $B = \text{pr}(A)$ and $\|B\| = \|A\|$ hold when U is the restriction to an invariant subspace of any unitary W .

(2) Let $B = \eta(A)$ with $A \in K^\infty$. It follows easily from the proof of Theorem 3.2 that $A \in \underline{A}$ if and only if

$$\lim_{n \rightarrow \infty} \|BU^n\| = 0 .$$

Suppose now that L is separable . We show below how we can interpret the symbol set \underline{A} in terms of a familiar function space representation for U .

The following discussion of function space models for shifts of countable multiplicity is essentially the development given by Fillmore [14] and the details we omit can be found there. These representations and the characterisation of the reducing subspaces of a unilateral shift will prove useful in chapter 5 where we develop models for doubly commuting contractions in terms of the Hardy spaces on the torus.

Let L be a separable Hilbert space. A function f defined on the unit circle and taking values in L is called measurable if the complex valued function $(f(\cdot), x)$ is measurable for each x in L . Since L is separable, if f and g are measurable then so is $(f(\cdot), g(\cdot))$. $L^2[L]$ is the Hilbert space of functions f on the unit circle to L which are measurable and such that

$$\|f\|^2 = \frac{1}{2\pi} \int \|f(\theta)\|^2 d\theta < \infty,$$

functions which differ only on a set of measure zero being identified. (The ambiguous notation for norms should cause no confusion in practise). The inner product on $L^2[L]$ is given by

$$(f, g) = \frac{1}{2\pi} \int (f(\theta), g(\theta)) d\theta.$$

Each f in $L^2[L]$ admits a unique orthogonal expansion

$$f(\theta) = \sum_{-\infty < k < \infty} x_k e^{ik\theta},$$

with x_k in L and $\|f\|^2 = \sum \|x_k\|^2$. It follows that the mapping

$$\oplus_{-\infty < n < \infty} W^n c_n \rightarrow \sum_{-\infty < n < \infty} c_n e^{in\theta}$$

is a unitary mapping of K onto $L^2[L]$.

Moreover this mapping sends H onto $H^2[L]$ where

$$H^2[L] = \{f \in L^2[L] ; x_k = 0, k < 0\}.$$

If χ is the identity function on the unit circle then the above unitary establishes a unitary equivalence between W and the multiplication operator $\bar{\chi} : f \rightarrow \chi f$ on $L^2[L]$.

An operator valued function A from the unit circle to $B(L)$ is measurable if $A(\cdot)x$ is measurable for all x in L . Let $L^\infty[B(L)]$ denote the space of measurable functions $A : T \rightarrow B(L)$ such that $\|A(\cdot)\|$ is essentially bounded. For such A the multiplication operator \bar{A} on $L^2[L]$ is defined by

$$(\bar{A}f)(\theta) = A(\theta)f(\theta),$$

and we have $\|\bar{A}\| = \|A\|_\infty = \text{ess sup } \|A(\theta)\|$.

As in the classical single multiplicity case, the commutant of the bilateral shift $\bar{\chi}$ is precisely the space of operators \bar{A} for A in $L^\infty[B(L)]$.

Let $H^\infty[B(L)]$ be the space of those Φ in $L^\infty[B(L)]$ such that $(\Phi(e^{i\theta})x, y)$ is in H^∞ for each pair of vectors x and y in L and let $C[B(L)]$ denote the subspace of functions Φ in $L^\infty[B(L)]$ such that $\|\Phi(\cdot)\|$ is continuous. $\Phi : T \rightarrow B(L)$ is continuous.

It is routine to check that, under the equivalence above, the operators in \mathcal{A} correspond to those operators on $L^2[L]$ arising from the functions Φ in $\tilde{\mathcal{A}}$ where $\tilde{\mathcal{A}}$ is the closure in $L^\infty[B(L)]$ of

$$\bigcup_{n \in \mathbb{N}} \chi^{-n} H^\infty[B(L)].$$

We now use Rudin's theorem (Theorem 1.1) to show that $\tilde{\mathcal{A}} = H^\infty[B(L)] + C[B(L)]$.

Cesaro Means. Let $\Phi \in L^\infty[B(L)]$. Then

$$(x, y) \rightarrow \frac{1}{2\pi} \int e^{-ik\theta} (\Phi(\theta)x, y) d\theta,$$

is a bounded sesquilinear form on $L \times L$, $k \in \mathbb{Z}$, with bound not greater than $\|\phi\|$. Thus there exists an operator $\phi_k \in B(L)$ such that

$$(\phi_k x, y) = \frac{1}{2\pi} \int e^{-ik\theta} (\phi(\theta)x, y) d\theta .$$

ϕ_k is called the k^{th} Fourier coefficient of ϕ . Let

$$S_n(\theta) = \sum_{-n \leq k \leq n} e^{ik\theta} \phi_k , \quad n=0,1,2 \dots , \text{ and}$$

$$T_n(\theta) = \frac{1}{n} \sum_{0 \leq j < n} S_j(\theta) , \quad n=1,2,\dots .$$

The function $T_n(\theta)$ is the n^{th} Cesaro mean of the Fourier series for ϕ .

PROPOSITION 3.6. Let $\phi \in C[B(L)]$. Then T_n converges to ϕ uniformly on T .

Proof. The proof parallels the classical proof for Fejer's convergence theorem. For x and y in L and ϕ in $C[B(L)]$ let

$$\phi(e^{i\theta}) = (\phi(\theta)x, y)$$

and
$$\sigma_n(e^{i\theta}) = (T_n(\theta)x, y) , \quad n=1,2,\dots .$$

The functions σ_n , $n=1,2,\dots$, are the usual Cesaro means for the Fourier series of ϕ . Let

$$K_n(t) = \left[\frac{\sin(nx)}{\frac{x}{2}} / \frac{\sin(\frac{x}{2})}{\frac{x}{2}} \right]^2 / n \quad (x \in [-\pi, \pi])$$

be the usual Fejer kernel so that

$$\sigma_n(e^{it}) = \frac{1}{2\pi} \int \phi(e^{is}) K_n(t-s) ds .$$

Recall the following properties of K_n (See Hoffman [17] p17) for each n .

- (i) $K_n \geq 0$.
- (ii) K_n has mean equal to 1.
- (iii) If I is any open interval about $x=0$ then

$$\lim_{n \rightarrow \infty} \sup_{x \notin I} |K_n(x)| = 0 \quad (|x| < \pi) .$$

We have

$$\sigma_n(e^{it}) - \phi(e^{it}) = \frac{1}{2\pi} \int [\phi(e^{i(t-s)}) - \phi(e^{it})] K_n(s) ds$$

Thus for $\delta > 0$

$$|\sigma_n(e^{it}) - \phi(e^{it})| \leq \sup_{-\delta \leq s \leq \delta} |\phi(e^{i(t-s)}) - \phi(e^{it})| + 2\|\phi\|_\infty \sup_{|s| \geq \delta} K_n(s) \quad (1)$$

Since $\phi \in C[B(L)]$, ϕ is uniformly continuous, and given $\epsilon > 0$

there exists $\delta' > 0$ such that

$$\begin{aligned} |\phi(e^{i(t-s)}) - \phi(e^{it})| &= |(\phi(e^{i(t-s)}) - \phi(e^{it}))_{x,y}| \\ &\leq \epsilon \|x\| \|y\|, \end{aligned}$$

when $|s| < \delta'$. Moreover, note that δ' is independent of x and y .

Thus, by (1) above,

$$|\sigma_n(e^{it}) - \phi(e^{it})| \leq \epsilon \|x\| \|y\| + 2\|\phi\|_\infty \sup_{|s| \geq \delta'} K_n(s)$$

Using property (iii) of the Fejer kernel it now follows that

$$|\sigma_n(e^{it}) - \phi(e^{it})| \leq \epsilon \|x\| \|y\| + 2\|\phi\| \|x\| \|y\| \epsilon$$

for $n > N(\epsilon)$, where $N(\epsilon)$ is a natural number independent of x and y . Thus

$$\begin{aligned} \|(\sigma_n - \phi)(e^{it})\| &= \sup\{ |(\sigma_n(e^{it}) - \phi(e^{it}))| ; \|x\|, \|y\| \leq 1 \} \\ &\leq \epsilon(1 + 2\|\phi\|) \end{aligned}$$

when $n > N(\epsilon)$. Thus $T_n \rightarrow \phi$ uniformly as $n \rightarrow \infty$ completing the proof.

THEOREM 3.7 $\tilde{A} = H^\infty[B(L)] + C[B(L)]$.

Proof. Let $X = L^\infty[B(L)]$, $Y = C[B(L)]$ and $Z = H^\infty[B(L)]$. Let

$\Psi = \{\Delta_n ; \Delta_n : \phi \rightarrow T_n, \phi \in X\}$. We intend to apply Rudin's theorem, ie.

Theorem 1.1. Clearly a) holds and b) follows easily from the definition

of $H^\infty[B(L)]$. Proposition 3.6 shows that d) holds. Assuming the

well known scalar case we also have $\|\Delta_n\| \leq 1$ $n = 1, 2, \dots$. In fact

if x, y, ϕ, σ_n and T_n are related as in the proof of

Proposition 3.6, except with $\phi \in L^\infty[B(L)]$, then

$$\begin{aligned}
|((\Lambda_n \phi)(e^{it})_{x,y})| &= |(T_n(e^{it})_{x,y})| \\
&= |\sigma_n(e^{it})| \leq \|\phi\|_\infty \\
&= \|(\phi(\cdot)_{x,y})\|_\infty \leq \|\phi\|_\infty \|x\| \|y\|
\end{aligned}$$

for x and y in L . Thus $\|\Lambda_n\|_\infty \leq 1$ and part c) of Rudin's theorem holds. It now follows that $H^\infty[B(L)] + C[B(L)]$ is closed and also contains \tilde{A} . But Proposition 3.6 shows that $\tilde{A} \supset H^\infty[B(L)] + C[B(L)]$ and the theorem follows.

Remarks. 1. When U has infinite multiplicity some of the Hankel operators $\eta(A)$ with $A \in \underline{A}$ fail to be compact. Page [24] has characterised the compact Hankel operators in this case, for countable multiplicity shifts, in terms of the function space model above. He shows that a Hankel operator is compact if and only if its symbol is in $H^\infty[B(L)] + C[K(L)]$, where $C[K(L)]$ is the space of functions in $C[B(L)]$ taking values in $K(L)$ the ideal of compact operators on L . The proof that this symbol set is closed uses duality techniques similar to Sarason's original proof that $H^\infty + C$ is closed [33]. Such methods do not appear to be applicable to \underline{A} except when L is finite dimensional.

2. It is amusing to note that Hartman's theorem implies that $H^\infty + C$ is closed, a fact not used in the original proof.

Problem. Let \underline{S} be a closed linear space of operators on L . Is $H^\infty[B(L)] + C[\underline{S}]$ closed?

We complete this chapter by giving a characterisation of the reducing subspaces of a unilateral shift (as given in [14]). We shall use this result in chapter 5.

Let $V = \bar{\chi}|H^2[L]$ be the unilateral shift on $H^2[L]$. The commutant of V is given by

$$\{\bar{A}_+ = \bar{A}|H^2[L], A \in H^\infty[B(L)]\}$$

In the following lemma we identify L with the subspace of $H^2[L]$ consisting of constant L valued functions on the unit circle.

LEMMA 3.8. Let $A \in H^\infty[B(L)]$. Then the following statements are equivalent.

- (i) \bar{A}_+ and V^* commute.
- (ii) $\bar{A}_+L \subset L$.
- (iii) A is constant almost everywhere.

Proof. If \bar{A}_+ and V^* commute then $V^*\bar{A}_+L \subset \bar{A}_+V^*L = \{0\}$, so \bar{A}_+L is contained in L , the null space of V^* . Conversely suppose $\bar{A}_+L \subset L$. Then $\bar{A}_+V^* = 0 = V^*\bar{A}_+$ on L , while for x in L and $n > 0$,

$$\bar{A}_+V^*V^n x = \bar{A}_+V^{n-1}x = V^*V\bar{A}_+V^{n-1}x = V^*\bar{A}_+V^n x,$$

so \bar{A}_+ and V^* commute on V^nL .

Suppose $\bar{A}_+L \subset L$, and define an operator C on L by $C = \bar{A}_+|L$. If $\{b_\alpha\}$ is an orthonormal basis of L , then for each α there is a set N_α of measure zero such that $A(\theta)b_\alpha = Cb_\alpha$, $\theta \notin N_\alpha$. Since L is separable $N = \cup N_\alpha$ is of measure zero and $A(\theta) = C$ for $\theta \notin N$. Conversely, if $A(\theta) = C$ a.e. then for all $x \in L$, $(\bar{A}x)(\theta) = Cx$ a.e. so that $\bar{A}x \in L$.

PROPOSITION 3.9 A subspace M of $H^2[L]$ reduces V if and only if $M = H^2[G]$ for some subspace G of L .

Proof. Let Q be the projection on a reducing subspace M , so that $QV = VQ$, $QV^* = V^*Q$ and, by Lemma 3.8, $QL = L$. This implies

$G = QL$ is closed . Since $G \subset M$ and M is invariant it follows that

$H^2(G) \subset M$. On the other hand , if $\sum_{0 \leq n < \infty} v^n x_n \in M$ then

$$\sum v^n x_n = q(\sum v^n x_n) = \sum v^n q x_n \in H^2(G) .$$

CHAPTER 4

HANKEL OPERATORS ON THE LINE

In this chapter we prove Nehari and Hartman theorems for Hankel operators on $H^2(\mathbb{R})$. We also give the essential spectrum of certain (non compact) Hankel operators and show that $H^{\infty}(\mathbb{R}) + AP$ is closed where AP is the space of uniformly almost periodic functions on the real line.

We first give two applications of Theorem 2.7.

Let $L^2(\mathbb{R})$ be the Hilbert space of square integrable functions on the real line \mathbb{R} , with subspace $H = L^2(\mathbb{R}_+)$ of functions vanishing on the non positive real numbers. For $\alpha \in \mathbb{R}$, $W(\alpha)$ is the unitary operator on $L^2(\mathbb{R})$ defined by

$$(W(\alpha)f)(x) = f(x - \alpha) \quad (x \in \mathbb{R}).$$

Clearly H is invariant for $W(\alpha)$, $\alpha \geq 0$, and, if $U(\alpha) = W(\alpha)|_H$ for $\alpha \geq 0$ then $\underline{U} = \{U(\alpha); \alpha \in \mathbb{R}_+\}$ is a strongly continuous semigroup of isometries with minimal unitary dilation $\underline{W} = \{W(\alpha); \alpha \in \mathbb{R}_+\}$. Let F be the unitary on $L^2(\mathbb{R})$ defined by

$$(Ff)(x) = f(-x) \quad (x \in \mathbb{R}).$$

The following theorem characterises Hankel operators relative to \underline{U}

THEOREM 4.1. $\text{twi}(\underline{U}) = \text{pr}(\text{twi}(\underline{W})) = \text{pr}\{FA; A \in \{\underline{W}\}'\}$. Moreover for each $S \in \text{twi}(\underline{U})$ there exists $A \in \{\underline{W}\}'$ such that $S = \text{pr}(FA)$ and $\|A\| = \|S\|$.

Proof. For $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} (FW(\alpha)f)(x) &= ((W(\alpha)f)(-x) = f(-x - \alpha) \\ &= (Ff)(x + \alpha) = (W(-\alpha)Ff)(x) \end{aligned}$$

Hence $FW(\alpha) = W(-\alpha)F = W(\alpha)^*F$ for $\alpha \in \mathbb{R}$. Thus $F \in \text{twi}(W)$ and $F \in \text{twi}(W^*)$. For $C \in \{W\}'$, the commutant of W , we have $FCW(\alpha) = FW(\alpha)C = W(\alpha)^*FC$, $\alpha \geq 0$. Thus $FC \in \text{twi}(W)$ and $F\{W\}' \subset \text{twi}(W)$.

Conversely let $T \in \text{twi}(W)$. Then $FTW(\alpha) = FW(\alpha)^*T = W(\alpha)FT$ for $\alpha \geq 0$ and $FT \subset \{W\}'$. Since $F^2 = I$, $T \in F\{W\}'$ and hence $F\{W\}' = \text{twi}\{W\}$.

If $S \in \text{twi}(U)$ by Theorem 2.7 there exists $A \in \{W\}'$ such that $S = \text{pr}(FA)$ and $\|FA\| = \|S\|$, ie. $\|A\| = \|S\|$.

It remains to show that $\text{pr}(\text{twi}(W)) \subset \text{twi}(U)$ and this follows in an elementary way just as in the last part of the proof of Proposition 3.1.

Let $H^2(\mathbb{R})$ be the Hardy subspace of $L^2(\mathbb{R})$ consisting of boundary value functions of those functions f , analytic in the open upper half plane, such that

$$\sup_{0 < y < \infty} \int |f(x + iy)|^2 dx < \infty.$$

Let Q be the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. For $\phi \in L^\infty(\mathbb{R})$, M_ϕ is the multiplication operator on $L^2(\mathbb{R})$ defined by

$$M_\phi f = \phi f \quad (f \in L^2(\mathbb{R})).$$

DEFINITION 4.2. For $\phi \in L^\infty(\mathbb{R})$, $T_\phi = QM_\phi|_{H^2(\mathbb{R})}$ is the Toeplitz operator on $H^2(\mathbb{R})$ with symbol ϕ .

Such Toeplitz operators have been studied by Coburn and Douglas [7].

DEFINITION 4.3. For $\phi \in L^\infty(\mathbb{R})$, $S_\phi = QFM_\phi|_{H^2(\mathbb{R})}$ is the Hankel operator on $H^2(\mathbb{R})$ with symbol ϕ .

Let T_λ (resp. M_λ) denote the Toeplitz (resp. multiplication) operator with symbol $e^{i\lambda x}$, $\lambda \in \mathbb{R}$. Then $\mathcal{T} = \{T_\lambda; \lambda \geq 0\}$ is a

strongly continuous semigroup of isometries on $H^2(\mathbb{R})$ with minimal unitary dilation $\underline{M} = \{M_\lambda; \lambda \geq 0\}$ on $L^2(\mathbb{R})$. The following Nehari type theorem characterises Hankel operators relative to \underline{T} .

THEOREM 4.4. $\text{twi}(\underline{T}) = \text{pr}(\text{twi}(\underline{M})) = \{S_\phi; \phi \in L^\infty(\mathbb{R})\}$. Moreover for each $S \in \text{twi}(\underline{T})$ there exists $\phi \in L^\infty(\mathbb{R})$ such that $S = S_\phi$ and $\|\phi\| = \|S\|$.

Proof. It is well known that $\{\underline{M}\}' = \{M_\phi; \phi \in L^\infty(\mathbb{R})\}$. Since $F \in \text{twi}(\underline{M})$ and $F \in \text{twi}(\underline{M}^*)$ the theorem is proved in the same way as Theorem 4.1.

Remark. The Paley Wiener theorem ([17] p131) shows that the Plancherel transform ([30]) provides a unitary equivalence between \underline{U} acting on H and \underline{T} acting on $H^2(\mathbb{R})$. It follows that Theorems $\S 4.1$ and $\S 4.4$ are equivalent in the sense that each can be readily deduced from the other.

We now consider Hartman theorems for Hankel operators on $H^2(\mathbb{R})$.

Notation. Let ν be the symbol map defined by $\nu(\phi) = S_\phi$ for $\phi \in L^\infty(\mathbb{R})$. Let $H^\infty(\mathbb{R})$ be the usual space of boundary functions of functions bounded and analytic in the upper half plane. Since

$$H^\infty(\mathbb{R}) = \{\phi \in L^\infty(\mathbb{R}); \phi H^2(\mathbb{R}) \subset H^2(\mathbb{R})\}$$

and since

$$FH^2(\mathbb{R}) = H^2(\mathbb{R})^\perp$$

it follows that

$$\text{kern } \nu = H^\infty(\mathbb{R}).$$

It is not at first sight obvious which symbol sets may be appropriate for theorems of a Hartman type. We shall consider the following

- $C_0(\mathbb{R})$: The space of continuous functions on the real line vanishing at $+\infty$ and $-\infty$.
- $C_\infty(\mathbb{R})$: The space of continuous functions on the real line possessing limits (possibly different) at $+\infty$ and $-\infty$.
- BUC : The space of bounded uniformly continuous functions on the real line.
- AP : The space of uniformly almost periodic functions on the real line.

It is well known ([9] p200) that the Toeplitz operators on the line are unitarily equivalent to the Toeplitz operators on the circle. We show that the corresponding result is true for the Hankel operators. Using this equivalence we shall see that S_ϕ is compact if and only if $\phi \in H^\infty(\mathbb{R}) + C_0(\mathbb{R})$. We shall also use this equivalence to compute the essential spectrum of S_ϕ when $\phi \in H^\infty(\mathbb{R}) + C_\infty(\mathbb{R})$.

It is natural to consider the following symbol set \underline{A} , associated with \underline{T} , which corresponds to the \underline{A} defined in chapter 3

$$\underline{A} = \left(\bigcup_{\lambda > 0} e^{-i\lambda x} H^\infty(\mathbb{R}) \right)^{-}$$

Sarason [34] has shown that $\underline{A} = H^\infty(\mathbb{R}) + \text{BUC}$. We shall characterise the Hankel operators with symbol in \underline{A} in terms of a compact action on an algebra of operators, in a similar way to that of Theorem 3.2.

We finish this chapter by showing that $H^\infty(\mathbb{R}) + \text{AP}$ is closed (and thus the Hankel operators with almost periodic symbol form a closed subspace). We also pose an abstract 'H[∞] + C closure' problem which contains the above and several other such theorems in the literature.

The Unitary Equivalence between $H^2(\mathbb{T})$ and $H^2(\mathbb{R})$.

The linear fractional map from the open unit disc D to the upper half plane is given by

$$\omega = \frac{i(1-z)}{1+z} \quad (z \in D).$$

The inverse transformation

$$z = \frac{1+i\omega}{1-i\omega}$$

shows that

$$e^{i\theta} = \frac{1+it}{1-it} \quad (\theta \in [-\pi, \pi], t \in \mathbb{R}).$$

Thus

$$\frac{d\theta}{2\pi} = \frac{dt}{\pi(1+t^2)}. \quad (1)$$

Define Z from $L^2(\mathbb{T})$ to $L^2(\mathbb{R})$ by the map $Zf = g$ where

$$g(t) = \frac{1}{\sqrt{\pi(1-it)}} f\left(\frac{1+it}{1-it}\right) \quad (t \in \mathbb{R}) \quad (2)$$

By (1) Z is an isometry and, since the inverse is well defined, Z is unitary. Moreover Z maps $H^2(\mathbb{T})$ onto $H^2(\mathbb{R})$.

Let J be the unitary on $L^2(\mathbb{T})$ defined by

$$(Jf)(e^{i\theta}) = f(e^{-i\theta}) \quad (f \in L^2(\mathbb{T})).$$

For $f \in L^2(\mathbb{T})$ we have, by (2),

$$\begin{aligned} (FZf)(t) &= \frac{1}{\sqrt{\pi(1+it)}} f\left(\frac{1-it}{1+it}\right) \\ &= \frac{1-it}{1+it} \frac{1}{\sqrt{\pi(1-it)}} (Jf)\left(\frac{1+it}{1-it}\right) \\ &= (ZM_Z Jf)(t) = (ZJM_Z f)(t). \end{aligned}$$

Thus

$$JM_Z = Z^* FZ. \quad (3)$$

Let P be the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ so that for $\phi \in L^\infty(\mathbb{T})$ the Toeplitz and Hankel operators with symbol ϕ acting on $H^2(\mathbb{T})$ are defined by

$$T_\phi = PM_\phi|_{H^2(\mathbb{T})} \quad \text{and} \quad S_\phi = PJM_\phi|_{H^2(\mathbb{T})} \quad \text{respectively.}$$

For $\phi \in L^\infty(\mathbb{T})$ we define $\tilde{\phi} \in L^\infty(\mathbb{R})$ by

$$\tilde{\phi}(t) = \phi\left(\frac{1+it}{1-it}\right)$$

Since $M_{\tilde{\phi}} = ZM_\phi Z^*$, $\phi \in L^\infty(\mathbb{T})$, it now follows that

$$T_{\tilde{\phi}} = ZT_\phi Z^*|_{H^2(\mathbb{R})} \quad (4)$$

and

$$S_{\tilde{\phi}} = ZS_{z\phi} Z^*|_{H^2(\mathbb{R})} \quad (5)$$

(The dual use of the notation S_ψ should cause no confusion in practise as it will be clear from the context whether ψ is a function on \mathbb{R} or \mathbb{T}).

THEOREM 4.5. Let $\psi \in L^\infty(\mathbb{R})$. Then S_ψ is compact if and only if $\psi \in H^\infty(\mathbb{R}) + C_0(\mathbb{R})$.

Proof. Let $\phi \in L^\infty(\mathbb{T})$ be such that $\tilde{\phi} = \psi$. Then by (5) S_ψ is compact if and only if $S_{z\phi}$ is compact. Thus by Hartman's theorem (Cor.3.5 and Theorem 3.7) S_ψ is compact if and only if $z\phi \in H^\infty(\mathbb{T}) + C(\mathbb{T})$. But $z\phi \in H^\infty(\mathbb{T}) + C(\mathbb{T})$ if and only if $\tilde{\phi} \in H^\infty(\mathbb{R}) + C_0(\mathbb{R})$, completing the proof.

We now find the essential spectrum of S_ϕ when $\phi \in H^\infty(\mathbb{R}) + C_\infty(\mathbb{R})$. As one might hope, this can be expressed neatly in terms of the limits at $+\infty$ and $-\infty$ of the $C_\infty(\mathbb{R})$ component.

Notation. Let ϕ_0 be the function on the unit circle defined by

$$\phi_0(e^{it}) = \begin{cases} i(t + \pi)e^{it} & \pi \leq t < 2\pi \\ i(t - \pi)e^{it} & 0 \leq t < \pi \end{cases}$$

Then ϕ_0 is continuous apart from a jump discontinuity at 1.

In fact $\lim_{t \rightarrow 0^+} \phi_0(e^{it}) = -\pi i$ and $\lim_{t \rightarrow 0^-} \phi_0(e^{it}) = \pi i$. A simple

computation shows that

$$(S_{\phi_0} e_n, e_0) = (n + 1)^{-1} \quad n = 0, 1, 2, \dots$$

where $\{e_n; n \in \mathbb{Z}_+\}$ is the standard orthonormal basis for $H^2(\mathbb{T})$.

It follows that S_{ϕ_0} is the Hankel operator given by Hilbert's matrix $(a_{ij}) = (i + j + 1)^{-1}$ ($i, j = 0, 1, 2, \dots$).

Magnus [41] has shown that the spectrum of S_{ϕ_0} is purely continuous and is the interval $[0, \pi]$. Thus, denoting the essential spectrum of an operator T by $\sigma_e(T)$, we have the following theorem.

THEOREM 4.6 (Magnus) $\sigma_e(S_{\phi_0}) = [0, \pi]$.

Notation. For $\psi \in C_\infty(\mathbb{R})$ let $\psi_\infty = \frac{1}{2} \lim_{x \rightarrow \infty} (\psi(x) - \psi(-x))$.

For $\phi \in L^\infty(\mathbb{T})$ and $\alpha \in \mathbb{T}$ let $\phi^{(\alpha)} = \lim_{t \rightarrow \alpha^+} \frac{1}{2} (\phi(e^{it}) - \phi(e^{-it}))$

when this limit exists.

THEOREM 4.7. Let $\psi \in C_\infty(\mathbb{R})$. Then $\sigma_e(S_\psi) = \{it\psi_\infty; 0 \leq t \leq 1\}$.

Proof. Since $\sigma_e(S_{\phi_0}) = [0, \pi]$, and since, by Hartman's theorem

S_θ is compact when θ is a continuous function on \mathbb{T} , it follows that

$$\sigma_e(S_{\alpha\phi_0 + \theta}) = \{\alpha t \pi; 0 \leq t \leq 1\}, \quad \alpha \in \mathbb{C}. \quad (6)$$

Now if $C_1(T) = \{\alpha\phi_0 + \theta; \alpha \in \mathbb{C}, \theta \in C(T)\}$, then $C_1(T)$ is the space of all functions continuous on T apart from possibly a jump discontinuity at 1 . Moreover, if $\theta \in C_1(T)$, then (6) shows that

$$\sigma_e(S_\theta) = \{it\theta(0); 0 \leq t \leq 1\} \quad (7)$$

(since $\phi_0(0) = -\pi$).

Let W be the unitary operator on $H^2(T)$ defined by $(Wf)(z) = f(-z)$. Then $WJ = JW$ and for $\phi \in L^\infty(T)$ $W^*S_\phi W = S_{\phi'}$, where $\phi'(e^{it}) = \phi(-e^{it})$. If ϕ is continuous apart from a jump discontinuity at -1 then plainly $\phi' \in C_1(T)$ and also

$$\phi'(-\pi) = \phi'(0) \quad (8)$$

Thus by (7) and (8)

$$\sigma_e(S_\phi) = \{it\phi'(-\pi); 0 \leq t \leq 1\} \quad (9)$$

Let ϕ be the function on the unit circle such that $\tilde{\phi} = \psi$, where $\psi \in C_\infty(\mathbb{R})$. Then ϕ is continuous apart from (possibly) a jump discontinuity at -1 and $\phi(\pi) = -\psi_\infty$. Thus $(z\phi)(\pi) = \psi_\infty$, and from (5) and (9) it follows that

$$\sigma_e(S_\psi) = \{it\psi_\infty; 0 \leq t \leq 1\}.$$

Notation. For $\lambda > 0$ let B_λ denote the real norm closed algebra generated by I and T_λ .

The following theorem is a Hartman type theorem for Hankel operators on $H^2(\mathbb{R})$ with symbol in $\underline{A} = H^\infty(\mathbb{R}) + BUC$.

THEOREM 4.8. Let S be a Hankel operator on $H^2(\mathbb{R})$, then the following conditions are equivalent.

- (i) $S = S_\phi$ for some $\phi \in \underline{A}$.
- (ii) S acts compactly on B_λ for some $\lambda > 0$.
- (iii) S acts compactly on B_λ for all $\lambda > 0$.
- (iv) $\|ST_\lambda^n\| \rightarrow 0$ as $n \rightarrow \infty$ for some $\lambda > 0$.
- (v) $\|ST_\lambda^n\| \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$.

Proof. (ii) \Rightarrow (iv). Suppose that S acts compactly on B_λ for some $\lambda > 0$. Since $T_\lambda^n \in B_\lambda$ ($n = 1, 2, \dots$) there exists a subsequence $\{T_\lambda^{n_k}\}$ and $T \in B(H^2(\mathbb{R}))$ such that $ST_\lambda^{n_k} - T \rightarrow 0$ as $k \rightarrow \infty$. Since S is in $\text{twi}(\mathbb{T})$ this gives $T_\lambda^{*n_k} S - T \rightarrow 0$ as $k \rightarrow \infty$. However, T_λ^{*n} converges to zero in the strong operator topology (since T_λ^* is unitarily equivalent to $U(\lambda)^*$ on $L^2(\mathbb{R}_+)$). Therefore $T = 0$, and so $ST_\lambda^{n_k} \rightarrow 0$, as $k \rightarrow \infty$. Since this argument applies with the sequence $\{T_\lambda^n\}$ replaced by each of its subsequences, it follows that $ST_\lambda^n \rightarrow 0$ as $n \rightarrow \infty$ and (iv) holds. Similarly (iii) \Rightarrow (v).

(iv) \Rightarrow (i). Let $S = S_\phi$ for some $\phi \in L^\infty(\mathbb{R})$ and suppose that (iv) holds. Since $ST_\lambda^n \in \text{twi}(\mathbb{T})$, Theorem 4.4 gives the existence of $\phi_n \in L^\infty(\mathbb{R})$ with $ST_\lambda^n = S_{\phi_n}$ and $\|\phi_n\| = \|ST_\lambda^n\|$, $n \in \mathbb{N}$. It follows that $\phi e^{in\lambda x} - \phi_n = \psi_n \in \ker v = H^\infty(\mathbb{R})$. Thus $\|\phi - e^{-in\lambda x} \psi_n\| = \|\phi_n\| \rightarrow 0$ as $n \rightarrow \infty$, which shows that $\phi \in \underline{A}$ and (i) holds.

(i) \Rightarrow (iii). Let $\lambda > 0$. Suppose that $\phi = e^{-i\lambda n x} \psi$ with $\psi \in H^\infty(\mathbb{R})$, $n \in \mathbb{N}$. Then $S_\phi T_\lambda^n = S_\psi = 0$ and so $S_\phi = S_\psi (I - T_\lambda^n T_\lambda^{*n})$. Since $(I - T_\lambda^n T_\lambda^{*n})B_\lambda$ has finite dimension, $S_\psi B_\lambda$ has finite dimension and so S_ϕ acts compactly on B_λ . Given $\theta \in \underline{A}$ and $\epsilon > 0$ there exists ϕ of this form with $\|\theta - \phi\| < \epsilon$. Since the symbol map is linear and contractive and ϵ arbitrary we see that S_θ acts compactly on B_λ for $\lambda > 0$, $\theta \in \underline{A}$ and (i) holds.

Since (iii) \Rightarrow (ii) and (v) \Rightarrow (iv) trivially, the other implications follow.

We now show that if AP is the space of uniformly almost periodic functions on \mathbb{R} , that is, the uniform closure of the trigonometric polynomials, then $H^\infty(\mathbb{R}) + AP$ is closed. The proof uses Rudin's theorem (Theorem 1.1) and certain Bochner-Fejér type maps which play the role of the linear maps $\Lambda \in \Psi$.

The usual Fejér kernel on \mathbb{R} is given by

$$K_n(t) = \sum_{|\nu| < n} \left(1 - \frac{|\nu|}{n}\right) e^{-i\nu t} = \frac{1}{n} \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2, \quad n \in \mathbb{N}$$

Let $K_{\underline{n}, \underline{\beta}}(t) = \prod_{1 \leq i \leq p} K_{n_i}(\beta_i t)$, where $\beta_1, \beta_2, \dots, \beta_p$ are real linearly independent numbers and $n_i \in \mathbb{N}$, $1 \leq i \leq p$. These are the Bochner-Fejér kernels ([2] p46). over the
rationals

For $f \in AP$ the mean

$$M(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(s) ds$$

always exists ([2] p12). Thus for $f \in AP$ we can consider

$$\sigma_{\underline{n}, \underline{\beta}}(x) = M(f_x K_{\underline{n}, \underline{\beta}})$$

where $f_x(t) = f(x + t)$. We have

$$\sigma_{\underline{n}, \underline{\beta}}(x) = \sum_{\substack{|\nu_1| < n_1 \\ \vdots \\ |\nu_p| < n_p}} \left(1 - \frac{|\nu_1|}{n_1}\right) \dots \left(1 - \frac{|\nu_p|}{n_p}\right) M\{f(t) e^{-i(\nu_1 \beta_1 + \dots + \nu_p \beta_p)t}\}_x e^{i(\nu_1 \beta_1 + \dots + \nu_p \beta_p)x}.$$

$\sigma_{\underline{n}, \underline{\beta}}$ is the Bochner Fejér polynomial for f corresponding to $\underline{n}, \underline{\beta}$. The following approximation theorem can be found in [2] p50.

THEOREM 4.9. For each $f \in AP$ there exist Bochner Fejér polynomials for f converging uniformly to f .

THEOREM 4.10. $H^\infty(\mathbb{R}) + AP$ is closed.

Proof. Let $\underline{M} = \{f \in L^\infty(\mathbb{R}) ; M(f) \text{ exists}\}$. Then the mean, $M(\cdot)$, is a bounded linear translation invariant functional on \underline{M} such that $M(1) = \|M\| = 1$. Let \tilde{M} be a translation invariant functional on $L^\infty(\mathbb{R})$ extending M such that $\tilde{M} = \tilde{M}(1) = 1$. (See p96b).

Define $\Lambda_{n,\beta}$ on $L^\infty(\mathbb{R})$ by

$$(\Lambda_{n,\beta} f)(x) = \tilde{M}\{f_x K_{n,\beta}\} \quad (f \in L^\infty(\mathbb{R})) .$$

Since \tilde{M} is a state on $L^\infty(\mathbb{R})$ and $K_{n,\beta}(t) \geq 0$, $t \in \mathbb{R}$,

$$|(\Lambda_{n,\beta} f)(x)| \leq \|f\| \tilde{M}\{K_{n,\beta}\} = \|f\| . \quad (1)$$

Since \tilde{M} is translation invariant,

$$\begin{aligned} (\Lambda_{n,\beta} f)(x) &= \sum_{\substack{|\nu_1| < n_1 \\ \vdots \\ |\nu_p| < n_p}} \left(1 - \frac{|\nu_1|}{n_1}\right) \cdots \left(1 - \frac{|\nu_p|}{n_p}\right) \tilde{M}\{f(t) e^{-i(\nu_1 \beta_1 + \cdots + \nu_p \beta_p)t}\}_x \\ &\quad e^{i(\nu_1 \beta_1 + \cdots + \nu_p \beta_p)x} . \quad (2) \end{aligned}$$

Thus $\Lambda_{n,\beta} : L^\infty(\mathbb{R}) \rightarrow AP$. (3)

For $f \in AP$, by Theorem 4.9, there exist $\Lambda^{(k)} \in \{\Lambda_{n,\beta}\}$ such that $\Lambda^{(k)} f \rightarrow f$ in $L^\infty(\mathbb{R})$. In view of this and (1) and (3) above, to show that $H^\infty(\mathbb{R}) + AP$ is closed, it suffices, by Theorem 1.1, to show that $\Lambda_{n,\beta} : H^\infty(\mathbb{R}) \rightarrow H^\infty(\mathbb{R})$. Thus by (2) above, it suffices to show that $\tilde{M}\{f(t) e^{i\lambda t}\} = 0$ for $\lambda > 0$, $f \in H^\infty(\mathbb{R})$.

For $f \in H^\infty(\mathbb{R})$ and $\lambda > 0$,

$$\int_{-T}^T f(t) e^{i\lambda t} dt = \int_C f(z) e^{i\lambda z} dz$$

where C is the semicircular arc from $-T$ to T in the upper half plane.

Thus

$$\begin{aligned} \left| \int_{-T}^T f(t) e^{i\lambda t} dt \right| &\leq \|f\|_\infty \int_C |e^{i\lambda z}| |dz| \\ &\leq \|f\|_\infty \pi/\lambda , \end{aligned}$$

the last inequality following from Jordan's lemma. Thus $M(f(t) e^{i\lambda t})$ exists and is zero, completing the proof.

COROLLARY 4.11. The Hankel operators on $H^2(\mathbb{R})$ with uniformly almost periodic symbol form a closed subspace of $B(H^2(\mathbb{R}))$.

Proof. Immediate from Theorem 4.10 / and Theorem 4.4.

We close this chapter by posing the following abstract $H^\infty + C$ closure problem. It is easy to check that special cases of this problem include 1) the result above 2) Sarason's classical result and 3) the generalisations to functions on the polydisc and the ball in \mathbb{C}^n discussed by Rudin in [32] .

Problem . Let F be a commutative family of operators on a Hilbert space. Let H^∞ be the weakly closed algebra generated by F and let C be the C^* -algebra generated by F . Is $H^\infty + C$ closed ?

CHAPTER 5

DOUBLY COMMUTING CONTRACTIONS

In this chapter we develop some canonical models for doubly commuting contractions in terms of the Hardy spaces on the torus . We shall also discuss the lifting problem for operators which intertwine families of doubly commuting contractions and related problems.

The following lemma shows how certain compressions of doubly commuting contractions are again doubly commuting .

LEMMA 5.1. Let T_1 and T_2 be doubly commuting operators on a Hilbert space. Let M_1 (resp. M_2) be a reducing subspace for T_1 (resp. T_2). If the projections onto M_1 and M_2 commute and if $S_i = \text{pr}_{M_1 \cap M_2}(T_i)$, $i = 1, 2$, then S_1 and S_2 doubly commute .

Proof. For $i = 1, 2$, let P_i be the projection onto M_i so that $P_i T_i = T_i P_i$. For x, y in $M_1 \cap M_2$ we have

$$\begin{aligned} (S_1 S_2 x, y) &= (S_2 x, S_1^* y) = (P_1 P_2 T_2 x, T_1^* y) \\ &= (T_2 P_2 x, T_1^* P_1 y) = (T_2 x, T_1^* y) \\ &= (T_1 T_2 x, y) . \end{aligned}$$

That is $S_1 S_2 = \text{pr}(T_1 T_2)$ (compression to $M_1 \cap M_2$ understood) .

Similar computations show that $S_2 S_1 = \text{pr}(T_2 T_1)$, $S_1^* S_2 = \text{pr}(T_1^* T_2)$ and $S_2 S_1^* = \text{pr}(T_2 T_1^*)$. Since the T_i doubly commute so do the S_i .

Notation. Let \underline{R} be a Hilbert space and let $K(\underline{R})$ be the Hilbert space of square summable functions from \mathbb{Z}_+^2 to \underline{R} . That is

$$K(\underline{R}) = \{f; \mathbb{Z}_+^2 \rightarrow \underline{R}; \sum_{n, m \in \mathbb{Z}_+} \|f(n, m)\|^2 < \infty\} .$$

For convenience vectors in $K(\underline{R})$ will be denoted by a double sequence

$$\underline{x} = (x_{i,j})_{i,j=0}^{\infty}, \quad x_{i,j} \in \underline{R}.$$

Let U_1 (resp. U_2) be the unilateral shift on $K(\underline{R})$ defined by $(U_1 \underline{x})_{n,m} = x_{n+1,m}$ (resp. $(U_2 \underline{x})_{n,m} = x_{n,m+1}$) for $\underline{x} \in K(\underline{R})$.

One easily checks that U_1 and U_2 are doubly commuting unilateral shifts of infinite multiplicity.

Remarks. The restriction of doubly commuting contractions to a common invariant subspace need not doubly commute. For example let M be the subspace of $K(\underline{R})$ consisting of those vectors \underline{x} such that $x_{0,0} = 0$. Then M is invariant for U_1 and U_2 , but if $S_i = U_i|_M$ $i = 1, 2$, then S_1 and S_2 do not doubly commute. To see this take a vector $(x_{i,j})$ in M such that $x_{i,j} \neq 0$ if and only if $i=1, j=0$. Then $S_2 S_1^* \underline{x} = 0$ and $S_1^* S_2 \underline{x} \neq 0$.

The following theorem is a two dimensional version of a result of deBranges and Rovnyak (see for example [14]).

THEOREM 5.2 Let T_1 and T_2 be doubly commuting contractions on a Hilbert space H such that $T_i^n \rightarrow 0$ strongly as $n \rightarrow \infty$. Then there exists a Hilbert space \underline{R} , a subspace G of $K(\underline{R})$ invariant for U_1^* and U_2^* and a unitary W from H to G such that

$$W T_i W^* = U_i^*|_G \quad (i = 1, 2).$$

Proof. Let $R_i = (I - T_i^* T_i)^{\frac{1}{2}}$, $i = 1, 2$. Then $R_1 R_2 = R_2 R_1$, $R_1 T_2 = T_2 R_1$ and $R_2 T_1 = T_1 R_2$. Let $R = R_1 R_2$ and $\underline{R} = (R_1 R_2 H)^{\overline{}}$. Define Z from H to $K(\underline{R})$ by

$$(Zx)_{i,j} = R T_1^i T_2^j x \quad (i, j \in \mathbb{Z}_+).$$

Since $\|R_1 T_1^n y\|^2 = \|T_1^n y\|^2 - \|T_1^{n+1} y\|^2$, $n = 0, 1, 2, \dots$, $y \in H$, we have

$$\begin{aligned}
\sum_{0 \leq k < \infty} \|R_i T_i^k y\|^2 &= \lim_{n \rightarrow \infty} \sum_{0 \leq k \leq n} \|R_i T_i^k y\|^2 \\
&= \lim_{n \rightarrow \infty} \{ \|y\|^2 - \|T_i^{n+1} y\|^2 \} \\
&= \|y\|^2 .
\end{aligned}$$

Thus

$$\begin{aligned}
\|Zx\|^2 &= \sum_{i, j \in \mathbb{Z}_+} \|R_1 R_2 T_1^i T_2^j x\|^2 \\
&= \sum_{j \in \mathbb{Z}_+} \sum_{i \in \mathbb{Z}_+} \|R_1 T_1^i R_2 T_2^j x\|^2 \\
&= \sum_{j \in \mathbb{Z}_+} \|R_2 T_2^j x\|^2 = \|x\|^2, x \in H .
\end{aligned}$$

It follows that Z is a well defined isometry from H to $K(\underline{R})$.

Moreover $ZT_i = U_i^* Z$ for $i = 1, 2$. Thus $G = ZH$ is invariant for U_1^* and U_2^* . Let Z_0 be the unitary from H to G such that $Z_0^* = Z^*|_G$. Then $Z_0 T_i Z_0^* = U_i^*|_G$, $i = 1, 2$, and thus taking $W = Z_0$ completes the proof.

Notation. Let $T_1, T_2, R_1, R_2, R, \underline{R}, U_1, U_2$ and G be as in the proof of Theorem 5.2. In addition let $\underline{R}_i = (R_i H)^-$ and let G_i be the closed linear span of $\{U_i^n G; n \in \mathbb{Z}_+\}$.

$$\text{LEMMA 5.3 } G_1 = \{ \underline{x} \in K(\underline{R}); x_{n,m} = R_2 T_2^m y_n \text{ with } y_n \in \underline{R}_1 \} \quad (1)$$

$$\text{and } G_2 = \{ \underline{x} \in K(\underline{R}); x_{n,m} = R_1 T_1^n y_m \text{ with } y_m \in \underline{R}_2 \} . \quad (2)$$

Proof. Let $\underline{x}^{(k)} \in \text{lin span } \{U_1^n G; n \in \mathbb{Z}_+\}$ be a Cauchy sequence converging to y in G_1 . Then $x_{n,m}^{(k)} = R_2 T_2^m x_n^{(k)}$ where $x_n^{(k)}$ is in $\text{lin span } \{R_1 T_1^i H, i \in \mathbb{Z}_+\}$. For fixed n

$$\omega_n^{(k)} = (R_2 x_n^{(k)}, R_2 T_2 x_n^{(k)}, R_2 T_2^2 x_n^{(k)}, \dots)$$

is a Cauchy sequence in $\bigoplus_{n \geq 0} \underline{R}$ (since $\omega_n^{(k)}$ is the image of $\underline{x}^{(k)}$)

under the projection onto $\{\underline{x}; x_{i,j} = 0 \text{ } i \neq n\}$. As we have seen in the proof of Theorem 5.2

$$y \rightarrow (R_2 y, R_2 T_2 y, R_2 T_2^2 y, \dots)$$

is an isometry, Thus $x_n^{(k)}$ is a Cauchy sequence (in k) converging to y_n say, in \underline{R}_1 , as $k \rightarrow \infty$. Thus

$$y_{n,m} = \lim_k x_n^{(k)} = R_2 T_2^m y_n.$$

It follows that G_1 is contained in the right hand side of (1).

To show the reverse inclusion, since G_1 is closed and invariant under U_1 , it suffices to show that $x' \in G_1$ when

$$x'_{n,m} = \begin{cases} R_2 T_2^m y & n=0, y \in \underline{R}_1 \\ 0 & n>0. \end{cases}$$

In fact we may assume that $y = R_1 y'$ where $y' \in H$. But then

$$x' = Z y' - U_1 Z T_1 y' \in G_1.$$

Hence (1) holds and (2) is proved in a similar fashion.

Remark. The closed linear span of $\{U_2^n G_1\} = K(\underline{R})$ and thus $\{U_1, U_2\}$ is a minimal dilation of $\{T_1, T_2\}$ consisting of doubly commuting isometries.

PROPOSITION 5.4. Let $\frac{R}{\underline{R}}$ have closed range. Then $G = G_1 \cap G_2$. G_1 is reducing for U_1 and invariant for U_2^* and G_2 is reducing for U_2 and invariant for U_1^* .

Proof. If $R = R_1 R_2$ has closed range, since

$$R_1 R_2 H \subset R_1 R_2 \cap R_2 R_1 \subset (R_1 R_2 H)^\perp = \underline{R}$$

we have $R_1 R_2 H = R_1 R_2 \cap R_2 R_1$. Let $y \in G_1 \cap G_2$. By the above and Lemma 5.3 $y_{0,0} = R_1 R_2 y_0$ for some y_0 in H . Since $y \in G_1$ by Lemma 5.3

we have $y_{0,m} = R_2 T_2^m y_0$. Since $y \in G_2$ this implies

$$y_{n,m} = R_1 T_1^n R_2 T_2^m y_0$$

and thus $y \in G$. Clearly $G \subset G_1 \cap G_2$ and the proposition follows.

Remarks. There are natural generalisations of the above results to finite families of doubly commuting contractions.

Is $G_1 \cap G_2 = G$ always?

The theorem of de Branges and Rovnyak shows that if T is a contraction such that $T^{*n} \rightarrow 0$ strongly and $I - T^*T$ is rank one then T is the restriction of the unilateral shift of multiplicity one to one of its invariant subspaces. If we combine this with Beurling's characterisation [3] of the invariant subspaces of the shift we obtain an explicit function representation for T in terms of the Hardy space $H^2(T)$ on the circle.

We obtain a similar representation below, using the results already proved, in terms of the Hardy space on the torus.

Notation. Let U^2 (resp. T^2) be the cartesian product of two copies of the open unit disc (resp. circle). $L^2(T^2)$ is the Hilbert space of square integrable functions on T^2 with respect to normalised Lebesgue measure. If z_1 and z_2 are the coordinate functions (restricted to T^2) then

$$\{z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}, \alpha_i \in \mathbb{Z}\}$$

is a complete orthonormal basis for $L^2(T^2)$.

$H^2(T^2)$ is the closed subspace of $L^2(T^2)$ spanned by $\{z^\alpha; (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2\}$. If P is the orthogonal projection of $L^2(T^2)$ onto $H^2(T^2)$ and if $\phi \in L^\infty(T^2)$, then $T_\phi = PM_\phi|_{H^2(T^2)}$ is the Toeplitz operator on $H^2(T^2)$ with symbol ϕ , where M_ϕ denotes the

multiplication operator on $L^2(T^2)$ defined by $M_\phi f = \phi f$, $f \in L^2(T^2)$.

One readily sees that T_{z_1} and T_{z_2} are doubly commuting isometries of infinite multiplicity. Using Proposition 5.1 we can obtain from these operators a wide class of doubly commuting contractions (see Proposition 5.6).

Let ϕ be an inner function on the unit circle T , that is, $\phi \in H^\infty(T)$ and $|\phi(e^{i\theta})| = 1$ a.e. on T . The following theorem, due to Beurling [3], is well known and we omit the proof.

THEOREM 5.5. A closed subspace M of $H^2(T)$ is invariant for T_z^* if and only if there exists an inner function ϕ such that $M = H^2(T) \ominus \phi H^2(T)$.

We will use the following notation: For an inner function ϕ in $H^\infty(T)$ let ϕ_i , $i = 1, 2$, be the functions on T^2 defined by

$$\phi_i(z_1, z_2) = \phi(z_i) \quad (i = 1, 2, z_i \in T).$$

PROPOSITION 5.6. Let ϕ, ψ be inner functions in $H^\infty(T)$. Let

$$M_1 = H^2(T^2) \ominus \phi_1 H^2(T^2), \quad M_2 = H^2(T^2) \ominus \psi_2 H^2(T^2) \quad \text{and}$$

$$S_i = T_{z_i}^* |_{M_1 \cap M_2} \quad (i = 1, 2).$$

Then S_1 and S_2 are doubly commuting contractions such that S_1^n and S_2^n converge to zero strongly and such that $(I - S_1^* S_1)(I - S_2^* S_2)$ is a finite rank operator of at most rank one.

Proof. T_{ϕ_1} is in the weakly closed algebra generated by I and T_{z_1} . For if $\phi_1(r)(e^{is}, e^{it}) = \phi(re^{is})$ for $0 < r < 1$, then $T_{\phi_1}(r)$ is in the uniform algebra generated by T_{z_1} and I , and $T_{\phi_1}(r)$ converges to T_{ϕ_1} as $r \rightarrow 1$ in the weak operator topology. Similarly T_{ψ_2} is in the weakly closed algebra generated by T_{z_2} and I .

Since T_{z_1} and T_{z_2} doubly commute it follows that $P_1 = T_{\phi_1} T_{\psi_2}^*$

and $P_2 = T_{\psi_2} T_{\psi_2}^*$ commute. Since P_1 (resp. P_2) is the orthogonal projection onto $\phi_1 H^2(T^2)$ (resp. $\psi_2 H^2(T^2)$) Proposition 5.1 shows that S_1 and S_2 are doubly commuting contractions.

Plainly $S_1^n \rightarrow 0$ strongly as $n \rightarrow \infty$ since the same is true of the backward shift $T_{z_1}^*$.

Finally if $Q = (I - T_{z_1} T_{z_1}^*)(I - T_{z_2} T_{z_2}^*)$ the proof of Proposition 5.1 shows that

$$(I - S_1^* S_1)(I - S_2^* S_2) = \text{pr}(Q).$$

Since Q is rank one, the proof is complete.

In order to prove a converse to the above proposition we will consider the equivalence between the spaces $H^2(T^2)$, $H^2[H^2(T)]$, $H^2(T) \otimes H^2(T)$ and $K(C)$.

Let $H^2[H^2(T)]$ be the Hilbert space defined in chapter 3 where we replace L by $H^2(T)$. Let U be the canonical unilateral shift on $H^2[H^2(T)]$ and V the operator defined by the analytic operator valued function $V(\theta) = T_z$.

Each $f \in H^2[H^2(T)]$ admits an orthogonal expansion

$$f = \sum_{0 \leq k < \infty} g_k e^{ik\theta}, \quad g_k \in H^2(T).$$

Moreover each g_k has an orthogonal expansion

$$g_k = \sum_{0 \leq j < \infty} a_{kj} e^j$$

with respect to the usual orthonormal basis $\{e_j\}_{j=0}^{\infty}$ of $H^2(T)$. Thus we can define a unitary operator Φ from $H^2[H^2(T)]$ to $H^2(T^2)$ by

$$\Phi f = \sum_{0 \leq j, k < \infty} a_{kj} z_1^k z_2^j.$$

Moreover, $\Phi^* T_{z_1} \Phi = U$ and $\Phi^* T_{z_2} \Phi = V$.

$H^2(T^2)$ is naturally isomorphic to $H^2(T) \otimes H^2(T)$ and the operator Ψ defined by

$$\psi_{z_1}^n \psi_{z_2}^m = e_n \otimes e_m$$

and linearity extends to a unitary, also denoted Ψ , which implements the equivalence. We have $\Psi T_{z_1} \Psi^* = T_z \otimes I$ and $\Psi T_{z_2} \Psi^* = I \otimes T_z$.

Thus, pictorially, we have the following equivalence,

$$\begin{array}{ccc} \{H^2[H^2(T)], U, V\} & \xrightarrow{\Theta} & \{H^2(T) \otimes H^2(T), T_z \otimes I, I \otimes T_z\} \\ & \searrow \Phi & \nearrow \Psi \\ & & \{H^2(T^2), T_{z_1}, T_{z_2}\} \end{array} \quad (1)$$

where $\Theta = \Psi\Phi$ is the unitary given by

$$\Theta f = \sum_{k, j \in \mathbb{Z}_+} a_{kj} e_k \otimes e_j \quad (f \in H^2[H^2(T)]),$$

(using the notation introduced for f above).

Note that if L is a closed subspace of $H^2(T)$ then

$$\Theta H^2[L] = H^2(T) \otimes L \quad (2)$$

Note also that if ϕ is an inner function in $H^\infty(T)$ then, using the notation above,

$$\Psi(\phi_2 H^2(T^2)) = H^2(T) \otimes \phi H^2(T) \quad (3)$$

Using the equivalences above we can now see what Proposition 3.9 says about $H^2(T^2)$.

PROPOSITION 5.7. Let M be a subspace of $H^2(T^2)$ which is invariant for $T_{z_2}^*$ and reducing for T_{z_1} . Then there exists an inner function $\phi \in H^\infty(T)$ such that

$$M = H^2(T^2) \ominus \phi_2 H^2(T^2).$$

Proof. Since $\Phi^* M$ is a reducing subspace of $H^2[H^2(T)]$ for U , by

Proposition 3.9 $\phi^* M = H^2[L]$ for some subspace L of $H^2(T)$. Also since $\phi^* M$ is invariant for V^* , plainly, by considering constant L valued functions in $H^2[L]$, we see that L is invariant for T_z^* . Thus by Theorem 5.5 $L = H^2(T) \ominus \phi H^2(T)$ for some inner function $\phi \in H^\infty(T)$. By (2) above

$$\phi H^2[L] = H^2(T) \otimes [H^2(T) \ominus \phi H^2(T)] .$$

Thus

$$\psi M = (H^2(T) \otimes H^2(T)) \ominus (H^2(T) \otimes \phi H^2(T))$$

and thus by (3)

$$M = H^2(T^2) \ominus \phi_2 H^2(T^2) .$$

THEOREM 5.8. Let S_1 and S_2 be doubly commuting contractions on H such that $S_i^n \rightarrow 0$ strongly as $n \rightarrow \infty$, $i = 1, 2$, and such that $(I - S_1^* S_1)(I - S_2^* S_2)$ is a rank one operator. Then there exist inner functions ϕ and ψ in $H^\infty(T)$ and a unitary Z from H to

$$M = [H^2(T^2) \ominus \phi_1 H^2(T^2)] \cap [H^2(T^2) \ominus \psi_2 H^2(T^2)]$$

such that

$$Z S_i Z^* = T_{z_i}^* | M \quad (i = 1, 2) .$$

Proof. $R = (I - S_1^* S_1)(I - S_2^* S_2)$ is a rank one operator and has closed range which we identify with \mathbb{C} . Thus by Theorem 5.2 there exists a subspace G of $K(\mathbb{C})$, invariant for U_1^* and U_2^* , and a unitary operator W from H to G such that $W S_i W^* = U_i^* | G$, $i = 1, 2$. By Proposition 5.4, $G \cong G_1 \cap G_2$ where G_1 is reducing for U_1 and invariant for U_2^* and G_2 is reducing for U_2 and invariant for U_1^* . Using the natural identification of $K(\mathbb{C})$, U_1 , U_2 and $H^2(T^2)$, T_{z_1} , T_{z_2} and Proposition 5.7 above, the theorem follows.

Lifting Problems.

In chapter 2 we have seen that every family \underline{T} of doubly commuting contractions possesses a unitary dilation \underline{U} . It is natural to ask whether we have a lifting theorem for the commutant of \underline{T} .

Problem a) Let \underline{T} be a family of doubly commuting contractions with a minimal unitary dilation \underline{U} . Does there exist a constant $\kappa > 0$ such that for each X in $\{\underline{T}\}'$ there exists an operator Y in $\{\underline{U}\}'$ such that $X = \text{pr}(Y)$ and $\|Y\| \leq \kappa \|X\|$?

Whilst it may be optimistic to think that this is true in general, we may have lifting theorems for special cases other than trivial extensions of the classical case. A special case of this problem concerns the doubly commuting contractions we have characterised above. Let S_1 and S_2 be as in Proposition 5.6 and let $A_i = S_i^*$, $i = 1, 2$. It follows that for $\phi \in H^\infty(\mathbb{T}^2)$ that $\text{pr}_{M_1 \cap M_2}(\underline{T}_\phi)$ commutes with A_1 and A_2 . Do we have the converse ?

Problem b) Let X commute with A_1 and A_2 . Does there exist $\phi \in H^\infty(\mathbb{T}^2)$ such that $X = \text{pr}_{M_1 \cap M_2}(\underline{T}_\phi)$?

In fact the single variable analogue of problem b) was solved by Sarason [33], using function theory techniques, and it was this result which led Sz-Nagy and Foias to prove the lifting theorem. Similarly here, problem b) might be settled by considering the function theory of the torus and in particular the relationship between $H^2(\mathbb{T}^2)$ and $H^1(\mathbb{T}^2)$.

Another special case of problem a) (more precisely of its intertwining formulation) and more in line with the theme of this thesis, is whether we have a Nehari theorem for Hankel operators on $H^2(\mathbb{T}^2)$. We consider this in more detail below.

Notation. Let L be the unitary operator on $L^2(\mathbb{T}^2)$ defined by

$$Lz_1^n z_2^m = z_1^{-n} z_2^{-m} \quad (n, m \in \mathbb{Z})$$

and for $\phi \in L^\infty(\mathbb{T}^2)$ let S_ϕ be the operator defined by

$$S_\phi = PLM_\phi|_{H^2(\mathbb{T}^2)},$$

where P is the orthogonal projection of $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{T}^2)$.

It is straightforward to verify, as in Proposition 3.1, that S_ϕ is Hankel relative to $\{T_{z_1}, T_{z_2}\}$, but do all such Hankel operators arise in this way?

Problem c) Does there exist $\kappa > 0$ such that for each $S \in \text{twi}\{T_{z_1}, T_{z_2}\}$ there exists $\phi \in L^\infty(\mathbb{T}^2)$ such that $S = S_\phi$ and $\|\phi\| \leq \kappa \|S\|$?

As in problem b), this may be settled by considering the relationship between $H^1(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$. In fact the classical Nehari theorem may be proved using the fact that each f in $H^1(\mathbb{T})$ can be written as a product of functions g and h in $H^2(\mathbb{T})$ such that

$$\|g\|_2^2 = \|h\|_2^2 = \|f\|_1 \quad (\text{see for example [9] p204 problem 7.9}).$$

However such factorisation theorems do not hold for $H^1(\mathbb{T}^n)$ if $n > 1$ (see [31], [28]). Nevertheless, one is able to show that the answer to c) is yes if the unit ball of $H^1(\mathbb{T}^2)$ is the closed convex hull of $\{fg; f, g \in H^2(\mathbb{T}^2)_1\}$.

The following example shows that if \underline{T} is a family of commuting isometries with minimal commuting unitary dilation \underline{U} , then we do not necessarily have a lifting theorem for $\text{twi}(\underline{T})$

Let $\underline{T} = \{T_{z_1}, T_{z_1 z_2}\} \subset B(H^2(\mathbb{T}^2))$. \underline{T} is a family of commuting isometries (not doubly commuting) with minimal commuting unitary dilation $\underline{W} = \{M_{z_1}, M_{z_1 z_2}\} \subset B(L^2(\mathbb{T}^2))$. Let $M = \ker(T_{z_1}^*)$ and for Y in $B(M)$ let



$$S = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $H^2(T^2) = M \oplus M^\perp$. We have $ST_{z_1} = T_{z_1}^* S = 0$. Thus $ST_{z_1 z_2} = T_{z_1 z_2}^* S = 0$ and $S \in \text{twi}(\underline{T})$. Note however that S need not be in $\text{twi}(T_{z_2})$. (For example let Y be the projection of M onto the linear span of $\{1, z_2\}$). It is easy to verify that $\text{twi}\{\underline{W}\} = \text{twi}\{M_{z_1}, M_{z_2}\} = \{LM_\phi; \phi \in L^\infty\}$. It follows that if $T \in \text{twi}\{\underline{W}\}$ then $\text{pr}_{H^2(T^2)}(T) \in \text{twi}(T_{z_2})$. Thus $\text{twi}(\underline{T}) \not\subset \text{pr}(\text{twi}\{\underline{W}\})$.

It is clear what is going 'wrong' here ; \underline{T}^* has a 'large' kernel on which we can construct arbitrary operators. This leads us to the following interesting problem.

Problem d) Let \underline{U} be a commutative family of isometries with minimal unitary dilation \underline{W} . If \underline{U} has a cyclic vector is $\text{twi}\{\underline{U}\} = \text{pr}(\text{twi}\{\underline{W}\})$?

Remarks . 1. A family of commuting isometries always possesses a minimal commuting unitary dilation which is unique up to isomorphism (see [22]) .

2. A simple closed graph argument shows that if Problem d) has an affirmative answer then there exists $\kappa > 0$ such that for each $X \in \text{twi}\{\underline{U}\}$ there exists $Y \in \text{twi}\{\underline{W}\}$ such that $X = \text{pr}(Y)$ and $\|Y\| \leq \kappa \|X\|$. If this holds do we in fact have $\kappa = 1$?

3. Note that problem d) contains the problem mentioned at the end of Chapter 2 .

Lifting Theorems and Complemented Spaces .

We finish this chapter by showing the following. If there exists a bounded weakly continuous projection from $B(H^2)$ onto the space of (classical) Hankel operators then we have a Nehari theorem for Hankel operators on the torus.

This result is unsatisfactory in that we do not know whether such a projection exists, but perhaps the result and the techniques used in its proof may contribute to the eventual solution of the problem.

In what follows it will be convenient to work with $H^2[H^2(T)]$ instead of $H^2(T^2)$.

Notation. H^2 is the classical Hardy subspace of L^2 . $H^2[H^2]$, $L^2[H^2]$, $L^2[L^2]$ are the usual Hilbert spaces of L^2 valued functions as defined in chapter 3.

Let U be the canonical bilateral shift on $L^2[L^2]$,

$$U' = U|_{L^2[H^2]}, \text{ a bilateral shift,}$$

and $U'' = U|_{H^2[H^2]}$, a unilateral shift.

Let W be the bilateral shift on $L^2[L^2]$ defined by the operator valued function $W(e^{i\theta}) = M_z$ a.e.

$$W' = W|_{L^2[H^2]}, \text{ a unilateral shift,}$$

$$W'' = W|_{H^2[H^2]}, \text{ a unilateral shift.}$$

We have $\{U, W\}^c = L^\infty[\{M_z\}^c]$, where $\{\cdot\}^c$ denotes the commutant, and we denote this space by $L^\infty[L^\infty]$.

Let J be the unitary operator on $L^2[L^2]$ defined by

$$(Jf)(e^{i\theta}) = f(e^{-i\theta}) \quad (f \in L^2[L^2])$$

and $J' = J|_{L^2[H^2]}$, also a unitary. Let R be the unitary operator on L^2 defined by $(Rf)(e^{i\theta}) = f(e^{-i\theta})$ and let F be the unitary on $L^2[L^2]$ defined by

$$(Fg)(e^{i\theta}) = R(g(e^{+i\theta})) \quad (g \in L^2[L^2]).$$

We have $J \in \text{twi}(U)$, $F \in \text{twi}(W)$ and $JF = FJ \in \text{twi}\{U,W\}$. Since $\{U,W\}^c = L^\infty[L^\infty]$ it follows that

$$\text{twi}\{U,W\} = JFL^\infty[L^\infty]$$

Remarks 1. Under the equivalence between $L^2[L^2]$ and $L^2(T^2)$ given earlier in this chapter JF corresponds to the operator L .

2. Without any real confusion and for notational convenience , we identify operator valued functions and the operators they define .

The following proposition which we denote by (P) is equivalent to problem c) .

(P) There exists a constant $\kappa > 0$ such that for each $S \in \text{twi}\{U',W'\}$ there exists $\theta \in L^\infty[L^\infty]$ such that

$$S = S_\theta = \text{pr}(JF\theta) \quad \text{and} \quad \|\theta\| \leq \kappa \|S\| .$$

Notation. For $\phi \in L^\infty[B(H^2)] \subset B(L^2[H^2])$, $S_\phi = \text{pr}_{H^2[H^2]}(J'\phi)$

and for $\theta \in L^\infty[B(L^2)] \subset B(L^2[L^2])$, $S_\theta = \text{pr}_{H^2[H^2]}(JF\theta)$.

Also let $H_0^\infty[B(H^2)] = \{ \phi \in L^\infty[B(H^2)] ; \phi_k = 0 \text{ for } k \leq 0 \}$

ie. $= \{ \phi \in L^\infty[B(H^2)] ; (\phi(x,y)) \in H_0^\infty \forall x,y \in H^2 \}$

PROPOSITION 5.9. Let $S \in \text{twi}\{U'\}$. Then there exists $\phi \in L^\infty[B(H^2)]$ such that $\|\phi\| = \|S\|$ and $S = \text{pr}_{H^2[H^2]}(J'\phi) = S_\phi$. Also $S_\phi \in \text{twi}\{W'\}$ if and only if $\phi W' - W'^* \phi \in H_0^\infty[B(H^2)]$.

Proof. Since $\{U'\}^c = L^\infty[B(H^2)]$ and $J' \in \text{twi}(U')$, as in Proposition 3.1 we see that $\text{twi}(U') = J'L^\infty[B(H^2)]$. Since U' is a unitary dilation of U'' the first part of the proposition now follows from Theorem 2.6.

Let $S = S_\phi$ with $\phi \in L^\infty[B(H^2)]$. For f and g in $H^2[H^2]$ the following equations are equivalent .

$$(SW'f, g) = (W'^* Sf, g)$$

$$(SW'f, g) = (Sf, W'g)$$

$$(J' \Phi W'f, g) = (J' \Phi f, W'g)$$

$$(\Phi W'f, J'g) = (\Phi f, W'J'g)$$

$$((\Phi W' - W'^* \Phi)f, J'g) = 0$$

Thus $S \in \text{twi}(W')$ if and only if $\Phi W' - W'^* \Phi \in H_0^\infty[B(H^2)]$.

We will need the following technical lemma for Proposition 5.11.

Notation. Since H^2 is separable, the weak operator topology (τ_w topology) on the unit ball of $B(H^2)$ is metrizable. Thus there exists a sequence $\{V_m\}_{m=1}^\infty$ of open neighbourhoods of the origin such that if $T_n \in B(H^2)_1$, then $T_n \rightarrow 0$ (τ_w) if and only if for each m there exists N such that $T_n \in V_m$ for $n \geq N$.

DEFINITION. Let $\Phi, \Phi_n \in L^\infty[B(H^2)]$. Then $\Phi_n \rightarrow \Phi$ uniformly in the weak operator topology, or uniformly τ_w , if for each M there exists N such that

$$\Phi_n(e^{i\theta}) - \Phi(e^{i\theta}) \in V_M \quad \text{a.e.} \quad \text{for } n > N.$$

Remarks. 1. This definition is independent of the choice of $\{V_m\}_{m=1}^\infty$

2. The uniform τ_w topology on $L^\infty[B(H^2)]$ is weaker than the norm topology but stronger than the weak operator topology.

LEMMA 5.10. Let S be a subspace of $B(H^2)$ which is closed in the weak operator topology. Then for each $\Phi \in L^\infty[S]$ there exists $\Phi^{(n)} \in L^\infty[S]$ $n=1, 2, \dots$ such that

(i) $\Phi^{(n)}$ is simple, $n \in \mathbb{N}$,

(ii) $\|\Phi^{(n)}\| \leq \|\Phi\|$, $n \in \mathbb{N}$,

(iii) $\Phi^{(n)} \rightarrow \Phi$ uniformly τ_w .

In particular $\Phi^{(n)} \rightarrow \Phi$ in the weak operator topology.

Proof. Since $B(H^2)_1$ is compact in the τ_w topology there exists, for each n , a finite collection of translates of V_n , say $\{E_1^n, \dots, E_{r_n}^n\}$, covering $B(H^2)_1$.

Clearly it is sufficient to prove the lemma for those ϕ of norm ~~not greater than~~ ^{equal to} 1. So suppose that $\|\phi\| \leq 1$ and let

$$F_k^n = \{e^{i\theta}; \phi(e^{i\theta}) \in E_k^n, \phi(e^{i\theta}) \notin E_1^n, 1 \leq k \leq r_n\}, \quad 1 \leq k \leq r_n.$$

Then $\{F_i^n; 1 \leq i \leq r_n\}$ is a partition of the unit circle into

α measurable subsets. Let χ_i^n be the characteristic function of F_i^n

for $1 \leq i \leq r_n$ and $n \in \mathbb{N}$. If F_i^n has measure zero set $S_i^n = 0$,

otherwise choose $S_i^n \in S \cap E_i^n$ such that $\|S_i^n\| \leq 1$. Now let

$$\phi^{(n)} = \sum_{1 \leq i \leq r_n} \chi_i^n S_i^n$$

so that the $\phi^{(n)}$ satisfy (i) and (ii). From the way we have

constructed the $\phi^{(n)}$ we see that $\phi^{(n)}(e^{i\theta}) - \phi(e^{i\theta}) \in V_n - V_n$ for

all θ apart from a set of measure zero. It follows that $\phi^{(n)} \rightarrow \phi$

uniformly in the weak operator topology and (iii) holds.

PROPOSITION 5.11. Let $\phi \in L^\infty[B(H^2)]$ be Hankel operator valued almost everywhere. Then there exists $\theta \in L^\infty[L^\infty]$ such that $\|\theta\| = \|\phi\|$ and

$$\phi(e^{i\theta}) = \text{pr}_{H^2}(\mathcal{F}\theta(e^{i\theta})) \quad \text{a.e.}$$

$$\text{or} \quad \phi = \text{pr}_{L^2[H^2]}(\mathcal{F}\theta)$$

Proof. By Lemma 5.10 there exist Hankel operator valued simple functions

$$\phi^{(n)} = \sum_{1 \leq i \leq r_n} \chi_i^n S_i^n \quad \text{such that} \quad \|\phi^{(n)}\| \leq \|\phi\| \quad \text{and} \quad \phi^{(n)} \text{ converges}$$

to ϕ in the weak operator topology. Here the S_i^n are Hankel operators

and, for each n , $\{\chi_i^n; 1 \leq i \leq r_n\}$ is a set of characteristic functions of a partition of the unit circle.

By the Sz-Nagy Foias lifting theorem or the classical Nehari theorem, for each S_i^n there exists $\phi_i^n \in L^\infty(\mathbb{T})$ such that

$S_i^n = \text{pr}_{H^2}(\mathcal{R}\phi_i^n)$ and $\|\phi_i^n\| = \|S_i^n\|$. Let

$$\theta^{(n)} = \sum_{1 \leq i \leq n} \chi_i^n \phi_i^n$$

so that $\Phi^{(n)} = \text{pr}(\mathcal{F}\theta^{(n)})$ and $\|\theta^{(n)}\| \leq \|\Phi\|$. $\theta^{(n)}$ is clearly measurable and in $L^\infty[L^\infty]$. Let θ be a τ_w limit point of $\{\theta^{(n)}\}$.

It is now routine to check that θ has the required properties.

Unfortunately we cannot combine Propositions 5.9 and 5.11 to deduce that $S = S_\Phi = S_\theta$. The Φ of Proposition 5.9 is not necessarily Hankel operator valued almost everywhere. What we have is $\Phi W' - W'^* \Phi \in H_0^\infty[B(H^2)]$ (instead of $\Phi W' - W'^* \Phi = 0$). Theorem 5.13 shows that the existence of a τ_w continuous bounded projection from $B(H^2)$ onto the space of classical operators can bridge this gap.

First we show that proposition (P) is equivalent to the following proposition (Q) concerning operator valued functions whose negative Fourier coefficients are Hankel operators.

(Q) There exists a constant $\kappa > 0$ such that for each $\Phi \in L^\infty[B(H^2)]$ whose negative Fourier coefficients $\Phi_k, k \leq 0$, are Hankel operators there exists $\Psi \in L^\infty[B(H^2)]$ such that

- (i) Ψ_k is a Hankel operator for all k .
- (ii) $\Psi_k = \Phi_k$ for all $k \leq 0$.
- (iii) $\|\Psi\| \leq \kappa \|\Phi\|$.

Remark. Of course one can ask whether similar propositions hold when the Hankel operators are replaced by an arbitrary τ_w closed space of operators

~~PROPOSITION 5.12. (P) holds if and only if (Q) holds.~~

PROPOSITION 5.12. (P) holds if and only if (Q) holds .

Proof. (Q) \Rightarrow (P) . Let $S \in \text{twi}\{U'', W''\}$ and let $\Phi \in L^\infty[B(H^2)]$ be the symbol of Proposition 5.9 so that $S = S_\Phi$ and

$$\Phi W' - W'^* \Phi \in H_0^\infty[B(H^2)] .$$

Since $(\Phi W' - W'^* \Phi)_k = \Phi_k T_z - T_z^* \Phi_k$ it follows that $\Phi W' - W'^* \Phi \in H_0^\infty[B(H^2)]$ if and only if Φ_k is a Hankel operator for $k \leq 0$. Thus if (Q) holds there exists $\Psi \in L^\infty[B(H^2)]$ such that Ψ is Hankel operator valued almost everywhere , (ie. Ψ_k is Hankel for all k), $\|\Psi\| \leq \kappa \|\Phi\|$ and $\Psi_k = \Phi_k$ for $k \leq 0$. This last condition implies $\Phi - \Psi \in H_0^\infty[B(H^2)]$ and thus $S = S_\Phi = S_\Psi$. Moreover , Proposition 5.11 applies to Ψ and there exists $\Theta \in L^\infty[L^\infty]$ such that $\|\Theta\| = \|\Psi\| \leq \kappa \|\Phi\| = \kappa \|S\|$ and $\Psi = \text{pr}_{L^2[H^2]}(F\Theta)$. Thus $S = S_\Psi = S_\Theta$, and (P) holds.

(P) \Rightarrow (Q) . Let $\Phi \in L^\infty[B(H^2)]$ be such that Φ_k is a Hankel operator for $k \leq 0$. It follows that $\Phi W' - W'^* \Phi \in H_0^\infty[B(H^2)]$ (as we noted above). By Proposition 5.9 $S_\Phi \in \text{twi}\{U'', W''\}$. Thus if (P) holds there exists Θ in $L^\infty[L^\infty]$ such that $\|\Theta\| \leq \kappa \|\Phi\|$ and $S_\Phi = \text{pr}_{H^2[H^2]}(JF\Theta)$. Thus $S_\Phi = \text{pr}_{H^2[H^2]}(J'\Psi) = S_\Psi$ where $\Psi = \text{pr}_{L^2[H^2]}(F\Theta)$. Since Ψ is Hankel operator valued , Ψ_k is a Hankel operator for all k . Moreover $\|\Psi\| \leq \|\Theta\| \leq \kappa \|\Phi\|$ and $S_\Phi = S_\Psi$ implies $\Phi - \Psi \in H_0^\infty[B(H^2)] \Rightarrow \Phi_k = \Psi_k$ for $k \leq 0$ and (Q) holds.

THEOREM 5.13. Suppose there exists a τ_w continuous bounded projection π of $B(H^2)$ onto the space of Hankel operators. Then (P) and (Q) hold with $\kappa \leq \|\pi\|$

Proof. Let \mathcal{S} denote the space of Hankel operators on H^2 . Let $\tilde{\pi}$ be the projection of $L^\infty[B(H^2)]$ onto $L^\infty[\mathcal{S}]$ defined by

$$\tilde{\pi}(\Phi)(e^{i\theta}) = \pi\Phi(e^{i\theta}) \quad (\theta \in [-\pi, \pi]).$$

Then $\tilde{\pi}\Phi \in L^\infty[\mathcal{S}]$ and $\|\tilde{\pi}\Phi\| \leq \|\pi\| \|\Phi\|$. In fact if $\Phi^{(n)}$, $n = 1, 2, \dots$ are the simple \mathcal{S} valued functions converging to Φ uniformly τ_w as in Lemma 5.11, then $\tilde{\pi}\Phi$ is the uniform τ_w limit of $\tilde{\pi}\Phi^{(n)}$ and thus is in $L^\infty[\mathcal{S}]$.

$\tilde{\pi}$ is also continuous in the τ_w topology on $L^\infty[B(H^2)]_1$. For suppose $\Phi^{(n)}$, $\Phi \in L[B(H^2)]_1$ and $\Phi^{(n)} \rightarrow \Phi$ (τ_w). To show that $\tilde{\pi}\Phi^{(n)} \rightarrow \tilde{\pi}\Phi$, τ_w , it suffices to show

$$(\tilde{\pi}\Phi^{(n)} \chi_E f, \chi_E g) \rightarrow (\tilde{\pi}\Phi \chi_E f, \chi_E g) \quad \text{as } n \rightarrow \infty,$$

for arbitrary characteristic functions $\chi_E, \chi_{E'}$, and $f, g \in H^2 \subset L^2[H^2]$.

Since χ_E commutes with $\tilde{\pi}\Phi^{(n)}$ we must show that

$$\frac{1}{2\pi} \int_0^{2\pi} \chi_E \chi_{E'} (\tilde{\pi}\Phi^{(n)}(\theta) f, g) d\theta \rightarrow \frac{1}{2\pi} \int_0^{2\pi} \chi_E \chi_{E'} (\tilde{\pi}\Phi(\theta) f, g) d\theta. \quad (1)$$

Since $\Phi^{(n)} \rightarrow \Phi$ (τ_w) it follows that $\Phi^{(n)}(e^{i\theta}) \rightarrow \Phi(e^{i\theta})$ and thus that $\tilde{\pi}\Phi^{(n)}(e^{i\theta}) \rightarrow \tilde{\pi}\Phi(e^{i\theta})$ (τ_w) almost everywhere. Thus by dominated convergence (1) holds and we see that $\tilde{\pi}$ is τ_w continuous on $L^\infty[B(H^2)]_1$.

To show that (Q) (and therefore (P)) holds, it will be sufficient to show that if the negative Fourier coefficients Φ_k , $k \leq 0$ are Hankel operators then $(\tilde{\pi}\Phi)_k = \Phi_k$. (For if this is so we may take $\Psi = \tilde{\pi}\Phi$ and $\kappa = \|\pi\|$ in (Q)).

Let $\Phi \in L^\infty[B(H^2)]$ and let Φ_k be Hankel for $k \leq 0$. Let T_n be the Cesaro means of Φ as defined in chapter 3. Then, as in the classical case, $\|T_n\| \leq \|\Phi\|$ and $T_n \rightarrow \Phi$ in the τ_w topology. Thus by the above $\tilde{\pi}T_n \rightarrow \tilde{\pi}\Phi$ and $(\tilde{\pi}T_n)_k \rightarrow (\tilde{\pi}\Phi)_k$. But

$$T_n(\theta) = \sum_{|k| < n} \left(1 - \frac{|k|}{n}\right) \Phi_k e^{ik\theta}$$

and so

$$(\tilde{\pi}_n)_k = \left(1 - \frac{|k|}{n}\right) \pi \phi_k, \quad k \in \mathbb{Z}.$$

Thus for $k \leq 0$

$$(\tilde{\pi}_n)_k = \left(1 - \frac{|k|}{n}\right) \phi_k \rightarrow \phi_k \quad \text{as } n \rightarrow \infty.$$

Hence $(\tilde{\pi}\Phi)_k = \phi_k$ for $k \leq 0$ completing the proof.

CHAPTER 6

HANKEL OPERATORS WITH DISCONTINUOUS SYMBOL

In this chapter we show that the closed linear span of the positive Hankel operators is a proper subspace of the Hankel operators and contains all the compact Hankel operators .

There is an elementary formulae which relates products of Hankel operators to products of Toeplitz operators which in particular shows that each positive Hankel operator and the product of any two Hankel operators lie in the C*-algebra \mathcal{T} generated by the Toeplitz operators. We shall use the elegant localisation theory of Douglas [9] to show that there exist Hankel operators not in \mathcal{T} . This may be regarded as an extension of the result above. The proof rests on the fact that the adjoint of a Hankel operator does not depend locally on its symbol .

Notation. For $0 \leq \alpha < 2\pi$ let $\phi_\alpha(e^{it}) = \begin{cases} i(t + \pi - \alpha)e^{it} & , \alpha - \pi \leq t < \alpha \\ i(t - \pi - \alpha)e^{it} & , \alpha \leq t < \alpha + \pi \end{cases}$

Then ϕ_α is continuous apart from a jump discontinuity at $e^{i\alpha}$, and $\|\phi_\alpha\| \leq \pi$. An elementary computation shows that

$$(S_{\phi_\alpha} e_n, e_0) = e^{i\alpha(n+1)} / (n+1) = \int_{R_\alpha} z^n dz \quad (1)$$

where R_α is the ray from 0 to $e^{i\alpha}$.

We have already noted in chapter 4 that S_{ϕ_0} is the positive Hankel operator given by Hilbert's matrix $(i + j + 1)^{-1}$ $i, j = 0, 1, \dots$.

(1) shows that $(S_{\phi_0} e_n, e_0)_{n=0}^\infty$ is the moment sequence given by Lebesgue measure on $[0, 1]$.

In fact in order for the infinite Hankel matrix $T = (c_{i+j})$ to be positive ie,

$$\sum_{0 \leq j, k \leq n} c_{j+k} a_k \bar{a}_j \geq 0$$

for all complex numbers a_0, a_1, \dots, a_n , it is necessary and sufficient that there exists a non decreasing function μ on $(-\infty, \infty)$ such that

$$c_j = \int_{-\infty}^{\infty} x^j d\mu(x) \quad (j = 0, 1, \dots)$$

(for example see [43]).

The following theorem due to Widom [39] characterises the measures μ which correspond to Hankel operators.

If T is a bounded operator it is necessary that the c_j be bounded and hence that $\mu(x)$ be constant for $x < -1$ and $x > 1$. Thus we may assume that μ is a non decreasing function on $[-1, 1]$.

THEOREM 6.1. (Widom) The following are equivalent .

- a) T represents a bounded operator .
- b) $c_j = O(j^{-1})$, $j \rightarrow \infty$.
- c) $\mu(1) - \mu(x) = O(1-x)$ as $x \rightarrow 1$ and
 $\mu(x) - \mu(-1) = O(1+x)$ as $x \rightarrow -1$.

Proof. We may assume that $\mu = \mu_1 + \mu_2$ where μ_1 is constant for $x < 0$ and μ_2 is constant for $x > 0$. Let

$$a_k = \int_{-1}^1 t^k d\mu_1(t) \quad , \quad b_k = \int_{-1}^1 t^k d\mu_2(t)$$

so that $c_k = a_k + b_k$, $k \geq 0$. Note that $a_k \geq 0$ for all k and $b_k \geq 0$ for k even .

c) \Rightarrow b) Integrating by parts gives

$$\begin{aligned} a_j &= j \int_0^1 x^{j-1} [\mu_1(1) - \mu_1(x)] dx \leq Aj \int_0^1 x^{j-1} (1-x) dx \\ &= A/(j+1) \end{aligned}$$

for some constant A .

Similarly $|b_j| \leq B / (j+1)$ for some constant B and hence $b)$ holds.

$b) \Rightarrow c)$. If $b)$ holds then since a_k, b_k are positive for k even $a_k \leq Ck^{-1}$ and $b_k \leq Ck^{-1}$ for k even and for some positive constant C .

Now

$$a_j = \int_0^1 x^j d\mu_1(x) \geq (1-\delta)^j (\mu_1(1) - \mu_1(1-\delta)) .$$

Let j be the smallest even integer not less than δ^{-1} , so that $\delta^{-1} \leq j < \delta^{-1} + 2$. Then

$$\mu_1(1) - \mu_1(1-\delta) \leq a_j (1-\delta)^{-j} \leq C\delta (1-\delta)^{-(\delta^{-1}+2)} = o(\delta)$$

as $\delta \rightarrow 0$. A similar argument shows that $\mu_2(x) - \mu_2(-1) = o(1+x)$ as $x \rightarrow -1$ and $c)$ follows.

$b) \Rightarrow a)$. This follows from an elementary argument using the fact that Hilbert's matrix defines a bounded operator.

$a) \Rightarrow c)$. Since

$$\sum c_{j+k} a_j \bar{a}_k = \int_{-1}^1 |\sum a_j x^j|^2 d\mu(x)$$

for finite sums, we have, if T is bounded,

$$\int_{-1}^1 |\sum a_j x^j|^2 d\mu(x) \leq \|T\| \sum |a_j|^2 ,$$

and this holds whenever the power series is uniformly convergent on $[-1, 1]$.

If $a_j = r^j$ ($0 < r < 1$) we conclude

$$\int_{-1}^1 \frac{d\mu(x)}{(1-rx)^2} \leq \frac{\|T\|}{1-r^2}$$

and so

$$\frac{\mu(1) - \mu(r)}{(1-r^2)^2} \leq \int_r^1 \frac{d\mu(x)}{(1-rx)^2} \leq \frac{\|T\|}{1-r^2}$$

and a similar inequality for $\mu(-r) - \mu(-1)$ giving $c)$.

DEFINITION 6.2. A function f on the unit circle has a jump discontinuity at $e^{i\alpha}$ if $\lim_{t \rightarrow \alpha^-} f(e^{it})$ and $\lim_{t \rightarrow \alpha^+} f(e^{it})$ exist and are different.

Let \tilde{P} be the norm closed linear span of the positive Hankel operators.

THEOREM 6.3. Let ϕ be a function on the unit circle which is continuous apart from possibly a jump discontinuity at 1 and -1 .

Then $S_\phi \in \tilde{P}$.

Proof. Since $S_\phi^* = S_{\tilde{\phi}}$ where $\tilde{\phi}(z) = \overline{\phi(\bar{z})}$ (see Lemma 6.8) it will be sufficient to consider ϕ such that ϕ is continuous on $T^1 = \{e^{it}; 0 \leq t \leq \pi\}$ and vanishes on $T \setminus T^1$. By the Stone Weierstrass theorem it suffices to show that $S_{\psi_n} \in \tilde{P}$ where

$$\psi_n(e^{it}) = \begin{cases} e^{int} & 0 \leq t \leq \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

for $n \in \mathbb{Z}$. Fix $n \in \mathbb{Z}$ and $\epsilon > 0$ and let

$$\theta_\epsilon(z) = \begin{cases} z(\epsilon i + z)^{-(n+1)} & , z \in T^1 \\ 0 & , z \in T \setminus T^1 \end{cases}$$

Since θ_ϵ converges uniformly to z^{-n} on T^1 as $\epsilon \rightarrow 0$ it will be sufficient to show that $S_{\theta_\epsilon} \in \tilde{P}$. We have

$$\begin{aligned} (S_{\theta_\epsilon} e_m, e_0) &= \frac{1}{2\pi} \int_0^{2\pi} \theta_\epsilon(e^{it}) e^{imt} dt \\ &= \frac{1}{2\pi} \int_0^\pi e^{it} (\epsilon i + e^{it})^{-(n+1)} e^{imt} dt \\ &= \frac{1}{2\pi i} \int_1^{-1} (\epsilon i + z)^{-(n+1)} z^m dz \quad (1) \end{aligned}$$

Now $(\epsilon i + s)^{-(n+1)} = h_1 - h_2 + i(h_3 - h_4)$ where $0 \leq h_i(s) \leq \kappa$ for

$s \in [-1, 1]$ and some constant κ . Let

$$\mu_i(s) = \int_{-1}^s h_i(s) ds, \quad s \in [-1, 1],$$

for $1 \leq i \leq 4$. Theorem 6.1 applies and thus there exist positive Hankel operators S_i , $1 \leq i \leq 4$, such that

$$(S_i e_n, e_0) = \int_{-1}^1 s^n d\mu_i(s), \quad 1 \leq i \leq 4.$$

It follows that $S_{\theta_\epsilon} = -(2\pi i)^{-1} \{S_1 - S_2 + i(S_3 - S_4)\} \in \underline{\mathcal{P}}$ and this completes the proof.

COROLLARY 6.4. $\underline{\mathcal{P}}$ contains all the compact Hankel operators.

Proof. This follows immediately from Theorem 6.3 and Hartman's theorem.

Hankel Operators not in $\underline{\mathcal{P}}$.

THEOREM 6.5. Let S be a Hankel operator such that $\{(S e_n, e_0)\}_0^\infty$ is the moment sequence of a finite regular Borel measure on a closed set $W \subset \bar{D} \setminus V$ where V is an open neighbourhood of $e^{i\alpha}$ in the closed unit disc \bar{D} . Then $\|S_{\phi_\alpha} - S\| \geq 1$.

Proof. Let $\{(S e_n, e_0)\}_{n=0}^\infty$ be the moment sequence of the finite regular Borel measure μ , that is

$$(S e_n, e_0) = \int_W s^n d\mu(s).$$

Let $x = \sum_{0 \leq n \leq N} \xi_n e_n$, $\tilde{x} = \sum_{0 \leq n \leq N} \bar{\xi}_n e_n$ and $p(\lambda) = \sum_{0 \leq n \leq N} \xi_n \lambda^n$, $\lambda \in \bar{D}$.

Then if we let $S_\alpha = S_{\phi_\alpha}$ we have

$$\begin{aligned}
((S_\alpha - S)x, \tilde{x}) &= \sum_{j,k} \xi_j \xi_k (S_\alpha e_j, e_k) - \sum_{j,k} \xi_j \xi_k (S e_j, e_k) \\
&= \sum_{j,k} \xi_j \xi_k \int_{R_\alpha} z^{j+k} dz - \sum_{j,k} \xi_j \xi_k \int_W z^{j+k} d\mu(z) \\
&= \int_{R_\alpha} p(\lambda)^2 d\lambda - \int_W p(\lambda)^2 d\mu(\lambda) \quad (1)
\end{aligned}$$

Clearly (1) also holds for p analytic in a neighbourhood of \bar{D} with x the corresponding H^2 function.

Let $p^\epsilon(z) = (z - (1+\epsilon)e^{i\alpha})^{-1}$, $\epsilon > 0$. Then $|p^\epsilon(z)| \leq \kappa$ for all $t \in \bar{D} \setminus V$ and $\epsilon > 0$, for some constant κ . Thus

$$\left| \int_W p^\epsilon(\lambda)^2 d\mu(\lambda) \right| \leq M \quad \text{for all } \epsilon > 0 \quad (2)$$

for some M . Let x^ϵ be the H^2 function corresponding to p^ϵ . Since $\|x^\epsilon\| \rightarrow \infty$ as $\epsilon \rightarrow 0$ it follows from (1) and (2) that it will be sufficient to prove that there exists an $\epsilon_0 > 0$ such that

$$\left| \int_{R_\alpha} p^\epsilon(\lambda)^2 d\lambda \right| \geq \|x^\epsilon\|^2 \quad 0 < \epsilon < \epsilon_0 \quad (3)$$

For if (3) holds we have

$$\begin{aligned}
\|S_\alpha - S\| \|x^\epsilon\| \|\tilde{x}^\epsilon\| &\geq |((S_\alpha - S)x^\epsilon, \tilde{x}^\epsilon)| \\
&\geq \left| \int_{R_\alpha} p^\epsilon(\lambda)^2 d\lambda \right| - \left| \int_W p^\epsilon(\lambda)^2 d\mu(\lambda) \right| \\
&\geq \|x^\epsilon\|^2 - M, \quad 0 < \epsilon < \epsilon_0.
\end{aligned}$$

Since $\|\tilde{x}^\epsilon\| = \|x^\epsilon\|$ and $\|x^\epsilon\| \rightarrow \infty$ as $\epsilon \rightarrow 0$, $\|S_\alpha - S\| \geq 1$. *we have*

We prove (3) and thus complete the proof of the theorem by proving the following two lemmas.

LEMMA A. There exists $\epsilon_1 > 0$ such that

$$\left| \int_{R_\alpha} p^\epsilon(\lambda)^2 d\lambda \right| > \frac{3}{4\epsilon} \quad \text{for } 0 < \epsilon < \epsilon_1 .$$

LEMMA B. There exists $\epsilon_2 > 0$ such that

$$\|x^\epsilon\|^2 < \frac{3}{4\epsilon} \quad \text{for } 0 < \epsilon < \epsilon_2 .$$

Proof of Lemma A .

$$\begin{aligned} \left| \int_{R_\alpha} p^\epsilon(\lambda)^2 d\lambda \right| &= \left| \int_0^1 \frac{e^{i\alpha} dt}{e^{i\alpha}(t - (1+\epsilon))^2} \right| \\ &= \int_0^1 \frac{dt}{(t - (1+\epsilon))^2} = \frac{1}{\epsilon} - \frac{1}{(1+\epsilon)} \\ &> \frac{3}{4\epsilon} \quad \text{for } \epsilon < \epsilon_1 . \end{aligned}$$

Proof of Lemma B .

$$\|x^\epsilon\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|e^{it} - (1+\epsilon)e^{i\alpha}|^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|e^{it} - (1+\epsilon)|^2}$$

With $\delta > 0$ fixed note that $\frac{1}{2\pi} \int_C \frac{dt}{|e^{it} - (1+\epsilon)|^2}$ is bounded

as $\epsilon \rightarrow 0$ where $C = \{e^{is} ; \delta \leq s \leq 2\pi - \delta\}$. Thus

$$\|x^\epsilon\|^2 = \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{dt}{|e^{it} - (1+\epsilon)|^2} + o(1) \quad \text{as } \epsilon \rightarrow 0 .$$

$$\begin{aligned} \text{Now } \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{dt}{|e^{it} - (1+\epsilon)|^2} &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{dt}{(1 + (1+\epsilon)^2 - 2(1+\epsilon)\cos t)} \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{dt}{\epsilon^2 + 2(1+\epsilon)(1 - \cos t)} \end{aligned}$$

Choose δ' such that $1 - \cos t > t^2/4$, $t \in [-\delta', \delta']$, then

$$\frac{1}{2\pi} \int_{-\delta'}^{\delta'} < \frac{1}{2\pi} \int_{-\delta'}^{\delta'} \frac{dt}{\epsilon^2 + t^2/2} = \frac{\sqrt{2}}{\pi} \epsilon^{-1} \tan^{-1} \left(\frac{\delta'}{\sqrt{2}\epsilon} \right)$$

As $\epsilon \rightarrow 0$ $\frac{\sqrt{2}}{\pi} \tan^{-1} \left(\frac{\delta'}{\sqrt{2}\epsilon} \right) \rightarrow \frac{1}{\sqrt{2}} < \frac{3}{4}$. This completes the proof of the lemma.

COROLLARY 6.6. $S_{\phi_\alpha} \notin \mathcal{P}$ for $\alpha \in (0, \pi) \cup (\pi, 2\pi)$.

Proof. It follows from Theorem 6.1 that if S is a Hankel operator in the linear span of the positive Hankels then $\{(S e_n, e_0)\}_{n=0}^{\infty}$ is the moment sequence of a finite regular Borel measure on $[-1, 1]$. By Theorem 6.5 $\|S_{\phi_\alpha} - S\| \geq 1$ for α such that $e^{i\alpha} \notin \{-1, 1\}$. Thus $\|S_{\phi_\alpha} - T\| \geq 1$ for all $T \in \mathcal{P}$.

COROLLARY 6.7. The linear space of Hankel operators is non separable.

Proof. Theorem 6.5 implies that $\{S_{\phi_\alpha}; 0 \leq \alpha < 2\pi\}$ is a non denumerable set of Hankel operators such that $\|S_{\phi_\alpha} - S_{\phi_\beta}\| \geq 1$ for $\alpha \neq \beta$.

Remark. Nehari's theorem shows that the space of Hankel operators is isomorphic to L^∞ / H_0^∞ (as a Banach space). It follows from Corollary 6.7 that L^∞ / H_0^∞ is non separable. See [1] for seven more proofs of this.

We now use the elegant localisation theory of Douglas [9] to show that there are Hankel operators which do not lie in the C*-algebra \mathcal{T} generated by the Toeplitz operators on H^2 . We shall see that this may be regarded as a generalisation of Corollary 6.6 (since $\mathcal{P} \subset \mathcal{T}$). Moreover, here we avoid Widom's characterisation of the positive Hankel operators (Theorem 6.1) and the rather technical Theorem 6.4.

Notation. Let T_ϕ denote the Toeplitz operator on H^2 , for $\phi \in L^\infty$, i.e. $T_\phi = PM_\phi|_{H^2}$, and let \mathcal{T} be the C*-algebra generated by the Toeplitz operators.

For $\phi \in L^\infty$ let $\phi^*(z) = \phi(\bar{z})$ and $\tilde{\phi}(z) = \overline{\phi^*(z)}$ for $z \in T$, and where T denotes the unit circle. We have

$$JM_\phi = M_{\phi^*}J$$

where J is the usual 'flip' unitary defined by $Je_n = e_{-n}$, $n \in \mathbb{Z}$, and where $\{e_n; n \in \mathbb{Z}\}$ is the standard basis of L^2 .

LEMMA 6.8. $S_\phi^* = S_{\tilde{\phi}}$, $\phi \in L^\infty$.

Proof. For $n, m \in \mathbb{Z}_+$ $(S_\phi e_n, e_m) = (JM_\phi e_n, e_m) = (M_{\phi^*} J e_n, e_m)$
 $= (J e_n, M_\phi e_m) = (e_n, S_\phi e_m)$.

LEMMA 6.9. $S_\phi^* S_\psi = T_{\tilde{\phi}\psi} - T_{z\tilde{\phi}} T_{\bar{z}\psi}$, $\phi, \psi \in L^\infty$.

Proof. $S_\phi^* S_\psi = PJM_\phi^* PJM_\psi|_{H^2} = PM_\phi^* JPJM_\psi|_{H^2}$
 $= PM_{\tilde{\phi}} M_z^* JPJM_z M_{\bar{z}\psi}|_{H^2}$.

Since $M_z^* JPJM_z = I - P$ we have

$$S_\phi^* S_\psi = PM_{\tilde{\phi}} (I - P) M_{\bar{z}\psi}|_{H^2}$$

$$= T_{\tilde{\phi}\psi} - T_{z\tilde{\phi}} T_{\bar{z}\psi}.$$

COROLLARY 6.10 $\underline{\mathbb{T}} \subset \underline{\mathbb{T}}$.

Proof. It suffices to show that if S is a positive Hankel operator then $S \in \underline{\mathbb{T}}$. Lemma 6.9 shows that $S^2 \in \underline{\mathbb{T}}$, therefore, since S is the unique positive square root of S^2 , and $\underline{\mathbb{T}}$ is a C^* -algebra , $S \in \underline{\mathbb{T}}$.

COROLLARY 6.11. If $\phi, \psi \in L^\infty$ and if ϕ or ψ is continuous then $T_{\phi\psi} - T_\phi T_\psi$ is compact .

Proof. Immediate from Lemma 6.9 and Hartman's theorem .

Notation. Let J_λ be the closed ideal in $\underline{\mathbb{T}}$ generated by

$$\{ T_\phi ; \phi \in C(T) , \phi(\lambda) = 0 \} .$$

Let $\underline{\mathbb{T}}_\lambda = \underline{\mathbb{T}} / J_\lambda$ and let $\phi_\lambda : \underline{\mathbb{T}} \rightarrow \underline{\mathbb{T}}_\lambda$ be the natural quotient $*$ -homomorphism . Let Φ be the $*$ -homomorphism from $\underline{\mathbb{T}} \rightarrow \sum_{\lambda \in T} \oplus \underline{\mathbb{T}}_\lambda$ defined by

$$\Phi = \sum_{\lambda \in T} \oplus \phi_\lambda$$

The following two theorems can be found in [9] .

THEOREM 6.12. $\ker \Phi$ contains the ideal of compact operators K and $\underline{\mathbb{T}}$ induces a $*$ isometrical isomorphism Φ_c

$$\Phi_c : \underline{\mathbb{T}} / K \rightarrow \sum_{\lambda \in T} \oplus \underline{\mathbb{T}}_\lambda$$

Notation. Let Λ be the maximal ideal space of L^∞ and let

$$F_\lambda = \{ m \in \Lambda ; m(z) = \lambda \}$$

be the fibre in Λ over λ for $\lambda \in T$.

THEOREM 6.13. Let $\{\phi_{ij}\}_{i,j=1}^n$ be functions in L^∞ with Gelfand transforms ϕ_{ij}^* on Λ and let $\lambda \in T$. Then $\Phi_\lambda(\prod_{i=1}^n \prod_{j=1}^n T_{\phi_{ij}})$ depends only on the functions $\{\phi_{ij}|_{F_\lambda}\}_{i,j=1}^n$.

COROLLARY 6.14. If $\phi, \psi \in L^\infty$ and ψ is continuous on a neighbourhood of $\lambda \in T$ then $\Phi_\lambda(T_{\phi\psi} - T_\phi T_\psi) = 0$.

Proof. Theorem 6.13 implies that we can assume ψ to be continuous on T . Apply Corollary 6.11 and Theorem 6.12.

THEOREM 6.15. Let $\alpha \in T$, $\alpha \notin \{1, -1\}$, and let ϕ be a function in L^∞ which is continuous on T apart from a proper discontinuity at α . Then $S_\phi \notin \tilde{T}$.

Proof. By Lemma 6.9 $S_\phi^* S_\phi = T|\phi|^2 - T_{z\bar{\phi}} T_{\bar{z}\phi}$. By Corollary 6.14

$$\Phi_\lambda(S_\phi^* S_\phi) = 0, \lambda \neq \alpha.$$

Suppose now that $S_\phi \in \tilde{T}$. Since Φ_λ is a $*$ homomorphism $\Phi_\lambda(S_\phi)^* \Phi_\lambda(S_\phi) = 0$ for $\lambda \neq \alpha$. Similarly, since by Lemma 6.8 $S_\phi^* = S_{\bar{\phi}}$ and $\bar{\phi}$ is continuous apart from a discontinuity at $\bar{\alpha}$, it follows that $\Phi_\lambda(S_\phi) \Phi_\lambda(S_\phi)^* = 0$ for $\lambda \neq \bar{\alpha}$. Thus $\Phi_\lambda(S_\phi) = 0$ for all λ . Therefore by Theorem 6.12 S_ϕ is compact (since $\text{ker } \Sigma = \text{compact}$). However, by Hartman's theorem, S_ϕ is not compact. Thus we conclude that $S_\phi \notin \tilde{T}$.

COROLLARY 6.16. Let ϕ be a function in L^∞ which is continuous apart from a proper discontinuity at $\alpha \neq 1$ or -1 . Then $S_\phi \notin \tilde{P}$.

Proof. Immediate from Theorem 6.15 and Corollary 6.10.

CHAPTER 7

SPACES OF OPERATORS WHOSE SUMS ARE C*-ALGEBRAS

§1 Introduction.

Let X be an infinite dimensional Banach space with closed subspaces Y and Z . Then $Y + Z$ need not be closed. In this chapter we obtain algebraic hypotheses on closed spaces of operators on a Hilbert space which ensure that their sum is closed.

Conditions on Y and Z which ensure the automatic closure of their sum have been investigated by various authors. Not surprisingly, because of the generality of this problem, the results are rather scattered in the literature over various topics, and the techniques used peculiar to the context. For example, conditions on ideals in Banach algebras have been considered by Reiter [29] (for ideals of $L^1(G)$), Detraz [40] (for function algebras), Stegenga [35] (for the disc algebra) and Rudin [32]. Rudin's paper also contains generalisations of Sarason's result that $H^\infty + C(T)$ is closed (using Theorem 1.1 of chapter 1). Further results concerned with this circle of ideas have been considered by Stegenga ([36] and [37]) and in chapters 3 and 4 of this thesis.

The following result ([42] cor 1.17.5) is nearer to what we shall be considering here: Let A be a C*-algebra, B a C*-subalgebra of A and I a closed two sided ideal of A . Then $B + I$ is a C*-algebra. In section 3 we show that I can be replaced by certain C*-modules. Of particular interest to us will be C*-modules which arise as spaces of intertwining operators.

The general philosophy of this chapter is that 'algebraic and adjoint closure' imply topological closure. We finish with two open questions which reflect this belief and contain many of the theorems we shall prove.

We first show that some conditions on spaces of operators, other than just being self-adjoint, are necessary for closed sums. The following lemma is an elementary corollary of the open mapping theorem and we omit its proof.

LEMMA. Let X and Y be disjoint closed subspaces of a Banach space, such that $X \cap Y = \{0\}$. Then $X + Y$ is not closed if and only if there exist x_n in X and y_n in Y , $n = 1, 2, \dots$, such that $\|x_n\|$ and $\|y_n\|$ are bounded above and below and $\|x_n + y_n\| \rightarrow 0$, $n \rightarrow \infty$.

PROPOSITION 7.1 There exist C^* -algebras of operators A and B on a Hilbert space K such that $A + B$ is not closed.

Proof. Let M and N be closed subspaces of an infinite dimensional Hilbert space K such that $M + N$ is a non closed direct sum which is dense in K . (eg. see [15]). Let A (resp. B) be the C^* -algebra of operators a (resp. b) such that M (resp. N) reduces a (resp. b) and $a|_M = 0$ (resp. $b|_N = 0$). Since $M + N$ is not closed by the lemma there exist real constants κ_1 and κ_2 , and $x_n \in M$, $y_n \in N$ such that $\kappa_1 \leq \|x_n\|$, $\|y_n\| \leq \kappa_2$, $n = 1, 2, \dots$, and $\|x_n + y_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let a_n (resp. b_n) be the operator in A (resp. B) defined by $a_n z = (z, x_n)x_n$ (resp. $b_n z = (z, y_n)y_n$) for z in K , $n = 1, 2, \dots$. We have

$$\begin{aligned} (a_n - b_n)z &= (z, x_n)x_n - (z, y_n)y_n \\ &= (z, x_n + y_n)x_n - (z, y_n)(x_n + y_n) \end{aligned}$$

Thus

$$\|a_n - b_n\| \leq \|x_n + y_n\| \|x_n\| + \|y_n\| \|x_n + y_n\|$$

and so $\|a_n - b_n\| \rightarrow 0$ as $n \rightarrow \infty$. Since $A \cap B = \{0\}$, and

$\kappa_1^2 \leq \|a_n\|$, $\|b_n\| \leq \kappa_2^2$, $n = 1, 2, \dots$, the lemma shows that $A + B$

is not norm closed.

Notation and Definitions.

Throughout this chapter, unless otherwise stated, we shall be concerned with spaces of operators acting on a fixed Hilbert space K . Spaces of operators are denoted by capital letters and the operators themselves by lower case letters.

Given a set S of operators on K , let S_+ denote the positive operators in S . ^{We say that} S is self-adjoint if $a \in S$ implies $a^* \in S$. The weak operator topology and strong operator topology are denoted by τ_w and τ_s respectively.

DEFINITION Let A be a C^* -algebra acting on K . A C^* -bimodule of A is a space of operators B on K such that $ab, ba \in B$ for all $a \in A, b \in B$.

Let $F \subset B(K)$.

DEFINITION 7.2 An even (resp. odd) product of F is an operator of the form $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}$ where $a_i \in F, \alpha_i \in \mathbb{N}$ and $\sum \alpha_i$ is even (resp. odd).

DEFINITION 7.3. For $\lambda \in \mathbb{C}$ $\text{com}_\lambda(F) = \{b \in B(K); ba = \lambda ab, a \in F\}$
and $\text{twi}_\lambda(F) = \{b \in B(K); ba = \lambda a^*b, a \in F\}$.

Let A_e be the linear span of the identity and the even products of F and A_o the linear span of the odd products of F .
 $A = A_e + A_o$ and A^n, A_e^n, A_o^n (resp. A^w, A_e^w, A_o^w) denote the norm closure (resp. τ_w closure) of A, A_e and A_o respectively. When F is self-adjoint it is quickly verified that A_o^n is a norm closed self-adjoint C^* -bimodule of A_e^n such that $(A_o^n)^2 \subset A_e^n$.

Similarly when F is self-adjoint, $\text{com}_{-1}(F)$ is a closed self-adjoint C^* -bimodule of $\text{com}(F)$ such that $(\text{com}_{-1}(F))^2 \subset \text{com}(F)$.

It is precisely this odd/even situation which we consider in the next section , where we also show how A_e^W and A_o^W may be obtained via the $com_{-1}()$ operation.

For convenience we will refer to C^* -bimodules of A as A -bimodules.

§2 Odd Even Sums.

The following results have also appeared in [26] .

THEOREM 7.4. Let A be a C^* -algebra of operators on K , B a ^{closed} self-adjoint A -bimodule of operators on K such that $B^2 \subset A$. Then the following conditions are equivalent :

- (i) $A \cap B = \{0\}$,
- (ii) $B_+ = \{0\}$,
- (iii) $a \in A$, $b \in B$ and $a + b \geq 0$ imply $a \geq 0$,
- (iv) $a \in A$, $b \in B$ imply $\|a + b\| \geq \max\{\|a\|, \|b\|\}$.

If the above conditions hold then $A + B$ is a C^* -algebra . Moreover if A and B are τ_w closed then $A + B$ is τ_w closed.

Proof. (i) \Rightarrow (ii). Let $b \in B_+$. Since $b^2 \in A$ and b is the unique positive square root of b^2 , $b \in A$. Thus $b \in A \cap B$ and $b = 0$.

(ii) \Rightarrow (i). It is easy to see that $A \cap B$ is a C^* -algebra .

$(A \cap B)_+ \subset B_+ = \{0\}$, thus $A \cap B = \{0\}$

(ii) \Rightarrow (iii). Let $a \in A$, $b \in B$ and $a + b \geq 0$, then $a - a^* = b^* - b$. Therefore by (i) $a = a^*$, hence $a = a^+ - a^-$, $a^+ \geq 0$, $a^- \geq 0$ and $a^+ a^- = 0$. Since $a^- a a^- = -(a^-)^3 \leq 0$ and $a^-(a + b)a^- \geq 0$, it follows that $a^- b a^- \geq 0$. Thus (ii) implies $a^- b a^- = 0$ and so $-(a^-)^3 = a^- a a^- = a^-(a + b)a^- \geq 0$. Therefore $(a^-)^3 = 0$, $a^- = 0$ and $a \geq 0$ as required .

To prove (iii) \Rightarrow (iv) we first show that (iii) holds with

A replaced by the C^* -algebra A_1 generated by A and I . Since A_1 and B also satisfy the hypotheses of the theorem, it is enough to prove (ii). If $a \in (A \cap B)_+$, by (iii) $-a + a = 0$ implies $-a \geq 0$ and thus $a = 0$. Since $A \cap B$ is a C^* -algebra and $(A \cap B)_+ = \{0\}$, $A \cap B = \{0\}$. This implies $B_+ = \{0\}$ as required.

Let $a \in A$, $b \in B$ and $\|a + b\| \leq 1$. Then

$$0 \leq I - (a + b)^*(a + b) = I - a^*a - b^*b - (b^*a + a^*b).$$

By the above, $I - a^*a - b^*b \geq 0$. Thus $a^*a \leq I$ and $b^*b \leq I$, that is $\|a\| \leq 1$, $\|b\| \leq 1$ and hence (iv).

(iv) \Rightarrow (i) is clear.

Suppose now that (i)-(iv) hold. Since $B = B^*$, $B^2 \subset A$ and B is an A -bimodule, it follows that $A + B$ is a $*$ -algebra of operators, and since A and B are norm closed, (iv) shows that $A + B$ is a C^* -algebra.

Suppose further that A and B are τ_w closed. Let $a_\alpha + b_\alpha$ be a τ_s convergent net in $(A + B)_1$, the unit ball of $A + B$, with $a_\alpha \in A$, $b_\alpha \in B$. By (iv) $\|a_\alpha\| \leq 1$, $\|b_\alpha\| \leq 1$, and thus there exists a subnet α' and $a \in A$, $b \in B$ such that $a_{\alpha'}$ and $b_{\alpha'}$ are τ_w convergent to a , b respectively. But $a_\alpha + b_\alpha$ is also τ_w convergent and so the τ_w limit is $a + b$. Thus $a + b$ is the τ_s limit of $a_\alpha + b_\alpha$ and $(A + B)_1$ is therefore τ_s closed. Therefore by Kaplansky's density theorem ([8] p43), $(A + B)_1$ is the unit ball of the τ_s closure of $A + B$. It follows that $A + B$ is τ_s closed (and thus τ_w closed).

THEOREM 7.5. Let A be a C^* -algebra of operators on K and let B be a norm closed self-adjoint A -bimodule of operators such that $B^2 \subset A$. Then $A + B$ is a C^* -algebra. Moreover if A and B are τ_w closed then so is $A + B$.

Proof. $A + B$ is a $*$ -algebra of operators and $(A + B)^{-}$, its norm closure, a C^* -algebra. It is easy to show that $A \cap B$ is an $A + B$ bi-ideal and thus an $(A + B)^{-}$ bi-ideal. Let $C = (A + B)^{-} / A \cap B$, $A_0 = A / A \cap B$ and $B_0 = B / A \cap B$. By the Gelfand Naimark theorem [42], the C^* -algebra C may be represented as an operator algebra on a Hilbert space. A_0 is a C^* -subalgebra of C with a norm closed self-adjoint A_0 -bimodule $B_0 \subset C$, such that $B_0^2 \subset A_0$.

Let $[x]$ denote the coset of x in C , for $x \in A + B$. If $[x] \in A_0 \cap B_0$ then $[x] = [a] = [b]$ for some $a \in A, b \in B$. Thus $a - b \in A \cap B$ and $a \in B$. Hence $a \in A \cap B, [x] = [a] = [0]$ and $A_0 \cap B_0 = \{0\}$. By Theorem 7.4 $A + B / A \cap B = A_0 + B_0$ is a C^* -algebra. Thus $A + B / A \cap B = (A + B)^{-} / A \cap B$ and $A + B = (A + B)^{-}$.

Now suppose that A and B are τ_w closed. Let A_1, B_1 and $(A + B)_1$ be the unit balls of A, B and $A + B$ respectively. From the open mapping theorem, applied to the mapping $(a,b) \rightarrow a + b$ from $A \oplus B$ to $A + B$, it follows that $A_1 + B_1 \not\subset \delta(A + B)_1$ for some positive δ . Since A_1 and B_1 are τ_w compact, so is $(A + B)_1$. Thus $A + B$ is τ_w -closed ([8], Theorem 2, p42).

COROLLARY 7.6. Let A be a C^* -algebra of operators (resp. a von Neumann algebra) on K and let v be an isometry such that $Av^* = vA$. Then $A + vA$ is a C^* -algebra (resp. a von Neumann algebra).

Proof. Since v is an isometry, vA is norm closed (resp τ_w closed). Also $Av^* = vA$ implies that vA is a self-adjoint A -bimodule of operators. Since $(vA)^2 = Av^*vA \subset A$ we may apply Theorem 7.5.

COROLLARY 7.7. Let B be a norm closed (resp. τ_w closed) self-adjoint space of operators such that $B^3 \subset B$. Let $A = \{a \in B(K) ; aB \subset B, Ba \subset B\}$. Then $A + B$ is a C^* -algebra (resp. a von Neumann algebra).

Proof. It is elementary to prove that A is a C^* -algebra (resp. a von Neumann algebra) and that B is an A -bimodule. Since $B^3 \subset B$, $B^2 \subset A$ and we may apply Theorem 7.5.

COROLLARY 7.8. Let S be a self-adjoint set of operators on K . Then $\{S\}' + \text{twi}_1(S)$ is a von Neumann algebra.

Proof. Let $x, y \in \text{twi}_1(S)$, $a' \in \{S\}'$ and $a \in S$. Then the following hold :

$$\begin{aligned} a^* x^* &= x^* a^* , \\ a' x a &= a' a^* x = a^* a' x , \\ x a' a &= x a a' = a^* x a' , \\ x y a &= x a^* y = a x y . \end{aligned}$$

These equations show that $\text{twi}_1(S)$ is a self-adjoint $\{S\}'$ -bimodule such that $(\text{twi}_1(S))^2 \subset \{S\}'$. Since $\text{twi}_1(S)$ is weakly closed, Theorem 7.5. applies.

COROLLARY 7.9 Let a be a normal operator on K . Then the following are von Neumann algebras.

- (i) $\{a\}' + \text{com}_{-1}(a)$,
- (ii) $\{a\}' + \text{twi}_\lambda(a)$ for $|\lambda| = 1$.

Proof: The Putnam Fuglede theorem ([27] p20) shows that $\text{com}_{-1}(a)$ and $\{a\}'$ are self-adjoint. It is easy to check that $\{a\}'$ and $\text{com}_{-1}(a)$ satisfy the hypotheses of Theorem 7.5., hence (i).

Let $x \in \text{twi}_\lambda(a)$, then $xa = \lambda a^* x$ implies $a x^* = \bar{\lambda} x^* a$. Since $|\lambda| = 1$, $x^* \in \text{twi}_\lambda(a)$. By Fuglede's theorem $\{a\}' = \{a^*\}'$, thus if $a' \in \{a\}'$

$$a'xa = \lambda a' a^* x = \lambda a^* a' x,$$

and

$$xa'a = xaa' = \lambda a^* xa'.$$

Thus $\text{twi}_\lambda(a)$ is a self-adjoint $\{a\}'$ -bimodule. By the Putnam Fuglede theorem if $y \in \text{twi}_\lambda(a)$ then $ya^* = \bar{\lambda} ay$. Thus

$$yxa = \lambda ya^* x = \lambda \bar{\lambda} ayx = ayx.$$

That is, $(\text{twi}_\lambda(a))^2 \subset \{a\}'$, and Theorem 7.5 completes the proof.

The following theorem may be regarded as a non commutative generalisation of the fact that a continuous function on $[-1,1]$ is the sum of an odd and an even function. The notation of the theorem is that given in section 1 for the case when F is self-adjoint.

THEOREM 7.10. (i) $A^n = A_e^n + A_o^n$,
(ii) $A^w = A_e^w + A_o^w$.

Proof. It is easy to check that A_e^n is a C^* -algebra of operators on K and A_o^n a norm closed self-adjoint A_e^n -bimodule of operators such that $(A_o^n)^2 \subset A_e^n$. By Theorem 7.5 $A_e^n + A_o^n$ is a C^* -algebra and so $A^n = A_e^n + A_o^n$.

Let a_α, b_α be τ_w convergent nets in A_o^n with limits a, b in A_o^w . Then $a_\alpha b = \lim_\beta a_\alpha b_\beta$ and $ab = \lim_\alpha a_\alpha b$ show that $ab \in A_e^w$. That is, $(A_o^w)^2 \subset A_e^w$. Similarly A_o^w is a self-adjoint A_e^w -bimodule. By Theorem 7.5, $A_e^w + A_o^w$ is τ_w -closed and thus equal to A^w .

When the sum of Theorem 7.5 is direct the mapping θ defined by

$$\theta(a + b) = a - b \quad (a \in A_e^w, b \in A_o^w)$$

is a *-automorphism of A^W such that $\theta^2 = I$. Conversely, every such (non trivial) automorphism arises in this way. We now find a sufficient condition under which the sum of Theorem 7.10 is direct and a necessary and sufficient condition that θ be spatial in this case. Moreover A_0^W and A_e^W can be defined in terms of the $\text{com}_{-1}(\)$ operation.

DEFINITION 7.11. A set S of operators on K is separating if the intersection of the null spaces of the operators in S is $\{0\}$.

THEOREM 7.12. Let S be a self-adjoint set of operators on K . Then $\{S\}' + \text{com}_{-1}(S)$ is a von Neumann algebra and the sum is direct if and only if S is separating.

Proof. If $A = \{S\}'$ and $B = \text{com}_{-1}(S)$, then A and B satisfy the hypotheses of Theorem 7.5 and therefore $A + B$ is a von Neumann algebra.

Let e be the orthogonal projection of K onto the intersection of the null spaces of the operators in S . Since $S = S^*$, we have $xe = ex = 0$ for all x in S . Thus $e \in \{S\}' \cap \text{com}_{-1}(S)$, and so, if the sum is direct, $e = 0$ and S is separating.

Conversely let S be separating and suppose $x \in \{S\}' \cap \text{com}_{-1}(S)$. Then $xs = sx$ and $xs = -sx$, and so $sx = 0$ for all s in S . Thus $x = 0$ and the sum is direct.

$F = F^*$ and

THEOREM 7.13. If $\wedge \text{com}_{-1}(F)$ is separating then the following hold.

- (i) $A^W = A_0^W + A_e^W$ is a direct sum.
- (ii) $A_e^W = \{\text{com}_{-1}(F)\}'$ and $A_0^W = \text{com}_{-1}(\text{com}_{-1}(F))$.
- (iii) θ is spatial if and only if $\text{com}_{-1}(F)$ contains a unitary.

Proof. By Theorem 7.12 , $C = \{com_{-1}(F)\}' + com_{-1}(com_{-1}(F))$ is a von Neumann algebra and the sum is direct if $com_{-1}(F)$ is separating .

It is easy to check that $(com_{-1}(F))'$ contains the even products of F and hence A_e^W , and similarly that $com_{-1}(com_{-1}(F)) \subset A_o^W$. Thus $A^W \neq A$ $A^W = A_e^W + A_o^W$ is a direct sum . To show that these inclusions are in fact equalities , it will be sufficient to prove that $C = A^W$ that is , by von Neumann's double commutant theorem , that $C = \{F\}''$.

Let $x \in \{F\}'$, $a \in \{com_{-1}(F)\}'$, $s \in com_{-1}(F)$. Then $sx \in com_{-1}(F)$ and thus $a(sx) = (sx)a$. Therefore $sax = asx = sxa$. Since $com_{-1}(F)$ is separating , $ax = xa$. Thus $\{com_{-1}(F)\}' \subset \{F\}''$.

Similarly if $b \in com_{-1}(com_{-1}(F))$, x , s as before , $sx \in com_{-1}(F)$, thus $b(sx) = -(sx)b$. Therefore $-sbx = bsx = -sxb$ and since $com_{-1}(F)$ is separating $bx = xb$. Hence $com_{-1}(com_{-1}(F)) \subset \{F\}''$ and $C \subset \{F\}''$ as required.

If θ is spatial , $-a = \theta(a) = uau^*$ for all a in F for some unitary on K . Clearly $u \in com_{-1}(F)$. Conversely if $com_{-1}(F)$ contains a unitary u it is easy to see that u implements the automorphism θ

As an immediate corollary we have the following double commutant theorem

COROLLARY 7.14. If $F = F^* \subset B(K)$ and $com_{-1}(F)$ is separating then $\{F\}'' = \{com_{-1}(F)\}' + com_{-1}(com_{-1}(F))$ and the sum is direct .

Remark. Let F contain just the 3×3 diagonal matrix with eigenvalues 1 , 1 and -1 . Then $com_{-1}(F)$ is separating but does not contain a unitary .

3 Roots of unity sums .

In this section we consider some generalisations of the results of section 2 to several spaces of operators .

Let E_0, E_1, \dots, E_{k-1} be k r_n closed linear subspaces of $B(K)$ such that

$$(i) \quad E_i E_j \subset E_{i+j \pmod k} \quad (0 \leq i, j < k)$$

and $(ii) \quad E_i^* \subset E_{-i \pmod k} \quad (0 \leq i < k) .$

Then $A = E_0 + E_1 + \dots + E_{k-1}$ is a self-adjoint algebra of operators .

Multiplicatively , the E_i behave like the spectral subspaces of a *-automorphism θ of a C*-algebra such that $\theta^k = I$. We shall see that as in the case $k = 2$, such spaces arise naturally as certain spaces of intertwining operators and as subspaces of a C*-algebra associated with its generators . We conjecture that A is always closed (and thus a C*-algebra). We prove this for certain special cases involving direct sums which will be sufficient to yield a 'roots of unity' generalisation of Theorems 7.13 and 7.14 .

Assumption. We assume that $E_0 \cap (E_1 + E_2 + \dots + E_{k-1}) = \{0\}$ and that $I \in E_0$.

Let $B = E_1 + E_2 + \dots + E_{k-1}$. The following theorem generalises part of Theorem 7.4 .

THEOREM 7.15. The following conditions are equivalent .

- (i) $B_+ = \{0\}$.
- (ii) $a \in E_0, b \in B$ and $a + b \geq 0$ imply $a \geq 0$.
- (iii) If $a_i \in E_i, 0 \leq i < k$, then $\|a_0 + a_1 + \dots + a_{k-1}\| \leq \max \{ \|a_j\| ; 0 \leq j < k \}$.
- (iv) A is a C*-algebra .

Moreover if the above conditions hold and if the E_i are r_w closed then A is a von Neumann algebra .

Proof. (i) \Rightarrow (ii). Let $a \in E_0$, $b \in B$ and suppose $a + b \geq 0$. Then $a - a^* = b^* - b$ and our assumption implies that $a = a^*$. Thus $a = a^+ - a^-$ with $a^+ \geq 0$, $a^- \geq 0$ and $a^+ a^- = 0$. Moreover a^+ , $a^- \in E_0$ since E_0 is a C^* -algebra. Since $a^- b a^- = (-a^-)^3 \leq 0$, and $a^- (a + b) a^- \geq 0$, it follows that $a^- b a^- \geq 0$. But $a^- b a^- \in B$ and so $a^- b a^- = 0$ by (i). Thus $a^- a a^- = a^- (a + b) a^- \geq 0$. Since we also have $a^- a a^- \leq 0$, $a^- a a^- = 0$. Hence $(a^-)^3 = 0$, $a^- = 0$ and $a \geq 0$ as required .

(ii) \Rightarrow (iii). We may suppose that $a_i \in E_i$, $0 \leq i < k$, and that $\|a_0 + a_1 + \dots + a_{k-1}\| \leq 1$. Thus

$$I - (a_0 + a_1 + \dots + a_{k-1})^* (a_0 + a_1 + \dots + a_{k-1}) \geq 0 .$$

Now $a_i^* a_i \in E_0$, $0 \leq i < k$, and $a_i^* a_j \in B$ for $i \neq j$, thus

$$I - (a_0^* a_0 + a_1^* a_1 + \dots + a_{k-1}^* a_{k-1}) + b \geq 0$$

for some $b \in B$. By (ii) it follows that

$$I - (a_0^* a_0 + a_1^* a_1 + \dots + a_{k-1}^* a_{k-1}) \geq 0 .$$

Thus $a_i^* a_i \leq I$ or $\|a_i\| \leq 1$ for $0 \leq i < k$ and (iii) now follows .

(iii) \Rightarrow (iv). This is clear.

(iv) \Rightarrow (i). Let $a_i \in E_i$, $1 \leq i < k$, and $b = a_1 + a_2 + \dots + a_{k-1} \geq 0$.

Since A is a C^* -algebra the positive square root c of b is in A .

Let $c = c_0 + c_1 + \dots + c_{k-1}$ with $c_i \in E_i$, $0 \leq i < k$. Then

$$b = c^2 = c^* c = c_0^* c_0 + c_1^* c_1 + \dots + c_{k-1}^* c_{k-1} + d$$

where $d \in B$. By our assumption $b \in B$ implies

$$c_0^* c_0 + c_1^* c_1 + \dots + c_{k-1}^* c_{k-1} = 0$$

Thus $c_i = 0$, $0 \leq i < k$, $c = 0$ and $b = 0$. That is , $B_+ = \{0\}$.

Suppose now that (i) - (iv) hold and the E_i are τ_w -closed. Let $(E_i)_1$ be the unit ball of E_i , $0 \leq i < k$. From the open mapping theorem applied to the mapping $(a_0, a_1, \dots, a_{k-1}) \rightarrow a_0 + a_1 + \dots + a_{k-1}$ from $E_0 \oplus E_1 \oplus \dots \oplus E_{k-1}$ to $E_0 + E_1 + \dots + E_{k-1}$, it follows that $(E_0)_1 + (E_1)_1 + \dots + (E_{k-1})_1 \supset \delta(E_0 + E_1 + \dots + E_{k-1})_1$ for some positive δ . Since $(E_i)_1$ is τ_w compact for $0 \leq i < k$, it follows that $(E_0 + E_1 + \dots + E_{k-1})_1 = \delta^{-1}((E_0)_1 + (E_1)_1 + \dots + (E_{k-1})_1) \cap B(K)_1$ is also τ_w compact. Thus $E_0 + E_1 + \dots + E_{k-1}$ is τ_w closed ([8] p41).

COROLLARY 7.16. Let E_0 contain finite rank projections P_n such that P_n converges to I in the strong operator topology as $n \rightarrow \infty$. Then A is a C^* -algebra. Moreover if the E_i are τ_w closed then A is a von Neumann algebra.

Proof. $P_n A P_n = P_n E_0 P_n + P_n E_1 P_n + \dots + P_n E_{k-1} P_n$ is a C^* -algebra of finite rank operators for each n . By Theorem 7.15,

$$(P_n E_1 P_n + P_n E_2 P_n + \dots + P_n E_{k-1} P_n)_+ = \{0\}.$$

Let $b \in (E_1 + E_2 + \dots + E_{k-1})_+$. Since $P_n b P_n \geq 0$ for all n , $P_n b P_n = 0$ for all n . But $P_n b P_n$ converges to b in the strong operator topology and so $b = 0$. Theorem 3.1 now completes the proof.

THEOREM 7.17. Let $k = 2^n$ for some $n \in \mathbb{N}$. Then A is a C^* -algebra. Moreover if the E_i are τ_w closed then A is a von Neumann algebra.

Proof. We prove the theorem by induction on n .

By Theorem 7.4 the theorem holds for $n = 1$.

Suppose now that the theorem is true for $n = m-1$, $m > 1$, and we are given $E_0, E_1, \dots, E_{2^m-1}$ satisfying the hypotheses at the beginning of this section.

Let $A_0 = E_0 + E_2 + \dots + E_{2^{m-2}}$ and $A_1 = E_1 + E_3 + \dots + E_{2^{m-1}}$.

It is easy to check that the 2^{m-1} spaces of operators $E_0, E_2, \dots, E_{2^{m-2}}$ satisfy the hypotheses of the theorem and so by the induction hypothesis

A_0 is a C*-algebra and τ_w closed if the E_i are .

We now show that A_1 is τ_n closed . Let $a = a_1 + a_3 + \dots + a_{2^{m-1}}$ where $a_i \in E_i$, $i = 1, 3, \dots, 2^{m-1}$. It will be sufficient to show that $\|a\| \leq 1$ implies $\|a_i\| \leq 1$, for $i = 1, 3, \dots, 2^{m-1}$. We have

$$0 \leq I - a^*a = I - (a_1^*a_1 + a_3^*a_3 + \dots + a_{2^{m-1}}^*a_{2^{m-1}}) + c$$

where $c \in E_2 + E_4 + \dots + E_{2^{m-2}}$. It follows from Theorem 7.15 (ii)

for A_0 that $I - (a_1^*a_1 + a_3^*a_3 + \dots + a_{2^{m-1}}^*a_{2^{m-1}}) \geq 0$. Thus

$a_i^*a_i \leq I$ and $\|a_i\| \leq 1$ for $i = 1, 3, \dots, 2^{m-1}$. Thus A_1 is τ_n closed.

Since A_0 and A_1 satisfy the hypotheses of the theorem for $n=1$ $A_0 + A_1$ is a C*-algebra. Theorem 7.15 now shows that $A_0 + A_1$ is τ_w closed if the E_i are . Thus the theorem is true for $n = m$, and the proof is completed.

Remark. 1. We have seen in Theorem 7.4 that for $k = 2$ $E_0 \cap B = \{0\}$ is equivalent to $B_+ = \{0\}$. Using the induction argument of Theorem 7.17 one can show that for $k = 2^n$ $E_0 \cap B = \{0\}$ is equivalent to $B_+ = \{0\}$. It would be interesting to prove this for arbitrary k and hence deduce that A is always a C*-algebra under our assumptions .

2. It is not clear how the quotient techniques of Theorem 7.5 may generalise to deal with non direct sums .

Nilpotent Automorphisms.

An immediate corollary of Theorem 7.17 is that if A is a C*-algebra on which is defined a *-automorphism θ such that $\theta^k = 1$ and if E_0, E_1, \dots, E_{k-1} are the k spectral subspaces of θ , then $(E_1 + E_2 + \dots + E_{k-1})_+ = \{0\}$. In fact it is easy to say much more .

Notation. Let A be a complex unital Banach algebra and let θ be an isometric linear isomorphism of A such that $\theta(1) = 1$, $\theta^k = I$. Let $\omega_0, \omega_1, \dots, \omega_{k-1}$ be the k^{th} roots of unity in their usual order.

It is a simple algebraic fact that $A = A_0 + A_1 + \dots + A_{k-1}$ where $A_i = \{a \in A ; \theta(a) = \omega_i a\}$, $0 \leq i < k$. In fact

$$A_i = \{a + \bar{\omega}_i \theta(a) + \bar{\omega}_i^2 \theta^2(a) + \dots + \bar{\omega}_i^{k-1} \theta^{k-1}(a) ; a \in A\} .$$

Let $B = A_1 + A_2 + \dots + A_{k-1}$. For $a \in A$, let $V(a)$ denote the algebra numerical range of a (see [4]).

THEOREM 7.18. For $b \in B$ $V(b) \not\subset -(k-1)^{-1}V(b)$. Moreover $k-1$ is the best such constant.

Proof. Let f be a state on A , that is, a linear functional such that $\|f\| = f(1) = 1$. Since θ is isometric and $\theta(1) = 1$ it follows that the equation

$$f_i(a) = f(\theta^i(a)) \quad (a \in A)$$

defines a state f_i on A , $0 \leq i < k$. Let $b = a_1 + a_2 + \dots + a_{k-1}$ with $a_i \in A_i$, $1 \leq i < k$, and let $f(a_i) = \lambda_i$ so that $\lambda_1 + \lambda_2 + \dots + \lambda_{k-1} = f(b)$. Clearly $f_i(a_j) = \omega_j^i f(a_j)$, $1 \leq j < k$. Thus

$$f_i(b) = \omega_1^i \lambda_1 + \omega_2^i \lambda_2 + \dots + \omega_{k-1}^i \lambda_{k-1} \in V(b)$$

for $0 \leq i < k$. Since

$$\sum_{1 \leq j < k} \omega_j^i = -1 \quad (1 \leq j < k),$$

we see that

$$\sum_{1 \leq i < k} f_i(b) = -(\lambda_1 + \lambda_2 + \dots + \lambda_{k-1}) = -f(b) .$$

Since $V(b)$ is convex $-(k-1)^{-1}f(b) \in V(b)$ and hence the first part of the theorem follows.

To see that $k - 1$ is the best such constant consider the linear isomorphism θ of the $k \times k$ matrices defined by

$$\theta(a)_{ij} = \omega_1^{j-i} a_{ij} \quad (1 \leq i, j \leq k) .$$

It is easily checked that θ is in fact an algebra $*$ -automorphism and isometric. Here $B = \{ (a_{ij}) ; a_{11} = 0 \text{ for } 1 \leq i \leq k \}$. Let b be the matrix in B whose entries are 1 off the main diagonal and 0 on the main diagonal. Then $V(b) = [-1, k-1]$.

Remarks. In view of Theorem 7.18 it is natural to ask when $\text{co } \sigma(b) \subset -(k-1)^{-1} \text{co } \sigma(b)$ when θ is an algebra automorphism. This is true for commutative Banach algebras since the above proof works with states replaced by multiplicative linear functionals (since f_i is again a multiplicative linear functional etc.). This result also holds for the automorphism of the $k \times k$ matrices defined in the proof above and is a simple consequence of the fact that $\text{trace}(b) = 0$ for $b \in B$.

§4 Sums of Intertwining Spaces.

The following definition generalises the odd and even products of Definition 7.2 .

DEFINITION 7.19. Let k be a fixed positive integer and let F be a family of operators on a Hilbert space K .

a) A j -product of F is an operator of the form $u_1^{\alpha_1} u_2^{\alpha_2} \dots u_r^{\alpha_r}$ where $u_i \in F$, $1 \leq i \leq r$, and $\sum_{1 \leq i \leq r} \alpha_i \equiv j \pmod{k}$.

b) If $F \cap F^* = \emptyset$, a j^* -product of F is an operator of the form $v_1^{\alpha_1} v_2^{\alpha_2} \dots v_r^{\alpha_r}$ where $v_i \in F \cup F^*$, $1 \leq i \leq r$, and $\sum_{1 \leq i \leq r} \beta_i \equiv j \pmod{k}$ where $\beta_i = \alpha_i$ for $v_i \in F$ and $\beta_i = -\alpha_i$ for $v_i \in F^*$.

When $F \cap F^* = \emptyset$ let E_i be the norm closed linear span of the i^* -products of F , $0 \leq i < k$. One can verify that the E_i satisfy the conditions (i) and (ii) at the beginning of section 3. Thus if our conjecture there is true, the C^* -algebra generated by F is precisely $E_0 + E_1 + \dots + E_{k-1}$. We prove this below for certain special situations and also identify the E_j in terms of the $\text{com}_\lambda(\cdot)$ operation where λ is a k^{th} root of unity.

As before let $\omega_0, \omega_1, \dots, \omega_{k-1}$ be the k^{th} roots of unity in their usual order. We first prove some elementary properties of $\text{com}_{\omega_0}(F), \text{com}_{\omega_1}(F), \dots, \text{com}_{\omega_{k-1}}(F)$.

- PROPOSITION 7.20. (i) $(\text{com}_{\omega_j}(F))^* = \text{com}_{\bar{\omega}_j}(F^*)$, $0 \leq j < k$.
(ii) $\text{com}_{\omega_i}(F)\text{com}_{\omega_j}(F) \subset \text{com}_{\omega_i\omega_j}(F)$, $0 \leq i, j < k$.

Proof. Elementary.

PROPOSITION 7.21. Let F be a family of normal operators. Then $(\text{com}_{\omega_j}(F))^* = \text{com}_{\bar{\omega}_j}(F)$ for $0 \leq j < k$.

Proof. Let $x \in \text{com}_{\omega_j}(F)$ so that $xs = \omega_j sx$, $s \in F$. By the Putnam-Fuglede theorem ([27] p20) $xs^* = \omega_j s^*x$, $s \in F$. Thus $sx^* = \omega_j x^*s$ and $x^*s = \omega_j sx^*$, $s \in F$. That is $x^* \in \text{com}_{\bar{\omega}_j}(F)$. Thus $(\text{com}_{\omega_j}(F))^* \subset \text{com}_{\bar{\omega}_j}(F)$. Similarly $(\text{com}_{\bar{\omega}_j}(F))^* \subset \text{com}_{\omega_j}(F)$ and hence $(\text{com}_{\omega_j}(F))^* = \text{com}_{\bar{\omega}_j}(F)$.

- PROPOSITION 7.22. (i) $\text{com}_{\omega_j}(\text{com}_{\bar{\omega}_1}(F))$ contains the j -products of F .
(ii) If F consists of normal operators and $F \cap F^* = \emptyset$ then $\text{com}_{\omega_j}(\text{com}_{\bar{\omega}_1}(F))$ contains the j^* -products of F .

Proof. (i) We have $xu = \bar{\omega}_1 ux$ for $x \in \text{com}_{\bar{\omega}_1}(F)$, $u \in F$. Thus $ux = \omega_1 xu$ for $x \in \text{com}_{\bar{\omega}_1}(F)$, $u \in F$. If y is a j -product of F then $yx = \omega_1^j xy$, $x \in \text{com}_{\bar{\omega}_1}(F)$. That is $y \in \text{com}_{\omega_j}(\text{com}_{\bar{\omega}_1}(F))$.

(ii) As in (i) we have $ux = \omega_1 xu$ for $x \in \text{com}_{\bar{\omega}_1}(F)$, $u \in F$. By the Putnam-Fuglede theorem $u^*x = \omega_1^{-1}xu^*$. From the definition of a j^* -product, y say, it follows that $yx = \omega_1^j xy$, $x \in \text{com}_{\bar{\omega}_1}(F)$. That is $y \in \text{com}_{\omega_j}(\text{com}_{\bar{\omega}_1}(F))$ as required.

The following theorem is an analogue of Corollary 7.12 involving several spaces of operators.

DEFINITION. A family F of operators on a Hilbert space is strongly separating if for each operator T such that $(Tf, f) \geq 0$ for $f \in \text{Range}(u)$, $u \in F$ it follows that $T \geq 0$.

Remark. 1. The definition says that if T is locally positive on the ranges of operators in F then T is positive.

2. If F contains an operator with dense range then clearly F is strongly separating.

3. If F is strongly separating then F^* is separating, but the converse statement is false.

THEOREM 7.23. Let F^* be a strongly separating family of normal operators.

Then
$$\text{com}_{\omega_0}(F) + \text{com}_{\omega_1}(F) + \dots + \text{com}_{\omega_{k-1}}(F)$$

is a von Neumann algebra and the sum is direct.

Proof. Let $E_i = \text{com}_{\omega_i}(F)$, $0 \leq i < k$. Propositions 7.20 and 7.21 show that (i) and (ii) of section 3 hold. Let $a_i \in E_i$ for some i , and let $u \in F$. We have $a_i u = \omega_i u a_i$ and, by the Putnam-Fuglede theorem,

$a_i u^* = \bar{\omega}_i u^* a_i$. Thus $u^* a_i = \omega_i a_i u^*$ and

$$u^* a_i u = \omega_i a_i u^* u = \omega_i u^* u a_i.$$

Let $a = a_1 + a_2 + \dots + a_{k-1}$ and $a^{(j)} = \omega_1^j a_1 + \omega_2^j a_2 + \dots + \omega_{k-1}^j a_{k-1}$ for $0 \leq j < k$, where $a_i \in E_i$ for $1 \leq i < k$.

Suppose first that $a \geq 0$. Then by the above $0 \leq u^* a u = a^{(1)} u^* u = u^* u a^{(1)}$. Since $a^{(1)}$ commutes with $u^* u$ it follows that $(a^{(1)})_{f,f} \geq 0$ for $f \in \text{Range}(u^* u)^{\frac{1}{2}} = \text{Range}(u^*)$. Since F^* is strongly separating we see that $a^{(1)} \geq 0$. Now $0 \leq u^* a^{(1)} u = a^{(2)} u^* u = u^* u a^{(2)}$, and similar computations show that $a^{(2)} \geq 0$. By induction $a^{(j)} \geq 0$ for $0 \leq j < k$. Thus $-a = \sum_{1 \leq j < k} a^{(j)} \geq 0$ which implies $a = 0$. Thus

if $B = E_1 + E_2 + \dots + E_{k-1}$ then $B_+ = \{0\}$. In view of Theorem 7.15 to show that the sum is τ_w closed it will suffice to show that $E_0 \cap B = \{0\}$.

Let $a, a^{(j)}$, $0 \leq j < k$, be as above and suppose that $a \in E_0 \cap B$. We have $u^* a u = u^* u a$, $u \in F$, since $a \in E_0$. On the other hand $u^* a u = u^* u a^{(1)}$, $u \in F$. Since F is separating $a = a^{(1)}$. Similarly by induction, since $u^* a^{(j)} u = a^{(j+1)}$, we see that $a = a^{(j)}$ for $0 \leq j < k$. Thus $(k-1)a = \sum_{1 \leq j < k} a^{(j)} = -a$ and so $a = 0$.

Finally suppose that $a = a_0 + a_1 + \dots + a_{k-1} = 0$. Then $a_i^* a_0 + a_i^* a_1 + \dots + a_i^* a_{k-1} = 0$. Since $E_0 \cap B = \{0\}$ it follows that $a_i^* a_i = 0$ and $a_i = 0$. Thus the sum is direct.

THEOREM 7.24. Let F be a family of normal operators and let $\text{com}_{\omega_1}(F)$ contain a family S of normal operators such that S^* is strongly separating. Then

$$F^{**} = \text{com}_{\omega_0}(\text{com}_{\omega_1}(F)) + \text{com}_{\omega_1}(\text{com}_{\omega_1}(F)) + \dots + \text{com}_{\omega_{k-1}}(\text{com}_{\omega_1}(F)).$$

Moreover the sum is direct and $\text{com}_{\omega_j}(\text{com}_{\omega_1}(F))$ is the τ_w closed linear span of the j^* -products of F .

Proof. We first show that $E_i = \text{com}_{\omega_i}(\text{com}_{\omega_1}(F)) \subset F''$, $0 \leq i < k$.

Let $x \in F'$, $a \in E_i$ and $s \in S \subset \text{com}_{\omega_1}(F)$. Then $sx \in \text{com}_{\omega_1}(F)$ and thus

$$(\omega_j sa)x = (as)x = a(sx) = \omega_j(sx)a.$$

Since S^* is strongly separating S is separating and $ax \equiv xa$.

Thus $E_i \subset F''$.

Let $F_i = \text{com}_{\omega_1}(S)$ so that $F_i \supset E_i$, $0 \leq i < k$. By Theorem 7.23

$A = F_0 + F_1 + \dots + F_{k-1}$ is a von Neumann algebra and the sum is direct. Thus by Theorem 7.15 if $a_j \in F_j$, $0 \leq j < k$, then

$$\|a_0 + a_1 + \dots + a_{k-1}\| \geq \|a_j\|, \quad 0 \leq j < k.$$

In particular $C = E_0 + E_1 + \dots + E_{k-1}$ is norm closed. Since E_i is τ_w closed, the open mapping argument of Theorem 7.23 shows that C_1 , the unit ball of C , is τ_w compact. Thus C is τ_w closed ([8] p38).

By Proposition 7.22 E_j contains the j^* -products of F , $0 \leq j < k$, and therefore $C \subset F''$, by von Neumann's double commutant theorem.

Thus $C = F''$. The sum is direct since, by Theorem 7.23,

$A = F_0 + F_1 + \dots + F_{k-1}$ is a direct sum. This completes the proof.

Remarks. With the notation of section 3, if we replace condition (ii) with (ii) $E_i^* = E_i$ then A is again a self-adjoint algebra of operators. Is it already closed? Such spaces of operators arise when we let F be a self adjoint set of operators and take $E_i = \text{com}_{\omega_i}(F)$.

§ 5 Further Generalisations and Open Questions.

The following theorem generalises part of Theorem 7.4.

THEOREM 7.25. Let A be a C^* -algebra and B a self-adjoint A -bimodule such that $B_+ = \{0\}$. If $A + B$ is an algebra then it is a C^* -algebra.

Proof. Since $A \cap B$ is a C^* -algebra and $A \cap B_+ \subset B_+ = \{0\}$ we have $A \cap B = \{0\}$. Suppose that $a + b \geq 0$ with $a \in A$, $b \in B$. Then since $A \cap B = \{0\}$, $a = a^*$ and $a = a^+ - a^-$ with $a^+, a^- \in A_+$ and $a^+ a^- = 0$. Since $a^-(a + b)a^- \geq 0$ implies $a^- b a^- \geq -a^- a a^- = (a^-)^3 \geq 0$ by our hypotheses $a^- b a^- = 0$, and $a^- = 0$. Thus we have shown that $a + b \geq 0$, $a \in A$, $b \in B$ implies $a \geq 0$.

Without loss we may suppose that A has an identity. Let $a \in A$, $b \in B$ $\|a + b\| \leq 1$. Then

$$0 \leq I - (a + b)^*(a + b) = I - a^*a - b^*b - b^*a - a^*b.$$

Since $A + B$ is an algebra $b^*b = a' + b'$ with $a' \in A$, $b' \in B$. Since $b^*b \geq 0$, by what we have shown above $a' \geq 0$. Thus

$$0 \leq I - a^*a - a' + d$$

where $d \in B$. Hence $0 \leq I - a^*a - a'$, $a^*a \leq I$ and $\|a\| \leq 1$. Thus we have shown that $\|a\| \leq \|a + b\|$ for $a \in A$ and $b \in B$ and it follows that $A + B$ is closed and therefore a C^* -algebra.

In view of this result it would be interesting to settle the following question.

Problem. Let A be a C^* -algebra of operators and let B be a self-adjoint A -bimodule. Is $A + B$ closed?

The following question arises naturally from the considerations of section 4 .

Problem. Let G be a finite group and let $\{E_g ; g \in G\}$ be a family of closed spaces of operators such that

(i) $E_g E_{g'} \subset E_{gg'}$, $(g , g' \in G)$,

and (ii) $E_g^* = E_{g^{-1}}$, $(g \in G)$.

Is the linear span of $\{E_g ; g \in G\}$ closed ?

One can construct such spaces as follows . Let G be a finite group of order k with elements $g_0 , g_1 , \dots \dots , g_{k-1}$. Let $FG(k)$ be the free group on k symbols $v_0 , v_1 , \dots \dots , v_{k-1}$ and define $\phi:FG(k) \rightarrow G$ by $\phi(w) = w'$ where w is a word of $FG(k)$ and w' is an element of G obtained by substituting g_i for v_i , $0 \leq i < k$.

Let F be a family of k unitaries $\{u_0 = I , u_1 , u_2 , \dots \dots , u_{k-1}\}$ acting on K , and define $\Phi : FG(k) \rightarrow B(K)$ as the map which substitutes u_i for v_i .

Let $E_g =$ closed linear span of $\Phi\phi^{-1}(g_i)$, $0 \leq i < k$. Then $\{E_g : g \in G\}$ satisfies (i) and (ii) .

One can ask when we may have an odd even decomposition for certain Banach algebras. Not surprisingly the appropriate generalisation of Theorem 7.4 fails even for semisimple commutative Banach algebras as the following simple example demonstrates.

Notation. D is a domain in the complex plane which contains the origin and has a simply connected compact closure \bar{D} . Let A be the function algebra on \bar{D} generated by the identity I and the function z . Let C (resp. B) be the closed linear span in A of the non negative even (resp. odd) powers of z .

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PROPOSITION 7.26. $C + B$ is closed if and only if $D = -D$.

Proof. First note that if $f \in C \cap B$ then for $z \in D \cap -D$ we have $f(z) = f(-z)$ (since $f \in C$) and $f(z) = -f(-z)$ (since $f \in B$). Thus $f(z) = 0$ for z in the non void open set $D \cap -D$. Since D is simply connected and f is analytic on D $f = 0$, and so $C \cap B = \{0\}$.

If $C + B$ is closed then since the sum is direct we can define an automorphism θ on $A = C + B$ by

$$\theta(f + g) = f - g \quad (f \in C, g \in B)$$

We have $\bar{D} = \sigma(z) = \sigma(\theta(z)) = \sigma(-z) = -\bar{D}$ and thus $D = -D$.

Conversely suppose that $D = -D$. Given a polynomial p define $(\theta p)(z) = p(-z)$. $D = -D$ implies $\|\theta p\| = \|p\|$. If $f \in A$ and p_n are polynomials converging to f in A then $\frac{1}{2}(p_n + \theta(p_n))$ converges to $h \in C$ and $\frac{1}{2}(p_n - \theta(p_n))$ converges to $g \in B$ and $f = h + g$. That is, $A = C + B$ is closed.

PROBLEMS

1. (Ch.2) Let $\underline{W} = \{W(\alpha) ; \alpha \in \mathbb{R}_+\}$ be a semigroup of unitary operators and let H be an invariant subspace for \underline{W} . If $\underline{U} = \{U(\alpha) = W(\alpha)|_H\}$ then is $\text{twi}(\underline{U}) = \text{pr}(\text{twi}(\underline{W}))$?
2. (Ch. 3) Let S be a norm closed subspace of $B(H)$. Is $H^\infty[B(H)] + C[S]$ closed ?
3. (Ch.4) Let F be a commutative family of operators on a Hilbert space. Let C be the C^* -algebra generated by F and let H^∞ be the weakly closed operator algebra generated by F . Is $H^\infty + C$ closed ?
4. (Ch.5) Let \underline{U} be a family of doubly commuting isometries with unitary dilation \underline{W} . Is $\text{twi}(\underline{U}) = \text{pr}(\text{twi}(\underline{W}))$?
5. (Ch.5) Does there exist a constant $\kappa > 0$ such that for each $\Phi \in L^\infty[B(H)]$ whose negative Fourier coefficients $\Phi_k, k \leq 0$, are Hankel operators, there exists $\Psi \in L^\infty[B(H)]$ such that
 - (i) Ψ_k is a Hankel operator for all k ,
 - (ii) $\Psi_k = \Phi_k$ for all $k \leq 0$,
 - (iii) $\|\Psi\| \leq \kappa \|\Phi\|$?
6. (Ch.5) Is the space of Hankel operators complemented by a weakly closed space of operators ?
7. (Ch 7) Let A be a C^* -algebra of operators and let B be a self-adjoint C^* -bimodule of A . Is $A + B$ closed ?

8. (Ch.7) Let G be a finite group and let $\{E_g ; g \in G\}$ be a family of closed spaces of operators on a Hilbert space such that

$$(i) \quad E_g E_h \subset E_{gh} \quad (g, h \in G) ,$$

$$(ii) \quad E_g^* = E_{g^{-1}} \quad (g \in G) .$$

Is $\text{lin span}\{E_g ; g \in G\}$ closed ?

Appendix.

Notation. $B(\mathbb{R})$ denotes the commutative C^* -algebra of bounded functions on the real line \mathbb{R} . For f in $B(\mathbb{R})$ and x in \mathbb{R} let f_x be the translate of f defined by $f_x(y) = f(y - x)$, $y \in \mathbb{R}$.

DEFINITION (i) A functional m on $B(\mathbb{R})$ is translation invariant if $m(f_x) = m(f)$ for all $x \in \mathbb{R}$.

(ii) A state f on $B(\mathbb{R})$ (or $L^\infty(\mathbb{R})$) is a bounded linear functional such that $\|f\| = f(1) = 1$.

The following theorem is a corollary of Theorem 17.5 in 'Abstract Harmonic Analysis vol I' by E. Hewitt and K.A. Ross (Springer Verlag).

THEOREM. There exists a translation invariant state on $B(\mathbb{R})$.

COROLLARY. Let \underline{M} be a closed translation invariant subspace of $L^\infty(\mathbb{R})$ and let M be a translation invariant state on \underline{M} . Then there exists a translation invariant state \tilde{M} on $L^\infty(\mathbb{R})$ extending M .

Proof. Let m be a translational invariant state on $B(\mathbb{R})$.

By the Hahn Banach theorem there exists a state M_1 on $L^\infty(\mathbb{R})$ extending M . For $g \in L^\infty(\mathbb{R})$ define $\phi_g \in B(\mathbb{R})$ by $\phi_g(y) = M_1(g_y)$ for $y \in \mathbb{R}$. Define \tilde{M} on $L^\infty(\mathbb{R})$ by $\tilde{M}(g) = m(\phi_g)$. Then \tilde{M} is a translation invariant state on $L^\infty(\mathbb{R})$ extending M .

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