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# Fano Varieties: Positivity, K-stability and more.

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Doctor of Philosophy  
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# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

NIVEDITA VISWANATHAN

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# Abstract

This thesis is about Fano varieties and their properties. We will determine the K-stability of certain singular del Pezzo surfaces and smooth Fano 3-folds, the existence of cylinders in singular del Pezzo surfaces, and also classify higher dimensional Fano varieties with certain properties. In dimension 2, many new examples of K-stable polarized singular del Pezzo surfaces with du Val singular points have been introduced and the existence of polarized cylinders in many of these surfaces has been determined. We also completely solve the K-stability problem for singular del Pezzo surfaces that are index 2 hypersurfaces in weighted projective space. In dimension 3, all deformation families of smooth three-dimensional Fano varieties that contain K-polystable elements have been described. In higher dimensions, a complete classification of smooth Fano varieties of large index that have positive second and third Chern characters has been given, and all rational homogeneous spaces of Picard rank 1 having positive second Chern character have been described. In particular, we prove that the only rational homogeneous spaces of Picard rank 1 with positive second and third Chern characters are projective spaces and quadric hypersurfaces. This thesis also contains few auxiliary results, which are closely related to K-stability of Fano varieties. For instance, for a reduced plane curve of degree  $d$ , the sixth worst log canonical threshold that it can have, has been determined.

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# Chapter 1

## Introduction

Fano varieties are varieties with ample anticanonical class. These have been intensively studied in the last few decades. In this thesis, we will study some properties of Fano varieties, in detail. The thesis is split into chapters based on the dimension of the variety that we study.

Throughout this thesis, the base field we work over is the complex numbers  $\mathbb{C}$ .

### 1.1 Curves

We begin by looking at plane curves of degree  $d$  in  $\mathbb{P}^2$  in Chapter 3. The main motivation behind this chapter is to study singularities of such curves and to explicitly have a way of measuring how singular a given curve is. This sets the groundwork for understanding singularities of varieties in higher dimensions.

The main aim is to address the following question:

**Question.** *Given a curve  $C_d$  of fixed degree  $d$ , what is the worst singularity that the curve can have at the point  $P$  on  $C_d$ ?*

We can use various parameters to measure the singularity at the point  $P$ , such as multiplicity of the curve at the point  $P$ ,  $\text{mult}_P(C_d)$ , Milnor number,  $\mu(P)$  and log canonical threshold of the curve at the point  $P$ ,  $\text{lct}_P(\mathbb{P}^2, C_d)$ .

All three parameters mentioned earlier give the same answer to the above question, since  $\text{mult}_P(C_d) \leq d$  and  $\mu(P) \leq (d-1)^2$  with  $\text{mult}_P(C_d) = d$  and  $\mu(P) = (d-1)^2$  if and only if  $C_d$  is a union of  $d$  lines. The following theorem proves that computing the log canonical threshold of the curve at  $P$  also gives the same answer to the above question.

**Theorem 1.1.** *([C01, Theorem 4.1]) One has  $\text{lct}_P(\mathbb{P}^2, C_d) \geq \frac{2}{d}$  and  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2}{d}$  if and only if  $C_d$  is a union of  $d$  lines passing through  $P$ .*

We can then ask the following question:

**Question.** *What is the second worst singularity at the point  $P$ ?*

To present this answer, we introduce certain types of singularities in Section 3.1 and we call these types of singularities  $\mathbb{K}_n, \mathbb{T}_n, \tilde{\mathbb{K}}_n, \tilde{\mathbb{T}}_n, \mathbb{M}_n, \tilde{\mathbb{M}}_n, \hat{\mathbb{M}}_n$ , where  $n = \text{mult}_P(C_d)$ .

In [C17], the following result was obtained.

**Theorem 1.2.** *Suppose  $d \geq 5$  and  $\frac{2}{d} < \text{lct}_P(\mathbb{P}^2, C_d) \leq \frac{2d-3}{d(d-2)}$ . Then the curve  $C_d$  has singularity of type  $\mathbb{T}_{d-1}, \mathbb{K}_{d-1}, \tilde{\mathbb{T}}_{d-1}, \tilde{\mathbb{K}}_{d-1}$  at the point  $P$  and the values of their log canonical thresholds at the point  $P$  are  $\frac{2d-3}{(d-1)^2} < \frac{2d-1}{d(d-1)} < \frac{2d-5}{d^2-3d+1} < \frac{2d-3}{d(d-2)}$  respectively.*

This result and Theorem 1.1 give the five worst singularities of the curve  $C_d$ . In Section 3.2, we will determine the sixth worst singularity that a curve can have at the point  $P$ . That is, in Section 3.2, we prove the following.

**Main Result 1.** *Suppose  $d \geq 6$  and  $\frac{2d-3}{d(d-2)} < \text{lct}_P(\mathbb{P}^2, C_d) \leq \frac{2d-7}{d^2-4d+1}$ . Then the curve  $C_d$  has singularity of type  $\mathbb{M}_{d-1}, \tilde{\mathbb{M}}_{d-1}$  or  $\hat{\mathbb{M}}_{d-1}$  at the point  $P$  with  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-7}{d^2-4d+1}$ .*

## 1.2 Surfaces

In Chapter 4, Section 4.1 and Section 4.2, we study the notion of K-stability of Fano varieties of dimension 2, which are also called del Pezzo surfaces. In particular, we are interested in the K-stability of del Pezzo surfaces with du Val singular points (Section 4.1) and that of hypersurfaces in weighted projective space (Section 4.2). In Section 4.3, we also explicitly construct  $H$ -polar cylinders for any ample divisor  $H$ , on singular del Pezzo surfaces.

### 1.2.1 K-stability of del Pezzo surfaces

The notion of K-stability was introduced to algebraic geometers very recently and there has been enormous progress in this field of study since the development of the minimal model program. Since concepts and techniques from minimal model program were found to be closely related to the study of K-stability of Fano varieties, there has been significant progress in the study of the same. Since then, there have been numerous equivalent definitions of K-stability of Fano varieties by Fujita, Li and many others. We study few such notions in Section 2.2.

One of the main motivations behind studying the notion of K-stability of Fano varieties is the following conjecture:

**Conjecture.** *(Yau-Tian-Donaldson) Let  $X$  be a smooth variety and let  $L$  be an ample line bundle on  $X$ . Then  $X$  admits a constant scalar curvature Kähler (cscK) metric in  $c_1(L)$  if and only if the pair  $(X, L)$  is K-polystable.*

This gave an algebro-geometric method of establishing the existence of constant scalar curvature metrics on varieties  $X$ .

In the case of smooth del pezzo surfaces, a lot is known about their K-stability. For instance, the above conjecture is proven for smooth Fano varieties polarised by anticanonical line bundles by Chen, Donaldson and Sun in [CDS15] and [T15].

Smooth del Pezzo surfaces are Kähler-Einstein and therefore K-stable for the anticanonical polarization ([T90]). But determining if a given del Pezzo surface is K-stable for any polarisation other than the anti-canonical is challenging. In this regard, the Dervan's criterion (Theorem 1.3) comes in handy.

### 1.2.1.1 Applicability of Dervan's criterion

The criterion stated below was established by Dervan and gives a condition for the K-stability of a polarised variety.

**Theorem 1.3.** [D15, Theorem 1.1] *Let  $X$  be a Fano variety of dimension  $n$ , and let  $L$  be an ample  $\mathbb{Q}$ -divisor on it. Let  $\nu(L) = \frac{-K_X \cdot L^{n-1}}{L^n}$ . Then  $(X, L)$  is K-stable if*

1. *the  $\mathbb{Q}$ -divisor  $-K_X - \frac{n}{n+1}\nu(L)L$  is nef,*
2. *and  $\alpha(X, L) > \frac{n}{n+1}\nu(L)$ .*

Note that  $\nu(L)$  in Theorem 1.3 is referred to as the slope and the quantity  $\alpha(X, L)$  is as defined in Section 2.2.2.

In Section 4.1.2, we study the applicability of the above theorem to singular del Pezzo surfaces,  $S$ .

**Main Result 2.** *Let  $S$  be a del Pezzo surface of degree 1 with  $A_1$ ,  $A_2$  or  $A_3$  singularities. Let  $L$  be any ample divisor and  $\nu(L) = \frac{-K_S \cdot L}{L^2}$  on  $S$ . If the  $\mathbb{Q}$ -divisor  $-K_S - \frac{2}{3}\nu(L)L$  is nef, then  $(S, L)$  is K-stable.*

### 1.2.1.2 K-stability of singular hypersurfaces

The main objects that we are interested in, in Section 4.2, are quasismooth and well-formed hypersurfaces  $S_d$  in  $\mathbb{P}(a_0, a_1, a_2, a_3)$  of degree  $d$  with index  $I=2$ .

Recall that the index of a Fano variety  $X$  is the largest integer that divides  $-K_X$  in  $\text{Pic}(X)$ . In the case of hypersurfaces  $S_d$ , index of  $S_d$  is given by  $I = a_0 + a_1 + a_2 + a_3 - d$ .

These surfaces  $S_d$  are given by a quasihomogeneous polynomial equation of degree  $d$

$$f(x, y, z, t) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3) \cong \text{Proj}(\mathbb{C}[x, y, z, t]),$$

where  $\text{wt}(x) = a_0$ ,  $\text{wt}(y) = a_1$ ,  $\text{wt}(z) = a_2$  and  $\text{wt}(t) = a_3$ .  $S_d$  is said to be *quasismooth* if the only singular point is at the origin in  $\mathbb{C}^4$ , which implies that  $S_d$  has at most cyclic quotient singularities. It is *well-formed* if

$$\gcd(a_0, a_2, a_3) = \gcd(a_0, a_1, a_2) = \gcd(a_0, a_1, a_3) = \gcd(a_1, a_2, a_3) = 1.$$

Note that being well-formed implies that

$$-K_{S_d} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(a_0, a_1, a_2, a_3)}(a_0 + a_1 + a_2 + a_3 - d).$$

So suppose  $I$  is positive, then  $S_d$  is a del Pezzo surface.

All possible quintuples  $(a_0, a_1, a_2, a_3, d, I)$ , such that a quasismooth and well-formed hypersurface  $S_d$  in  $\mathbb{P}(a_0, a_1, a_2, a_3)$ , of degree  $d$  and index  $I$ , exist, have

been described in [JK01, Theorem 8] for  $I = 1$ , [CS13, Corollary 1.13] for  $I = 2$  and a complete classification is given in [P18].

The main problem that we are interested in, is the following:

**Problem. Calabi Problem:** Describe all  $K$ -polystable, quasismooth and well-formed hypersurfaces  $S_d$  in  $\mathbb{P}(a_0, a_1, a_2, a_3)$  of degree  $d$  and index  $I$ .

When  $I = 1$ , this is solved in [JK01, Theorem 8], [A02, Theorem 4.1], [CPS10, Theorem 1.10] and [CPS21, Theorem 1.7]. In the case of  $I = 2$ , this has been studied in [BGN03], [CPS10], [CS13], [CPS21] and [KW21, Theorem 1.2]. In Section 4.2, we will solve the Calabi Problem for the remaining quasismooth, well-formed hypersurfaces of index  $I = 2$ , thus giving a complete answer in the case of Index 2. This is a joint work with Joonyeong Won and In-Kyun Kim. <sup>1</sup>

**Main Result 3.** Let  $S_d$  be a quasismooth, well-formed hypersurface with  $I = 2$ . The following hypersurfaces  $S_d$  in  $\mathbb{P}(a_0, a_1, a_2, a_3)$  of degree  $d$ , are  $K$ -polystable.

No.	$(a_0, a_1, a_2, a_3)$	$d$
1	$(1, 1, n, n), n \geq 2$	$2n$
2	$(1, 2, n + 2, n + 3), n \geq 0$	$2(n + 3)$
3	$(1, 3, 4, 6)$	12
4	$(1, 4, 5, 7)$	15
5	$(1, 4, 5, 8)$	16
6	$(1, 4, 6, 9)$	18
7	$(1, 5, 7, 11)$	22
8	$(1, 6, 10, 15)$	30
9	$(1, 7, 12, 18)$	36
10	$(1, 8, 13, 20)$	40

## 1.2.2 Existence of Cylinders

In Section 4.3, we study the geometric property of the existence of  $H$ -polar cylinders in  $S_d$  for any ample divisor  $H$ . This is joint work with Grigory Belousov.<sup>2</sup>

**Definition 1.** [CPW16, Definition 1.3] Let  $M$  be an  $\mathbb{Q}$ -divisor on a projective normal variety  $X$ . An  $M$ -polar cylinder in  $X$  is an open subset

$$U = X \setminus \text{Supp}(D)$$

defined by an effective  $\mathbb{Q}$ -divisor  $D$  in the  $\mathbb{Q}$ -linear equivalence class of  $M$  such that  $U$  is isomorphic to  $Z \times \mathbb{A}^1$  for some affine variety  $Z$ .

When  $H = -K_{S_d}$ , these have been studied in [CPW16].

<sup>1</sup>In this joint work, I independently proved Theorem 3 for the hypersurfaces in  $\mathbb{P}(1, 1, n, n)$  when  $n \geq 2$  and hypersurfaces in  $\mathbb{P}(1, 2, n + 2, n + 3)$ . The remaining cases were a joint work with my collaborators.

<sup>2</sup>In this work, I explicitly described all possible ample divisors for the surfaces with different singularities and jointly, with the help and guidance of Dr. Grigory Belousov, explicitly constructed cylinders for them, also having drawn figures to represent the same.

---

**Theorem 1.4.** [CPW16, Theorem 1.5] *Let  $S_d$  be a del Pezzo surface of degree  $d$  with at most  $du$  Val singularities.*

- I. *The surface  $S_d$  does not admit a  $(-K_{S_d})$ -polar cylinder when*
  1.  *$d = 1$  and the surface  $S_d$  allows only singular points of types  $A_1, A_2, A_3, D_4$  if any;*
  2.  *$d = 2$  and  $S_d$  allows only singular points of type  $A_1$  if any;*
  3.  *$d = 3$  and  $S_d$  allows no singular point.*
- II. *The surface  $S_d$  has a  $(-K_{S_d})$ -polar cylinder if it is not one of the del Pezzo surfaces listed in I.*

In Section 4.3, we prove the following.

**Main Result 4.** *Let  $S$  be a del Pezzo surface with at least 1 singular point of type  $E_6, E_7, E_8, D_4, D_5, D_6, D_7$  or  $A_7$ . Then  $S$  has a  $H$ -polar cylinder for any ample divisor  $H \neq -K_S$  on  $S$ .*

### 1.3 3-folds

Smooth Fano 3-folds have been classified in [Is77, Is78, MoMu83, MoMu03] into 105 families, which are labeled as №1.1, №1.2, №1.3, ..., №9.1, №10.1, which is the numeration followed in [IP99].

The main objective of this work is to answer the following question.

**Calabi Problem.** *Find all  $K$ -polystable smooth Fano threefolds in each family.*

This problem has already been solved for many families, and partial results are known in many cases. (See [ACCFKGSSV] for the list of works that have contributed to this problem.)

In particular, it has been proved in [F16] that all smooth threefolds in the 26 families

№2.23, №2.28, №2.30, №2.31, №2.33, №2.35, №2.36, №3.14,  
 №3.16, №3.18, №3.21, №3.22, №3.23, №3.24, №3.26, №3.28, №3.29,  
 №3.30, №3.31, №4.5, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2

are divisorially unstable (see Definition 8), which implies that *none* of them are  $K$ -polystable.

In [ACCFKGSSV], we show that all smooth Fano threefolds in the Family №2.26 are not  $K$ -polystable, and therefore proving the following. This is a joint work with Carolina Araujo, Ana-Maria Castravet, Ivan Cheltsov, Kento Fujita, Anne-Sophie Kaloghiros, Jesus Martinez-Garcia, Constantin Shramov and Hendrik Süß.<sup>3</sup>

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<sup>3</sup>My contribution to this project was to explicitly compute the  $\delta$ -invariant using the Abban-Zhuang Theory in few cases.

---

**Main Result 5.** *Let  $X$  be a general Fano threefold in the family  $\mathbb{N}^\circ\mathcal{N}$ . Then*

$$X \text{ is } K\text{-polystable} \iff \mathcal{N} \notin \left\{ \begin{array}{l} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, \\ 2.36, 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, \\ 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, 4.5, \\ 4.8, 4.9, 4.10, 4.11, 4.12, 5.2. \end{array} \right\}$$

As a consequence, we have

**Corollary.** *Let  $X$  be a general Fano threefold in the family  $\mathbb{N}^\circ\mathcal{N} \neq 2.26$ . Then*

$$X \text{ is } K\text{-polystable} \iff X \text{ is divisorially semistable} \iff X \text{ is } K\text{-semistable}.$$

In Section 2.2, we will introduce the notions of  $K$ -stability and the various results that we will need in order to prove the  $K$ -polystability of the 3-folds belonging to Families  $\mathbb{N}^\circ 2.16$ ,  $\mathbb{N}^\circ 2.24$ ,  $\mathbb{N}^\circ 3.10$ ,  $\mathbb{N}^\circ 4.13$ ,  $\mathbb{N}^\circ 4.3$ ,  $\mathbb{N}^\circ 5.1$ . Each case is dealt in detail in Chapter 5. This gives a rough idea of the proofs involved in the remaining cases. For further details and proofs of remaining cases, refer to [ACCFKGSSV].

## 1.4 Higher dimensions

The objective of Chapter 6 is to extend positivity conditions from that of Fano manifolds to higher dimensions. Recall that Fano manifolds are complex projective manifolds with positive first Chern class  $c_1(T_X)$  (See Section 2.3 for the definition). Examples include Projective Spaces, Grassmanians and other homogeneous spaces.

In [Mor79], Mori showed that any Fano manifold is covered by rational curves. Later, in [KMM92] and [Cam92], a much stronger property was proven for Fano manifolds, which is that any Fano manifold  $X$  is *rationally connected*, i.e. there are rational curves connecting any two points of  $X$ . It is expected that such interesting properties hold in Higher Fano manifolds too and thus, in recent years, there has been a lot of work done in defining suitable higher analogues of the Fano condition. Thus, this problem can be stated as follows:

**Problem.** *Find natural geometric conditions  $\mathfrak{R}_r$  on a manifold  $X$  such that if a complex projective manifold satisfies  $\mathfrak{R}_r$ , then  $X$  is covered by rational varieties of dimension  $r$ .*

For  $r = 1$ , this is the condition that  $X$  is rationally connected, or more restrictively, that  $X$  is a Fano manifold.

In [dJS06a], De Jong and Starr introduced the notion of *2-Fano manifolds* as a candidate for the condition  $\mathfrak{R}_2$ . A projective manifold  $X$  is said to be *2-Fano* if it is Fano and the second Chern character  $\text{ch}_2(T_X) = \frac{1}{2}c_1(T_X)^2 - c_2(T_X)$  is positive, i.e.,  $\text{ch}_2(T_X) \cdot S > 0$  for every surface  $S$  in  $X$ , where  $T_X$  is the tangent bundle of the manifold  $X$ . They also present some notions of *rational simple connectedness*.

In [dJS07], it is shown that 2-Fano manifolds satisfying some mild assumptions are covered by rational surfaces. In [AC12], Araujo and Castravet introduced

a new approach to study 2-Fano manifolds, via *polarized minimal families of rational curves*  $(H_x, L_x)$ . (See Section 2.4 for details.) In [AC12], many new examples of 2-Fano manifolds have been described and this was followed by the classification of 2-Fano manifolds with large index in [AC13].

In my joint work with Araujo, Beheshti, Castravet, Jabbusch, Makarova, Mazon and Taylor, we suggest the following definition as a candidate for the condition  $\mathfrak{A}_r$ .<sup>4</sup>

**Definition 2.** *We say that a Fano manifold  $X$  satisfies the condition  $\mathfrak{F}_r$  if its Chern characters  $ch_i(X)$  are positive for all  $1 \leq i \leq r$ . This positivity condition means that  $ch_i(T_X) \cdot Z > 0$  for every effective  $i$ -cycle  $Z$  in  $X$ .*

As an encouraging step forward, in [Suz20] and [Nag19], it has been shown that with further assumptions, Fano manifolds satisfying  $\mathfrak{F}_r$  are covered by rational  $r$ -folds. Given a Fano manifold  $X$ , with a polarized minimal family of rational curves  $(H_x, L_x)$  that satisfies the condition  $\mathfrak{F}_r$ , one can also ask if  $H_x$  satisfies condition  $\mathfrak{F}_{r-1}$ .

With these motivating questions and observations, in Chapter 6, we study examples of Fano manifolds that satisfy  $\mathfrak{F}_2$  and  $\mathfrak{F}_3$ .

**Main Result 6.** *The following is the complete list of rational homogeneous spaces of Picard rank 1 satisfying the condition  $\mathfrak{F}_2$ :*

- $A_n/P^k$ , for  $k = 1, n$  and for  $n = 2k - 1, 2k$  when  $2 \leq k \leq \frac{n+1}{2}$ ;
- $B_n/P^k$ , for  $k = 1, n$  and for  $2n = 3k + 1$  when  $2 \leq k \leq n - 1$ ;
- $C_n/P^k$ , for  $k = 1, n$  and for  $2n = 3k - 2$  when  $2 \leq k \leq n - 1$ ;
- $D_n/P^k$ , for  $k = 1, n - 1, n$  and for  $2n = 3k + 2$  when  $2 \leq k < n - 1$ ;
- $E_n/P^k$ , for  $n = 6, 7, 8$  and  $k = 1, 2, n$ ;
- $F_4/P^4$ ;
- $G_2/P^k$ , for  $k = 1, 2$ .

The notations used above to denote the various rational homogeneous spaces of Picard rank 1 are as explained in Section 2.6.

However, we get no new examples of Fano manifolds satisfying  $\mathfrak{F}_3$ .

**Main Result 7.** *The only rational homogeneous spaces of Picard rank 1 satisfying  $\mathfrak{F}_3$ , are projective spaces  $\mathbb{P}^n$ ,  $n \geq 3$ , and quadric hypersurfaces  $Q^n \subset \mathbb{P}^{n+1}$ ,  $n \geq 7$ .*

We also go through the list of 2-Fano manifolds with large index in [AC13] and check the  $\mathfrak{F}_3$  condition for those. We obtain the following classification.

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<sup>4</sup>In this project, I primarily contributed to computing the bounds for the dimension  $n$  in Main Result 8 and to also write up the results along with my collaborators.

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**Main Result 8.** *Let  $X$  be a Fano manifold of dimension  $n \geq 3$  and index  $i_X \geq n - 2$ . If  $X$  satisfies  $\mathfrak{F}_3$ , then  $X$  is isomorphic to one of the following.*

- $\mathbb{P}^n$ .
- *Complete intersections in projective spaces:*
  - *Quadric hypersurfaces  $Q^n \subset \mathbb{P}^{n+1}$  with  $n > 6$ ;*
  - *Complete intersections of quadrics  $X_{2,2} \subset \mathbb{P}^{n+2}$  with  $n > 13$ ;*
  - *Cubic hypersurfaces  $X_3 \subset \mathbb{P}^{n+1}$  with  $n > 25$ ;*
  - *Quartic hypersurfaces in  $\mathbb{P}^{n+1}$  with  $n > 62$ ;*
  - *Complete intersections  $X_{2,3} \subset \mathbb{P}^{n+2}$  with  $n > 32$ ;*
  - *Complete intersections  $X_{2,2,2} \subset \mathbb{P}^{n+3}$  with  $n > 20$ .*
- *Complete intersections in weighted projective spaces:*
  - *Degree 4 hypersurfaces in  $\mathbb{P}(2, 1, \dots, 1)$  with  $n > 55$ ;*
  - *Degree 6 hypersurfaces in  $\mathbb{P}(3, 2, 1, \dots, 1)$  with  $n > 181$ ;*
  - *Degree 6 hypersurfaces in  $\mathbb{P}(3, 1, \dots, 1)$  with  $n > 188$ ;*
  - *Complete intersections of two quadrics in  $\mathbb{P}(2, 1, \dots, 1)$  with  $n > 6$ .*

Note that here  $X_{d_1, d_2, \dots, d_s}$  denotes the smooth complete intersection of hypersurfaces of degree  $d_i$ .

# Chapter 2

## Preliminaries

In this chapter, we will go through all the basic results that we will need in this thesis, to prove the main results. Note that throughout this thesis, the base field that we work with is the set of complex numbers.

### 2.1 Singularities of Pairs

Let  $X$  be a normal variety such that  $K_X$  is a  $\mathbb{Q}$ -Cartier divisor, let  $\pi: \hat{X} \rightarrow X$  be its resolution of singularities. Denote the  $\pi$ -exceptional divisors by  $E_1, \dots, E_m$ . Then

$$K_{\hat{X}} + \sum_{i=1}^m e_i E_i = \pi^*(K_X) \quad (2.1)$$

for some rational numbers  $e_1, \dots, e_m$ . For each  $i \in \{1, \dots, m\}$ , we let  $A_X(E_i) = 1 - e_i$  and say that  $A_X(E_i)$  is the log discrepancy of the divisor  $E_i$ . We say that

- $X$  has terminal singularities if each  $e_i < 0$ ,
- $X$  has canonical singularities if each  $e_i \leq 0$ ,
- $X$  has Kawamata log terminal singularities if each  $e_i < 1$ ,
- $X$  has log canonical singularities if each  $e_i \leq 1$ .

One can show that these definitions do not depend on the choice of the morphism  $\pi$ .

If  $X$  is smooth, then its singularities are terminal. Moreover, if  $X$  is a surface, then  $X$  is smooth if and only if it has terminal singularities. Similarly, if  $X$  is a surface, then it has canonical singularities if and only if  $X$  has Du Val singularities. Likewise, if  $X$  is a surface, then it follows from [K97, Theorem 3.6] that  $X$  has Kawamata log terminal singularities if and only if  $X$  has quotient singularities. In all dimensions, Kawamata log terminal singularities are rational by [K97, Theorem 11.1]. Starting from now, we assume that the variety  $X$  has Kawamata log terminal singularities.

Let  $B_X$  be an effective  $\mathbb{Q}$ -divisor on  $X$ . Then

$$B_X = \sum_{i=1}^r a_i B_i, \quad (2.2)$$

where each  $B_i$  is a prime Weil divisor on  $X$ , and each  $a_i$  is a non-negative rational number. We say that  $(X, B_X)$  is a log pair,  $B_X$  is its boundary, and  $K_X + B_X$  is its log canonical divisor. Let us define singularity classes for the log pair  $(X, B_X)$  following [K97, KM98].

Let  $\widehat{B}_1, \dots, \widehat{B}_r$  be the proper transforms on  $\widehat{X}$  of the divisors  $B_1, \dots, B_r$ , respectively. Let us also replace (if necessary) the resolution of singularities  $\pi: \widehat{X} \rightarrow X$  by a slightly better one such that the divisor

$$\sum_{i=1}^r \widehat{B}_i + \sum_{i=1}^m E_i$$

has simple normal crossing singularities. Such resolution of singularities exists [H64, K07], and it is often called a log resolution of the log pair  $(X, B_X)$ . Suppose, in addition, that the divisor  $B_X$  is a  $\mathbb{Q}$ -Cartier divisor. Then there are rational numbers  $d_1, \dots, d_m$  such that

$$K_{\widehat{X}} + \sum_{i=1}^r a_i \widehat{B}_i + \sum_{i=1}^m d_i E_i = \pi^*(K_X + B_X). \quad (2.3)$$

Using this, we define the log pull back of the pair  $(X, B_X)$  as follows:

$$\left( \widehat{X}, \sum_{i=1}^r a_i \widehat{B}_i + \sum_{i=1}^m d_i E_i \right)$$

This new log pair is often denoted as  $(\widehat{X}, B^{\widehat{X}})$ .

**Definition 3.** *We say that*

- $(X, B_X)$  has Kawamata log terminal singularities if each  $a_i < 1$  and each  $d_j < 1$ ,
- $(X, B_X)$  has log canonical singularities if each  $a_i \leq 1$  and each  $d_j \leq 1$ .

Both these definitions do not depend on the choice of the log resolution  $\pi: \widehat{X} \rightarrow X$ . Note that  $B^{\widehat{X}}$  is not always effective. Nevertheless, our definition still works in this case.

**Remark 2.1.** *Moreover, it is easy to check (using the definition) that  $(X, B_X)$  has log canonical singularities if and only if  $(\widehat{X}, B^{\widehat{X}})$  has log canonical singularities. Similarly, one can show that the log pair  $(X, B_X)$  has Kawamata log terminal singularities if and only if the log pair  $(\widehat{X}, B^{\widehat{X}})$  has Kawamata log terminal singularities.*

Let  $P$  be a point in  $X$ . Then we can localize our definitions of singularities at this point.

---

**Definition 4.** We say that the pair  $(X, B_X)$  has log canonical singularities at  $P$  if the following two conditions are satisfied:

- for every  $\widehat{B}_i$  in (2.3) such that  $P \in B_i$ , one has  $a_i \leq 1$ ,
- for every  $E_i$  in (2.3) such that  $P \in \pi(E_i)$ , one has  $d_i \leq 1$ .

Likewise, we say that the log pair  $(X, B_X)$  has Kawamata log terminal singularities at the point  $P$  if the following two conditions are satisfied:

- for every  $\widehat{B}_i$  in (2.3) such that  $P \in B_i$ , one has  $a_i < 1$ ,
- for every  $E_i$  in (2.3) such that  $P \in \pi(E_i)$ , one has  $d_i < 1$ .

**Lemma 2.2.** Suppose that  $X$  is smooth at  $P$ . Then the following assertions hold:

- (i) if  $\text{mult}_P(B_X) \leq 1$ , then  $(X, B_X)$  is log canonical at  $P$ ;
- (ii) if  $\text{mult}_P(B_X) < 1$ , then  $(X, B_X)$  is Kawamata log terminal at  $P$ ;
- (iii) if  $\text{mult}_P(B_X) > \dim(X)$ , then  $(X, B_X)$  is not log canonical at  $P$ ;
- (iv) if  $\text{mult}_P(B_X) \geq \dim(X)$ , then  $(X, B_X)$  is not Kawamata log terminal at  $P$ .

*Proof.* This is [K97, Lemma 8.10] and [CKS03, Exercise 6.18].  $\square$

To measure how far the log pair  $(X, B_X)$  is from being log canonical, we can use the following number, which is called log canonical threshold:

$$\text{lct}(X, B_X) = \sup \left\{ \lambda \in \mathbb{Q}_{>0} \mid (X, \lambda B_X) \text{ has log canonical singularities} \right\}.$$

We can localize it at point  $P \in X$  as follows:

$$\text{lct}_P(X, B_X) = \sup \left\{ \lambda \in \mathbb{Q}_{>0} \mid (X, \lambda B_X) \text{ has log canonical singularities at } P \right\}.$$

Similarly, if  $Z$  is an irreducible subvariety of the variety  $X$ , we let

$$\text{lct}_Z(X, B_X) = \sup \left\{ \lambda \in \mathbb{Q}_{>0} \mid (X, \lambda B_X) \text{ is log canonical at every point in } Z \right\}.$$

Now, let us denote by  $\text{Nklt}(X, B_X)$  the subset in  $X$  consisting of all points where the singularities of the pair  $(X, B_X)$  are not Kawamata log terminal. To be precise, let

$$\text{Nklt}(X, B_X) = \left( \bigcup_{a_i \geq 1} B_i \right) \cup \left( \bigcup_{d_i \geq 1} \pi(E_i) \right) \subsetneq X.$$

This locus has been introduced in [S93, Definition 3.14] as the locus of log canonical singularities of the log pair  $(X, B_X)$ . Because of this, it is often denoted by  $\text{LCS}(X, B_X)$ . Observe that  $\text{Nklt}(X, B_X) = \emptyset \iff (X, B_X)$  has Kawamata log terminal singularities.

Let  $X$  be a normal projective variety that has Kawamata log terminal singularities, and let  $B_X$  be an effective  $\mathbb{Q}$ -divisor on the variety  $X$  that is given by (2.2). The following result is commonly known as the inversion of adjunction.

---

**Theorem 2.3** ([KM98, Theorem 5.50]). *Suppose that  $a_1 = 1$ ,  $B_1$  is a Cartier divisor, and  $B_1$  has Kawamata log terminal singularities. The following assertions are equivalent:*

- $(X, B_X)$  is log canonical at every point of the divisor  $B_1$ ;
- the singularities of the log pair  $(B_1, \sum_{i=2}^r a_i B_i|_{B_1})$  are log canonical.

Let  $Z$  be a proper irreducible subvariety of the variety  $X$ . Following [K97, Definition 1.3], we say that  $Z$  is a center of log canonical singularities or a log canonical center of the log pair  $(X, B_X)$  if one of the following conditions is satisfied:

- $Z = B_i$  for  $\widehat{B}_i$  in (2.3) such that  $a_i \geq 1$ ,
- $Z = \pi(E_i)$  for some  $E_i$  in (2.3) such that  $d_i \geq 1$ ,

for *some choice* of the log resolution  $\pi: \widehat{X} \rightarrow X$ . If  $Z$  is a log canonical center of the log pair  $(X, B_X)$ , then  $Z \subseteq \text{Nklt}(X, B_X)$ .

**Lemma 2.4** ([K97, Proposition 1.5]). *Let  $Z'$  be a proper irreducible subvariety in  $X$ . Suppose that  $Z$  and  $Z'$  are centers of log canonical singularities of the log pair  $(X, B_X)$ . Then every irreducible component of the intersection  $Z \cap Z'$  is a center of log canonical singularities of the log pair  $(X, B_X)$ .*

If  $Z$  is a log canonical center of the log pair  $(X, B_X)$ , we say that it is a minimal log canonical center if  $Z$  does not contain a proper irreducible subvariety that is also a center of log canonical singularities of the log pair  $(X, B_X)$ .

**Theorem 2.5** ([K98, Theorem 1]). *Suppose that  $Z$  is a minimal center of log canonical singularities of the log pair  $(X, B_X)$ . Then  $Z$  is normal and has rational singularities. Let  $H$  be an ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$ . Then  $(K_X + B_X + H)|_Z \sim_{\mathbb{Q}} K_Z + B_Z$  for an effective  $\mathbb{Q}$ -divisor  $B_Z$  on  $Z$  such that  $(Z, B_Z)$  has Kawamata log terminal singularities.*

This result is Kawamata's subadjunction theorem or Kawamata's subadjunction.

**Corollary 2.6.** *Suppose that  $-K_X$  is ample,  $B_X \sim \lambda(-K_X)$  for a rational number  $\lambda$ , and  $Z$  is a minimal log canonical center of  $(X, B_X)$ , and  $Z$  is a curve. Then  $Z$  is smooth. Moreover, if  $\lambda < 1$ , then  $-K_X \cdot Z \leq \frac{2}{1-\lambda}$  and  $Z$  is rational. If  $\lambda > 1$ , then  $-K_X \cdot Z \geq \frac{2g-2}{\lambda-1}$ , where  $g$  is the genus of the curve  $Z$ .*

*Proof.* By Theorem 2.5, the curve  $Z$  is smooth. Let  $g$  be its genus. Choose small rational number  $\epsilon > 0$ . Set  $H = \epsilon(-K_X)$ . Then  $(\lambda - 1 + \epsilon)(-K_X \cdot Z) = (K_X + B_X + H) \cdot Z \geq 2g - 2$  by Theorem 2.5. Since  $\epsilon$  can be arbitrary small, we get  $(\lambda - 1)(-K_X \cdot Z) \geq 2g - 2$ , which implies all required assertions.  $\square$

## Results for surfaces

Let  $X$  be a surface that is smooth at  $P$  and  $B_X$  be an effective divisor on  $X$ . As a consequence of Lemma 2.2, we have

**Corollary 2.7.** *Suppose  $(X, B_X)$  is not log canonical at some point  $P \in B_1$ , and the curve  $B_1$  is smooth at this point. If  $a_1 \leq 1$ , then*

$$\left( \sum_{i=2}^r a_i B_i \right) \cdot B_1 \geq \text{mult}_P \left( \left( \sum_{i=2}^r a_i B_i \right) \cdot B_1 \right) > 1.$$

The following result is very useful in the Sections that follow.

**Lemma 2.8.** *Suppose  $(X, B_X)$  is not log canonical at  $P$ . Let  $\pi : \tilde{X} \rightarrow X$  be the blow up at the point  $P$  with  $E_1$  as the exceptional divisor of the blow up. Suppose  $\text{mult}_P(B_X) \leq 2$ . Then there exists a unique point in  $E_1$  such that  $(\tilde{X}, B^{\tilde{X}})$  is not log canonical at it. Similarly, if  $(X, B_X)$  is not Kawamata log terminal at  $P$ , and  $\text{mult}_P(B_X) < 2$ , then there exists a unique point in  $E_1$  such that  $(\tilde{X}, B^{\tilde{X}})$  is not Kawamata log terminal at it.*

*Proof.* Suppose  $(X, B_X)$  is not log canonical at  $P$  and  $\text{mult}_P(B_X) \leq 2$  and suppose there exist 2 distinct points  $P_1$  and  $P_2$  in  $E_1$  at which  $(\tilde{X}, B^{\tilde{X}})$  is not log canonical. Then,

$$2 \geq \text{mult}_P(B_X) = B^{\tilde{X}} \cdot E_1 \geq \text{mult}_{P_1}(B^{\tilde{X}} \cdot E_1) + \text{mult}_{P_2}(B^{\tilde{X}} \cdot E_1) > 2$$

by Corollary 2.7. Thus, Remark 2.1 proves the first assertion. Similarly we can prove the second assertion.  $\square$

**Theorem 2.9.** *([C14, Theorem 13]) Suppose  $X$  is as described above. Let  $\Delta_1$  and  $\Delta_2$  be two irreducible curves on  $X$  that are both smooth at  $P$  and intersect transversally at  $P$  with  $a_1$  and  $a_2$ , rational numbers. Let  $\Omega$  be an effective  $\mathbb{Q}$ -divisor on the surface  $S$  whose support does not contain the curves  $\Delta_1$  and  $\Delta_2$ . Thus, the boundary  $B_X = a_1 \Delta_1 + a_2 \Delta_2 + \Omega$ . Suppose that the log pair  $(X, B_X)$  is not log canonical at  $P$ . Put  $m = \text{mult}_P(\Omega)$ . Suppose that  $m \leq 1$ . Then  $\text{mult}_P(\Omega \cdot \Delta_1) > 2(1 - a_2)$  or  $\text{mult}_P(\Omega \cdot \Delta_2) > 2(1 - a_1)$ .*

Suppose that  $X$  is a surface with cyclic quotient singularity at point  $P$  of type  $\frac{1}{n}(\mathbf{a}, \mathbf{b})$  where  $\mathbf{a}$  and  $\mathbf{b}$  are coprime integers that are also coprime to  $n$ . Then there is an orbifold chart  $\pi : \tilde{U} \rightarrow U$  for some open set  $P \in U$  on  $X$  such that  $\tilde{U}$  is smooth and  $\pi$  is a cyclic cover of degree  $n$  branched over  $P$ . Let  $\tilde{P} \in \tilde{U}$  be a point such that  $\pi(\tilde{P}) = P$ . Also let  $B_{X,U} = B_X|_U$  and  $B_{X,\tilde{U}} = \pi^{-1}(B_{X,U})$ .

**Lemma 2.10** ([K97]). *The log pair  $(U, B_{X,U})$  is log canonical at the point  $P$  if and only if the log pair  $(\tilde{U}, B_{X,\tilde{U}})$  is log canonical at the point  $\tilde{P}$ .*

Let  $Z$  be a  $\mathbb{Q}$ -Cartier divisor of  $X$ . If any component of  $Z$  is not contained in the support of  $B_X$  then we write

$$\text{mult}_P(B_X \cdot Z) = \text{mult}_{\tilde{P}}(B_{X,\tilde{U}} \cdot Z_{\tilde{U}})$$

where  $Z_U = Z|_U$  and  $Z_{\tilde{U}} = \pi^{-1}(Z_U)$ .

**Lemma 2.11.** *Suppose that no component of  $Z$  is contained in the support of  $B_X$ . Then the inequality*

$$Z \cdot B_X \geq \frac{\text{mult}_P(Z) \text{mult}_P(D_X)}{n}$$

*holds.*

*Proof.* It immediately follows from  $Z \cdot B_X = \sum_{Q \in X} \frac{\text{mult}_Q(Z \cdot B_X)}{n_Q}$ , where the singular type of  $Q$  is  $\frac{1}{n_Q}(a_Q, b_Q)$ .  $\square$

Let  $C$  be a reduced irreducible curve on the surface  $X$ . Suppose the point  $P$  is a smooth point of the curve  $C$  and the boundary of  $X$  is given by  $B_X = aC + \Delta$ , where  $a \in \mathbb{Q}$  such that  $a \geq 0$  and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C \not\subset \text{Supp}(\Delta)$ .

**Lemma 2.12.** *Suppose that  $a \leq 1$  and suppose that  $(X, B_X)$  is not log canonical at  $P$ . Then we have*

$$\frac{1}{n} < C \cdot \Delta.$$

*Proof.* See the proof of [CPS10, Lemma 2.5].  $\square$

## 2.2 K-stability

Let  $X$  be a Fano variety of dimension  $n \geq 2$  that has Kawamata log terminal singularities and set  $L = -K_X$ . A (normal) *test configuration* of the (polarized) pair  $(X; L)$  consists of

- a normal variety  $\mathcal{X}$  with a  $\mathbb{G}_m$  action,
- a flat  $\mathbb{G}_m$ -equivariant morphism  $p: \mathcal{X} \rightarrow \mathbb{P}^1$ , where  $\mathbb{G}_m$  acts naturally on  $\mathbb{P}^1$  by

$$(t, [x : y]) \mapsto [tx : y],$$

- a  $\mathbb{G}_m$ -invariant  $p$ -ample  $\mathbb{Q}$ -line bundle  $\mathcal{L} \rightarrow \mathcal{X}$  and a  $\mathbb{G}_m$ -equivariant isomorphism

$$\left( \mathcal{X} \setminus p^{-1}(0), \mathcal{L}|_{\mathcal{X} \setminus p^{-1}(0)} \right) \cong \left( X \times (\mathbb{P}^1 \setminus \{0\}), \text{pr}_1^*(L) \right).$$

where  $\text{pr}_1$  is the projection to the first factor, and  $0 = [0 : 1]$ .

For such test configuration, we let

$$\text{DF}(\mathcal{X}; \mathcal{L}) = \frac{1}{L^n} \left( \mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1} + \frac{n}{n+1} \mathcal{L}^{n+1} \right). \quad (2.4)$$

This number is called *Donaldson–Futaki invariant* of the test configuration  $(\mathcal{X}, \mathcal{L})$ .

Denote the central fibre  $p^{-1}(0)$  by  $\mathcal{X}_0$ , and denote the fibre at infinity  $p^{-1}(\infty)$  by  $\mathcal{X}_\infty$ , where  $\infty = [1 : 0]$ . The test configuration  $(\mathcal{X}, \mathcal{L})$  is said to be

- *trivial* if there is a  $\mathbb{G}_m$ -equivariant isomorphism

$$\left(\mathcal{X} \setminus \mathcal{X}_\infty, \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_\infty}\right) \cong \left(X \times (\mathbb{P}^1 \setminus \infty), \text{pr}_1^*(L)\right),$$

- *product-type* if we have an isomorphism  $\mathcal{X} \setminus \mathcal{X}_\infty \cong X \times (\mathbb{P}^1 \setminus \infty)$ ,
- *special* if the fiber  $\mathcal{X}_0$  is irreducible, reduced, and  $(\mathcal{X}, \mathcal{X}_0)$  has purely log terminal singularities, so that  $\mathcal{X}_0$  is a Fano variety with Kawamata log terminal singularities.

**Definition 5.** *The Fano variety  $X$  is said to be  $K$ -semistable if for every test configuration  $(\mathcal{X}, \mathcal{L})$  one has  $\text{DF}(\mathcal{X}; \mathcal{L}) \geq 0$ . Similarly, the Fano variety  $X$  is said to be  $K$ -stable if for every non-trivial test configuration  $(\mathcal{X}, \mathcal{L})$  one has  $\text{DF}(\mathcal{X}; \mathcal{L}) > 0$ . Finally, the Fano variety  $X$  is said to be  $K$ -polystable if it is  $K$ -semistable and*

$$\text{DF}(\mathcal{X}; \mathcal{L}) = 0 \iff (\mathcal{X}, \mathcal{L}) \text{ is of the product type.}$$

Thus, we have the following implications:

$$X \text{ is } K\text{-stable} \implies X \text{ is } K\text{-polystable} \implies X \text{ is } K\text{-semistable.}$$

If  $X$  is not  $K$ -semistable, we say that  $X$  is  $K$ -unstable. Similarly, if  $X$  is  $K$ -semistable, but the Fano variety  $X$  is not  $K$ -polystable, we say that  $X$  is strictly  $K$ -semistable.

The following results come in very handy, when determining the  $K$ -stability of a Fano variety  $X$ .

**Theorem 2.13** ([ABLX19, M57]). *If  $X$  is  $K$ -polystable, then  $\text{Aut}(X)$  is reductive.*

**Theorem 2.14** ([BX19, Corollary 1.3]). *If  $X$  is  $K$ -stable, then  $\text{Aut}(X)$  is finite.*

**Corollary 2.15.** *If  $\text{Aut}(X)$  is finite, then  $X$  is  $K$ -stable if and only if it is  $K$ -polystable.*

A longstanding conjecture is that a complex Fano variety with a compact group  $G \subset \text{Aut}(X)$  acting on it is  $K$ -semistable if and only if it is equivariantly  $K$ -semistable. This was initially confirmed in the case of smooth Fano varieties with the additional condition that the group is reductive ([DS16]), but has been later proven for non-reductive group actions too ([Z21]).

On the other hand, the  $K$ -polytability of a Fano variety  $X$  is implied by the  $G$ -equivariant  $K$ -polstability of  $X$ , only when  $G$  is reductive.

So let  $G$  be a reductive subgroup in  $\text{Aut}(X)$ . A given test configuration  $(\mathcal{X}, \mathcal{L})$  is said to be  $G$ -equivariant if the product  $G \times \mathbb{G}_m$  acts on  $(\mathcal{X}, \mathcal{L})$  such that

- $\{1\} \times \mathbb{G}_m$  acting on  $(\mathcal{X}, \mathcal{L})$  is the original  $\mathbb{G}_m$ -action,
- the  $\mathbb{G}_m$ -equivariant isomorphism

$$\left(\mathcal{X} \setminus p^{-1}(0), \mathcal{L}|_{\mathcal{X} \setminus p^{-1}(0)}\right) \cong \left(X \times (\mathbb{P}^1 \setminus \{0\}), \text{pr}_1^*(L)\right).$$

is  $G \times \mathbb{G}_m$ -equivariant.

**Definition 6.** *The Fano variety  $X$  is said to be  $G$ -equivariantly  $K$ -polystable if for every  $G$ -equivariant test configuration  $(\mathcal{X}, \mathcal{L})$  one has  $\text{DF}(\mathcal{X}; \mathcal{L}) \geq 0$ , and  $\text{DF}(\mathcal{X}; \mathcal{L}) = 0$  if and only if  $(\mathcal{X}, \mathcal{L})$  is of the product type.*

**Remark 2.16.** *It has been proved in [LX14, G19] that it is enough to consider only special test configurations in Definitions 5 and 6.*

If  $X$  is  $K$ -polystable, then  $X$  is  $G$ -equivariantly  $K$ -polystable. Surprisingly, we have

**Theorem 2.17** ([DS16, LWX, LZ20, Z21]). *Suppose that  $X$  is  $G$ -equivariantly  $K$ -polystable. Then  $X$  is  $K$ -polystable.*

The notions of  $K$ -stability of Fano variety  $X$  can also be defined in terms of various invariants that are now introduced in Section 2.2.1, Section 2.2.2 and Section 2.2.3.

## 2.2.1 Valuative criterion

Let  $X$  be a Fano variety with Kawamata log terminal singularities, let  $G$  be a reductive subgroup in  $\text{Aut}(X)$ , let  $f: \tilde{X} \rightarrow X$  be a  $G$ -equivariant birational morphism, let  $E$  be a  $G$ -invariant prime divisor in  $\tilde{X}$ , and let  $n = \dim(X)$ .

**Definition 7.** *We say that  $E$  is a  $G$ -invariant prime divisor over the Fano variety  $X$ . If  $E$  is  $f$ -exceptional, we say that  $E$  is an exceptional  $G$ -invariant prime divisor over  $X$ . We will denote the subvariety  $f(E)$  by  $C_X(E)$ . We say that  $E$  is dreamy if the algebra*

$$\bigoplus_{m, j \in \mathbb{Z}_{\geq 0}} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(f^*(-mK_X) - jE))$$

is a finitely generated  $\mathbb{C}$ -algebra.

Let

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\tau \text{vol}(f^*(-K_X) - xE) dx,$$

where  $\tau = \tau(E)$  is the pseudo-effective threshold of  $E$  with respect to  $-K_X$ , i.e. we have

$$\tau(E) = \sup \left\{ x \in \mathbb{Q}_{>0} \mid f^*(-K_X) - xE \text{ is big} \right\}.$$

In some cases, it is not easy to compute  $S_X(E)$ , but one can estimate it using basic properties of the volumes. That is, for  $a \in (0, \tau(E))$  we have

$$\int_0^{\tau(E)} \text{vol}(f^*(-K_X) - xE) dx \leq \int_0^a \text{vol}(f^*(-K_X) - xE) dx + (\tau(E) - a) \text{vol}(f^*(-K_X) - aE)$$

because  $\text{vol}(f^*(-K_X) - xE)$  is a decreasing function on  $x$ . This observation is very handy, since the volume function  $\text{vol}(f^*(-K_X) - xE)$  is often difficult to compute for large  $x \in (0, \tau(E))$ . Using log concavity of the volumes and

the restricted volumes [LM09, ELMNP09], we can improve the latter inequality. Namely, arguing as in the proof of [Fu19:2, Proposition 2.1], we get

$$\int_0^{\tau(E)} \text{vol}(f^*(-K_X) - xE) dx \leq \int_0^a \text{vol}(f^*(-K_X) - xE) dx \quad (2.5)$$

$$+ \frac{n}{n+1} (\tau(E) - a) \text{vol}(f^*(-K_X) - aE). \quad (2.6)$$

Let  $\beta(E) = A_X(E) - S_X(E)$ , where  $A_X(E)$  is the log discrepancy of the divisor  $E$ , defined as  $A_X(E) = 1 + \text{ord}_E(K_{\bar{X}} - f^*(K_X))$ .

**Theorem 2.18** ([Fu19:1, L17, BX19]). *The following assertions hold:*

- $X$  is  $K$ -stable  $\iff \beta(F) > 0$  for every prime divisor  $F$  over  $X$ ;
- $X$  is  $K$ -semistable  $\iff \beta(F) \geq 0$  for every prime divisor  $F$  over  $X$ .

This criterion leads to the notion of *divisorial stability*, which is weaker than  $K$ -stability, since it depends only on the value of  $\beta(E)$  for all prime divisors  $E$  in  $X$ . That is,

**Definition 8** ([F16, Definition 1.1]). *The Fano variety  $X$  is said to be divisorially stable (respectively, semistable) if  $\beta(F) > 0$  (respectively,  $\beta(F) \geq 0$ ) for every prime divisor  $F$  in  $X$ . We say that  $X$  is divisorially unstable if it is not divisorially semistable.*

To prove  $K$ -polystability, we can use the following handy criterion:

**Theorem 2.19** ([Z21, Corollary 4.14]). *Suppose that  $\beta(F) > 0$  for every  $G$ -invariant dreamy prime divisor  $F$  over  $X$ . Then  $X$  is  $K$ -polystable.*

## 2.2.2 Tian's criterion

Let  $X$  be a Fano variety with at most Kawamata log terminal singularities of dimension  $n \geq 2$ . Let  $G$  be a reductive subgroup in  $\text{Aut}(X)$ . Then

$$\alpha_G(X) = \sup \left\{ \epsilon \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } \left( X, \frac{\epsilon}{m} \mathcal{D} \right) \text{ is log canonical for any } m \in \mathbb{Z}_{>0} \\ \text{and every } G\text{-invariant linear system } \mathcal{D} \subset |-mK_X| \end{array} \right. \right\}.$$

This number, also known as the global log canonical threshold (see [CS11, Definition 3.1]) was defined by Tian ([T87],[T08, Appendix 2]). By [CS08, Theorem A.3], both definitions coincide.

If the group  $G$  is a trivial group, then  $\alpha(X) = \alpha_G(X)$  and so

$$\alpha(X) = \sup \left\{ \epsilon \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \epsilon D) \text{ is log canonical for any effective} \\ \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right. \right\}.$$

This can also be extended to give the definition of  $\alpha(X, L)$  for any ample divisor  $L$  on  $X$ .

$$\alpha(X, L) = \sup \left\{ \epsilon \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \epsilon D) \text{ is log canonical for any effective} \\ \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} L \end{array} \right. \right\}.$$

**Lemma 2.20.** *Suppose that  $G = \mathbb{G}_m^r \rtimes B$  for some finite group  $B$ . Then*

$$\alpha_G(X) = \sup \left\{ \epsilon \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \epsilon D) \text{ is log canonical for every} \\ G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right. \right\}.$$

*Proof.* See [ACCFKGSSV, Lemma 1.4.1].  $\square$

Observe that the invariant  $\alpha(X)$  has a global nature. It measures the singularities of effective  $\mathbb{Q}$ -divisors on  $X$  that are  $\mathbb{Q}$ -linearly equivalent to the anti-canonical divisor  $-K_X$ . We can also localize  $\alpha(X)$  as follows. Let  $Z$  be a proper irreducible subvariety in  $X$ . Let

$$\alpha_Z(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical at general point} \\ \text{of } Z \text{ for every effective } \mathbb{Q}\text{-divisor } D \text{ on } X \\ \text{such that } D \sim_{\mathbb{Q}} -K_X \end{array} \right. \right\}.$$

Clearly, we have

$$\alpha(X) = \inf_{P \in X} \alpha_P(X),$$

where the infimum is taken by all (closed) points in  $X$ . If the subvariety  $Z$  is  $G$ -invariant, we can also define the number  $\alpha_{G,Z}(X)$  as follows:

$$\alpha_{G,Z}(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is log canonical at general point} \\ \text{of } Z \text{ for any effective } G\text{-invariant } \mathbb{Q}\text{-divisor } D \text{ on } X \\ \text{such that } D \sim_{\mathbb{Q}} -K_X \end{array} \right. \right\}.$$

Then  $\alpha_G(X) \leq \alpha_{G,Z}(X)$ .

**Remark 2.21** ([F21, Lemma 2.5]). *Let  $f: \tilde{X} \rightarrow X$  be an arbitrary  $G$ -equivariant birational morphism, let  $F$  be a  $G$ -invariant prime divisor in  $\tilde{X}$  such that  $Z \subseteq f(F)$ , and let*

$$\tau(F) = \sup \left\{ x \in \mathbb{Q}_{>0} \mid f^*(-K_X) - xF \text{ is big} \right\}.$$

*Then  $\frac{A_X(F)}{\tau(F)} \geq \alpha_{G,Z}(X)$ .*

**Lemma 2.22.** *With the notations and assumptions of Remark 2.21, suppose in addition that  $X$  is smooth and  $\dim(Z) \geq 1$ . Then*

$$\frac{A_X(F)}{S_X(F)} > \frac{n+1}{n} \alpha_{G,Z}(X).$$

The  $\alpha$ -invariants are important because of the following results, that help one determine the K-stability of Fano variety  $X$ :

**Theorem 2.23** ([DS16, LZ20, Z21, T87]). *The Fano variety  $X$  is K-semistable if*

$$\alpha_G(X) \geq \frac{n}{n+1}$$

*Moreover, if  $\alpha_G(X) > \frac{n}{n+1}$ , then  $X$  is K-polystable.*

**Theorem 2.24.** *If  $X$  is smooth and  $\alpha_G(X) \geq \frac{n}{n+1}$ , then  $X$  is K-polystable.*

### 2.2.3 Stability Threshold

Let  $X$  be a normal projective variety of dimension  $n$ , let  $\Delta$  be an effective  $\mathbb{Q}$ -divisor on it such that the log pair  $(X, \Delta)$  has at most Kawamata log terminal singularities, and let  $L$  be an ample  $\mathbb{Q}$ -divisor on  $X$ . Let  $f: Y \rightarrow X$  be a projective birational morphism with normal variety  $Y$ , and let  $E$  be a (not necessarily  $f$ -exceptional) prime divisor in  $Y$ . Then  $E$  is a divisor over  $X$ . Let

$$A_{X,\Delta}(E) = 1 + \text{ord}_E(K_Y - f^*(K_X + \Delta)),$$

and we let

$$S_L(E) = \frac{1}{L^n} \int_0^\infty \text{vol}(L - xE) dx.$$

If  $(X, \Delta)$  is a log Fano variety and  $L = -(K_X + \Delta)$ , we set  $S_{X,\Delta}(E) = S_L(E)$  for simplicity. Note that this (infinite) integral is actually finite, since  $\text{vol}(L - xE) = 0$  for  $x > \tau_L(E)$ , where  $\tau_L(E)$  is the pseudo-effective threshold:

$$\tau_L(E) = \sup \left\{ \lambda \in \mathbb{R}_{>0} \mid \text{vol}(L - \lambda E) > 0 \right\}.$$

Then the  $\delta$ -invariant is defined as

$$\delta(X, \Delta; L) = \inf_{E/X} \frac{A_{X,\Delta}(E)}{S_L(E)},$$

where the infimum is taken over all prime divisors over  $X$ . The number  $\delta(X, \Delta)$  is also known as *the stability threshold*, because of the following result (cf. Theorem 2.18).

**Theorem 2.25** ([FO18, Fu19:1, L17, B20, CP, LXZ21]). *If  $(X, \Delta)$  is a log Fano variety, then*

- $\delta(X, \Delta) > 1 \iff (X, \Delta)$  is  $K$ -stable;
- $\delta(X, \Delta) \geq 1 \iff (X, \Delta)$  is  $K$ -semistable.

We will now describe the local analogues of the Stability threshold, which along with the Abban-Zhuang Theory in Section 2.2.3.2 is extremely handy in computing  $\delta(X, \Delta)$ .

#### 2.2.3.1 Local Analogues of the Stability Threshold

We will now define the local analogues of the  $\delta$ -invariant as follows,

$$\delta_P(X, \Delta; L) = \inf_{\substack{E/X \\ P \in C_X(E)}} \frac{A_{X,\Delta}(E)}{S_L(E)}.$$

We can also give an alternative definition to the  $\delta$ -invariant using log canonical thresholds. In order to do this, we need the following definition.

**Definition 9.** For effective  $\mathbb{Q}$ -divisor  $D$  such that  $D \sim_{\mathbb{Q}} L$ , we say that  $D$  is cool if the inequality  $\text{ord}_E(D) \leq S_L(E)$  holds for every prime Weil divisor  $E$  over  $X$ .

Using this, we can now formulate an alternative definition of the  $\delta$ -invariant.

**Proposition 2.26.** Let  $P$  be a point in  $X$ . Then

$$\delta_P(X, \Delta; L) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \Delta + \lambda D) \text{ is log canonical at } P \\ \text{for any effective cool } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right. \right\}.$$

Then, we have

**Corollary 2.27.**

$$\delta(X, \Delta; L) = \inf \left\{ \text{lct}(X, \Delta; D) \mid D \text{ is cool effective } \mathbb{Q}\text{-divisor such that } D \sim_{\mathbb{Q}} L \right\}.$$

Note that this notion has been mentioned previously in [CPS21], under the assumption that the divisor  $D$  is a  $k$ -basis type divisor (see [FO18] for definition).

Recently, there has been a lot of progress in understanding the applications of  $\delta$ -invariant. In this regard, we will briefly describe the theory developed by Abban (Ahmadinezhad) and Zhuang in [AZ20].

### 2.2.3.2 General Abban-Zhuang Theory

Let  $X$  be a normal variety of dimension  $n$  that has at most Kawamata log terminal singularities, let  $Z \subseteq X$  be an irreducible subvariety, let  $L$  be some big line bundle on  $X$ , and let  $M(L)$  be the set consisting of all positive integers  $m$  such that  $h^0(X, \mathcal{O}_X(mL)) \neq 0$ . The  $\delta$ -invariant  $\delta_Z(X; L)$  along  $Z$  is defined by

$$\delta_Z(X; L) = \inf_{\substack{E/X \\ Z \subseteq C_X(E)}} \frac{A_X(E)}{S_L(E)},$$

where the infimum runs over all prime divisors  $E$  over the variety  $X$  such that  $Z \subseteq C_X(E)$ . In the case when  $X$  is a Fano variety and  $L = -K_X$ , we let

$$\delta_Z(X) = \delta_Z(X; L).$$

In this section, we explain how to estimate  $\delta_Z(X; L)$  using the technique developed in [AZ20].

Let  $Y$  be a prime Cartier divisor in  $X$  such that  $Z \subset Y$ , and  $Y$  is not contained in the supports of the negative part of the  $\sigma$ -decomposition of  $L$ , see [N04, Definition III.1.12]. The latter condition always holds if  $L$  is nef. Then [AZ20, Theorem 3.3] implies the following

**Theorem 2.28.** Let  $\delta_Z(Y; W_{\bullet, \bullet}^Y)$  be the number defined by

$$\delta_Z(Y; W_{\bullet, \bullet}^Y) = \inf_{\substack{F/Y \\ Z \subseteq C_Y(F)}} \frac{A_Y(F)}{S(W_{\bullet, \bullet}^Y; F)}, \quad (2.7)$$

where the infimum is taken over prime divisors  $F$  over the variety  $Y$  with  $Z \subseteq C_Y(F)$ . Then

$$\delta_Z(X, L) \geq \min \left\{ \frac{1}{S_L(Y)}, \delta_Z(Y; W_{\bullet, \bullet}^Y) \right\}.$$

We will now define the number  $S(W_{\bullet, \bullet}^Y; F)$  and elaborate this computation in the case of surfaces and 3-folds. For a more general description of the theory, refer to [AZ20] and [ACCFKGSSV, Section 1.7].

### 2.2.3.3 Abban-Zhuang Theory for surfaces

Let  $S$  be a del Pezzo surface that has at most Kawamata log terminal singularities, let  $\mathfrak{p} \in S$  be a point. In this section, we will explicitly describe how to estimate  $\delta_{\mathfrak{p}}(S)$ , using the technique developed in [AZ20].

Let  $Y$  be an irreducible curve in  $S$  such that  $\mathfrak{p} \in Y$ . Assume that  $Y$  is such that it is either Cartier on  $S$  or  $(S, Y)$  is purely log terminal. Then [AZ20, Theorem 3.3] implies the following

**Theorem 2.29.** [ACCFKGSSV, Theorem 1.7.1], [AZ20, Lemma 2.21], [AZ20, Corollary 2.22] *Let  $S(W_{\bullet, \bullet}^Y; \mathfrak{p})$  be the number defined in Theorem 2.31. Then*

$$\delta_{\mathfrak{p}}(S) \geq \min \left\{ \frac{1}{S_S(Y)}, \frac{A_Y(\mathfrak{p})}{S(W_{\bullet, \bullet}^Y; \mathfrak{p})} \right\}.$$

**Remark 2.30.** [KMM87, Lemma 5.1.9] [K91, Proposition 16.6], [S93, Proposition 3.9] *Recall that we have the following adjunction formula*

$$(K_S + Y)|_Y = K_Y + \Delta$$

where  $\Delta$  is called the different. Then,

$$A_Y(\mathfrak{p}) = 1 - \text{coeff}(\Delta|_{\mathfrak{p}}).$$

Since  $K_S + Y$  is purely log terminal at  $\mathfrak{p}$ ,

- $\text{coeff}(\Delta|_{\mathfrak{p}}) = 0$  when  $\mathfrak{p}$  is a smooth point of  $S$ ,
- $\text{coeff}(\Delta|_{\mathfrak{p}}) = \frac{m-1}{m}$  when  $\mathfrak{p}$  is a singular point of  $S$  and is of type  $\mathbb{C}^2 \setminus \mathbb{Z}_m$ , for  $m \neq 1$ .

The number  $S(W_{\bullet, \bullet}^Y; \mathfrak{p})$  is defined in detail in [AZ20]. The following assertion helps in computing it.

**Theorem 2.31.** [ACCFKGSSV, Theorem 1.7.13], [Fu, Theorem 3.16] *For every point  $\mathfrak{p}$  over  $Y$ , let*

$$h(u) = (P(u) \cdot Y) \cdot \text{ord}_{\mathfrak{p}}(N(u)|_Y) + \int_0^{\infty} \text{vol}(P(u)|_Y - v\mathfrak{p}) \, dv,$$

where  $P(u)$  and  $N(u)$  are the positive and negative parts respectively of the Zariski Decomposition of the divisor  $-K_S - uY$ . Then one has

$$S(W_{\bullet, \bullet}^Y; \mathfrak{p}) = \frac{2}{(-K_S)^2} \int_0^{\tau} h(u) \, du.$$

### 2.2.3.4 Abban-Zhuang theory for 3-folds

In the case of 3-folds, we have the following result, that makes the computations easy. Refer to [AZ20, Theorem 3.3] and [ACCFKGSSV, Corollary 1.7.25, Corollary 1.7.26].

**Corollary 2.32.** *Let  $X$  be a smooth Fano threefold, let  $Y$  be an irreducible normal surface in the threefold  $X$ , let  $Z$  be an irreducible curve in  $Y$ , and let  $E$  be a prime divisor over the threefold  $X$  such that  $C_X(E) = Z$ . Then*

$$\frac{A_X(E)}{S_X(E)} \geq \min \left\{ \frac{1}{S_X(Y)}, \frac{1}{S(W_{\bullet, \bullet}^Y; Z)} \right\} \quad (2.8)$$

and

$$\begin{aligned} S(W_{\bullet, \bullet}^Y; Z) &= \frac{3}{(-K_X)^3} \int_0^\tau (P(u)^2 \cdot Y) \cdot \text{ord}_Z(N(u)|_Y) du \\ &\quad + \frac{3}{(-K_X)^3} \int_0^\tau \int_0^\infty \text{vol}(P(u)|_Y - vZ) dv du, \end{aligned}$$

where  $P(u)$  is the positive part of the Zariski decomposition of the divisor  $-K_X - uY$ , and  $N(u)$  is its negative part. Moreover, if the equality holds in (2.8), then

$$\frac{A_X(E)}{S_X(E)} = \frac{1}{S_X(Y)}.$$

Next, we will look at the various notions that we require to study Fano varieties in higher dimensions, as in Chapter 6.

## 2.3 Chern classes and Chern characters

Determining the intersection between subvarieties of a given variety, is vital in completely understanding the variety. The basic step towards this is Bezout's theorem which states that two curves of degree  $d$  and  $d'$  intersect in  $d \cdot d'$  points, when counted with multiplicities. In this regard, one defines a ring  $A(X)$ , called the Chow ring of a variety  $X$ , in which the intersection product between elements of the Chow ring, generalises Bezout's theorem to higher dimensions. (See [EH16] for detailed descriptions.)

Let  $X$  be a smooth projective variety of dimension  $n$ . The group of cycles on  $X$ , given by  $Z(X)$ , is a free abelian group generated by the set of subvarieties of  $X$ . That is,  $Z(X)$  is graded by dimension  $k$  of the subvariety, where every component of the group, denoted by  $Z_k(X)$ , is given by a formal linear combination of subvarieties of dimension  $k$  (these are called  $k$ -cycles) and thus,  $Z(X) = \bigoplus_k Z_k(X)$ .

**Definition 10.** [EH16, Definition 1.3] *Let  $Z(X)$  be defined as above. Two cycles are rationally equivalent if there is a rationally parametrized family of cycles interpolating between them. The Chow group of  $X$  is the quotient*

$$A(X) = Z(X)/\text{Rat}(X),$$

which is the group of rational equivalence classes of cycles on  $X$ .

The Chow group when given a ring structure (See [EH16, Definition 1.5] for details) is called the Chow ring of  $X$ , denoted by  $A^*(X)$ . The Chow ring is graded by codimension  $c$ , that is

$$A^*(X) = \bigoplus_{c=0}^{\dim(X)} A^c(X),$$

where  $A^c(X)$  is the group  $A_{\dim(X)-c}$  where  $A_k$  is the group of rational equivalence classes of  $k$ -cycles.

The most interesting classes in a Chow group are those of vector bundles, for which one can define what are called Chern classes.

Recall that the first Chern class of a line bundle  $\mathcal{L}$  is defined as follows:

**Definition 11.** [EH16, Section 1.4] *Let  $\mathcal{L}$  be any line bundle on  $X$ . Then the first Chern class*

$$c_1(\mathcal{L}) \in A_{n-1}(X)$$

*is the rational equivalence class of the divisor  $\sigma$ , for any nonzero rational section  $\sigma$  of  $\mathcal{L}$ .*

We then define the Chern classes of any rank  $r$  vector bundle  $\varepsilon$ .

**Definition 12.** [EH16, Section 5.2] *Let  $\varepsilon$  be a rank  $r$  vector bundle, with global sections  $\tau_0, \dots, \tau_{r-1}$ . Consider the bundle  $\bigwedge^r \varepsilon$ , which is of rank 1. Then*

$$c_1(\varepsilon) = c_1\left(\bigwedge^r \varepsilon\right) \in A_{n-1}(X),$$

*is the class of the vanishing locus of the section  $\tau_0 \wedge \dots \wedge \tau_{r-1} \in \bigwedge^r \varepsilon$ . In general, for any  $i$ ,*

$$c_i(\varepsilon) = [D] \in A_{n-i}(X),$$

*where  $[D]$  is the class of the vanishing locus of  $\tau_0 \wedge \dots \wedge \tau_{r-i} \in \bigwedge^{r-i+1} \varepsilon$ .*

A certain power series in the Chern classes is called the Chern character. For a detailed description of Chern characters of vector bundles, refer to [EH16, Section 14.2].

The following is the list of  $k$ -th Chern characters,  $\text{ch}_k(X)$ , of varieties  $X$ , ([AC13, Section 3]), that we will be using in the computations in Sections that follow. Note that  $\text{ch}_k(X) = \text{ch}_k(\mathcal{T}_X)$  where  $\mathcal{T}_X$  is the tangent bundle of  $X$ .

- **Projective Space:** Set  $h := c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ . Then

$$\text{ch}(\mathbb{P}^n) = n + \sum_{k=1}^n \frac{n+1}{k!} h^k. \quad (2.9)$$

- **Weighted Projective Space:** Let  $\mathbb{P} = \mathbb{P}(a_0, a_1, \dots, a_n)$  be weighted projective space with  $\gcd(a_0, a_1, \dots, a_n) = 1$  and let  $H$  be the effective generator of the class group  $\text{Cl}(\mathbb{P}) \cong \mathbb{Z}$ . From the Euler sequence, on the smooth locus of  $\mathbb{P}$ , we have

$$\text{ch}(\mathbb{P}) = n + \sum_{k=1}^n \frac{a_0^k + \dots + a_n^k}{k!} c_1(H)^k. \quad (2.10)$$

- **Complete intersections:**

If  $Y$  is a smooth variety and  $X$  is a smooth complete intersection of divisors  $D_1, \dots, D_c$  in  $Y$ , then

$$\mathrm{ch}_k(X) = \left( \mathrm{ch}_k(Y) - \frac{1}{k!} \sum_{i=1}^c D_i^k \right)_{|X}. \quad (2.11)$$

In particular, we have the following:

- **Complete intersections in  $\mathbb{P}^n$ :** If  $X$  is a complete intersection of hypersurfaces of degree  $d_1, \dots, d_c$  in  $\mathbb{P}^n$ , then

$$\mathrm{ch}_k(X) = \frac{1}{k!} \left( (n+1) - \sum d_i^k \right) h_{|X}^k, \quad (2.12)$$

where  $h := \mathcal{O}_{\mathbb{P}^n}(1)$ .

- **Complete intersections in weighted projective spaces:** Let  $\mathbb{P} = \mathbb{P}(a_0, a_1, \dots, a_n)$  be weighted projective space with  $\mathrm{gcd}(a_0, a_1, \dots, a_n) = 1$  and let  $X$  be a smooth complete intersection of hypersurfaces with classes  $d_1H, \dots, d_cH$  in  $\mathbb{P}$ . Assume  $X$  is smooth, and contained in the smooth locus of  $\mathbb{P}$ . Then the Chern character of  $X$  is given by

$$\mathrm{ch}(X) = (n-c) + \sum_{k=1}^n \frac{a_0^k + \dots + a_n^k - \sum d_i^k}{k!} c_1(H_{|X})^k. \quad (2.13)$$

Let  $\sigma_{a_1, a_2, \dots, a_k}$  denote the usual Schubert cycles on  $Gr(k, n)$  (See Section 2.5 for the definition) and let  $\sigma_1$  be the class of the hyperplane via the Plucker embedding that generates  $\mathrm{Pic}(Gr(k, n))$ .

- **Grassmanians:** Grassmanian  $Gr(k, n)$  parametrize  $k$ -dimensional subspaces of an  $n$ -dimensional vector space  $V$ . Recall from [AC13, Section 3.4] the following formula for the Chern character of  $Gr(k, n)$ .

$$\begin{aligned} \mathrm{ch}(Gr(k, n)) &= k(n-k) + n\sigma_1 + \left( \frac{n+2-2k}{2} \sigma_2 - \frac{n-2-2k}{2} \sigma_{1,1} \right) \\ &\quad + \frac{n-2k}{6} (\sigma_3 - \sigma_{2,1} + \sigma_{1,1,1}) + \dots \end{aligned} \quad (2.14)$$

- **Orthogonal Grassmanians:** Let  $OG(k, n)$  be the subvariety of the Grassmanian  $Gr(k, n)$  parametrizing linear subspaces that are isotropic with respect to a nondegenerate symmetric bilinear form on an  $n$ -dimensional vector space  $V$ .

If  $n = 2k$ ,  $OG(k, 2k)$  has two connected components: denoted by  $OG_+(k, 2k)$  and  $OG_-(k, 2k)$ . The two components are disjoint and isomorphic.

For any component  $X$  of  $OG(k, n)$ , we have the following description of the Chern character of  $OG(k, n)$  [AC13, Section 6.2].

$$\begin{aligned} \text{ch}(X) &= \frac{k(2n - 3k - 1)}{2} + (n - k - 1)\sigma_1 \\ &+ \left( \frac{n - 3k - 1}{2}\sigma_2 - \frac{n - 3k - 3}{2}\sigma_{1,1} \right) \\ &+ \left( \frac{n - 3k - 7}{6}\sigma_3 - \frac{n - 3k - 4}{6}\sigma_{2,1} + \frac{n - 3k - 1}{6}\sigma_{1,1,1} \right) + \dots \end{aligned} \tag{2.15}$$

- **Symplectic Grassmanians:** Let  $SG(k, n)$  be the subvariety of the Grassmanian  $Gr(k, n)$  parametrizing linear subspaces that are isotropic with respect to a nondegenerate anti-symmetric bilinear form on an  $n$ -dimensional vector space  $V$ , with  $n$  even.

From [AC13, Section 6.3], we have the following description of the Chern character of  $SG(k, n)$ .

$$\begin{aligned} \text{ch}(SG(k, n)) &= \frac{k(2n - 3k + 1)}{2} + (n - k + 1)\sigma_1 \\ &+ \left( \frac{n - 3k + 3}{2}\sigma_2 - \frac{n - 3k + 1}{2}\sigma_{1,1} \right) \\ &+ \left( \frac{n - 3k + 1}{6}\sigma_3 - \frac{n - 3k + 4}{6}\sigma_{2,1} + \frac{n - 3k + 7}{6}\sigma_{1,1,1} \right) + \dots \end{aligned} \tag{2.16}$$

## 2.4 Polarised family of minimal rational curves

In this subsection, we will study the family of minimal rational curves on a smooth projective variety  $X$ .

Let  $X$  be a Fano manifold and  $x \in X$  be a general point. Then we can construct a scheme  $\text{RatCurves}^n(X, x)$ , which is a scheme of rational curves on  $X$  through  $x$ . (See [Kol96, II.2.11] for a detailed description of this construction.)

Let  $H_x$  be a proper irreducible component of  $\text{RatCurves}^n(X, x)$ . We can take it to be an irreducible component of  $\text{RatCurves}^n(X, x)$  parametrizing rational curves through  $x$  and having minimal degree with respect to  $-K_X$ . Using the universal properties of  $A^*(X)$ , we get the following universal family diagram:

$$\begin{array}{ccc} U_x & \xrightarrow{\text{ev}} & X, \\ \pi \downarrow & & \\ H_x & & \end{array}$$

where  $\pi$  is a  $\mathbb{P}^1$ -bundle.

The variety  $H_x$  comes with a natural finite morphism  $\tau_x : H_x \rightarrow \mathbb{P}(T_x X^*)$  sending a curve that is smooth at  $x$  to its tangent direction at  $x$  (see [Keb02,

Theorems 3.3 and 3.4]). The morphism  $\tau_x$  is also birational by [HM04]. Therefore,  $H_x$  has a natural polarization  $L_x := \tau_x^* \mathcal{O}(1)$ . The pair  $(H_x, L_x)$  is called a *polarized minimal family of rational curves through  $x$* .

In [AC12], Araujo and Castravet computed all the Chern characters of  $H_x$  in terms of the Chern characters of  $X$  and  $c_1(L_x)$ . In order to state the result, first we describe the relation between the Chow ring of  $X$ ,  $A^*(X)$  and  $H_x$ .

Recall the definition of the Chow ring as described in Section 2.3. Let  $N^k(X)$  be the quotient of  $A^k(X)$  by numerical equivalence, where  $A^k(X)$  is the  $k$ -th graded piece of the Chow ring  $A^*(X)$ . Then, for any  $k \geq 1$ , we have

$$T := \pi_* ev^* : N^k(X)_{\mathbb{R}} \rightarrow N^{k-1}(H_x)_{\mathbb{R}}.$$

By [AC12, Proposition 1.3], we have

$$\mathrm{ch}_k(H_x) = \sum_{j=0}^k A_j c_1(L_x)^j \cdot T(\mathrm{ch}_{k+1-j}(X)) - \frac{1}{k!} c_1(L_x)^k \quad (2.17)$$

where  $A_j = \frac{(-1)^j B_j}{j!}$  and the  $B_j$ 's are the Bernoulli numbers.

Set  $d := \dim(H_x)$  and  $\eta := T(\mathrm{ch}_2(X)) - \frac{L_x}{2}$ . Then, for  $1 \leq k \leq 3$ , (2.17) becomes:

$$c_1(H_x) = T(\mathrm{ch}_2(X)) + \frac{d}{2} c_1(L_x) \quad (2.18)$$

$$\mathrm{ch}_2(H_x) = T(\mathrm{ch}_3(X)) + \frac{1}{2} \left( c_1(H_x) - \frac{d}{2} c_1(L_x) \right) L_x + \frac{d-4}{12} L_x^2 \quad (2.19)$$

$$\begin{aligned} \mathrm{ch}_3(H_x) &= T(\mathrm{ch}_4(X)) + \frac{1}{2} T(\mathrm{ch}_3(X)) \cdot L_x + \frac{1}{12} T(\mathrm{ch}_2(X)) \cdot L_x^2 - \frac{1}{6} L_x^3 \\ &= T(\mathrm{ch}_4(X)) + \frac{1}{2} T(\mathrm{ch}_3(X)) \cdot L_x + \frac{L_x^2}{12} \left( \eta - \frac{3}{2} L_x \right), \end{aligned} \quad (2.20)$$

The following theorem provides few necessary conditions for  $X$ , in terms of  $H_x$  on  $X$  for  $x \in X$ , to satisfy  $\mathfrak{F}_2$ ,  $\mathfrak{F}_3$  and  $\mathfrak{F}_4$  conditions.

**Theorem 2.33.** *Let  $X$  be a Fano manifold, and  $(H_x, L_x)$  a polarized minimal family of rational curves through a general point  $x \in X$ .*

1. ([AC12, Theorem 1.4 (2)]) *If  $X$  satisfies  $\mathfrak{F}_2$  and  $d \geq 1$ , then  $H_x$  satisfies  $\mathfrak{F}_1$  (i.e., it is a Fano manifold).*
2. ([AC12, Theorem 1.4 (3)]) *If  $X$  satisfies  $\mathfrak{F}_3$  and  $d \geq 2$ , then  $H_x$  satisfies  $\mathfrak{F}_2$  and  $\rho(H_x) = 1$ .*
3. *If  $X$  satisfies  $\mathfrak{F}_4$ ,  $d \geq 3$  and  $\eta \geq \frac{3}{2} L_x$ , then  $H_x$  satisfies  $\mathfrak{F}_3$  and  $\rho(H_x) = 1$ .*

Here  $\rho(H_x)$  denotes the Picard rank of  $H_x$ .

*Proof of (3).* Suppose that the Fano manifold  $X$  satisfies  $\mathfrak{F}_4$ ,  $d \geq 3$  and  $\eta \geq \frac{3}{2} L_x$ . We know from (2) that  $H_x$  is a Fano manifold satisfying  $\mathfrak{F}_2$  and  $\rho(H_x) = 1$ . Using the fact that the map  $T : N^k(X)_{\mathbb{R}} \rightarrow N^{k-1}(H_x)_{\mathbb{R}}$  preserves positivity for  $k-1 \leq d$ , it follows from (2.20) that  $\mathrm{ch}_3(H_x) > 0$ , and thus  $H_x$  satisfies  $\mathfrak{F}_3$ .  $\square$

A more precise description of the polarised family of minimal rational curves  $(H_x, L_x)$  when  $X$  is a Fano manifold satisfying  $\mathfrak{F}_2$ , can be given with the help of the following theorem.

**Theorem 2.34** ([AC12, Theorem 1.4 (2)]). *Let  $X$  be a Fano manifold,  $(H_x, L_x)$  a polarized minimal family of rational curves through a general point  $x \in X$ , and set  $d = \dim H_x$ .*

1. *If  $X$  is 2-Fano, then  $H_x$  is a Fano manifold with  $\text{Pic}(H_x) = \mathbb{Z} \cdot [L_x]$ , except if  $(H_x, L_x)$  is isomorphic to one of the following*

- (a)  $(\mathbb{P}^m \times \mathbb{P}^m, p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1))$ , with  $d = 2m$ ,
- (b)  $(\mathbb{P}^{m+1} \times \mathbb{P}^m, p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1))$ , with  $d = 2m + 1$ ,
- (c)  $(\mathbb{P}_{\mathbb{P}^{m+1}}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m}), \mathcal{O}_{\mathbb{P}}(1))$ , with  $d = 2m + 1$ ,
- (d)  $(\mathbb{P}^m \times \mathbb{Q}^{m+1}, p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1))$ , with  $d = 2m + 1$
- (e)  $(\mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}}), \mathcal{O}_{\mathbb{P}}(1))$ , with  $d = 2m + 1$ .
- (f)  $(\mathbb{P}^d, \mathcal{O}(2))$ , or
- (g)  $(\mathbb{P}^1, \mathcal{O}(3))$ .

2. *Let  $b_i(X)$  be the  $i$ -th Betti number of  $X$ . Suppose  $b_4(X) = 1$ . Then  $X$  is 2-Fano if and only if  $-2K_{H_x} - dL_x$  is ample.*

## 2.5 Schubert Calculus

In this subsection, we recall the necessary background on Schubert calculus that we will need, in order to study Grassmanians [C20, EH16, Ful97].

Let  $Gr(k, n)$  denote the Grassmanian that parametrizes  $k$ -dimensional linear subspaces of a fixed  $n$ -dimensional vector space  $V$ . This can also be considered as the parameter space of  $k - 1$  dimensional projective linear spaces in  $\mathbb{P}^{n-1}$ . This is a  $k(n - k)$  dimensional complex manifold. The cohomology ring of  $Gr(k, n)$  can be explicitly described using Schubert cycles. In order to define these, fix a ‘flag’ of subspaces in the vector space  $V$ , that is a nested sequence of subspaces,

$$F_{\bullet} : 0 = F_0 \subset F_1 \subset \dots \subset F_n = V$$

where  $F_i$  are vector subspaces such that  $\dim(F_i) = i$ . Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  be a partition satisfying

$$n - k \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0.$$

For a given partition  $\lambda$ , we can define a *Schubert Cycle*,  $\Sigma_{\lambda_1, \lambda_2, \dots, \lambda_k}(F_{\bullet})$ , to be the closed subset

$$\Sigma_{\lambda_1, \lambda_2, \dots, \lambda_k}(F_{\bullet}) := \{[\Omega] \in Gr(k, n) : \dim(\Omega \cap F_{n-k+i-\lambda_i}) \geq i\}.$$

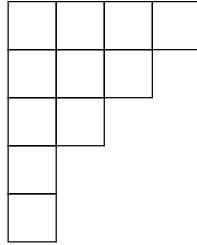
Note that  $\dim(\Sigma_{\lambda_1, \lambda_2, \dots, \lambda_k}(F_\bullet)) = k(n - k) - \sum_i \lambda_i$ .

Since any two flags in a vector space  $V$  differ by the action of  $GL_n$ , the class  $|\Sigma_{\lambda_1, \lambda_2, \dots, \lambda_k}(F_\bullet)| \in A^*(Gr(k, n))$  does not depend on the choice of the flag  $F_\bullet$  ([EH16, Theorem 1.7]). These classes,

$$\sigma_{\lambda_1, \lambda_2, \dots, \lambda_k} := |\Sigma_{\lambda_1, \lambda_2, \dots, \lambda_k}(F_\bullet)| \in A^*(Gr(k, n)),$$

are called *Schubert classes* and have codimension  $\sum_i \lambda_i$ . These form the basis of the Chow ring  $A^*(Gr(k, n))$  ([EH16, Corollary 4.7]). Note that when denoting the Schubert cycles, any  $\lambda_i$  that is 0 is omitted.

Any Schubert cycle  $\sigma_{\lambda_1, \lambda_2, \dots, \lambda_k}$  can be represented by a Young diagram, which is a collection of boxes placed from left to right, in rows, with the  $i$ -th row of length  $\lambda_i$ . That is, for example,  $\sigma_{4,3,2,1,1}$  can be represented as



The condition on the partitions  $\lambda_i$  means that the Young diagram fits into a box with  $k$  rows and  $n - k$  columns and the rows of the diagram are *non-increasing* in length from top to bottom.

Since Schubert cycles generate the cohomology ring of  $Gr(k, n)$ , the intersection of two Schubert cycles represents the intersection of elements in the cohomology ring and can in turn be written as a linear combination of Schubert cycles. See [EH16, Chapter 4] for more details on computing the intersections of Schubert cycles.

Here, we will recall a simple rule that can be used to find the intersection of any Schubert cycle with a special Schubert cycle, which is a Schubert cycle corresponding to a partition of the form  $\lambda = (\lambda_1, 0, 0, \dots, 0)$ . We will use this rule frequently in our computations. This is called Pieri's rule and is stated as follows:

**Theorem 2.35.** [EH16, Proposition 4.9] *Let  $\sigma_\lambda$  be a special Schubert cycle. Suppose  $\sigma_\mu$  is any Schubert cycle with parts  $\mu_1, \mu_2, \dots, \mu_k$ . Then,*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\mu_i \leq \nu_i \leq \mu_i - 1 \\ \sum \nu_i = \lambda + \sum \mu_i}} \sigma_\nu.$$

Pieri's rule can be described with the help of Young diagrams. The Young diagram of the product  $\sigma_\lambda \cdot \sigma_\mu$  is obtained from that of  $\sigma_\mu$ , by adding  $\lambda$  boxes,

such that no two boxes appear in the same column. For example,

$$\begin{aligned}
 \sigma_{2,1} \cdot \sigma_3 &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} \\
 &+ \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & & & & \\ \hline \square & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & \\ \hline \square & & & & & \\ \hline \end{array} \\
 &= \sigma_{3,2,1} + \sigma_{4,2} + \sigma_{4,1,1} + \sigma_{5,1}.
 \end{aligned}$$

In fact, this gives a complete algorithm for computing the product of any two Schubert cycles, since any Schubert cycle can be expressed as a linear combination of products of special Schubert cycles ([Ful97, Section 9.4], [EH16, Proposition 4.16]).

In the next subsection, we will recall the basic theory on Rational Homogeneous Varieties.

## 2.6 Rational Homogeneous Varieties

An algebraic variety is called homogeneous if there is an algebraic group acting transitively on it. The classical theorem by Borel and Rummert [Ak95, page 101] states that any projective homogeneous variety can be decomposed into a product of an abelian variety and a rational homogeneous space. That is, given  $X$ , a projective homogeneous variety, we have

$$X \cong Ab \times X',$$

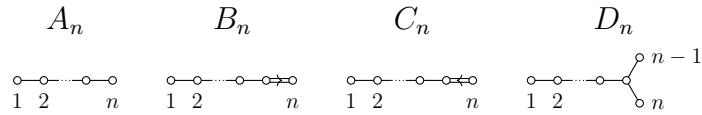
where  $Ab$  is an abelian variety and  $X'$ , a rational homogeneous space, described as the quotient  $G/P$  of a semi-simple algebraic group  $G$  by  $P$ , a parabolic subgroup.  $X'$  can be further decomposed into the following product,

$$X' \sim G_1/P_1 \times G_2/P_2 \times \dots \times G_l/P_l$$

where  $G_i$  are simple algebraic groups and  $P_i$  are parabolic subgroups. So it is enough if we consider  $G$ , simple algebraic groups and  $P$ , parabolic subgroups, in order to understand the complete classification of rational homogeneous varieties. The most known examples of rational homogeneous varieties are projective spaces, grassmanians and smooth quadric hypersurfaces.

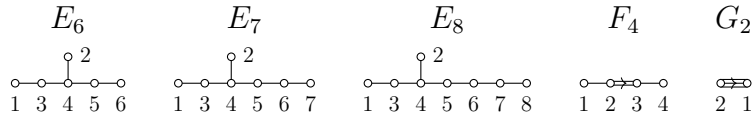
The homogeneous space that is given by  $G$  will only depend on its Lie algebra. Refer to [Bou68, FH91, Hum75, Mil13, Spr98] for more details on the theory of Lie algebras and their classification. This classification can be described pictorially, with the help of Dynkin diagrams.

That is, the list of simple Lie algebras can be classified into Classical and Exceptional. The *Classical Lie algebras* are given by:



If the diagram has a symmetry, we can fold it along the symmetry and denote this by drawing an arrow pointing away from the hinge.

The *Exceptional Lie algebras* are given as follows:



Note that  $B_n$  is a fold of  $D_{n+1}$ ,  $C_n$  is a fold of  $A_{2n-1}$ ,  $F_4$  is a fold of  $E_6$  and  $G_2$  is a fold of  $D_4$  (It is a double fold, i.e. it is  $B_3$  folded again!).

Let the nodes of the Dynkin diagram be labelled  $\{1, 2, \dots, l\}$ . Choose a set of labels  $I \subset \{1, 2, \dots, l\}$  such that the labels in  $\{1, 2, \dots, l\} \setminus I$  correspond to marked nodes in the Dynkin diagram. To a fixed set of nodes of the Dynkin diagram, corresponding to the labels in  $I$ , we can assign a Parabolic subgroup. (see [?, Section 3.2] for more details.) For example, if  $I = \emptyset$ , then  $P_I = B$ , the Borel subgroup (a maximal closed and connected solvable algebraic subgroup) and if  $I = \{1, 2, \dots, l\}$ , then  $P_I = G$ . If  $\{1, 2, \dots, l\} \setminus I = \{k\}$ , then we denote  $P_I = P^k$ , which is the *maximal parabolic subgroup*. Thus, from the Dynkin diagram of a particular Lie algebra, the corresponding homogeneous variety can be described and hence, the classification of Lie algebras in turn give us an explicit classification of all possible homogeneous spaces.

When  $G$  is simple, the quotients by maximal parabolic subgroups  $X = G/P^k$  are Fano varieties of Picard rank one. The following result comes in very handy when studying rational homogeneous varieties of Picard rank one.

**Lemma 2.36.** *When  $G$  is simple and  $P = P^k$ , we have  $b_2(G/P^k) = 1$  and  $b_4(G/P^k)$  equals the number of simple roots adjacent to  $\alpha_k$ , where  $b_i(G/P^k)$  is the  $i$ -th Betti number of  $G/P^k$ .*

*Proof.* See [?, Lemma 3.1]. □

For  $G$  simple and a parabolic subgroup  $P = P^k$  corresponding to a non-short root  $\alpha_k$ , i.e., a root such that no arrows in the diagram point in the direction of the corresponding node in the Dynkin diagram, Landsberg and Manivel describe the minimal family of rational curves  $H_x$  through a point of  $X = G/P^k$ :

**Theorem 2.37** ([LM03, Theorem 4.8], see also [LM04, Theorem 2.5] and the subsequent paragraph). *Let  $X = G/P^k$  be a rational homogeneous variety such that  $\alpha_k$  is not short. Then  $H_x$  is homogeneous and the associated marked diagram*

is determined as follows: remove the node corresponding to  $k$  and mark the nodes adjacent to  $k$ . Moreover, the embedding of  $H_x$  in  $\mathbb{P}(T_x X)$  is minimal if and only if the Dynkin diagram of  $G$  is simply laced, i.e., without multiple edges.

$$X = E_8/P^6 \quad H_x = \text{Seg}(\text{OG}_+(5, 10) \times \mathbb{P}^2)$$

**Example 1.**



In Section 6.1 we study Classical rational homogeneous varieties of Picard rank one and check if they satisfy the  $\mathfrak{F}_3$  condition and in Section 6.2, we study the Exceptional rational homogeneous varieties of Picard rank one, and check which ones satisfy  $\mathfrak{F}_2$  and  $\mathfrak{F}_3$  conditions.

# Chapter 3

## Curves

Let  $C_d \subset \mathbb{P}^2$  be a reduced plane curve of degree  $d$  over  $\mathbb{C}$  and  $P$  be a point on  $C_d$ . In this chapter, we study the singularities of such curves by computing the log canonical threshold of the curve  $C_d$  at a point  $P \in C_d$ .

Recall that

$$\text{lct}_P(\mathbb{P}^2, C_d) = \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (\mathbb{P}^2, \lambda C_d) \text{ is log canonical at } P\}.$$

By [KSC04, Exercise 6.18] and [KSC04, Lemma 6.35], we have

$$\frac{1}{\text{mult}_P(C_d)} \leq \text{lct}_P(\mathbb{P}^2, C_d) \leq \frac{2}{\text{mult}_P(C_d)}.$$

This implies that the smaller the value of  $\text{lct}_P(\mathbb{P}^2, C_d)$ , the worse the singularity of the curve  $C_d$  at  $P$ .

The values of log canonical threshold of a given reduced curve  $C_d$  of degree  $d \leq 4$  at a point  $P$  on  $C_d$  have been computed.

**Example 2.** If  $d = 1$  or  $d = 2$ , then  $\text{lct}_P(\mathbb{P}^2, C_d) = 1$ .

**Example 3.** If  $d = 3$ , then the log canonical threshold of curves  $C_3$  of degree 3 at  $P$ ,  $\text{lct}_P(\mathbb{P}^2, C_3)$  is one of  $\{1, \frac{5}{6}, \frac{3}{4}, \frac{2}{3}\}$ . The worst singularity corresponds to  $\text{lct}_P(\mathbb{P}^2, C_3) = \frac{2}{3}$  and in this case,  $C_d$  is a union of 3 lines intersecting at  $P$  (example of such a curve  $C_d$  is  $xy(x - y) = 0$ ). Examples of curves with the given values of log canonical threshold ( $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1$  resp.) are illustrated below:

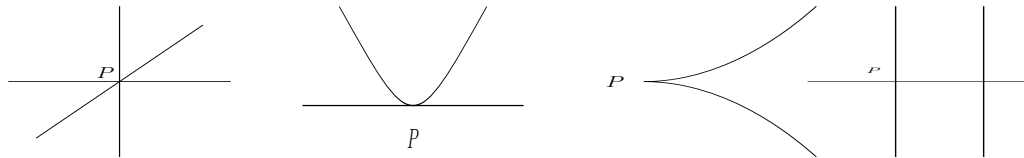


Figure 3.1: (i)  $C_3 : xy(x - y) = 0$ ; (ii)  $C_3 : y(x^2 - y) = 0$ ; (iii)  $C_3 : x^2 - y^3 = 0$ ; (iv)  $C_3 : xy(x - 1) = 0$

**Example 4** (Erik Paemurru). Let  $C_4$  be a quartic curve. The log canonical threshold of  $C_4$  at the point  $P$ ,  $\text{lct}_P(\mathbb{P}^2, C_4)$  is one of  $\{1, \frac{5}{6}, \frac{3}{4}, \frac{7}{10}, \frac{9}{14}, \frac{5}{8}, \frac{2}{3}, \frac{3}{5}, \frac{7}{12}, \frac{5}{9}, \frac{1}{2}\}$ . The worst singularity occurs when  $\text{lct}_P(\mathbb{P}^2, C_4) = \frac{1}{2}$  and in this case  $C_4$  is a

union of 4 lines passing through the point  $P$  (example of such a curve  $C_4$  is  $xy(x - y)(x + y) = 0$ ) as illustrated below.

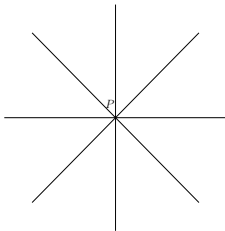


Figure 3.2:  $C_4 : xy(x^2 - y^2) = 0$

For curves  $C_d$  with  $d \leq 3$ ,  $\text{lct}_P(\mathbb{P}^2, C_d)$  correspond uniquely to a type of singularity of  $C_d$  at the point  $P$ . When  $d = 4$ , this is not the case. For example, both the curves illustrated below have  $\text{lct}_P(\mathbb{P}^2, C_4) = \frac{5}{8}$  but have very different singularities at the point  $P$ .



Figure 3.3: (i)  $C_4 : y(x^2 - y^3) = 0$  (ii)  $C_4 : (x + x^2 + y^2)(x + y^2) = 0$

In Section 3.1, we will study different types of singularities that a given curve  $C_d$  of degree  $d$  can have. Then in Section 3.2, we will prove

**Theorem 3.1.** (Also Main Result 1) Suppose  $d \geq 6$  and  $\frac{2d-3}{d(d-2)} < \text{lct}_P(\mathbb{P}^2, C_d) \leq \frac{2d-7}{d^2-4d+1}$ . Then the curve  $C_d$  has singularity of type  $\mathbb{M}_{d-1}$ ,  $\tilde{\mathbb{M}}_{d-1}$  or  $\hat{\mathbb{M}}_{d-1}$  at the point  $P$  with  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-7}{d^2-4d+1}$ .

### 3.1 Singularities of curves

Let  $C$  be a reduced curve on a smooth surface  $S$  and  $P$  be a point on  $C$ . We try to understand the singularities of the curve  $C$  at the point  $P$ . In this section, we introduce various types of singularities that we denote by  $\mathbb{T}_n$ ,  $\mathbb{K}_n$ ,  $\tilde{\mathbb{T}}_n$ ,  $\tilde{\mathbb{K}}_n$ ,  $\mathbb{M}_n$ ,  $\tilde{\mathbb{M}}_n$  and  $\hat{\mathbb{M}}_n$ , where  $n = \text{mult}_P(C)$ . We aim to describe the geometric properties of the curve  $C$  having one of these types of singularities at  $P$ .

Let  $f_1 : S_1 \rightarrow S$  be the blow-up of  $S$  at the point  $P$ . Let  $C^1$  be the proper transform in  $S_1$  of the curve  $C$  and  $E_1$  be the exceptional divisor of the blow-up.

### 3.1.1 Singularities of type $\mathbb{K}_n$ (cusps)

Curve  $C$  having singularity of type  $\mathbb{K}_n$  can be defined with the help of its geometric properties as given below. These singularities are also called cusps.

1.  $\text{mult}_P(C) = n \geq 2$ ,
2.  $C^1 \cap E_1 = P_1$ ,
3.  $C^1$  intersects  $E_1$  tangentially at  $P_1$  and is smooth at this point.

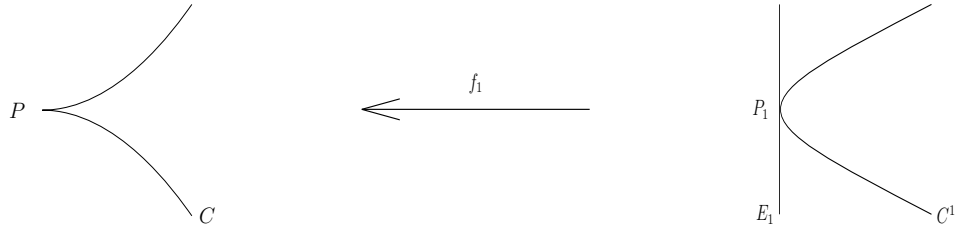


Figure 3.4:  $C$  with  $\mathbb{K}_n$  singularity at  $P$  and its blow-up at  $P$

Recall from [K99, Theorem 1.1], the log canonical threshold of a cuspidal curve is

$$\text{lct}_P(S, C) = \frac{1}{n} + \frac{1}{n+1}.$$

**Remark 3.2.** Suppose  $S = \mathbb{P}^2$ . Let  $C$  be a curve of degree  $d \geq 3$  having  $\mathbb{K}_n$  singularity at the point  $P$ . Then  $n \leq d - 1$ . If  $n = d - 1$ , then the curve  $C$  is irreducible. Such curves do exist. For example, the curve  $C$  given by  $zx^{d-1} + y^d = 0$  has singularity of type  $\mathbb{K}_{d-1}$  at the point  $P = [0 : 0 : 1]$ .

### 3.1.2 Singularities of type $\mathbb{T}_n$

Curve  $C$  having singularity of type  $\mathbb{T}_n$  at  $P$  can be defined using the following geometric properties.

1.  $\text{mult}_P(C) = n \geq 3$ ,
2.  $C^1 \cap E_1 = P_1$ ,
3. The point  $P_1$  is an ordinary double point of  $C^1$ .

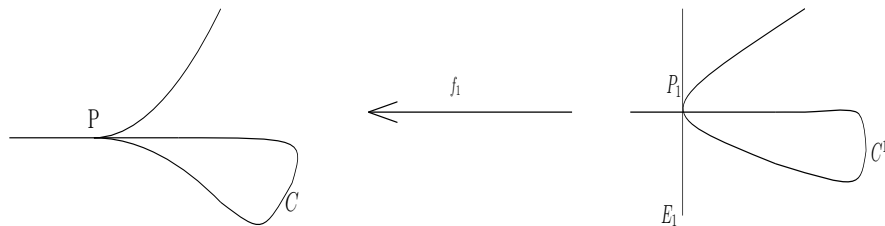


Figure 3.5:  $C$  with  $\mathbb{T}_n$  singularity at  $P$  and its blow-up at  $P$

**Remark 3.3.** Suppose  $S = \mathbb{P}^2$  and  $C$  is a curve of degree  $d$ . Let  $L$  be a line in  $\mathbb{P}^2$  passing through the point  $P$ , whose proper transform  $L^1$  in  $S_1$  passes through the point  $P_1$ . If the curve  $C$  has singularity of type  $\mathbb{T}_n$ , then  $C = L + Z$ , where  $Z$  is a reduced curve of degree  $d - 1$  that does not contain  $L$  as an irreducible component. Since if not, then

$$d \geq L.C = d - 1 + 2 = d + 1$$

which is absurd. Thus,  $C = L + Z$  and  $L \cap Z = P$  where  $Z$  has singularity of type  $\mathbb{K}_{d-2}$  at the point  $P$ .

### 3.1.3 Singularities of type $\tilde{\mathbb{T}}_n$

We will define curve  $C$  having singularity of type  $\tilde{\mathbb{T}}_n$  at  $P$  using the following geometric properties.

1.  $\text{mult}_P C = n \geq 4$ ,
2.  $C^1 \cap E_1 = \{P_1, Q_1\}$ ,
3. The point  $P_1$  is an ordinary double point of  $C^1$ ,
4.  $C^1$  intersects  $E_1$  transversally at  $Q_1$  and is smooth at this point.

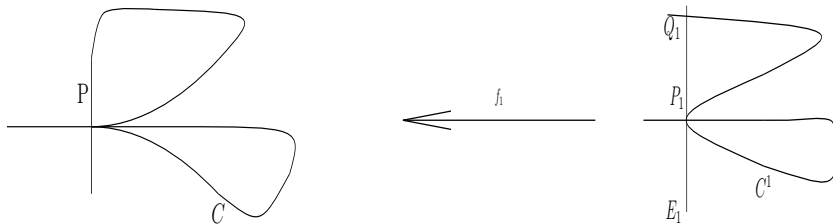


Figure 3.6:  $C$  with  $\tilde{\mathbb{T}}_n$  singularity at  $P$  and its blow-up at  $P$

**Remark 3.4.** Suppose  $S = \mathbb{P}^2$  and  $C$  is a curve of degree  $d$ . Let  $L$  be a line in  $\mathbb{P}^2$  passing through the point  $P$ , whose proper transform  $L^1$  in  $S_1$  passes through the point  $P_1$ . Similar computations as the one in Remark 3.3 imply  $C = Z + L$  such that  $L \cap Z = P$ , where  $Z$  is a reduced curve of degree  $d - 1$  that does not contain  $L$  as an irreducible component and  $Z$  has singularity of type  $\tilde{\mathbb{K}}_{d-2}$  at the point  $P$ , which is introduced in the following subsection.

### 3.1.4 Singularities of type $\tilde{\mathbb{K}}_n$

Curve  $C$  with  $\tilde{\mathbb{K}}_n$  singularity can be defined using its geometric properties as given below:

1.  $\text{mult}_P(C) = n \geq 3$ ,

2.  $C^1 \cap E_1 = \{P_1, Q_1\}$ ,
3.  $C^1$  intersects  $E_1$  tangentially at the point  $P_1$  and is smooth at this point,
4.  $C^1$  is smooth at  $Q_1$  and intersects  $E_1$  transversally at this point.

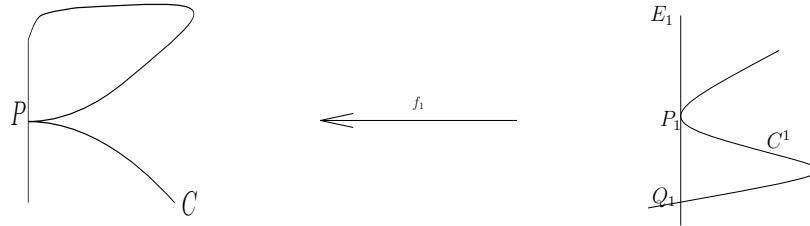


Figure 3.7:  $C$  with  $\tilde{\mathbb{K}}_n$  singularity at  $P$  and its blow-up at  $P$

**Remark 3.5.** Suppose  $S = \mathbb{P}^2$  and  $C$  be a curve of degree  $d$ . Then  $C$  with  $\tilde{\mathbb{K}}_{d-1}$  singularity at the point  $P$  exists. Such a  $C$  can be reducible, for example  $y(x^{d-2} - y^{d-1}) = 0$  or can be irreducible, for example  $x^{d-2}y + y^d + x^d = 0$ . If  $C$  is reducible, then  $C = L + Z$  where  $Z$  is curve of degree  $d-1$  which does not contain  $L$  as an irreducible component and has singularity of type  $\mathbb{K}_{d-2}$  at the point  $P$ .

### 3.1.5 Singularities of type $\mathbb{M}_n$

Curve  $C$  that has singularity of type  $\mathbb{M}_n$  at  $P$  can be defined using the following geometric properties:

1.  $\text{mult}_P(C) = n \geq 5$ ,
2.  $C^1 \cap E_1 = \{P_1, Q_1, R_1\}$ ,
3.  $C^1$  is smooth at the points  $Q_1$  and  $R_1$  where it intersects transversally with  $E_1$ ,
4. The point  $P_1$  is an ordinary double point of  $C^1$ .

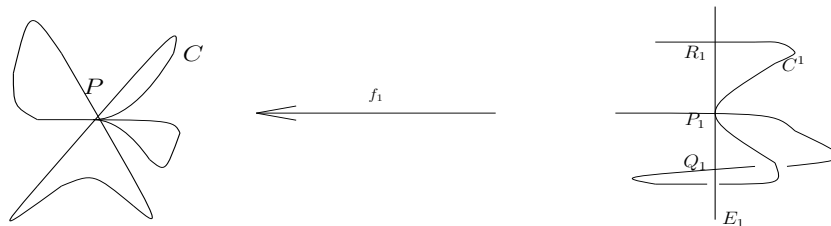


Figure 3.8:  $C$  with  $\mathbb{M}_n$  singularity at  $P$  and its blow-up at  $P$

**Remark 3.6.** Suppose  $S = \mathbb{P}^2$  and  $C$  is a curve of degree  $d$ .  $C$  having singularity of type  $\mathbb{M}_{d-1}$  at the point  $P$  exists, for example,  $x(x^2 - y^2)(x^{d-4} - y^{d-3})$ . It is reducible and thus  $C = Z + L$  where  $L$  is a line in  $S$  that contains the point  $P$  such that its proper transform  $L^1$  in  $S_1$  contains the point  $P_1$  and  $Z$  is a reduced curve of degree  $d - 1$  which does not contain  $L$  as an irreducible component.

### 3.1.6 Singularities of type $\tilde{\mathbb{M}}_n$

Curve  $C$  that has singularity of type  $\tilde{\mathbb{M}}_n$  at the point  $P$  can be defined using its geometric properties.

1.  $\text{mult}_P(C) = n \geq 5$ ,
2.  $C^1 \cap E_1 = \{P_1, Q_1\}$ ,
3.  $P_1$  is an ordinary double point of  $C^1$  with  $(C^1.E_1)_{P_1} = n - 2$ ,
4.  $C^1$  intersects  $E_1$  tangentially at the point  $Q_1$  with  $(C^1.E_1)_{Q_1} = 2$  and is smooth at this point.

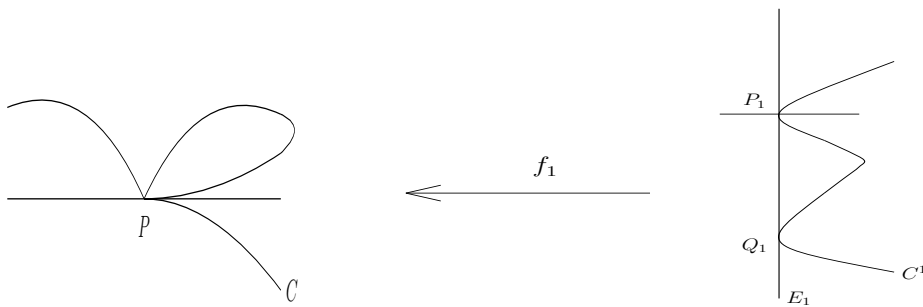


Figure 3.9:  $C$  with  $\tilde{\mathbb{M}}_n$  singularity at  $P$  and its blow-up at  $P$

**Remark 3.7.** Suppose  $S = \mathbb{P}^2$  and  $C$  be a curve of degree  $d$ . Then  $n = d - 1$  is possible and curve  $C$  with  $\tilde{\mathbb{M}}_{d-1}$  exists, for example,  $y(zx^2 + y^3)(zy^{d-4} + x^{d-3}) = 0$  has  $\tilde{\mathbb{M}}_{d-1}$  singularity at the point  $P = [0 : 0 : 1]$ . In this case,  $C$  is reducible and thus,  $C = L + Z$  where  $L$  is the line in  $S$  containing  $P$  such that its proper transform  $L^1$  passes through the point  $P_1$  in  $S_1$  and  $Z$  is a  $d - 1$  degree reduced curve that does not contain  $L$  as an irreducible component.

### 3.1.7 Singularities of type $\hat{\mathbb{M}}_n$

Curve  $C$  having singularity of type  $\hat{\mathbb{M}}_n$  at the point  $P$  can be defined using the following geometric properties.

1.  $\text{mult}_P(C) = n \geq 5$ ,
2.  $C^1 \cap E_1 = \{P_1, Q_1\}$ ,

3.  $P_1$  is an ordinary double point of  $C^1$  with  $(C_1.E_1)_{P_1} = n - 2$ ,
4.  $Q_1$  is an ordinary double point of  $C^1$  with  $(C_1.E_1)_{Q_1} = 2$ .

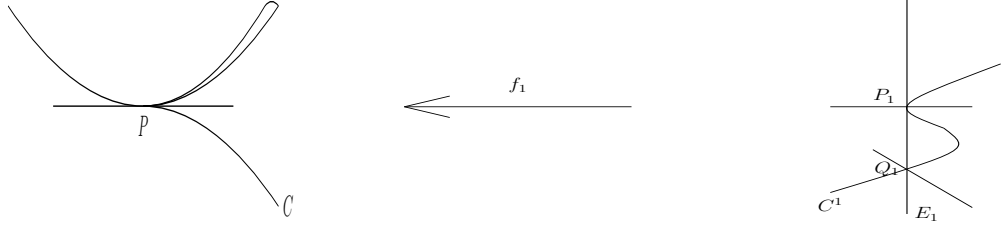


Figure 3.10:  $C$  with  $\widehat{\mathbb{M}}_n$  singularity at  $P$  and its blow-up at  $P$

**Remark 3.8.** Suppose  $S = \mathbb{P}^2$  and  $C$  be a curve of degree  $d$  having  $\widehat{\mathbb{M}}_n$  singularity at the point  $P$ . Then  $n = d - 1$  is possible, for example  $C$  given by  $x(zx^{d-4} + y^{d-3})(z^2y^2 + x^4) = 0$  has  $\widehat{\mathbb{M}}_{d-1}$  singularity at the point  $P = [0 : 0 : 1]$ . That is,  $C = L + Z$  where  $L$  is a line in  $S$  that passes through the point  $P$  whose proper transform contains the point  $P_1$  and  $Z$  is a reduced curve in  $S$  of degree  $d - 1$  which does not contain  $L$  as an irreducible component.

### 3.1.8 Defining equations

In this subsection, we describe curve  $C$  having any of the above types of singularities using local equations. But these descriptions are indeed not essential to prove Theorem 3.1. Upto analytic change of coordinates, the equation of curve  $C$  with the respective singularities are given below:

1.  $\mathbb{K}_n: x^n + y^{n+1} + \sum_{i=1}^{n+1} a_i x^i y^{n+1-i} + \text{H.O.T.} = 0$
2.  $\mathbb{T}_n: x(x^{n-1} - y^n + \sum_{i=2}^{n+1} a_i x^{i-1} y^{n+1-i} + \text{H.O.T.}) = 0$
3.  $\widetilde{\mathbb{T}}_n: x(y(y^{n-1} - x^{n-2}) + \sum_{i=2}^{n+1} a_i x^{i-1} y^{n+1-i} + \text{H.O.T.}) = 0$
4.  $\widetilde{\mathbb{K}}_n: y(x^{n-1} - y^n) + \sum_{i=1}^{n+1} a_i x^{i-1} y^{n+1-i} + \text{H.O.T.} = 0$
5.  $\mathbb{M}_n: x((x^2 - y^2)(x^{n-3} - y^{n-2}) + \sum_{\substack{i=2 \\ i \neq 3}}^n a_i x^i y^{n+1-i} + \text{H.O.T.}) = 0$
6.  $\widetilde{\mathbb{M}}_n: y((x^2 + y^3)(y^{n-3} + x^{n-2}) + \sum_{\substack{i=1 \\ i \neq n}}^n a_i x^i y^{n-i} + \sum_{\substack{i=0 \\ i \neq n-2}}^{n+1} b_i x^i y^{n+1-i} + \text{H.O.T.}) = 0$

$$\begin{aligned}
7. \hat{\mathbb{M}}_n : & x((x^{n-3} + y^{n-2})(y^2 + x^4) + \sum_{\substack{i=0 \\ i \neq 1}}^n a_i x^{i-1} y^{n+1-i} + \sum_{i=0}^n b_i x^{i-1} y^{n+2-i} \\
& + \sum_{\substack{i=0 \\ i \neq 5}}^{n+2} c_i x^{i-1} y^{n+3-i} + \text{H.O.T}) = 0
\end{aligned}$$

The above set of equations comprise an exhaustive list of curves  $C$  of a given degree with the various types of singularities, upto analytic change of coordinates and include the curves missing from the list in [C17, Definition 1.9].

## 3.2 Proof of Main Result 1

Let  $C_d$  be a reduced curve of degree  $d \geq 6$  on a smooth surface  $S$  such that  $P \in C_d$  and let  $m_0 = \text{mult}_P(C_d)$ . Suppose  $\frac{2d-3}{d(d-2)} < \text{lct}_P(S, C_d) < \frac{2d-7}{d^2-4d+1}$ . This means that there exists a  $\lambda < \frac{2d-7}{d^2-4d+1}$  such that  $(S, \lambda C_d)$  is not Kawamata log terminal at a point  $P$ . Let us also assume that  $m_0 \neq d$  and thus  $C_d$  is not a union of  $d$ -lines. We need to show that the curve  $C_d$  has singularity of type  $\mathbb{M}_{d-1}$ ,  $\tilde{\mathbb{M}}_{d-1}$  or  $\hat{\mathbb{M}}_{d-1}$  at the point  $P$ . It is important to notice that the arguments presented below are very similar to the arguments in [C17]. Also, these are local arguments i.e. it is not necessary for the curve  $C_d$  to be smooth everywhere outside of  $P$ . We assume that the respective divisors on the surface  $S$  at various levels is Kawamata log terminal (or log canonical) at a punctured neighbourhood of  $P$ .

**Lemma 3.9.** *The following inequalities are used in the proof of the main result:*

1.  $\lambda < \frac{2}{d-1}$
2.  $\lambda < \frac{2k+1}{kd}$ , for  $k \in \mathbb{Z}_{>0}$  such that  $k \leq d-3$
3.  $\lambda < \frac{2k+1}{kd+1}$  for  $k \in \mathbb{Z}_{>0}$  such that  $k \leq d-5$
4.  $\lambda < \frac{3}{d}$
5.  $\lambda < \frac{2}{d-2}$
6.  $\lambda < \frac{6}{3d-4}$
7.  $\lambda < \frac{5}{2d}$

*Proof.* Straightforward. □

We will now introduce the notations used in the proof of the main theorem. Let  $S = S_0 = \mathbb{P}^2$  and  $D = (\mathbb{P}^2, \lambda C_d)$ . Consider a sequence of blow-ups  $f_i : S_i \rightarrow S_{i-1}$  such that  $f_1$  is the blow-up of  $P_0 = P$ ,  $f_2$  is the blow-up of the point  $P_1$  and so on, i.e.  $f_i$  is the blow-up of the point  $P_{i-1} \in S_{i-1}$ . We have,

$$\dots S_{k+1} \xrightarrow{f_{k+1}} S_k \dots \xrightarrow{f_4} S_3 \xrightarrow{f_3} S_2 \xrightarrow{f_2} S_1 \xrightarrow{f_1} S_0 \quad (3.1)$$

Also, let  $f : S_{k+1} \rightarrow S$  be the composition of the blow ups i.e.  $f = f_1 \circ f_2 \circ \dots \circ f_{k+1}$ . The  $f_i$ -exceptional divisors during each blow up is denoted by  $E_i$ .

The proper transform of the exceptional divisors  $E_j$  in  $S_i$  are denoted by  $E_j^i$  for all  $j < i$ . Also, after the  $f_i$  blow-up, the curve  $C_d$  is denoted by  $C_d^i$  in  $S_i$ . The divisors comprising of the curve and the exceptional curves on every floor  $S_i$  is together denoted by  $D^{S_i}$ . We will explicitly describe how each of these points of blow-up are chosen.

Since  $(S, \lambda C_d)$  is not Kawamata log terminal at the point  $P \in C_d$ , by Remark 2.1 one has that  $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$  is not Kawamata log terminal at some point in  $E_1$ . Let this point be  $P_1$  and let  $m_1 = \text{mult}_{P_1}(C_d^1.E_1)$ .

**Lemma 3.10.**  $\lambda m_0 < 2$

*Proof.* Since  $m_0 \leq d - 1$ , we have  $\lambda m_0 \leq \lambda(d - 1)$ . Using Lemma 3.9(1), we get  $\lambda m_0 < 2$ .  $\square$

From Lemma 2.8 this implies that the point  $P_1$  is a unique point on  $E_1$  at which  $(S_1, D^{S_1})$ , that is,  $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$  is not Kawamata log terminal.

Let  $L$  be the line in  $\mathbb{P}^2$  whose proper transform,  $L^1$  in  $S_1$ , contains the point  $P_1$ .

**Lemma 3.11.** *Suppose  $m_0 = d - 1$ . Then  $L$  is an irreducible component of the curve  $C_d$ .*

*Proof.* Observe that

$$C_d^1 \sim_{\mathbb{Q}} f_1^*(C_d) - m_0 E_1$$

If  $L$  is not an irreducible component of the curve  $C_d$ , then we have

$$m_1 \leq C_d^1.L^1 = C_d.L - m_0 E_1.L^1 = d - m_0. \quad (3.2)$$

Thus we have  $m_0 + m_1 \leq d$ .  $P_1 \in C_d^1$ , since if not then  $(S_1, (\lambda m_0 - 1)E_1)$  is not log canonical at the point  $P_1$ , which is not possible since  $\lambda m_0 - 1 < 1$  from Lemma 3.10. Since  $m_0 = d - 1$ ,  $m_1 = 1$ . Therefore,  $C_d^1$  is smooth at  $P_1$ .

Let  $k = \text{mult}_{P_1}(C_d^1.E_1)$ . We claim that  $k > d - 3$ . Indeed, suppose  $k \leq d - 3$ . Since  $(S_1, D^{S_1})$  is not Kawamata log terminal at the point  $P_1 \in E_1$ , by Definition 4, we have  $(S_2, D^{S_2})$  not Kawamata log terminal at some point in  $E_2$ , where

$$D^{S_2} = \lambda C_d^2 + (\lambda(d - 1) - 1)E_1^2 + (\lambda d - 2)E_2.$$

Let this point be  $P_2$ . Note that all the coefficients of the curves in  $D^{S_2}$  are less than 1. In particular, we have

$$\lambda d - 2 < \frac{d - 2}{d^2 - 4d + 1} < 1.$$

Thus, by Definition 4,  $(S_3, D^{S_3})$  is not Kawamata log terminal at some point in  $E_3$ . Let this point be  $P_3$ . Note that  $P_3 \in C_d^3 \cap E_1^3 \cap E_3$ . Thus,  $f_2$  is a blow up of  $P_1 \in C_d^1 \cap E_1$ ,  $f_3$  is the blow up of  $P_2 \in C_d^2 \cap E_1^2 \cap E_2$  and so on. One then has the sequence as mentioned in Eq. (3.1).

We require  $k + 1$  blow-ups to ensure simple normal crossing of the elements of the divisor over the point  $P$ . Here the points of blow up are such that  $P_i = C^i \cap E_i$  and using the notations described earlier, we have

$$\begin{aligned} K_{S_{k+1}} + \lambda C_d^{k+1} + (\lambda(d-1) - 1)E_1^{k+1} + (\lambda d - 2)E_2^{k+1} + \dots + (\lambda k d - 2k)E_{k+1} \\ \sim_{\mathbb{Q}} f^*(K_S + \lambda C_d). \end{aligned} \quad (3.3)$$

Let the coefficients of  $E_i^{k+1}$  in (3.3) be denoted by  $b_i$ . Then since

$$\lambda C_d^{k+1} + \sum_{i=1}^k b_i E_i^{k+1} + b_{k+1} E_{k+1},$$

is a divisor with simple normal crossings over  $P$ , at least one of the  $b_i > 1$  or  $b_{k+1} > 1$ . But the coefficients  $b_i$  are such that  $b_j < b_i$  for all  $j < i$  and we have

$$b_{k+1} = \lambda k d - 2k \leq \frac{d^2 - 5d + 6}{d^2 - 4d + 1} < 1,$$

from Lemma 3.9(2). That is, in particular  $b_j < b_{k+1} < 1$  for all  $j < k + 1$ .

This contradiction implies that  $k > d - 3$ . We also know that,

$$k = \text{mult}_{P_1}(C_d^1.E_1) \leq (C_d^1.E_1) = m_0 = d - 1.$$

Therefore, these inequalities imply

$$k = d - 1 \text{ or } d - 2$$

Thus, when

- $k = d - 1$ , then  $C_d$  has singularity  $\mathbb{K}_{d-1}$  at  $P$  (See Section 3.1.1).
- $k = d - 2$ , then  $C_d$  has singularity  $\tilde{\mathbb{K}}_{d-1}$  at the point  $P$  (See Section 3.1.4).

If the curve  $C_d$  has either of the above singularities at the point  $P$ , then  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-1}{d(d-1)}$  or  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-3}{d(d-2)}$  respectively.

Since we assume  $\text{lct}_P(\mathbb{P}^2, C_d) > \frac{2d-3}{d(d-2)}$  neither of these values for  $k$  are possible. Therefore, this contradiction implies that  $L$  is an irreducible component of the curve  $C_d$ .  $\square$

**Lemma 3.12.** *Suppose  $m_0 = d - 1$ . Then  $C_d$  has singularity  $\mathbb{M}_{d-1}$ ,  $\tilde{\mathbb{M}}_{d-1}$  or  $\hat{\mathbb{M}}_{d-1}$  at the point  $P$ .*

*Proof.* From Lemma 3.11 we know that  $L$  is an irreducible component of the curve  $C_d$  i.e. we have  $C_d = C_{d-1} + L$  where  $C_{d-1}$  is a reduced curve of degree  $d - 1$  which does not contain  $L$  as an irreducible component. Let  $n_0 = \text{mult}_P(C_{d-1})$ . Since  $m_0 = \text{mult}_P(C_d) = d - 1$ , we have  $n_0 = m_0 - 1 = d - 2$ .

Let  $f_1 : S_1 \rightarrow S$  be the blow-up at the point  $P$  and  $n_1 = \text{mult}_{P_1}(C_{d-1}^1)$ . We have,  $n_1 = m_1 - 1$ . We also have  $P_1 \in C_{d-1}^1$  since if not it would mean that  $(S_1, \lambda L^1 + (\lambda(d-1) - 1)E_1)$  is not log canonical at the point  $P_1$  which is a contradiction since  $\lambda < 1$  and  $\lambda(d-1) - 1 < 1$  and  $L^1, E_1$  are simple normal crossing divisors over  $P_1$ . Thus,  $n_1 \geq 1$ .

Consider

$$n_1 \leq L^1.C_{d-1}^1 = d - 1 - n_0,$$

that is,  $n_0 + n_1 \leq d - 1$  and since  $n_0 = d - 2$ , we have  $n_1 = 1$ . Thus the curve  $C_{d-1}^1$  is smooth at  $P_1$ .

Let  $k = \text{mult}_{P_1}(C_{d-1}^1.E_1)$ . We claim  $k > d - 5$ . Instead, suppose  $k \leq d - 5$ , then using similar computations as in Lemma 3.11, after  $k + 1$  blow-ups, we get

$$\begin{aligned} K_{S_{k+1}} + \lambda C_{d-1}^{k+1} + \lambda L^{k+1} + (\lambda(d-1) - 1)E_1^{k+1} + \dots + (\lambda(kd+1) - 2k)E_{k+1} \\ \sim_{\mathbb{Q}} f^*(K_S + \lambda C_d) \end{aligned}$$

where  $(S_{k+1}, D^{S_{k+1}})$  is not Kawamata log terminal at some point in  $E_{k+1}$ , which we take to be  $P_{k+1}$ . Here again,  $f$  is a composition of the  $k + 1$  blow ups and  $b_i$  are the coefficients of  $E_i^{k+1}$  in the above equation.

Since the curves in the divisor

$$\lambda C_{d-1}^{k+1} + \lambda L^{k+1} + \sum_{i=1}^k b_i E_i^{k+1} + b_{k+1} E_{k+1}$$

intersect at simple normal crossing at the point  $P$  after the  $k + 1$  blow ups, one of these coefficients should be such that  $b_i > 1$  or  $b_{k+1} > 1$  but we have

$$b_{k+1} = \lambda(kd+1) - 2k = \frac{(k+2)d - 2k - 7}{d^2 - 4d + 1} < 1,$$

from Lemma 3.9(3) and the coefficients are such that  $b_j < b_i$  for all  $j < i$ . In particular,  $b_j < b_{k+1} < 1$  for all  $j < k + 1$ . This contradiction implies  $k > d - 5$ . We also know,

$$k = \text{mult}_{P_1}(C_{d-1}^1.E_1) \leq (C_{d-1}^1.E_1) = \text{mult}_P(C_{d-1}) = n_0 = d - 2.$$

These inequalities imply that

$$k = d - 2 \text{ or } d - 3 \text{ or } d - 4.$$

Thus, when

- $k = d - 2$ ,  $C_d$  has the singularity of type  $\mathbb{T}_{d-1}$  at  $P$  (see Section 3.1.2),
- $k = d - 3$ ,  $C_d$  has the singularity of type  $\tilde{\mathbb{T}}_{d-1}$  at  $P$  (see Section 3.1.3).

If  $C_d$  has either one of the above singularities at the point  $P$ , then  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-3}{(d-1)^2}$  or  $\text{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-5}{d^2-3d+1}$  respectively. Since we assume that  $\text{lct}_P(\mathbb{P}^2, C_d) > \frac{2d-3}{d(d-2)}$ , these values of  $k$  are not possible. Thus  $k = d - 4$ , i.e.  $C_d$  has the singularity of type  $\mathbb{M}_{d-1}$ ,  $\tilde{\mathbb{M}}_{d-1}$  or  $\hat{\mathbb{M}}_{d-1}$  at  $P$ .  $\square$

Observe that Lemma 3.11 and Lemma 3.12 complete the proof of the main result if  $m_0 = d - 1$ . In the remaining part of the section, we will prove that  $m_0 \leq d - 2$  is not possible. In particular, we prove the following proposition.

**Proposition 3.13.** *If  $m_0 \leq d - 2$ , then  $\text{lct}_P(S, C_d) \geq \frac{2}{d-1}$ .*

This in turn proves that for our choice of  $\lambda$  and the assumption that  $(S, \lambda C_d)$  is not Kawamata log terminal at a point  $P$ ,  $m_0 \leq d - 2$  is not possible, since  $\lambda < \frac{2}{d-1}$ . Let us prove this proposition by the method of contradiction.

*Proof.* Suppose  $m_0 \leq d - 2$  and  $\text{ct}(S, C_d) < \frac{2}{d-1}$ . Let  $\mu = \frac{2}{d-1}$ . Then  $(S, \mu C_d)$  is not log canonical, in particular, is not Kawamata log terminal at a point, say  $P$ .

Let us now obtain the necessary contradiction.

**Lemma 3.14.** *The line  $L$  is not an irreducible component of the curve  $C_d$ .*

*Proof.* We shall prove this by contradiction. Suppose  $L$  is an irreducible component of the curve  $C_d$ . Then  $C_d = L + C_{d-1}$ , where  $C_{d-1}$  is a reduced curve of degree  $d - 1$  in  $\mathbb{P}^2$  and does not contain  $L$  as an irreducible component. Let  $f_1 : S_1 \rightarrow S$  be the blow-up at the point  $P$  in  $C_d$ . Let  $n_0 = \text{mult}_P(C_{d-1})$ .

Since  $(S, \mu C_{d-1} + \mu L)$  is not log canonical at  $P$ , we have that  $(S_1, \mu C_{d-1}^1 + \mu L^1 + (\mu(n_0 + 1) - 1)E_1)$  is not log canonical at some point in  $E_1$ . We choose this point to be  $P_1$ . Let  $n_1 = \text{mult}_{P_1}(C_{d-1}^1)$ .

Consider

$$d - 1 - n_0 = C_{d-1}^1 \cdot L^1 \geq n_1$$

which implies that  $n_0 + n_1 \leq d - 1$ . But  $n_0 = m_0 - 1 \leq d - 3$ , using our assumption.

Also,  $2n_1 \leq n_0 + n_1$  which implies  $2n_1 \leq d - 1$ . We can then conclude that  $\mu n_1 \leq 1$ . We also have  $L^1$  and  $E_1$  smooth at  $P_1$  and intersecting transversally at  $P_1$ . Thus applying Theorem 2.9, we get

$$\mu(d - 1 - n_0) = \mu C_{d-1}^1 \cdot L^1 > 2(2 - \mu(n_0 + 1)) \quad (3.4)$$

which implies that  $\mu(d - 1) > 2$  or

$$\mu n_0 = \mu C_{d-1}^1 \cdot E_1 > 2(1 - \mu) \quad (3.5)$$

which implies that  $\mu(n_0 + 2) > 2$ .

These two inequalities imply that  $\mu(d - 1) > 2$  which is absurd. Thus,  $L$  is not an irreducible component of  $C_d$ .  $\square$

Since  $L$  is not an irreducible component of the curve  $C_d$ , from the computations in (3.2) we can also assume

$$m_0 + m_1 \leq d.$$

Since  $(S, \mu C_d)$  is not log canonical at the point  $P$  and since  $\mu < 1$ , we have that  $(S_1, \mu C_d^1 + (\mu m_0 - 1)E_1)$  is not log canonical at some point in  $E_1$ , say  $P_1$ .

We also have that,

$$\mu m_0 - 1 \leq \mu(d - 2) - 1 = \frac{2}{d-1}(d - 2) - 1 = \frac{d-3}{d-1} < 1,$$

for  $d \geq 5$ . Thus, from Lemma 2.8 there exists a unique point in  $E_2$ , say  $P_2$  such that  $(S_2, \mu C_d^2 + (\mu m_0 - 1)E_1^2 + (\mu(m_0 + m_1) - 2)E_2)$  is not log canonical at  $P_2$ .

We know  $P_2 \in C_d^2$  since, if not, it would imply  $(S_2, (\mu m_0 - 1)E_1^2 + (\mu(m_0 + m_1) - 2)E_2)$  is not log canonical at the point  $P_2$ . This is not possible since

$$\begin{aligned}\mu m_0 - 1 &< 1, \\ \mu(m_0 + m_1) - 2 &\leq \mu d - 2 < 1,\end{aligned}$$

and  $E_1^2, E_2$  are *simple normal crossing* divisors at  $P_2$ .

**Lemma 3.15.**  $P_2 \notin E_1^2$

*Proof.* Suppose  $P_2 \in E_1^2$ .

Observe that

$$C_d^2 \sim f_2^*(C_d^1) - m_1 E_2$$

so that

$$C_d^2 \cdot E_1^2 = C_d^1 \cdot E_1 - m_1 E_2 \cdot E_1^2 = m_0 - m_1$$

Also, since  $P_2 \in C_d^2 \cap E_1^2 \cap E_2$  we have

$$m_2 = \text{mult}_{P_2}(C_d^2) \leq (C_d^2 \cdot E_1^2)$$

Therefore,  $m_2 \leq m_0 - m_1$ . Since  $m_2 \leq m_1$ , we have

$$2m_2 \leq m_1 + m_2 \leq m_0$$

which implies

$$m_2 \leq \frac{m_0}{2}$$

From Lemma 3.9(5), we have

$$\mu m_2 \leq \mu \frac{m_0}{2} \leq \mu \frac{d-2}{2} < 1$$

We also know that  $E_1^2$  and  $E_2$  are smooth at  $P_2$  and intersect transversally at  $P_2$ .

Since  $(S_2, \mu C_d^2 + (\mu m_0 - 1)E_1^2 + (\mu(m_0 + m_1) - 2)E_2)$  is not log canonical at the point  $P_2$ , Theorem 2.9 implies,

$$\mu(m_0 - m_1) = \mu C_d^2 \cdot E_1^2 > 2(3 - \mu(m_0 + m_1))$$

or

$$\lambda m_1 = \lambda C_d^2 \cdot E_2 > 2(2 - \lambda m_0)$$

That is,

$$\mu(3m_0 + m_1) > 6 \text{ or } \mu(2m_0 + m_1) > 4$$

We have

$$3m_0 + m_1 = 2m_0 + m_0 + m_1 \leq 2(d-2) + d = 3d-4, \quad (3.6)$$

$$2m_0 + m_1 \leq d-2 + d = 2d-2 \quad (3.7)$$

Since the above mentioned inequalities obtained from using Theorem 2.9 result in contradiction, it proves our claim that  $P_2 \notin E_1^2$ .  $\square$

**Lemma 3.16.**  $P_2 \notin L^2$

*Proof.* Suppose  $P_2 \in L^2$ . Since  $L$  is not an irreducible component of  $C_d$ , we have

$$d - m_0 - m_1 = L^2.E_2 \geq m_2$$

and this implies that  $m_0 + m_1 + m_2 \leq d$ . Also, applying Lemma 2.2 we get,

$$\mu d \geq \mu(m_0 + m_1 + m_2) > 3$$

which results in a contradiction since  $\mu d < 3$ . Thus  $P_2 \neq L^2 \cap E_2$ .  $\square$

Thus, we have

$$(S_2, \mu C_d^2 + (\mu(m_0 + m_1) - 2)E_2)$$

not log canonical at the point  $P_2$ . Then from Remark 2.1,

$$(S_3, \mu C_d^3 + (\mu(m_0 + m_1) - 2)E_2^3 + (\mu(m_0 + m_1 + m_2) - 3)E_3)$$

is not log canonical at some point in  $E_3$ , say  $P_3$ .

We have

$$2m_1 \leq m_0 + m_1 \leq d,$$

and

$$\mu(m_0 + m_1 + m_2) \leq \mu(m_0 + 2m_1) \leq \mu(d - 2 + d) = 4. \quad (3.8)$$

Therefore

$$\mu(m_0 + m_1 + m_2) - 3 \leq 1.$$

$P_3 \in C_d^3$  since if not, then this would imply that

$$(S_3, (\mu(m_0 + m_1) - 2)E_2^3 + (\mu(m_0 + m_1 + m_2) - 3)E_3)$$

is not log canonical at the point  $P_3$ . But since the coefficients of the  $E_i \leq 1$  and  $E_2^3, E_3$  are *simple normal crossing* divisors over the point  $P_3$ , this is not possible.

**Lemma 3.17.**  $P_3 \notin E_2^3$

*Proof.* Suppose  $P_3 \in E_2^3$ . Observe that

$$C_d^3 \sim_{\mathbb{Q}} f_3^*(C_d^2) - m_2 E_3$$

We thus have

$$C_d^3.E_2^3 = f_3^*(C_d^2.E_2) - m_2(E_3.E_2^3) = m_1 - m_2$$

Therefore Theorem 2.3 implies

$$\mu(m_1 - m_2) + (\mu(m_0 + m_1 + m_2) - 3) = (\mu C_d^3 + (\mu(m_0 + m_1 + m_2) - 3)E_3).E_2^3 > 1,$$

which implies

$$\mu(m_0 + 2m_1) > 4.$$

From (3.8) we know that  $\mu(m_0 + 2m_1) \leq 4$ . Thus this contradiction proves our claim.  $\square$

Therefore, the log pair

$$(S_3, \mu C_d^3 + (\mu(m_0 + m_1 + m_2) - 3)E_3)$$

is not log canonical, at the point  $P_3$ . Thus from Remark 2.1,

$$(S_4, \mu C_d^4 + (\mu(m_0 + m_1 + m_2) - 3)E_3^4 + (\mu(m_0 + m_1 + m_2 + m_3) - 4)E_4)$$

is not log canonical at a point  $P_4 \in E_4$ , where  $m_3 = \text{mult}_{P_3}(C_d^3)$ . We have

$$m_2 + m_3 \leq 2m_2 \leq m_0 + m_1 \leq d.$$

Thus,

$$\mu(m_0 + m_1 + m_2 + m_3) \leq \mu(2(m_0 + m_1)) \leq \frac{2}{d-1}2d < 5. \quad (3.9)$$

**Lemma 3.18.**  $P_4 \notin E_3^4$

*Proof.* Suppose  $P_4 \in E_3^4$ . From the inequality (3.9), we know  $m_0 + m_1 + 2m_2 < \frac{5}{\mu}$ . Also, observe that

$$C_d^4 \sim_{\mathbb{Q}} f_4^*(C_d^3) - m_3 E_4$$

so that

$$C_d^4 \cdot E_3^4 = m_2 - m_3.$$

Then from Theorem 2.3 we have

$$(\mu C_d^4 + (\mu(m_0 + m_1 + m_2 + m_3) - 4)E_4) \cdot E_3^4 > 1,$$

which implies

$$m_0 + m_1 + 2m_2 > \frac{5}{\mu}.$$

This contradicts the inequality in (3.9). Thus  $P_4 \notin E_3^4$ .  $\square$

Thus,  $(S_4, \mu C_d^4 + (\mu(m_0 + m_1 + m_2 + m_3) - 4)E_4)$  is not log canonical at the point  $P_4$ . From Theorem 2.3, we have

$$\mu \text{mult}_{P_4}(C_d^4) + (\mu(m_0 + m_1 + m_2 + m_3) - 4) > 1$$

which implies

$$\text{mult}_{Q_4}(C_d^4) + m_0 + m_1 + m_2 + m_3 > \frac{5}{\mu}. \quad (3.10)$$

Now using (3.10) and a geometric construction of a special curve in  $S_4$  we will try to arrive at a contradiction. We may assume that the line  $L$  is given by  $x = 0$ ,  $P = [0 : 0 : 1]$ . Let  $\mathcal{C}$  be the conic in  $\mathbb{P}^2$  that is given by

$$xz + Axy + By^2 = 0,$$

where  $A, B \in \mathbb{C}$  and  $B \neq 0$ . Then  $\mathcal{C}$  is smooth and is tangent to the line  $L$ . Denote the proper transform of  $\mathcal{C}$  in  $S^i$  by  $C^i$ . It follows from Lemma 3.15, Lemma 3.16, Lemma 3.17, Lemma 3.18, that there exists  $A$  and  $B \neq 0$  such that  $C^i$  on  $S^i$

contains  $P_i$  for  $i = 1, 2, 3$ . So we can assume that  $A, B$  are chosen this way. Then we have

$$\mathcal{C}^4 \sim 2L^4 + E_1^4 + 2E_2^4 + E_3^4.$$

Thus this pencil  $|\mathcal{C}^4|$  does not have base points.

Also, let  $\mathcal{L}$  be a pencil of conics in  $\mathbb{P}^2$  given by the following

$$sx^2 + t(xz + Axy + By^2) = 0,$$

where  $s, t \in \mathbb{C}$ . This is generated by  $2L$  and  $\mathcal{C}$ .

Let  $\phi_{|\mathcal{C}^4|} : S^4 \rightarrow \mathbb{P}^1$  be the morphism defined by the pencil  $|\mathcal{C}^4|$ . Similarly, let  $\phi_{\mathcal{L}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  be the rational map defined by the pencil  $\mathcal{L}$ .

These make the following diagram commutative.

$$\begin{array}{ccccccc} \mathbb{P}^2 & \xleftarrow{f_1} & S_1 & \xleftarrow{f_2} & S_2 & \xleftarrow{f_3} & S_3 & \xleftarrow{f_4} & S_4 \\ & \searrow \phi_{\mathcal{L}} & & & & & & & \swarrow \phi_{|\mathcal{C}^4|} \\ & & & & & & & & \mathbb{P}^1 \end{array}$$

Choose a curve  $Z^4$  in  $|\mathcal{C}^4|$  that passes through the point  $P_4$ . Then  $Z^4$  is a smooth irreducible curve. Let the proper transform of  $Z^4$  in  $\mathbb{P}^2$  be denoted by  $Z$ . Thus  $Z$  is a smooth conic in the pencil  $\mathcal{L}$ . Suppose  $Z$  is not an irreducible component of the curve  $C_d$ , then we have

$$2d - (m_0 + m_1 + m_2 + m_3) = C_d^4 \cdot Z^4 \geq \text{mult}_{P_4}(C_d^4) \quad (3.11)$$

(3.10) and (3.11) result in a contradiction since  $\mu < \frac{5}{2d}$ .

Thus,  $C_d = Z + C_{d-2}$  where  $C_{d-2}$  is an irreducible curve of degree  $d-2$  which does not contain the conic  $Z$  as an irreducible component.

Let  $C_{d-2}^1, C_{d-2}^2, C_{d-2}^3, C_{d-2}^4$  be the proper transform of the curve  $C_{d-2}$  on the surfaces  $S_1, S_2, S_3$  and  $S_4$  respectively.

Denote by  $n_0 = \text{mult}_P(C_{d-2})$ ;  $n_1 = \text{mult}_{P_1}(C_{d-2}^1)$ ;  $n_2 = \text{mult}_{P_2}(C_{d-2}^2)$ ;  $n_3 = \text{mult}_{P_3}(C_{d-2}^3)$  and  $n_4 = \text{mult}_{P_4}(C_{d-2}^4)$ . Thus we have,

$$(S_4, \mu C_{d-2}^4 + \mu Z^4 + ((\mu(n_0 + n_1 + n_2 + n_3 + 4) - 4)E_4))$$

is not log canonical at  $P_4$ .

Thus, applying Theorem 2.3 to the above gives us

$$\mu(2(d-2) - n_0 - n_1 - n_2 - n_3) + \mu(n_0 + n_1 + n_2 + n_3 + 4) - 4 > 1$$

which implies

$$\mu > \frac{5}{2d}.$$

But  $\mu < \frac{5}{2d}$  and thus this contradiction proves the proposition.  $\square$

This in turn proves that  $m_0 \leq d-2$  is not possible for the chosen value of  $\lambda$ , hence completing the proof of Theorem 3.1.

# Chapter 4

## Surfaces

In this chapter, the main objects of study are del pezzo surfaces  $S_d$ , of degree  $d$  with at most du Val singular points. Recall that del Pezzo surfaces are Fano varieties of dimension 2, that is, these are varieties of dimension 2 whose anti-canonical divisor is ample. We are interested in determining the K-stability of these surfaces and also, in constructing  $H$ -polar cylinders for any ample divisor  $H$ . For a brief introduction about both these notions, see Section 1.2.1 and 1.2.2.

### 4.1 Applicability of Dervan's criterion

#### 4.1.1 Foundations

The applicability of Dervan's criterion (Section 1.2.1.1) for smooth del Pezzo surfaces, has been studied by Cheltsov and Martinez-Garcia in [CM20]. We briefly recall the results.

**Theorem 4.1.** [CM20, Theorem 1.1] *Let  $S$  be a smooth del Pezzo surface such that  $K_S^2 \leq 2$ , and let  $L$  be an ample  $\mathbb{Q}$ -divisor on  $S$ . If  $-K_S - \frac{2}{3} \frac{-K_S \cdot L}{L^2} L$  is nef, then  $(S, L)$  is K-stable.*

The following negative result is also proven in [CM20].

**Theorem 4.2.** [CM20, Theorem 1.5] *Let  $S$  be a smooth del Pezzo surface such that  $K_S^2 \geq 4$ , and let  $L$  be an ample  $\mathbb{Q}$ -divisor on  $S$ . Then  $\alpha(S, L) \leq \frac{2}{3} \frac{-K_S \cdot L}{L^2}$ . Moreover, the equality holds if and only if  $K_S^2 = 4$  and  $L \in \mathbb{Q}_{>0}[-K_S]$ .*

The following is a partial result for smooth cubic surfaces from [CM20].

**Theorem 4.3.** [CM20, Theorem 1.6] *Let  $S$  be a smooth cubic surface in  $\mathbb{P}^3$ , and let  $E_1, E_2, E_3, E_4, E_5, E_6$  be disjoint lines on the surface  $S$ . Let  $L = -K_S + x \sum_{i=1}^6 E_i$ , where  $x$  is a non-negative rational number such that  $x < 1$ . Then  $L$  is ample. Furthermore, if  $0 < x \leq \frac{1}{10}$ , then  $(S, L)$  is K-stable.*

Theorem 4.1, Theorem 4.2 and Theorem 4.3, have been proven using the Dervan's criterion, which we recall here (also Theorem 1.3).

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**Theorem 4.4.** [D15, Theorem 1.1] *Let  $X$  be a Fano variety of dimension  $n$ , and let  $L$  be an ample  $\mathbb{Q}$ -divisor on it. Let  $\nu(L) = \frac{-K_X \cdot L^{n-1}}{L^n}$ . Then  $(X, L)$  is  $K$ -stable if*

1. *the  $\mathbb{Q}$ -divisor  $-K_X - \frac{n}{n+1}\nu(L)L$  is nef,*
2. *and  $\alpha(X, L) > \frac{n}{n+1}\nu(L)$ .*

Note that if  $\nu(L) = 0$ , that is,  $-K_X \cdot L^{n-1} = 0$ , then the first condition requires that  $-K_X$  is nef. But from [D15, Remark 11], we know that  $-K_X$  cannot be nef and not be numerically zero: in other words,  $X$  has to be numerically either a Calabi-Yau or Fano.

In Section 4.1.2, we establish the applicability of the above criterion to singular del Pezzo surface of degree 1. In particular, we prove

**Theorem 4.5.** (Also Main Result 2) *Let  $S$  be a del Pezzo surface of degree 1 with  $A_1$ ,  $A_2$  or  $A_3$  singularities. Let  $L$  be any ample divisor and  $\nu(L) = \frac{-K_S \cdot L}{L^2}$ . If the  $\mathbb{Q}$ -divisor  $-K_S - \frac{2}{3}\nu(L)L$  is nef, then  $(S, L)$  is  $K$ -stable.*

## 4.1.2 Proof of Main Result 2

In this section, we prove Theorem 4.5.

Let  $S$  be a del Pezzo surface of degree 1 with singular points of type  $A_1$ ,  $A_2$  or  $A_3$ . Let  $L$  be an ample divisor on  $S$  such that  $-K_S - \frac{2}{3}\nu(L)L$  is nef, where  $\nu(L) = \frac{-K_S \cdot L}{L^2}$  is the slope of  $L$ .

Let us now scale  $L$  by  $\frac{2}{3}\frac{-K_S \cdot L}{L^2}L$ . Scaling  $L$  will result in the slope,  $\nu(L)$  being scaled accordingly, that is, after scaling,  $\nu(L) = \frac{-K_S \cdot L}{L^2} = \frac{3}{2}$ .

This can be written as

$$(L - \frac{2}{3}(-K_S)).L = 0 \tag{4.1}$$

Let  $B \equiv -K_S - L$  and  $\epsilon = -K_S \cdot B$ . Since  $B \not\sim_{\mathbb{Q}} 0$ , we have  $\epsilon > 0$ . We now need to prove that  $\alpha(S, L) > 1$ .

We will briefly describe the method used to prove this. First, for the various singular del Pezzo surfaces  $S$ , we choose a  $\delta > 1$  and prove that  $\alpha(S, L) \geq \delta > 1$  by contradiction. That is, suppose  $\alpha(S, L) < \delta$ . Then  $\exists \lambda < \delta$  and  $D \sim_{\mathbb{Q}} L$  such that  $(S, \lambda D)$  is not log canonical at some point  $P \in S$ . Then we get a contradiction to this implication, which would in turn imply that  $\alpha(S, L) \geq \delta > 1$ .

Let

$$\delta = \begin{cases} \frac{3}{3-\epsilon}; & \text{if } \epsilon < 1 \\ \frac{11}{10}; & \text{if } \epsilon \geq 1. \end{cases} \tag{4.2}$$

Let  $D = aC + \Delta$ , where  $a$  is a non-negative rational number,  $C$  is a curve in the pencil  $| -K_S |$  that passes through the point  $P$ . Let  $\Delta = \sum_j c_j C_j$  be an effective  $\mathbb{Q}$ -divisor whose support does not contain the curve  $C$ . Thus we have that  $(S, \lambda aC + \lambda \Delta)$  is not log canonical at  $P \in C$ .

**Lemma 4.6.**  $L \not\sim_{\mathbb{Q}} \frac{2}{3}(-K_S)$

*Proof.* If  $L \sim_{\mathbb{Q}} \frac{2}{3}(-K_S)$ , then consider

$$\alpha(S, L) = \alpha\left(S, \frac{2}{3}(-K_S)\right) = \frac{3}{2}\alpha(S, -K_S) \geq \frac{3}{2} \cdot \frac{3}{4} = \frac{9}{8}$$

([CK14, Example 1.27]) and this is a contradiction since  $\alpha(S, L) < \delta \leq \frac{11}{10} < \frac{9}{8}$  from our assumption.  $\square$

**Lemma 4.7.**  $a < 2/3$

*Proof.* Suppose  $a \geq \frac{2}{3}$ , then

$$0 = L \cdot (L - \frac{2}{3}(-K_S)) \equiv L \cdot (D - \frac{2}{3}(-K_S)) = L \cdot ((a - \frac{2}{3})C + \Delta) \geq (a - \frac{2}{3})L \cdot C$$

Since  $L \cdot C > 0$ ,  $a = \frac{2}{3}$  since if otherwise it contradicts the above inequality and since  $L$  is ample,  $L \cdot \Delta \geq 0$  and this implies that  $\Delta = 0$ . This contradicts Lemma 4.6.  $\square$

#### 4.1.2.1 Smooth point of surface

Suppose this point  $P$  on  $C$  is not a point of singularity of the surface. Then the curve  $C$  can either be smooth at  $P$  or singular at  $P$ .

Let us consider each one of these cases.

**$C$  smooth at  $P$ :** Since  $C$  is smooth at  $P$  and  $\lambda a < \frac{9}{8} \frac{2}{3} < 1$ , Corollary 2.7 gives

$$\frac{1}{\lambda} < C \cdot \Delta = C \cdot (D - aC) = 1 - B \cdot C - a < 1 - B \cdot C = 1 - \epsilon.$$

This implies that  $\delta > \lambda > \frac{1}{1-\epsilon} > \frac{3}{3-\epsilon}$  which is a contradiction to our initial assumption and (4.2). Therefore this is not possible.

**$C$  singular at  $P$ :** Suppose  $C$  is singular at  $P$ . Since  $C \in |-K_S|$ , this implies that  $C$  is either a cuspidal cubic or a nodal cubic curve and therefore,  $\text{mult}_P C = 2$ . Let  $f : S^1 \rightarrow S$  be the blowup at  $P$  and  $E_1$  be the exceptional divisor. Let  $C^1$  and  $\Delta^1$  be the proper transforms of  $C$  and  $\Delta$  respectively in  $S^1$ . Let  $m = \text{mult}_P(\Delta)$ .

Also,

$$\begin{aligned} C^1 &= f^*(C) - 2E_1 \\ \Delta^1 &= f^*(\Delta) - mE_1 \\ -K_{S^1} &= f^*(-K_S) - E_1. \end{aligned}$$

We can estimate a bound for the value of  $m$  using these equations, as given below.

$$0 \leq \Delta^1 \cdot C^1 = \Delta \cdot C - 2m = 1 - \epsilon - a - 2m \tag{4.3}$$

Since  $(S, \lambda aC + \lambda \Delta)$  is not log canonical at  $P$ ,  $(S^1, \lambda aC^1 + \lambda \Delta^1 + (\lambda(2a + m) - 1)E_1)$  is not log canonical at some point  $P_1 \in E_1$ , by Remark 2.1.

**Lemma 4.8.**  $\lambda(2a + m) - 1 < 1$

*Proof.* Since  $a < \frac{2}{3}$ , and  $a + 2m < 1$  from (4.3),  $2a + m < \frac{3}{2}$  and therefore,  $\lambda(2a + m) - 1 < 1$ .  $\square$

- $P_1 \in C^1$ . Suppose  $P_1 \notin C^1$ . Since  $E_1$  is smooth at  $P_1$ , and  $\lambda(2a + m) - 1 < 1$ , by Corollary 2.7 we have,

$$\frac{1}{2} \geq \frac{1-a}{2} \geq m = (\Delta^1 \cdot E_1) > \frac{1}{\lambda} > \frac{8}{9}$$

from (4.3) and (4.2). This contradiction proves the claim.

This implies that  $P_1 \in C^1 \cap E_1$  is the only possibility.

**Remark 4.9.** (1)  $\lambda < \frac{9}{8}$  and  $a < \frac{2}{3}$  which implies  $\lambda a < 1$ . (2)  $C^1$  is smooth at  $P$  and  $C^1 \cdot E_1 = 2$ .

- $P_1 \notin C^1 \cap E_1$ . Suppose  $P_1 \in C^1 \cap E_1$ . Then Corollary 2.7 implies,

$$\begin{aligned} 1 < C^1 \cdot (\lambda \Delta^1 + \lambda(2a + m) - 1)E_1 &= \lambda(\Delta \cdot C - 2m) + 2\lambda(2a + m) - 2 \\ &= \lambda(1 - \epsilon - a - 2m + 4a + 2m) - 2 \\ &= \lambda(1 - \epsilon + 3a) - 2 \end{aligned}$$

Since  $\epsilon > 0$  and  $a < \frac{2}{3}$ ,  $3 < \lambda(1 + 3a - \epsilon) < \lambda(3 - \epsilon) < 3$  which is absurd, therefore proving our claim.

This contradiction proves that such a curve  $C$  is not possible, when  $P$  is a smooth point of the surface.

We will now look at the different singular points.

#### 4.1.2.2 $A_1$ singularity

Suppose  $S$  has at least 1 singular point of type  $A_1$  and  $P$  is the point of  $A_1$  singularity of  $S$ . Suppose  $C$  is singular at the point  $P$ . Then  $C$  could be one of the following.

1.  $C$  is a nodal curve with node at  $P$ .
2.  $C$  is a cuspidal curve with cusp at  $P$ .

We will now look at both these cases and prove that neither is possible.

Let  $C$  be any one of the above curves and let  $f : S^1 \rightarrow S$ , be the blow up of the point  $P$  which is a point of singularity of type  $A_1$  of  $S$ , and is the singular point of the curve  $C$  in  $|-K_S|$ . Since it is of  $A_1$  type, we have

$$K_{S^1} = f^*(K_S), \quad (4.4)$$

$$C^1 = f^*(C) - E_1, \quad (4.5)$$

$$\Delta^1 = f^*(\Delta) - m_1 E_1, \quad (4.6)$$

where  $m_1 \in \mathbb{Q}_{\geq 0}$ . From the above we can estimate  $m_1$ ,

$$\begin{aligned}
0 &\leq \Delta^1.C^1 = \Delta.C - 2m_1, \\
\therefore m_1 &\leq \frac{\Delta.C}{2} = \frac{1-a-\epsilon}{2}.
\end{aligned} \tag{4.7}$$

Note that the log pullback is given by

$$K_{S^1} + \lambda a C^1 + \lambda \Delta^1 + \lambda(a + m_1)E_1 = f^*(K_S + \lambda a C + \lambda \Delta).$$

Since  $(S, \lambda a C + \lambda \Delta)$  is not log canonical at  $P$ ,  $(S^1, \lambda a C^1 + \lambda \Delta^1 + \lambda(a + m_1)E_1)$  is not log canonical at some point  $P_1 \in E_1$ , by Remark 2.1.

**Lemma 4.10.**  $\lambda(a + m_1) < 1$

*Proof.* Using (4.7) and (4.2), we have

$$\lambda(a + m_1) < \lambda \left( a + \frac{1-a}{2} \right) = \lambda \left( \frac{1+a}{2} \right) < \lambda \left( \frac{5}{6} \right) < 1.$$

□

- $P_1 \in C^1$ . Suppose  $P_1 \in E_1$  but  $P_1 \notin C^1$ . Since  $E_1$  is smooth at  $P_1$  and  $\lambda(a + m_1) < 1$ , from Corollary 2.7 we have,

$$1 - \epsilon > 2m_1 = \Delta^1.E_1 > \frac{1}{\lambda}.$$

We get  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  which is a contradiction by (4.2). Therefore,  $P_1 \in C^1$ .

- $P_1 \notin C^1 \cap E_1$ . We will consider the different possibilities of the curve  $C$ .

**Case 1:** Suppose  $C$  is the nodal curve with node at  $P$ . We have shown above that if the point  $P_1$  exists, then it is such that  $P_1 \in C^1 \cap E_1$ . We will now prove that this is not possible.

Suppose  $P_1 \in C^1 \cap E_1$ . Since  $E_1$  is smooth at  $P_1$  and  $\lambda(a + m_1) < 1$ , applying Corollary 2.7, we get,

$$((aC^1 + \Delta^1).E_1)_{P_1} > \frac{1}{\lambda}$$

But,

$$\frac{1}{\lambda} < ((aC^1 + \Delta^1).E_1)_{P_1} = a + 2m_1 \leq a + 1 - a - \epsilon = 1 - \epsilon$$

since  $\epsilon > 0$ . This again contradicts (4.2), thus proving our claim.

This contradiction proves that such a  $P_1$  does not exist, which contradicts our assumption that  $(S, \lambda a C + \lambda \Delta)$  is not log canonical at the point  $P$ . Therefore  $C$  cannot be a nodal curve with node at  $P$ .

- **Case 2.** Suppose  $C$  is the cuspidal curve  $C$  in  $|-K_X|$  and  $P_1 \in C^1 \cap E_1$ .

Let  $f_2 : S^2 \longrightarrow S^1$  be the blow up of  $S^1$  at the point  $P_1$ . We get that

$$C^2 = \pi^*(C) - E_1^2 - 2E_2, \quad (4.8)$$

$$\Delta^2 = \pi^*(\Delta) - m_1 E_1^2 - (m_1 + m_2)E_2, \quad (4.9)$$

$$-K_{S^2} = \pi^*(K_S) - E_2, \quad (4.10)$$

where  $\pi = f_1 \circ f_2$ ,  $m_1, m_2 \in \mathbb{Q}_{\geq 0}$ .

Now we have that the log pair,

$$(S^2, \lambda a C^2 + \lambda \Delta^2 + \lambda(a + m_1)E_1^2 + (\lambda(2a + m_1 + m_2) - 1)E_2),$$

is not log canonical at some point  $P_2 \in E_2$ , by Remark 2.1.

Note that we have  $\lambda a < 1$ ,  $\lambda(a + m_1) < 1$  and

$$\begin{aligned} 0 &\leq \Delta^2.E_1^2 = 2m_1 - m_2, \\ 0 &\leq \Delta^2.C^2 = 1 - \epsilon - a - 2m_1 - m_2. \end{aligned}$$

We will use these inequalities in the results that follow.

**Lemma 4.11.**  $\lambda(2a + m_1 + m_2) - 1 < 1$

*Proof.*

$$\lambda(2a + m_1 + m_2) \leq \lambda(2a + 1 - a - \epsilon - m_1) < \lambda(1 + a - \epsilon) < \lambda\left(\frac{5 - 3\epsilon}{3}\right) < 2.$$

□

Now we will look at the various possible positions of the point  $P_2$ .

- $P_2 \in E_2 \cap E_1^2 \cap C^2$ . Suppose  $P_2 \notin E_1^2 \cup C^2$ . Since  $E_1^2$  and  $C^2$  intersect  $E_2$  at the same point, this would then imply that  $P_2 \notin E_1^2$  and  $P_2 \notin C^2$ . Then, since  $E_2$  is smooth at  $P_2$  and  $\lambda(2a + m_1 + m_2 - 1) < 1$ , Corollary 2.7 gives the following:

$$\frac{1}{\lambda} < \Delta^2.E_2 = m_2 \leq 2m_1 \leq 1 - \epsilon - a - m_2 < 1 - \epsilon.$$

This contradicts (4.2) since the above argument gives  $\lambda > \frac{1}{1-\epsilon} > \frac{3}{3-\epsilon}$ . Thus  $P_2 \in E_1^2 \cap E_2 \cap C^2$ .

Let  $f_3 : S^3 \longrightarrow S^2$  be the blow up of  $S^2$  at the point  $P_2$ . We have

$$C^3 = \psi^*(C) - E_1^3 - 2E_2^3 - 4E_3,$$

$$\Delta^3 = \psi^*(\Delta) - m_1 E_1^3 - (m_1 + m_2)E_2^3 - (2m_1 + m_2 + m_3)E_3,$$

$$-K_{S^3} = \psi^*(-K_S) - E_2^3 - 2E_3,$$

where  $\psi = f_1 \circ f_2 \circ f_3$ ,  $m_1, m_2, m_3 \in \mathbb{Q}_{\geq 0}$ .

Then

$$(S^3, \lambda a C^3 + \lambda \Delta^3 + \lambda(a + m_1)E_1^3 + (\lambda(2a + m_1 + m_2) - 1)E_2^3 + (\lambda(4a + 2m_1 + m_2 + m_3) - 2)E_3)$$

is not log canonical at some point  $P_3 \in E_3$ .

Note that we have the following inequalities

$$\begin{aligned} 0 &\leq \Delta^3.C^3 = 1 - a - \epsilon - 2m_1 - m_2 - m_3, \\ 0 &\leq \Delta^3.E_1^3 = 2m_1 - m_2 - m_3, \\ 0 &\leq \Delta^3.E_2^3 = m_2 - m_3, \\ 0 &\leq \Delta^3.E_3 = m_3. \end{aligned} \tag{4.11}$$

**Lemma 4.12.**  $\lambda(4a + 2m_1 + m_2 + m_3) - 2 < 1$

*Proof.* Using (4.11) we get,

$$\lambda(4a + 2m_1 + m_2 + m_3) \leq \lambda(1 + 3a - \epsilon) < \lambda(3 - \epsilon) < 3.$$

□

We now work through all the various possibilities for the position of the point  $P_3$ .

–  $P_3 \in E_1^3 \cup E_2^3 \cup C^3$ . Suppose  $P_3 \in E_3$  and  $P_3 \notin E_1^3 \cup E_2^3 \cup C^3$ . Since  $E_3$  is smooth at  $P_3$  and  $\lambda(4a + 2m_1 + m_2 + m_3) - 2 < 1$ , using Corollary 2.7, we have

$$\frac{1}{\lambda} < \Delta^3.E_3 = m_3 \leq 1 - a - \epsilon - 2m_1 - m_2 < 1 - \epsilon$$

This implies  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$ . This contradicts (4.2). This proves that  $P_3 \in E_1^3 \cup E_2^3 \cup C^3$ .

–  $P_3 \notin E_2^3$ . Suppose  $P_3 \in E_2^3 \cap E_3$ .

$$\begin{aligned} 1 &< E_3.((\lambda(2a + m_1 + m_2) - 1)E_2^3 + \lambda\Delta^3) = \lambda(2a + m_1 + m_2 + m_3) - 1 \\ &\leq \lambda(1 + a - \epsilon - m_1) - 1 \\ &< \lambda\left(\frac{5 - \epsilon}{3}\right) - 1. \end{aligned}$$

This implies that  $\frac{3}{3-\epsilon} < \frac{6}{5-3\epsilon} < \lambda$ , which contradicts (4.2).

–  $P_3 \notin E_1^3$ . Suppose  $P_3 \in E_1^3 \cap E_3$ . We have that  $(S^3, \lambda\Delta^3 + \lambda(a + m_1)E_1^3 + (\lambda(4a + 2m_1 + m_2 + m_3) - 2)E_3)$  is not log canonical at a point  $P_3 \in E_3$ . Since  $\lambda(4a + 2m_1 + m_2 + m_3) - 2 < 1$ , Corollary 2.7 gives

$$\frac{1}{\lambda} < E_3.((a + m_1)E_1^3 + \Delta^3) = a + m_1 + m_3 < 1 - \epsilon - m_1 - m_2 < 1 - \epsilon.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$ , which contradicts (4.2).

–  $P_3 \notin C^3$ . Suppose  $P_3 \in C^3 \cap E_3$ . We then have

$$(S^3, \lambda a C^3 + \lambda \Delta^3 + (\lambda(4a + 2m_1 + m_2 + m_3) - 2)E_3)$$

not log canonical at  $P_3$ . Again since  $\lambda(4a + 2m_1 + m_2 + m_3) - 2 < 1$ , Corollary 2.7 implies that,

$$\frac{1}{\lambda} < E_3.(aC^3 + \Delta^3) \leq (a + m_3) \leq 1 - \epsilon - 2m_1 - m_2 < 1 - \epsilon.$$

This contradicts (4.2) since the above inequality implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$ . This proves that  $P_3 \notin C^3$ .

From the above claims, we can now conclude that a point such as  $P_3$  does not exist. This contradicts our initial assumption that  $(S, \lambda a C + \lambda \Delta)$  is not log canonical at the point  $P$  and hence  $C$  cannot be this curve either.

Thus, in this subsection we have proved that if the surface  $S$  has  $A_1$  singularity, then the point  $P$  at which the log pair  $(S, \lambda D)$  is not log canonical is not the singular point of type  $A_1$ . Hence, if the surface  $S$  has only singular points of type  $A_1$ , then this completes the proof by contradiction. Thus, in this case, we have now proven that  $\alpha(S, D) > \delta > 1$ .

#### 4.1.2.3 $A_2$ singularity

Suppose  $S$  has at least 1 singular point of type  $A_2$  and  $P$  is the singular point of type  $A_2$ . Then  $C$  can be one of the following.

1.  $C$  is a nodal curve with node at  $P$ ,
2.  $C$  is a cuspidal curve with cusp at  $P$ .

We have that  $(S, \lambda D)$  is not log canonical at the point  $P$ . Recall that  $D = aC + \Delta$  where  $C \in |-K_S|$ . So if  $C$  is a cuspidal curve with cusp at  $P$ , then

$$\alpha(S, D) \leq \alpha(S, -K_S) = \frac{2}{3},$$

by [P03, Theorem 3.4]. Therefore, in this case, Dervan's criterion is no longer applicable. Hence we only look at  $C$  which is a nodal curve with node at  $P$ .

Let  $f_1 : S^1 \rightarrow S$  be the blow-up of  $S$  at the point  $P$ . From Remark 2.1, we have that

$$(S^1, \lambda a C^1 + \lambda \Delta^1 + \lambda(a + m_1)E_1 + \lambda(a + m_2)E_2)$$

is not log canonical at some point  $P_1 \in E_1 \cup E_2$ . Here,  $m_1$  and  $m_2$  are given by the following:

$$\begin{aligned} C^1 &= f_1^*(C) - E_1 - E_2, \\ \Delta^1 &= f_1^*(\Delta) - m_1 E_1 - m_2 E_2 \end{aligned} \tag{4.12}$$

where  $m_1, m_2 \in \mathbb{Q}_{\geq 0}$ . Thus, we have

$$\begin{aligned} 0 &\leq \Delta^1.C^1 = \Delta.C - m_1 - m_2 = 1 - a - \epsilon - m_1 - m_2, \\ 0 &\leq \Delta^1.E_1 = 2m_1 - m_2, \\ 0 &\leq \Delta^1.E_2 = 2m_2 - m_1. \end{aligned} \tag{4.13}$$

**Lemma 4.13.**  $\lambda(a + m_1) < 1$

*Proof.*  $\lambda(a + m_1) \leq \lambda(1 - \epsilon - m_2) < \lambda(1 - \epsilon) < \frac{3}{3-\epsilon}(1 - \epsilon) < 1$  from (4.13) and (4.2).  $\square$

Similar argument proves the following.

**Lemma 4.14.**  $\lambda(a + m_2) < 1$

We will now look at the various possibilities for the position of  $P_1$ .

- $\underline{P_1 \in C^1 \cup E_2}$ . Suppose not, i.e. suppose  $P_1 \in E_1$  and  $P_1 \notin C^1$  and  $P_1 \notin E_2$ . Then, we have that  $(S^1, \lambda\Delta^1 + \lambda(a + m_1)E_1)$  is not log canonical at the point  $P_1$ . Since  $E_1$  is smooth at  $P_1$  and  $\lambda(a + m_1) < 1$  from Lemma 4.13, Corollary 2.7 implies that

$$\frac{1}{\lambda} < \Delta^1.E_1 = 2m_1 - m_2 \leq m_1 + m_2 \leq 1 - a - \epsilon < 1 - \epsilon$$

from (4.13). This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  and this contradicts (4.2). Thus, this proves our claim that  $P_1 \in C^1 \cup E_2$ .

- $\underline{P_1 \in C^1 \cup E_1}$ . Suppose  $P_1 \in E_2$  and  $P_1 \notin C^1$  and  $P_1 \notin E_1$ . Similar argument as above leads to a contradiction.

But we also know that  $P_1 \in E_1 \cup E_2$ . The following now proves that  $P_1 \notin E_1 \cap E_2$ .

- $\underline{P_1 \in C^1}$ . Suppose  $P_1 \notin C^1$ . This would imply that  $P_1 \in E_1 \cap E_2$  and therefore

$$(S^1, \lambda\Delta^1 + \lambda(a + m_1)E_1 + \lambda(a + m_2)E_2)$$

is not log canonical at the point  $P_1$ , by Remark 2.1. Let  $f_2 : S^2 \rightarrow S^1$  be the blow up of the point  $P_1$ , with the exceptional divisor given by  $E_3$ . We then have

$$\begin{aligned} C^2 &= \pi^*(C) - E_1^2 - E_2^2 - 2E_3, \\ \Delta^2 &= \pi^*(\Delta) - m_1E_1^2 - m_2E_2^2 - (m_1 + m_2 + m_3)E_3, \\ -K_{S^2} &= \pi^*(-K_S) - E_3, \end{aligned} \tag{4.14}$$

where  $\pi := f_1 \circ f_2$ ,  $m_1, m_2, m_3 \in \mathbb{Q}_{\geq 0}$ . With this, we have the following inequalities.

$$\begin{aligned} 0 &\leq \Delta^2.C^2 = 1 - a - \epsilon - m_1 - m_2, \\ 0 &\leq \Delta^2.E_1^2 = 2m_1 - m_2 - m_3, \\ 0 &\leq \Delta^2.E_2^2 = 2m_2 - m_1 - m_3. \end{aligned} \tag{4.15}$$

This implies that

$$(S^2, \lambda a C^2 + \lambda \Delta^2 + \lambda(a + m_1)E_1^2 + \lambda(a + m_2)E_2^2 + (\lambda(2a + m_1 + m_2 + m_3) - 1)E_3)$$

is not log canonical at some point  $P_2 \in E_3$ , by Remark 2.1.

**Lemma 4.15.**  $\lambda(2a + m_1 + m_2 + m_3) - 1 < 1$

*Proof.*  $\lambda(2a + m_1 + m_2 + m_3) \leq \lambda(2a + 3 - 3a - 3\epsilon - 3m_1) < 3\lambda(1 - \epsilon) < 2$ .  $\square$

We will look at different positions of  $P_2$ .

–  $P_2 \notin E_1^2$  Suppose not. That is, suppose  $P_2 \in E_1^2 \cap E_3$ . Since  $\lambda(a + m_1) < 1$ , Corollary 2.7 gives

$$1 < E_1^2 \cdot (\lambda\Delta^2 + (\lambda(2a + m_1 + m_2 + m_3) - 1)E_3) < \lambda(3m_1 + 2a) - 1 < 3\lambda(1 - \epsilon) - 1.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{2}{3(1-\epsilon)} < \lambda$ , which contradicts (4.2).

–  $P_2 \notin E_2^2$ : Similar argument as above shows this.

–  $P_2 \notin E_3 \setminus (E_1^2 \cup E_2^2)$ : Suppose  $P_2 \in E_3 \setminus (E_1^2 \cup E_2^2)$ . Then,  $(S^2, \lambda\Delta^2 + (\lambda(2a + m_1 + m_2 + m_3) - 1)E_3)$  is not log canonical at  $P_2$ . Using Corollary 2.7, we get

$$\frac{1}{\lambda} < E_3 \cdot \Delta^2 = m_3 \leq 2m_2 - m_1 \leq m_2 + m_1 - m_3 \leq 1 - a - \epsilon - m_3 < 1 - \epsilon.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  which is a contradiction.

•  $P_1 \notin C^1 \cap E_2$ : Suppose  $P_1 \in C^1 \cap E_2$ . Then  $(S^1, \lambda C^1 + \lambda\Delta^1 + \lambda(a + m_2)E_2)$  is not log canonical at  $P_1$ . Let  $f_2 : S^2 \rightarrow S^1$  be the blow up of  $P_1$  with the exceptional divisor given by  $E_3$ . Then we have

$$\begin{aligned} C^2 &= \pi^*(C) - E_1^2 - E_2^2 - 2E_3, \\ \Delta^2 &= \pi^*(\Delta) - m_1 E_1^2 - m_2 E_2^2 - (m_2 + m_3)E_3, \\ -K_{S^2} &= \pi^*(-K_S) - E_3, \end{aligned} \tag{4.16}$$

where  $\pi := f_1 \circ f_2$ ,  $m_1, m_2, m_3 \in \mathbb{Q}_{\geq 0}$ . Using this, we have the following inequalities.

$$\begin{aligned} 0 &\leq \Delta^2 \cdot C^2 = 1 - a - \epsilon - m_1 - m_2 - m_3, \\ 0 &\leq \Delta^2 \cdot E_1^2 = 2m_1 - m_2, \\ 0 &\leq \Delta^2 \cdot E_2^2 = 2m_2 - m_1 - m_3, \\ 0 &\leq \Delta^2 \cdot E_3 = m_3. \end{aligned} \tag{4.17}$$

Thus,

$$(S^2, \lambda a C^2 + \lambda\Delta^2 + \lambda(a + m_1)E_1^2 + \lambda(a + m_2)E_2^2 + (\lambda(2a + m_2 + m_3) - 1)E_3)$$

is not log canonical at some point  $P_2 \in E_3$ , by Remark 2.1.

**Lemma 4.16.**  $\lambda(2a + m_2 + m_3) - 1 < 1$ .

*Proof.*  $\lambda(2a + m_2 + m_3) < \lambda(1 + a - \epsilon - m_1) < \frac{\lambda(5-3\epsilon)}{3} < 2$ .  $\square$

We will now look at different positions of the point  $P_2$ .

- $P_2 \notin E_2^2$ . Suppose  $P_2 \in E_3 \cap E_2^2$ . Using Corollary 2.7, we get

$$\frac{1}{\lambda} < E_3 \cdot ((a + m_2)E_2^2 + \Delta^2) = a + m_2 + m_3 < 1 - \epsilon - m_1 < 1 - \epsilon.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  which is a contradiction.

- $P_2 \notin C^2$  : Suppose  $P_2 \in E_3 \cap C^2$ . Using Corollary 2.7 gives

$$\frac{1}{\lambda} < E_3 \cdot (aC^2 + \Delta^2) = a + m_3 < 1 - \epsilon - m_1 - m_2 < 1 - \epsilon.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  which is a contradiction.

- $P_2 \notin E_3 \setminus (E_2^2 \cup C^2)$ . Suppose  $P_3 \in E_3 \setminus (E_2^2 \cup C^2)$ . Again, using Corollary 2.7, we have

$$\frac{1}{\lambda} < E_3 \cdot \Delta^2 = m_3 < 1 - a - \epsilon - m_1 - m_2 < 1 - \epsilon.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  which is a contradiction.

We can thus conclude that such a point  $P_1$  does not exist. But this contradicts our initial assumption that  $(S, \lambda aC + \lambda \Delta)$  is not log canonical at the point  $P$ . Therefore the point  $P$  at which the log pair  $(S, \lambda D)$  is not log canonical cannot be the singular point of type  $A_2$ . Thus, if the surface has only  $A_2$  singular points, or if  $S$  has singular points which are of type  $A_1$  and  $A_2$  *only*, then Section 4.1.2.2 and Section 4.1.2.3 proves our claim that  $\alpha(S, D) > \delta > 1$ .

#### 4.1.2.4 $A_3$ singularity

Suppose  $S$  has at least 1 singular point of type  $A_3$  and  $P$  is the singular point of type  $A_3$ . Then, the curve  $C$  can only be a nodal curve with the node at the point  $P$ .

Recall that we have  $(S, \lambda aC + \lambda \Delta)$  not log canonical at the point  $P$ . Let  $f_1 : S^1 \rightarrow S$  be the minimal resolution of  $S$  at the point  $P$ . This implies that

$$(S^1, \lambda aC^1 + \lambda \Delta^1 + \lambda(a + m_1)E_1 + \lambda(a + m_2)E_2 + \lambda(a + m_3)E_3)$$

is not log canonical at a point  $P_1 \in E_1 \cup E_2 \cup E_3$  by Remark 2.1, where  $m_1, m_2, m_3 \in \mathbb{Q}_{\geq 0}$ . It satisfies the following equivalences.

$$\begin{aligned} C^1 &= f_1^*(C) - E_1 - E_2 - E_3, \\ \Delta^1 &= f_1^*(\Delta) - m_1 E_1 - m_2 E_2 - m_3 E_3. \end{aligned} \tag{4.18}$$

Thus we have,

$$\begin{aligned} 0 &\leq \Delta^1 \cdot C^1 = 1 - a - \epsilon - m_1 - m_2, \\ 0 &\leq \Delta^1 \cdot E_1 = 2m_1 - m_3, \\ 0 &\leq \Delta^1 \cdot E_2 = 2m_2 - m_3, \\ 0 &\leq \Delta^1 \cdot E_3 = 2m_3 - m_1 - m_2. \end{aligned} \tag{4.19}$$

---

**Lemma 4.17.**  $\lambda(a + m_1) < 1$

*Proof.* From (4.19) and (4.2),  $\lambda(a + m_1) \leq \lambda(1 - \epsilon - m_2) < \lambda(1 - \epsilon) < 1$ .  $\square$

Similar argument proves the following.

**Lemma 4.18.**  $\lambda(a + m_2) < 1$

**Lemma 4.19.**  $\lambda(a + m_3) < 1$

*Proof.* Using the inequalities in (4.19), we can prove the claim in the 2 possible cases.

- If  $m_1 \leq m_2$ ,  $\lambda(a + m_3) \leq \lambda(a + 2m_1) \leq \lambda(a + m_1 + m_2) \leq \lambda(a + 1 - a - \epsilon) = \lambda(1 - \epsilon) < 1$ .
- If  $m_2 \leq m_1$ ,  $\lambda(a + m_3) \leq \lambda(a + 2m_2) < \lambda(a + 1 - a - \epsilon) = \lambda(1 - \epsilon) < 1$ .

$\square$

We will now look at the various possible positions of the point  $P_1$ .

- $P_1 \notin E_3 \setminus E_1 \cup E_2$ . Suppose  $P_1 \in E_3 \setminus E_1 \cup E_2$ , then we have that  $(S^1, \lambda\Delta^1 + \lambda(a + m_3)E_3)$  is not log canonical at the point  $P_1$ . Since  $\lambda(a + m_3) < 1$ , applying Corollary 2.7 we get that

$$\frac{1}{\lambda} < (\Delta^1.E_3)_{P_1} \leq \Delta^1.E_3 = 2m_3 - m_1 - m_2 \leq m_1 + m_2 < 1 - \epsilon.$$

Thus we have  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < 1$  which contradicts (4.2).

- $P_1 \notin E_3 \cap E_2$ . Suppose  $P_1 \in E_3 \cap E_2$ . Then we have that

$$(S^1, \lambda(a + m_2)E_2 + \lambda(a + m_3)E_3 + \lambda\Delta^1)$$

is not log canonical at the point  $P_1$ .

- If  $m_1 \leq m_2$  consider

$$\text{mult}_{P_1}(\lambda\Delta^1) = \lambda \text{mult}_{P_1}\Delta^1 \leq \lambda m_3 \leq 2\lambda m_1 \leq \lambda(m_1 + m_2) < \lambda(1 - \epsilon) < 1.$$

- If  $m_2 \leq m_1$ , consider

$$\text{mult}_{P_1}(\lambda\Delta^1) = \lambda \text{mult}_{P_1}\Delta^1 \leq \lambda m_3 \leq 2\lambda m_2 \leq \lambda(m_1 + m_2) < \lambda(1 - \epsilon) < 1.$$

Similarly, we can show that  $P_1 \notin E_3 \cap E_1$ . From Theorem 2.9 we have that

$$\text{mult}_{P_1}(\lambda\Delta^1.E_3) > 2(1 - \lambda(a + m_2))$$

which can be simplified as follows, If  $m_1 \leq m_2$ , using inequalities in (4.19) we have

$$\frac{2}{\lambda} < 2m_3 - m_1 - m_2 + 2a + 2m_2 \leq 3m_1 + m_2 + 2a \leq 1 + a - \epsilon + 2m_1 \leq 2 - 2\epsilon.$$

Thus this implies  $\frac{1}{\lambda} < 1 - \epsilon$  which contradicts (4.2).

If  $m_2 \leq m_1$ , using inequalities in (4.19) we have

$$\frac{2}{\lambda} < 2m_3 - m_1 - m_2 + 2a + 2m_2 \leq 1 + a - \epsilon + 2m_2 \leq 2 - 2\epsilon.$$

Thus this implies  $\frac{1}{\lambda} < 1 - \epsilon$  which contradicts (4.2).

Or

$$\text{mult}_{P_1}(\lambda\Delta^1.E_2) > 2(1 - \lambda(a + m_3))$$

which when simplified using (4.19) gives

$$\begin{aligned} \frac{2}{\lambda} < 2m_2 - m_3 + 2a + 2m_3 &= 2m_2 + 2a + m_3 \leq 2m_2 + 2a + 2m_1 \\ &\leq 2(1 - a - \epsilon + a) = 2(1 - \epsilon). \end{aligned}$$

Thus this implies  $\frac{1}{\lambda} < 1 - \epsilon$  which contradicts (4.2). Since both the inequalities are not possible, this contradicts Theorem 2.9, thus proving our claim.

- $P_1 \notin E_2 \setminus E_3 \cup C^1$ . Suppose  $P_1 \in E_2 \setminus E_3 \cup C^1$ . Then we have  $(S^1, \lambda\Delta^1 + \lambda(a + m_2)E_2)$  is not log canonical at the point  $P_1$ . Since  $\lambda(a + m_2) < 1$ , from Corollary 2.7 we have

$$\begin{aligned} \frac{1}{\lambda} < (\Delta^1.E_2)_P &\leq \Delta^1.E_2^1 = 2m_2 - m_3 \leq m_3 + m_2 - m_1 \leq 2m_1 + m_2 - m_1 \\ &\leq 1 - a - \epsilon < 1 - \epsilon. \end{aligned}$$

This contradicts (4.2).

- $P_1 \notin C^1$ . Suppose  $P_1 \in C^1 \cap E_2$ . Then  $(S^1, \lambda aC^1 + \lambda\Delta^1 + \lambda(a + m_2)E_2)$  is not log canonical at the point  $P_1$ . Applying Corollary 2.7, we get that

$$\frac{1}{\lambda} < E_2.(aC^1 + \Delta^1) = a + 2m_2 - m_3 \leq a + m_2 + m_3 - m_1 \leq 1 - \epsilon - 2m_1 + m_3 \leq 1 - \epsilon.$$

This implies that  $\frac{3}{3-\epsilon} < \frac{1}{1-\epsilon} < \lambda$  which contradicts (4.2). Similarly, we can prove that  $P_1 \notin C^1 \cap E_1$ .

We can thus conclude that such a point  $P_1$  does not exist. But this contradicts our initial assumption that  $(S, \lambda aC + \lambda\Delta)$  is not log canonical at the point  $P$ .

Therefore the point  $P$  at which the log pair  $(S, \lambda D)$  is not log canonical cannot be the singular point of type  $A_3$ . Thus, if the surface has only  $A_3$  singular points, or if  $S$  has singular points which are of type  $A_1$ ,  $A_2$  and  $A_3$  *only*, then Section 4.1.2.2, Section 4.1.2.3 and Section 4.1.2.4 proves our claim that  $\alpha(S, D) > \delta > 1$ . This completes the proof of Theorem 4.5 (also Main Result 2).

## 4.2 K-stability of hypersurfaces

In this section, we will determine the K-stability of quasismooth, well-formed hypersurfaces  $S_d$ , of degree  $d$ , in weighted projective Space  $\mathbb{P}(a_0, a_1, a_2, a_3)$ , which are of Index  $I = 2$ . To recall the main result that we will be proving in this section,

**Theorem 4.20.** *(Also Main Result 3) Let  $S_d$  be a quasismooth, well-formed hypersurface with  $I = 2$ . The following hypersurfaces  $S_d$  in  $\mathbb{P}(a_0, a_1, a_2, a_3)$  of degree  $d$ , are K-polystable.*

No.	$(a_0, a_1, a_2, a_3)$	$d$
1	$(1, 1, n, n), n \geq 2$	$2n$
2	$(1, 2, n + 2, n + 3), n \geq 0$	$2(n + 3)$
3	$(1, 3, 4, 6)$	12
4	$(1, 4, 5, 7)$	15
5	$(1, 4, 5, 8)$	16
6	$(1, 4, 6, 9)$	18
7	$(1, 5, 7, 11)$	22
8	$(1, 6, 10, 15)$	30
9	$(1, 7, 12, 18)$	36
10	$(1, 8, 13, 20)$	40

The method we adopt is to compute  $\beta(S)$  for Family No. 1 in Table 4.20 using the Valuative criterion (Section 2.2.1) and for Family No. 2, we use the Abban-Zhuang theory (Section 2.2.3.2) to compute  $\delta(S)$ . For Families No.3-No.10, we estimate  $\delta(S)$  using the method adopted in [CPS21]. Let us briefly describe this method. Here, we study log pairs  $(S_d, D)$ , where  $S_d$  are hypersurfaces of degree  $d$  in weighted projective space and  $D$  is a  $k$ -basis type divisor for  $k \gg 1$  (see [FO18] for definition) such that  $D \sim_{\mathbb{Q}} -K_{S_d}$ . We then use Theorem 4.22 and Corollary 4.23 to get bounds on the multiplicities of  $D$  along certain curves on  $S_d$ . This then enables us to estimate  $\delta(S)$ .

### 4.2.1 Foundations

In this subsection, we will present the notions and results that we will need, in addition to results described in Section 2.1 and Section 2.2, in order to prove Theorem 4.20.

Since  $S_d$  is quasi-smooth, it can have at most cyclic quotient singularities. Therefore, let  $S_d$  have a cyclic quotient singularity of type  $\frac{1}{n}(a, b)$  at the point  $\mathfrak{p}$  and let the weighted coordinates around this point  $\mathfrak{p}$  be  $x$  and  $y$ . Let  $D$  be an effective  $\mathbb{Q}$ -divisor on  $S_d$ .

Let  $\phi: \widetilde{S}_d \rightarrow S_d$  be the weighted blow-up at  $\mathfrak{p}$  of  $S_d$  with weights  $\text{wt}(x) = a$  and  $\text{wt}(y) = b$ . Then we have the following:

$$K_{\widetilde{S}_d} \sim_{\mathbb{Q}} \phi^*(K_{S_d}) + \left(-1 + \frac{a}{n} + \frac{b}{n}\right) E$$

where  $E$  is the exceptional divisor of  $\phi$  and

$$E^2 = -\frac{n}{ab}.$$

Let  $C$  be a curve on  $S_d$  that is locally given by  $x = 0$  near  $\mathfrak{p}$ . Then we have

$$\tilde{C} \sim_{\mathbb{Q}} \phi^*(C) - \frac{a}{n}E$$

where  $\tilde{C}$  is the strict transform of  $C$ .

Let  $\tilde{D}$  be the proper transform of  $D$  on  $\tilde{S}_d$ . We have

$$\tilde{D} \sim_{\mathbb{Q}} \phi^*(D) - mE,$$

for some non-negative rational number  $m$ .

If  $C$  is not contained in the support of the divisor  $D$ , we can bound  $m$  using

$$0 \leq \tilde{D} \cdot \tilde{C} = (\phi^*(D) - mE) \cdot \tilde{C} = D \cdot C - mE \cdot \tilde{C}.$$

The log pullback is given by

$$K_{\tilde{S}_d} + \tilde{D} + \left(m - \frac{a+b-n}{n}\right)E \sim f^*(K_{S_d} + D).$$

This implies that

**Proposition 4.21.** *The log pair  $(S_d, D)$  is log canonical at  $P$  if and only if the log pair*

$$\left(\tilde{S}_d, \tilde{D} + \left(m - \frac{a+b-n}{n}\right)E\right)$$

*is log canonical along the curve  $E$ .*

If  $C$  is in the support of the divisor  $D$ , write  $D = aC + \Delta$  where  $a$  is a non-negative rational number and is such that  $C \not\subset \text{Supp}(\Delta)$ . Then we can use the following results to bound the value of  $a$ .

**Theorem 4.22** ([CPS21, Theorem 2.9]). *Suppose that  $D$  is a big  $\mathbb{Q}$ -divisor of  $k$ -basis type for  $k \gg 1$ . Then*

$$a \leq \frac{1}{D^2} \int_0^{\tau(C)} \text{vol}(D - uC) du + \epsilon_k$$

where  $\epsilon_k$  is a small constant depending on  $k$  such that  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ .

The above theorem implies the following.

**Corollary 4.23** ([CPS21, Corollary 2.10]). *Suppose that  $D$  is a big  $\mathbb{Q}$ -divisor of  $k$ -basis type for  $k \gg 1$ , and*

$$C \sim_{\mathbb{Q}} \mu D$$

*for some positive rational number  $\mu$ . Then*

$$a \leq \frac{1}{3\mu} + \epsilon_k,$$

where  $\epsilon_k$  is a small constant depending on  $k$  such that  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ .

From the above results, it is evident that for a  $k$ -basis type divisor  $D = aC + \Delta$ , one can bound  $a$  by computing the volume of the pseudoeffective divisor  $D - uC$ . Here, we recall how the  $\text{vol}(D)$  of an  $\mathbb{R}$  divisor  $D$  is calculated. If  $D$  is not pseudoeffective, then  $\text{vol}(D) = 0$  ([L04, Definition 2.2.31]). If  $D$  is pseudoeffective, then there exists a nef  $\mathbb{R}$  divisor  $P$  on the surface  $S$  such that

$$D \sim_{\mathbb{R}} P + \sum_{i=1}^r a_i C_i,$$

where each  $C_i$  is an irreducible curve on  $S$  with  $P \cdot C_i = 0$ , each  $a_i$  is a non-negative real number and the intersection form of the curves  $C_1, C_2, \dots, C_r$  is negative definite ([L04, Theorem 2.3.19]). Such a decomposition is unique and  $\text{vol}(D) = \text{vol}(P) = P^2$ .

In the next subsection, we will detail the proof of the K-polystability of the hypersurfaces  $S_d$  enumerated in Theorem 4.20.

## 4.2.2 Proof of Main Result 3

### 4.2.2.1 $S_{2n}$ in $\mathbb{P}(1, 1, n, n)$

In this section, we consider a quasismooth, well-formed hypersurface,  $S_{2n}$  of degree  $2n$  in  $\mathbb{P}(1, 1, n, n)$ . After suitable changes of coordinates, we can assume that the surface  $S_{2n}$  is given by

$$zt + f_{2n}(x, y) = 0,$$

where  $f_{2n}(x, y)$  is a polynomial of degree  $2n$  in  $x$  and  $y$ . Since  $S$  is quasismooth,  $f_{2n}(x, y)$  does not have multiple roots. We are going to prove that  $S_{2n}$  is K-polystable for  $n > 1$ , using Theorem 2.19.

In our case, we can explicitly describe the action of the group  $G = \mathbb{C}^* \times \mu_2$  on  $S_{2n}$  as follows

$$\begin{aligned} \lambda \cdot [x : y : z : t] &\longmapsto [x : y : \lambda z : \lambda^{-1} t], \text{ for } \lambda \in \mathbb{C}^* \\ \mu_2 \cdot [x : y : z : t] &\longmapsto [x : y : t : z]. \end{aligned}$$

Here  $[0 : 0 : 1 : 0]$  and  $[0 : 0 : 0 : 1]$  are the two singular points of the surface  $S_{2n}$ . Let  $\mathcal{P}$  be the pencil of  $G$ -invariant curves in  $S_{2n}$  generated by the curves  $C_x$  and  $C_y$ . Then  $-K_{S_{2n}} \sim 2C_x$  and  $C_x^2 = \frac{2}{n}$ .

From the definition of the  $\beta$ -invariant of a prime divisor  $E$  over  $S_{2n}$ , we have that

$$\beta(E) = A(E) - \frac{1}{(-K_S)^2} \int_0^\infty \text{vol}(-K_S - uE) du,$$

where  $A(E)$  is the log discrepancy of the divisor  $E$ .

So, we need to check  $\beta(E)$  for all  $E$  that are  $G$ -invariant prime divisors in  $S_{2n}$  and all  $G$ -invariant prime divisors over  $S_{2n}$ .

All  $G$ -invariant irreducible prime divisors in  $S_{2n}$  are irreducible curves in the pencil  $\mathcal{P}$ . Let  $C$  be one such curve in  $S_{2n}$ , in the pencil  $\mathcal{P}$ , that is,  $C : ax + by = 0$  for some  $[a : b] \in \mathbb{P}^1$ . Consider the divisor  $-K_S - xC$ . Then we have that  $\tau(C) = 2$  and  $\sigma(C) = 2$ .

Then

$$\beta(C) = 1 - \frac{1}{4} \int_0^2 (2-u)^2 du = \frac{1}{3}.$$

We now need to consider only the  $G$ -invariant prime divisors over  $S_{2n}$ . These divisors will be mapped to  $G$ -fixed points on  $S_{2n}$ . These  $G$ -fixed points are the singular points of the reducible curves in  $\mathcal{P}$ . Let us now understand what these  $G$ -fixed points are.

Observe that the reducible curves in  $\mathcal{P}$  are cut out on  $S_{2n}$  by the linear factors of the polynomial  $f_{2n}(x, y)$ . Since there are  $2n$  such linear factors of the polynomial  $f_{2n}(x, y)$ , we have  $2n$  reducible curves in  $\mathcal{P}$ . After a suitable change of coordinates, we may assume that one such linear factor is the curve  $C_x$  that is cut out by  $x = 0$ , so that  $f_{2n} = xg_{2n-1}$  for some polynomial  $g_{2n-1}(x, y)$  of degree  $2n - 1$ .

This implies that the equation of  $S$  looks like

$$zt + xg_{2n-1}(x, y) = 0.$$

Substituting for  $x = 0$ , we see that we get reducible components  $z = x = 0; t = x = 0$ . So the singular point of the curve  $C_x : x = 0$ , is  $[0 : 1 : 0 : 0]$ . Similarly, for each of the linear factors  $h(x, y)$  of the polynomial  $g_{2n-1}(x, y)$ , we get the reducible components of the curve in  $\mathcal{P}$  to be  $z = h(x, y) = 0$  and  $t = h(x, y) = 0$  and we can then compute the singular points in each case. In total, we get  $2n$  singular points. These  $2n$  points are all the  $G$ -invariant points on  $S_{2n}$ .

Consider one such reducible curve and let the irreducible components of it be given by  $C_1$  and  $C_2$ . The point of intersection of  $C_1$  and  $C_2$  is  $[0 : 1 : 0 : 0]$ . We will call this point  $\mathfrak{p}$ . Let  $\pi : \tilde{S}_{2n} \rightarrow S_{2n}$  be the blow up of  $\mathfrak{p}$  with the exceptional divisor of the blow up denoted by  $F$ . Observe that the action of  $G$  lifts to  $\tilde{S}_{2n}$  and thus the morphism  $\pi$  is  $G$ -equivariant. Also observe that  $G$  doesn't fix any point on  $F$ . Thus, the exceptional divisors upon blow up of the singular points of the reducible curves in the pencil  $\mathcal{P}$ , are the only irreducible  $G$ -invariant prime divisors over  $S_{2n}$ , and in this case, this is  $F$ .

Consider the divisor

$$D = \pi^*(-K_{S_{2n}}) - uF = 2(\tilde{C}_1 + \tilde{C}_2) + (4 - u)F,$$

where  $\tilde{C}_1$  and  $\tilde{C}_2$  are the proper transforms of  $C_1$  and  $C_2$  on  $\tilde{S}$  and the pseudoeffective threshold,  $\tau(D) = 4$ . Let us now compute the Seshadri constant,  $\sigma(D)$ ,

$$D \cdot \tilde{C}_1 = 2 \left( \frac{1 - 2n}{n} \right) + (4 - u) \geq 0.$$

This implies that  $\sigma(D) = \frac{2}{n}$ . For  $u \in [0, \frac{2}{n}]$ , we have the following

$$\text{vol}(D) = \text{vol}(\pi^*(-K_{S_{2n}}) - uF) = \frac{8}{n} - u^2.$$

For  $u \in [\frac{2}{n}, 4]$ , the Zariski decomposition is given by

$$D = \left( D - \left( \frac{2-nu}{1-2n} \right) (\tilde{C}_1 + \tilde{C}_2) \right) + \left( \frac{2-nu}{1-2n} \right) (\tilde{C}_1 + \tilde{C}_2).$$

Therefore,

$$\text{vol}(D) = \text{vol}(\pi^*(-K_{S_{2n}}) - uF) = (4-u)^2 \left( \frac{1}{2n-1} \right).$$

Thus,

$$\beta(F) = 2 - \frac{1}{4C_x^2} \left( \int_0^{\frac{2}{n}} \left( \frac{8}{n} - u^2 \right) du + \int_{\frac{2}{n}}^4 (4-u)^2 \frac{1}{2n-1} du \right) = \frac{2n-2}{3n}.$$

Therefore, by Theorem 2.19, for all  $n > 1$ , this proves that all quasismooth hypersurfaces  $S_{2n}$  of degree  $2n$  in  $\mathbb{P}(1, 1, n, n)$  are K-polystable.

#### 4.2.2.2 $S_{2n+6}$ in $\mathbb{P}(1, 2, n+2, n+3)$

Let  $S_{2n+6}$  be a quasismooth, well-formed hypersurface of degree  $2n+6$  in  $\mathbb{P}(1, 2, n+2, n+3)$ . In this subsection, we will denote  $S_{2n+6}$  with  $S$ . We may assume that the surface  $S$  is given by

$$t^2 + z^2y + f_{2n+6}(x, y) = 0,$$

where  $f_{2n+6}(x, y)$  is a polynomial of degree  $2n+6$  in  $x$  and  $y$ . When  $n$  is even, the surface  $S$  is singular at the points  $\mathbf{p}_z := [0 : 0 : 1 : 0]$  of type  $\frac{1}{n+2}(1, 1)$  and  $Q_1 := [0 : 1 : \alpha : 0]$ ;  $Q_2 := [0 : 1 : \beta : 0]$  of type  $\frac{1}{2}(1, 1)$ , where  $\alpha$  and  $\beta$  are distinct real numbers and is smooth away from these points. Note that when  $n = 0$ , the points  $\mathbf{p}_z$ ,  $Q_1$  and  $Q_2$  are of type  $\frac{1}{2}(1, 1)$ .

When  $n$  is odd, the surface is singular at the points  $\mathbf{p}_t := [0 : 0 : 0 : 1]$  of type  $\frac{1}{n+2}(1, 1)$  and the points  $Q_1 := [0 : 1 : 0 : \alpha]$ ;  $Q_2 := [0 : 1 : 0 : \beta]$  of type  $\frac{1}{2}(1, 1)$ , where  $\alpha$  and  $\beta$  are distinct real numbers and is smooth away from these points. Note that all the singular points lie on the curve  $C_x$ . We also have

$$-K_S^2 = \frac{4}{n+2}; \quad C_x^2 = \frac{1}{n+2}.$$

We use the stability threshold from Theorem 2.25 to conclude that the surface is K-stable for all  $n \geq 0$ .

In order to prove the above claim, we will estimate the value of  $\delta_{\mathbf{p}}(S)$  using Theorem 2.29 for different positions of the point  $\mathbf{p} \in S$ .

**Lemma 4.24.** *If  $\mathbf{p} \in C_x$ , then  $\delta_{\mathbf{p}}(S) \geq \frac{3}{2}$ .*

*Proof.* Since  $\mathbf{p} \in C_x$  we will take  $Y = C_x$  according to the notations of Theorem 2.29. Note that  $A_S(C_x) = 1$ . We will now compute  $S_{-K_S}(C_x)$ . Consider the divisor  $-K_S - uC_x = (2-u)C_x$ . With the help of intersection numbers of the

divisor with  $C_x$ , we can conclude that  $\tau(C_x) = 2$  and that the divisor  $-K_S - uC_x$  is nef when  $u \in [0, 2]$ . We then have that

$$\begin{aligned} S_{-K_S}(C_x) &= \frac{1}{(-K_S)^2} \int_0^\infty \text{vol}(-K_S - uC_x) du, \\ &= \frac{n+2}{4} \int_0^2 (2-u)^2 du, \\ &= \frac{2}{3}. \end{aligned}$$

Therefore, for any point  $\mathbf{p}$  on  $C_x$ ,  $\frac{A_S(C_x)}{S_{-K_S}(C_x)} = \frac{3}{2}$ . We will now use Theorem 2.31 to compute  $S(W_{\bullet, \bullet}^Y; \mathbf{p})$  for different points  $\mathbf{p}$  on  $C_x$ .

**Case 1:** Suppose the point  $\mathbf{p} = \mathbf{p}_z$  on  $C_x$ . Recall that  $\mathbf{p}_z$  and  $\mathbf{p}_t$  are the singular points of type  $\frac{1}{n+2}(1, 1)$  when  $n$  is even and odd respectively. Since both  $\mathbf{p}_z$  and  $\mathbf{p}_t$  lie on  $C_x$ , the proof below works for both. So without loss of generality, let us take  $\mathbf{p} = \mathbf{p}_z$ . Note that  $A_{C_x}(\mathbf{p}) = \frac{1}{n+2}$ . Therefore, using Theorem 2.31, we get that

$$S(W_{\bullet, \bullet}^{C_x}; \mathbf{p}) = \frac{2}{(-K_S)^2} \int_0^{\tau(C_x)} h(u) du,$$

where

$$h(u) = (P(u) \cdot C_x) \cdot \text{ord}_{\mathbf{p}}(N(u)|_{C_x}) + \int_0^\infty \text{vol}(P(u)|_{C_x} - v\mathbf{p}) dv.$$

Since the divisor  $-K_S - uC_x = (2-u)C_x$  is nef when  $u \in [0, \tau(C_x) = 2]$ ,  $P(u) = (2-u)C_x$  and  $N(u) = 0$ . Therefore,

$$S(W_{\bullet, \bullet}^{C_x}; \mathbf{p}) = \frac{n+2}{2} \int_0^2 \int_0^{\frac{2-u}{n+2}} \left( \frac{2-u}{n+2} - v \right) dv du = \frac{2}{3(n+2)}.$$

This implies that

$$\frac{A_{C_x}(\mathbf{p})}{S(W_{\bullet, \bullet}^{C_x}; \mathbf{p})} = \frac{\frac{1}{n+2}}{\frac{2}{3(n+2)}} = \frac{3}{2}.$$

From Theorem 2.29, we then get that for  $\mathbf{p} = \mathbf{p}_z$ ,

$$\delta_P(S) \geq \frac{3}{2}.$$

**Case 2:** Suppose the point  $\mathbf{p} = Q_1$ . Since  $Q_1$  and  $Q_2$  are singular points of type  $\frac{1}{2}(1, 1)$ , the proof below works for both points. So without loss of generality, let us take  $\mathbf{p} = Q_1$ . Note that  $A_{C_x}(\mathbf{p}) = \frac{1}{2}$ . Similar to Case 1, we get that  $S(W_{\bullet, \bullet}^{C_x}; \mathbf{p}) = \frac{2}{3(n+2)}$ . This implies that

$$\frac{A_{C_x}(\mathbf{p})}{S(W_{\bullet, \bullet}^{C_x}; \mathbf{p})} = \frac{\frac{1}{2}}{\frac{2}{3(n+2)}} = \frac{3(n+2)}{4}.$$

From Theorem 2.29, we then get that for  $\mathbf{p} = Q_1$  and  $\mathbf{p} = Q_2$ ,

$$\delta_{\mathbf{p}}(S) \geq \frac{3}{2}.$$

**Case 3:** Suppose the point  $\mathbf{p} \in C_x \setminus \{\mathbf{p}_z, Q_1, Q_2\}$ . This implies that the point  $\mathbf{p}$  is a smooth point. Then  $A_{C_x}(\mathbf{p}) = 1$ . Again, similar to Case 1, we get that  $S(W_{\bullet, \bullet}^{C_x}; \mathbf{p}) = \frac{2}{3(n+2)}$ . This implies that

$$\frac{A_{C_x}(\mathbf{p})}{S(W_{\bullet, \bullet}^{C_x}; \mathbf{p})} = \frac{1}{\frac{2}{3(n+2)}} = \frac{3(n+2)}{2}.$$

From Theorem 2.29, we then get that for  $\mathbf{p} \in C_x \setminus \{\mathbf{p}_z, Q_1, Q_2\}$ ,

$$\delta_{\mathbf{p}}(S) \geq \frac{3}{2}.$$

This completes proof of the claim.  $\square$

**Lemma 4.25.** *If  $\mathbf{p} \in S \setminus C_x$ , then  $\delta_{\mathbf{p}}(S) > 1$ .*

*Proof.* Since  $\mathbf{p} \notin C_x$ , the point  $\mathbf{p}$  is a smooth point of the surface. Note that there is a curve from the pencil  $C_\mu : y = \mu x^2$  for some  $\mu \in \mathbb{R}$  that passes through the point  $\mathbf{p}$ .

**Case 1:** Suppose the curve  $C_\mu$  that passes through  $\mathbf{p}$  is irreducible. Note that  $A_S(C_\mu) = 1$ . Consider the divisor  $-K_S - uC_\mu = (1-u)C_\mu$ . This implies that  $\tau(C_\mu) = 1$  and that the divisor  $-K_S - uC_\mu$  is nef when  $u \in [0, 1]$ . We then have that

$$\begin{aligned} S_{-K_S}(C_\mu) &= \frac{1}{(-K_S)^2} \int_0^\infty \text{vol}(-K_S - uC_\mu) du \\ &= \int_0^1 (1-u)^2 du = \frac{1}{3}. \end{aligned}$$

Therefore, for any point  $\mathbf{p} \in C_\mu$ ,  $\frac{A_S(C_\mu)}{S_{-K_S}(C_\mu)} = 3$ .

Let us now try to bound  $\delta_{\mathbf{p}}(S)$  using Theorem 2.29. According to the notations of Theorem 2.29,  $Y = C_\mu$ . Note that  $A_{\mathbf{p}}(C_\mu) = 1$  since  $\mathbf{p}$  is a smooth point. When  $u \in [0, \tau(C_\mu) = 1]$ ,  $P(u) = (1-u)C_\mu$  and  $N(u) = 0$  since the divisor  $-K_S - uC_\mu = (1-u)C_\mu$  is nef. Therefore,

$$S(W_{\bullet, \bullet}^{C_\mu}; \mathbf{p}) = \frac{n+2}{2} \int_0^1 \int_0^{\frac{4(1-u)}{n+2}} \left( \frac{4(1-u)}{n+2} - v \right) dv du = \frac{4}{3(n+2)}.$$

This implies that

$$\frac{A_{C_\mu}(\mathbf{p})}{S(W_{\bullet, \bullet}^{C_\mu}; \mathbf{p})} = \frac{1}{\frac{4}{3(n+2)}} = \frac{3(n+2)}{4}.$$

From Theorem 2.29, we then get that for  $\mathbf{p} \in C_\mu \setminus C_x$  where  $C_\mu$  is irreducible,

$$\delta_{\mathbf{p}}(S) \geq \min \left\{ 3, \frac{3(n+2)}{4} \right\} > 1,$$

for all  $n \geq 0$ .

**Case 2:** Suppose the curve  $C_\mu$  is reducible with  $C_\mu = Z_1 + Z_2$  where  $Z_1$  and  $Z_2$  are the irreducible components of the curve  $C_\mu$ .

Note that  $A_S(Z_1) = 1$ . Consider the divisor  $-K_S - uZ_1 = (1-u)Z_1 + Z_2$ . Since  $Z_1^2 = Z_2^2 = \frac{-(n+1)}{n+2}$ , we can conclude that  $\tau(Z_1) = 1$ . From intersections of the divisor with  $Z_1$  and  $Z_2$  we get that the divisor  $-K_S - uZ_1$  is nef when  $u \in [0, \frac{2}{n+3}]$ . When  $u \in [\frac{2}{n+3}, 1]$ , the Zariski decomposition is given by

$$-K_S - uZ_1 = \left( (1-u)Z_1 + \frac{(1-u)(n+3)}{n+1}Z_2 \right) + \frac{u(n+3)-2}{n+1}Z_2.$$

Therefore,

$$\text{vol}(-K_S - uZ_1) = \begin{cases} \frac{4(1-u) - (n+1)u^2}{n+2} & \text{if } u \in \left[0, \frac{2}{n+3}\right], \\ \frac{4(u-1)^2}{n+1} & \text{if } u \in \left[\frac{2}{n+3}, 1\right]. \end{cases}$$

Using this, we get that

$$\begin{aligned} S_{-K_S}(Z_1) &= \frac{1}{(-K_S)^2} \int_0^\infty \text{vol}(-K_S - uZ_1) du \\ &= \frac{n+2}{4} \left( \int_0^{\frac{2}{n+3}} \frac{4(1-u) - (n+1)u^2}{n+2} du + \int_{\frac{2}{n+3}}^1 \frac{4(u-1)^2}{n+1} du \right) \\ &= \frac{n+4}{3(n+3)}. \end{aligned}$$

Therefore, for any point  $\mathbf{p} \in Z_1$ ,  $\frac{A_S(Z_1)}{S_{-K_S}(Z_1)} = \frac{3(n+3)}{n+4}$ .

**Case 2a:** Suppose the point  $\mathbf{p}$  is on one of the components. Without loss of generality, let us assume that  $\mathbf{p} \in Z_1 \setminus Z_2$ .

We will now bound  $\delta_{\mathbf{p}}(S)$  using Theorem 2.29. In this case, we will take  $Y = Z_1$ . Note that  $A_{Z_1}(\mathbf{p}) = 1$ . Recall that we have

$$-K_S - uZ_1 = P(u) + N(u)$$

where

$$P(u) = \begin{cases} (1-u)Z_1 + Z_2 & \text{if } u \in \left[0, \frac{2}{n+3}\right], \\ (1-u)Z_1 + \frac{(1-u)(n+3)}{n+1}Z_2 & \text{if } u \in \left[\frac{2}{n+3}, 1\right] \end{cases}$$

and

$$N(u) = \begin{cases} 0 & \text{if } u \in \left[0, \frac{2}{n+3}\right], \\ \frac{u(n+3)-2}{n+1}Z_2 & \text{if } u \in \left[\frac{2}{n+3}, 1\right]. \end{cases}$$

Therefore,

$$\begin{aligned}
S(W_{\bullet, \bullet}^{Z_1}; \mathbf{p}) &= \frac{n+2}{2} \left[ \int_0^{\frac{2}{n+3}} \int_0^{\frac{2+u(n+1)}{n+2}} \left( \frac{2+u(n+1)}{n+2} - v \right) dv du \right. \\
&\quad \left. + \int_{\frac{2}{n+3}}^1 \int_0^{\frac{4(1-u)}{n+1}} \left( \frac{4(1-u)}{n+1} - v \right) dv du \right] \\
&= \frac{2(2n+5)}{3(n+2)(n+3)}.
\end{aligned}$$

This implies that

$$\frac{A_{Z_1}(\mathbf{p})}{S(W_{\bullet, \bullet}^{Z_1}; \mathbf{p})} = \frac{1}{\frac{2(2n+5)}{3(n+2)(n+3)}} = \frac{3(n+2)(n+3)}{2(2n+5)}.$$

From Theorem 2.29, we then get that for  $\mathbf{p} \in (Z_1 \cup Z_2) \setminus C_x$  where  $Z_1, Z_2$  are irreducible components of the curve  $C_\mu$ ,

$$\delta_{\mathbf{p}}(S) \geq \min \left\{ \frac{3(n+3)}{n+4}, \frac{3(n+2)(n+3)}{2(2n+5)} \right\} = \frac{3(n+2)(n+3)}{2(2n+5)} > 1,$$

for all  $n \geq 0$ .

**Case 2b:** Suppose the point  $\mathbf{p}$  is on both components of the curve  $C_\mu$ , i.e.  $\mathbf{p} \in Z_1 \cap Z_2$ .

Let us consider the divisor  $-K_S - uZ_1$ . From the computations in Case 2a, we see that  $\frac{A_S(Z_1)}{S_{-K_S}(Z_1)} = \frac{3(n+3)}{n+4}$ . We will now bound  $\delta_{\mathbf{p}}(S)$  using Theorem 2.29. In order to use this theorem, let  $Y = Z_1$  by the notations of Theorem 2.29. Note that  $A_{Z_1}(\mathbf{p}) = 1$ . From the Zariski decompositions calculated earlier for the divisor  $-K_S - uZ_1$ , we can compute the following.

$$\begin{aligned}
S(W_{\bullet, \bullet}^{Z_1}; \mathbf{p}) &= \frac{n+2}{2} \left[ \int_0^{\frac{2}{n+3}} \int_0^{\frac{2+u(n+1)}{n+2}} \left( \frac{2+u(n+1)}{n+2} - v \right) dv du \right. \\
&\quad + \int_{\frac{2}{n+3}}^1 \left( \frac{4(1-u)(u(n+3)-2)(n+3)}{(n+1)^2(n+2)} \right) du \\
&\quad \left. + \int_{\frac{2}{n+3}}^1 \int_0^{\frac{4(1-u)}{n+1}} \left( \frac{4(1-u)}{n+1} - v \right) dv du \right] \\
&= \frac{n+4}{3(n+2)}.
\end{aligned}$$

This implies that

$$\frac{A_{Z_1}(\mathbf{p})}{S(W_{\bullet, \bullet}^{Z_1}; \mathbf{p})} = \frac{1}{\frac{n+4}{3(n+2)}} = \frac{3(n+2)}{n+4}.$$

From Theorem 2.29, we then get that for  $\mathbf{p} \in (Z_1 \cap Z_2) \setminus C_x$  where  $Z_1, Z_2$  are irreducible components of the curve  $C_\mu$ ,

$$\delta_{\mathbf{p}}(S) \geq \min \left\{ \frac{3(n+3)}{n+4}, \frac{3(n+2)}{n+4} \right\} = \frac{3(n+2)}{n+4} > 1,$$

for all  $n \geq 0$ . This completes the proof of the claim.  $\square$

*Proof of Main Theorem.* From Lemma 4.24 and Lemma 4.25, we can conclude that

$$\delta(S) = \inf_{p \in S} \delta_p(S) > 1.$$

Therefore, Theorem 2.25 implies that  $S$  is K-stable for all  $n \geq 0$ .  $\square$

#### 4.2.2.3 $S_{12}$ in $\mathbb{P}(1, 3, 4, 6)$

Let  $S_{12}$  be a quasismooth, well-formed hypersurface of degree 12 in  $\mathbb{P}(1, 3, 4, 6)$ . By a suitable coordinate change we can assume that  $S_{12}$  can be given by a quasi-homogeneous polynomial

$$t^2 + z^3 + y^4 + xf(x, y, z) = 0$$

where  $f(x, y, z)$  is a quasihomogeneous polynomial in  $x, y, z$  of degree 11.

In order to determine the singular points of the surface, we will take the equation of the surface, obtained using coordinate change, to be

$$t(t + y^2) + z^3 + xf(x, y, z) = 0.$$

Therefore the points  $[0 : i : 0 : 1]$  and  $[0 : -i : 0 : 1]$  are singular points of type  $\frac{1}{3}(1, 1)$  and the point  $Q_1 := [0 : 0 : -1 : 1]$  is of type  $\frac{1}{2}(1, 1)$ . Without loss of generality take  $Q_2 := [0 : i : 0 : 1]$ .

Note that all singular points lie on the curve  $C_x$  which is isomorphic to the variety given by

$$t^2 + z^3 + y^4 = 0$$

in  $\mathbb{P}(3, 4, 6)$ . From this, we can see that  $C_x$  is irreducible and smooth at these singular points of the surface  $S_{12}$ .

**Proposition 4.26.** *Let  $D$  be a  $\mathbb{Q}$  effective divisor on  $S_{12}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{12}}$  and  $D = aC_x + \Delta$  where  $a$  is non-negative number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Let  $\lambda = \frac{6}{5}$  and suppose also that  $a \leq \frac{3}{4}$ . Then  $(S_{12}, \lambda D)$  is log canonical.*

**Corollary 4.27.**  $\delta(S_{12}) \geq \frac{6}{5}$ .

*Proof.* Let  $D$  be a  $\mathbb{Q}$ -divisor of  $k$ -basis type on  $S_{12}$  with  $k \gg 0$ . We write  $D = aC_x + \Delta$  where  $a$  is non-negative number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . By Corollary 4.23,  $a \leq \frac{3}{4}$  for  $k \gg 0$ . It implies that the log pair  $(S_{12}, \frac{6}{5}D)$  is log canonical by Proposition 4.26. Since  $\delta(S_{12}) \geq \frac{3}{2}\alpha(S_{12})$ , in particular, we have that  $\delta(S_{12}) \geq \frac{6}{5}$ .  $\square$

We will now prove Proposition 4.26.

**Lemma 4.28.** *The log pair  $(S_{12}, \lambda D)$  is log canonical along  $S_{12} \setminus C_x$ .*

*Proof.* Suppose that the log pair  $(S_{12}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in S_{12} \setminus C_x$ . This implies that  $\mathbf{p}$  is a smooth point of the surface  $S_{12}$ . Since  $\mathbf{p} \notin C_x$ ,  $\mathbf{p}$  is in the chart given by  $x \neq 0$  and by a suitable coordinate change we can assume that  $\mathbf{p} := \mathbf{p}_x$ .

For the purposes of this lemma, we consider the equation of the surface  $S_{12}$  in the patch  $x \neq 0$ , as given by

$$G(1, y, z, t) + a_1 y + a_2 z + a_3 t = 0,$$

where  $a_1, a_2, a_3$  are complex constants,  $G(1, y, z, t)$  is a quasihomogeneous polynomial in  $y, z, t$ .

We will consider different curves through the point  $\mathbf{p}_x$  depending on the linear terms in the equation of the surface  $S_{12}$ .

Suppose  $a_1 \neq 0$  and  $a_2 = a_3 = 0$ . Then consider the curve  $C_y$  which is isomorphic to the variety given by

$$t^2 + z^3 = 0$$

in  $\mathbb{P}(1, 4, 6)$ . Since the coefficient of the monomial  $t^2$  is nonzero,  $\text{mult}_{\mathbf{p}}(C_y) = 2$ . We can see that  $C_y$  is irreducible. Write

$$D = bC_y + \Omega$$

where  $b$  is nonnegative constant and  $\Omega$  is an effective  $\mathbb{Q}$ -divisor such that  $C_y \not\subset \text{Supp}(\Omega)$ . By Corollary 4.23 we have  $b \leq \frac{2}{9} + \epsilon_k$ . By Lemma 2.11 we have

$$2 \left( \frac{1}{\lambda} - 2b \right) < \text{mult}_{\mathbf{p}}(C_y) \text{mult}_{\mathbf{p}}(D - bC_y) \leq C_y \cdot (D - bC_y) = 1 - \frac{3}{2}b.$$

It implies that  $\frac{4}{15} < b$ . This is impossible. Thus the linear term in the equation of the surface in the chart  $x = 1$  cannot be  $y$ .

Suppose  $a_2 \neq 0$  and  $a_3 = 0$ . Consider the curve  $C_z$  which is isomorphic to the variety given by

$$t^2 + y^4 = 0.$$

Since the coefficient of the monomial  $t^2$  is nonzero,  $\text{mult}_{\mathbf{p}}(C_z) = 2$ .

We first consider the case when  $C_z$  is irreducible. Write

$$D = cC_z + \Psi$$

where  $c$  is a non-negative constant and  $\Psi$  is effective  $\mathbb{Q}$ -divisor such that  $C_z \not\subset \text{Supp}(\Psi)$ . By Corollary 4.23 we have  $c \leq \frac{1}{6} + \epsilon_k$ .

By Lemma 2.11 we have

$$2 \left( \frac{1}{\lambda} - 2c \right) < \text{mult}_{\mathbf{p}}(C_z) \text{mult}_{\mathbf{p}}(D - cC_z) \leq C_z \cdot (D - cC_z) = \frac{4}{3} - \frac{8}{3}c.$$

It implies that  $\frac{1}{4} < c$ . This is impossible. This implies that  $C_z$  is reducible. Let us rewrite the equation of the surface  $S_{12}$  in such a way that it is given by

$$zg(x, y, z, t) + (t + a_1 y^2)(t + a_2 y^2 + a_3 x^6) = 0$$

where  $a_1, a_2$  and  $a_3$  are constants and  $g(x, y, z, t)$  is a quasihomogeneous polynomial of degree 8. Then, the curve  $C_z =: M$  is given by

$$M = L + R$$

where  $L$  is the curve that is given by  $z = t + a_1 y^2 = 0$  in  $\mathbb{P}(1, 3, 4, 6)$  and  $R$  is the curve that is given by  $z = t + a_2 y^2 + a_3 x^6 = 0$  in  $\mathbb{P}(1, 3, 4, 6)$ . We then have the following intersection numbers.

$$L \cdot R = 2, \quad L^2 = R^2 = -\frac{2}{3}, \quad -K_{S_{12}} \cdot L = -K_{S_{12}} \cdot R = \frac{2}{3}.$$

Since  $\mathfrak{p} \in C_z$ , w.l.o.g, we can assume that  $\mathfrak{p} \in L$  and we write

$$D = dL + \Lambda$$

where  $d$  is non-negative constant and  $\Lambda$  is an effective  $\mathbb{Q}$ -divisor such that  $L \not\subset \text{Supp}(\Lambda)$ . This gives us that  $(S_{12}, \lambda dL + \lambda \Lambda)$  is not log canonical at  $\mathfrak{p}$ .

We now bound  $d$ , using Theorem 4.22. Consider the following divisor,

$$-K_{S_{12}} - uL \sim_{\mathbb{Q}} \left( \frac{1}{2} - u \right) L + \frac{1}{2} R.$$

This divisor is pseudoeffective for  $u < \frac{1}{2}$  and when  $u \geq \frac{1}{2}$ ,  $\text{vol}(-K_{S_{12}} - uL) = 0$ , since the divisor  $-K_{S_{12}} - uL$  is no longer big. Thus  $\tau(L) = \frac{1}{2}$ .

Using the above intersection numbers we have that  $-K_{S_{12}} - xL$  is nef for  $x \leq \frac{1}{3}$ . Therefore, when  $x \in [\frac{1}{3}, \frac{1}{2}]$ , the Zariski decomposition of the divisor is given as

$$-K_{S_{12}} - uL = \left( \left( \frac{1}{2} - u \right) (L + 3R) \right) + (3u - 1)R.$$

Thus we have

$$\text{vol}(-K_{S_{12}} - uL) = \begin{cases} \frac{2}{3} - \frac{4}{3}u - \frac{2}{3}u^2 & \text{for } u \leq \frac{1}{3}, \\ \frac{16}{3} \left( \frac{1}{2} - u \right)^2 & \text{for } \frac{1}{3} \leq u \leq \frac{1}{2}. \end{cases}$$

From this we have

$$\begin{aligned} d &\leq \frac{3}{2} \int_0^{\frac{1}{2}} \text{vol}(-K_{S_{12}} - uL) du + \epsilon_k \\ &= \frac{3}{2} \left( \int_0^{\frac{1}{3}} \left( \frac{2}{3} - \frac{4}{3}u - \frac{2}{3}u^2 \right) du + \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{16}{3} \left( \frac{1}{2} - u \right)^2 du \right) + \epsilon_k \\ &= \frac{2}{9} + \epsilon_k. \end{aligned}$$

This implies that  $\lambda d \leq 1$ . Applying Corollary 2.7 to  $L$ , we have

$$\frac{1}{\lambda} < \text{mult}_{\mathfrak{p}}(\Lambda \cdot L) \leq \Lambda \cdot L = (D - dL) \cdot L = \frac{2}{3} + \frac{2}{3}d.$$

Thus we have  $d > \frac{1}{4}$  and this is impossible. Thus, the only remaining case is when  $a_3 \neq 0$ . Consider the linear system of curves cut by  $\mathcal{M} := \alpha xy + \beta z = 0$  where  $[\alpha : \beta] \in \mathbb{P}^1$ . Any curve  $M \in \mathcal{M}$  passes through the point  $\mathfrak{p}_x$ .

Let  $\phi: \bar{S}_{12} \rightarrow S_{12}$  be a blow-up at  $\mathfrak{p}_x$ . Then we have

$$\bar{D} \sim_{\mathbb{Q}} \phi^*(D) - \eta E$$

where  $\bar{D}$  is the strict transform of  $D$ ,  $\eta$  is a constant and  $E$  is the exceptional divisor of  $\phi$ . Since  $\phi$  is a blow up of a smooth point of the surface  $S_{12}$ , the log pullback is given by

$$K_{\bar{S}_{12}} + \lambda \bar{D} + (\lambda \eta - 1)E \sim_{\mathbb{Q}} \phi^*(K_X + \lambda D).$$

Then the log pair  $(\bar{S}_{12}, \lambda \bar{D} + (\lambda \eta - 1)E)$  is not log canonical at some point  $\mathfrak{q} \in E$ . Let  $\bar{M}$  be the strict transform of  $M$  in  $\bar{S}_{12}$  and be such that it passes through the point  $\mathfrak{q}$ .

Suppose that  $M$  is irreducible. We write

$$D = mM + \Delta$$

where  $m$  is non-negative constant and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $M \not\subset \text{Supp}(\Delta)$ . By Corollary 4.23 we have  $\lambda m \leq 1$ . Thus the log pair  $(\bar{S}_{12}, \lambda m \bar{M} + \lambda \bar{\Delta} + (\lambda \eta - 1)E)$  is not log canonical at  $\mathfrak{q}$ . We have the following intersection numbers.

$$\bar{\Delta} \cdot \bar{M} = (\bar{D} - a\bar{M}) \cdot \bar{M} = \frac{4}{3} - \eta - \frac{5}{3}m, \quad E \cdot \bar{M} = 1.$$

By Lemma 2.2 and Corollary 2.7, we have

$$1 < \text{mult}_{\mathfrak{q}}(\lambda \bar{\Delta} + (\lambda \eta - 1)E) \cdot \bar{M} \leq (\lambda \bar{\Delta} + (\lambda \eta - 1)E) \cdot \bar{M} = \lambda \left( \frac{4}{3} - \frac{5}{3}m \right) - 1$$

This implies that  $-\frac{1}{5} > m$  and this is impossible. If the curve  $M$  is reducible, then the same argument as given for the curve  $C_z$ , when reducible, works, thus completing the proof of our claim.  $\square$

**Lemma 4.29.** *The log pair  $(S_{12}, \lambda D)$  is log canonical along  $C_x \setminus \text{Sing}(S_{12})$ .*

*Proof.* Suppose the log pair  $(S_{12}, \lambda D)$  is not log canonical at some point  $\mathfrak{p} \in C_x \setminus \text{Sing}(S_{12})$ . Note that this is a smooth point of the surface  $S_{12}$ .

Since  $\lambda a < 1$ , applying Corollary 2.7 to  $C_x$ , we get

$$\frac{1}{\lambda} < \text{mult}_{\mathfrak{p}}(C_x \cdot \Delta) \leq C_x \cdot \Delta = \frac{1}{3} - \frac{1}{6}a.$$

This implies that  $a < 0$  and therefore this is impossible.  $\square$

**Lemma 4.30.** *The log pair  $(S_{12}, \lambda D)$  is log canonical at  $Q_1 \in \text{Sing}(S_{12})$ .*

*Proof.* Let  $Q_1$  be a singular point of  $S_{12}$  with type  $\frac{1}{2}(1, 1)$ . Suppose that  $(S_{12}, \lambda D)$  is not log canonical at  $Q_1$ . Since  $\lambda a < 1$  and  $C_x$  is smooth at  $Q_1$ , using Lemma 2.12, we have

$$\frac{1}{2\lambda} < \text{mult}_{\mathfrak{p}}(C_x \cdot \Delta) \leq (C_x \cdot \Delta) = \frac{1}{3} - \frac{1}{6}a.$$

This implies that  $a < 0$  which is impossible, thus proving our claim.  $\square$

**Lemma 4.31.** *The log pair  $(S_{12}, \lambda D)$  is log canonical at  $Q_2 \in \text{Sing}(S_{12})$ .*

*Proof.* Let  $Q_2$  be a singular point of  $S_{12}$  with type  $\frac{1}{3}(1, 1)$ . Suppose that  $(S_{12}, \lambda D)$  is not log canonical at  $Q_2$ .

Let  $\pi_1: \tilde{S}_{12} \rightarrow S_{12}$  be the weighted blow-up at  $\mathfrak{p}$  with weight  $(1, 1) = (x, z)$ . Then we have

$$\begin{aligned} K_{\tilde{S}_{12}} &\sim_{\mathbb{Q}} \pi_1^*(K_{S_{12}}) - \frac{1}{3}E, & \tilde{D} &\sim_{\mathbb{Q}} \pi_1^*(D) - mE, \\ \tilde{C}_x &\sim_{\mathbb{Q}} \pi_1^*(C_x) - \frac{1}{3}E, \end{aligned}$$

where  $E$  is the exceptional divisor of  $\pi$  and  $m$  is a non-negative constant, and  $\tilde{C}_x$  and  $\tilde{\Delta}$  denote the strict transforms of  $C_x$  and  $\Delta$  on  $\tilde{S}_{12}$ , respectively. Note that we denote  $\tilde{D} = a\tilde{C}_x + \tilde{\Delta}$ .

The log pullback is given by

$$K_{\tilde{S}_{12}} + \lambda a\tilde{C}_x + \lambda\tilde{\Delta} + \left(\lambda m + \frac{1}{3}\right)E \sim_{\mathbb{Q}} \pi_1^*(K_{S_{12}} + \lambda D).$$

Thus, we have that  $(\tilde{S}_{12}, \lambda a\tilde{C}_x + \lambda\tilde{\Delta} + (\lambda m + \frac{1}{3})E)$  is not log canonical at a point  $\mathfrak{q} \in E$ . Let us bound the value of  $m$ .

Consider

$$\pi_1^*(-K_{S_{12}}) - uE \sim_{\mathbb{Q}} \pi_1^*(2C_x) - uE \sim_{\mathbb{Q}} 2\tilde{C}_x + \left(\frac{2}{3} - u\right)E.$$

Since  $\tilde{C}_x^2 = -\frac{1}{6} < 0$ ,  $\tau(E) = \frac{2}{3}$ . This divisor is nef when  $u \in [0, \frac{1}{3}]$ . When  $u \in [\frac{1}{3}, \frac{2}{3}]$ , we have the following Zariski decomposition

$$\pi_1^*(-K_{S_{12}}) - uE = \left((4 - 6u)\tilde{C}_x + \left(\frac{2}{3} - u\right)E\right) + (6u - 2)\tilde{C}_x.$$

Therefore,

$$\text{vol}(\pi_1^*(-K_{S_{12}}) - uE) = \begin{cases} (\pi_1^*(-K_{S_{12}}) - uE)^2 = \frac{2}{3} - 3u^2 & \text{for } 0 \leq u \leq \frac{1}{3} \\ \left(\frac{2}{3} - u\right)^2 (6\tilde{C}_x + E)^2 = 3\left(\frac{2}{3} - u\right)^2 & \text{for } \frac{1}{3} \leq u \leq \frac{2}{3}. \end{cases}$$

Then, by Theorem 4.22,

$$\begin{aligned}
m &\leq \frac{1}{(-K_{S_{12}})^2} \int_0^{\frac{2}{3}} \text{vol}(\pi_1^*(-K_{S_{12}}) - uE) \, du + \epsilon_k \\
&= \frac{3}{2} \left( \int_0^{\frac{1}{3}} \frac{2}{3} - 3u^2 \, du + \int_{\frac{1}{3}}^{\frac{2}{3}} 3 \left( \frac{2}{3} - u \right)^2 \, du \right) + \epsilon_k \\
&= \frac{1}{3} + \epsilon_k.
\end{aligned}$$

This implies that  $\lambda m + \frac{1}{3} < 1$ . We will now look at various possible positions of the point  $\mathbf{q}$ . Suppose that  $\mathbf{q} \in \tilde{C}_x$ . Since  $\lambda a < 1$ , by Corollary 2.7 we have

$$1 < \left( \lambda \tilde{\Delta} + \left( \lambda m + \frac{1}{3} \right) E \right) \cdot \tilde{C}_x = \lambda \left( \frac{1}{3} + \frac{a}{6} \right) + \frac{1}{3}.$$

It implies that  $\lambda(a+2) > 4$  and this is a contradiction. Thus  $\mathbf{q} \notin \tilde{C}_x$ . This implies that the log pair

$$\left( \tilde{S}_{12}, \lambda \tilde{\Delta} + \left( \lambda m + \frac{1}{3} \right) E \right)$$

is not log canonical at  $\mathbf{q} \in E \setminus \tilde{C}_x$ .

Let  $\pi_2: \hat{S}_{12} \rightarrow \tilde{S}_{12}$  be the blow-up at  $\mathbf{q}$  with exceptional divisor  $F$ . Let  $\hat{\Delta}$  and  $\hat{E}$  denote the strict transforms of  $\Delta$  and  $E$ , in  $\hat{S}_{12}$  respectively. Then we have

$$\begin{aligned}
K_{\hat{S}_{12}} &\sim_{\mathbb{Q}} \pi_2^*(K_{\tilde{S}_{12}}) - \frac{1}{3}\hat{E} + \frac{2}{3}F, & \hat{D} &\sim_{\mathbb{Q}} \pi_2^*(D) - m\hat{E} - (m+n)F, \\
\hat{C}_x &\sim_{\mathbb{Q}} \pi_2^*(C_x) - \frac{1}{3}\hat{E} - \frac{1}{3}F,
\end{aligned}$$

where  $n = \text{mult}_{\mathbf{q}}(\tilde{D})$  and  $\pi = \pi_1 \circ \pi_2$ .

Since

$$0 \leq \hat{\Delta} \cdot \hat{E} = 3m - n - a,$$

we have that  $\lambda(m+n) - \frac{2}{3} = \lambda(4m-a) - \frac{2}{3} < 1$ . Therefore, the log pair

$$\left( \hat{S}_{12}, \lambda a \hat{C}_x + \lambda \hat{\Delta} + \left( \lambda m + \frac{1}{3} \right) \hat{E} + \left( \lambda(m+n) - \frac{2}{3} \right) F \right)$$

is not log canonical at some point  $\mathbf{t} \in F$ . We will now look at different positions of the point  $\mathbf{t}$ . If  $\mathbf{t} \in \hat{E} \cap F$  then applying Corollary 2.7 to  $\hat{E}$ , we get

$$1 < \left( \lambda \hat{\Delta} + \left( \lambda(m+n) - \frac{2}{3} \right) F \right) \cdot \hat{E} = \lambda(4m-a) - \frac{2}{3}.$$

This implies that  $m > \frac{5}{12\lambda} + \frac{a}{4}$ . This is a contradiction and thus  $\mathbf{t} \in F \setminus \hat{E}$ . Then the log pair

$$\left( \hat{S}_{12}, \lambda \hat{\Delta} + \left( \lambda(m+n) - \frac{2}{3} \right) F \right)$$

is not log canonical at  $\mathbf{t}$ . Since  $(\lambda(m+n) - \frac{2}{3}) < 1$ , using Corollary 2.7, we get

$$\frac{1}{\lambda} < \widehat{\Delta} \cdot F = n.$$

Consider the linear system  $\mathcal{L} := \alpha(\zeta_1 x^3 y + \zeta_2 x^2 z + t) + \beta x^6 = 0$  where  $[\alpha : \beta] \in \mathbb{P}^1$ . Since the base locus of  $\mathcal{L}$  is a finite set, there is a member  $M \in \mathcal{L}$  such that  $M \not\subset \text{Supp}(D)$ . Let  $D = \nu M + \Delta$  where  $\Delta$  is an effective divisor such that  $M \not\subset \text{Supp}(\Delta)$  and  $\nu$  is a non-negative constant. Note that  $Q_2 \in M$  and we can choose  $M$  such that  $\widetilde{M} \in \widetilde{S}_{12}$  passes through the point  $\mathbf{q} \in E \setminus \widetilde{C}_x$  and  $\widehat{M} \in \widehat{S}_{12}$  passes through the point  $\mathbf{t} \in F \setminus \widehat{E}$ .

Let  $U$  be an open neighborhood at  $Q_2$ . Then the local equation of  $S$  is given by

$$t + t^2 + z^3 + a_1 z^2 x + a_2 z x^2 + a_3 x^3 + g(x, z) = 0$$

where  $g(x, z)$  is a polynomial with  $\deg(g(x, y)) > 3$ . Since  $M$  is given by  $t = -\zeta_1 x^3 - \zeta_2 x^2 z + b x^6$  on  $U$  the local equation of  $M$  is

$$z^3 + a_1 z^2 x + (a_2 - \zeta_2) x^2 z + (a_3 - \zeta_1) x^3 + h(x, z) = 0$$

where  $h(x, z)$  is a polynomial with  $\deg(h(x, z)) > 3$ . Thus we have  $\text{mult}_{Q_2}(M) = 3$ . It implies that

$$\pi_1^*(M) \sim_{\mathbb{Q}} \widetilde{M} + \frac{3}{3}E = \widetilde{M} + E$$

Note that  $M$  is a special member of  $\mathcal{M}$  since  $M$  has to satisfy  $\text{mult}_{\mathbf{q}}(\widetilde{M}) = 2$  and  $\widetilde{M} \cdot \widetilde{C}_x = 0$ .

For the above local equation  $a_1, a_2$  and  $a_3$  are fixed and  $\zeta_1$  and  $\zeta_2$  are free to be chosen. This shows that we can choose  $M$  such that the above conditions are satisfied.

Thus we have

$$\pi^*(M) \sim_{\mathbb{Q}} \widehat{M} + \widehat{E} + (2+1)F = \widehat{M} + \widehat{E} + 3F.$$

and we have the following intersection numbers,

$$\widehat{M}^2 = \widetilde{M}^2 + (2F)^2 = M^2 + E^2 + (2F)^2 = \frac{6 \cdot 6 \cdot 12}{1 \cdot 3 \cdot 4 \cdot 6} - 3 - 4 = -1,$$

$$\widehat{M} \cdot \widehat{E} = \widetilde{M} \cdot E + 2 \cdot 1F^2 = 3 - 2 = 1,$$

$$\widehat{M} \cdot F = 2.$$

Using this, we then obtain

$$0 \leq \widehat{\Delta} \cdot \widehat{M} = \widetilde{\Delta} \cdot \widetilde{M} - 2n = (\widetilde{D} - \widetilde{C}_x) \cdot \widetilde{M} - 2n = 2 - 3m - 2n.$$

This implies that  $2 > \frac{2}{\lambda} + 3m$ . However we have  $m > \frac{1}{3\lambda}$ . It is impossible.  $\square$

This completes the proof of Proposition 4.26 and thus by Theorem 2.25, surface  $S_{12}$  is K-polystable.

#### 4.2.2.4 $S_{15}$ in $\mathbb{P}(1, 4, 5, 7)$

Let  $S_{15}$  be a quasismooth, well-formed hypersurface in  $\mathbb{P}(1, 4, 5, 7)$ . By suitable change of coordinates,  $S_{15}$  can be given by a quasihomogeneous polynomial

$$t^2x + ty^2 + z^3 + xf(x, y, z) = 0$$

of degree 15 where  $f(x, y, z)$  is quasihomogeneous polynomial of degree 14. The surface  $S_{15}$  is singular at  $\mathbf{p}_y := [0 : 1 : 0 : 0]$  of type  $\frac{1}{4}(1, 1)$  and at  $\mathbf{p}_t := [0 : 0 : 0 : 1]$  of type  $\frac{1}{7}(4, 5)$ . Observe that both singular points lie on the curve  $C_x$ . Note that  $C_x$  is isomorphic to the variety given by

$$ty^2 + z^3 = 0$$

in  $\mathbb{P}(4, 5, 7)$ . This shows that the curve  $C_x$  is only singular at  $\mathbf{p}_t$  and is smooth at  $\mathbf{p}_y$ .

**Proposition 4.32.** *Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $S_{15}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{15}}$ . We can write that*

$$D = aC_x + \Delta$$

where  $a$  is non-negative rational number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Let  $\lambda = \frac{11}{10}$  and  $a \leq \frac{3}{4}$ . Then the log pair  $(S_{15}, \lambda D)$  is log canonical.

**Corollary 4.33.**  $\delta(S_{15}) \geq \frac{11}{10}$ .

Let us now prove Proposition 4.32.

**Lemma 4.34.** *The log pair  $(S_{15}, \lambda D)$  is log canonical along  $S_{15} \setminus C_x$ .*

*Proof.* Suppose the log pair  $(S_{15}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in S_{15} \setminus C_x$ . Then this point  $\mathbf{p}$  is a smooth point of the surface  $S_{15}$ . There is a divisor  $M \in |\mathcal{O}_{S_{15}}(4)|$  passing through  $\mathbf{p} \in M$ . Then  $M$  is given by the hyperplane section of  $y - \zeta x^4 = 0$  in  $S_{15}$  where  $\zeta$  is a constant. It is isomorphic to the variety given by

$$t^2x + t(\zeta x^4)^2 + z^3 + xf(x, \zeta x^4, z) = 0$$

in  $\mathbb{P}(1, 5, 7)$ . Note that the above quasihomogeneous polynomial is irreducible. Thus  $M$  is irreducible. Since  $M$  has the monomial  $t^2x$ , we have  $\text{mult}_{\mathbf{p}}(M) \leq 2$ . Let

$$D = bM + \Omega$$

where  $b$  is nonnegative number and  $\Omega$  is an effective divisor such that  $M \not\subset \text{Supp}(\Omega)$ . Note that  $b \leq \frac{1}{4}$  from Corollary 4.23.

If  $M$  is smooth at  $\mathbf{p}$  then we have

$$\frac{1}{\lambda} < (\Omega \cdot M)_{\mathbf{p}} \leq (D - bM) \cdot M = \frac{6}{7} - \frac{12}{7}b.$$

This implies that  $b < 0$  and therefore, this is impossible. Thus  $\text{mult}_{\mathbf{p}}(M) = 2$ . Then we have

$$\left(\frac{1}{\lambda} - 2b\right) \cdot 2 < \text{mult}_{\mathbf{p}}(D - bM) \text{mult}_{\mathbf{p}}(M) \leq (D - bM) \cdot M = \frac{6}{7} - \frac{12}{7}b.$$

It implies that  $b > \frac{7}{8\lambda} - \frac{3}{8}$ . This is impossible. Therefore the log pair  $(S_{15}, \lambda D)$  is log canonical along  $S_{15} \setminus C_x$ .  $\square$

**Lemma 4.35.** *The log pair  $(S_{15}, \lambda D)$  is log canonical along  $C_x \setminus \text{Sing}(S_{15})$ .*

*Proof.* Suppose that the log pair  $(S_{15}, \lambda D)$  is not log canonical at some point  $\mathfrak{p} \in S_{15} \setminus \text{Sing}(S_{15})$ . Since  $\lambda a < 1$ , using Corollary 2.7, we get

$$\frac{1}{\lambda} < \text{mult}_{\mathfrak{p}}(\Delta \cdot C_x) \leq \Delta \cdot C_x = (D - aC_x) \cdot C_x = \frac{3}{14} - \frac{3}{28}a. \quad (4.20)$$

It implies that  $a < 0$  which is impossible. Thus the log pair  $(S_{15}, \lambda D)$  is log canonical along  $\mathfrak{p} \in S_{15} \setminus \text{Sing}(S_{15})$ .  $\square$

We will now show that the log pair  $(S_{15}, \lambda D)$  is log canonical at the singular points  $\mathfrak{p}_y$  and  $\mathfrak{p}_t$ .

**Lemma 4.36.** *The log pair  $(S_{15}, \lambda D)$  is log canonical at  $\mathfrak{p}_y$ .*

*Proof.* Suppose not, i.e. suppose that the log pair  $(S_{15}, \lambda D)$  is not log canonical at  $\mathfrak{p}_y$ . Since  $\mathfrak{p}_y \in C_x$  with  $C_x$  smooth at  $\mathfrak{p}_y$  and  $\lambda a < 1$ , by Lemma 2.12 we have

$$\frac{1}{4\lambda} < \text{mult}_{\mathfrak{p}}(\Delta \cdot C_x) \leq \Delta \cdot C_x = (D - aC_x) \cdot C_x = \frac{3}{14} - \frac{3}{28}a.$$

It implies that  $a < 0$ . It is impossible.  $\square$

**Lemma 4.37.** *The log pair  $(S_{15}, \lambda D)$  is log canonical at  $\mathfrak{p}_t$ .*

*Proof.* Let  $\pi: \tilde{S}_{15} \rightarrow S_{15}$  be the weighted blow-up at  $\mathfrak{p}_t$  with weights  $\text{wt}(y) = 3$  and  $\text{wt}(z) = 2$ . Then we have the following.

$$K_{\tilde{S}_{15}} \sim_{\mathbb{Q}} \pi^*(K_{S_{15}}) - \frac{2}{7}E, \quad \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - mE, \quad \tilde{C}_x \sim_{\mathbb{Q}} \pi^*(C_x) - \frac{6}{7}E,$$

where  $E$  is the exceptional divisor and  $m$  is some non-negative number. Using the above equivalences, we have

$$0 \leq \tilde{\Delta} \cdot \tilde{C}_x = (\tilde{D} - a\tilde{C}_x) \cdot \tilde{C}_x = \frac{3}{14} - m + \frac{3}{4}a. \quad (4.21)$$

From the above equations we have

$$K_{\tilde{S}_{15}} + \lambda\tilde{D} + \left(\lambda m + \frac{2}{7}\right)E = \pi^*(K_{S_{15}} + \lambda D).$$

Thus the log pair  $(\tilde{S}_{15}, \lambda\tilde{D} + (\lambda m + \frac{2}{7})E)$  is not log canonical at some point  $\mathfrak{q} \in E$ .

We will now bound  $m$  by computing the volume of the pseudoeffective divisor  $\pi^*(-K_{S_{15}}) - uE$ . Consider

$$\pi^*(-K_{S_{15}}) - uE = \pi^*(2C_x) - uE = 2\tilde{C}_x + \left(\frac{12}{7} - u\right)E.$$

Note that if  $0 \leq u \leq \frac{3}{14}$  then  $\pi^*(2C_x) - uE$  is nef. Moreover if  $\frac{3}{14} \leq u \leq \frac{12}{7}$  then the Zariski Decomposition is given by

$$\pi^*(-K_{S_{15}}) - uE = \pi^*(2C_x) - uE = \left( \left( \frac{16}{7} - \frac{4}{3}u \right) \tilde{C}_x + \left( \frac{12}{7} - u \right) E \right) + \left( \frac{4x}{3} - \frac{2}{7} \right) \tilde{C}_x.$$

We then have the following.

$$\text{vol}(\pi^*(-K_{S_{15}}) - uE) = \begin{cases} \frac{3}{7} - \frac{7}{6}u^2 & \text{for } 0 \leq u \leq \frac{3}{14}, \\ \frac{u^2}{6} - \frac{4u}{7} + \frac{24}{49} & \text{for } \frac{3}{14} \leq u \leq \frac{12}{7}. \end{cases}$$

Therefore,

$$\begin{aligned} m &\leq \frac{1}{(-K_{S_{15}})^2} \int_0^{\frac{12}{7}} \text{vol}(\pi^*(-K_{S_{15}}) - uE) \, du + \epsilon_k \\ &= \frac{7}{3} \left( \int_0^{\frac{3}{14}} \left( \frac{3}{7} - \frac{7}{6}u^2 \right) \, du + \int_{\frac{3}{14}}^{\frac{12}{7}} \left( \frac{u^2}{6} - \frac{4u}{7} + \frac{24}{49} \right) \, du \right) + \epsilon_k, \\ &= \frac{9}{14} + \epsilon_k \end{aligned}$$

It implies that  $\lambda m + \frac{2}{7} < 1$ . Now we will consider different positions of the point  $\mathfrak{q}$ .

Suppose  $\mathfrak{q} \in E \setminus \tilde{C}_x$ . Then the point  $\mathfrak{q}$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  for  $r \leq 3$ . We have that the log pair

$$\left( \tilde{S}_{15}, \lambda \tilde{\Delta} + \left( \lambda m + \frac{2}{7} \right) E \right)$$

is not log canonical at  $\mathfrak{q}$ . Since  $\lambda m + \frac{2}{7} < 1$ , using Lemma 2.12, we have

$$\frac{1}{3\lambda} < \text{mult}_{\mathfrak{q}}(\tilde{\Delta} \cdot E) \leq (\tilde{\Delta} \cdot E) = (\tilde{D} - a\tilde{C}_x) \cdot E = \frac{7}{6}m - a.$$

Using (4.21) and that  $m \leq \frac{9}{14} + \epsilon_k$ , we get a contradiction. Hence  $\mathfrak{q} \in E \cap \tilde{C}_x$ . This implies that  $\mathfrak{q}$  is a smooth point of  $\tilde{S}_{15}$ . Then we have

$$\frac{1}{\lambda} < \text{mult}_{\mathfrak{q}}(\tilde{D} \cdot E) = \frac{7}{6}m \leq \frac{7}{6} \left( \frac{3}{14} + \frac{3a}{4} \right) = \frac{1}{4} + \frac{7a}{8}.$$

Note that we have used the bound for  $m$  from (4.21).

This implies that  $a > \frac{8}{7\lambda} - \frac{2}{7}$ , but  $a \leq \frac{3}{4}$  by assumption. Thus, this gives a contradiction, hence proving our claim.  $\square$

This completes the proof of Proposition 4.32 and thus by Theorem 2.25, surface  $S_{15}$  is K-polystable.

#### 4.2.2.5 $S_{16}$ in $\mathbb{P}(1, 4, 5, 8)$

Let  $S_{16} \subset \mathbb{P}(1, 4, 5, 8)$  be a quasismooth, well-formed hypersurface of degree 16. By a suitable coordinate change we can assume that  $S_{16}$  is given by a quasihomogeneous polynomial

$$t^2 - y^4 + xz^3 + x^2 f_{14}(x, y, z) = 0$$

where  $f_{14}(x, y, z)$  is a quasihomogeneous polynomial of degree 14. The surface  $S_{16}$  is singular at  $\mathbf{p}_z := [0 : 0 : 1 : 0]$  of type  $\frac{1}{5}(4, 3)$  and at  $Q_1 := [0 : 1 : 0 : -1]$  and  $Q_2 := [0 : 1 : 0 : 1]$  of type  $\frac{1}{4}(1, 1)$ . Note that both these singular points lie on the curve  $C_x$  which is isomorphic to the variety given by  $(t + a_1 y^2)(t + a_2 y^2) = 0$  in  $\mathbb{P}(4, 5, 8)$ , where  $a_1, a_2$  are distinct non-zero complex numbers. Thus we have

$$C_x = L_1 + L_2$$

where  $L_i$  is given by  $x = t + a_i y^2 = 0$  in  $\mathbb{P}(4, 5, 8)$ . In particular,  $C_x$  is smooth at the points  $Q_1, Q_2$  and is singular at  $\mathbf{p}_z$ . We have the following intersection numbers

$$L_1 \cdot L_2 = \frac{2}{5}, \quad L_i^2 = -\frac{7}{20}, \quad C_x \cdot L_i = \frac{1}{20}, \quad C_x^2 = \frac{1}{10}.$$

**Proposition 4.38.** *Let  $D$  be an effective  $\mathbb{Q}$ -divisor on  $S$  such that*

$$D \sim_{\mathbb{Q}} -K_{S_{16}}.$$

*Let  $D = \alpha L_1 + \beta L_2 + \Delta$ , where  $\alpha$  and  $\beta$  are non-negative constants and  $\Delta$  is an effective  $\mathbb{Q}$  divisor on the surface  $S$  whose support does not contain the curves  $L_1$  and  $L_2$ . Let  $\lambda = \frac{40}{39}$ . Then the log pair  $(S_{16}, \lambda D)$  is log canonical.*

Since  $L_2 \not\subset \text{Supp}(\Delta)$ , we have

$$\frac{2}{5}\alpha = \alpha L_1 \cdot L_2 \leq (D - \beta L_2) \cdot L_2 = \frac{1}{10} + \frac{7}{20}\beta.$$

It implies that

$$\alpha \leq \frac{1}{4} + \frac{7}{8}\beta. \tag{4.22}$$

This will come in handy for computations later.

We will now bound the values of  $\alpha$  and  $\beta$ . Consider

$$2C_x - uL_2 = 2L_1 + (2 - u)L_2.$$

Note that since  $L_2^2 < 0$ , in order for the divisor  $2C_x - uL_2$  to be pseudo-effective, we have  $u \leq 2$ . By multiplying the above equation by  $L_1$ , we have

$$(2C_x - uL_2) \cdot L_1 = \frac{1}{10} - \frac{2}{5}u.$$

Thus  $\frac{1}{4} \geq u$  implies that  $2C_x - uL_2$  is nef. For the case that  $2 \geq u \geq \frac{1}{4}$ , we have

$$2C_x - uL_2 = (2 - u) \left( \frac{8}{7}L_1 + L_2 \right) + \frac{8u - 2}{7}L_1.$$

Thus

$$\text{vol}(2C_x - uL_2) = \begin{cases} -\frac{7}{20}u^2 - \frac{1}{5}u + \frac{2}{5} & \text{for } 0 \leq u \leq \frac{1}{4}, \\ \frac{3}{28}(2-u)^2 & \text{for } \frac{1}{4} \leq u \leq 2. \end{cases}$$

We have

$$\begin{aligned} \beta &\leq \frac{1}{(-K_X)^2} \int_0^2 \text{vol}(2L_1 + (2-u)L_2) du + \epsilon_k \\ &= \frac{5}{2} \left( \int_0^{\frac{1}{4}} -\frac{7}{20}u^2 - \frac{1}{5}u + \frac{2}{5} du + \int_{\frac{1}{4}}^2 \frac{3}{28}(2-u)^2 du \right) + \epsilon_k, \\ &= \frac{5}{2} \left( \frac{353}{3840} + \frac{49}{256} \right) + \epsilon_k = \frac{17}{24} + \epsilon_k. \end{aligned}$$

Similar computations show that  $\alpha \leq \frac{17}{24} + \epsilon_k$ . This implies that  $\lambda\alpha < 1$  and  $\lambda\beta < 1$ .

Let us now prove Proposition 4.38.

**Lemma 4.39.** *The log pair  $(S_{16}, \lambda D)$  is log canonical along  $S_{16} \setminus C_x$ .*

*Proof.* Suppose the log pair  $(S_{16}, \lambda D)$  is not log canonical at a point  $\mathfrak{p} \in S_{16} \setminus C_x$ . This implies that  $\mathfrak{p}$  is a smooth point of the surface  $S_{16}$ . Note that  $C_y$  is isomorphic to the variety given by

$$t^2 + xz^3 + x^2 f_{14}(x, 0, z) = 0 \quad (4.23)$$

in  $\mathbb{P}(1, 5, 8)$ . Since the quasihomogeneous polynomial (4.23) has the monomial term  $xz^3$ , it is irreducible. It implies that  $C_y$  is irreducible. Write

$$D = bC_y + \Omega$$

where  $b$  is a nonnegative real number and  $\Omega$  is an effective  $\mathbb{Q}$ -divisor such that  $C_y \not\subset \text{Supp}(\Omega)$ . From Corollary 4.23, we see that  $b \leq \frac{1}{5}$  and this implies that  $\lambda b < 1$ .

We know that  $\text{mult}_{\mathfrak{p}}(C_y) \leq 2$ . Suppose that  $C_y$  is smooth at  $\mathfrak{p}$ , that is  $\text{mult}_{\mathfrak{p}}(C_y) = 1$ . Then we have

$$\frac{1}{\lambda} < \Omega \cdot C_y = (D - bC_y) \cdot C_y = \frac{4}{5} - \frac{8}{5}b.$$

This implies that  $b < 0$  and this is a contradiction. Thus we have  $\text{mult}_{\mathfrak{p}}(C_y) = 2$ . Since  $\mathfrak{p}$  is a smooth point of  $S_{16}$ , we have that

$$\frac{1}{\lambda} < \text{mult}_{\mathfrak{p}}(D) = \text{mult}_{\mathfrak{p}}(bC_y + \Omega) = 2b + \text{mult}_{\mathfrak{p}}(\Omega)$$

Using this we have,

$$\left( \frac{1}{\lambda} - 2b \right) 2 < \text{mult}_{\mathfrak{p}}(\Omega) \text{mult}_{\mathfrak{p}}(C_y) \leq \Omega \cdot C_y = \frac{4}{5} - \frac{8}{5}b.$$

This implies that

$$\left(\frac{2}{\lambda} - \frac{4}{5}\right) < b \leq \frac{1}{5}.$$

From this we have  $\frac{25}{16} < \lambda = \frac{5}{4}$ , which is absurd.  $\square$

**Lemma 4.40.** *The log pair  $(S_{16}, \lambda D)$  is log canonical along  $C_x \setminus \text{Sing}(S_{16})$ .*

*Proof.* We consider the case that  $\mathbf{p} \in C_x \setminus \text{Sing}(S_{16})$ . Without loss of generality we can assume that  $\mathbf{p} \in L_1$ . Then the log pair

$$(S_{16}, \lambda \alpha L_1 + \lambda \Delta)$$

is not log canonical at  $\mathbf{p}$ . Since  $\lambda \alpha < 1$ , using Corollary 2.7 we get that

$$\frac{1}{\lambda} < \Delta \cdot L_1 = (D - \alpha L_1 - \beta L_2) \cdot L_1 < \frac{1}{10} + \frac{7\alpha}{20}$$

Thus we have  $\frac{20}{7}(\frac{1}{\lambda} - \frac{1}{10}) < \alpha \leq \frac{1}{5}$ . This is a contradiction.  $\square$

Now we will prove that the log pair  $(S_{16}, \lambda D)$  is log canonical at the singular points of  $S_{16}$ .

**Lemma 4.41.** *The log pair  $(S_{16}, \lambda D)$  is log canonical at  $Q_1$  and  $Q_2$ .*

*Proof.* Observe that  $Q_1$  and  $Q_2$  lie on one of the components of  $C_x$ . Since  $Q_1$  and  $Q_2$  are both of type  $\frac{1}{4}(1, 1)$ , the following proof works for both singular points. Without loss of generality, we may assume that  $Q_1 \in L_1$ . Suppose that the log pair  $(S_{16}, \lambda \alpha L_1 + \lambda \Delta)$  is not log canonical at  $Q_1$ .

Since  $\alpha L_1 < 1$ , using Lemma 2.12 and the inequality (4.22), we get that

$$\begin{aligned} \frac{1}{4\lambda} &< (\Delta \cdot L_1)_{Q_1} \leq \Delta \cdot L_1 = (D - \alpha L_1 - \beta L_2) \cdot L_1 \\ &= \left(\frac{1}{10} + \frac{7}{20}\alpha - \frac{2}{5}\beta\right) \\ &\leq \left(\frac{1}{10} + \frac{7}{20}\left(\frac{1}{4} + \frac{7\beta}{20}\right) - \frac{2}{5}\beta\right) \\ &= \frac{3}{16}\left(1 - \frac{\beta}{2}\right). \end{aligned}$$

This implies that  $\beta < 0$ , which is absurd.  $\square$

**Lemma 4.42.** *The log pair  $(S_{16}, \lambda D)$  is log canonical at the point  $\mathbf{p}_z$ .*

*Proof.* Suppose that the log pair  $(S_{16}, \lambda D)$  is not log canonical at the point  $\mathbf{p}_z$ . Since  $\mathbf{p}_z \in L_1 \cap L_2$ ,  $L_1, L_2$  are smooth at  $\mathbf{p}_z$  and  $\lambda \alpha < 1$ ,  $\lambda \beta < 1$ , using Lemma 2.12, we have

$$\frac{1}{5\lambda} < (D - \alpha L_1) \cdot L_1 = (\beta L_2 + \Delta) \cdot L_1 = \frac{7\alpha}{20} + \frac{1}{10}, \quad (4.24)$$

$$\frac{1}{5\lambda} < (D - \beta L_2) \cdot L_2 = (\alpha L_1 + \Delta) \cdot L_2 = \frac{7\beta}{20} + \frac{1}{10}. \quad (4.25)$$

These inequalities will come in handy in the computations that follow.

Let  $\pi: \tilde{S}_{16} \rightarrow S_{16}$  be the weighted blow-up at  $\mathfrak{p}_z$  with weights  $\text{wt}(y) = 4$  and  $\text{wt}(t) = 3$ . Then we have

$$\begin{aligned} K_{\tilde{S}_{16}} &\sim_{\mathbb{Q}} \pi^*(K_{S_{16}}) + \frac{2}{5}E, & \tilde{D} &\sim_{\mathbb{Q}} \pi^*(D) - mE, \\ \tilde{L}_1 &\sim_{\mathbb{Q}} \pi^*(L_1) - \frac{3}{5}E, & \tilde{L}_2 &\sim_{\mathbb{Q}} \pi^*(L_2) - \frac{3}{5}E, \end{aligned}$$

where  $E$  is the exceptional divisor of  $\pi$ . The log pair

$$\left( \tilde{S}_{16}, \lambda \tilde{D} + \left( \lambda m - \frac{2}{5} \right) E \right)$$

is not log canonical at some point  $\mathfrak{q} \in E$ .

In order to bound the value of  $m$ , we will compute the volume of the pseudo-effective divisor  $\pi^*(-K_{S_{16}}) - uE$ . This is given by

$$\pi^*(2C_x) - uE \sim_{\mathbb{Q}} 2\tilde{L}_1 + 2\tilde{L}_2 + \left( \frac{12}{5} - u \right) E.$$

Since  $\tilde{L}_1$  and  $\tilde{L}_2$  are negative definite  $\tau(E) = \frac{12}{5}$ . This divisor is nef when  $u \in [0, \frac{2}{5}]$  and the Zariski Decomposition of the divisor when  $u \in [\frac{2}{5}, \frac{12}{5}]$  is given by

$$2\tilde{L}_1 + 2\tilde{L}_2 + \left( \frac{12}{5} - u \right) E = \left( \tilde{L}_1 + \tilde{L}_2 + E \right) \left( \frac{12}{5} - u \right) + \left( u - \frac{2}{5} \right) \left( \tilde{L}_1 + \tilde{L}_2 \right).$$

Using this we then have

$$\text{vol}(\pi^*(-K_{S_{16}}) - uE) = \begin{cases} (\pi^*(-K_{S_{16}}) - uE)^2 = \frac{2}{5} - \frac{5}{12}u^2 & \text{for } 0 \leq u \leq \frac{2}{5}, \\ \left( \tilde{L}_1 + \tilde{L}_2 + E \right)^2 \left( \frac{12}{5} - u \right)^2 = \frac{1}{60} \left( \frac{12}{5} - u \right)^2 & \text{for } \frac{2}{5} \leq u \leq \frac{12}{5}. \end{cases}$$

Then

$$m \leq \frac{1}{(-K_{S_{16}})^2} \int_0^{\tau(E)} \text{vol}(-K_{S_{16}} - uE) du + \epsilon_k = \frac{14}{15} + \epsilon_k.$$

It implies that  $\lambda m - \frac{2}{5} < 1$ . The point  $\mathfrak{q} \in E$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  where  $r \leq 4$ . Since  $\tilde{L}_i \cdot E = \frac{1}{4}$ ,  $\tilde{L}_i \cap E$  is the point with type  $\frac{1}{4}(1, 1)$ .

Suppose  $\mathfrak{q} \in E \setminus (\tilde{L}_1 \cup \tilde{L}_2)$ . Then  $\mathfrak{q}$  is either a smooth point or a singular point of type  $\frac{1}{3}(1, 1)$ .

Since we have that the log pair  $(\tilde{S}_{16}, \lambda \tilde{\Delta} + (\lambda m - \frac{2}{5})E)$  is not log canonical at  $\mathfrak{q}$  and  $(\lambda m - \frac{2}{5}) < 1$ , using Lemma 2.12, we get

$$\frac{1}{3\lambda} < \frac{1}{r\lambda} < \tilde{\Delta} \cdot E = \frac{5m}{12} - \frac{\alpha}{4} - \frac{\beta}{4}.$$

From (4.24), this gives a contradiction, thus showing that  $\mathbf{q} \in \tilde{L}_1 \cap \tilde{L}_2 \cap E$ . Note that  $E \cap \tilde{L}_1 = E \cap \tilde{L}_2$  and hence this is the only remaining position of the point  $\mathbf{q}$ .

Then we have that the log pair  $(\tilde{S}_{16}, \lambda\alpha\tilde{L}_1 + \beta\tilde{L}_2 + \lambda\tilde{\Delta} + (\lambda m - \frac{2}{5})E)$  is not log canonical at  $\mathbf{q}$ . Since  $\lambda\alpha < 1$ , we can use Lemma 2.12 on  $\tilde{L}_1$ , and we get that

$$\frac{1}{4} < \left( \lambda\beta\tilde{L}_2 + \lambda\tilde{\Delta} + (\lambda m - \frac{2}{5})E \right) \cdot \tilde{L}_1 = \lambda \left( \frac{1}{10} + \frac{\alpha}{2} \right) - \frac{1}{10}. \quad (4.26)$$

This implies that  $\alpha > \frac{7}{10\lambda} - \frac{1}{5}$ . Similarly, using Lemma 2.12, we get that  $\beta > \frac{7}{10\lambda} - \frac{1}{5}$ .

Again using Lemma 2.12, we get that

$$\frac{1}{4\lambda} < \left( \alpha\tilde{L}_1 + \beta\tilde{L}_2 + \tilde{\Delta} \right) \cdot E = \frac{5m}{12}.$$

Note that we don't get a contradiction in this case and thus we will consider a weighted blow-up  $\pi_2: \hat{S}_{16} \rightarrow S_{16}$  at  $\mathbf{p}_z$  with different weights, that is, let  $\text{wt}(y) = 1$  and  $\text{wt}(z) = 2$ . From this we have

$$\begin{aligned} K_{\hat{S}_{16}} &\sim_{\mathbb{Q}} \pi_2^*(K_{S_{16}}) - \frac{2}{5}F, & \hat{D} &\sim_{\mathbb{Q}} \pi_2^*(D) - nF, \\ \hat{L}_1 &\sim_{\mathbb{Q}} \pi_2^*(L_1) - \frac{2}{5}F, & \hat{L}_2 &\sim_{\mathbb{Q}} \pi_2^*(L_2) - \frac{2}{5}F, \end{aligned}$$

where  $F$  is the exceptional divisor of  $\pi_2$ . The log pair

$$\left( \hat{S}_{16}, \lambda \left( \alpha\hat{L}_1 + \beta\hat{L}_2 + \hat{\Delta} \right) + \left( \lambda n + \frac{2}{5} \right) F \right)$$

is not log canonical at some point  $\mathbf{q} \in F$ .

We will now bound  $n$  using Corollary 4.23 for the divisor  $\pi_2^*(-K_{\hat{S}_{16}}) - uF$ .

$$\pi_2^*(-K_{\hat{S}_{16}}) - uF \sim_{\mathbb{Q}} 2\hat{L}_1 + 2\hat{L}_2 + \left( \frac{8}{5} - u \right) F.$$

Since  $\hat{L}_1$  and  $\hat{L}_2$  are negative definite we have  $\tau(F) = \frac{8}{5}$ . This divisor is nef when  $u \in [0, \frac{1}{10}]$ . Therefore, the Zariski decomposition of the divisor when  $u \in [\frac{1}{10}, \frac{8}{5}]$  is given by

$$\pi_2^*(-K_{\hat{S}_{16}}) - uF = \left( \frac{32}{15} - \frac{4u}{3} \right) (\hat{L}_1 + \hat{L}_2) + \left( \frac{8}{5} - u \right) F + \left( \frac{4u}{3} - \frac{2}{15} \right) (\hat{L}_1 + \hat{L}_2).$$

We also have

$$\text{vol}(-K_{S_{16}} - uF) = \begin{cases} (-K_{S_{16}} - uF)^2 = \frac{2}{5} - \frac{5}{2}u^2 & \text{for } 0 \leq u \leq \frac{1}{10} \\ \left( \frac{4}{3}\hat{L}_1 + \frac{4}{3}\hat{L}_2 + F \right)^2 \left( \frac{8}{5} - u \right)^2 = \frac{1}{30} \left( \frac{8}{5} - u \right)^2 & \text{for } \frac{1}{10} \leq u \leq \frac{8}{5}. \end{cases}$$

Then

$$n \leq \frac{1}{(-K_{S_{16}})^2} \int_0^{\tau(F)} \text{vol}(-K_{S_{16}} - uF) du + \epsilon_k = \frac{17}{30} + \epsilon_k.$$

It implies that  $\lambda n + \frac{2}{5} < 1$ .

If  $\mathfrak{q} \in F \setminus (\widehat{L}_1 \cup \widehat{L}_2)$ , then we have that the log pair  $(\widehat{S}_{16}, \lambda \widehat{\Delta} + (\lambda n + \frac{2}{5}) F)$  is not log canonical at the point  $\mathfrak{q}$ . Using Lemma 2.12, we get

$$\frac{1}{2\lambda} < \widehat{\Delta} \cdot F = \frac{5n}{2} - \alpha - \beta$$

This, along with (4.26) and (4.22) give a contradiction.

Therefore,  $F \in \widehat{L}_1 \cup \widehat{L}_2$ . Without loss of generality, let us assume that  $\mathfrak{q} \in F \cup \widehat{L}_1$ . Then the log pair  $(\widehat{S}_{16}, \lambda(\alpha \widehat{L}_1 + \widehat{\Delta}) + (\lambda n + \frac{2}{5}) F)$  is not log canonical at  $\mathfrak{q}$ . Using Lemma 2.12, we get

$$\frac{1}{\lambda} < (\alpha \widehat{L}_1 + \widehat{\Delta}) \cdot F = \frac{5n}{2} - \beta$$

Again, this using the inequalities in (4.26) and (4.22) give a contradiction. This completes the proof of our claim.  $\square$

This completes the proof of Proposition 4.38 and thus by Theorem 2.25, surface  $S_{16}$  is K-polystable.

#### 4.2.2.6 $S_{18}$ in $\mathbb{P}(1, 4, 6, 9)$

Let  $S_{18}$  be a quasismooth weighted hypersurface of degree 18 in  $\mathbb{P}(1, 4, 6, 9)$ . By a suitable coordinate change  $S_{18}$  is given by a quasihomogeneous polynomial

$$t^2 + z^3 + y^3 z + x f(x, y, z) = 0$$

where  $f(x, y, z)$  is a quasihomogeneous polynomial of degree 17. The singular points of the surface are  $\mathfrak{p}_y$ ,  $[0 : -1 : 1 : 0]$ ,  $[0 : 0 : -1 : 1]$ , of types  $\frac{1}{4}(1, 1)$ ,  $\frac{1}{2}(1, 1)$  and  $\frac{1}{3}(1, 1)$  respectively. Note that all the singular points lie on  $C_x$  which is isomorphic to the variety given by the quasihomogeneous polynomial

$$t^2 + z^3 + y^3 z = 0$$

in  $\mathbb{P}(4, 6, 9)$ . From this we can check that  $C_x$  is quasismooth.

**Proposition 4.43.** *Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $S_{18}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{18}}$ . We can write that*

$$D = aC_x + \Delta$$

where  $a$  is non-negative rational number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Let  $\lambda = \frac{4}{3}$  and  $a \leq \frac{3}{4}$ . Then the log pair  $(S_{18}, \lambda D)$  is log canonical.

**Corollary 4.44.**  $\delta(S_{18}) \geq \frac{4}{3}$ .

Let us now prove Proposition 4.43.

**Lemma 4.45.** *The log pair  $(S_{18}, \lambda D)$  is log canonical along  $S_{18} \setminus C_x$ .*

*Proof.* Suppose that the log pair  $(S_{18}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in S_{18} \setminus C_x$ . This implies that the point  $\mathbf{p}$  is a smooth point of the surface. By a suitable coordinate change we can assume that  $\mathbf{p} = \mathbf{p}_x$ . Note that  $C_y$  is isomorphic to the variety given by the equation

$$t^2 + a_1 x^9 t + z^3 + a_2 x^6 z^2 + a_3 x^{12} z = 0$$

in  $\mathbb{P}(1, 6, 9)$  where  $a_1, a_2$  and  $a_3$  are constants. It is irreducible and  $\text{mult}_{\mathbf{p}}(C_y) \leq 2$ . Write

$$D = bC_y + \Omega$$

where  $b$  is non-negative constant and  $\Omega$  is an effective  $\mathbb{Q}$ -divisor such that  $C_y \not\subset \text{Supp}(\Omega)$ . From Corollary 4.23, we have that  $b \leq \frac{1}{5}$ .

If  $\text{mult}_{\mathbf{p}}(C_y) = 1$  then, by Corollary 2.7 we have

$$1 < \lambda \Omega \cdot C_y = \lambda(D - bC_y) \cdot C_y = \frac{8}{9} - \frac{16}{9}b.$$

This implies that  $b < 0$ .

Thus  $\text{mult}_{\mathbf{p}}(C_y) = 2$ . Then we have

$$\frac{8}{3} - \frac{16}{3}b < \text{mult}_{\mathbf{p}}(\lambda \Omega) \text{mult}_{\mathbf{p}}(C_y) \leq \lambda \Omega \cdot C_y = \frac{8}{9} - \frac{16}{9}b.$$

It implies that  $\frac{1}{2} < b$ . By Corollary 4.23, it is impossible. Therefore the log pair  $(S_{18}, \lambda D)$  is log canonical along  $S_{18} \setminus C_x$ .  $\square$

**Lemma 4.46.** *The log pair  $(S_{18}, \lambda D)$  is log canonical along  $C_x$ .*

*Proof.* Suppose that the log pair  $(S_{18}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in C_x$ . Then the singular point  $\mathbf{p}$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  where  $r \leq 4$ . Since  $\lambda a \leq 1$ , by Lemma 2.12, we have

$$\frac{1}{4\lambda} \leq \frac{1}{r\lambda} < \Delta \cdot C_x = (D - aC_x) \cdot C_x = \frac{1}{6} - \frac{a}{12}.$$

This implies that  $a < 0$  which is absurd. Thus the log pair  $(S, \lambda D)$  is log canonical along  $C_x$ .  $\square$

This completes the proof of Proposition 4.43 and thus by Theorem 2.25, surface  $S_{18}$  is K-polystable.

#### 4.2.2.7 $S_{22}$ in $\mathbb{P}(1, 5, 7, 11)$

Let  $S_{22}$  be a quasismooth weighted hypersurface in  $\mathbb{P}(1, 5, 7, 11)$  of degree 22. By a suitable coordinate change  $S_{22}$  is given by a quasihomogeneous polynomial

$$t^2 + y^3 z + x f(x, y, z) = 0$$

where  $f(x, y, z)$  is a quasihomogeneous polynomial of degree 21. The surface  $S_{22}$  is singular at points  $\mathbf{p}_y$  and  $\mathbf{p}_z$ , with singularities of type  $\frac{1}{5}(1, 1)$  and  $\frac{1}{7}(5, 4)$ , respectively. Note that both points lie on  $C_x$  which is isomorphic to the variety given by the quasihomogeneous polynomial

$$t^2 + y^3 z = 0$$

in  $\mathbb{P}(5, 7, 11)$ . From this we can check that  $C_x$  is irreducible and is singular at  $\mathbf{p}_z$ .

**Proposition 4.47.** *Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $S_{22}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{22}}$ . We can write that*

$$D = aC_x + \Delta$$

where  $a$  is non-negative rational number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Suppose that  $a \leq \frac{353}{504}$  and let  $\lambda = \frac{18}{17}$ . Then the log pair  $(S_{22}, \lambda D)$  is log canonical.

**Corollary 4.48.**  $\delta(S_{22}) \geq \frac{18}{17}$ .

Let us now prove Proposition 4.47.

**Lemma 4.49.** *The log pair  $(S_{22}, \lambda D)$  is log canonical along  $S_{22} \setminus C_x$ .*

*Proof.* Suppose that the log pair  $(S_{22}, \lambda D)$  is not log canonical at some point  $\mathfrak{p} \in S_{22} \setminus C_x$ . This implies that the point  $\mathfrak{p}$  is a smooth point of the surface. By a suitable coordinate change we can assume that  $\mathfrak{p} = \mathfrak{p}_x$ . Let  $\mathcal{L}$  be the linear system that is given by  $\alpha x^2 y + \beta z = 0$  with  $[\alpha : \beta] \in \mathbb{P}^1$ . Since the base locus of  $\mathcal{L}$  is the finite points set, there is an effective divisor  $M \in \mathcal{L}$  such that  $M \not\subset \text{Supp}(D)$ . We have the inequality

$$1 < \lambda D \cdot M = \frac{72}{85}.$$

It is impossible. Therefore the log pair  $(S_{22}, \lambda D)$  is log canonical along  $S_{22} \setminus C_x$ .  $\square$

**Lemma 4.50.** *The log pair  $(S_{22}, \lambda D)$  is log canonical along  $C_x \setminus \{\mathfrak{p}_z\}$ .*

*Proof.* Suppose that the log pair  $(S_{22}, \lambda D)$  is not log canonical at some point  $\mathfrak{p} \in C_x \setminus \{\mathfrak{p}_z\}$ . Then the point  $\mathfrak{p}$  is of type  $\frac{1}{r}(\mathfrak{a}, \mathfrak{b})$  where  $r \leq 5$ . Since  $\lambda a < 1$ , by Corollary 2.7, we have

$$\frac{1}{5} \leq \frac{1}{r} < \lambda \Delta \cdot C_x = \lambda(D - aC_x) \cdot C_x = \frac{72}{595} - \frac{36}{595}a.$$

It implies that  $a < 0$  which is impossible. Thus the log pair  $(S, \lambda D)$  is log canonical along  $C_x \setminus \{\mathfrak{p}_z\}$ .  $\square$

**Lemma 4.51.** *The log pair  $(S_{22}, \lambda D)$  is log canonical at  $\mathfrak{p}_z$ .*

*Proof.* Let  $\pi: \tilde{S}_{22} \rightarrow S_{22}$  be the weighted blow-up at  $\mathfrak{p}_z$  with weights  $\text{wt}(y) = 2$  and  $\text{wt}(t) = 3$ . Then we have

$$K_{\tilde{S}_{22}} \sim_{\mathbb{Q}} \pi^*(K_{S_{22}}) - \frac{2}{7}E, \quad \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - mE, \quad \tilde{C}_x \sim_{\mathbb{Q}} \pi^*(C_x) - \frac{6}{7}E$$

where  $E$  is the exceptional divisor of  $\pi$ . From the above equations we obtain

$$K_{\tilde{S}_{22}} + \lambda \tilde{D} + \left( \lambda m + \frac{2}{7} \right) E \sim_{\mathbb{Q}} \pi^*(K_{S_{22}} + \lambda D).$$

Then the log pair  $\left( \tilde{S}_{22}, \lambda \tilde{D} + \left( \lambda m + \frac{2}{7} \right) E \right)$  is not log canonical at some point  $\mathfrak{q} \in E$ .

We have the following intersection numbers:

$$E^2 = -\frac{7}{6}, \quad \tilde{C}_x^2 = -\frac{4}{5}, \quad \tilde{D} \cdot \tilde{C}_x = \frac{4}{35} - m, \quad \tilde{D} \cdot E = \frac{7}{6}m.$$

Meanwhile, the inequality

$$0 \leq \tilde{\Delta} \cdot \tilde{C}_x = (\tilde{D} - a\tilde{C}_x) \cdot \tilde{C}_x = \frac{4}{35} - m + \frac{4}{5}a$$

implies that

$$m \leq \frac{4}{35} + \frac{4}{5}a. \quad (4.27)$$

If  $\mathbf{q} \in \tilde{C}_x$  then, by Corollary 2.7 we have the following inequality.

$$1 < \left( \lambda \tilde{\Delta} + \left( \lambda m + \frac{2}{7} \right) E \right) \cdot \tilde{C}_x = \lambda \left( \frac{4}{35} + \frac{4}{5}a \right) + \frac{2}{7} \leq 1,$$

which is absurd. Thus  $\mathbf{q} \notin \tilde{C}_x$ .

The point  $\mathbf{q} \in E \setminus \tilde{C}_x$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  with  $r \leq 3$ . Since  $\lambda m + \frac{2}{7} \leq 1$ , by Corollary 2.7 and the inequality (4.27) we have the following:

$$\frac{1}{3\lambda} \leq \frac{1}{r\lambda} < \tilde{\Delta} \cdot E = \frac{7}{6}m - a \leq \frac{2}{15} - \frac{a}{15}.$$

This implies that  $a < 0$ , which is absurd. Thus the log pair  $(S_{22}, \lambda D)$  is log canonical at  $\mathbf{p}_z$ .  $\square$

This completes the proof of Proposition 4.47 and thus by Theorem 2.25, surface  $S_{22}$  is K-polystable.

#### 4.2.2.8 $S_{30}$ in $\mathbb{P}(1, 6, 10, 15)$

Let  $S_{30}$  be a quasismooth weighted hypersurface in  $\mathbb{P}(1, 6, 10, 15)$  of degree 30. By a suitable coordinate change  $S_{30}$  is given by a quasihomogeneous polynomial

$$t^2 + z^3 + y^5 + xf(x, y, z) = 0$$

where  $f(x, y, z)$  is a quasihomogeneous polynomial of degree 29. The surface is singular at points  $[0 : -1 : 1 : 0]$  (of type  $\frac{1}{2}(1, 1)$ ),  $[0 : -1 : 0 : 1]$  (of type  $\frac{1}{3}(1, 1)$ ),  $[0 : 0 : -1 : 1]$  (of type  $\frac{1}{5}(1, 1)$ ). Observe that all the singular points lie on the curve  $C_x$  which is isomorphic to the variety given by the quasihomogeneous polynomial

$$t^2 + z^3 + y^5 = 0$$

in  $\mathbb{P}(6, 10, 15)$ . From this equation of  $C_x$ , we can check that  $C_x$  is quasismooth.

**Proposition 4.52.** *Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $S_{30}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{30}}$ . We can write that*

$$D = aC_x + \Delta$$

where  $a$  is non-negative rational number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Suppose  $a \leq \frac{3}{4}$  and let  $\lambda = \frac{4}{3}$ . Then the log pair  $(S_{30}, \lambda D)$  is log canonical.

**Corollary 4.53.**  $\delta(S_{30}) \geq \frac{4}{3}$ .

Let us now prove Proposition 4.52.

**Lemma 4.54.** *The log pair  $(S_{30}, \lambda D)$  is log canonical along  $S_{30} \setminus C_x$ .*

*Proof.* Suppose that the log pair  $(S_{30}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in S_{30} \setminus C_x$ . This implies that the point  $\mathbf{p}$  is a smooth point of the surface. By a suitable coordinate change we can assume that  $\mathbf{p} = \mathbf{p}_x$ . Let  $\mathcal{L}$  be the linear system that is given by  $\alpha x^4 y + \beta z = 0$  with  $[\alpha : \beta] \in \mathbb{P}^1$ . Since the base locus of  $\mathcal{L}$  is the finite points set, there is an effective divisor  $M \in \mathcal{L}$  such that  $M \not\subset \text{Supp}(D)$ . We have the inequality

$$1 < \lambda D \cdot M = \frac{8}{9}.$$

It is impossible. Therefore the log pair  $(S_{30}, \lambda D)$  is log canonical along  $S_{30} \setminus C_x$ .  $\square$

**Lemma 4.55.** *The log pair  $(S_{30}, \lambda D)$  is log canonical along  $C_x$ .*

*Proof.* Suppose that the log pair  $(S_{30}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in C_x$ . Then the point  $\mathbf{p}$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  where  $r \leq 5$ . Since  $\lambda a \leq 1$ , by Lemma 2.12, we have the inequality

$$\frac{1}{5\lambda} \leq \frac{1}{r\lambda} < \Delta \cdot C_x = (D - aC_x) \cdot C_x = \left( \frac{1}{15} - \frac{1}{30}a \right).$$

It implies that  $a < 0$ . It is impossible. Thus the log pair  $(S, \lambda D)$  is log canonical along  $C_x$ .  $\square$

This completes the proof of Proposition 4.52 and thus by Theorem 2.25, surface  $S_{30}$  is K-polystable. .

#### 4.2.2.9 $S_{36}$ in $\mathbb{P}(1, 7, 12, 18)$

Let  $S_{36}$  be a quasismooth weighted hypersurface in  $\mathbb{P}(1, 7, 12, 18)$  of degree 36. By a suitable coordinate change  $S_{36}$  is given by a quasihomogeneous polynomial

$$t^2 + z^3 + xf(x, y, z) = 0$$

where  $f(x, y, z)$  is a quasihomogeneous polynomial of degree 35. The surface  $S_{36}$  is singular at points  $\mathbf{p}_y$  and  $[0 : 0 : -1 : 1]$ , of type  $\frac{1}{7}(2, 3)$  and  $\frac{1}{6}(1, 1)$  respectively. Observe that both points lie on the curve  $C_x$ , which is isomorphic to the variety given by the quasihomogeneous polynomial

$$t^2 + z^3 = 0$$

in  $\mathbb{P}(7, 12, 18)$ . From this we can check that  $C_x$  is irreducible and singular at  $\mathbf{p}_y$ .

**Proposition 4.56.** *Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $S_{36}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{36}}$ . We can write that*

$$D = aC_x + \Delta$$

where  $a$  is non-negative rational number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Suppose also that  $a \leq \frac{11}{16}$  and let  $\lambda = \frac{8}{7}$ . Then the log pair  $(S_{36}, \lambda D)$  is log canonical.

**Corollary 4.57.** *One has  $\delta(S) \geq \frac{8}{7}$ .*

Let us now prove Proposition 4.56.

**Lemma 4.58.** *The log pair  $(S_{36}, \lambda D)$  is log canonical along  $S_{36} \setminus C_x$ .*

*Proof.* Suppose that the log pair  $(S_{36}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in S_{36} \setminus C_x$ . This implies that the point  $\mathbf{p}$  is a smooth point of the surface. By a suitable coordinate change we can assume that  $\mathbf{p} = \mathbf{p}_x$ . Let  $\mathcal{L}$  be the linear system that is given by  $\alpha x^5 y + \beta z = 0$  with  $[\alpha : \beta] \in \mathbb{P}^1$ . Since the base locus of  $\mathcal{L}$  is the finite points set, there is an effective divisor  $M \in \mathcal{L}$  such that  $M \not\subset \text{Supp}(D)$ . We have the inequality

$$1 < \lambda D \cdot M = \frac{32}{49}.$$

It is impossible. Therefore the log pair  $(S_{36}, \lambda D)$  is log canonical along  $S_{36} \setminus C_x$ .  $\square$

**Lemma 4.59.** *The log pair  $(S_{36}, \lambda D)$  is log canonical along  $C_x \setminus \{\mathbf{p}_y\}$ .*

*Proof.* Suppose that the log pair  $(S_{36}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in C_x \setminus \{\mathbf{p}_y\}$ . Then the point  $\mathbf{p}$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  where  $r \leq 6$ . Since  $\lambda a \leq 1$ , by Corollary 2.7, we have the inequality

$$\frac{1}{6} \leq \frac{1}{r} < \lambda \Delta \cdot C_x = \lambda(D - aC_x) \cdot C_x = \lambda \left( \frac{1}{21} - \frac{1}{42}a \right).$$

It implies that  $a < 0$ . It is impossible. Thus the log pair  $(S, \lambda D)$  is log canonical along  $C_x \setminus \{\mathbf{p}_y\}$ .  $\square$

**Lemma 4.60.** *The log pair  $(S_{36}, \lambda D)$  is log canonical at  $\mathbf{p}_y$ .*

*Proof.* Let  $\pi: \tilde{S}_{36} \rightarrow S_{36}$  be the weighted blow-up at  $\mathbf{p}_y$  with weights  $\text{wt}(z) = 2$  and  $\text{wt}(t) = 3$ . Then we have

$$K_{\tilde{S}_{36}} \sim_{\mathbb{Q}} \pi^*(K_{S_{36}}) - \frac{2}{7}E, \quad \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - mE, \quad \tilde{C}_x \sim_{\mathbb{Q}} \pi^*(C_x) - \frac{6}{7}E$$

where  $E$  is the exceptional divisor of  $\pi$ . From above equations we obtain

$$K_{\tilde{S}_{36}} + \lambda \tilde{D} + \left( \lambda m + \frac{2}{7} \right) E \sim_{\mathbb{Q}} \pi^*(K_{S_{36}} + \lambda D).$$

Then the log pair  $(\tilde{S}_{36}, \lambda \tilde{D} + (\lambda m + \frac{2}{7}) E)$  is not log canonical at a point  $\mathbf{q} \in E$ .

We have the following intersection numbers:

$$E^2 = -\frac{7}{6}, \quad \tilde{C}_x^2 = -\frac{5}{6}, \quad \tilde{D} \cdot \tilde{C}_x = \frac{1}{21} - m, \quad \tilde{D} \cdot E = \frac{7}{6}m.$$

Meanwhile, the inequality

$$(\tilde{D} - a\tilde{C}_x) \cdot \tilde{C}_x = \frac{1}{21} - m + \frac{5}{6}a \geq 0$$

implies that

$$\frac{1}{21} + \frac{5}{6}a \geq m. \quad (4.28)$$

If  $\mathbf{q} \in \tilde{C}_x$  then, by Corollary 2.7 we have the following inequality.

$$1 < \tilde{C}_x \cdot \left( \lambda \tilde{\Delta} + \left( \lambda m + \frac{2}{7}E \right) \right) = \lambda \left( \frac{1}{21} + \frac{5}{6}a \right) + \frac{2}{7} \leq \frac{195}{196},$$

which is absurd. Thus  $\mathbf{q} \notin \tilde{C}_x$ .

Since  $\mathbf{q} \in E \setminus \tilde{C}_x$ , we have that the log pair  $(\tilde{S}_{36}, \lambda \tilde{\Delta} + (\lambda m + \frac{2}{7}E))$  is not log canonical at  $\mathbf{q}$ . The point  $\mathbf{q}$  is a point of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  with  $r \leq 3$ . Since  $\lambda m + \frac{2}{7} \leq 1$ , by Corollary 2.7 and the inequality (4.28) we have the following:

$$\frac{1}{3\lambda} \leq \frac{1}{r\lambda} < \tilde{\Delta} \cdot E = (\tilde{D} - a\tilde{C}_x) \cdot E = \frac{7}{6}m - a \leq \frac{1}{18} - \frac{a}{36},$$

which implies that  $a < 0$ . Thus the log pair  $(S_{36}, \frac{8}{7}D)$  is log canonical at  $\mathbf{p}_y$ .  $\square$

This completes the proof of Proposition 4.56 and thus by Theorem 2.25, surface  $S_{36}$  is K-polystable. .

#### 4.2.2.10 $S_{40}$ in $\mathbb{P}(1, 8, 13, 20)$

Let  $S_{40}$  be a quasismooth weighted hypersurface in  $\mathbb{P}(1, 8, 13, 20)$  of degree 40. By a suitable coordinate change  $S_{40}$  is given by a quasihomogeneous polynomial

$$t^2 + y^5 + xf(x, y, z) = 0$$

where  $f(x, y, z)$  is a quasihomogeneous polynomial of degree 39. The surface  $S_{40}$  is singular at points  $\mathbf{p}_z$  of type  $\frac{1}{13}(2, 5)$  and at  $Q := [0 : -1 : 0 : 1]$  of type  $\frac{1}{4}(1, 1)$ . Note that the hyperplane section  $C_x$  that is cut out by the equation  $x = 0$  in  $S$ , is isomorphic to the variety given by the quasihomogeneous polynomial

$$t^2 + y^5 = 0$$

in  $\mathbb{P}(8, 13, 20)$ , and contains both the singular points of the surface  $S_{40}$ . From the equation of  $C_x$  we can also check that  $C_x$  is irreducible and singular at  $\mathbf{p}_z$ .

**Proposition 4.61.** *Let  $D$  be an effective  $\mathbb{Q}$ -Cartier divisor of  $S_{40}$  such that  $D \sim_{\mathbb{Q}} -K_{S_{40}}$ . We can write that*

$$D = aC_x + \Delta$$

where  $a$  is non-negative rational number and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor such that  $C_x \not\subset \text{Supp}(\Delta)$ . Suppose that  $a \leq \frac{11}{16}$  and  $\lambda = \frac{79}{78}$ . Then the log pair  $(S, \lambda D)$  is log canonical.

**Corollary 4.62.**  $\delta(S_{40}) \geq \frac{79}{78}$ .

Let us now prove Proposition 4.61.

**Lemma 4.63.** *The log pair  $(S_{40}, \lambda D)$  is log canonical along  $S_{40} \setminus C_x$ .*

*Proof.* Suppose that the log pair  $(S_{40}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in S_{40} \setminus C_x$ . This implies that  $\mathbf{p}$  is a smooth point of the surface. By a suitable coordinate change we can assume that  $\mathbf{p} = \mathbf{p}_x$ . Let  $\mathcal{L}$  be the linear system that is given by  $\alpha x^5 y + \beta z = 0$  with  $[\alpha : \beta] \in \mathbb{P}^1$ . Since the base locus of  $\mathcal{L}$  is the finite points set, there is an effective divisor  $M \in \mathcal{L}$  such that  $M \not\subset \text{Supp}(D)$ . We have the inequality

$$1 < \lambda D \cdot M = \frac{79}{156}.$$

It is impossible. Therefore the log pair  $(S_{40}, \lambda D)$  is log canonical along  $S_{40} \setminus C_x$ .  $\square$

**Lemma 4.64.** *The log pair  $(S_{40}, \lambda D)$  is log canonical along  $C_x \setminus \{\mathbf{p}_z\}$ .*

*Proof.* Suppose that the log pair  $(S_{40}, \lambda D)$  is not log canonical at some point  $\mathbf{p} \in C_x \setminus \{\mathbf{p}_z\}$ . Then the point  $\mathbf{p}$  is of type  $\frac{1}{r}(\mathbf{a}, \mathbf{b})$  where  $r \leq 4$ . Since  $\lambda a \leq 1$ , by Corollary 2.7, we have the inequality

$$\frac{1}{4} \leq \frac{1}{r} < \lambda \Delta \cdot C_x = \lambda(D - aC_x) \cdot C_x = \lambda \left( \frac{1}{26} - \frac{a}{52} \right).$$

It implies that  $a < 0$ . It is impossible. Thus the log pair  $(S, \lambda D)$  is log canonical along  $C_x \setminus \{\mathbf{p}_z\}$ .  $\square$

**Lemma 4.65.** *The log pair  $(S_{40}, \lambda D)$  is log canonical at  $\mathbf{p}_z$ .*

*Proof.* Let  $\pi: \tilde{S}_{40} \rightarrow S_{40}$  be the weighted blow-up at  $\mathbf{p}_z$  with weights  $\text{wt}(y) = 2$  and  $\text{wt}(t) = 5$ . Then

$$K_{\tilde{S}_{40}} \sim_{\mathbb{Q}} \pi^*(K_{S_{40}}) - \frac{6}{13}E, \quad \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - mE, \quad \tilde{C}_x \sim_{\mathbb{Q}} \pi^*(C_x) - \frac{10}{13}E.$$

where  $m$  is a non-negative constant and  $E$  is the exceptional divisor of  $\pi$ . From the above equations, we have

$$K_{\tilde{S}_{40}} + \lambda \tilde{D} + \left( \lambda m + \frac{6}{13} \right) E \sim_{\mathbb{Q}} \pi^*(K_{S_{40}} + \lambda D).$$

Then the log pair  $(\tilde{S}_{40}, \lambda \tilde{D} + (\lambda m + \frac{6}{13}) E)$  is not log canonical at some  $\mathbf{q} \in E$ .

We have the following intersection numbers:

$$E^2 = -\frac{13}{10}, \quad \tilde{C}_x^2 = -\frac{3}{4}, \quad \tilde{D} \cdot \tilde{C}_x = \frac{1}{26} - m, \quad \tilde{D} \cdot E = \frac{13}{10}m.$$

By Corollary 4.23 we have  $a \leq \frac{3}{4}$  for  $k \gg 1$ . To find a bound of the constant  $m$  we compute the volume of the pseudoeffective divisor  $\pi^*(-K_{S_{40}}) - uE$  where  $u$  is a non-negative real number. Consider

$$\pi^*(-K_{S_{40}}) - uE \sim_{\mathbb{Q}} 2\tilde{C}_x + \left( \frac{20}{13} - u \right) E.$$

Since  $\tilde{C}_x^2 < 0$ , we have  $\tau(E) = \frac{20}{13}$ , and  $\text{vol}(\pi^*(-K_{S_{40}}) - uE) = 0$  for  $u > \tau(E)$ . Since

$$\left( 2\tilde{C}_x + \left( \frac{20}{13} - u \right) E \right) \cdot \tilde{C}_x = \frac{1}{26} - u,$$

$\pi^*(-K_{S_{40}}) - uE$  is nef for  $0 \leq u \leq \frac{1}{26}$ . Thus we have

$$\text{vol}(\pi^*(-K_{S_{40}}) - uE) = (\pi^*(-K_{S_{40}}) - uE)^2 = \frac{1}{13} - \frac{13}{10}u^2$$

for  $0 \leq u \leq \frac{1}{26}$ . We consider the case when  $\frac{1}{26} \leq u \leq \tau(E)$ . Since

$$\left(\frac{20}{13} - u\right) \left(\frac{4}{3}\tilde{C}_x + E\right) \cdot \tilde{C}_x = 0,$$

the Zariski decomposition of  $\pi^*(-K_{S_{40}}) - uE$ , in this interval, is

$$\left(\frac{20}{13} - u\right) \left(\frac{4}{3}\tilde{C}_x + E\right) + \left(-\frac{2}{39} + \frac{4}{3}u\right) \tilde{C}_x.$$

From this we have

$$\text{vol}(\pi^*(-K_{S_{40}}) - uE) = \left(\frac{20}{13} - u\right)^2 \left(\frac{4}{3}\tilde{C}_x + E\right)^2 = \left(\frac{20}{13} - u\right)^2 \frac{1}{30}.$$

Consequently, we have

$$\begin{aligned} m &\leq \frac{1}{(-K_{S_{40}})^2} \int_0^{\frac{20}{13}} \text{vol}(\pi^*(-K_{S_{40}}) - uE) du + \epsilon_k \\ &= 13 \left( \int_0^{\frac{1}{26}} \frac{1}{13} - \frac{13}{10}u^2 du + \int_{\frac{1}{26}}^{\tau(E)} \left(\frac{20}{13} - u\right)^2 \frac{1}{30} du \right) + \epsilon_k = \frac{41}{78} + \epsilon_k \end{aligned}$$

where  $\epsilon_k$  is a small constant depending on  $k$  such that  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . For  $k \gg 1$  we can assume that

$$m \leq \frac{42}{79}.$$

It implies that  $\lambda m + \frac{6}{13} \leq 1$ .

Meanwhile, the inequality

$$(\tilde{D} - a\tilde{C}_x) \cdot \tilde{C}_x = \frac{1}{26} - m + \frac{3}{4}a \geq 0$$

implies that

$$\frac{1}{26} + \frac{3}{4}a \geq m.$$

If  $\mathbf{q} \in \tilde{C}_x$  then Corollary 2.7 we have

$$\frac{1}{\lambda} < \tilde{D} \cdot E = \frac{13}{10} \leq \frac{13}{10} \left( \frac{1}{26} + \frac{3}{4}a \right).$$

This contradicts  $0 \leq a \leq \frac{11}{16}$ . Thus  $\mathbf{q} \notin \tilde{C}_x$ . Thus the log pair  $(\tilde{S}_{40}, \lambda\tilde{\Delta} + (\lambda m + \frac{6}{13})E)$  is not log canonical at the point  $\mathbf{q} \in E \setminus \tilde{C}_x$ .

Since the point  $\mathbf{q}$  is of type  $\frac{1}{r}(a, b)$  with  $r \leq 5$ , by Corollary 2.7 and the above inequality, we have

$$\frac{1}{5\lambda} \leq \frac{1}{r\lambda} < \tilde{\Delta} \cdot E = \frac{13}{10}m - a \leq \frac{1}{20} - \frac{a}{40}.$$

It is impossible. Therefore the log pair  $(S_{40}, \lambda D)$  is log canonical at  $\mathbf{p}_z$ .  $\square$

This completes the proof of Proposition 4.61 and thus by Theorem 2.25, surface  $S_{40}$  is K-polystable.

## 4.3 Existence of polar cylinders

In Section 4.3, we explicitly construct  $H$ -polar cylinders for any ample divisor  $H$  on a del Pezzo surface  $S$  with du Val singular points.

Recall from Theorem 1.4, that we have a complete description of the existence of  $(-K_S)$ -polar cylinders on del Pezzo surfaces with du Val singularities. We consider the surfaces on which  $(-K_S)$ -polar cylinders exist and in this Section, we prove

**Theorem 4.66.** *(Also Main Result 4) Let  $S$  be a del Pezzo surface with at least 1 singular point of type  $E_6, E_7, E_8, D_4, D_5, D_6, D_7$  or  $A_7$ . Then  $S$  has a  $H$ -polar cylinder for any ample divisor  $H \not\equiv -K_S$  on  $S$ .*

### 4.3.1 Foundations

#### 4.3.1.1 Ample divisors on surface $S$

Suppose  $S$  is a del Pezzo surface with du Val singular points and  $H$  is an ample divisor on  $S$ .

Let us denote the type of singularity on  $S$  by  $\text{Type}(S)$ . Take  $\phi : \tilde{S} \rightarrow S$  to be the minimal resolution of the surface and let  $\#(S)$  be the number of  $(-1)$ -curves on the surface  $S$ .

Then, observe that the surface  $S$  is completely determined by the  $\text{Type}(S)$  and  $\#(S)$ , which are in fact finite.

Adopting the required terminology from [CP21], consider the following.

**Definition 13.** *An irreducible curve  $L \subset S$  is a line if  $L = \phi(\tilde{L})$  for a  $(-1)$ -curve  $\tilde{L} \subset \tilde{S}$ .*

We now recall some of the useful results that will help us in describing the ample divisors on  $S$ .

**Lemma 4.67.** *[CP21, Lemma 2.8] Suppose that the degree of  $S$  is  $d \leq 7$ . For any singular point of  $S$  there is a line passing through it.*

**Lemma 4.68.** *[CP21, Lemma 2.9] Suppose that  $d \leq 7$ , where  $d$  is the degree of the surface  $S$ . Then the following assertions hold.*

1. *The group  $\text{Cl}(S)$  is generated by the classes of lines in  $S$ .*
2. *If  $\rho(S) = 1$  and  $S$  contains two distinct lines  $L$  and  $L'$ , then  $L \not\sim L'$  and  $L \sim_{\mathbb{Q}} L'$ .*
3. *Every extremal ray of the Mori cone  $\overline{\text{NE}}(S)$  is generated by the class of a line.*
4. *For every effective divisor  $D \in \text{Cl}(S)$ , there are  $a_0, a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}$  such that*

$$D \sim a_0(-K_S) + \sum_{i=1}^r a_i L_i$$

*where  $L_1, \dots, L_r$  are lines in  $S$ ,  $r = \#(S)$ , and  $a_0 = 0$  if  $d \neq 1$ .*

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**Corollary 4.69.** [CP21, Corollary 2.10] *One has  $\#(S) \geq \rho(S)$ . Moreover, if  $\#(S) = \rho(S)$ , then  $\text{Cl}(S)$  is torsion free, and every line in  $S$  generates an extremal ray of the Mori cone  $\overline{\text{NE}}(S)$ .*

Observe that a line  $L$  on the surface  $S$  generates an extremal ray of  $\overline{\text{NE}}(S)$   
 $\iff L^2 \leq 0$ .

**Lemma 4.70** ([M85, Proposition 1.2],[P01, § 7.1]). *Let  $S$  be a surface that has Du Val singularities, and let  $\psi: S \rightarrow Y$  be an extremal Mori contraction. Then one of the following holds:*

1. *either  $\psi$  is a weighted blow up of a smooth point in  $Y$  with weights  $(1, n)$ , the exceptional curve  $E$  is smooth and rational, one has  $E^2 = -\frac{1}{n}$ , and  $E \cap \text{Sing}(S)$  consists of one point which is of type  $A_{n-1}$  on  $S$ ;*
2. *or  $\psi$  is a conic bundle, one has  $-K_S \cdot F = 2$  and  $F_{\text{red}} \cong \mathbb{P}^1$  for any its scheme fiber  $F$ , and if  $F$  is not reduced, then one of the following three cases holds:*
  - *$F \cap \text{Sing}(S)$  consists of two singular points of type  $A_1$ ;*
  - *$F \cap \text{Sing}(S)$  consists of one singular point of type  $A_3$ ;*
  - *$F \cap \text{Sing}(S)$  consists of one singular point of type  $D_n$ , where  $n \geq 4$ .*

*In the case 1, we say that  $\psi$  is a  $(1, n)$ -contraction.*

Applying this lemma to our du Val del Pezzo surface  $S$ , we get

**Corollary 4.71.** [CP21, Corollary 2.13] *Let  $E$  be an irreducible curve on  $S$  such that  $E^2 < 0$ . Then  $E$  is a line on  $S$ , and  $E$  is an exceptional divisor of a  $(1, n)$ -contraction for some  $n \geq 1$ .*

**Corollary 4.72.** [CP21, Corollary 2.14] *Suppose there exists a birational morphism  $\psi: S \rightarrow Y$  that is a  $(1, n)$ -contraction, and let  $E$  be the exceptional curve of the morphism  $\psi$ . Then*

- *the point  $\psi(E)$  is a smooth point of the surface  $Y$ ;*
- *$Y$  is a Du Val del Pezzo surface,  $K_Y^2 = d + n$  and  $\rho(Y) = \rho(S) - 1$ ;*
- *the point  $\psi(E)$  is not contained in a line in  $Y$ .*

**Corollary 4.73.** [CP21, Corollary 2.15] *Let  $\psi: S \rightarrow Y$  be a contraction of a proper face of the cone  $\overline{\text{NE}}(S)$ . Then*

- *either the morphism  $\psi$  is birational,  $Y$  is a Du Val del Pezzo surface, and  $\psi$  contracts a disjoint union of lines on the surface  $S$ ,*
- *or the morphism  $\psi$  is a conic bundle and  $Y \cong \mathbb{P}^1$ .*

**Notations:** If  $\psi$  is a birational contraction of a proper face of the cone  $\overline{\text{NE}}(S)$  that is generated by lines  $L_i$  on  $S$  for  $i = 1, \dots, (\rho(S) - 1)$ , then ample divisor  $H$ , upto scaling, is given by

$$H \equiv -K_S + \sum_{i=1}^{\rho(S)-1} a_i L_i,$$

where  $a_i$  are non-negative constants. These curves  $L_i$  are mutually disjoint and their intersection numbers are negative definite.

*Note: Not all  $a_i = 0$ .*

If the morphism  $\psi$  is a conic bundle, then, upto scaling,

$$H \equiv -K_S + bC + \sum_{i=1}^{\rho(S)-2} a_i L_i,$$

where  $a_i$  are non-negative constants,  $b$  is a positive constant,  $C$  is a generic fibre of  $|C|$ ;  $L_i$  are mutually disjoint irreducible components of fibres of the conic bundle given by  $|C|$  such that  $L_i \cdot C = 0$ .

*Note: At least one of  $a_i$  or  $b$  is non-zero.*

In Subsection 4.3.2, given a surface  $S$  of  $\text{Type}(S)$ , we will explicitly give all possible descriptions of ample divisors  $H$  on  $S$  depending on the geometry of  $S$  which in turn determines the existence of the lines  $L_i$  on  $S$ . Using these descriptions for  $H$ , we will then construct an example of a  $H$ -polar cylinder for every case.

#### 4.3.1.2 Constructing the cylinder

We will now describe the method used to construct a  $H$ -polar cylinder on a del Pezzo surface  $S$  with at most du Val singular points where  $H$  is any ample divisor on  $S$  such that  $H \notin \mathbb{R}_+[-K_S]$ .

**Step 1:** Consider either  $\mathbb{F}_1$  or  $\mathbb{F}_2$  and take curves  $M, F_1, \dots, F_r$  where  $M$  is the unique negative section on  $\mathbb{F}_n$ ,  $F_1, \dots, F_r$  are fibres of  $\mathbb{F}_n$ , with  $r \leq 4$ . Note that we have

$$\mathbb{F}_n \setminus (M + F_1 + \dots + F_r) \cong \mathbb{A}^1 \times (\mathbb{A}^*)^{r-1}.$$

This is the cylinder that we are going to work with. Observe that this is a cylinder on  $\mathbb{F}_n$ . So we next *birationally modify* this cylinder, to give us the required cylinder on the surface  $S$ .

**Step 2:** The birational modification of the curves can be described as below:

$$S^i = \bar{S} \xrightarrow{\pi_i} S^{i-1} \xrightarrow{\pi_{i-1}} \dots \xrightarrow{\pi_2} S^1 \xrightarrow{\pi_1} \mathbb{F}_n = S^0$$

and where we denote  $\pi = \pi_i \circ \pi_{i-1} \circ \dots \circ \pi_1$ .

Here,  $\pi_j : S^j \rightarrow S^{j-1}$  for  $1 \leq j \leq r$  are the blow ups of points on fibres  $F_1, \dots, F_r$  of  $\mathbb{F}_n$  and for  $r+1 \leq j \leq i$ , are the blow ups on the proper transforms of these fibres  $F_1, \dots, F_r$ .

To be explicit, we first blow up points on the fibres of  $\mathbb{F}_n$  and then on the proper transforms of these fibres, until we get the configuration of curves on  $\bar{S}$

and we do this in such a way that we do not destroy the cylinder from Step 1. This is achieved by performing these blow ups away from the negative section  $M$ .

**Step 3:** Now, observe that we can choose non-negative rational constants  $\alpha, \beta, \gamma, \delta$  such that  $\alpha + \beta + \gamma + \delta = 3$  (and  $\alpha + \beta + \gamma + \delta = 4$ ) if  $n = 1$  ( $n = 2$  respectively) and thus have

$$-K_{\mathbb{F}_n} \equiv 2M + \alpha F_1 + \beta F_2 + \gamma F_3 + \delta F_4.$$

After the birational modifications performed in Step 2, we then take the log pullback of  $D_{\mathbb{F}_n}$ . That is, we now have

$$-K_{\bar{S}} \equiv \pi^*(-K_{\mathbb{F}_n}) + \sum_{k=1}^i b_k C_k$$

where  $b_k$  are non-zero constants and  $C_k$  are  $(-1)$  or  $(-2)$  curves in the *boundary/complement* of the cylinder in Step 1.

**Step 4:** Let  $\Gamma : \bar{S} \rightarrow \tilde{S}$  be the blow down of curves  $C_k \in \bar{S}$  such that we get the configuration on  $\tilde{S}$ , which is the minimal resolution of  $S$ . For simplicity, we will order the curves  $C_k$  such that  $\Gamma$  contracts  $C_k$  for  $1 \leq k \leq m$ . This implies that we have

$$-K_{\tilde{S}} \equiv \Gamma_*(\pi^*(-K_{\mathbb{F}_n})) + \sum_{k=m+1}^i b_k \tilde{C}_k,$$

where  $\tilde{C}_k$  are the proper transforms of  $C_k$  on  $\tilde{S}$ . A priori, the divisor  $-K_{\tilde{S}}$  may not be effective.

**Remark 4.74.** Note that  $\Gamma : \bar{S} \rightarrow \tilde{S}$  could be identity.

**Step 5:** Since any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + \sum_{l=1}^{\rho(S)-1} L_l,$$

we choose the proper transforms of  $\tilde{L}_l$  on  $\tilde{S}$  to be boundary curves. That is, for every  $1 \leq l \leq \rho(S) - 1$  one can write  $\tilde{L}_l \sim \sum_{k=m+1}^i c_{k,l} C_k$  where  $c_{k,l}$  are non-negative constants, due to *equivalence of fibres*. Also, since  $\tilde{S}$  is the minimal resolution of  $S$ , we have

$$\tilde{L}_l \sim \phi^*(L_l) - \sum_{k=m+1}^i d_{k,l} C_k,$$

where  $d_{k,l}$  are non-negative constants.

This then implies that we have

$$-K_{\tilde{S}} + \sum_{l=1}^{\rho(S)-1} a_l \phi^*(L_l) \equiv \Gamma_*(\pi^*(-K_{\mathbb{F}_n})) + \sum_{l=1}^{\rho(S)-1} \left( \sum_{k=m+1}^i (b_k + a_l(c_{k,l} + d_{k,l})) \tilde{C}_k \right).$$

This is done such that the divisor  $\phi^*(H) = -K_{\tilde{S}} + \sum_{l=1}^{\rho(S)-1} a_l \phi^*(L_l)$  is effective.

**Step 6:** Next, let  $\phi : \tilde{S} \rightarrow S$  be the contraction of  $-2$  curves among the curves  $\tilde{C}_k$  for  $m+1 \leq k \leq i$ . Since these curves are in the *boundary* of the cylinder from Step 1, this then gives a  $H$ -polar cylinder on  $S$ .

**Remark 4.75.** *The cylinder that we obtain at the end of this process in Step 6, is the same cylinder that we started with on  $\mathbb{F}_n$ , in Step 1! This is because the birational modifications from Step 2 to Step 5 are done carefully using curves in the boundary of this cylinder, so that in the end this cylinder remains untouched.*

This process is represented in the following diagram.

$$\begin{array}{ccc}
 & (\bar{S}, D_{\bar{S}}) & \\
 \pi \swarrow & & \searrow \Gamma \\
 (\mathbb{F}_n, D_{\mathbb{F}_n}) & & (\tilde{S}, D_{\tilde{S}}) \\
 & & \downarrow \phi \\
 & & (S, D_S)
 \end{array}$$

where  $\phi$  is the minimal resolution of the surface  $S$ ,  $\pi$  is the blow up of the fibres of  $\mathbb{F}_n$  for  $n = 0, 1$  which results in  $\bar{S}$ , and  $\Gamma$  are contractions of curves from  $\bar{S}$  to give  $\tilde{S}$ . Existence of such birational morphisms  $\Gamma$  and  $\pi$  shows that the given surface  $S$  admits a  $H$ -polar cylinder since

$$\begin{aligned}
 \mathbb{A}^1 \times (\mathbb{A}^*)^{(r-1)} = \mathbb{F}_n \setminus \text{Supp}(-K_{\mathbb{F}_n}) &\cong \bar{S} \setminus \text{Supp}((\phi \circ \Gamma)^*(H)) \cong \tilde{S} \setminus \text{Supp}(\phi^*(H)) \\
 &\cong S \setminus \text{Supp}(H).
 \end{aligned}$$

**Reversability of construction:** A subtle detail in the above construction, is that the construction of curves on  $\tilde{S}$  using  $(\pi \circ \Gamma^*) : \tilde{S} \rightarrow \mathbb{F}_n$ , is an *example* of a construction of curves such that we get a  $H$ -polar cylinder for a given ample divisor  $H$  on  $S$ . Therefore, we need to prove that *given a surface  $S$  of a given Type, we can always construct a cylinder as shown in each construction!*

This is done, by taking the given surface  $S$ , its minimal resolution  $\tilde{S}$ , and the exceptional curves of  $\phi : \tilde{S} \rightarrow S$ , and proving the existence of the remaining curves on  $\tilde{S}$ , that form a part of the construction of the cylinder in each example. That is, we prove that the construction obtained to get an example of a  $H$ -polar cylinder in each case, is in fact *reversible!*

To illustrate the procedure described above, we will explicitly explain the steps involved in constructing an example of  $H$ -polar cylinder when the surface has  $E_n$  singular point and also prove that such an example can always be constructed, given a surface  $S$  of Type  $E_n$ . These are in Section 4.3.2.1 and Section 4.3.2.2.

### 4.3.2 Proof of Main Result 4

In this section, we will give a detailed proof of Main Result 4 by explicitly constructing the cylinder on the surface, as described in Section 4.3.1.

For the remainder of this section, fix  $S$  to be a del Pezzo surface with at most du Val singularities and let  $H$  be any ample divisor on  $S$  such that  $H \neq -K_S$ .

**How to read the Figures?** The figures in this section describe the configuration of curves on  $\tilde{S}$  or on  $\bar{S}$  (as mentioned under each figure.)

- The proper transforms of the section  $M$  and fibres  $F_1, F_2, F_3, F_4$  of  $\mathbb{F}_n$  on  $\tilde{S}$  (on  $\bar{S}$ ) are denoted by  $\tilde{M}, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, \tilde{F}_4$  respectively ( $\bar{M}, \bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{F}_4$  respectively).
- The proper transform of curves  $L, L_i, C$  on  $\tilde{S}$  are given by  $\tilde{L}, \tilde{L}_i, \tilde{C}$  and  $\bar{L}, \bar{L}_i, \bar{C}$  on  $\bar{S}$ , respectively .
- The exceptional curves of the blow up  $\pi : \bar{S} \rightarrow \mathbb{F}_n$  (Step 2 described above) and the subsequent blow down  $\Gamma : \bar{S} \rightarrow \tilde{S}$  (Step 4 described above), are denoted by  $E_{\circ}, E_{\circledast}, \dots, E_{\circledast}$  on  $\tilde{S}$  in the order of blow ups.
- The solid and dashed curves without superscripts are  $(-2)$  and  $(-1)$  curves respectively; the solid curves with the circled numbers as superscripts have self intersection given by  $(-\text{circled number})$ . That is, for example  $\bar{F}_1^{\circ}$  denotes the curve  $\bar{F}_1$  which is a  $(-4)$ -curve.

*Note: Non-exceptional curves drawn in the Figures could intersect more curves than as drawn in the corresponding configurations. The intersections that are shown in the Figures are the ones required to prove that the example is indeed  $H$ -polar. The rest of the intersections are not drawn in order to maintain clarity of configurations.*

#### 4.3.2.1 $E_7$ singular point

Suppose  $S$  has exactly one singular point and it is of type  $E_7$ . Since we want  $S$  such that  $\rho(S) \geq 2$ , the only possible surface is of degree 1. Using Lemma 4.70, we can conclude that  $\psi$  is a  $(1,1)$  contraction of the birational type. Therefore, any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

where  $L$  is a  $(-1)$  curve that does not pass through the singular point and  $0 < a < 1$ .

Take  $\mathbb{F}_2$  with the  $(-2)$ -section given by  $M$  and 2 fibres  $F_1$  and  $F_2$ . So the cylinder that we consider on  $\mathbb{F}_2$  is

$$\mathbb{F}_2 \setminus (M + F_1 + F_2) \cong \mathbb{A}^1 \times \mathbb{A}^*.$$

We will now *birationally modify* this cylinder, such that this remains a cylinder on the surface  $S$ . This birational modification can be described as follows:

$$S^7 = \bar{S} = \tilde{S} \xrightarrow{\tilde{\pi}_7} S^6 \xrightarrow{\pi_6} \dots \xrightarrow{\pi_2} S^1 \xrightarrow{\pi_1} \mathbb{F}_2 = S^0$$

Here,  $\pi_1 : S^1 \rightarrow \mathbb{F}_2$  and  $\pi_2 : S^2 \rightarrow S^1$  are blow ups of points on the fibres  $F_1$  and  $F_2$ , of  $\mathbb{F}_2$  respectively. The morphisms  $\pi_i$  for  $3 \leq i \leq 7$ , are blow ups of points on the proper transforms of  $F_1$  on  $S^{i-1}$ . This gives the following configuration on  $\tilde{S}$ .

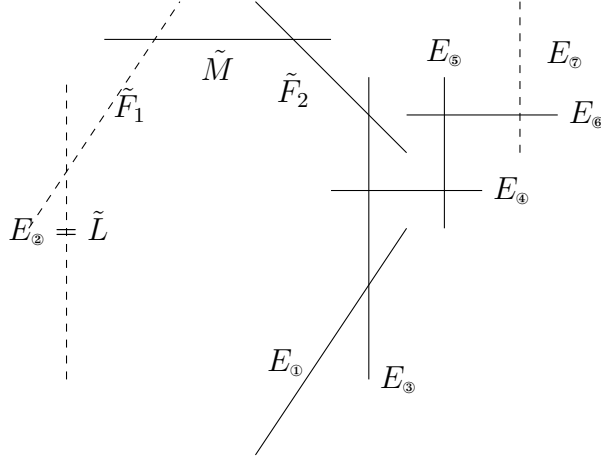


Figure 4.1:  $\tilde{S}$

Note that here the configuration on  $\bar{S}$  is the same as on  $\tilde{S}$  and hence  $\Gamma : \bar{S} \rightarrow \tilde{S}$  is an isomorphism.

We now prove that this cylinder is in fact a  $H$ -polar cylinder. Choose non-negative constants  $\alpha, \beta$ , such that

$$-K_{\mathbb{F}_2} \equiv 2M + \alpha F_1 + \beta F_2,$$

where  $\alpha + \beta = 4$ . After the blow ups described above, we get

$$\begin{aligned} -K_{\tilde{S}} &\equiv 2\tilde{M} + \beta\tilde{F}_2 + \alpha\tilde{F}_1 + (\alpha - 1)E_{\circledast} + (\beta - 1)E_{\circledcirc} + (2\beta - 2)E_{\circledcirc} \\ &\quad + (2\beta - 3)E_{\circledcirc} + (2\beta - 4)E_{\circledcirc} + (2\beta - 5)E_{\circledcirc} + (2\beta - 6)E_{\circledcirc}. \end{aligned}$$

We can now choose  $\tilde{L} = E_{\circledast}$ , where  $\tilde{L}$  is the proper transform of  $L$  on  $\tilde{S}$ , since  $L$  on  $S$  is a  $(-1)$  curve that does not pass through the singular point. That is, on  $S$ , we choose  $L = \phi(E_{\circledast})$ . This then implies that we have,

$$\begin{aligned} \phi^*(H) &\equiv 2\tilde{M} + \beta\tilde{F}_2 + \alpha\tilde{F}_1 + (\alpha - 1 + a)\tilde{L} + (\beta - 1)E_{\circledcirc} + (2\beta - 2)E_{\circledcirc} \\ &\quad + (2\beta - 3)E_{\circledcirc} + (2\beta - 4)E_{\circledcirc} + (2\beta - 5)E_{\circledcirc} + (2\beta - 6)E_{\circledcirc}. \end{aligned}$$

Since  $a > 0$ , this divisor is effective and hence the cylinder that we have constructed is an example of a  $H$ -polar cylinder, for any ample divisor  $H$  on  $S$ .

We will now prove that this cylinder can be obtained on any such given surface  $S$  of Type  $E_7$ .

That is, suppose  $S$  is a del Pezzo surface of degree 1 of Type  $E_7$  and  $H$ , an ample divisor on  $S$ . Then as explained above,  $H \equiv -K_S + aL$ , for  $L$  a  $(-1)$  curve on  $S$  that does not pass through the singular point and  $0 < a < 1$ .

Then on  $\tilde{S}$ , the minimal resolution of  $S$ , we have the curves  $\tilde{M}, \tilde{F}_2, E_{\circledcirc}, E_{\circledcirc}, E_{\circledcirc}, E_{\circledcirc}, E_{\circledcirc}$ , as these are the exceptional curves of the blow up  $\phi : \tilde{S} \rightarrow S$ . We also have the curve  $E_{\circledast}$  as in the configuration above, as this is  $\tilde{L} = \phi^*(L)$ , the proper transform of  $L$  on  $\tilde{S}$ .

So, in order to get the complete configuration in Fig. 4.1, we need to prove the existence of the curves  $\tilde{F}_1$  and  $E_{\circledcirc}$ , as in the configuration.

Observe that  $\tilde{F}_1$  is the proper transform of the anticanonical curve  $C \in |-K_S|$  that intersects  $\tilde{M}$  as in Fig. 4.1. Also, since  $\tilde{L} \cdot -K_{\tilde{S}} = 1$ , we have that  $\tilde{F}_1 \cdot \tilde{L} = 1$ .

In order to prove the existence of  $E_{\mathfrak{v}}$ , contract  $\tilde{F}_1$ ,  $\tilde{M}$ ,  $\tilde{F}_2$ ,  $E_{\mathfrak{3}}$ ,  $E_{\mathfrak{4}}$ ,  $E_{\mathfrak{5}}$ , in order. Then we have a smooth del Pezzo surface of degree 7, which has 3  $(-1)$  curves. This implies the existence of  $E_{\mathfrak{v}}$  as in the configuration in Fig. 4.1.

This shows that the given example of a  $H$ -polar cylinder, can always be constructed on any given del Pezzo surface  $S$  of degree 1 of Type  $E_7$ .

### 4.3.2.2 $E_6$ singular point

Suppose  $S$  has at least 1 singular point of type  $E_6$ . We will now consider all possible singularities that  $S$  can have, in addition to the  $E_6$  singular point and construct  $H$ -polar cylinders for every case.

1.  $E_6$ : Suppose  $S$  has exactly 1 singular point and it is of type  $E_6$ . This implies that  $\rho(S) \leq 3$ , that is the surface  $S$  is of degree 1 with  $\rho(S) = 3$  or  $S$  is of degree 2 with  $\rho(S) = 2$ .

Using Lemma 4.70, we can conclude that any ample divisor  $H$  on  $S$  is given by

- (a)  $H \equiv -K_S + a_1L_1 + a_2L_2$  with  $\rho(S) = 3$ ,  $0 < a_1, a_2 < 1$ ,
- (b)  $H \equiv -K_S + a_1L_1$  with  $\rho(S) = 3$ ,  $0 < a_1 < 1$ ,
- (c)  $H \equiv -K_S + a_1L_1$  with  $\rho(S) = 2$ ,  $0 < a_1 < 1$ ,

where  $L_1$  and  $L_2$  are  $(-1)$  curves on  $S$  that do not pass through the singular point. One can understand the above construction of surfaces by considering del Pezzo of degree 1 of Type  $E_6$  to be the blow up of del Pezzo surface of degree 2 of Type  $E_6$ , at a general point.

- [1] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + a_1L_1 + a_2L_2$$

where  $0 \leq a_1, a_2 < 1$ ,  $L_1$ ,  $L_2$  are  $(-1)$ -curves that do not pass through any singular point.

Without loss of generality, let us assume  $a_1 \leq a_2$ . Take  $\mathbb{F}_2$  with the  $(-2)$ -section  $M$  and fibres  $F_1$  and  $F_2$ . The cylinder that we consider on  $\mathbb{F}_2$  is given by

$$\mathbb{F}_2 \setminus (M + F_1 + F_2) \cong \mathbb{A}^1 \times \mathbb{A}^*.$$

We will now *birationally modify* this cylinder on  $\mathbb{F}_2$  to then give us a  $H$ -polar cylinder on the surface  $S$ .

This birational modification can be described as follows,

$$S^7 = \bar{S} \xrightarrow{\pi_7} S^6 \xrightarrow{\pi_6} \dots \xrightarrow{\pi_2} S^1 \xrightarrow{\pi_1} \mathbb{F}_2 = S^0$$

Here,  $\pi_1 : S^1 \rightarrow \mathbb{F}_2$  and  $\pi_3 : S^3 \rightarrow S^2$  are blow ups of points on  $F_2$  and  $\pi_2 : S^2 \rightarrow S^1$  is a blow up of a point on  $F_1$ . The morphisms  $\pi_i$  for  $4 \leq i \leq 7$ , are blow ups of points on the proper transforms of  $F_2$  on  $S^{i-1}$ . This gives the following configuration on  $\bar{S}$ .

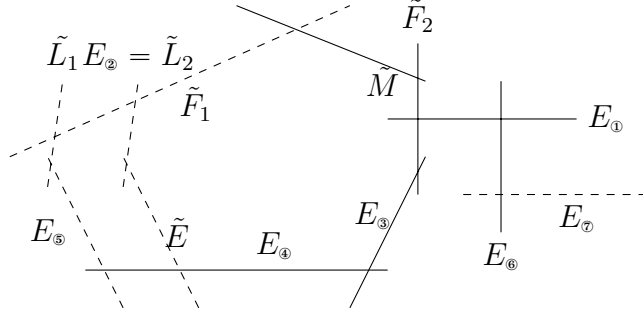


Figure 4.2:  $\tilde{S}$

In order to prove the existence of the curve  $\tilde{E}$  and  $\tilde{L}_1$  on  $\tilde{S}$ , contract  $E_7, E_6, E_1, \tilde{F}_2$ , in order, which gives a del Pezzo surface of degree 5 of Type  $A_1$ . Using [CT88, Proposition 8.5], we can show the existence of the curves  $\tilde{E}$  and  $\tilde{L}_1$ .

Note that here the configuration on  $\tilde{S}$  is the same as on  $\tilde{S}$  and hence  $\Gamma : \tilde{S} \rightarrow \tilde{S}$  is an isomorphism.

Consider the following non-negative constants,  $\alpha, \beta$  such that

$$-K_{\mathbb{F}_2} \equiv 2M + \alpha F_1 + \beta F_2,$$

with  $\alpha + \beta = 4$ . This then implies that

$$\begin{aligned} -K_{\tilde{S}} &\equiv 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_2 + \beta\tilde{F}_2 + (\beta - 1)E_3 + (\beta - 2)E_4 + (\beta - 3)E_5 \\ &\quad + (\beta - 1)E_6 + (\beta - 2)E_7 + (\beta - 3)E_8. \end{aligned}$$

Let us now prove that this is in fact a  $H$ -polar cylinder.

Let the curves  $\tilde{L}_1, \tilde{L}_2$  on  $\tilde{S}$  be the proper transforms of  $L_1, L_2$  on  $\tilde{S}$ . In order to show that this is a  $H$ -polar cylinder, we can take  $E_2 = \tilde{L}_2$ , since  $L_2$  on  $S$  is a  $(-1)$  curve that does not pass through the singular point.

Consider the linear system  $|\tilde{L}_1 + E_5|$ . Since the curves in the divisor  $2E_7 + 2E_6 + 2E_1 + 2\tilde{F}_2 + E_3 + \tilde{M}$  do not intersect the curves in the divisor  $\tilde{L}_1 + E_5$ , and

$$\begin{aligned} &\left(2E_7 + 2E_6 + 2E_1 + 2\tilde{F}_2 + E_3 + \tilde{M}\right)^2 = 0, \\ &\left(2E_7 + 2E_6 + 2E_1 + 2\tilde{F}_2 + E_3 + \tilde{M}\right) \cdot \mathcal{C} = 0, \text{ for } \mathcal{C} \in \{E_7, E_6, E_1, \tilde{F}_2, E_3, \tilde{M}\}, \end{aligned}$$

we can conclude that  $2E_7 + 2E_6 + 2E_1 + 2\tilde{F}_2 + E_3 + \tilde{M}$  is a  $(0)$ -curve and is a fibre of the conic bundle  $|\tilde{L}_1 + E_5|$ . Therefore, we have the following equivalence,

$$\tilde{L}_1 + E_5 \sim 2E_7 + 2E_6 + 2E_1 + 2\tilde{F}_2 + E_3 + \tilde{M}.$$

Therefore,

$$\begin{aligned} \phi^*(H) &\equiv (2 + a_1)\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1 + a_2)\tilde{L}_2 + (\beta + 2a_1)\tilde{F}_2 \\ &\quad + (\beta - 1 + a_1)E_3 + (\beta - 2)E_4 + (\beta - 3 - a_1)E_5 + (\beta - 1 + 2a_1)E_6 \\ &\quad + (\beta - 2 + 2a_1)E_7 + (\beta - 3 + 2a_1)E_8. \end{aligned}$$

If  $a_1 < a_2$ , then the above divisor is effective. This shows that

$$\mathbb{A}^1 \times (\mathbb{A}^*) = \mathbb{F}_2 \setminus \text{Supp}(-K_{\mathbb{F}_2}) \cong \tilde{S} \setminus \text{Supp}(\phi^*(H)) \cong S \setminus \text{Supp}(H).$$

If  $a_1 = a_2 = a$ , then using the conic bundle given by the linear system  $|\tilde{F}_2 + \tilde{M} + \tilde{F}_1 + E_{\textcircled{1}} + E_{\textcircled{2}} + E_{\textcircled{3}}|$ , we have

$$E_{\textcircled{5}} + E_{\textcircled{4}} + \tilde{E} \sim \tilde{F}_2 + \tilde{M} + \tilde{F}_1 + E_{\textcircled{1}} + E_{\textcircled{2}} + E_{\textcircled{3}}.$$

Therefore,

$$\begin{aligned} \phi^*(H) \equiv & (2 + \epsilon)\tilde{M} + (\alpha - a)\tilde{F}_1 + (a - \epsilon)\tilde{L}_2 + (3 + 2\epsilon + a)\tilde{F}_2 + (2 + \epsilon + a)E_{\textcircled{3}} \\ & + (1 + a)E_{\textcircled{4}} + (2 + 2\epsilon + a)E_{\textcircled{1}} + (1 + 2\epsilon + a)E_{\textcircled{2}} + (2\epsilon + a)E_{\textcircled{7}} \\ & + (a - \epsilon)\tilde{E}. \end{aligned}$$

In this case, the divisor  $\phi^*(H)$  is effective and this proves that the same cylinder is again a  $H$ -polar cylinder in this case too!

We will now show that the example that we have constructed, can be obtained on any surface  $S$  of Type  $E_6$  and  $H$ , ample divisor given as in Case 1(a).

Consider  $S$  and  $\tilde{S}$ , the minimal resolution of  $S$ . The curves  $\tilde{M}, \tilde{F}_2, E_{\textcircled{3}}, E_{\textcircled{4}}, E_{\textcircled{1}}, E_{\textcircled{2}}$  are the exceptional curves of  $\phi : \tilde{S} \rightarrow S$  and thus exist on  $\tilde{S}$  for any  $S$  of Type  $E_6$ . For the given  $H$ , we also have the curves  $\tilde{L}_1$  and  $E_{\textcircled{7}} = \tilde{L}_2$  as given in the above configuration.

Therefore, we only need to prove the existence of the curves  $E_{\textcircled{5}}, \tilde{E}, E_{\textcircled{2}}$  and  $\tilde{F}_1$ . Observe that  $\tilde{F}_1$  is the proper transform of the anticanonical curve  $C \in |-K_S|$ , that passes through the singular point on  $S$ . In order to prove the existence of the remaining curves,  $\tilde{E}, E_{\textcircled{2}}$  and  $E_{\textcircled{5}}$ , contract  $E_{\textcircled{7}}, E_{\textcircled{6}}, E_{\textcircled{1}}, \tilde{F}_2$  in order. This gives a del Pezzo surface of degree 5 of Type  $A_1$ . Using [CT88, Proposition 8.5], we can prove the existence of the curves  $\tilde{E}, E_{\textcircled{2}}$  and  $E_{\textcircled{5}}$ , as in the configuration above.

Thus, this proves that given any surface  $S$  of Type  $E_6$ , we can always obtain the configuration of curves as in Fig. 4.2 and thus, can obtain a  $H$ -polar cylinder for  $H$  given as in Case 1(a).

**Remark 4.76.** *Note that the same construction above works for Case 1(c). Take the configuration on  $\tilde{S}$  as in Fig. 4.2. Contract the curve  $E_{\textcircled{2}}$  on  $\tilde{S}$ . Now this is a configuration of curves on  $\tilde{S}$  which is the minimal resolution of  $S$  of degree 2 and the same arguments as above prove that the constructed example is a  $H$ -polar cylinder, for  $H$  as described in Case 1(c).*

[II] Suppose that the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + a_1 L_1,$$

where  $0 < a_1 < 1$ ,  $\rho(S) = 3$ ,  $L_1$  is a  $(-1)$  curve that does not pass through the singular point. Let us now construct an example of a  $H$ -polar cylinder.

Here, we again take the same cylinder on  $\mathbb{F}_2$  as in Case 1(a) and *birationally modify* it as follows:

$$S^7 = \bar{S} \xrightarrow{\pi_7} S^6 \xrightarrow{\pi_6} \dots \xrightarrow{\pi_2} S^1 \xrightarrow{\pi_1} \mathbb{F}_2 = S^0$$

Here, the morphisms  $\pi_i : S^i \rightarrow S^{i-1}$ , for  $1 \leq i \leq 3$  are blow ups of points on  $F_1$  and  $F_2$  and  $\pi_i : S^i \rightarrow S^{i-1}$ , for  $3 \leq i \leq 7$ , are blow ups of points on the proper transforms of the fibre  $F_2$  on  $S^{i-1}$ . This gives the following configuration on  $\tilde{S}$ .

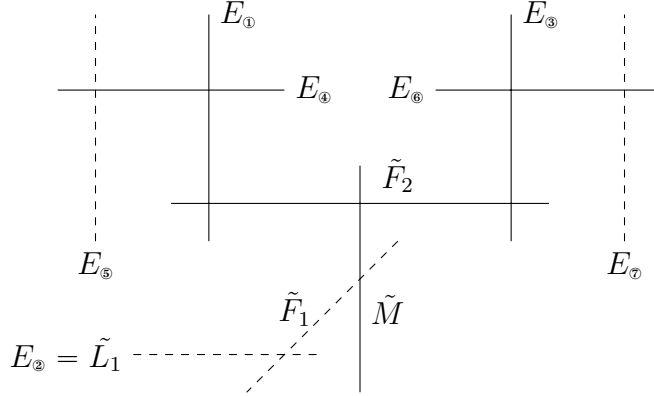


Figure 4.3:  $\tilde{S}$

Since the configuration on  $\bar{S}$  is the same as on  $\tilde{S}$ ,  $\Gamma : \bar{S} \rightarrow \tilde{S}$  is an isomorphism. Consider non-negative constants,  $\alpha, \beta$  such that

$$-K_{\mathbb{F}_2} \equiv 2M + \alpha F_1 + \beta F_2,$$

with  $\alpha + \beta = 4$ . This then implies that

$$\begin{aligned} -K_{\tilde{S}} &\equiv 2\tilde{M} + \beta\tilde{F}_2 + \alpha\tilde{F}_1 + (\beta - 1)E_{\textcircled{1}} + (\beta - 2)E_{\textcircled{2}} + (\beta - 3)E_{\textcircled{3}} + (\beta - 1)E_{\textcircled{4}} \\ &\quad + (\beta - 2)E_{\textcircled{5}} + (\beta - 3)E_{\textcircled{7}} + (\alpha - 1)E_{\textcircled{2}}. \end{aligned}$$

In order to prove that this is in fact a  $H$ -polar cylinder, for any ample divisor  $H$  on  $S$ , we can take the curve  $\tilde{L}_1 = E_{\textcircled{2}}$ , where  $\tilde{L}_1$  is the proper transform of  $L_1$  on  $\tilde{S}$ , since  $L$  on  $S$  is a  $(-1)$  curve that does not pass through the singular point. Note that  $\phi^*(L_1) \sim \tilde{L}_1$ . Therefore, we get

$$\begin{aligned} \phi^*(H) &\equiv 2\tilde{M} + \beta\tilde{F}_2 + \alpha\tilde{F}_1 + (\beta - 1)E_{\textcircled{1}} + (\beta - 2)E_{\textcircled{2}} + (\beta - 3)E_{\textcircled{3}} + (\beta - 1)E_{\textcircled{4}} \\ &\quad + (\beta - 2)E_{\textcircled{5}} + (\beta - 3)E_{\textcircled{7}} + (\alpha - 1 + a_1)\tilde{L}_1, \end{aligned}$$

which is effective if we choose  $\epsilon < a$ . This shows that

$$\mathbb{A}^1 \times (\mathbb{A}^*) = \mathbb{F}_2 \setminus \text{Supp}(-K_{\mathbb{F}_2}) \cong \tilde{S} \setminus \text{Supp}(\phi^*(H)) \cong S \setminus \text{Supp}(H)$$

and therefore, the cylinder constructed above is a  $H$ -polar cylinder, for any ample divisor  $H$  on  $S$ .

We will now prove that the cylinder constructed above, can be obtained on any given surface  $S$  of degree 1 of Type  $E_6$  and ample divisor  $H$  on  $S$  given by Case 1(b).

Consider surface  $S$  and  $\tilde{S}$ , its minimal resolution. The curves  $E_{\textcircled{1}}, E_{\textcircled{2}}, \tilde{F}_2, \tilde{M}, E_{\textcircled{3}}, E_{\textcircled{4}}, E_{\textcircled{5}}$ , are the exceptional curves of the minimal resolution  $\phi : \tilde{S} \rightarrow S$ . Since  $H$  is given as in Case 1(b), the curve  $\tilde{L}_1$  exists as in the configuration Fig. 4.3. Therefore, we only need to prove the existence of the curves  $E_{\textcircled{2}}, E_{\textcircled{7}}$  and  $\tilde{F}_1$ .

Observe that  $\tilde{F}_1$  is the proper transform of the anticanonical curve  $C \in |-K_S|$  and thus exists as in Fig. 4.3. In order to prove the existence of the curves  $E_{\circledast}$  and  $E_{\circledcirc}$ , contract  $\tilde{F}_1, \tilde{M}, \tilde{F}_2$  in order. This gives a del Pezzo surface of degree 4 of Type  $2A_1$ . Using [CT88, Proposition 6.1], we can prove the existence of the curves  $E_{\circledast}$  and  $E_{\circledcirc}$ , as in the configuration in Fig. 4.3.

This proves that we can always get the configuration of curves as in Fig. 4.3 on any surface  $S$  of degree 1 of Type  $E_6$  and thus construct a  $H$ -polar cylinder for any ample divisor  $H$  on  $S$ , as given in Case 1(b).

2.  $E_6 + A_1$ : Consider surface  $S$  of degree 1 that contains  $E_6 + A_1$  singularity. This implies that  $\rho(S) \leq 2$ .

Using Lemma 4.70 and the known result that there does not exist a degree 2 del Pezzo surface of Type  $E_6 + A_1$ , we can conclude that any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL,$$

where  $L$  is a  $(\frac{-1}{2})$ -curve on  $S$  that passes through the  $A_1$  singular point and  $0 < a < 2$ .

Let us now construct an example of a  $H$ -polar cylinder on  $S$ . Take  $\mathbb{F}_2$  with the  $(-2)$  section  $M$  and fibres  $F_1$  and  $F_2$ . Taking the same cylinder on  $\mathbb{F}_2$  as in Case 1(a), we can *birationally modify* the cylinder as follows,

$$S^7 = \bar{S} \xrightarrow{\pi_7} S^6 \xrightarrow{\pi_6} \dots \xrightarrow{\pi_2} S^1 \xrightarrow{\pi_1} \mathbb{F}_2 = S^0$$

Here, the morphisms  $\pi_i : S^i \rightarrow S^{i-1}$ , for  $1 \leq i \leq 2$  are blow ups of points on  $F_1$  and  $F_2$  and  $\pi_i : S^i \rightarrow S^{i-1}$ , for  $3 \leq i \leq 7$ , are blow ups of points on the proper transforms of the fibre  $F_2$  on  $S^{i-1}$ .

This gives the following configuration on  $\bar{S}$ .

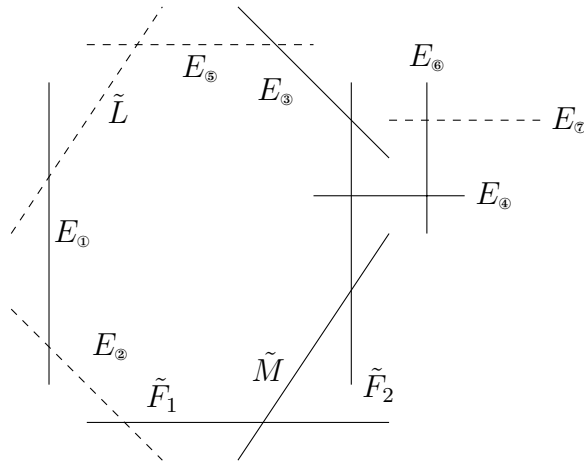


Figure 4.4:  $\tilde{S}$

Contracting  $E_{\circledcirc}, E_{\circledast}, E_{\circledcirc}, \tilde{F}_2$  gives a del Pezzo surface of degree 5 of Type  $2A_1$ . Using [CT88, Proposition 8.5], we can prove the existence of the curve  $\tilde{L}$  as in the configuration above.

Taking  $|\tilde{L} + E_{\mathfrak{5}}|$ , we see that  $E_{\mathfrak{3}}$  and  $E_{\mathfrak{1}}$  are sections, and thus we have

$$\tilde{L} + E_{\mathfrak{5}} \sim E_{\mathfrak{2}} + \tilde{F}_1 + \tilde{M} + \tilde{F}_2 + E_{\mathfrak{3}} + E_{\mathfrak{6}} + E_{\mathfrak{7}}. \quad (4.29)$$

Similarly, taking  $|\tilde{L} + E_{\mathfrak{2}} + E_{\mathfrak{1}}|$ , the sections are given by  $E_{\mathfrak{5}}$  and  $\tilde{F}_1$ , and we get

$$\tilde{L} + E_{\mathfrak{1}} + E_{\mathfrak{2}} \sim E_{\mathfrak{3}} + \tilde{M} + 2\tilde{F}_2 + 2E_{\mathfrak{4}} + 2E_{\mathfrak{6}} + 2E_{\mathfrak{7}}. \quad (4.30)$$

Then, using (4.29) and (4.30), we get

$$\tilde{L} \sim 3E_{\mathfrak{2}} - 2E_{\mathfrak{5}} + E_{\mathfrak{1}} + 2\tilde{F}_1 + \tilde{M} - E_{\mathfrak{3}}. \quad (4.31)$$

Using the conic bundle given by the linear system  $|E_{\mathfrak{5}} + E_{\mathfrak{3}} + \tilde{F}_2 + E_{\mathfrak{4}} + E_{\mathfrak{6}} + E_{\mathfrak{7}}|$ , we get

$$E_{\mathfrak{1}} + 2E_{\mathfrak{2}} + \tilde{F}_1 \sim E_{\mathfrak{5}} + E_{\mathfrak{3}} + \tilde{F}_2 + E_{\mathfrak{4}} + E_{\mathfrak{6}} + E_{\mathfrak{7}}. \quad (4.32)$$

Note that we have  $\tilde{L} \sim_{\mathbb{Q}} \phi^*(L) - \frac{1}{2}E_{\mathfrak{1}}$ . Using this and (4.29), (4.30), (4.32), we get

$$\phi^*(L) \equiv \frac{1}{2} \left[ \tilde{F}_1 - E_{\mathfrak{5}} + 2\tilde{M} + E_{\mathfrak{3}} + 3\tilde{F}_2 + 3E_{\mathfrak{4}} + 3E_{\mathfrak{6}} + 3E_{\mathfrak{7}} \right].$$

Therefore, we have

$$\begin{aligned} \phi^*(H) = -K_{\tilde{S}} + a\phi^*(L) &\equiv (2+a)\tilde{M} + \left(\alpha + \frac{a}{2}\right)\tilde{F}_1 + (\alpha-1)E_{\mathfrak{1}} + (2\alpha-2)E_{\mathfrak{2}} \\ &\quad + \left(\beta + \frac{3a}{2}\right)\tilde{F}_2 + \left(\beta-1 + \frac{a}{2}\right)E_{\mathfrak{3}} + \left(\beta-1 + \frac{3a}{2}\right)E_{\mathfrak{4}} \\ &\quad + \left(\beta-2 + \frac{3a}{2}\right)E_{\mathfrak{6}} + \left(\beta-3 + \frac{3a}{2}\right)E_{\mathfrak{7}} + \left(\beta-2 - \frac{a}{2}\right)E_{\mathfrak{5}}. \end{aligned}$$

This is an effective divisor.

We now prove that the above example of  $H$ -polar cylinder, for  $H$  given by  $H \equiv -K_S + aL$ , where  $L$  a  $(\frac{-1}{2})$ -curve that passes through the  $A_1$  singular point, can always be constructed on a given degree 1 del Pezzo surface  $S$  with any ample divisor  $H$ .

Consider  $S$  and its minimal resolution  $\tilde{S}$ . Then the curves  $\tilde{M}$ ,  $\tilde{F}_2$ ,  $E_{\mathfrak{3}}$ ,  $E_{\mathfrak{1}}$ ,  $E_{\mathfrak{4}}$ ,  $E_{\mathfrak{6}}$ ,  $\tilde{F}_1$  exist on  $\tilde{S}$ , since these are the exceptional curves of the minimal resolution  $\phi : \tilde{S} \rightarrow S$ . Also the curve  $\tilde{L} = \tilde{L}$  exists as in Fig. 4.4, since  $L$  on  $S$  passes through the  $A_1$  singular point.

Therefore, we only need to prove the existence of  $E_{\mathfrak{7}}$ ,  $E_{\mathfrak{5}}$  and  $E_{\mathfrak{2}}$ , as in the configuration.  $E_{\mathfrak{7}}$  is the proper transform of the unique curve  $C \in |-K_S|$ . In order to prove the existence of  $E_{\mathfrak{5}}$  and  $E_{\mathfrak{2}}$ , contract  $E_{\mathfrak{7}}$ ,  $E_{\mathfrak{6}}$ ,  $E_{\mathfrak{4}}$ ,  $\tilde{F}_2$  in order and this gives a del Pezzo surface of degree 5 of Type  $2A_1$ . Using [CT88, Proposition 8.5], we can prove the existence of the curves  $E_{\mathfrak{5}}$  and  $E_{\mathfrak{2}}$  as in the configuration in Fig. 4.4. This completes the proof.

### 4.3.2.3 $D_7$ singular point

Suppose the surface  $S$  has at least 1 singular point of type  $D_7$ . Then  $\rho(S) \leq 2$ . Consider del Pezzo surface  $S$  of degree 1 of Type  $D_7$ . Then  $\rho(S) = 2$ .

Using Lemma 4.70 and the known result that there does not exist a degree 2 del Pezzo surface of Type  $D_7$ , we can conclude that any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC,$$

where  $C$  is a fibre of the conic bundle given by  $|C|$  and  $b > 0$ .

Blowing up fibres of  $\mathbb{F}_2$ , we get the following configuration on  $\tilde{S}$ .

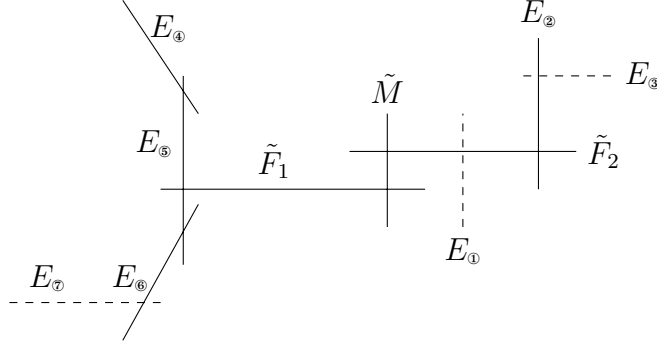


Figure 4.5:  $\tilde{S}$

Note that we can choose  $C$  such that

$$\tilde{C} = 2E_{\mathfrak{3}} + 2E_{\mathfrak{2}} + 2\tilde{F}_2 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\mathfrak{5}} + E_{\mathfrak{6}} + E_{\mathfrak{4}},$$

where  $\tilde{C}$  is the proper transform of  $C$  in  $\tilde{S}$  and  $\phi^*(C) = \tilde{C}$ . Here, we use the conic bundle given by the linear system  $|2E_{\mathfrak{3}} + 2E_{\mathfrak{2}} + 2\tilde{F}_2 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\mathfrak{5}} + E_{\mathfrak{6}} + E_{\mathfrak{4}}|$ .

Therefore, we have

$$\begin{aligned} \phi^*(H) &\equiv (2 + 2b)\tilde{M} + (\alpha + 2b)\tilde{F}_1 + (\alpha - 1 + b)E_{\mathfrak{3}} + (2\alpha - 2 + 2b)E_{\mathfrak{5}} \\ &\quad + (2\alpha - 3 + b)E_{\mathfrak{6}} + (2\alpha - 4)E_{\mathfrak{7}} + (\beta + 2b)\tilde{F}_2 + (\beta - 1)E_{\mathfrak{1}} + (\beta - 1 + 2b)E_{\mathfrak{2}} \\ &\quad + (\beta - 2 + 2b)E_{\mathfrak{3}} \end{aligned}$$

The above divisor is effective and hence proves that the above construction gives an example of a  $H$ -polar cylinder for  $H$  as described above.

### 4.3.2.4 $D_6$ singular point

Suppose the surface  $S$  has at least 1 singular point of type  $D_6$ . Then  $\rho(S) \leq 3$ .

1.  **$D_6$  singularity:** If the surface  $S$  has exactly 1 singular point and it is of type  $D_6$ , then  $\rho(S) \leq 3$ . The surface  $S$  can be either of degree 1, with  $\rho(S) = 3$  or of degree 2, with  $\rho(S) = 2$ .

Since there does not exist a cubic del Pezzo surface  $S$  of Type  $D_6$ , using Lemma 4.70, we can conclude that any ample divisor  $H$  on  $S$  is given by

- (a)  $H \equiv -K_S + aL + bC$  with  $\rho(S) = 3$ ,  $b > 0$ ,  $0 < a < 1$ ,
- (b)  $H \equiv -K_S + aL$  with  $\rho(S) = 3$ ,  $0 < a < 1$ ,
- (c)  $H \equiv -K_S + bC$  with  $\rho(S) = 3$ ,  $b > 0$ ,
- (d)  $H \equiv -K_S + bC$  with  $\rho(S) = 2$ ,  $b > 0$ ,

where  $L$  is a  $(-1)$ -curve that does not pass through the singular point and  $C$  is a  $(0)$ -curve.

[I] Consider

$$H \equiv -K_S + aL + bC,$$

with  $\rho(S) = 3$ , where  $0 < a < 1$  and  $b \geq 0$ .

Constructing  $\tilde{S}$  from  $\mathbb{F}_2$ , we get the following configuration.

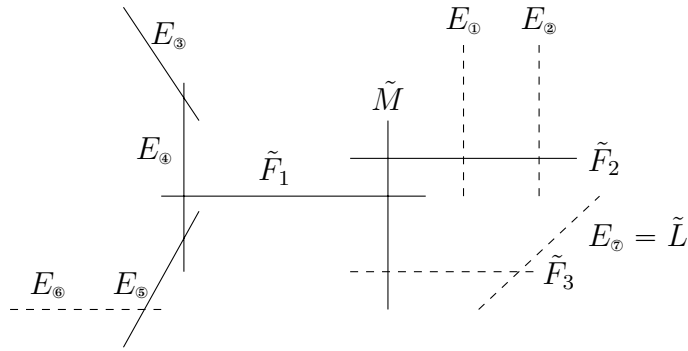


Figure 4.6:  $\tilde{S}$

Note that we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim 2E_{\textcircled{1}} + 2\tilde{F}_2 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\textcircled{4}} + E_{\textcircled{5}} + E_{\textcircled{3}},$$

where  $\tilde{C}$  is the proper transform of  $C$  in  $\tilde{S}$  and  $\phi^*(C) = \tilde{C}$ . Here, we use the conic bundle given by the linear system  $|2E_{\textcircled{1}} + 2\tilde{F}_2 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\textcircled{4}} + E_{\textcircled{5}} + E_{\textcircled{3}}|$ . Therefore, we have

$$\begin{aligned} \phi^*(H) &\equiv (2 + b)\tilde{M} + (\alpha + 2b)\tilde{F}_1 + (\alpha - 1 + b)E_{\textcircled{3}} + (2\alpha - 2 + 2b)E_{\textcircled{4}} \\ &\quad + (2\alpha - 3 + b)E_{\textcircled{5}} + (2\alpha - 4)E_{\textcircled{6}} + (\beta + 2b)\tilde{F}_2 + (\beta - 1 + 2b)E_{\textcircled{1}} \\ &\quad + (\beta - 1)E_{\textcircled{2}} + \gamma\tilde{F}_3 + (\gamma - 1 + a)E_{\textcircled{7}}. \end{aligned}$$

Since  $a > 0$ , this divisor is effective.

**Remark 4.77.** Note that contracting  $\tilde{L}$  in the above configuration, gives an example of a  $H$ -polar cylinder, when  $H \equiv -K_S + bC$  for a del Pezzo surface of degree 2 of Type  $D_6$ , as in Case 1(d), described above.

[II] Suppose the ample divisor  $H \equiv -K_S + bC$ , where  $S$  is a degree 1 del Pezzo surface of Type  $D_6$ , with  $\rho(S) = 3$ .

In this case, constructing  $\tilde{S}$  from  $\mathbb{F}_2$ , we get the following configuration.

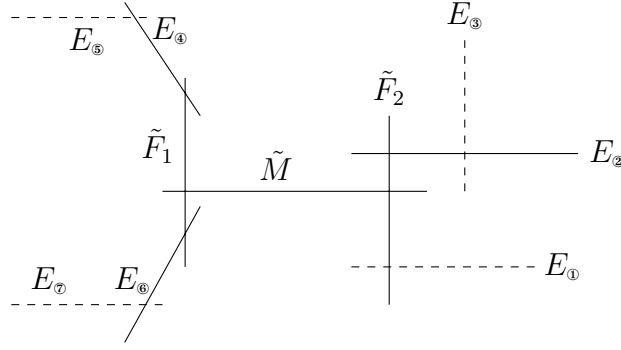


Figure 4.7:  $\tilde{S}$

Using the conic bundle given by the linear system  $|E_4 + E_6 + 2\tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_2 + 2E_3|$ , we can choose  $C$  on  $S$  such that

$$\phi^*(C) \sim E_4 + E_6 + 2\tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_2 + 2E_3.$$

Therefore, we get the following:

$$\begin{aligned} \phi^*(H) \equiv & (2 + 2b)\tilde{M} + (\alpha + 2b)\tilde{F}_1 + (\alpha - 1 + b)E_4 + (\alpha - 2)E_5 + (\alpha - 1 + b)E_6 \\ & + (\alpha - 2)E_7 + (\beta + 2b)\tilde{F}_2 + (\beta - 1 + 2b)E_2 + (\beta - 1)E_1 \\ & + (\beta - 2 + 2b)E_3. \end{aligned}$$

This divisor is effective and hence the above construction is an example of a  $H$ -polar cylinder for  $H$  as described above.

2.  $D_6 + A_1$  **singularity**: Consider del Pezzo surfaces of degree 1 of Type  $D_6 + A_1$ . Then  $\rho(S) = 2$ .

Using Lemma 4.70 and the result that there does not exist cubic del Pezzo surface of Type  $D_6$ , we can conclude that any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

with  $0 < a < 1$ , where  $L$  is a  $(-1)$ -curve on  $S$  that does not pass through any singular point.

Constructing  $\tilde{S}$  from  $\mathbb{F}_2$  by blowing up the fibres of  $\mathbb{F}_2$  appropriately, we get the following configuration,

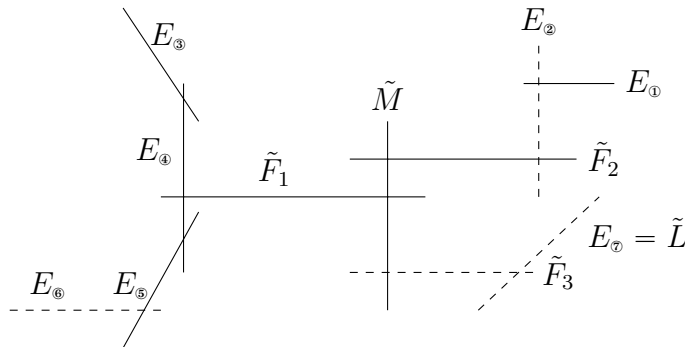


Figure 4.8:  $\tilde{S}$

Note that  $\phi^*(L) \sim \tilde{L}$  and therefore, we get

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{3}} + (2\alpha - 2)E_{\textcircled{4}} + (2\alpha - 3)E_{\textcircled{5}} \\ & + (2\alpha - 4)E_{\textcircled{6}} + \beta\tilde{F}_2 + (\beta - 1)E_{\textcircled{1}} + (2\beta - 2)E_{\textcircled{2}} + \gamma\tilde{F}_3 + (\gamma - 1 + a)\tilde{L}. \end{aligned}$$

This is an effective divisor.

#### 4.3.2.5 $D_5$ singular point

Suppose the surface  $S$  has at least 1 singular point of type  $D_5$ . This implies that  $\rho(S) \leq 4$ .

1.  **$D_5$  singularity:** Suppose  $S$  has at most 1 singular point and this is of type  $D_5$ . Then  $\rho(S) \leq 4$ . The surface  $S$  can be of degree 1 ( $\rho(S) = 4$ ), of degree 2 ( $\rho(S) = 3$ ) or of degree 3 ( $\rho(S) = 2$ ).

Using Lemma 4.70, we get that any ample divisor  $H$  on  $S$  can be given by

- (a)  $H \equiv -K_S + a_1L_1 + a_2L_2 + a_3L_3$ , with  $\rho(S) = 4$ ,  $0 \leq a_1, a_2, a_3 < 1$ ,
- (b)  $H \equiv -K_S + bC + a_1L_1 + a_2L_2$ , with  $\rho(S) = 4$ ,  $0 \leq a_1, a_2 < 1$ ,  $L_1, L_2$  are irreducible components of  $|C|$  such that  $L_1 \cdot C = L_2 \cdot C = L_1 \cdot L_2 = 0$ ,
- (c)  $H \equiv -K_S + a_1L_1 + a_2L_2$ , with  $\rho(S) = 3$ ,  $0 < a_1 < 1$ ,  $0 \leq a_2 < 1$ ,
- (d)  $H \equiv -K_S + bC + a_1L_1$ , with  $\rho(S) = 3$ ,  $0 \leq a_1 < 1$ ,  $L_1$  is an irreducible component of a fibre of the conic bundle given by  $|C|$  such that  $L_1 \cdot C = 0$ ,
- (e)  $H \equiv -K_S + aL$ , with  $\rho(S) = 2$ ,  $0 \leq a < 1$ ,
- (f)  $H \equiv -K_S + bC$ , with  $\rho(S) = 2$ ,  $0 < b$ ,

where  $L_1, L_2, L_3$  are  $(-1)$ -curves on  $S$  that do not pass through the singular point,  $C$  is a fibre of the conic bundle given by  $|C|$ .

- Let  $\rho(S) = 4$ . That is, the surface  $S$  is of degree 1.

[I] Suppose ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL,$$

where  $\rho(S) = 4$ ,  $L$  is a  $(-1)$ -curve on  $S$  that does not pass through the singular point and  $0 < a < 1$ .

We blow up fibres of  $\mathbb{F}_2$  to get the following configuration.

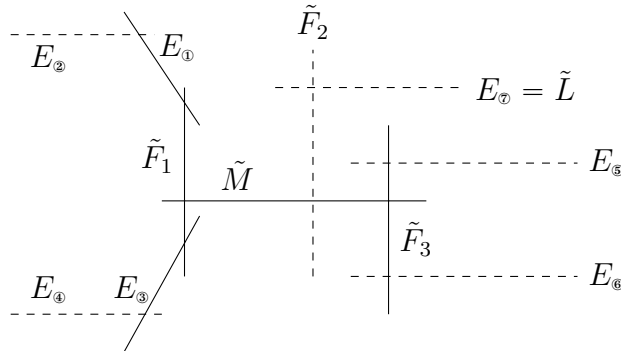


Figure 4.9:  $\tilde{S}$

Therefore, we get

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{1}} + (\alpha - 1)E_{\textcircled{3}} + (\alpha - 2)E_{\textcircled{4}} + (\alpha - 2)E_{\textcircled{2}} + \gamma\tilde{F}_3 \\ & + (\gamma - 1)E_{\textcircled{5}} + (\gamma - 1)E_{\textcircled{6}} + \beta\tilde{F}_2 + (\beta - 1 + a)\tilde{L}. \end{aligned}$$

This is an effective divisor for a sufficiently small  $\epsilon$ .

[II] Suppose ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + a_1L_1 + a_2L_2 + bC,$$

with  $\rho(S) = 4$ ,  $b > 0$  and  $0 < a_1 \leq a_2 < 1$ . Blowing up the fibres of  $\mathbb{F}_2$  accordingly, we get the following configuration on  $\tilde{S}$ ,

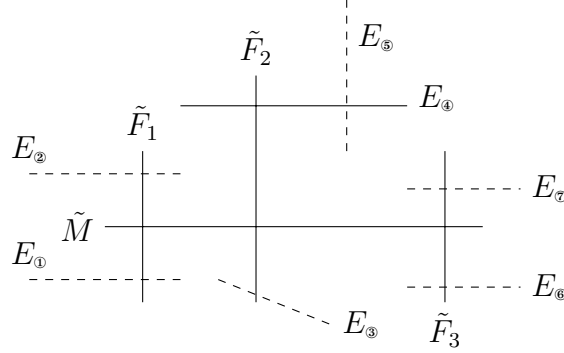


Figure 4.10:  $\tilde{S}$

In order to understand how the curve  $\tilde{L}_1$  is, in the above configuration, consider the linear system  $|E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}|$ . Note that this linear system is such that

$$\begin{aligned} (E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}})^2 &= 0, \\ (E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}) \cdot \mathcal{C} &= 0, \text{ for } \mathcal{C} \in \{E_{\textcircled{7}}, \tilde{F}_3, \tilde{M}, \tilde{F}_2, E_{\textcircled{4}}, E_{\textcircled{5}}\}. \end{aligned}$$

This implies that this linear system gives a conic bundle morphism from  $\tilde{S} \rightarrow \mathbb{P}^1$ . Observe that  $E_{\textcircled{1}} \cdot (E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}) = 0$  and this implies that  $E_{\textcircled{1}}$  is an irreducible component of a fibre of conic bundle given by  $|E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}|$ . Since  $E_{\textcircled{1}}^2 = -1$ , there exists a  $(-1)$ -curve, which is also an irreducible component of the fibre, that intersects  $E_{\textcircled{1}}$ , say  $\tilde{L}_1$  and is such that  $\tilde{L}_1 \cdot (E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}) = 0$ . This implies that we have

$$E_{\textcircled{1}} + \tilde{L}_1 \sim \tilde{M} + \tilde{F}_2 + \tilde{F}_3 + E_{\textcircled{7}} + E_{\textcircled{4}} + E_{\textcircled{5}}.$$

Similarly, using  $|E_{\textcircled{6}} + E_{\textcircled{4}} + \tilde{F}_2 + \tilde{M} + \tilde{F}_1 + E_{\textcircled{1}}|$ , we can prove the existence of the curve  $\tilde{L}_2$  in the above configuration, wherein  $\tilde{L}_2$  intersects  $E_{\textcircled{7}}$  and is such that

$$E_{\textcircled{7}} + \tilde{L}_2 \sim E_{\textcircled{6}} + E_{\textcircled{4}} + \tilde{F}_2 + \tilde{M} + \tilde{F}_1 + E_{\textcircled{1}}.$$

With this, we have

$$\begin{aligned} a_1\tilde{L}_1 + a_2\tilde{L}_2 &\equiv (a_1 - a_2)E_{\mathbb{7}} + a_1(\tilde{M} + \tilde{F}_2 + \tilde{F}_3 + E_{\mathbb{4}}) + (a_1 + a_2)E_{\mathbb{5}} \\ &\quad + (a_2 - a_1)E_{\mathbb{1}} + a_2(\tilde{M} + \tilde{F}_2 + \tilde{F}_1 + E_{\mathbb{4}}). \end{aligned}$$

Note that  $\phi^*(L_1) \sim \tilde{L}_1$  and  $\phi^*(L_2) \sim \tilde{L}_2$ . Therefore, we get

$$\begin{aligned} \phi^*(H) &\equiv (2 + a_1 + a_2)\tilde{M} + (\alpha + b + a_1)\tilde{F}_1 + (\alpha - 1)E_{\mathbb{2}} + (\alpha - 1 + a_2 - a_1)E_{\mathbb{1}} \\ &\quad + (\gamma + b + a_1)\tilde{F}_3 + (\gamma - 1 + a_1 - a_2)E_{\mathbb{7}} + (\gamma - 1)E_{\mathbb{6}} + (\beta + 2b + a_1 + a_2)\tilde{F}_2 \\ &\quad + (\beta - 1)E_{\mathbb{3}} + (\beta - 1 + 2b + a_1 + a_2)E_{\mathbb{4}} + (\beta - 2 + a_1 + a_2)E_{\mathbb{5}}. \end{aligned}$$

This divisor is effective.

- [III] Suppose  $a_1 = a_2 = 0$  and  $H \equiv -K_S + bC$ . Consider the configuration in Fig. 4.10. Using the conic bundle morphism that is given by the linear system  $|2E_{\mathbb{5}} + 2E_{\mathbb{4}} + 2\tilde{F}_2 + 2\tilde{M} + \tilde{F}_3 + \tilde{F}_1|$ , we can take  $C$  on  $S$  such that

$$\phi^*(C) \sim_{\mathbb{Q}} \tilde{F}_1 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\mathbb{4}} + 2E_{\mathbb{5}}.$$

Therefore, we get,

$$\begin{aligned} \phi^*(H) &\equiv 2\tilde{M} + (\alpha + b)\tilde{F}_1 + (\alpha - 1)E_{\mathbb{2}} + (\alpha - 1)E_{\mathbb{1}} + (\gamma + b)\tilde{F}_3 + (\gamma - 1)E_{\mathbb{7}} \\ &\quad + (\gamma - 1)E_{\mathbb{6}} + (\beta + 2b)\tilde{F}_2 + (\beta - 1)E_{\mathbb{3}} + (\beta - 1 + 2b)E_{\mathbb{4}} \\ &\quad + (\beta - 2 + 2b)E_{\mathbb{5}}. \end{aligned}$$

This divisor is effective and this shows that the constructions above gives an example of a  $H$ -polar cylinder, for  $H$  as described in 1(b).

- [IV] Suppose  $H$  is given by

$$H \equiv -K_S + a_1L_1 + a_2L_2 + a_3L_3$$

where  $\rho(S) = 4$ ,  $L_1, L_2, L_3$  are  $(-1)$ -curves that do not pass through the singular point and  $0 < a_3 \leq a_2 \leq a_1 < 1$ .

Blowing up the fibres of  $\mathbb{F}_1$  suitably, we get the following configuration on  $\bar{S}$ ,

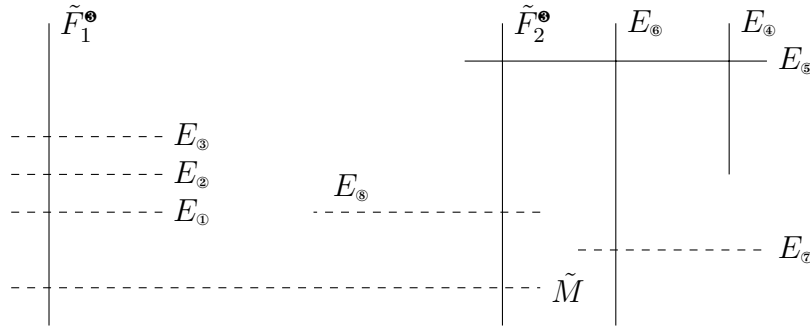


Figure 4.11:  $\bar{S}$

Contracting  $\tilde{M}$ , we get the following configuration on  $\tilde{S}$ .

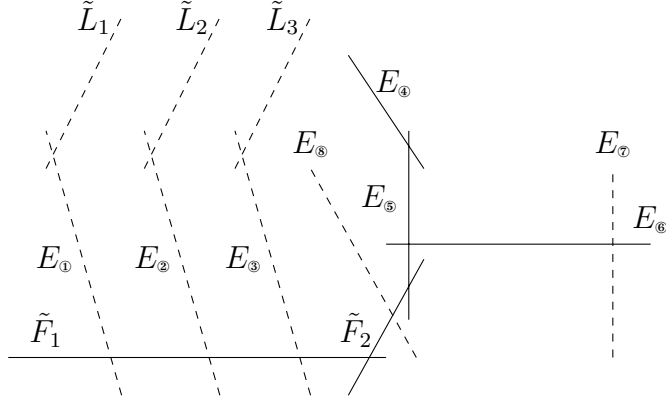


Figure 4.12:  $\tilde{S}$

In order to prove the existence of the curves  $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ , contract  $E_7, E_6, E_5$  in order. This gives a del Pezzo surface of degree 4 with  $A_1$  singularity. Therefore, from [CT88, Proposition 6.1], we can show that curves  $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$  exist as in the above configuration.

Consider the conic bundle given by the linear system  $|2E_7 + 2E_6 + 2E_5 + E_4 + \tilde{F}_2|$ , we have

$$\tilde{L}_1 + E_1 \sim \tilde{L}_2 + E_2 \sim \tilde{L}_3 + E_3 \sim 2E_7 + 2E_6 + 2E_5 + E_4 + \tilde{F}_2.$$

Therefore, on  $\tilde{S}$ , we have

$$\begin{aligned} \phi^*(H) &\equiv \alpha\tilde{F}_1 + (\alpha - 1 - a_1)E_1 + (\alpha - 1 - a_2)E_2 + (\alpha - 1 - a_3)E_3 \\ &\quad + (\beta + a_1 + a_2 + a_3)\tilde{F}_2 + (\beta - 1)E_4 + (\beta - 1 + a_1 + a_2 + a_3)E_5 \\ &\quad + (2\beta - 2 + 2a_1 + 2a_2 + 2a_3)E_6 + (2\beta - 3 + 2a_1 + 2a_2 + 2a_3)E_7 \\ &\quad + (2\beta - 4 + 2a_1 + 2a_2 + 2a_3)E_8 \end{aligned}$$

This divisor is effective.

- Let  $\rho(S) = 3$ . That is, the surface  $S$  is of degree 2.

[1] Suppose ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL + bC$$

where  $\rho(S) = 3$ ,  $0 \leq a < 1$  and  $b \geq 0$ , and  $L$  is a  $(-1)$ -curve on  $S$  that does not pass through the singular point.

Consider the configuration in Fig. 4.15. Blowing up a point on the fibre  $\tilde{F}_1$  on  $\tilde{S}$  in Fig. 4.15, gives the following configuration for  $\tilde{S}$ , for Case 1(d).

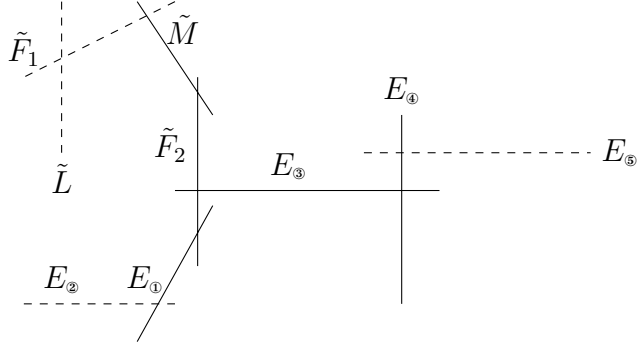


Figure 4.13:  $\tilde{S}$

Then, since  $\phi^*(L) \sim \tilde{L}$ , just as in Case 1(f), we get

$$\begin{aligned} \phi^*(H) \equiv & (2+b)\tilde{M} + (\beta+2b)\tilde{F}_2 + \alpha\tilde{F}_1 + (\alpha-1+a)\tilde{L} + (\beta-1+b)E_{\textcircled{1}} + (\beta-2)E_{\textcircled{2}} \\ & + (\beta-1+2b)E_{\textcircled{3}} + (\beta-2+2b)E_{\textcircled{4}} + (\beta-3+2b)E_{\textcircled{5}}. \end{aligned}$$

This divisor is effective and is therefore an example of a  $H$ -polar cylinder on  $S$  for  $H$  as described in 1(d).

[II] Suppose ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + a_1L_1 + a_2L_2$$

where  $\rho(S) = 3$ ,  $L_1$  and  $L_2$  are  $(-1)$ -curves that do not pass through the singular point and without loss of generality  $0 \leq a_1 < 1$ ,  $0 \leq a_2 < 1$ .

Blowing up fibres of  $\mathbb{F}_2$  appropriately, we get the following configuration

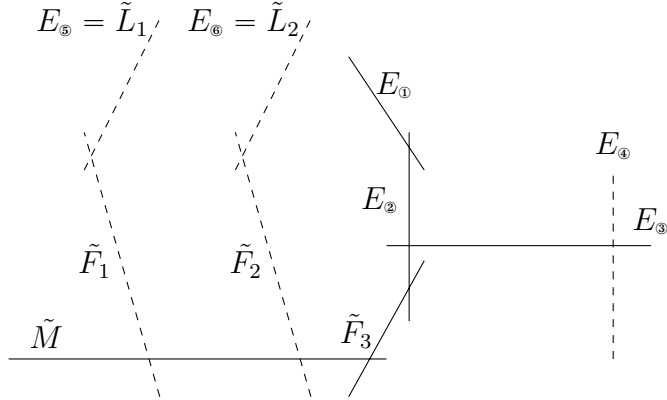


Figure 4.14:  $\tilde{S}$

Therefore, on  $\tilde{S}$ , we get

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha-1+a_1)\tilde{L}_1 + \beta\tilde{F}_2 + (\beta-1+a_2)\tilde{L}_2 + \gamma\tilde{F}_3 + (\gamma-1)E_{\textcircled{1}} \\ & + (2\gamma-2)E_{\textcircled{2}} + (2\gamma-3)E_{\textcircled{3}} + (2\gamma-4)E_{\textcircled{4}}. \end{aligned}$$

This divisor is effective.

- Let  $\rho(S) = 2$ . That is, the surface  $S$  is of degree 3.

[I] Suppose ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $\rho(S) = 2$ ,  $b > 0$ , and  $C$  is a  $(0)$ -curve. We can blow up fibres of  $\mathbb{F}_2$  to get the following configuration.

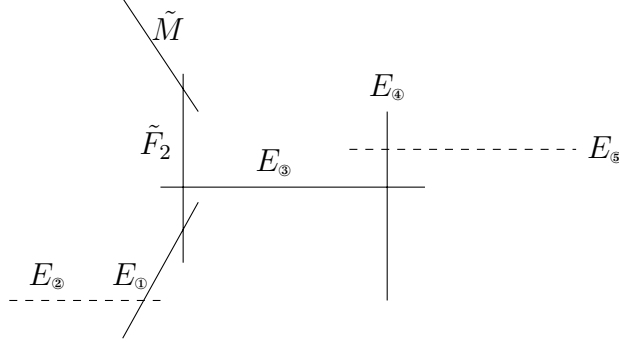


Figure 4.15:  $\tilde{S}$

Note that  $\tilde{F}_1$  on  $\tilde{S}$  is the  $(0)$ -curve which is the proper transform of  $F_1$  on  $\tilde{S}$ . Therefore, using the conic bundle given by the linear system  $|\tilde{F}_2 + E_1 + E_2 + E_3 + E_4 + E_5|$ , we can write

$$\tilde{F}_1 \sim \tilde{F}_2 + E_1 + E_2 + E_3 + E_4 + E_5.$$

We can choose  $C$  on  $S$  such that, on  $\tilde{S}$ , we have

$$\phi^*(C) \equiv E_1 + \tilde{M} + 2\tilde{F}_2 + 2E_3 + 2E_4 + 2E_5, .$$

where we use the conic bundle given by the linear system  $|E_1 + \tilde{M} + 2\tilde{F}_2 + 2E_3 + 2E_4 + 2E_5|$ .

Therefore, we get

$$\begin{aligned} \phi^*(H) \equiv & (2 + b)\tilde{M} + (\alpha + 2b)\tilde{F}_2 + \beta\tilde{F}_1 + (\alpha - 1 + b)E_1 + (\alpha - 2)E_2 \\ & + (\alpha - 1 + 2b)E_3 + (\alpha - 2 + 2b)E_4 + (\alpha - 3 + 2b)E_5. \end{aligned}$$

This divisor is effective.

[II] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

where  $0 < a < 1$  and  $L$  is a  $(-1)$ -curve that does not pass through the singular point.

Using the configuration of curves on  $\tilde{S}$  in Fig. 4.15, we can prove the existence of the curve  $\tilde{L}$  such that  $\tilde{L} \cdot E_5 = 1$  and does not intersect any other curve on  $\tilde{S}$ . Contracting  $E_2$  gives a del Pezzo surface of degree 4 of Type  $A_4$  and using [CT88, Proposition 6.1], we can conclude that a curve such as  $\tilde{L}$  exists on  $\tilde{S}$ .

Then, using the conic bundle given by  $|\tilde{L} + E_5|$ , we have

$$\tilde{L} + E_5 \sim 2E_2 + 2E_1 + 2\tilde{F}_2 + \tilde{M} + E_3.$$

Therefore, on  $\tilde{S}$ , we have

$$\begin{aligned} \phi^*(H) \equiv & (2+a)\tilde{M} + (\alpha+2a)\tilde{F}_2 + \beta\tilde{F}_1 + (\alpha-1+2a)E_{\circledast} + (\alpha-2+2a)E_{\circledcirc} \\ & + (\alpha-1)E_{\circledast} + (\alpha-2)E_{\circledcirc} + (\alpha-3-a)E_{\circledcirc}. \end{aligned}$$

Since  $a < 1$ , this is effective and hence proves that the construction in Fig. 4.15 is an example of a  $H$ -polar cylinder for  $H$  as described above.

2.  **$D_5 + A_1$  singularity:** Suppose the surface  $S$  has singularity of type  $D_5 + A_1$ . This implies that  $\rho(S) \leq 3$ . The surface  $S$  is either of degree 1 ( $\rho(S) = 3$ ) or of degree 2 ( $\rho(S) = 2$ ).

Using Lemma 4.70 and the result that there are no cubic del Pezzo surfaces of type  $D_5 + A_1$ , from [BW79], we can conclude that any ample divisor  $H$  on  $S$  is given by one of the following,

- (a)  $H \equiv -K_S + a_1L_1 + a_2L_2$  with  $\rho(S) = 3$ ,  $0 \leq a_1 < 2$ ,  $0 \leq a_2 < 1$ ,
- (b)  $H \equiv -K_S + bC + aL$  with  $\rho(S) = 3$ ,  $0 \leq a < 2$ ,  $b > 0$ ,
- (c)  $H \equiv -K_S + aL$  with  $\rho(S) = 2$ ,  $0 < a < 2$ , where  $L$ ,  $L_1$  are  $(\frac{-1}{2})$ -curves that pass through the singular point of type  $A_1$ ,  $L_2$  is a  $(-1)$ -curve on  $S$  that does not pass through any singular point,  $C$  is a fibre of  $|C|$  and for Case 2(b),  $L$  is an irreducible component of a fibre of the conic bundle given by  $|C|$  such that  $L \cdot C = 0$ .

[I] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL + bC$$

where  $\rho(S) = 3$ ,  $0 \leq a < 2$ ,  $b > 0$ ,  $L$  is a  $(\frac{-1}{2})$ -curve that passes through singular point of type  $A_1$  and  $C$  is a  $(0)$ -curve.

We get the following configuration on  $\tilde{S}$  by blowing up the fibres of  $\mathbb{F}_2$ .

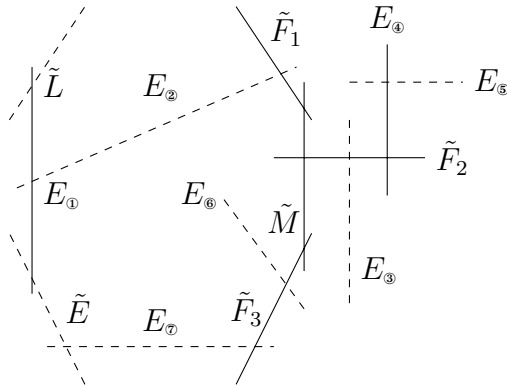


Figure 4.16:  $\tilde{S}$

Contracting the curves  $E_{\circledast}$ ,  $E_{\circledcirc}$ ,  $\tilde{F}_2$ ,  $\tilde{M}$  gives a del Pezzo surface of degree 5 of Type  $A_1$ . Using [CT88, Proposition 8.5], we can prove the existence of curves  $\tilde{E}$  and  $\tilde{L}$ . Note that this also gives us that  $E_{\circledcirc} \cdot \tilde{L} = 1$ .

Consider the conic bundle given by  $|\tilde{L} + E_{\textcircled{1}} + \tilde{E}|$ . Since the curves in the divisor,  $\tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}}$  do not intersect the curves in the divisor  $\tilde{L} + E_{\textcircled{1}} + \tilde{E}$ , and

$$\begin{aligned} & \left( \tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}} \right)^2 = 0, \\ & \left( \tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}} \right) \cdot \mathcal{C} = 0, \text{ for } \mathcal{C} \in \{\tilde{F}_3, \tilde{F}_1, \tilde{M}, \tilde{F}_2, E_{\textcircled{4}}, E_{\textcircled{5}}\}, \end{aligned}$$

we can conclude that  $\tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}}$  is a (0)-curve and a fibre of the conic bundle given by  $|\tilde{L} + E_{\textcircled{1}} + \tilde{E}|$  and therefore, we can choose  $C$  on  $S$  such that

$$\phi^*(C) \sim_{\mathbb{Q}} \tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}}.$$

Similarly, using the conic bundle that is given by  $|E_{\textcircled{2}} + E_{\textcircled{1}} + \tilde{L}|$ , we have

$$\tilde{L} + E_{\textcircled{1}} + E_{\textcircled{2}} \sim_{\mathbb{Q}} E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}.$$

Note that we have  $\phi^*(L) \sim \tilde{L} + \frac{1}{2}E_{\textcircled{1}}$ . Therefore, we choose  $L$  on  $S$  such that

$$\phi^*(L) \sim -E_{\textcircled{2}} - \frac{1}{2}E_{\textcircled{1}} + E_{\textcircled{7}} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\textcircled{4}} + E_{\textcircled{5}}.$$

Note that  $L$  is an irreducible component of a fibre of the conic bundle given by  $|\tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}}|$  and therefore  $L \cdot C = 0$ .

Once we choose  $L$  and  $C$  as described above, we now prove that the example of the cylinder that is constructed in the above configuration, is indeed a  $H$ -polar cylinder. From the above construction, we have

$$\begin{aligned} \phi^*(H) = -K_{\tilde{S}} + a\phi^*(L) + b\phi^*(C) & \equiv (2 + a + 2b)\tilde{M} + (\alpha + b)\tilde{F}_1 + (\gamma + a + b)\tilde{F}_3 \\ & + (\beta + a + 2b)\tilde{F}_2 + \left(\alpha - 1 - \frac{a}{2}\right)E_{\textcircled{1}} + (2\alpha - 2 - a)E_{\textcircled{2}} \\ & + (\gamma - 1 + a)E_{\textcircled{7}} + (\gamma - 1)E_{\textcircled{6}} + (\beta - 1)E_{\textcircled{3}} \\ & + (\beta - 1 + a + 2b)E_{\textcircled{4}} + (\beta - 2 + 2b)E_{\textcircled{5}} \end{aligned}$$

This divisor is effective.

Let us now prove that the example of the  $H$ -polar cylinder that we have constructed above, can always be constructed on any surface  $S$  of degree 1 of Type  $D_5 + A_1$  with the ample divisor  $H$  on  $S$ , as described in this case.

Consider the minimal resolution of  $S$ ,  $\tilde{S}$ . The curves on  $\tilde{S}$  that we know exist are  $E_{\textcircled{4}}, \tilde{F}_2, \tilde{M}, \tilde{F}_1, \tilde{F}_3, E_{\textcircled{1}}$ , since these are the exceptional curves of  $\phi : \tilde{S} \rightarrow S$ . Since we choose the conic bundle given by  $|\tilde{L} + E_{\textcircled{1}} + \tilde{E}|$ , we also have the curves  $\tilde{L}$  and  $\tilde{E}$ . The curve  $E_{\textcircled{5}}$  is the pullback of  $C \in |-K_S|$  and therefore intersects  $\tilde{F}_2$  as in the figure.

Since the curves  $\tilde{F}_3, \tilde{M}, \tilde{F}_1, \tilde{F}_2, E_{\textcircled{4}}$  intersect each other, they form a part of the same fibre of the conic bundle given by  $|\tilde{L} + E_{\textcircled{1}} + \tilde{E}|$ . Therefore, there exists a curve  $E_{\textcircled{5}}$  as in the figure, so that  $\tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\textcircled{4}} + 2E_{\textcircled{5}}$  forms a fibre of the conic bundle.

In order to prove the existence of the remaining curves,  $E_{\mathbb{2}}$ ,  $E_{\mathbb{7}}$ ,  $E_{\mathbb{6}}$ , in the Figure Fig. 4.17, contract  $E_{\mathbb{5}}$ ,  $E_{\mathbb{4}}$ ,  $\tilde{F}_2$  in order. This gives a del Pezzo surface of degree 4 of Type  $3A_1$ . Using [CT88, Proposition 6.1], we can prove the existence of the curves  $E_{\mathbb{2}}$ ,  $E_{\mathbb{7}}$ ,  $E_{\mathbb{6}}$  as in the Fig. 4.17, with  $E_{\mathbb{6}} \cdot \tilde{L} = 1$ .

Thus, this proves our claim that we can always construct the configuration on  $\tilde{S}$  as in Fig. 4.17 and hence obtain the above example of a cylinder on  $S$ .

[II] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + a_1 L_1 + a_2 L_2$$

where  $\rho(S) = 3$ ,  $L_1$  is a curve that passes through the singular point of type  $A_1$ ,  $L_2$  is a curve that does not pass through any singular point and  $0 \leq a_1 < 2$ ,  $0 \leq a_2 < 1$ .

Blowing up the fibres of  $\mathbb{F}_1$  appropriately to get the configuration on  $\tilde{S}$  and then blowing down the  $(-1)$  section,  $\tilde{M}$  on  $\tilde{S}$ , we get the following configuration on  $\tilde{S}$ ,

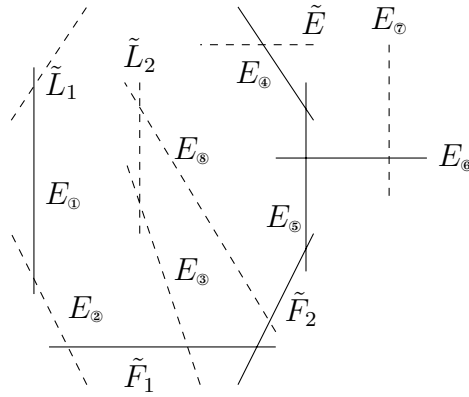


Figure 4.17:  $\tilde{S}$

Contracting  $E_{\mathbb{7}}$ ,  $E_{\mathbb{6}}$ ,  $E_{\mathbb{5}}$  gives a del Pezzo surface of degree 4 of Type  $2A_1$ . The existence of the curve  $E_{\mathbb{2}}$  shows that this surface is unique and using [CT88, Proposition 6.1], we can prove the existence of the curves  $\tilde{L}_1$ ,  $\tilde{L}_2$  and  $\tilde{E}$ .

Using the conic bundle given by  $|\tilde{L}_1 + E_{\mathbb{1}} + E_{\mathbb{2}}|$ , we have

$$\tilde{L}_1 + E_{\mathbb{1}} + E_{\mathbb{2}} \sim \tilde{L}_2 + E_{\mathbb{3}} \sim \tilde{F}_2 + E_{\mathbb{4}} + 2E_{\mathbb{5}} + 2E_{\mathbb{6}} + 2E_{\mathbb{7}}.$$

Note that  $\phi^*(L_1) \sim \tilde{L}_1 + \frac{1}{2}E_{\mathbb{6}}$  and  $\phi^*(L_2) \sim \tilde{L}_2$ . Therefore

$$\begin{aligned} \phi^*(L_1) &\sim_{\mathbb{Q}} -E_{\mathbb{2}} - \frac{1}{2}E_{\mathbb{1}} + \tilde{F}_2 + E_{\mathbb{4}} + 2E_{\mathbb{5}} + 2E_{\mathbb{6}} + 2E_{\mathbb{7}}, \\ \phi^*(L_2) &\sim_{\mathbb{Q}} -E_{\mathbb{3}} + \tilde{F}_2 + E_{\mathbb{4}} + 2E_{\mathbb{5}} + 2E_{\mathbb{6}} + 2E_{\mathbb{7}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi^*(H) &\equiv (\alpha - 1 - \frac{a_1}{2})E_{\mathbb{1}} + \alpha\tilde{F}_1 + (\beta + a_1 + a_2)\tilde{F}_2 + (2\beta - 2 + 2a_1 + 2a_2)E_{\mathbb{5}} \\ &\quad + (\beta - 1 + a_1 + a_2)E_{\mathbb{4}} + (2\beta - 3 + 2a_1 + 2a_2)E_{\mathbb{6}} + (2\alpha - 2 - a_1)E_{\mathbb{2}} \\ &\quad + (\alpha - 1 - a_2)E_{\mathbb{3}} + (2\beta - 4 + 2a_1 + 2a_2)E_{\mathbb{7}} + (\beta - 1)E_{\mathbb{8}} \end{aligned}$$

This divisor is effective.

[III] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

where  $\rho(S) = 2$ ,  $L$  is a  $(\frac{-1}{2})$ -curve that passes through the singular point of type  $A_1$  and  $0 < a < 2$ .

We can get the following configuration on  $\tilde{S}$  by blowing up fibres on  $\mathbb{F}_2$ ,

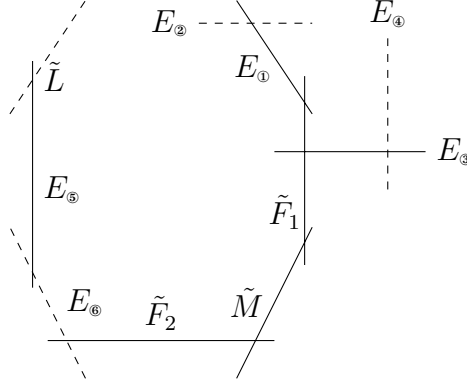


Figure 4.18:  $\tilde{S}$

Contracting  $E_4$ ,  $E_3$ ,  $\tilde{F}_1$  in order, gives a del Pezzo surface of degree 4 of Type  $2A_1$  (of degree 5 if  $\rho(S) = 2$ ). Using [CT88, Proposition 6.1] ([CT88, Proposition 8.5]), we can prove the existence of the curve  $\tilde{L}$  in the above configuration.

From the conic bundle given by  $|E_6 + \tilde{L} + E_5|$ , we have

$$\tilde{L} + E_5 + E_6 \sim \tilde{M} + E_1 + 2\tilde{F}_1 + 2E_3 + 2E_4.$$

Since  $\phi^*(L) \equiv \tilde{L} + \frac{1}{2}E_5$ , we have,

$$\phi^*(L) \equiv -E_6 - \frac{1}{2}E_5 + \tilde{M} + E_1 + 2\tilde{F}_1 + 2E_3 + 2E_4,$$

and therefore

$$\begin{aligned} \phi^*(H) \equiv & (2 + a)\tilde{M} + \beta\tilde{F}_2 + (\alpha + 2a)\tilde{F}_1 + (\beta - 1 - \frac{a}{2})E_5 + (2\beta - 2 - a)E_6 \\ & + (\alpha - 1 + a)E_1 + (\alpha - 1 + 2a)E_3 + (\alpha - 2)E_2 + (\alpha - 2 + 2a)E_4. \end{aligned}$$

This divisor is effective.

3.  $D_5 + 2A_1$  **singularity** This implies that  $\rho(S) \leq 2$ . Consider surface  $S$  of degree 1 with  $D_5 + 2A_1$ . Then  $\rho(S) = 2$ .

Using Lemma 4.70 and the result that there are no cubic del Pezzo surfaces of Type  $D_5 + A_1$ , from [BW79], we can conclude that every ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_X + bC$$

where  $C$  is a  $(0)$ -curve and  $b > 0$ . Note that Lemma 4.70 implies that there exists an irreducible fibre  $C' \sim C$  that passes through both singular points of type  $A_1$ .

Blowing up the fibres of  $\mathbb{F}_2$ , we get the following configuration.

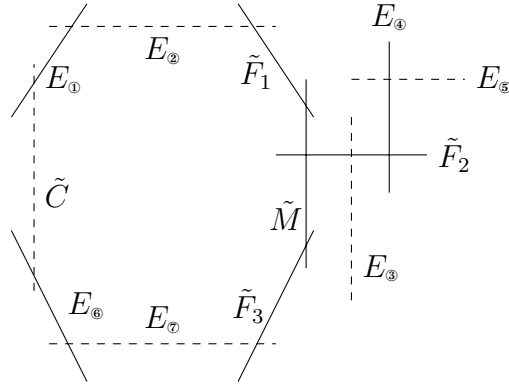


Figure 4.19:  $\tilde{S}$

Here  $\tilde{C}$  is the proper transform of  $C'$  on  $\tilde{S}$ .

In order to prove the existence of  $\tilde{C}$  on  $\tilde{S}$ , from the construction from  $\mathbb{F}_2$ , as shown above, contract  $E_6$ ,  $E_4$ ,  $\tilde{F}_2$ ,  $\tilde{M}$ , in order, we get a del Pezzo surface of degree 5 of Type  $2A_1$ . Then [CT88, Proposition 8.5] implies the existence of the curve  $\tilde{C}$  as in the configuration in Fig. 4.19.

Now, consider the conic bundle given by  $|E_6 + \tilde{C} + E_1|$ . This implies that we can choose  $C$  on  $S$  such that

$$\phi^*(C) \sim \tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_4 + 2E_5.$$

Since  $\tilde{C} = \phi^*(C) - \frac{1}{2}E_6 - \frac{1}{2}E_1$ , we get

$$\begin{aligned} \phi^*(H) &= 2\tilde{M} + (\alpha + b)\tilde{F}_1 + (\alpha - 1)E_1 + (2\alpha - 2)E_2 + (\gamma + b)\tilde{F}_3 + (\gamma - 1)E_6 \\ &\quad + (2\gamma - 2)E_7 + (\beta + 2b)\tilde{F}_2 + (\beta - 1)E_3 + (\beta - 1 + 2b)E_4 \\ &\quad + (\beta - 2 + 2b)E_5. \end{aligned}$$

Thus, the divisor is effective, thus proving that this construction is an example of a  $H$ -polar cylinder, for any ample divisor  $H$  on  $S$ .

4.  $D_5 + A_2$  **singularity:** Suppose a surface  $S$  has singularity of type  $D_5 + A_2$ . Then  $\rho(S) \leq 2$ . So consider surface  $S$  of degree 1 with  $D_5 + A_2$ . Then  $\rho(S) = 2$ .

Lemma 4.70 implies that any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

such that  $0 < a < 3$  and  $L$  is a  $(\frac{-1}{3})$ -curve that passes through the singular point of  $A_2$ -type.

Consider the following configuration of  $\bar{S}$  obtained by blowing up the fibres of  $\mathbb{F}_1$  suitably.

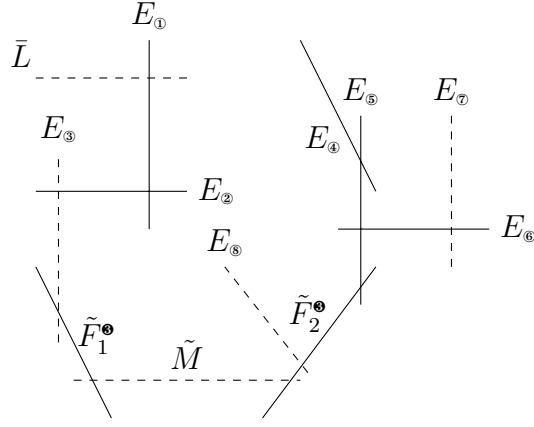


Figure 4.20:  $\tilde{S}$

Contract  $\tilde{M}$  to get the following configuration on  $\tilde{S}$ .

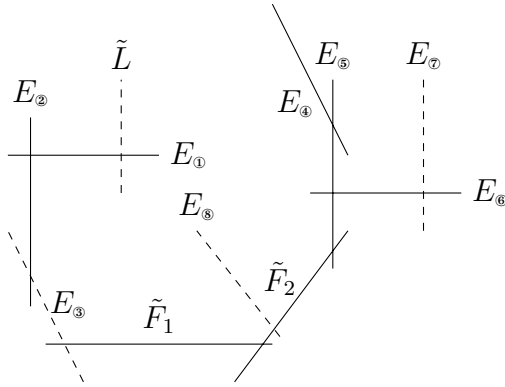


Figure 4.21:  $\tilde{S}$

In order to prove the existence of the curve  $\tilde{L}$  as in the above configuration on  $\tilde{S}$ , contract  $E_7, E_6, E_5$ , in order. This gives a del Pezzo surface of degree 4 of Type  $A_2 + A_1$ . Using [CT88, Proposition 6.1], we can conclude that such a curve exists. Note that this also shows that  $\tilde{L} \cdot E_8 = 1$ .

Using the conic bundle given by the linear system  $|\tilde{L} + E_1 + E_2 + \tilde{F}_1|$ , we can conclude that

$$\tilde{L} + E_1 + E_2 + \tilde{F}_1 \sim 2E_7 + 2E_6 + 2E_5 + E_4 + \tilde{F}_2.$$

Note that  $\tilde{L} \sim \phi^*(L) - \frac{2}{3}E_1 - \frac{1}{3}E_2$ . This then gives us that

$$\phi^*(L) = -E_3 - \frac{2}{3}E_1 - \frac{1}{3}E_2 + 2E_7 + 2E_6 + E_4 + 2E_5 + \tilde{F}_2.$$

On  $\tilde{S}$ , we get

$$\begin{aligned} \phi^*(H) &= \alpha\tilde{F}_1 + (\alpha - 1 - \frac{2a}{3})E_1 + (2\alpha - 2 - \frac{a}{3})E_2 + (3\alpha - 3 - a)E_3 + (\beta + a)\tilde{F}_2 \\ &\quad + (\beta - 1)E_8 + (2\beta - 2 + 2a)E_5 + (\beta - 1 + a)E_4 + (2\beta - 3 + 2a)E_6 \\ &\quad + (2\beta - 4 + 2a)E_7. \end{aligned}$$

The divisor is effective and hence proves that the construction done above is an example of a  $H$ -polar cylinder for  $H$  as described above.

#### 4.3.2.6 $D_4$ singular point

Suppose the surface  $S$  has at least 1 singular point of type  $D_4$ .

1.  **$D_4$  singularity:** Suppose the surface  $S$  has exactly one singular point, and this is of type  $D_4$ . Then  $\rho(S) \leq 5$ . The surface  $S$  can be of degree 1 ( $\rho(S) = 5$ ), of degree 2 ( $\rho(S) = 4$ ), of degree 3 ( $\rho(S) = 3$ ) or of degree 4 ( $\rho(S) = 2$ ).

Using Lemma 4.70 and the result that there are no del Pezzo surfaces of degree 5 of Type  $D_4$ , from [CT88], we can conclude that any ample divisor  $H$  on  $S$  is given by one of the following:

- (a)  $H \equiv -K_S + bC + a_1L_1 + a_2L_2 + a_3L_3$  with  $\rho(S) = 5$ ,  $0 \leq a_1, a_2, a_3 < 1$ ,  $b \geq 0$ ,
- (b)  $H \equiv -K_S + bC + a_1L_1 + a_2L_2$  with  $\rho(S) = 4$ ,  $0 \leq a_1, a_2 < 1$ ,  $b \geq 0$ ,
- (c)  $H \equiv -K_S + bC + aL$  with  $\rho(S) = 3$ ,  $0 < a < 1$ ,  $b > 0$ ,
- (d)  $H \equiv -K_S + aL$ , with  $\rho(S) = 3$ ,  $0 < a < 1$ ,
- (e)  $H \equiv -K_S + bC$ , with  $\rho(S) = 3$ ,  $b > 0$ ,
- (f)  $H \equiv -K_S + bC$  with  $\rho(S) = 2$ ,  $b > 0$ ,

where  $C$  is a fibre of  $|C|$  that gives a conic bundle extremal contraction,  $L_1, L_2, L_3, L$  are  $(-1)$ -curves that do not pass through the singular point, and are each mutually disjoint irreducible components of fibres of  $|C|$  chosen such that  $L_1 \cdot C = L_2 \cdot C = L_3 \cdot C = L \cdot C = 0$ .

- Let  $\rho(S) = 5$ , that is consider  $S$  of degree 1. Recall that in this case,  $S$  does not have a  $(-K_S)$ -polar cylinder. Here, for any  $H \neq -K_S$ , we construct a  $H$ -polar cylinder.

[I] Suppose the ample divisor is given by

$$H \equiv -K_S + aL$$

where  $L$  is a  $(-1)$ -curve in  $S$  which does not pass through the singular point and  $0 < a < 1$ .

Blowing up the fibres of  $\mathbb{F}_2$  suitably, we get the following configuration on  $\tilde{S}$ ,

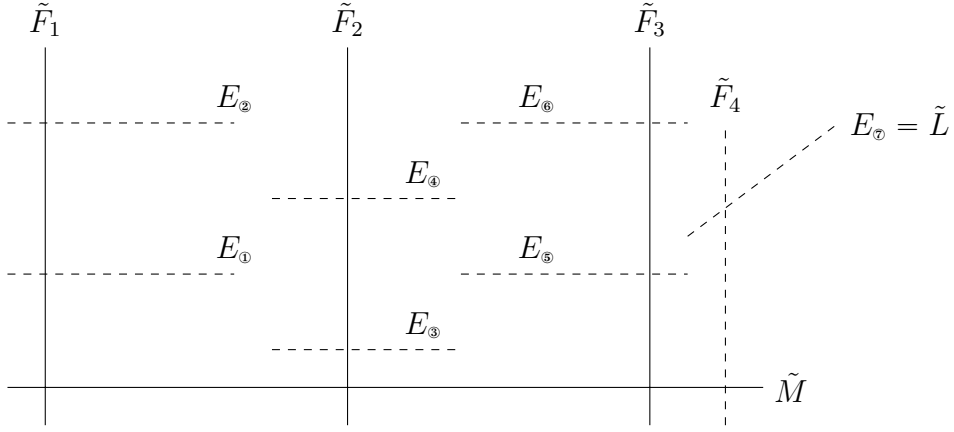


Figure 4.22:  $\tilde{S}$

Note that  $\phi^*(L) = \tilde{L}$ . Taking non-negative constants  $\alpha, \beta, \gamma, \delta$ , we get

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{1}} + (\alpha - 1)E_{\textcircled{2}} + \beta\tilde{F}_2 + (\beta - 1)E_{\textcircled{3}} + (\beta - 1)E_{\textcircled{4}} \\ & + \gamma\tilde{F}_3 + (\gamma - 1)E_{\textcircled{5}} + (\gamma - 1)E_{\textcircled{6}} + \delta\tilde{F}_4 + (\delta - 1 + a)\tilde{L}. \end{aligned}$$

This divisor is effective.

[II] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $0 < b$  and  $C$  is a  $(0)$ -curve.

Blowing up the fibres of  $\mathbb{F}_1$  and then contracting the unique  $(-1)$  section, we get the following configuration on  $\tilde{S}$ ,

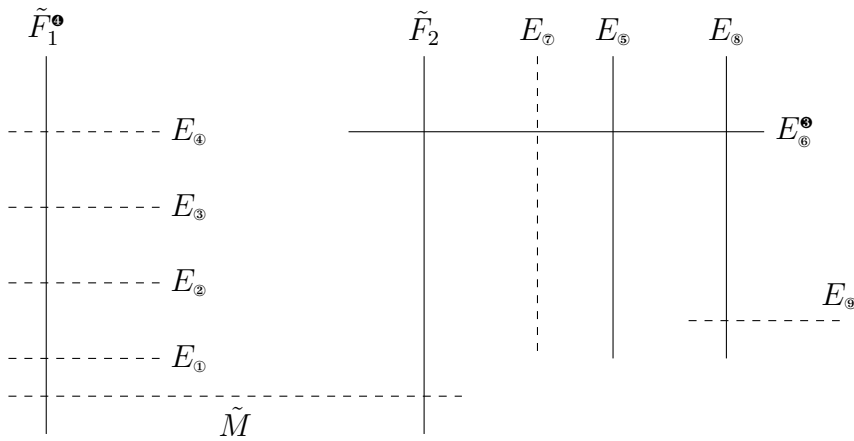


Figure 4.23:  $\tilde{S}$

Contracting  $\tilde{M}$  and  $\tilde{F}_2$  in order, we get the configuration on  $\tilde{S}$  (Figure 4.24).

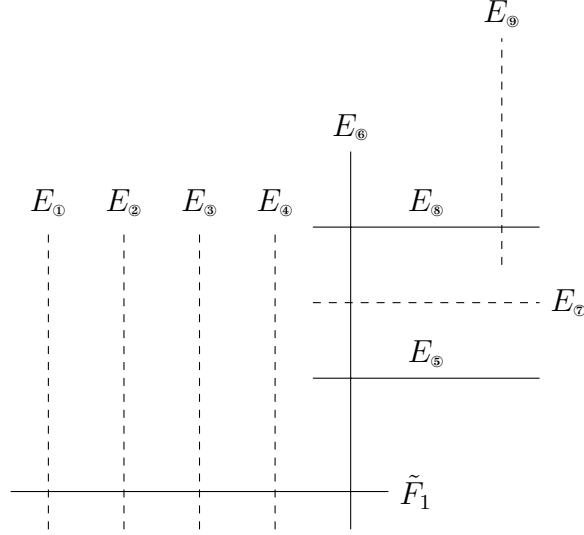


Figure 4.24:  $\tilde{S}$

Consider the conic bundle given by the linear system  $|\tilde{F}_1 + E_5 + 2E_6 + 2E_8 + 2E_9|$ . Using this, we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{F}_1 + E_5 + 2E_6 + 2E_8 + 2E_9.$$

Using this, we then get that

$$\begin{aligned} \phi^*(H) \equiv & (\alpha + b)\tilde{F}_1 + (\alpha - 1)(E_1 + E_2 + E_3 + E_4) + (2\beta - 2 + 2b)E_5 \\ & + (\beta - 1 + b)E_6 + (2\beta - 3 + 2b)E_8 + (2\beta - 4 + 2b)E_9 + (2\beta - 3)E_7, \end{aligned}$$

and this divisor is effective.

[III] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_X + aL + bC$$

where  $0 < a < 1$  and  $b > 0$ , and the curves  $L$  and  $C$  are  $(-1)$  and  $(0)$ -curves respectively, that do not pass through the singular point.

Blowing up fibres of  $\mathbb{F}_2$  suitably, we get the configuration on  $\tilde{S}$  as in Figure 4.22. Consider the conic bundle given by  $|\tilde{F}_2 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_1 + 2E_2|$ . Then, we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{F}_2 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_1 + 2E_2.$$

Since  $\text{Supp}(\tilde{C}) \subset -K_{\tilde{S}} + a\tilde{L}$  as computed in Case [I] above, the same  $H$ -polar cylinder as in Case 1(a), works here too.

[IV] Suppose the ample divisor is given by

$$H = -K_S + a_1L_1 + a_2L_2 + a_3L_3$$

where  $L_1, L_2, L_3$  are  $(-1)$ -curves on  $S$  that do not pass through the singular point and  $0 \leq a_1 < 1$ ,  $a_1 \leq a_2 \leq a_3 < 1$ .

Blowing up fibres of  $\mathbb{F}_1$  suitably to get  $\tilde{S}$  and then contracting the unique  $(-1)$  section  $\tilde{M}$ , we get the following configuration on  $\tilde{S}$  (Fig. 4.25),

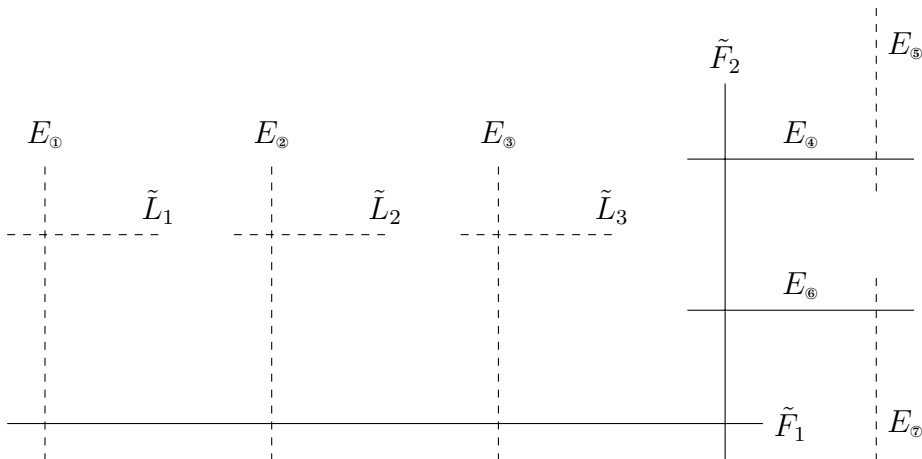


Figure 4.25:  $\tilde{S}$

In order to prove the existence of the curves  $\tilde{L}_2$ ,  $\tilde{L}_3$  as in the configuration above, contract  $E_1$ ,  $E_5$ ,  $E_4$ , in order. This gives a del Pezzo surface of degree 4 of Type  $A_1$ . Using [CT88, Proposition 6.1], we can show the existence of the curves  $\tilde{L}_2$ ,  $\tilde{L}_3$  as in Fig. 4.25. Similarly, instead of contracting  $E_1$ , contract  $E_2$  and the same argument proves the existence of the curve  $\tilde{L}_1$  as in Fig. 4.25.

Using the conic bundle given by  $|\tilde{L}_1 + E_1|$ , we get

$$E_1 + \tilde{L}_1 \sim E_2 + \tilde{L}_2 \sim E_3 + \tilde{L}_3 \sim \tilde{F}_2 + E_4 + E_6 + E_5 + E_7.$$

Using this and  $\phi^*(L_i) = \tilde{L}_i$ , we have

$$\begin{aligned} \phi^*(H) \equiv & \alpha \tilde{F}_1 + (\alpha - 1 - a_1)E_1 + (\alpha - 1 - a_2)E_2 + (\alpha - 1 - a_3)E_3 + (\beta + a)\tilde{F}_2 \\ & + (\beta - 1 + a)E_4 + (\beta - 1 + a)E_6 + (\beta - 2 + a)E_7 + (\beta - 2 + a)E_5, \end{aligned}$$

where  $a = a_1 + a_2 + a_3$ . This divisor is effective and shows that the construction done above gives an example of a  $H$ -polar cylinder on  $S$ .

[V] Suppose the ample divisors are given by

$$H \equiv -K_S + a_1L_1 + a_2L_2 + a_3L_3 + bC,$$

where  $L_i$  for  $1 \leq i \leq 3$  and  $C$  are  $(-1)$  and  $(0)$ -curves, respectively, that do not pass through the singular point.

Blowing up fibres of  $\mathbb{F}_1$  suitably we get the configuration as in Fig. 4.25. Consider the conic bundle given by the linear system  $|\tilde{F}_1 + E_4 + 2\tilde{F}_2 + 2E_6 + 2E_7|$ . Then you can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{F}_1 + E_4 + 2\tilde{F}_2 + 2E_6 + 2E_7.$$

Since  $\text{Supp}(\tilde{C}) \subset -K_{\tilde{S}} + a_1\tilde{L}_1 + a_2\tilde{L}_2 + a_3\tilde{L}_3$  as computed in Case [IV] above, the same  $H$  polar cylinder as in Case [IV], works here too.

- Let  $\rho(S) = 4$ , that is, consider  $S$  to be of degree 2.  
 [I] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC + a_1L_1 + a_2L_2$$

where  $0 < a_1 < 1$ ,  $0 \leq a_2 < 1$ ,  $b \geq 0$ ,  $C$  is a fibre of  $|C|$  that gives a conic bundle extremal contraction, and  $L_1, L_2$  are  $(-1)$ -curves that do not pass through the singular point, and are each mutually disjoint irreducible components of fibres of  $|C|$  chosen such that  $L_1 \cdot C = L_2 \cdot C = 0$ .

Blowing up fibres of  $\mathbb{F}_2$  suitable, we get the following configuration on  $\tilde{S}$ .

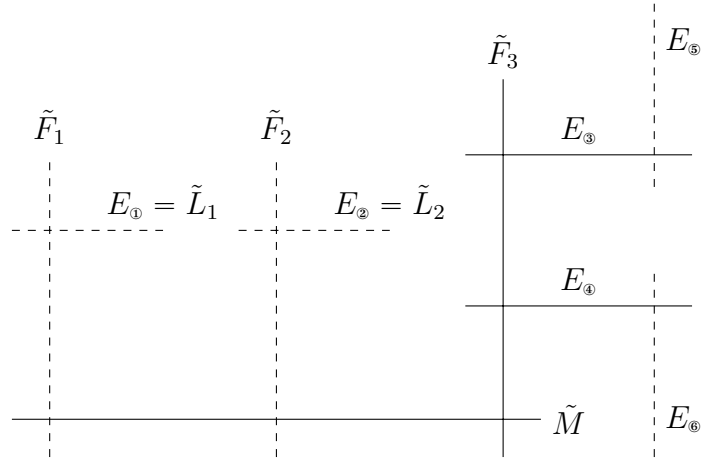


Figure 4.26:  $\tilde{S}$

By the conic bundle given by the linear system  $|\tilde{M} + E_4 + 2\tilde{F}_3 + 2E_3 + 2E_5|$ , we have that

$$\phi^*(C) \sim \tilde{M} + E_4 + 2\tilde{F}_3 + 2E_3 + 2E_5.$$

Therefore, on  $\tilde{S}$ , we have

$$\begin{aligned} \phi^*(H) \equiv & (2 + b)\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1 + a_1)\tilde{L}_1 + \beta\tilde{F}_2 + (\beta - 1 + a_2)\tilde{L}_2 \\ & + (\gamma + 2b)\tilde{F}_3 + (\gamma - 1 + b)E_4 + (\gamma - 1 + 2b)E_3 + (\gamma - 2 + 2b)E_5 \\ & + (\gamma - 2)E_6. \end{aligned}$$

Since this divisor is effective, it proves that the above constructed example is a  $H$ -polar cylinder for  $H$  as described above.

- [II] Suppose  $a_1 = a_2 = 0$ . Then any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $b > 0$ ,  $C$  is a fibre of the conic bundle given by the linear system  $|C|$ .

Blowing up the fibres of  $\mathbb{F}_2$  suitably, gives the following configuration on  $\tilde{S}$ .

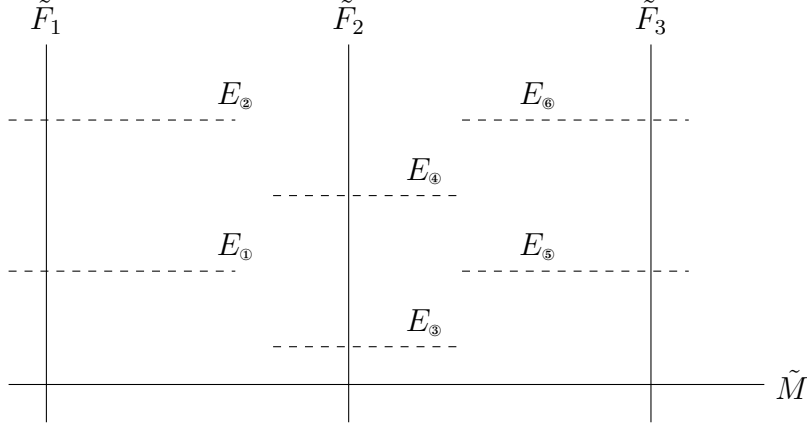


Figure 4.27:  $\tilde{S}$

Using the conic bundle given by the linear system  $|\tilde{F}_2 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\circledast}|$ , we can take  $C$  on  $S$  such that

$$\phi^*(C) \sim \tilde{F}_2 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\circledast}.$$

Therefore, on  $\tilde{S}$ , we get

$$\begin{aligned} \phi^*(C) \sim & (2 + 2b)\tilde{M} + (\alpha + 2b)\tilde{F}_1 + (\alpha - 1 + 2b)E_{\circledast} + (\alpha - 1)E_{\circledcirc} + (\beta + b)\tilde{F}_2 \\ & + (\beta - 1)E_{\circledast} + (\beta - 1)E_{\circledcirc} + (\gamma + b)\tilde{F}_3 + (\gamma - 1)E_{\circledast} + (\gamma - 1)E_{\circledcirc}. \end{aligned}$$

Since this divisor is effective, it proves that the above constructed example is a  $H$ -polar cylinder for  $H$  as described above.

- Let  $\rho(S) = 3$ . That is, consider  $S$  to be of degree 3. Suppose ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC + aL,$$

where  $0 \leq a < 1$ ,  $0 \leq b$ ,  $C$  is a fibre of the conic bundle given by the linear system  $|C|$ ,  $L$  is a  $(-1)$ -curve not passing through any singular point and is an irreducible component of a fibre of  $|C|$  such that  $L \cdot C = 0$ .

Blowing up fibres of  $\mathbb{F}_2$ , we get the following configuration on  $\tilde{S}$ .

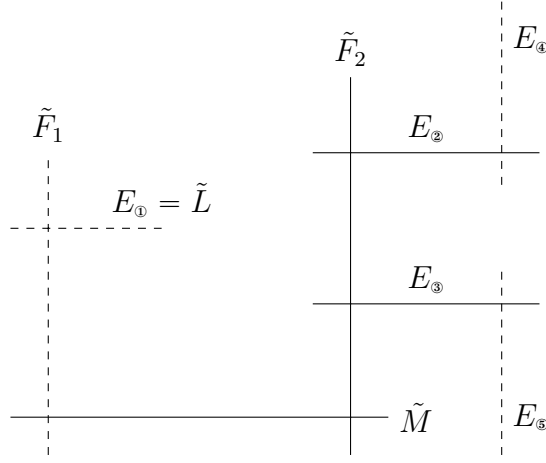


Figure 4.28:  $\tilde{S}$

Using the conic bundle given by the linear system  $|\tilde{M} + E_3 + 2\tilde{F}_2 + 2E_2 + 2E_4|$ , we can choose  $C$  on  $S$  such that

$$\phi^*(C) \sim \tilde{M} + E_3 + 2\tilde{F}_2 + 2E_2 + 2E_4.$$

Therefore, on  $\tilde{S}$ , we have

$$\begin{aligned} \phi^*(C) \equiv & (2 + b)\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1 + a)E_1 + (\beta + 2b)\tilde{F}_2 + (\beta - 1 + b)E_3 \\ & + (\beta - 1 + 2b)E_2 + (\beta - 2 + 2b)E_4 + (\beta - 2)E_5. \end{aligned}$$

Since this divisor is effective, it proves that the above constructed example is a  $H$ -polar cylinder for  $H$  as described above.

- Let  $\rho(S) = 2$ . That is, suppose  $S$  is of degree 4. Then any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $b > 0$ ,  $C$  is a fibre of the conic bundle given by  $|C|$ .

Blowing up fibres of  $\mathbb{F}_2$  suitably, we get the following configuration on  $\tilde{S}$ .

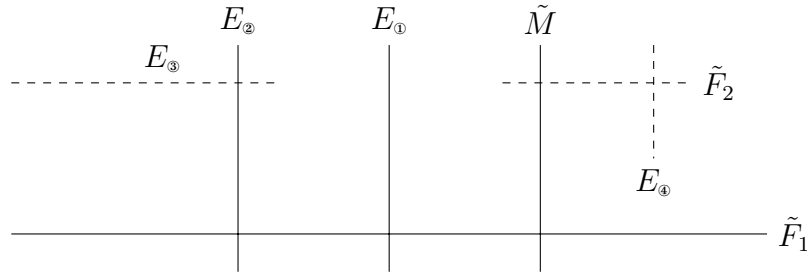


Figure 4.29:  $\tilde{S}$

Using the conic bundle given by the linear system  $|\tilde{M} + E_1 + 2\tilde{F}_1 + 2E_2 + 2E_3|$ , we can choose  $C$  on  $S$  such that

$$\phi^*(C) \sim \tilde{M} + E_1 + 2\tilde{F}_1 + 2E_2 + 2E_3.$$

Therefore, on  $\tilde{S}$ , we have

$$\begin{aligned} \phi^*(H) \equiv & (2 + b)\tilde{M} + (\alpha + 2b)\tilde{F}_1 + (\alpha - 1 + b)E_{\textcircled{1}} + (\alpha - 1 + 2b)E_{\textcircled{2}} \\ & + (\alpha - 2 + 2b)E_{\textcircled{3}} + \beta\tilde{F}_2 + (\beta - 1)E_{\textcircled{4}}. \end{aligned}$$

Since this divisor is effective, it proves that the above constructed example is a  $H$ -polar cylinder for  $H$  as described above.

2.  $D_4 + A_1$  **singularity**: Suppose the surface  $S$  has singularity of type  $D_4 + A_1$ . Then  $\rho(S) \leq 4$ . The surface  $S$  can be of degree 1 ( $\rho(S) = 4$ ) or of degree 2 ( $\rho(S) = 3$ ).

Since the following surfaces

- del Pezzo surface of degree 5 of Type  $D_4$ ,
- del Pezzo surface of degree 4 of Type  $D_4 + A_1$ ,
- cubic del Pezzo surface of Type  $D_4 + A_1$ .

do not exist, from [BW79] and [CT88], using Lemma 4.70, we can conclude that any ample divisor  $H$  on  $S$  can be given by

- (a)  $H \equiv -K_S + bC + a_1L_1 + a_2L_2$  with  $\rho(S) = 4$ ,  $b \geq 0$ ,  $0 \leq a_1 < 2$ ,  $0 \leq a_2 < 1$ ,
- (b)  $H \equiv -K_S + bC + aL$  with  $\rho(S) = 3$ ,  $0 \leq a < 2$ ,  $b \geq 0$ ,

where  $C$  is a fibre of  $|C|$  which gives the conic bundle extremal contraction from  $S$ ;  $L$ ,  $L_1$  are  $(-\frac{1}{2})$  curves that pass through the  $A_1$  singular point;  $L_2$  is a  $(-1)$ -curve on  $S$  that does not pass through any singular point and these curves are such that they are mutually disjoint.

For Cases 2(a) and 2(b),  $L_1$ ,  $L_2$ ,  $L$  are irreducible components of fibres of  $|C|$  such that  $L_1 \cdot C = L_2 \cdot C = L \cdot C = 0$ .

- Let  $\rho(S) = 4$ , that is the surface we now consider is  $S$  **of degree 1**.

[I] Suppose ample divisor on  $S$  is given by

$$H \equiv -K_S + aL$$

where  $L$  is a  $(-1)$ -curve that does not pass through the singular point and  $0 < a < 1$ .

Blowing up the fibres of  $\mathbb{F}_2$  accordingly, we get the following configuration on  $\tilde{S}$ ,

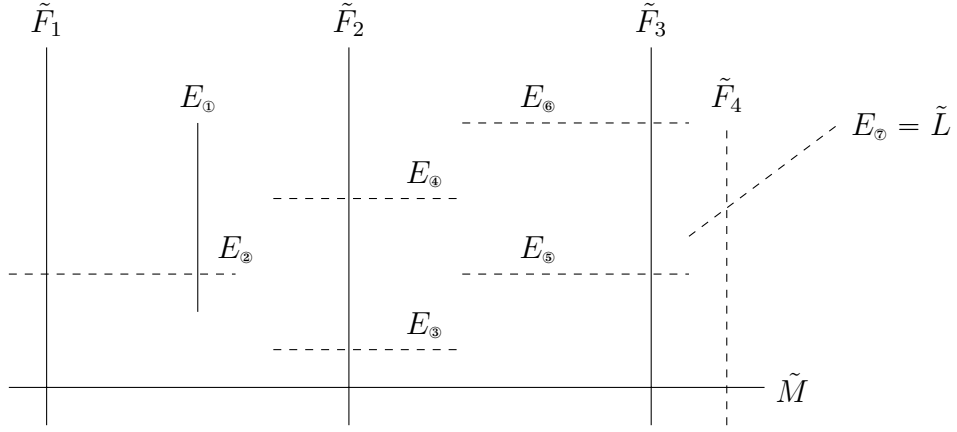


Figure 4.30:  $\tilde{S}$

Since  $\phi^*(L) \sim \tilde{L}$ , we get

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{1}} + (2\alpha - 2)E_{\textcircled{2}} + \beta\tilde{F}_2 + (\beta - 1)E_{\textcircled{3}} + (\beta - 1)E_{\textcircled{4}} \\ & + \gamma\tilde{F}_3 + (\gamma - 1)E_{\textcircled{5}} + (\gamma - 1)E_{\textcircled{6}} + \delta\tilde{F}_4 + (\delta - 1 + a)\tilde{L}, \end{aligned}$$

and the divisor is effective.

[II] Suppose ample divisor on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $b > 0$  and  $C$  is a (0)-curve.

Blowing up the fibres of  $\mathbb{F}_1$  suitably to get  $\bar{S}$ , we get the following configuration.

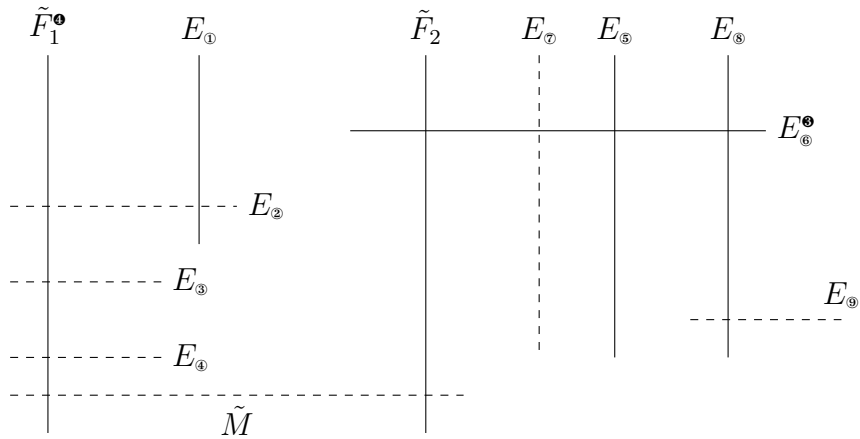


Figure 4.31:  $\bar{S}$

Contract  $\tilde{M}$ ,  $\tilde{F}_2$  in order, to get the following configuration on  $\tilde{S}$ .

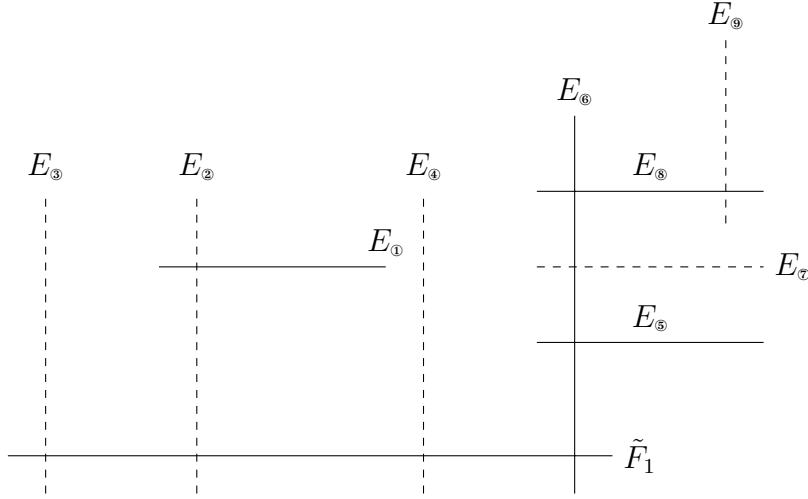


Figure 4.32:  $\tilde{S}$

Consider the conic bundle given by the linear system  $|\tilde{F}_1 + E_5 + 2E_6 + 2E_8 + 2E_9|$ . Then, we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{F}_1 + E_5 + 2E_6 + 2E_8 + 2E_9.$$

Therefore, we have

$$\begin{aligned} \phi^*(H) &= (\alpha + b)\tilde{F}_1 + (\alpha - 1)E_3 + (\alpha - 1)E_2 + (\alpha - 1)E_1 + (2\alpha - 2)E_2 \\ &\quad + (\beta - 1 + b)E_5 + (2\beta - 2 + 2b)E_6 + (2\beta - 3 + 2b)E_8 + (2\beta - 4 + 2b)E_9 \\ &\quad + (2\beta - 3)E_7, \end{aligned}$$

where  $\alpha + \beta = 3$ .

This divisor is effective and this shows that the above constructed example is a  $H$ -polar cylinder for  $H$  as described above.

[III] Suppose ample divisor  $H$  is given by

$$H \equiv -K_S + a_1L_1 + a_2L_2$$

where  $L_1$  is a  $(-1)$ -curve that does not pass through any singular point,  $L_2$  is a negative curve that passes through the singular point of type  $\mathbb{A}_1$  and  $0 \leq a_1 < 1$  and  $0 \leq a_2 < 2$ .

Blowing up the fibres of  $\mathbb{F}_1$  suitably, we get the following configuration on  $\tilde{S}$ ,

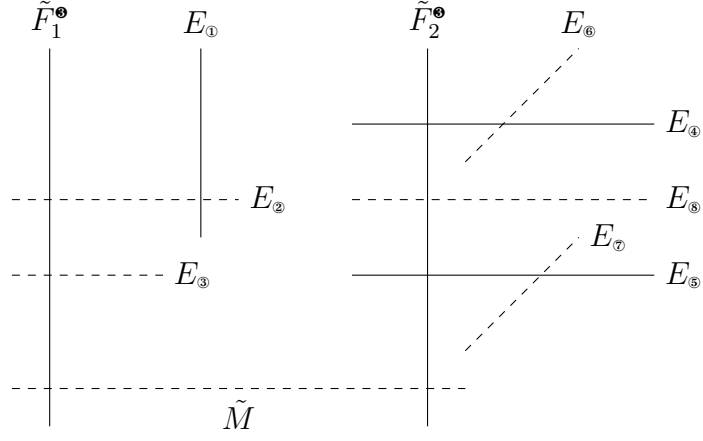


Figure 4.33:  $\bar{S}$

Contracting  $\tilde{M}$ , we get the following configuration on  $\tilde{S}$ .

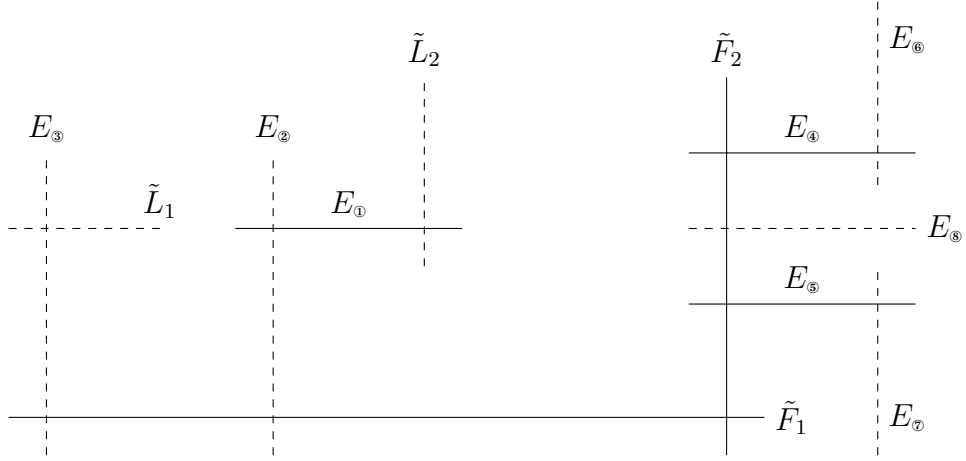


Figure 4.34:  $\tilde{S}$

Consider the conic bundle that is given by the linear system  $|E_3 + \tilde{L}_1|$ . Then we have

$$E_3 + \tilde{L}_1 \sim E_2 + E_1 + \tilde{L}_2 \sim \tilde{F}_2 + E_4 + E_5 + E_6 + E_7.$$

Using the above, we get

$$\begin{aligned} a_1 \tilde{L}_1 &\sim a_1(-E_3 + \tilde{F}_2 + E_4 + E_5 + E_6 + E_7), \\ a_2 \tilde{L}_2 + \frac{a_2}{2} E_1 &\sim a_2(-E_2 - \frac{1}{2} E_1 + \tilde{F}_2 + E_4 + E_5 + E_6 + E_7). \end{aligned}$$

Since  $\tilde{L}_1 \sim \phi^*(L_1)$  and  $\tilde{L}_2 \sim \phi^*(L_2) - \frac{1}{2} E_1$ , we get

$$\begin{aligned} \phi^*(H) &\equiv \alpha \tilde{F}_1 + (\alpha - 1 - a_1) E_3 + (\alpha - 1 - \frac{a_1}{2}) E_1 + (2\alpha - 2 - a_2) E_2 \\ &\quad + (\beta + a_1 + a_2) \tilde{F}_2 + (\beta - 1 + a_1 + a_2) E_4 + (\beta - 1 + a_1 + a_2) E_5 \\ &\quad + (\beta - 1) E_6 + (\beta - 2 + a_1 + a_2) E_7. \end{aligned}$$

This divisor is effective and therefore, this construction is an example of a  $H$ -polar cylinder for  $H$  as described above.

[IV] Suppose the ample divisors on  $S$  is given by

$$H \equiv -K_S + aL + bC$$

where  $C$  is a (0)-curve that does not pass through the singular point,  $L$  is a (-1)-curve that does not pass through any singular point and  $0 < a \leq 1$ ,  $b > 0$ . Note that  $\phi^*(C) = \tilde{C}$ . Consider  $\pi : \bar{S} \rightarrow \mathbb{F}_1$ , the blow up of the fibres of  $\mathbb{F}_1$  and let  $\psi : \bar{S} \rightarrow \tilde{S}$  be the blow down of the -1 section, to get the same configuration as in Figure 4.30. On  $\tilde{S}$ , consider the conic bundle given by the linear system  $|\tilde{F}_2 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\circledast}|$ . Then using this, we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{F}_2 + \tilde{F}_3 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\circledast}.$$

Since  $\text{Supp}(\tilde{C}) \subset -K_{\tilde{S}} + a\tilde{L}$  as computed in Case 2(a) above, the same  $H$  polar cylinder as in Case 2(a), works here too.

[V] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + a_1L_1 + a_2L_2 + bC$$

where  $L_1$  is a (-1)-curve that does not pass through the singular point,  $L_2$  is a  $(\frac{-1}{2})$ -curve on  $X$  that passes through the singular point of type  $\mathbb{A}_1$ ,  $C$  is a (0)-curve not passing through any singular point and  $0 \leq a_1 < 1$ ,  $0 \leq a_2 < 2$ ,  $b > 0$ .

Consider the configuration obtained in Fig. 4.34. Using the conic bundle given by the linear system  $|\tilde{M} + E_{\circledast} + 2\tilde{F}_3 + 2E_{\circledast} + 2E_{\circledast}|$ , we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{M} + E_{\circledast} + 2\tilde{F}_3 + 2E_{\circledast} + 2E_{\circledast}.$$

where  $\tilde{C}$  is the proper transform of  $C$  in  $\tilde{S}$ . This shows that  $\text{Supp}(\phi^*(C)) \subset \text{Supp}(-K_{\tilde{S}} + a_1\phi^*(L_1) + a_2\phi^*(L_2))$ . This implies that the same  $H$ -polar cylinder from Case [III] done above, will work for the ample divisor described here.

- Let  $\rho(S) = 3$ . That is consider surface  $S$  of degree 2.

[I] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL + bC$$

where  $C$  is a (0)-curve that does not pass through the singular point,  $L$  is a  $(\frac{-1}{2})$ -curve that passes through the  $A_1$ , singular point and  $0 \leq a < 2$ ,  $0 \leq b$ . We blow up the fibres of  $\mathbb{F}_1$  appropriately to get the following configuration.

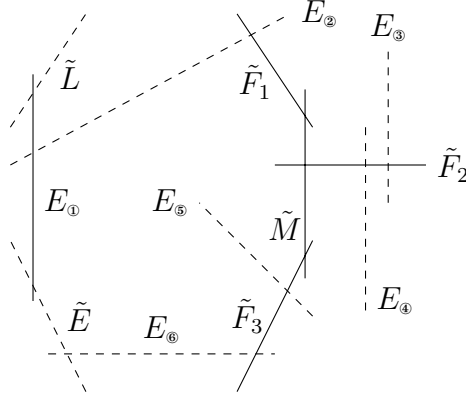


Figure 4.35:  $\tilde{S}$

Contracting  $E_{\circledast}$ ,  $\tilde{F}_2$ ,  $\tilde{M}$ , we get a del Pezzo surface of degree 5 of Type  $A_1$ . Using [CT88, Proposition 8.5], we can prove the existence of the curves  $\tilde{L}$  and  $\tilde{E}$ . Note that this also implies that  $\tilde{L} \cdot E_{\circledast} = 1$ .

From the conic bundle given by the linear system  $|\tilde{L} + E_{\circledast} + E_{\circledast}|$ , we have

$$\tilde{L} + E_{\circledast} + E_{\circledast} \sim E_{\circledast} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\circledast}.$$

This implies that

$$\phi^*(L) \equiv -\frac{1}{2}E_{\circledast} - E_{\circledast} + E_{\circledast} + \tilde{F}_3 + \tilde{M} + \tilde{F}_2 + E_{\circledast}.$$

Using the conic bundle given by  $|\tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\circledast}|$ , we can choose  $C$  on  $S$  such that, on  $\tilde{S}$

$$\tilde{C} \sim \tilde{F}_3 + \tilde{F}_1 + 2\tilde{M} + 2\tilde{F}_2 + 2E_{\circledast}.$$

Since  $\tilde{L} \sim \phi^*(L) + \frac{1}{2}E_{\circledast}$ , we have

$$\begin{aligned} \phi^*(H) &\equiv (2 + a + 2b)\tilde{M} + (\beta + a + 2b)\tilde{F}_2 + (\beta - 1)E_{\circledast} + (\beta - 1 + a + 2b)E_{\circledast} \\ &\quad + (\gamma + a + b)\tilde{F}_3 + (\gamma - 1 + a)E_{\circledast} + (\gamma - 1)E_{\circledast} + (\alpha + b)\tilde{F}_1 \\ &\quad + (\alpha - 1 - \frac{a}{2})E_{\circledast} + (2\alpha - 2 - a)E_{\circledast}. \end{aligned}$$

This divisor is effective and this shows that the above construction is an example of a  $H$ -polar cylinder for  $H$  as described above.

3.  $D_4 + 2A_1$  **singularity:** Suppose the surface  $S$  has singularity of type  $D_4 + 2A_1$ . Then  $\rho(S) \leq 3$ . The surface  $S$  can be of degree 1 ( $\rho(S) = 3$ ) or of degree 2 ( $\rho(S) = 2$ ).

Using Lemma 4.70 and the following we can explicitly describe all ample divisors  $H$  on  $S$ . From [BW79] and [CT88], the following surfaces do not exist:

- del Pezzo surface of degree 4 of Type  $D_4 + A_1$ ,
- del Pezzo surface of degree 3 of Type  $D_4 + 2A_1$ ,

- del Pezzo surface of degree 5 of Type  $D_4$ .

This then implies that every ample divisor  $H$  on  $S$  can be given by one of the following:

- (a)  $H \equiv -K_S + bC + aL$  with  $\rho(S) = 3$ ,  $0 \leq a < 1$ ,  $b \geq 0$ ,  $L$  not through any singular point,
- (b)  $H \equiv -K_S + bC$  with  $\rho(S) = 2$ ,  $b > 0$ , where  $C$  is a fibre of  $|C|$  that gives a conic bundle extremal contraction, and in Cases 3(b)&(c),  $L$  is an irreducible component of a fibre of  $|C|$  such that  $L \cdot C = 0$ .

[I] Suppose the ample divisor is given by

$$H \equiv -K_S + aL$$

where  $\rho(S) = 3$ ,  $0 < a < 1$ ,  $L$  is a  $(-1)$ -curve that does not pass through the singular point. This implies that  $\phi^*(L) = \tilde{L}$ . Blowing up fibres of  $\mathbb{F}_2$  suitably, we get the following configuration  $\tilde{S}$ , (Figure 4.36).

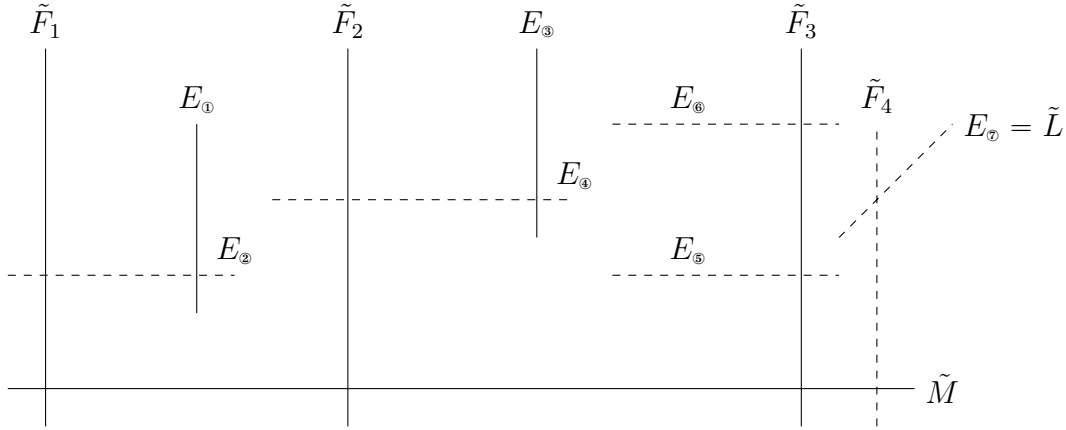


Figure 4.36:  $\tilde{S}$

Therefore, on  $\tilde{S}$ , we get that

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{3}} + (2\alpha - 2)E_{\textcircled{2}} + \beta\tilde{F}_2 + (\beta - 1)E_{\textcircled{3}} \\ & + (2\beta - 2)E_{\textcircled{4}} + \gamma\tilde{F}_3 + (\gamma - 1)E_{\textcircled{5}} + (\gamma - 1)E_{\textcircled{6}} + \delta\tilde{F}_4 + (\delta - 1 + a)\tilde{L}. \end{aligned}$$

This is an effective divisor.

[II] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC,$$

where  $\rho(S) = 3$ ,  $C$  is a  $(0)$ -curve that does not pass through the singular point and  $0 < b$ .

Blowing up fibres of  $\mathbb{F}_1$  suitably, we get the following configuration on  $\bar{S}$ .

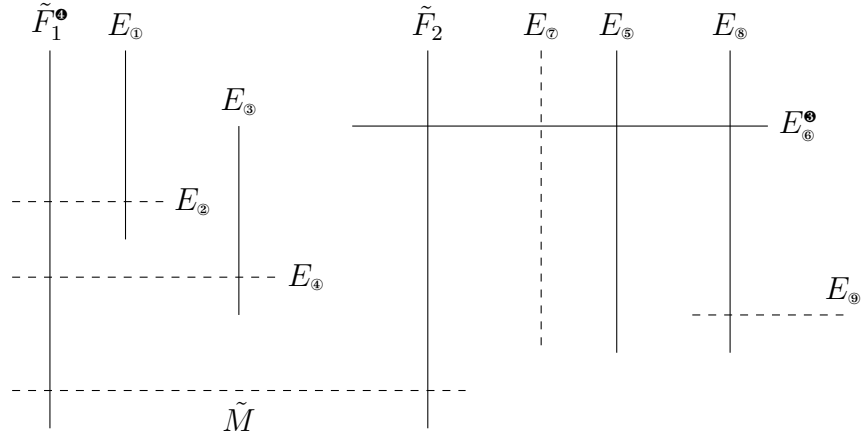


Figure 4.37:  $\bar{S}$

Contracting  $\tilde{M}$ ,  $\tilde{F}_2$  in order, we get the following configuration on  $\tilde{S}$ .

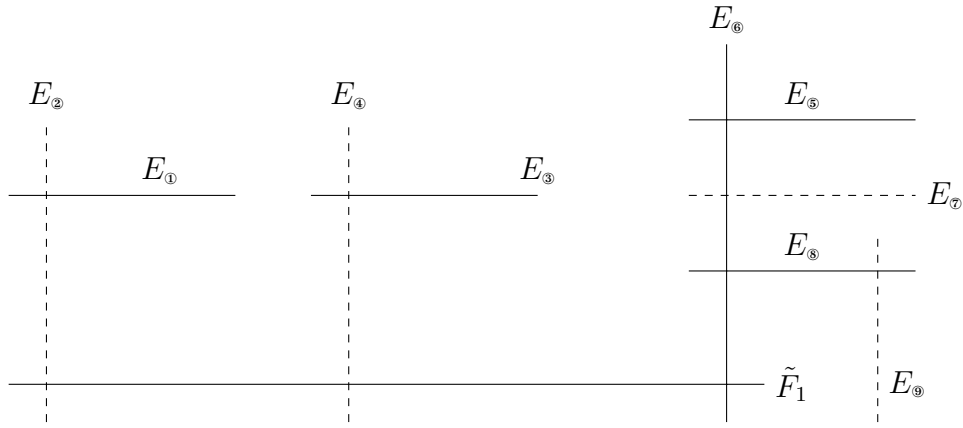


Figure 4.38:  $\tilde{S}$

Consider the linear system  $|2E_9 + 2E_8 + 2E_6 + E_5 + \tilde{F}_1|$ . Since this gives a conic bundle, we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim E_5 + \tilde{F}_1 + 2E_6 + 2E_8 + 2E_9.$$

Therefore, on  $\tilde{S}$ ,

$$\begin{aligned} \phi^*(H) \equiv & (\alpha + b)\tilde{F}_1 + (\alpha - 1)E_1 + (\alpha - 1)E_3 + (2\alpha - 2)E_4 + (2\alpha - 2)E_2 \\ & + (\beta - 1 + b)E_5 + (2\beta - 2 + 2b)E_6 + (2\beta - 3 + 2b)E_8 + (2\beta - 4 + 2b)E_9 \\ & + (2\beta - 3)E_7, \end{aligned}$$

where  $\alpha + \beta = 3$ .

This is an effective divisor and thus proves that the above constructed example is a  $H$ -polar cylinder, for  $H$  as described above.

[III] Suppose the ample divisor on  $S$  is given by

$$H \equiv -K_S + aL + bC$$

where  $\rho(S) = 3$ ,  $L$  is a  $(-1)$ -curve that does not pass through the singular point,  $C$  is a  $(0)$ -curve,  $0 < a < 1$  and  $0 < b$ .

Blowing up fibres of  $\mathbb{F}_2$  suitably, we get the same configuration as Figure 4.36.

Consider the conic bundle given by the linear system  $|\tilde{F}_3 + \tilde{F}_2 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\circledast}|$ . Then we have that

$$\tilde{C} \sim \tilde{F}_3 + \tilde{F}_2 + 2\tilde{M} + 2\tilde{F}_1 + 2E_{\circledast}.$$

This implies that  $\text{Supp}(\phi^*(C)) = \text{Supp}(\tilde{C}) \subset \text{Supp}(-K_{\tilde{S}} + a\tilde{L})$  and therefore, the same  $H$ -polar cylinder that exists in Case 3(a), works for this ample divisor  $H$ .

[IV] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $\rho(S) = 2$ ,  $C$  is a  $(0)$ -curve that does not pass through the singular point and  $b > 0$ . We blow up the fibres of  $\mathbb{F}_1$  accordingly to get the following configuration,

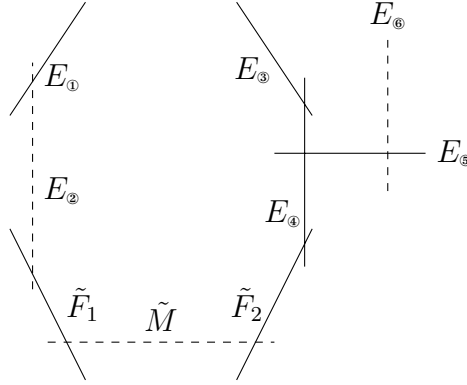


Figure 4.39:  $\tilde{S}$

Using the conic bundle given by the linear system  $|\tilde{F}_1 + E_{\circledast} + 2E_{\circledast}|$ , we can choose  $C$  on  $S$  such that

$$\tilde{C} \equiv \tilde{F}_2 + E_{\circledast} + 2E_{\circledast} + 2E_{\circledast} + 2E_{\circledast}.$$

Note that this is because the divisor  $\tilde{F}_2 + E_{\circledast} + 2E_{\circledast} + 2E_{\circledast} + 2E_{\circledast}$  is a fibre of the conic bundle given by  $|\tilde{F}_1 + E_{\circledast} + 2E_{\circledast}|$ .

Then, we get

$$\begin{aligned} \phi^*(H) \equiv & 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\circledast} + (2\alpha - 2)E_{\circledast} + (\beta + b)\tilde{F}_2 + (\beta - 1 + b)E_{\circledast} \\ & + (2\beta - 2 + 2b)E_{\circledast} + (2\beta - 3 + 2b)E_{\circledast} + (2\beta - 4 + 2b)E_{\circledast}. \end{aligned}$$

This divisor is effective.

4.  **$D_4 + 3A_1$  singularity:** Suppose the surface  $S$  has singularity of type  $D_4 + 3A_1$ . Then  $\rho(S) \leq 2$ . Consider the surface  $S$  of degree 1, and  $\rho(S) = 2$ .

Using Lemma 4.70 and the result that there does not exist a cubic del Pezzo surface of Type  $D_4 + 2A_1$ , from [BW79], we can conclude that any ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

where  $L$  is a  $(-1)$ -curve on  $S$  that does not pass through any singular point and  $0 < a < 1$ .

Blowing up fibres of  $\mathbb{F}_2$  suitably, we get the following configuration.

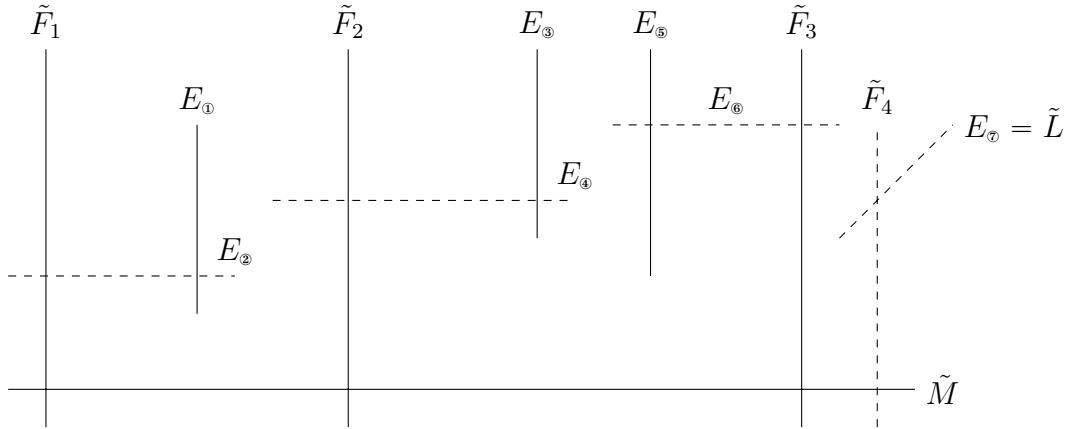


Figure 4.40:  $\tilde{S}$

Then, we get

$$\begin{aligned} \phi^*(-K_S + aL) &\equiv 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{1}} + (2\alpha - 2)E_{\textcircled{2}} + \beta\tilde{F}_2 + (\beta - 1)E_{\textcircled{3}} \\ &\quad + (2\beta - 2)E_{\textcircled{4}} + \gamma\tilde{F}_3 + (\gamma - 1)E_{\textcircled{5}} + (2\gamma - 2)E_{\textcircled{6}} + \delta\tilde{F}_4 \\ &\quad + (\delta - 1 + a)\tilde{L}. \end{aligned}$$

This divisor is effective.

5.  **$D_4 + A_2$  singularity:** Suppose the surface  $S$  has a singular point of type  $A_2$  in addition to the  $D_4$  singular point. Then,  $\rho(S) \leq 3$ . The surface  $S$  can only be of degree 1 ( $\rho(S) = 3$ ) since using Lemma 4.70 and the following results, we can conclude that there is no surface of degree 2 of Type  $D_4 + A_2$ .

From [BW79] and [CT88], the following surfaces do not exist.

- del Pezzo surface of degree 3 of Type  $D_4 + A_2$ ,
- del Pezzo surface of degree 5 of Type  $D_4$ .

Therefore, any ample divisor  $H$  on  $S$  can be given by

$$H \equiv -K_S + bC + aL$$

where  $0 \leq a < 3$ ,  $b \geq 0$ ,  $C$  is a fibre of  $|C|$  that gives the conic bundle extremal contraction from  $S$ ,  $L$  is a  $(\frac{-1}{3})$ -curve that passes through the  $A_2$  singular point and is an irreducible component of a fibre of  $|C|$  such that  $L \cdot C = 0$ .

[1] Suppose the ample divisor is given by

$$H \equiv -K_S + aL$$

where  $L$  is a  $(\frac{-1}{3})$ -curve that passes through the point of type  $A_2$  and  $0 < a < 3$ . Blowing up the fibres of  $\mathbb{F}_1$  suitably to get the configuration on  $\tilde{S}$  and then blowing down the unique  $(-1)$  section  $\tilde{M}$ , we get the following configuration (Fig. 4.41) on  $\tilde{S}$ ,

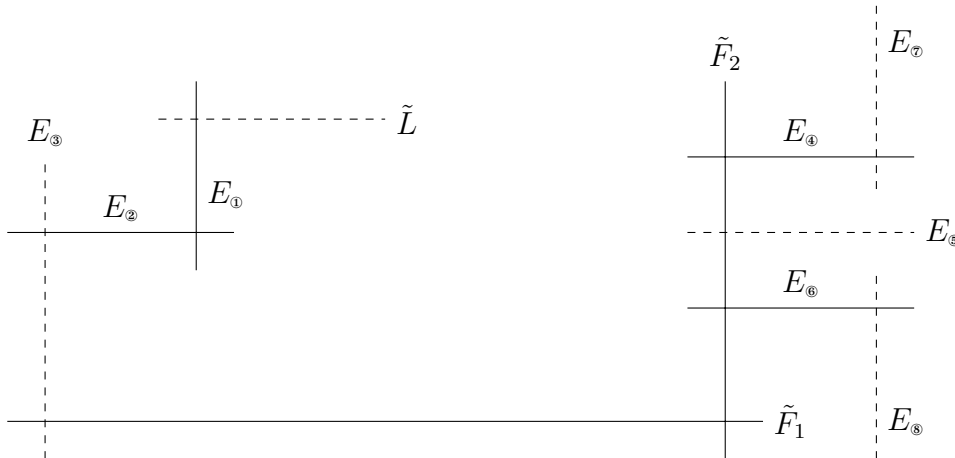


Figure 4.41:  $\tilde{S}$

In the above configuration (Fig. 4.41), contracting  $E_7$ ,  $E_4$ ,  $E_8$ , gives a del Pezzo surface of degree 4, of Type  $A_2 + A_1$ . Using [CT88, Proposition 6.1], we can prove the existence of the curve  $\tilde{L}$  as in the configuration Fig. 4.41. Note that we also have  $\tilde{L} \cdot E_8 = 1$ , from this!

Using the conic bundle given by  $|E_3 + E_2 + E_1 + \tilde{L}|$ , we can write

$$E_3 + E_2 + E_1 + \tilde{L} \sim \tilde{F}_2 + E_4 + E_6 + E_7 + E_8.$$

Note that  $\tilde{L} \sim \phi^*(L) - \frac{2}{3}E_1 - \frac{1}{3}E_2$  and using the above equivalences, we have

$$a(\tilde{L} + \frac{2}{3}E_1 + \frac{1}{3}E_2) \sim a(-\frac{2}{3}E_2 - \frac{1}{3}E_1 - E_3 + \tilde{F}_2 + E_4 + E_6 + E_7 + E_8)$$

Therefore,

$$\begin{aligned} \phi^*(H) &= \alpha\tilde{F}_1 + (\alpha - 1 - \frac{a}{3})E_1 + (2\alpha - 2 - \frac{2a}{3})E_2 + (3\alpha - 3 - a)E_3 + (\beta + a)\tilde{F}_2 \\ &\quad + (\beta - 1 + a)E_4 + (\beta - 1 + a)E_6 + (\beta - 1)E_5 + (\beta - 2 + a)E_7 \\ &\quad + (\beta - 2 + a)E_8. \end{aligned}$$

This is an effective divisor and the above construction gives an example of a  $H$ -polar cylinder, for  $H$  as described above.

[II] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + bC$$

where  $C$  is a (0)-curve that does not pass through the singular point and  $b > 0$ . Blowing up the fibres of  $\mathbb{F}_1$  suitably, we get the following configuration on  $\tilde{S}$ ,

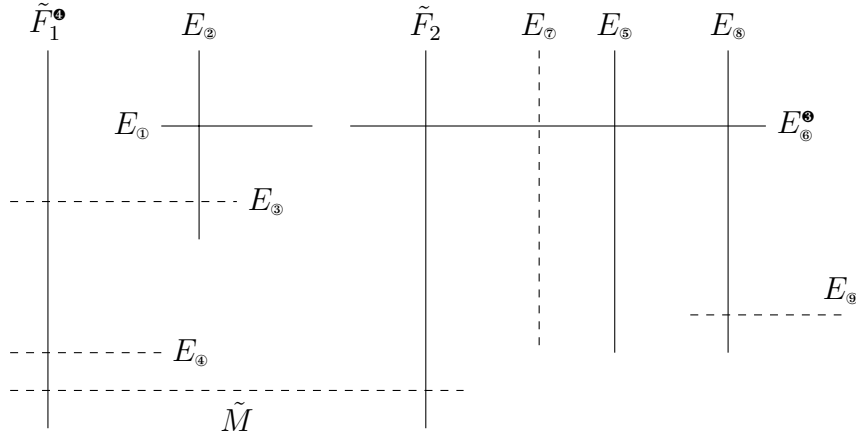


Figure 4.42:  $\tilde{S}$

Contracting  $\tilde{M}$ ,  $\tilde{F}_2$  in order, we get following configuration on  $\tilde{S}$ ,

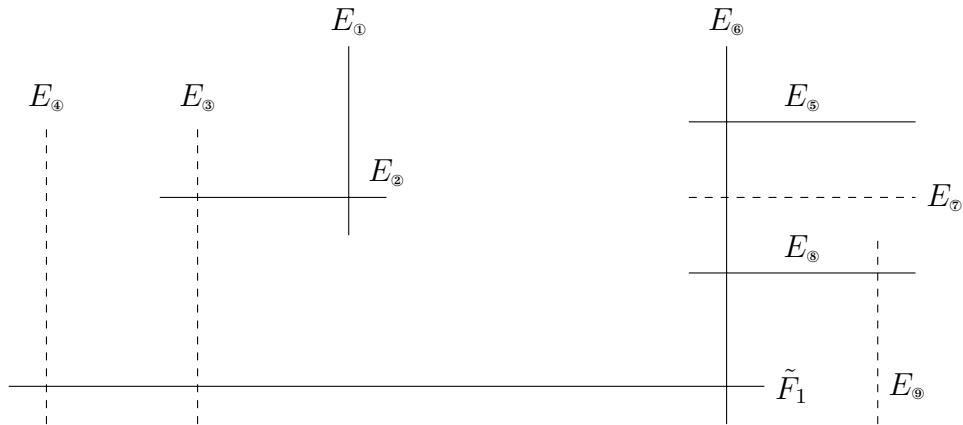


Figure 4.43:  $\tilde{S}$

Using the conic bundle that is given by the linear system  $|\tilde{F}_1 + E_5 + 2E_6 + 2E_8 + 2E_9|$ , we get that

$$\tilde{C} \sim \tilde{F}_1 + E_5 + 2E_6 + 2E_8 + 2E_9.$$

Therefore, on  $\tilde{S}$  we have that

$$\begin{aligned} \phi^*(H) &\equiv (\alpha + b)\tilde{F}_1 + (\alpha - 1)E_4 + (\alpha - 1)E_1 + (2\alpha - 2)E_2 + (3\alpha - 3)E_3 \\ &\quad + (\beta - 1 + b)E_5 + (2\beta - 2 + 2b)E_6 + (2\beta - 3 + 2b)E_8 \\ &\quad + (2\beta - 4 + 2b)E_9 + (2\beta - 3)E_7. \end{aligned}$$

This is an effective divisor and this shows that the above constructed example is indeed a  $H$ -polar cylinder.

[III] Suppose the ample divisor is given by

$$H \equiv -K_X + aC + bL$$

where  $C$  is a (0)-curve on  $S$  and  $L$  is a  $(\frac{-1}{3})$ -curve that passes through the  $A_2$  singular point.

Blowing up fibres of  $\mathbb{F}_2$  just as in Case 5(a), we get the exact configuration as in Figure 4.41.

Using the conic bundle that is given by  $|\tilde{F}_1 + E_{\circledast} + 2\tilde{F}_2 + 2E_{\circledast} + 2E_{\circledast}|$ , we can choose  $C$  on  $S$  such that

$$\tilde{C} \sim \tilde{F}_1 + E_{\circledast} + 2\tilde{F}_2 + 2E_{\circledast} + 2E_{\circledast}.$$

Since  $\text{Supp}(\tilde{C}) = \text{Supp}(\phi^*(C)) \subset \text{Supp}(-K_{\tilde{S}} + a\tilde{L})$ , the same  $H$ -polar cylinder as in Case 5(a) works here too.

#### 4.3.2.7 $A_7$ singular point

Suppose the surface has at least one singular point of type  $A_7$ . Then  $\rho(S) \leq 2$ . Consider del Pezzo surfaces of degree 1 with exactly one singular point and which is of  $A_7$  singularity type. Then  $\rho(S) = 2$ .

Using Lemma 4.70, we can conclude that any ample divisor  $H$  on  $S$  is given by one of the following,

- (a)  $H = -K_S + aL$ ,  $0 < a < 1$ ,  $L$  not through the singular point,
- (b)  $H = -K_S + aL$ ,  $0 < a < 8$ ,  $L$  through singular point.

[I] Suppose the ample divisor  $H$  on  $S$  is given by

$$H \equiv -K_S + aL$$

where  $0 < a < 1$ ,  $L$  is a  $(-1)$  curve on  $S$  that does not pass through the singular point.

Blowing up the fibres of  $\mathbb{F}_2$  suitably, we get the following configuration on  $\tilde{S}$ ,

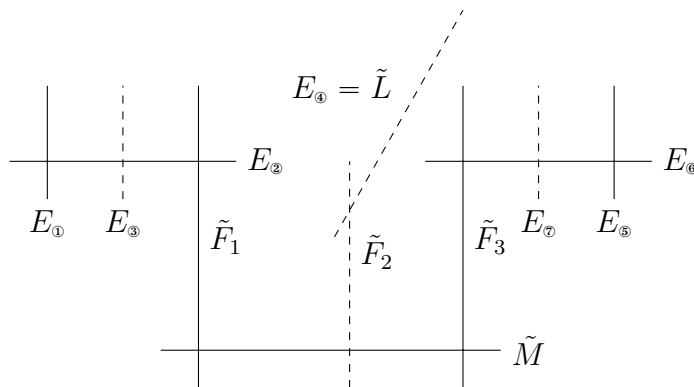


Figure 4.44:  $\tilde{S}$

Since  $\phi^*(L) = \tilde{L}$ , we get

$$\begin{aligned} \phi^*(H) = -K_{\tilde{S}} + a\phi^*(L) &\equiv 2\tilde{M} + \alpha\tilde{F}_1 + (\alpha - 1)E_{\textcircled{1}} + (2\alpha - 2)E_{\textcircled{2}} + (2\alpha - 3)E_{\textcircled{3}} \\ &\quad + \gamma\tilde{F}_3 + (\gamma - 1)E_{\textcircled{5}} + (2\gamma - 2)E_{\textcircled{6}} + (2\gamma - 3)E_{\textcircled{7}} + \beta\tilde{F}_2 \\ &\quad + (\beta - 1 + a)\tilde{L}. \end{aligned}$$

This is an effective divisor and therefore shows that this is indeed a  $H$ -polar cylinder on  $S$  for any ample divisor  $H$ , as described above.

[II] Suppose the ample divisor is given by

$$H \equiv -K_S + aL$$

where  $L$  is a  $(\frac{-1}{8})$ -curve on  $S$  that passes through the singular point and  $0 < a < 8$ . Blowing up the fibres of  $\mathbb{F}_2$  suitably, we get the following configuration on  $\tilde{S}$ ,

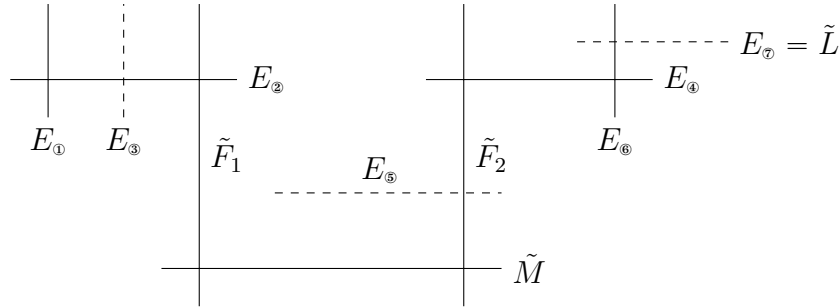


Figure 4.45:  $\tilde{S}$

Note that

$$\tilde{L} \sim \phi^*(L) - \frac{1}{8}E_{\textcircled{1}} - \frac{1}{4}E_{\textcircled{2}} - \frac{3}{8}\tilde{F}_1 - \frac{1}{2}\tilde{M} - \frac{5}{8}\tilde{F}_2 - \frac{3}{4}E_{\textcircled{4}} - \frac{7}{8}E_{\textcircled{6}}.$$

Taking  $\gamma = 0$ ,  $\delta = 0$  we get that

$$\begin{aligned} \phi^*(H) &\equiv -K_{\tilde{S}} + a\phi^*(L) = (2 + \frac{1}{2}a)\tilde{M} + (\alpha + \frac{3}{8}a)\tilde{F}_1 + (\alpha - 1 + \frac{1}{8}a)E_{\textcircled{1}} \\ &\quad + (2\alpha - 2 + \frac{1}{4}a)E_{\textcircled{2}} + (2\alpha - 3)E_{\textcircled{3}} + (\beta + \frac{5}{8}a)\tilde{F}_2 + (\beta - 1)E_{\textcircled{5}} \\ &\quad + (\beta - 1 + \frac{3}{4}a)E_{\textcircled{6}} + (\beta - 2 + \frac{7}{8}a)E_{\textcircled{6}} + (\beta - 3 + a)\tilde{L}. \end{aligned}$$

The above divisor is effective when  $\beta > 3 - a$  and  $\alpha > \frac{3}{2}$  which implies that when  $a > \frac{1}{2}$ , the above construction gives the  $H$ -polar cylinder on the surface.

When  $a \leq \frac{1}{2}$ , consider the following configuration on  $\tilde{S}$  which is obtained by first blowing up fibres of  $\mathbb{F}_1$  to get the configuration on  $\tilde{S}$  and then contracting the unique  $(-1)$  section,  $\tilde{M}$  to get the following on  $\tilde{S}$ .

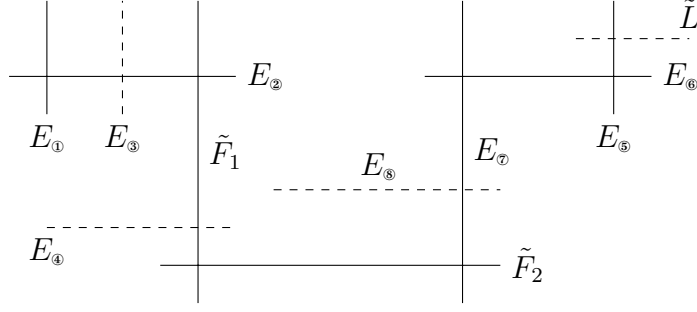


Figure 4.46:  $\tilde{S}$

In order to show the existence of the curve  $\tilde{L}$  as in the configuration above, contract  $E_3$ ,  $E_2$ ,  $E_8$  in order. This gives a del Pezzo surface of degree 4 of Type  $A_2 + A_1$ . Using [CT88, Proposition 6.1], we can conclude that a curve  $\tilde{L}$  as in the configuration exists. We also get that  $\tilde{L} \cdot E_4 = 1$ .

Using the conic bundle given by the linear system  $|\tilde{L} + E_5 + E_6 + E_7 + E_8|$ , we have

$$\tilde{L} + E_5 + E_6 + E_7 + E_8 \sim_{\mathbb{Q}} E_1 + \tilde{F}_1 + 2E_2 + 2E_3.$$

This implies that

$$\phi^*(L) \sim_{\mathbb{Q}} \frac{9}{8}E_1 + \frac{9}{4}E_2 + \frac{11}{8}\tilde{F}_1 + \frac{1}{2}\tilde{F}_2 - \frac{3}{8}E_7 - \frac{1}{4}E_6 - \frac{1}{8}E_5 + 2E_3 - E_8.$$

Therefore,

$$\begin{aligned} \phi^*(H) &= \left(\alpha + \frac{11a}{8}\right)\tilde{F}_1 + \left(\alpha - 1 + \frac{9a}{8}\right)E_1 + \left(2\alpha - 2 + \frac{9a}{4}\right)E_2 + (2\alpha - 3 + 2a)E_3 \\ &\quad + (\alpha - 1)E_4 + \left(\beta + \frac{a}{2}\right)\tilde{F}_2 + \left(\beta - 1 - \frac{a}{8}\right)E_5 + \left(2\beta - 2 - \frac{a}{4}\right)E_6 \\ &\quad + \left(3\beta - 3 - \frac{3a}{8}\right)E_7 + (3\beta - 4 - a)E_8. \end{aligned}$$

This divisor is effective and therefore proves that the above constructed example is a  $H$ -polar cylinder for  $H$  as described above.

*Proof of Main Result 4:* Consider all possible del Pezzo surfaces  $X$  such that  $\rho(X) > 1$  and  $X$  has at least one du Val singular point of type  $E_n$ ,  $D_n$  or  $A_7$ . In this section, we have then explicitly described all possible ample divisors  $H$  on  $X$  and for each such ample divisor, we have constructed a  $H$ -polar cylinder on  $X$ , thus proving Main Result 4. □

# Chapter 5

## 3-Folds

The main objects of study in this chapter are Fano 3-folds. Recall Main Result 5, that is

**Theorem 5.1.** [ACCFKGSSV] *Let  $X$  be a general Fano threefold in the family № $\mathcal{N}$ . Then*

$$X \text{ is } K\text{-polystable} \iff \mathcal{N} \notin \left\{ \begin{array}{l} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, \\ 2.36, 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, \\ 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, 4.5, \\ 4.8, 4.9, 4.10, 4.11, 4.12, 5.2. \end{array} \right\}$$

In this Chapter, we prove the  $K$ -polystability of smooth Fano 3-folds belonging to Families №2.16, №2.24, №3.10, №4.13, №4.3, №5.1. For detailed proofs of remaining cases, and thus Main Result 5, refer to [ACCFKGSSV].

### 5.1 Foundations

Let  $X$  be a Fano variety of dimension  $n \geq 2$  that has Kawamata log terminal singularities. In most of cases we consider, the variety  $X$  will be smooth. Set  $L = -K_X$ . Let  $(\mathcal{X}, \mathcal{L})$  (normal) *test configuration* of the (polarized) pair  $(X; L)$ . In addition to the results mentioned in Section 2.2, the following results are used in proving the  $K$ -polystability of the aforementioned families.

First, let us explore how  $K$ -stability behaves in Families.

**Theorem 5.2** ([O13, D15, BL, BLX19, LXZ21]). *Let  $\eta: \mathcal{X} \rightarrow Z$  be projective surjective morphism such that  $\mathcal{X}$  is  $\mathbb{Q}$ -Gorenstein,  $Z$  is a normal, and all fibers of  $\eta$  are Fano varieties with at most Kawamata log terminal singularities. For every closed point  $P \in Z$ , let  $X_P$  be the fiber of the morphism  $\eta$  over  $P$ . Then the set*

$$\{P \in Z \mid X_P \text{ is } K\text{-stable}\}$$

*is a Zariski open subset of the variety  $Z$ . Similarly, the set*

$$\{P \in Z \mid X_P \text{ is } K\text{-semistable}\}$$

is a Zariski open subset of the variety  $Z$ . Furthermore, the set

$$\left\{ P \in Z \mid X_P \text{ is } K\text{-polystable} \right\}$$

is a constructible subset of the variety  $Z$ .

Thus, if  $X$  is a  $K$ -polystable smooth Fano threefold such that the group  $\text{Aut}(X)$  is finite, then  $X$  is  $K$ -stable by Corollary 2.15, so that general Fano threefolds in the deformation family of  $X$  are  $K$ -stable. We will use this observation often in the proof of Main Result 5 to prove that a general member of a given family is  $K$ -stable. Vice versa, to prove that a given Fano threefold is not  $K$ -polystable, we will use the following result (cf. [C09, O13]).

**Theorem 5.3** ([BX19, Theorem 1.1]). *Let  $\eta: \mathcal{X} \rightarrow Z$  and  $\eta': \mathcal{X}' \rightarrow Z$  be projective surjective morphisms such that both  $\mathcal{X}$  and  $\mathcal{X}'$  are  $\mathbb{Q}$ -Gorenstein,  $Z$  is a smooth curve, and all fiber of  $\eta$  and  $\eta'$  are Fano varieties with at most Kawamata log terminal singularities. Let  $P$  be a point in  $Z$ , and let  $X_P$  and  $X'_P$  be the fibers of the morphism  $\eta$  and  $\eta'$  over  $P$ , respectively. Suppose that there is an isomorphism  $\mathcal{X} \setminus X_P \cong \mathcal{X}' \setminus X'_P$  that fits the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{X} \setminus X_P & \xrightarrow{\cong} & \mathcal{X}' \setminus X'_P \\ \eta|_{\mathcal{X} \setminus X_P} \downarrow & & \downarrow \eta'|_{\mathcal{X}' \setminus X'_P} \\ Z \setminus P & \xlongequal{\quad} & Z \setminus P \end{array}$$

If both  $X_P$  and  $X'_P$  are  $K$ -polystable, then they are isomorphic.

Together with Theorem 5.2, this result gives

**Corollary 5.4.** *Let  $p: \mathcal{X} \rightarrow \mathbb{P}^1$  be a test configuration for the Fano variety  $X$  such that the fiber  $p^{-1}(0)$  is a  $K$ -polystable Fano variety with at most Kawamata log terminal singularities that is not isomorphic to  $X$ . Then  $X$  is strictly  $K$ -semistable.*

In order to determine the  $K$ -stability of Fano threefolds using the  $\alpha$ -invariant of Tian, that was introduced in Section 2.2.2, we will use the following result, which is a refinement of [N90, Theorem 0.1] for threefolds.

**Theorem 5.5.** *Let  $X$  be a Fano threefold that has canonical Gorenstein singularities, let  $G$  be a reductive subgroup of  $\text{Aut}(X)$ , and let  $\mu$  be a positive number such that  $\mu \leq 1$ . Suppose that  $\alpha_G(X) < \mu$ . Then one of the following assertions holds:*

- (1) *There exists a  $G$ -invariant irreducible normal surface  $S$  on  $X$  such that*

$$-K_X \sim_{\mathbb{Q}} \lambda S + \Delta,$$

*where  $\Delta$  is effective  $\mathbb{Q}$ -divisor, and  $\lambda \in \mathbb{Q}$  such that  $\lambda > \frac{1}{\mu}$ .*

- (2) *There exists a  $G$ -invariant point  $P \in X$ . Moreover, the following holds:*

(2.1) if there is a del Pezzo fibration  $\pi: X \rightarrow \mathbb{P}^1$ , and  $F$  is its scheme fiber that contains the point  $P$ , then

$$\alpha(F) \leq \alpha_\Gamma(F) < \mu,$$

where  $\Gamma$  is the image in  $\text{Aut}(F)$  of the stabilizer of the fiber  $F$  in the group  $G$ , and we assume that  $\alpha_\Gamma(F) = 0$  in the case when  $F$  is not a del Pezzo surface with Du Val singularities.

(3) There exists a smooth rational  $G$ -invariant curve  $C \subset X$  such that

$$-K_X \cdot C \leq \frac{(-K_X)^3}{2} + 2.$$

Moreover, in this case, the following additional assertions hold:

(3.1) if  $\mu < 1$ , then  $-K_X \cdot C < \frac{2}{1-\mu}$ , e.g. if  $\mu = \frac{3}{4}$ , then  $-K_X \cdot C < 8$ ;

(3.2) if there is a del Pezzo fibration  $\pi: X \rightarrow \mathbb{P}^1$ , then  $F \cdot C \in \{0, 1\}$  and

$$\alpha(F) \leq \alpha_\Gamma(F) < \mu,$$

where  $F$  is any fiber of the fibration  $\pi$  that intersects (or contains) the curve  $C$ , and  $\Gamma$  is the image in  $\text{Aut}(F)$  of the stabilizer of  $F$  in the group  $G$ ;

(3.3) if in (3.2) we have  $F \cdot C = 1$ , then

$$\alpha(F_\pi) \leq \alpha_\Gamma(F_\pi) < \mu,$$

where  $F_\pi$  is the (scheme) generic fiber of the fibration  $\pi$ , which is a del Pezzo surface with Du Val singularities defined over the function field of the line  $\mathbb{P}^1$ , and  $\Gamma$  is the image in  $\text{Aut}(F_\pi)$  of the stabilizer of the fiber  $F_\pi$  in the group  $G$ .

*Proof.* See [ACCFKGSSV, Theorem 1.4.11]. □

Throughout the proofs of K-polystability in various cases, we use the valuative criterion and the stability threshold, introduced in Section 2.2.1 and Section 2.2.3, in addition to the Tian's criterion (Section 2.2.2). In this regard, the following result regarding divisorially stable Fano threefolds, is very frequently used in the proofs in Chapter 5. Recall the definition of divisorial stability from Definition 8.

**Theorem 5.6** ([F16, Theorem 10.1]). *Let  $X$  be any smooth Fano threefold that is not contained in the following 41 deformation families:*

*N<sup>o</sup>1.17, N<sup>o</sup>2.23, N<sup>o</sup>2.26, N<sup>o</sup>2.28, N<sup>o</sup>2.30, N<sup>o</sup>2.31, N<sup>o</sup>2.33, N<sup>o</sup>2.34, N<sup>o</sup>2.35, N<sup>o</sup>2.36,  
N<sup>o</sup>3.9, N<sup>o</sup>3.14, N<sup>o</sup>3.16, N<sup>o</sup>3.18, N<sup>o</sup>3.19, N<sup>o</sup>3.21, N<sup>o</sup>3.22, N<sup>o</sup>3.23, N<sup>o</sup>3.24, N<sup>o</sup>3.25,  
N<sup>o</sup>3.26, N<sup>o</sup>3.28, N<sup>o</sup>3.29, N<sup>o</sup>3.30, N<sup>o</sup>3.31, N<sup>o</sup>4.2, N<sup>o</sup>4.4, N<sup>o</sup>4.5, N<sup>o</sup>4.7, N<sup>o</sup>4.8, N<sup>o</sup>4.9,  
N<sup>o</sup>4.10, N<sup>o</sup>4.11, N<sup>o</sup>4.12, N<sup>o</sup>5.2, N<sup>o</sup>5.3, N<sup>o</sup>6.1, N<sup>o</sup>7.1, N<sup>o</sup>8.1, N<sup>o</sup>9.1, N<sup>o</sup>10.1.*

*Then  $S_X(E) < 1$  for every prime Weil divisor  $E \subset X$ , i.e.  $X$  is divisorially stable.*



section of the threefold  $V_4$ . Note that the discriminant curve of  $\eta$  is a smooth quartic curve.

**Lemma 5.7.** *One has  $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ .*

*Proof.* We have  $E \cong \mathbb{F}_n$  for some non-negative integer  $n$ , and  $-E|_E \sim s + af$  for some integer  $a$ , where  $s$  is a section of the projection  $E \rightarrow C$  with  $s^2 = -n$ , and  $f$  is a fiber of this projection. Then  $-2 = E^3 = (s + af)^2 = -n + 2a$ . Thus, we see that  $a = \frac{n-2}{2}$ . But  $(\pi^*(H) - E)|_E \sim s + \frac{n+2}{2}f$ . Thus, since  $|\pi^*(H) - E|$  is base point free, we get  $n \in \{0, 2\}$ . Since

$$(\pi^*(H) - E)|_E \cdot s = -n + \frac{n+2}{2} = \frac{2-n}{2},$$

if  $n = 2$ ,  $s$  is contracted by  $\eta$  to a point. This point has to be  $G$ -invariant in  $\mathbb{P}^2$ , since  $s$  is a  $-2$  curve in  $E \cong \mathbb{F}_2$ , which is unique and is hence  $G$ -invariant. But this is impossible, since  $G$  does not have fixed points in  $\mathbb{P}^2$ . Hence, we see that  $n = 0$ , so that  $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ .  $\square$

We will use the results from Section 2.2.3.2 to prove that  $X$  is K-polystable. Since  $\text{Aut}(X)$  is finite, this would imply that  $X$  is K-stable, by Corollary 2.15, so that any general member of the family  $\mathcal{N}^{\circ}2.16$  is K-stable.

**Lemma 5.8.** *Let  $C$  be a  $G$ -invariant irreducible curve in  $E$ . Then  $S(W_{\bullet, \bullet}^E; C) < 1$ .*

*Proof.* Let  $u$  be any non-negative real number. Then

$$-K_X - uE \sim_{\mathbb{R}} \pi^*(2H) - (1+u)E,$$

so that  $-K_X - uE$  is pseudo-effective  $\iff -K_X - uE$  is nef  $\iff u \leq 1$ .

It follows from Lemma 5.7 that  $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Now, using notations introduced in the proof of this lemma, we see that  $(-K_X - uE)|_E \sim_{\mathbb{R}} (1+u)s + (3-u)f$ .

Observe that  $|C - s| \neq \emptyset$ , since  $C \not\sim f$  as the conic  $C_1$  does not have  $G$ -fixed points. Thus, using Corollary 2.32, we get

$$\begin{aligned} S(W_{\bullet, \bullet}^E; C) &= \frac{3}{22} \int_0^1 \int_0^\infty \text{vol}((-K_X - uE)|_E - vC) dv du \\ &\leq \frac{3}{22} \int_0^1 \int_0^\infty \text{vol}((-K_X - uE)|_E - vs) dv du = \frac{3}{22} \int_0^1 \int_0^\infty \text{vol}((1+u-v)s + (3-u)f) dv du \\ &= \frac{3}{22} \int_0^1 \int_0^{1+u} 2(1+u-v)(3-u) dv du = \frac{67}{88} < 1 \end{aligned}$$

as required.  $\square$

**Lemma 5.9.** *Let  $C$  be any  $G$ -invariant irreducible smooth rational curve in  $X$  such that  $C \not\subset E$  and  $-K_X \cdot C < 8$ . Then  $\pi(C)$  is one of the conics  $C_2, C_3, C_4$ .*

*Proof.* Let  $\bar{C} = \pi(C)$ . Suppose that  $\bar{C}$  is not one of the conics  $C_2, C_3, C_4$ . Then

$$\pi^*(H) \cdot C = H \cdot \bar{C} \geq 3,$$

since  $V_4$  contains no  $G$ -invariant lines, and  $C_1, C_2, C_3, C_4$  are all  $G$ -invariant conics in  $V_4$ . Note also that  $\eta(C)$  is a curve, because  $G$  does not have fixed points in  $\mathbb{P}^2$ . Similarly, we see that  $\eta(C)$  is not a line. Hence, we conclude that  $(\pi^*(H) - E) \cdot C \geq 2$ . On the other hand, the number  $E \cdot C$  is even since  $C$  has no  $G$ -orbits of odd length. Moreover, we have

$$7 \geq -K_X \cdot C = \pi^*(H) \cdot C + (\pi^*(H) - E) \cdot C \geq 5,$$

so that  $-K_X \cdot C = 6$ ,  $\pi^*(H) \cdot C = 3$  and  $(\pi^*(H) - E) \cdot C = 3$ , which gives  $E \cdot C = 0$ . Hence, we conclude that  $\bar{C}$  is a smooth rational cubic curve. Then  $\eta(C)$  is a singular cubic curve. This is impossible, since  $G$  does not have fixed points in  $\mathbb{P}^2$ .  $\square$

Let  $\tilde{C}_2, \tilde{C}_3, \tilde{C}_4$  be the proper transforms on  $X$  of the conics  $C_2, C_3, C_4$ , respectively.

**Lemma 5.10.** *Let  $C$  be one of the curves  $\tilde{C}_2, \tilde{C}_3, \tilde{C}_4$ , let  $\bar{S}$  be a general hyperplane section of the threefold  $V_4$  that contains  $\pi(C)$ , and let  $S$  be its proper transform on  $X$ . Then  $S(W_{\bullet, \bullet}^S; C) < 1$ .*

*Proof.* Note that the surface  $\bar{S}$  is smooth, and it intersects  $C_1$  transversally in two points, so that the surface  $S$  is also smooth. Observe also that  $-K_S \sim (\pi^*(H) - E)|_S$  and  $K_S^2 = 2$ , so that  $S$  is a weak del Pezzo surface. Then  $\eta|_S: S \rightarrow \mathbb{P}^2$  is the anticanonical map.

Note that  $|H|_{\bar{S}} - \pi(C)$  is a base point free pencil. Let  $C'$  be the proper transform on the surface  $S$  of a general conic in this pencil. On  $S$ , we have  $(C')^2 = 0$  and  $C \cdot C' = 2$ . Moreover, we have  $\pi^*(H)|_S \sim C + C'$ .

Let  $u$  be a non-negative real number. Then  $-K_X - uS \sim_{\mathbb{R}} (2 - u)\pi^*(H) - E$ , which implies that  $-K_X - uS$  is pseudo-effective  $\iff -K_X - uS$  is nef  $\iff u \leq 1$ .

Suppose that  $u \in [0, 1]$ . Let  $v$  be a non-negative real number. Then

$$(-K_X - uS)|_S - vC \sim_{\mathbb{R}} -K_S + (1 - u - v)C + (1 - u)C',$$

which implies that  $(-K_X - uS)|_S - vC$  is nef for  $v \leq 1 - u$ . On the other hand, we have

$$\left( (-K_X - uS)|_S - vC \right) \cdot C' = \left( -K_S + (1 - u - v)C + (1 - u)C' \right) \cdot C' = 4 - 2u - 2v,$$

so that  $(-K_X - uS)|_S - vC$  is not pseudo-effective for  $v > 2 - u$ . Moreover, we have

$$\text{vol}\left( (-K_X - uS)|_S - (1 - u)C \right) = \left( (-K_X - uS)|_S - (1 - u)C \right)^2 = 6 - 4u,$$

Thus, using Corollary 2.32 and (2.5), we get

$$\begin{aligned} S(W_{\bullet, \bullet}^S; C) &= \frac{3}{22} \int_0^1 \int_0^{2-u} \text{vol}\left( (-K_X - uS)|_S - vC \right) dv du \leq \\ &\leq \frac{3}{22} \int_0^1 \int_0^{1-u} \left( (-K_X - uS)|_S - vC \right)^2 dv du + \frac{3}{22} \int_0^1 \frac{2}{3} (6 - 4u) dv du = \\ &= \frac{3}{22} \int_0^1 \int_0^{1-u} (14 - 16u - 8v + 4u^2 + 4uv) dv du + \frac{4}{11} = \frac{37}{44} < 1. \end{aligned}$$

$\square$

Now, we are ready to show that  $X$  is K-polystable. Suppose that it is not K-polystable. By Theorem 2.19, there exists a  $G$ -invariant prime divisor  $F$  over  $X$  such that  $\beta(F) \leq 0$ . Let  $Z = C_X(F)$ . Then  $Z$  is not a surface by Theorem 5.6, so that  $Z$  is a  $G$ -invariant irreducible curve, because  $X$  does not contain  $G$ -fixed points.

Using Lemma 2.22, we conclude that  $\alpha_{G,Z}(X) < \frac{3}{4}$ . Therefore, by Lemma 2.20, there exists a  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $X$  such that  $D \sim_{\mathbb{Q}} -K_X$  and the curve  $Z$  is contained in  $\text{Nklt}(X, \lambda D)$  for some positive rational number  $\lambda < \frac{3}{4}$ .

Since  $|\pi^*(H)|$  does not contain  $G$ -invariant surfaces, we see that  $\text{Nklt}(X, \lambda D)$  does not contain surfaces. Now, using [ACCFKGSSV, Corollary A.4.9] and 2.6, we conclude that  $Z$  is a smooth rational curve such that  $-K_X \cdot Z < 8$ .

By Corollary 2.32 and Lemma 5.8,  $Z \not\subset E$ , since  $S_X(E) < 1$  by Theorem 5.6. Then  $Z$  is one of the curves  $\tilde{C}_2, \tilde{C}_3, \tilde{C}_4$  by Lemma 5.9. Let  $S$  be a general surface in the linear system  $|\pi^*(H)|$  that contains the curve  $C$ . Then  $S_X(S) < 1$  by Theorem 5.6, so that  $S(W_{\bullet, \bullet}^S; C) \geq 1$  by Corollary 2.32. This contradicts Lemma 5.10.

## 5.2.2 Family №2.24

Let  $X$  be a smooth Fano threefold belonging to Family №2.24. Then  $X$  is a divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$  of degree  $(1, 2)$ . Let  $\text{pr}_1: X \rightarrow \mathbb{P}^2$  and  $\text{pr}_2: X \rightarrow \mathbb{P}^2$  be the projections to the first and the second factors, respectively. The morphism  $\text{pr}_1$  is a conic bundle, and  $\text{pr}_2$  is a  $\mathbb{P}^1$ -bundle. Let  $C$  be the discriminant curve of the conic bundle  $\text{pr}_1$ . Then  $C$  is a reduced cubic curve. Moreover, since  $X$  is smooth, the curve  $C$  is either smooth or nodal.

By [ACCFKGSSV, Lemma A.7.10], we can choose coordinates  $([x : y : z], [u : v : w])$  on  $\mathbb{P}^2 \times \mathbb{P}^2$  such that one of the following three cases hold:

- The threefold  $X$  is given by

$$(\mu vw + u^2)x + (\mu uw + v^2)y + (\mu vw + w^2)z = 0 \quad (5.1)$$

for some  $\mu \in \mathbb{C}$  such that  $\mu^3 \neq -1$ . In this case, the curve  $C$  is given by

$$\mu^2(x^3 + y^3 + z^3) = (\mu^3 + 4)xyz.$$

It is singular  $\iff \mu \in \{0, 2, -1 \pm \sqrt{3}i\} \iff C$  is a union of three lines.

- The threefold  $X$  is given by

$$(vw + u^2)x + (uw + v^2)y + w^2z = 0. \quad (5.2)$$

In this case, the curve  $C$  is given by  $x^3 + y^3 - 4xyz = 0$ . It is singular.

- The threefold  $X$  is given by

$$(vw + u^2)x + v^2y + w^2z = 0. \quad (5.3)$$

In this case, the curve  $C$  is given by  $x(x^2 - 4yz) = 0$ . It is reducible.

In the remaining part of this section, we will show that  $X$  is K-polystable in the first case, and  $X$  is strictly K-semistable in the other two cases.

**Lemma 5.11.** *The group  $\text{Aut}(X)$  is finite except the following cases:*

1.  $X$  is given by (5.1) with  $\mu \in \{0, 2, -1 \pm \sqrt{3}i\}$ ,
2.  $X$  is given by (5.3).

*In the first case, one has  $\text{Aut}^0(X) \cong \mathbb{G}_m^2$ . In the second case, one has  $\text{Aut}^0(X) \cong \mathbb{G}_m$ .*

*Proof.* The assertion follows from the proof of [CSP19, Lemma 10.2].  $\square$

The four threefolds given by (5.1) with  $\mu \in \{0, 2, -1 \pm \sqrt{3}i\}$  are all isomorphic to each other. These are known to be K-polystable [Su13]. The three singular Fano threefolds given by  $(\mu vw + u^2)x + (\mu uv + v^2)y + (\mu uv + w^2)z = 0$  with  $\mu^3 = -1$  are isomorphic to the threefold given by  $xvw + yuw + zuv = 0$ , see [Wa77]. This threefold has three isolated ordinary double points, and it is not  $\mathbb{Q}$ -factorial.

**Lemma 5.12.** *Let  $Y$  be a divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$  that is contained in the pencil*

$$\lambda(xu^2 + yv^2 + zw^2) + \mu(xvw + yuw + zuv) = 0,$$

*where  $[\lambda : \mu] \in \mathbb{P}^1$ . Then  $Y$  is a K-polystable Fano threefold.*

*Proof.* Let  $\mathcal{G}$  be the subgroup in  $\text{Aut}(\mathbb{P}^2 \times \mathbb{P}^2)$  generated by  $\alpha$ ,  $\beta$  and  $\gamma$  defined as follows:

$$\begin{aligned} \alpha &: ([x : y : z], [u : v : w]) \mapsto ([y : x : z], [v : u : w]), \\ \beta &: ([x : y : z], [u : v : w]) \mapsto ([y : z : x], [v : w : u]), \\ \gamma &: ([x : y : z], [u : v : w]) \mapsto ([\epsilon x : \epsilon^2 y : z], [\epsilon u : \epsilon^2 v : w]), \end{aligned}$$

where  $\epsilon$  is a primitive cube root of unity. Then  $\mathcal{G} \cong \mu_3 \rtimes \mathfrak{S}_3$ , it preserves  $Y$ , and it acts on the threefold  $Y$  faithfully, so that we can identify the group  $\mathcal{G}$  with a subgroup in  $\text{Aut}(Y)$ .

Let  $\pi_1 : Y \rightarrow \mathbb{P}^2$  and  $\pi_2 : Y \rightarrow \mathbb{P}^2$  be the projections to the first and the second factors, respectively. Then both  $\pi_1$  and  $\pi_2$  are  $\mathcal{G}$ -equivariant, and the induced  $\mathcal{G}$ -actions on both factors of  $\mathbb{P}^2 \times \mathbb{P}^2$  are faithful (cf. [DI09, Theorem 4.7]).

We claim that  $\alpha_{\mathcal{G}}(Y) \geq 1$ . To prove this claim, let us apply Theorem 5.5 with  $\mu = 1$ . First, we observe that  $Y$  has no  $\mathcal{G}$ -fixed points, because  $\mathbb{P}^2$  has no  $\mathcal{G}$ -fixed points.

Suppose that  $Y$  contains a  $\mathcal{G}$ -invariant irreducible rational curve  $C$ . Then  $\pi_1(C)$  is not a point and is not a line, since  $\mathbb{P}^2$  does not have  $\mathcal{G}$ -fixed points and  $\mathcal{G}$ -invariant lines. Then  $\pi_1(C)$  is an irreducible  $\mathcal{G}$ -invariant rational curve of degree at least 2, so that  $\mathcal{G}$  acts faithfully on its normalization, which is isomorphic to  $\mathbb{P}^1$ . But  $\text{Aut}(\mathbb{P}^1)$  does not contain a subgroup that is isomorphic to  $\mu_3 \rtimes \mathfrak{S}_3$ . This shows that  $Y$  does not contain  $\mathcal{G}$ -invariant irreducible rational curves.

To prove that  $\alpha_{\mathcal{G}}(Y) \geq 1$  it is enough to show that Theorem 5.5(1) does not hold. Suppose it does. Then  $Y$  contains a  $\mathcal{G}$ -irreducible surface  $S$  such that  $-K_Y \sim_{\mathbb{Q}} aS + \Delta$ , where  $a \in \mathbb{Q}$  such that  $a > 1$ , and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor on  $Y$ . If  $Y$  is smooth, then there are non-negative integers  $r$  and  $s$  such that

$$\frac{1}{a}\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(2)) + \frac{1}{a}\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \frac{1}{a}\Delta \sim_{\mathbb{Q}} \frac{1}{a}(-K_Y) - \frac{1}{a}\Delta \sim_{\mathbb{Q}} S \sim \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(r)) + \pi_2^*(\mathcal{O}_{\mathbb{P}^2}(s)),$$

which gives  $r = 1$  and  $s = 0$ , since  $a > 1$ . But  $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))|$  does not contain  $\mathcal{G}$ -invariant divisors. Thus, this proves the claim for smooth 3-folds  $Y$ . Thus, we may assume that  $Y$  is given by  $xvw + yuw + zwv = 0$ , whose discriminant curve is given by  $xyz = 0$ . This implies that the hyperplane sections  $x = 0$ ,  $y = 0$  and  $z = 0$ , of  $Y$ , are reducible surfaces, which are the reducible fibres of the conic bundle  $\text{pr}_1$ . Let us consider these surfaces.

Let  $S_{u,x}$ ,  $S_{v,y}$ ,  $S_{w,z}$  be the surfaces  $\{u = x = 0\}$ ,  $\{v = y = 0\}$ ,  $\{w = z = 0\}$ , respectively, let  $S'_{u,x}$ ,  $S'_{v,y}$ ,  $S'_{w,z}$  be the surfaces  $\{x = yw + zv = 0\}$ ,  $\{y = xw + zu = 0\}$ ,  $\{z = xv + yu = 0\}$ , respectively. Then  $S_{u,x} \cong S_{v,y} \cong S_{w,z} \cong \mathbb{P}^1 \times \mathbb{P}^1$ ,  $S'_{u,x} \cong S'_{v,y} \cong S'_{w,z} \cong \mathbb{F}_1$ , and these six surfaces are contained in  $Y$ . But  $S$  is not one of them, since they are not  $\mathcal{G}$ -invariant.

Let  $\ell$  be a general ruling of the surface  $S_{u,x} \cong \mathbb{P}^1 \times \mathbb{P}^1$  that is contracted by  $\pi_1$  to a point. Then  $\ell \cap \text{Sing}(Y) = \emptyset$  and  $1 = -K_Y \cdot \ell = aS \cdot \ell + \Delta \cdot \ell > S \cdot \ell$ , so that  $S \cdot \ell = 0$ , which implies that  $\ell$  and  $S$  are disjoint. Similarly, let  $\ell'$  be a general ruling of the surface  $S'_{u,x} \cong \mathbb{F}_1$ . Then  $\ell'$  and  $S$  must also be disjoint. Thus, if  $\mathcal{C}$  is a general fiber of the conic bundle  $\pi_1$ , then  $S \cdot \mathcal{C} = S \cdot (\ell + \ell') = 0$ , so that  $S$  is contracted by  $\pi_1$ .

Since  $\pi_1$  does not contract surfaces to points, we see that  $\pi_1(S)$  is an irreducible curve. Then  $\pi_1(S) \neq C$ , since  $C$  is reducible in this case. This implies that  $S \sim \pi_1^*(\mathcal{O}_{\mathbb{P}^2}(t))$  for some  $t \in \mathbb{Z}_{>0}$ . Arguing as above, we conclude that  $t = 1$ , which is impossible, because the linear system  $|\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))|$  does not contain  $\mathcal{G}$ -invariant surfaces.

Thus, we conclude that  $\alpha_{\mathcal{G}}(Y) \geq 1$ , so that  $Y$  is K-polystable by Theorem 2.23.  $\square$

**Corollary 5.13.** *If  $X$  is given by (5.2) or (5.3), then  $X$  is strictly K-semistable.*

*Proof.* Suppose that  $X$  is given by (5.2). Let  $X_s$  be the divisor in  $\mathbb{P}^2 \times \mathbb{P}^2$  given by

$$(svw + u^2)x + (suw + v^2)y + w^2z = 0.$$

where  $s \in \mathbb{C}$ . Then  $X_s$  is smooth for all  $s$ . Moreover, scaling coordinates  $x, y, z, u, v, w$ , we see that  $X_s \cong X$  for every  $s \neq 0$ . This gives us a test configuration for  $X$ , whose special fiber is the threefold  $X_0$ , which is a K-polystable smooth Fano threefold by Lemma 5.12. Then  $X$  is strictly K-semistable by Corollary 5.4.

Similarly, we see that the threefold given by (5.3) is also strictly K-semistable.  $\square$

A general threefold in the family №2.24 has finite automorphisms group by Lemma 5.11, so that it is K-stable by Theorem 5.2.

### 5.2.3 Family №3.10

Now, we solve Calabi Problem for all smooth Fano threefolds in the family №3.10 similar to what we did for the family №2.24.

Let  $Q$  be a smooth quadric threefold in  $\mathbb{P}^4$ , let  $C_1$  and  $C_2$  be two disjoint smooth irreducible conics in  $Q$ , and let  $X$  be the blow up of the quadric  $Q$  in these two conics. Then  $X$  is a smooth Fano threefold №3.10, and every smooth threefold in this family can be obtained in this way. Moreover, we may assume that

$$\begin{aligned} C_1 &= \{w^2 + zt = x = y = 0\}, \\ C_2 &= \{w^2 + xy = z = t = 0\}, \end{aligned}$$

where  $x, y, z, t, w$  are coordinates on  $\mathbb{P}^4$ . Then, using an appropriate coordinate change, we may assume that the quadric  $Q$  is given by one of the following three equations:

- (□)  $w^2 + xy + zt + a(xt + yz) + b(xz + yt) = 0$ , where  $a \in \mathbb{C} \ni b$  such that  $a \pm b \pm 1 \neq 0$ ;
- (⌈)  $w^2 + xy + zt + a(xt + yz) + xz = 0$ , where  $a \in \mathbb{C}$  such that  $a \neq \pm 1$ ;
- (⌋)  $w^2 + xy + zt + xt + xz = 0$ .

The goal of this section is to prove the following result:

**Proposition 5.14.** *The threefold  $X$  is  $K$ -polystable  $\iff Q$  is given by (□).*

In all three cases, we have the following commutative diagram:

$$\begin{array}{ccccc} & & X & & \\ & \alpha_1 \curvearrowright & \downarrow \eta & \curvearrowleft \alpha_2 & \\ Y_1 & \xleftarrow{\beta_1} & \mathbb{P}^1 & \xrightarrow{\beta_2} & Y_2 \\ & \xleftarrow{\gamma_1} & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\gamma_2} & \\ & \xleftarrow{\text{pr}_1} & \uparrow \omega & \xrightarrow{\text{pr}_2} & \\ & \xleftarrow{\delta_1} & Q & \xrightarrow{\delta_2} & \\ & \xleftarrow{\pi_1} & & \xrightarrow{\pi_2} & \end{array} \quad (5.4)$$

where  $\delta_1$  is a rational map given by  $[x : y : z : t : w] \mapsto [x : y]$ , the map  $\delta_2$  is a rational map given by  $[x : y : z : t : w] \mapsto [z : t]$ , the map  $\omega$  is a rational map

$$[x : y : z : t : w] \mapsto ([x : y], [z : t]),$$

the maps  $\pi_1$  and  $\pi_2$  are blow ups of the quadric  $Q$  at the conics  $C_1$  and  $C_2$ , respectively, the maps  $\alpha_1$  and  $\alpha_2$  are blow ups of the proper transforms of the conics  $C_2$  and  $C_1$ , respectively, both  $\beta_1$  and  $\beta_2$  are fibrations into quadric surfaces, both  $\gamma_1$  and  $\gamma_2$  are fibrations into sextic del Pezzo surfaces,  $\eta$  is a conic bundle, and  $\text{pr}_1$  and  $\text{pr}_2$  are natural projections. Occasionally, we will consider  $[x : y]$  and  $[z : t]$  as coordinates on  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $\mathcal{C}$  be the discriminant curve in  $\mathbb{P}^1 \times \mathbb{P}^1$  of the conic bundle  $\eta$ . Then  $\mathcal{C}$  has at most nodal singularities, and its degree is  $(2, 2)$ . If  $Q$  is given by  $(\beth)$ , then  $\mathcal{C}$  is given by

$$a^2(x^2t^2+y^2z^2)+2ab(xyz^2+xyt^2+ztx^2+zty^2)+b^2(x^2z^2+y^2t^2)+2(a^2+b^2-2)yzxt = 0.$$

If  $ab \neq 0$ , the curve  $\mathcal{C}$  is irreducible and smooth, which also implies that  $\text{Aut}(X)$  is finite. If  $a = 0$  or  $b = 0$  (but not both), the curve  $\mathcal{C}$  is reducible: it splits as a union of two smooth curves of degree  $(1, 1)$ , which meet at two points. In this case, we have  $\text{Aut}^0(X) \cong \mathbb{G}_m$ . Similarly, if  $a = 0$  and  $b = 0$ , then  $\text{Aut}^0(X) \cong \mathbb{G}_m^2$  and the curve  $\mathcal{C}$  is given by  $xyzt = 0$ , so that  $X$  is the unique smooth Fano threefold  $\text{No}3.10$  that admits an effective  $\mathbb{G}_m^2$ -action.

If the quadric  $Q$  is given by  $(\beth)$ , then  $\mathcal{C}$  is given by the following equation:

$$a^2t^2x^2 + (2a^2 - 4)xyzt + 2atzx^2 + a^2y^2z^2 + 2ayz^2x + z^2x^2 = 0.$$

If  $a \neq 0$ , this curve is irreducible and has one node, which implies that  $\text{Aut}(X)$  is finite. On the other hand, if  $a = 0$ , then the defining equation simplifies as  $zx(zx - 4yt) = 0$ , so that the curve  $\mathcal{C}$  splits as a union of 3 smooth curves of degree  $(0, 1)$ ,  $(1, 0)$  and  $(1, 1)$ , which meet transversally at 3 distinct points. In this subcase, we have  $\text{Aut}^0(X) \cong \mathbb{G}_m$ .

Finally, if  $Q$  is given by  $(\beth)$ , then the curve  $\mathcal{C}$  is given by  $2x(t^2x + 2txz - 4tyz + xz^2) = 0$ , so that  $\mathcal{C}$  is a union of a curve of degree  $(1, 0)$  and a smooth curve of degree  $(1, 2)$ , which implies that  $\text{Aut}(X)$  is also finite in this case.

Let  $H$  be the pull back on  $X$  of a general hyperplane section of the quadric threefold  $Q$ , let  $E_1$  be the  $\alpha_1$ -exceptional surface, and let  $E_2$  be the  $\alpha_2$ -exceptional surface. Then

$$\text{Eff}(X) = \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[H - E_1] + \mathbb{R}_{\geq 0}[H - E_2] + \mathbb{R}_{\geq 0}[2H - E_1 - E_2],$$

the del Pezzo fibration  $\gamma_1$  is given by  $|H - E_1|$ , the fibration  $\gamma_2$  is given by  $|H - E_2|$ , and the conic bundle  $\eta$  is given by the linear system  $|2H - E_1 - E_2|$ .

Let us show that  $X$  is  $K$ -polystable in the case when  $Q$  is given by  $(\beth)$ .

**Lemma 5.15** ([Su13, Theorem 1.1]). *Suppose that  $Q$  is given by  $(\beth)$  and  $a = b = 0$ . Then  $X$  is  $K$ -polystable.*

*Proof.* Let  $G$  be the subgroup in  $\text{Aut}(\mathbb{P}^4)$  generated by the following transformations:

$$\begin{aligned} [x : y : z : t : w] &\mapsto [z : t : x : y : w], \\ [x : y : z : t : w] &\mapsto [y : x : z : t : w], \\ [x : y : z : t : w] &\mapsto [x : y : t : z : w], \\ [x : y : z : t : w] &\mapsto [x : y : rz : t/r : w], \\ [x : y : z : t : w] &\mapsto [sx : y/s : z : t : w], \end{aligned}$$

where  $r \in \mathbb{C}^*$  and  $s \in \mathbb{C}^*$ . Then  $G \cong \mathbb{G}_m^2 \rtimes (\boldsymbol{\mu}_2^2 \rtimes \boldsymbol{\mu}_2)$ , the quadric  $Q$  is  $G$ -invariant, and the locus  $C_1 \cup C_2$  is  $G$ -invariant, so that the action of the group  $G$  lifts to the threefold  $X$ .

Therefore, we may identify  $G$  with a subgroup in  $\text{Aut}(X)$ . Now, applying Theorem 5.5, we obtain  $\alpha_G(X) \geq 1$ , so that  $X$  is  $K$ -polystable by Theorem 2.23.  $\square$

**Lemma 5.16.** *Suppose that  $Q$  is given by  $(\square)$  and  $ab = 0$ . Then  $X$  is  $K$ -polystable.*

*Proof.* By Lemma 5.15, we may assume that  $a \neq 0$  or  $b \neq 0$ . Without loss of generality, we may assume that  $a \neq 0$ . Then  $b = 0$ . Let  $G$  be the subgroup in  $\text{Aut}(\mathbb{P}^4)$  generated by

$$\begin{aligned} [x : y : z : t : w] &\mapsto [y : x : t : z : w], \\ [x : y : z : t : w] &\mapsto [z : t : x : y : w], \\ [x : y : z : t : w] &\mapsto [x/s : ys : z/s : ts : w], \end{aligned}$$

where  $s$  is any non-zero complex number. Then  $Q$  is  $G$ -invariant, and  $G \cong (\mathbb{G}_m \times \mu_2) \times \mu_2$ . Moreover, the locus  $C_1 \cup C_2$  is  $G$ -invariant, so that the  $G$ -action lifts to the threefold  $X$ . Therefore, we may identify  $G$  with a subgroup in  $\text{Aut}(X)$ . Note that  $\alpha_G(X) \leq \frac{2}{3}$ .

Observe that  $X$  does not have  $G$ -fixed points, because  $Q$  does not have  $G$ -fixed points. The conic bundle  $\eta$  in (5.4) is  $G$ -equivariant, and  $G$  acts on  $\mathbb{P}^1 \times \mathbb{P}^1$  such that  $\mathbb{P}^1 \times \mathbb{P}^1$  does not contain  $G$ -fixed points,  $\mathbb{P}^1 \times \mathbb{P}^1$  does not contain  $G$ -invariant curves of degree  $(1, 0)$  or  $(0, 1)$ , and the only  $G$ -invariant curves in  $\mathbb{P}^1 \times \mathbb{P}^1$  of degree  $(1, 1)$  are the curves given by  $xt + yz = 0$  and  $xt - yz = 0$ .

Suppose  $X$  is not  $K$ -polystable. By Theorem 2.19, there exists a  $G$ -equivariant birational morphism  $f: \tilde{X} \rightarrow X$  such that  $\beta(F) = A_X(F) - S_X(F) \leq 0$  for some  $G$ -invariant dreamy prime divisor  $F \subset \tilde{X}$ . Let  $Z = f(F)$ . Then  $Z$  is not a surface by Theorem 5.6. Thus, since  $X$  does not contain  $G$ -fixed points,  $Z$  is a  $G$ -invariant irreducible curve.

Using Lemma 2.22, we conclude that  $\alpha_{G,Z}(X) < \frac{3}{4}$ . Thus, by Lemma 2.20, there is a  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $X$  such that  $D \sim_{\mathbb{Q}} -K_X$  and  $Z$  is contained in  $\text{Nklt}(X, \lambda D)$  for some positive rational number  $\lambda < \frac{3}{4}$ . By [ACCFKGSSV, Corollary A.1.7], the locus  $\text{Nklt}(X, \lambda D)$  is connected.

Moreover, since  $D \sim_{\mathbb{Q}} 3H - E_1 - E_2$ , either the locus  $\text{Nklt}(X, \lambda D)$  is one-dimensional, or it contains one  $G$ -invariant surface, which is contained in  $|2H - E_1 - E_2|$ . In the former case, the  $G$ -invariant surface in  $\text{Nklt}(X, \lambda D)$  is mapped by the conic bundle  $\eta$  to a  $G$ -invariant curve in  $\mathbb{P}^1 \times \mathbb{P}^1$  of degree  $(1, 1)$ .

If  $Z$  is not contained in a two-dimensional component of the locus  $\text{Nklt}(X, \lambda D)$ , then applying [ACCFKGSSV, Corollary A.1.15] to  $(X, \lambda D)$ , we get  $(H - E_1) \cdot Z \leq 1$  and  $(H - E_2) \cdot Z \leq 1$ , so that either  $\eta(Z)$  is a point, or  $\eta(Z)$  is a  $G$ -invariant irreducible curve of degree  $(1, 1)$ . If  $Z$  is contained in a two-dimensional  $G$ -irreducible component of the locus  $\text{Nklt}(X, \lambda D)$ , then this component is mapped by  $\eta$  to a  $G$ -invariant curve of degree  $(1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Hence, either  $\eta(Z)$  is a  $G$ -invariant point, or  $\eta(Z)$  is a  $G$ -invariant curve of degree  $(1, 1)$ . Since  $\mathbb{P}^1 \times \mathbb{P}^1$  contains no  $G$ -fixed points, we see that  $\eta(Z)$  is a curve given by  $xt \pm yz = 0$ .

Let  $S$  be the unique surface in  $|2H - E_1 - E_2|$  that contains  $Z$ , let  $\bar{S}$  be its image in  $Q$ . Then  $\bar{S} = \{w^2 + xy + zt + a(xt + yz) = xt \pm yz = 0\}$ , so that  $\bar{S}$  is

a singular quartic del Pezzo surface, whose singular locus consist of 4 points. If  $\eta(Z)$  is given by  $xt + yz = 0$ , these points are  $[1 : 0 : -1 : 0 : 0]$ ,  $[1 : 0 : 1 : 0 : 0]$ ,  $[0 : 1 : 0 : -1 : 0]$ ,  $[0 : 1 : 0 : 1 : 0]$ . Similarly, if  $\eta(Z)$  is given by  $xt - yz = 0$ , then the surface  $\bar{S}$  is singular at the following points:  $[-a \pm \sqrt{a^2 - 1} : 0 : 1 : 0 : 0]$  and  $[0 : -a \pm \sqrt{a^2 - 1} : 0 : 0 : 1]$ . In both cases, the surface  $\bar{S}$  contains  $C_1$  and  $C_2$ , and  $\text{Sing}(\bar{S})$  is disjoint from these conics, so that  $S \cong \bar{S}$ .

Let  $\mathcal{H} = H|_S$ ,  $\mathcal{C}_1 = E_1|_S$ ,  $\mathcal{C}_2 = E_2|_S$ . Then  $|\mathcal{C}_1|$  and  $|\mathcal{H} - \mathcal{C}_1|$  are base point free pencils, and the surface  $S$  contains two curves  $\ell$  and  $\ell'$  such that  $\mathcal{C}_1 \sim \mathcal{C}_2 \sim 2\ell$  and  $\mathcal{H} - \mathcal{C}_1 \sim 2\ell'$ . Then  $\ell^2 = (\ell')^2 = 0$  and  $\ell \cdot \ell' = \frac{1}{2}$ . One has  $\mathcal{H} \sim 2\ell + 2\ell'$ . Moreover, there are non-negative integers  $n$  and  $m$  such that  $Z \sim_{\mathbb{Q}} n\ell + m\ell'$ . If  $n = 0$ , then  $(2H - E_1 - E_2) \cdot Z = 0$ , so that  $\eta(Z)$  is a point, which is impossible. Then  $n \geq 1$ , so that  $Z - \ell$  is pseudo-effective.

Let us apply results of Section 2.2.3.2 to  $S$  and  $Z$  using notations introduced in this section. First, we note that  $S_X(S) < 1$  by Theorem 5.6. Hence, using Corollary 2.32, we conclude that  $S(W_{\bullet, \bullet}^S; Z) \geq 1$ . Let us show that this is not the case.

Let  $u \in \mathbb{R}_{\geq 0}$ , let  $v \in \mathbb{R}_{\geq 0}$ , let  $P(u) = P(-K_X - uS)$  and let  $N(u) = N(-K_X - uS)$ . Then  $-K_X - uS$  is not pseudoeffective for  $u > \frac{3}{2}$ , since  $-K_X - uS \sim_{\mathbb{R}} (\frac{3}{2} - u)S + \frac{1}{2}(E_1 + E_2)$ . Moreover, if  $0 \leq u \leq 1$ , then  $P(u)|_S - vZ \sim_{\mathbb{R}} (2 - nv)\ell + (6 - 4u - mv)\ell'$  on the surface  $S$ , because  $N(u) = 0$  and  $P(u) = -K_X - uS$  in this case. Similarly, if  $1 \leq u \leq \frac{3}{2}$ , then we have  $P(u)|_S - vZ \sim_{\mathbb{R}} (6 - 4u - nv)\ell + (6 - 4u - mv)\ell'$ , because  $N(u) = (u - 1)(E_1 + E_2)$  and  $P(u) = (3 - 2u)H$  in this case. Thus, if  $Z = \mathcal{C}_1$  or  $Z = \mathcal{C}_2$ , then

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &= \frac{3}{26} \int_1^{\frac{3}{2}} (6 - 4u)^2 (u - 1) du + \frac{3}{26} \int_0^1 \int_0^{\infty} \text{vol}((2 - 2v)\ell + (6 - 4u)\ell') dv du + \\ &\quad + \frac{3}{26} \int_1^{\frac{3}{2}} \int_0^{\infty} \text{vol}((6 - 4u - 2v)\ell + (6 - 4u)\ell') dv du = \\ &= \frac{1}{104} + \frac{3}{26} \int_0^1 \int_0^1 (2 - 2v)(6 - 4u) dv du + \frac{3}{26} \int_1^{\frac{3}{2}} \int_0^{3 - 2u} (6 - 4u - 2v)(6 - 4u) dv du = \frac{1}{2}. \end{aligned}$$

Likewise, if  $Z \neq \mathcal{C}_1$  and  $Z \neq \mathcal{C}_2$ , then  $S(W_{\bullet, \bullet}^S; Z) \leq S(W_{\bullet, \bullet}^S; \ell) = \frac{51}{52}$ . Thus, in every case we have  $S(W_{\bullet, \bullet}^S; Z) < 1$ , which is a contradiction, since we proved earlier that  $S(W_{\bullet, \bullet}^S; Z) \geq 1$ . The obtained contradiction shows that  $X$  is K-polystable.  $\square$

**Lemma 5.17.** *Suppose that  $Q$  is given by  $(\square)$  and  $a = b$ . Then  $X$  is K-polystable.*

*Proof.* By Lemma 5.15, we may assume that  $a = b \neq 0$ . Then the curve  $\mathcal{C}$  is smooth, and the group  $\text{Aut}(X)$  is finite. Let  $G$  be the finite subgroup in  $\text{Aut}(\mathbb{P}^4)$  generated by

$$\begin{aligned} [x : y : z : t : w] &\mapsto [y : x : z : t : w], \\ [x : y : z : t : w] &\mapsto [x : y : t : z : w], \\ [x : y : z : t : w] &\mapsto [z : t : x : y : w], \\ [x : y : z : t : w] &\mapsto [x : y : z : t : -w]. \end{aligned}$$

Then  $G \cong \mu_2 \times (\mu_2^2 \rtimes \mu_2)$ , the quadric  $Q$  is  $G$ -invariant, and  $C_1 \cup C_2$  is  $G$ -invariant. The action of the group  $G$  lifts to  $X$ , and we may identify  $G$  with a subgroup in  $\text{Aut}(X)$ . Then  $X$  contains no  $G$ -fixed points,  $\eta$  is  $G$ -equivariant, and  $G$  acts on  $\mathbb{P}^1 \times \mathbb{P}^1$  such that the only  $G$ -fixed points in  $\mathbb{P}^1 \times \mathbb{P}^1$  are  $([1 : 1], [1 : 1])$  and  $([1 : -1], [1 : -1])$ ,  $\mathbb{P}^1 \times \mathbb{P}^1$  does not contain  $G$ -invariant curves of degree  $(1, 0)$  or  $(0, 1)$ , and the only  $G$ -invariant curves of degree  $(1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  are reducible curves  $(x - y)(z - t) = 0$  and  $(x + y)(z + t) = 0$ .

Suppose  $X$  is not  $K$ -polystable. By Theorem 2.19, there is a  $G$ -invariant prime divisor  $F$  over  $X$  such that  $\beta(F) = A_X(F) - S_X(F) \leq 0$ . Let  $Z = C_X(F)$ . Then  $Z$  is not a surface by Theorem 5.6, so that  $Z$  is a  $G$ -invariant curve, since  $X$  has no  $G$ -fixed points.

Arguing as in the proof of Lemma 5.16, we see that either  $\eta(Z)$  is a  $G$ -invariant point, or  $\eta(Z)$  is an irreducible  $G$ -invariant curve of degree  $(1, 1)$ . But  $\mathbb{P}^1 \times \mathbb{P}^1$  does not contain irreducible  $G$ -invariant curves of degree  $(1, 1)$ . Thus, we conclude that  $\eta(Z)$  is a point. Then either  $\eta(Z) = ([1 : 1], [1 : 1])$  or  $\eta(Z) = ([1 : -1], [1 : -1])$ , so that  $\eta(Z) \notin \mathcal{C}$ , which implies that  $Z$  is a smooth fiber of the conic bundle  $\eta$ .

Let  $S$  be the unique surface in the linear system  $|H - E_1|$  that contains the curve  $Z$ , and let  $\bar{S}$  be its image in  $Q$ . Then  $\bar{S}$  is a smooth quadric surface,  $C_1 \subset \bar{S}$ , and  $\bar{S}$  intersects the conic  $C_2$  transversally in two points, so that  $S$  is a smooth sextic del Pezzo surface, and  $\pi_1 \circ \alpha_1 = \pi_2 \circ \alpha_2$  induces a birational morphism  $\varphi: S \rightarrow \bar{S}$  that is a blow up of the intersection points  $\bar{S} \cap C_2$ . We have  $E_2|_S = \mathbf{e}_1 + \mathbf{e}_2$ , where  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are  $(-1)$ -curves in  $S$  contracted by  $\varphi$ . We also have  $E_1|_S \sim H|_S \sim \ell_1 + \ell_2 + \mathbf{e}_1 + \mathbf{e}_2$ , where  $\ell_1$  and  $\ell_2$  are  $(-1)$ -curves in  $S$  such that  $\varphi(\ell_1)$  and  $\varphi(\ell_2)$  are intersecting lines that pass through the points  $\varphi(\mathbf{e}_1)$  and  $\varphi(\mathbf{e}_2)$ , respectively. Then  $Z \sim \ell_1 + \ell_2$ .

As in the proof of Lemma 5.16, we are going to apply results of Section 2.2.3.2 to  $S$  and  $Z$ . By Theorem 5.6, we have  $S_X(S) < 1$ , so that  $S(W_{\bullet, \bullet}^S; Z) \geq 1$  by Corollary 2.32.

Let  $P(u) = P(-K_X - uS)$  and  $N(u) = N(-K_X - uS)$ , where  $u$  is a non-negative real number. Observe that  $-K_X - uS \sim_{\mathbb{R}} (2 - u)S + (H - E_2) + E_1 \sim_{\mathbb{R}} (3 - u)H - (1 - u)E_1 - E_2$ . Then  $-K_X - uS$  is nef  $\iff u \in [0, 1]$ , and  $-K_X - uS$  is pseudo-effective  $\iff u \in [0, 2]$ . Moreover, we have

$$P(u) = \begin{cases} (3 - u)H - (1 - u)E_1 - E_2 & \text{if } 0 \leq u \leq 1, \\ (3 - 2u)H & \text{if } 1 \leq u \leq 2, \end{cases}$$

and  $N(u) = (u - 1)E_1$  if  $1 \leq u \leq 2$ . Let  $v$  be a non-negative real number. If  $u \in [0, 1]$ , then  $P(u)|_S - vZ \sim_{\mathbb{R}} (2 - v)(\ell_1 + \ell_2) + \mathbf{e}_1 + \mathbf{e}_2$ , so that  $P(u)|_S - vZ$  is not pseudo-effective for every  $v > 2$ . In this case, if  $v \in [0, 1]$ , then the divisor  $P(u)|_S - vZ$  is nef. Furthermore, if  $v \in [1, 2]$ , then its Zariski decomposition is

$$P(u)|_S - vZ \sim_{\mathbb{R}} \underbrace{(2 - v)(\ell_1 + \ell_2 + \mathbf{e}_1 + \mathbf{e}_2)}_{\text{positive part}} + \underbrace{(v - 1)(\mathbf{e}_1 + \mathbf{e}_2)}_{\text{negative part}}.$$

Similarly, if  $u \in [1, 2]$ , then  $P(u)|_S - vZ \sim_{\mathbb{R}} (3 - u - v)(\ell_1 + \ell_2) + (2 - u)(\mathbf{e}_1 + \mathbf{e}_2)$ , so that the divisor  $P(u)|_S - vZ$  is not pseudo-effective for  $v > 3 - u$ . Moreover,

if  $v \in [0, 1]$ , then this divisor is nef. Finally, if  $1 \leq v \leq 3 - u$ , then its Zariski decomposition is

$$P(u)|_S - vZ \sim_{\mathbb{R}} \underbrace{(3 - u - v)(\ell_1 + \ell_2 + \mathbf{e}_1 + \mathbf{e}_2)}_{\text{positive part}} + \underbrace{(v - 1)(\mathbf{e}_1 + \mathbf{e}_2)}_{\text{negative part}}.$$

Thus, we have

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &= \frac{3}{26} \int_0^1 \int_0^1 \left( (2 - v)(\ell_1 + \ell_2) + \mathbf{e}_1 + \mathbf{e}_2 \right)^2 dvdu + \\ &\quad + \frac{3}{26} \int_0^1 \int_1^2 \left( (2 - v)(\ell_1 + \ell_2 + \mathbf{e}_1 + \mathbf{e}_2) \right)^2 dvdu + \\ &\quad + \frac{3}{26} \int_1^2 \int_0^1 \left( (3 - u - v)(\ell_1 + \ell_2) + (2 - u)(\mathbf{e}_1 + \mathbf{e}_2) \right)^2 dvdu + \\ &\quad + \frac{3}{26} \int_1^2 \int_1^{3-u} \left( (3 - u - v)(\ell_1 + \ell_2 + \mathbf{e}_1 + \mathbf{e}_2) \right)^2 dvdu = \\ &= \frac{3}{26} \int_0^1 \int_0^1 (6 - 4v) dvdu + \frac{3}{26} \int_0^1 \int_1^2 (2 - v)^2 dvdu + \\ &\quad + \frac{3}{26} \int_1^2 \int_0^1 2(2 - u)(4 - u - 2v) dvdu + \frac{3}{26} \int_1^2 \int_1^{3-u} 2(3 - u - v)^2 dvdu = \frac{3}{4}. \end{aligned}$$

The obtained contradiction completes the proof of the lemma.  $\square$

Now, combining the proofs of Lemma 5.16 and 5.17 together, we obtain

**Lemma 5.18.** *Suppose that  $Q$  is given by  $(\beth)$ . Then  $X$  is  $K$ -polystable.*

*Proof.* By Lemma 5.15, we may assume that  $a \neq 0$  and  $b \neq 0$ . Then  $\mathcal{C}$  is smooth, and the group  $\text{Aut}(X)$  is finite. Let  $G$  be the finite subgroup in  $\text{Aut}(\mathbb{P}^4)$  generated by

$$\begin{aligned} [x : y : z : t : w] &\mapsto [y : x : t : z : w], \\ [x : y : z : t : w] &\mapsto [z : t : x : y : w], \\ [x : y : z : t : w] &\mapsto [x : y : z : t : -w]. \end{aligned}$$

Then  $G \cong \mu_2^3$ , the quadric  $Q$  is  $G$ -invariant, and the locus  $C_1 \cup C_2$  is  $G$ -invariant, which implies that the  $G$ -action lifts to  $X$ , so that we may identify  $G$  with a subgroup in  $\text{Aut}(X)$ . Observe that  $X$  does not have  $G$ -fixed points, because  $Q$  does not have  $G$ -fixed points.

Recall that the conic bundle  $\eta$  in (5.4) is  $G$ -equivariant, and  $G$  acts on  $\mathbb{P}^1 \times \mathbb{P}^1$  such that  $([1 : 1], [1 : 1])$  and  $([1 : -1], [1 : -1])$  are the only  $G$ -fixed points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and  $\mathbb{P}^1 \times \mathbb{P}^1$  contains no  $G$ -invariant curves of degree  $(1, 0)$  or  $(0, 1)$ . Moreover, the  $G$ -invariant curves of degree  $(1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  can be described as follows:  $\{xt = yz\}$ ,  $\{xt = yz\}$ , and all curves in the pencil  $\mathcal{P}$  that is given by  $r(xt + yz) = s(xz + yt)$ , where  $[r : s] \in \mathbb{P}^1$ . Note that the pencil  $\mathcal{P}$  contains two reducible curves:  $\{(x - y)(z - t) = 0\}$  and  $\{(x + y)(z + t) = 0\}$ , which correspond to  $[r : s] = [1 : 1]$  and  $[r : s] = [1 : -1]$ , respectively.

Suppose  $X$  is not  $K$ -polystable. By Theorem 2.19, there exists a  $G$ -invariant prime divisor  $F$  over  $X$  with  $\beta(F) \leq 0$ . Let  $Z = C_X(F)$ . Then  $\dim(Z) \leq 1$  by Theorem 5.6, so that  $Z$  is a  $G$ -invariant irreducible curve, because  $X$  does not have  $G$ -fixed points.

Arguing as in the proof of Lemma 5.16, we see that either  $\eta(Z)$  is a  $G$ -invariant point, or  $\eta(Z)$  is an irreducible  $G$ -invariant curve of degree  $(1, 1)$ . Furthermore, if  $\eta(Z)$  is a point, then  $\eta(Z) \notin \mathcal{C}$ , so that  $Z$  is a smooth fiber of the conic bundle  $\eta$ . In this case, for all admissible  $a$  and  $b$ , the unique surface in  $|H - E_1|$  that contains the curve  $Z$  is a smooth sextic del Pezzo surface, so that we are exactly in the situation of the proof of Lemma 5.17 and, therefore, we can obtain a contradiction arguing exactly as in this proof. This shows that  $\eta(Z)$  is a curve of degree  $(1, 1)$ .

Let  $S$  be the surface in  $|2H - E_1 - E_2|$  that contains  $Z$ , and let  $\bar{S}$  be its image in  $Q$ . Then  $\bar{S}$  is a quartic del Pezzo surface that contains  $C_1$  and  $C_2$ . Since  $a \neq 0$  and  $b \neq 0$ , either the surface  $\bar{S}$  is smooth, or  $\bar{S}$  has exactly two isolated ordinary double points. Furthermore, if  $\bar{S}$  is singular, its singular locus is disjoint from the conics  $C_1$  and  $C_2$ . We will provide explicit computations in the end of the proof. In particular, one has  $S \cong \bar{S}$ . Now, we can proceed as we did in the proof of Lemma 5.16.

Namely, let us apply results of Section 2.2.3.4 to  $S$  and  $Z$  using notations introduced in this section. By Theorem 5.6, we have  $S(V_\bullet; S) < 1$ . Hence, using Corollary 2.32, we conclude that  $S(W_{\bullet, \bullet}^S; Z) \geq 1$ . Let us show that this is not the case.

Let  $\mathcal{H} = H|_S$ ,  $\mathcal{C}_1 = E_1|_S$  and  $\mathcal{C}_2|_S$ . Then  $\mathcal{C}_1 \sim \mathcal{C}_2$ , both  $|\mathcal{C}_1|$  and  $|\mathcal{H} - \mathcal{C}_1|$  are base point free pencils. Let  $\mathcal{C}'$  be a general curve in  $|\mathcal{H} - \mathcal{C}_1|$ . Then  $\mathcal{C}'^2 = 0$ ,  $(\mathcal{C}')^2 = 0$  and  $\mathcal{C}_1 \cdot \mathcal{C}' = 2$ .

Suppose that  $Z \sim_{\mathbb{Q}} \frac{n}{2}\mathcal{C}_1 + \frac{m}{2}\mathcal{C}'$  for some non-negative integers  $n$  and  $m$ . Then  $n \geq 1$ , since otherwise  $\eta(Z)$  would be a point, which is not the case. Thus, if  $Z \neq \mathcal{C}_1$  and  $Z \neq \mathcal{C}_2$ , then to estimate  $S(W_{\bullet, \bullet}^S; Z)$  from above we may assume that  $n = 1$  and  $m = 0$ . In this case, arguing as in the proof of Lemma 5.16, we see that

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &= \frac{3}{26} \int_0^1 \int_0^\infty \text{vol} \left( \left(1 - \frac{1}{2}v\right)\mathcal{C}_1 + (3 - 2u)\mathcal{C}' \right) dvdu + \\ &\quad + \frac{3}{26} \int_1^{\frac{3}{2}} \int_0^\infty \text{vol} \left( \left(3 - 2u - \frac{1}{2}v\right)\mathcal{C}_1 + (3 - 2u)\mathcal{C}' \right) dvdu = \\ &= \frac{3}{26} \int_0^1 \int_0^2 4 \left(1 - \frac{1}{2}v\right)(3 - 2u) dvdu + \frac{3}{26} \int_1^{\frac{3}{2}} \int_0^{6-4u} 4 \left(3 - 2u - \frac{1}{2}v\right)(3 - 4u) dvdu = \frac{51}{52}, \end{aligned}$$

Similarly, if  $Z = \mathcal{C}_1$  or  $Z = \mathcal{C}_2$ , then arguing as in the end of the proof of Lemma 5.16, we obtain  $S(W_{\bullet, \bullet}^S; Z) = \frac{1}{2}$ . Thus, we see that  $S(W_{\bullet, \bullet}^S; Z) < 1$ , so that  $X$  is  $K$ -polystable.

To complete the proof of the lemma, it is enough to show that every  $G$ -invariant curve on the surface  $S$  is  $\mathbb{Q}$ -rationally equivalent to  $\frac{1}{2}(n\mathcal{C}_1 + m\mathcal{C}')$  for some  $n \in \mathbb{Z}_{\geq 0}$  and  $m \in \mathbb{Z}_{\geq 0}$ . Since  $S \cong \bar{S}$ , we identify  $S = \bar{S}$ , so that now  $S$  is a quartic del Pezzo surface in  $\mathbb{P}^4$ .

Suppose that  $\eta(Z)$  is given by  $xt = yz$ . Then  $S = Q \cap \{xt = yz\}$ . Therefore, the projection  $[x : y : z : t : w] \mapsto [x : y : z : t]$  induces a  $G$ -equivariant double cover  $\varphi: S \rightarrow Y$  such that  $Y$  is the smooth quadric surface in  $\mathbb{P}^3$  that is given by  $xt = yz$ , and the ramification divisor of the double cover  $\varphi$  is the curve  $Y \cap \{xy + zt + a(xt + yz) + b(xz + yt) = 0\}$ , where we consider  $x, y, z, t$  as coordinates on  $\mathbb{P}^3$ . Explicit computations shows that  $R$  is smooth, since  $a \pm b \neq 1$ ,  $a \pm b \neq -1$  and  $b \neq 0$ . Then  $S$  is also smooth. Since the involution of the double cover  $\varphi$  is contained in  $G$ , every  $G$ -invariant curve in  $S$  is rationally equivalent to  $\phi^*(D)$  for some  $D \in \text{Pic}(Y)$ , which implies the required assertion.

Similarly, we see that the required assertion holds when  $\eta(Z)$  is given by  $xt = yz$ . Therefore, we can proceed to the case when  $\eta(Z)$  is an irreducible curve in the pencil  $\mathcal{P}$ . In this case, we have  $S = Q \cap \{r(xt + yz) = s(xz + yt)\}$ , where  $r$  and  $s$  are some numbers such that  $(r, s) \neq (0, 0)$ ,  $[r : s] \neq [1 : 1]$ ,  $[r : s] \neq [1 : -1]$ . As in the previous case, there exists a  $G$ -equivariant double cover  $\varphi: S \rightarrow Y$  such that  $Y$  is the quadric in  $\mathbb{P}^3$  given by

$$r(xt + yz) = s(xz + yt),$$

and the ramification divisor of  $\varphi$  is the curve  $R = Y \cap \{xy + zt + a(xt + yz) + b(xz + yt) = 0\}$ . Since  $[r : s] \neq [1 : 1]$  and  $[r : s] \neq [1 : -1]$ , one can check that the quadric  $Y$  is smooth. Thus, if the curve  $R$  is smooth, we obtain the required assertion as in the previous case. Therefore, we may assume that the curve  $R$  is singular.

Since  $R$  is singular, explicit computations show that  $br + (a \pm 1)s = 0$  or  $(b \pm 1)r + as = 0$ . In the former case, we have  $R = Y \cap \{(x \pm z)(t \pm y) = 0\}$ . Similarly, if  $(b \pm 1)r + as = 0$ , then  $R = Y \cap \{(y \pm z)(t \pm x) = 0\}$ . In each case, the curve  $R$  splits as a union of two smooth conics  $R_1$  and  $R_2$  that intersect transversally at two points, so that  $S$  has two isolated ordinary double points, which are disjoint from  $C_1 \cup C_2$ . As in the previous case, we see that every  $G$ -invariant Cartier divisor on  $S$  is rationally equivalent to  $\phi^*(D)$  for some  $D \in \text{Pic}(Y)$ . Since any Weil divisor on  $S$  becomes Cartier once it is multiplied by 2, the assertion follows. This completes the proof of the lemma.  $\square$

**Corollary 5.19.** *If  $Q$  is given by  $(\mathfrak{A})$  or  $(\mathfrak{B})$ , then  $X$  is strictly  $K$ -semistable.*

*Proof.* We only consider the case when  $Q$  is given by  $(\mathfrak{A})$ , because the other case is similar. Suppose that  $Q$  is given by  $(\mathfrak{A})$ . Let  $Q_s = \{w^2 + xy + zt + a(xt + yz) + sxz = 0\} \subset \mathbb{P}^4$ , where  $s \in \mathbb{C}$ . Then the quadric  $Q_s$  is smooth, and  $Q$  contains both conics  $C_1$  and  $C_2$ . Let  $X_s \rightarrow Q_s$  be the blow up of the conics  $C_1$  and  $C_2$ . Scaling coordinates  $x, y, z, t, w$ , we see that  $X_s \cong X$  for every  $s \neq 0$ . This gives us a test configuration for  $X$ , whose special fiber is  $X_0$ , which is a  $K$ -polystable smooth Fano threefold  $\mathbb{N}^{\circ}3.10$  by Lemma 5.18. Then  $X$  is strictly  $K$ -semistable by Corollary 5.4.  $\square$

Thus, Proposition 5.14 is completely proved.

### 5.2.4 Family №3.13

Let  $X$  be a smooth Fano threefold in the family №3.13. As it was observed in [CCGK16, § 66], the threefold  $X$  is a complete intersection in  $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  of 3 divisors of degrees  $(1, 1, 0)$ ,  $(0, 1, 1)$ ,  $(1, 0, 1)$ , respectively. Thus, the threefold  $X$  can be given by the following system of equations:

$$\begin{cases} f(x_0, x_1, x_2; y_0, y_1, y_2) = 0, \\ g(y_0, y_1, y_2; z_0, z_1, z_2) = 0, \\ h(x_0, x_1, x_2; z_0, z_1, z_2) = 0, \end{cases}$$

where  $f, g, h$  are bilinear forms, and  $[x_0 : x_1 : x_2]$ ,  $[y_0 : y_1 : y_2]$ ,  $[z_0 : z_1 : z_2]$  are coordinates on the first, the second and the third factor of  $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ , respectively. Observe that

$$\begin{aligned} f &= [x_0 \ x_1 \ x_2] M_{x,y} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}, \\ g &= [y_0 \ y_1 \ y_2] M_{y,z} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}, \\ h &= [x_0 \ x_1 \ x_2] M_{x,z} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

for some  $3 \times 3$  matrices  $M_{x,y}$ ,  $M_{y,z}$ ,  $M_{x,z}$ .

**Lemma 5.20.** *One has  $\det(M_{x,y}) \neq 0$ ,  $\det(M_{y,z}) \neq 0$  and  $\det(M_{x,z}) \neq 0$ .*

*Proof.* See [ACCFKGSSV, Lemma 5.19.1]. □

Let  $W_{x,y}$ ,  $W_{y,z}$ ,  $W_{x,z}$  be the threefolds in  $\mathbb{P}^2 \times \mathbb{P}^2$  that are given by  $f = 0$ ,  $g = 0$ ,  $h = 0$ , respectively. Then  $W_{x,y}$ ,  $W_{y,z}$ ,  $W_{x,z}$  are smooth by Lemma 5.20. Moreover, we have the following commutative diagram:

$$\begin{array}{ccccc} & & \mathbb{P}^2 & & \\ & \nearrow \text{pr}_y^{x,y} & & \nwarrow \text{pr}_y^{y,z} & \\ W_{x,y} & & & & W_{y,z} \\ & \nwarrow \pi_{x,y} & & \nearrow \pi_{y,z} & \\ & & X & & \\ & \nearrow \eta_x & & \nwarrow \eta_z & \\ & & \mathbb{P}^2 & & \mathbb{P}^2 \\ & \nwarrow \text{pr}_z^{x,z} & & \nearrow \text{pr}_z^{x,z} & \\ & & W_{x,z} & & \end{array} \quad (5.5)$$

where all morphisms are given by natural projections, e.g. the morphism  $\pi_{x,y}$  is given by

$$\left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) \mapsto \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2] \right),$$

the morphism  $\eta_z$  is given by  $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2]) \mapsto [z_0 : z_1 : z_2]$ , and the projection  $\text{pr}_y^{y,z}$  is given by  $([y_0 : y_1 : y_2], [z_0 : z_1 : z_2]) \mapsto [y_0 : y_1 : y_2]$ .

Note that the morphisms  $\pi_{x,y}$ ,  $\pi_{y,z}$ ,  $\pi_{x,z}$  are birational — they blow up smooth rational curves of degree  $(2, 2)$ . Let  $E_{x,y}$ ,  $E_{y,z}$ ,  $E_{x,z}$  be their exceptional surfaces, respectively. Then  $-K_X \sim E_{x,y} + E_{y,z} + E_{x,z}$ . Observe also that  $\eta_x$ ,  $\eta_y$  and  $\eta_z$  are (non-standard) conic bundles and  $-K_X \sim \eta_x^*(\mathcal{O}_{\mathbb{P}^2}(1)) + \eta_y^*(\mathcal{O}_{\mathbb{P}^2}(1)) + \eta_z^*(\mathcal{O}_{\mathbb{P}^2}(1))$ .

Let  $\Delta_x$ ,  $\Delta_y$ ,  $\Delta_z$  be the discriminant curves of the conic bundles  $\eta_x$ ,  $\eta_y$ ,  $\eta_z$ , respectively. Then the defining equations of the curves  $\Delta_x$ ,  $\Delta_y$ ,  $\Delta_z$  are

$$\begin{aligned} \begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix} D_x \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} &= 0, \quad \begin{bmatrix} y_0 & y_1 & y_2 \end{bmatrix} D_y \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = 0, \\ \begin{bmatrix} z_0 & z_1 & z_2 \end{bmatrix} D_z \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} &= 0 \end{aligned}$$

for some  $3 \times 3$  matrices  $D_x$ ,  $D_y$  and  $D_z$ .

**Lemma 5.21.** *One has*

$$\begin{aligned} D_x &= M_{x,z} (M_{y,z})^{-1} M_{x,y}^T, \\ D_y &= M_{y,z} (M_{x,z})^{-1} M_{x,y}, \\ D_z &= M_{y,z}^T (M_{x,y})^{-1} M_{x,z}. \end{aligned}$$

*Proof.* See [ACCFKGSSV, Lemma 5.19.3]. □

In particular, we see that the conics  $\Delta_x$ ,  $\Delta_y$ ,  $\Delta_z$  are smooth.

Linearly changing the coordinates  $([x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2])$ , we can simplify the shapes of the polynomials  $f$ ,  $g$  and  $h$ . To be precise, we have the following:

**Lemma 5.22** (cf. [M63, S14]). *One can choose coordinates on  $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$  such that one of the following two cases holds:*

(★) *the threefold  $X$  is given by*

$$\begin{cases} x_0 y_0 + x_1 y_1 + x_2 y_2 = 0, \\ y_0 z_0 + y_1 z_1 + y_2 z_2 = 0, \\ (1 + s)x_0 z_1 + (1 - s)x_1 z_0 - 2x_2 z_2 = 0, \end{cases}$$

where  $s \in \mathbb{C}$  such that  $s \neq \pm 1$ .

( $\blacklozenge$ ) the threefold  $X$  is given by

$$\begin{cases} x_0y_0 + x_1y_1 + x_2y_2 = 0, \\ y_0z_0 + y_1z_1 + y_2z_2 = 0, \\ x_0z_1 + x_1z_0 + x_1z_2 - x_2z_1 - 2x_2z_2 = 0. \end{cases}$$

*Proof.* See [ACCFKGSSV, Lemma 5.19.7].  $\square$

If  $X$  is given by ( $\star$ ) with  $s = 0$ , then  $X$  is isomorphic to the threefold given by

$$\begin{cases} x_0y_0 + x_1y_1 + x_2y_2 = 0, \\ y_0z_0 + y_1z_1 + y_2z_2 = 0, \\ x_0z_0 + x_1z_1 + x_2z_2 = 0, \end{cases}$$

which is the unique smooth Fano threefold  $\mathbb{N}^{\circ}3.13$  that admits an effective  $\mathrm{PGL}_2(\mathbb{C})$ -action. In this case, the Fano threefold  $X$  is K-polystable by [ACCFKGSSV, Example 1.6.17] and [ACCFKGSSV, Lemma 4.2.5].

On the other hand, if  $X$  is given by ( $\blacklozenge$ ), then  $X$  is not K-polystable by

**Lemma 5.23.** *Suppose that  $X$  is given by the equation ( $\blacklozenge$ ). Then  $\mathrm{Aut}(X) \cong \mathbb{G}_a \rtimes \mathfrak{S}_3$ . Moreover, the threefold  $X$  is strictly K-semistable.*

*Proof.* Suppose that the threefold  $X$  is given by ( $\blacklozenge$ ). For every  $a \in \mathbb{C}$ , let us consider the automorphism  $\phi_a \in \mathrm{Aut}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$  given by the following linear transformations:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \mapsto A \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} \mapsto (A^{-1})^T \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}, \quad \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} \mapsto A \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix},$$

where

$$A = \begin{pmatrix} 1 & a^2 & 2a \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix}.$$

Each such transformation  $\phi_a$  leaves  $X$  invariant, so that we can assume that  $\phi_a \in \mathrm{Aut}(X)$ . One can check that these transformations form a subgroup in  $\mathrm{Aut}(X)$  isomorphic to  $\mathbb{G}_a$ . Moreover, the group  $\mathrm{Aut}(X)$  also contains involutions  $\tau_{x,z}, \tau_{x,y}, \tau_{y,z}$  defined as

$$\begin{aligned} \tau_{x,z}: \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) &\mapsto \\ &\mapsto \left( [z_0 : z_1 : -z_2], [y_0 : y_1 : -y_2], [x_0 : x_1 : -x_2] \right), \end{aligned}$$

$$\begin{aligned} \tau_{x,y}: \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) &\mapsto \\ &\mapsto \left( [y_0 + 2y_1 + y_2 : 2y_0 : y_0 + y_2], [x_1 : x_0 - x_2 : 2x_2 - x_1], [z_0 : z_1 : -z_2] \right), \end{aligned}$$

$$\begin{aligned} \tau_{y,z} : & \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) \mapsto \\ & \mapsto \left( [x_0 : x_1 : -x_2], [z_1 : z_0 + z_2 : z_1 + 2z_2], [y_0 + 2y_1 - y_2 : 2y_0 : y_2 - y_0] \right). \end{aligned}$$

One can check that the involution  $\tau_{x,z}$ ,  $\tau_{x,y}$ ,  $\tau_{y,z}$  together with transformations  $\phi_a$  generate the group  $\text{Aut}(X)$ . Using this, we conclude that  $\text{Aut}(X) \cong \mathbb{G}_a \rtimes \mathfrak{S}_3$ . (For more details, check [ACCFKGSSV]).

By Theorem 2.13, the threefold  $X$  is not K-polystable. To show that  $X$  is K-semistable, observe that  $X$  is isomorphic to the threefold given by

$$\begin{cases} x_0y_0 + x_1y_1 + x_2y_2 = 0, \\ y_0z_0 + y_1z_1 + y_2z_2 = 0, \\ x_0z_1 + x_1z_0 - 2x_2z_2 + \epsilon(x_1z_2 - x_2z_1) = 0, \end{cases}$$

where  $\epsilon$  is any non-zero number. As we already mentioned, if  $\epsilon = 0$ , then these equations define the K-polystable smooth Fano threefold that admits an effective  $\text{PGL}_2(\mathbb{C})$ -action. Now, arguing as in the proof of Corollaries 5.13 and 5.19, we can construct a test configuration for the threefold  $X$ , whose special fiber is a K-polystable Fano threefold, so that  $X$  is strictly K-semistable by Corollary 5.4.  $\square$

In the remaining part of this section, we will prove the following result:

**Proposition 5.24.** *If  $X$  is given by  $(\star)$ , then  $X$  is K-polystable.*

To prove this result, we suppose that  $X$  is given by  $(\star)$ . Then  $\Delta_x$  is given by  $x_0x_1 = x_2^2$ , the curve  $\Delta_y$  is given by  $z_0z_1 = z_2^2$ , and  $\Delta_z$  is given by  $y_0y_1 = \frac{1-s^2}{4}y_2^2$ . Now, let us describe some automorphisms of the threefold  $X$ .

For every  $\lambda \in \mathbb{C}^*$ , the group  $\text{Aut}(X)$  contains the automorphism  $\varphi_\lambda : X \rightarrow X$  that is given by

$$\begin{aligned} \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) \mapsto & \left( \left[ \lambda x_0 : \frac{x_1}{\lambda} : x_2 \right], \left[ \frac{y_0}{\lambda} : \lambda y_1 : y_2 \right], \right. \\ & \left. \left[ \lambda z_0 : \frac{z_1}{\lambda} : z_2 \right] \right). \end{aligned}$$

These automorphisms form a proper subgroup  $\Gamma \subsetneq \text{Aut}(X)$ , which is isomorphic to  $\mathbb{G}_m$ . The full automorphism group  $\text{Aut}(X)$  also contains the involution  $\tau_{x,z}$  that is given by

$$\tau_{x,z} : \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) \mapsto \left( [z_1 : z_0 : z_2], [y_1 : y_0 : y_2], [x_1 : x_0 : x_2] \right),$$

the group  $\text{Aut}(X)$  also contains the involution  $\tau_{x,y}$  given by

$$\begin{aligned} \tau_{x,y} : & \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) \mapsto \\ & \mapsto \left( \left[ y_0 : \frac{y_1}{1-s^2} : -\frac{y_2}{2} \right], \left[ x_0 : (1-s^2)x_1 : -2x_2 \right], \left[ (s+1)z_1 : \frac{z_0}{s+1} : z_2 \right] \right), \end{aligned}$$

and it contains the involution  $\tau_{x,y}$  which is given by

$$\begin{aligned} \tau_{y,z} : & \left( [x_0 : x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1 : z_2] \right) \mapsto \\ & \mapsto \left( [x_1 : x_0 : -x_2], [(1-s)z_0 : (s+1)z_1 : 2z_2], \left[ \frac{y_0}{1-s} : \frac{y_1}{s+1} : \frac{y_2}{2} \right] \right). \end{aligned}$$

Let  $G$  be the subgroup in  $\text{Aut}(X)$  generated by  $\Gamma \cong \mathbb{G}_m$  and the involutions  $\tau_{x,y}, \tau_{x,z}, \tau_{y,z}$ . Then  $\Gamma$  is a normal subgroup in  $G$ . Note that  $G/\Gamma \cong \mathfrak{S}_3$ , so that we have  $G \cong \mathbb{G}_m \cdot \mathfrak{S}_3$ . Using the theory of splitting of extensions of groups, we get that  $G \cong \mathbb{G}_m \rtimes \mathfrak{S}_3$ .

**Remark 5.25.** *If  $s = 0$ , then we have  $\text{Aut}(X) \cong \text{PGL}_2(\mathbb{C}) \times \mathfrak{S}_3$ . Moreover, if  $s \neq 0$ , then one can show that  $\text{Aut}(X) = G$ . But we do not need this.*

To prove the K-polystability of the threefold  $X$ , we need to prove one technical lemma. To state it, we find it useful to replace the parameter  $s \in \mathbb{C} \setminus \{1, -1\}$  as  $s = \frac{r^3-1}{r^3+1}$  for a non-zero number  $r$  such that  $r^3 \neq -1$ . Then  $(1-s^2)(1+s) = \frac{8r^6}{(r^3+1)^3}$ , so that

$$\sqrt[3]{(1-s^2)(1+s)} = \left\{ \frac{2r^2}{r^3+1}, \frac{2\omega r^2}{r^3+1}, \frac{2\omega^2 r^2}{r^3+1} \right\},$$

where  $\omega$  is a primitive cube root of unity.

**Lemma 5.26.** *The following assertions holds:*

- (i) *one has  $\text{Pic}^G(X) = \mathbb{Z}[-K_X]$ ;*
- (ii) *the threefold  $X$  does not have  $G$ -fixed points;*
- (iii) *the threefold  $X$  contains exactly three distinct  $G$ -invariant irreducible curves, which can be parametrically described as follows:*

$$\begin{aligned} & \left( [u^2 : r(r^2 - r + 1)v^2 : ruv], \right. \\ & \quad [r(r^2 - r + 1)v^2 : ru^2 : -(r^3 + 1)uv], \\ & \quad \left. [ru^2 : (r^2 - r + 1)v^2 : ruv] \right), \quad (5.6) \end{aligned}$$

$$\begin{aligned} & \left( [ru^2 : \omega^2(r+1)(r+\omega^2)v^2 : ruv], \right. \\ & \quad [\omega(r+1)(r+\omega^2)v^2 : \omega r^2 u^2 : -(r^3 + 1)uv], \\ & \quad \left. [\omega^2 r^3 u^2 : (r+1)(r+\omega^2)v^2 : r^2 uv] \right), \quad (5.7) \end{aligned}$$

$$\begin{aligned} & \left( [ru^2 : \omega(r+1)(r+\omega)v^2 : ruv], \right. \\ & \quad [\omega^2(r+1)(r+\omega)v^2 : \omega^2 r^2 u^2 : -(r^3 + 1)uv], \\ & \quad \left. [\omega r^3 u^2 : (r+1)(r+\omega)v^2 : r^2 uv] \right), \quad (5.8) \end{aligned}$$

where  $[u : v] \in \mathbb{P}^1$ . All these three curves are smooth and rational.

*Proof.* See [ACCFKGSSV, Lemma 5.19.11]. □

Now, we are ready to prove

**Lemma 5.27.** *If  $s \neq 0$ , then  $\alpha_G(X) = 1$ . If  $s = 0$ , then  $\alpha_G(X) = \frac{2}{3}$ .*

*Proof.* First, let us recall that  $s = \frac{r^3-1}{r^3+1}$ , where  $r$  is a non-zero number such that  $r^3 \neq -1$ . If  $s = 0$ , we assume that  $r = 1$  to avoid repeating computations.

Since  $-K_X \sim E_{x,y} + E_{y,z} + E_{x,z}$ , we can conclude that  $\alpha_G(X) \leq 1$ . Moreover, if  $s = 0$ , then  $E_{x,y}$ ,  $E_{y,z}$  and  $E_{x,z}$  meet along the curve (5.6), which gives  $\alpha_G(X) \leq \frac{2}{3}$ . Set

$$\mu = \begin{cases} 1 & \text{if } s \neq 0, \\ \frac{2}{3} & \text{if } s = 0. \end{cases}$$

We see that  $\alpha_G(X) \leq \mu$ . Suppose that  $\alpha_G(X) < \mu$ . Let us seek for a contradiction.

Recall that  $\text{Pic}^G(X) = \mathbb{Z}[-K_X]$  and  $X$  has no  $G$ -fixed points by Lemma 5.26. Arguing as in the proof of Theorem 5.5 and using Lemma 2.20, we see that there exist an irreducible  $G$ -invariant curve  $C \subset X$  and a  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $X$  such that  $D \sim_{\mathbb{Q}} -K_X$ , the log pair  $(X, \lambda D)$  is strictly log canonical for some rational number  $\lambda < \mu$ , and  $C$  is its unique log canonical center. Then  $C$  is one of the curves (5.6), (5.7), (5.8) by Lemma 5.26.

Since  $\lambda < 1$  and  $C \subseteq \text{Nklt}(X, \lambda D)$ , we see that  $\text{mult}_C(D) \geq \frac{1}{\lambda} > \frac{1}{\mu} \geq 1$ .

Now, let us use assumptions and notations introduced in the proof of Lemma 5.26.

Let  $S_x, S_y, S_z$  be the surfaces in  $X$  that are cut out by  $x_0x_1 = q_x x_2^2$ ,  $y_0y_1 = q_y y_2^2$ ,  $z_0z_1 = q_z z_2^2$ , respectively. Then  $C \subset S_x \cap S_y \cap S_z$ , the divisor  $S_x + S_y + S_z$  is  $G$ -invariant and  $-K_X \sim_{\mathbb{Q}} \frac{1}{2}(S_x + S_y + S_z)$ . Moreover, if  $s = 0$  and  $C$  is the curve (5.6), then we have  $C = E_{x,y} \cap E_{y,z} \cap E_{x,z}$  and we have  $S_x = E_{x,y} + E_{x,z}$ ,  $S_y = E_{x,y} + E_{y,z}$ ,  $S_z = E_{x,z} + E_{y,z}$ . In all other cases, the surfaces  $S_x, S_y, S_z$  are smooth at general point of the curve  $C$ , and they meet each other pairwise transversally at general point of the curve  $C$ .

Indeed, to prove this claim, it is enough to check both assertions for  $S_x$  and  $S_y$ , because the group  $G$  acts two-transitively on  $\{S_x, S_y, S_z\}$ . Let us show that  $S_x$  and  $S_y$  are smooth at general point of the curve  $C$ , and they meet transversally at general point of the curve  $C$ . This can be explicitly checked at the point  $P \in C$  that corresponds to  $[u : v] = [1 : 1]$  in the parametrizations (5.6), (5.7) and (5.8). Thus, we can do this in the affine chart  $x_2 = y_2 = z_2 = 1$ . In this chart, the threefold  $X$  is given by

$$\begin{cases} x_0y_0 + x_1y_1 + 1 = 0, \\ y_0z_0 + y_1z_1 + 1 = 0, \\ (1+s)x_0z_1 + (1-s)x_1z_0 - 2 = 0, \end{cases}$$

the surface  $S_x$  is given by  $x_0x_1 = q_x$ , and the surface  $S_y$  is given by  $y_0y_1 = q_y$ , where we consider now  $x_0, x_1, y_0, y_1, z_0, z_1$  as coordinates on  $\mathbb{A}^6$ . If  $C$  is the curve (5.6), then

$$P = \left( \frac{1}{r}, r^2 - r + 1, -\frac{r}{r+1}, -\frac{r}{r^3+1}, 1, \frac{r^2 - r + 1}{r} \right),$$

so that the Zariski tangent space to the intersection  $S_x \cap S_y$  at the point  $P$  is given by

$$\begin{pmatrix} -\frac{r}{r+1} & -\frac{r}{r^3+1} & \frac{1}{r} & r^2-r+1 & 0 & 0 \\ 0 & 0 & 1 & \frac{r^2-r+1}{r} & -\frac{r}{r+1} & -\frac{r}{r^3+1} \\ r^2(r^2-r+1) & 1 & 0 & 0 & r^2-r+1 & r^2 \\ r^2-r+1 & \frac{1}{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r}{r^3+1} & -\frac{r}{r+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 - \frac{1}{r} \\ x_1 - r^2 + r - 1 \\ y_0 + \frac{r}{r+1} \\ y_1 + \frac{r}{r^3+1} \\ z_0 - 1 \\ z_1 - \frac{r^2-r+1}{r} \end{pmatrix} = 0.$$

The determinant of the matrix formed by the first 5 columns of this matrix is  $\frac{(r^2-r+1)(r-1)^2}{r+1}$ , and this vanishes if  $r = 1$ , since  $r^2 - r + 1 = 0$  implies that  $r^3 = -1$  and we assumed that this was not the case. So therefore, if this determinant vanishes, then  $s = 0$ .

Thus, if  $s \neq 0$  and  $C$  is the curve (5.6), then the Zariski tangent space to the intersection  $S_x \cap S_y$  at the point  $P$  is one-dimensional, so that both surfaces  $S_x$  and  $S_y$  are smooth at  $P$ , and intersect transversally at this point. This proves our claim in the case when  $C$  is the curve (5.6).

Similarly, if  $C$  is the curve (5.7), then

$$P = \left( 1, \frac{\omega^2(r+1)(r+\omega^2)}{r}, -\frac{\omega}{r+\omega}, -\frac{\omega r^2}{r^3+1}, \omega^2 r, \frac{(r+1)(r+\omega^2)}{r^2} \right),$$

and the dimension of the Zariski tangent space to the intersection  $S_x \cap S_y$  at this point equals the nullity of the following  $5 \times 6$  matrix:

$$\begin{pmatrix} -\frac{\omega}{r+\omega} & -\frac{\omega r^2}{r^3+1} & 1 & \frac{\omega^2(r+1)(r+\omega^2)}{r} & 0 & 0 \\ 0 & 0 & \omega^2 r & \frac{(r+1)(r+\omega^2)}{r^2} & -\frac{\omega}{r+\omega} & -\frac{\omega}{r^3+1} \\ r(r+1)(r+\omega^2) & \omega^2 r & 0 & 0 & \frac{\omega^2(r+1)(r+\omega^2)}{r} & r^3 \\ \frac{\omega^2(r+1)(r+\omega^2)}{r} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\omega r^2}{r^3+1} & -\frac{\omega}{r+\omega} & 0 & 0 \end{pmatrix}.$$

The determinant of its submatrix formed by the first 5 columns is  $\frac{\omega(r+1)(r-\omega)^2(r+\omega^2)}{r+\omega}$ , so that it never vanishes, because  $r^3 \neq -1$  and  $r \neq \omega$ . Therefore, the Zariski tangent space to  $S_x \cap S_y$  at the point  $P$  is always one-dimensional, so that both our surfaces  $S_x$  and  $S_y$  are smooth at  $P$ , and intersect transversally at  $P$ . This proves our claim in the case when  $C$  is the curve (5.7). Now, swapping  $\omega$  with  $\omega^2$ , we also obtain the proof of our claim in the case when  $C$  is the curve (5.8).

Thus, unless  $s = 0$  and  $C$  is the curve (5.6), the surfaces  $S_x, S_y, S_z$  are smooth at general point of the curve  $C$ , and they meet each other pairwise transversally at general point of the curve  $C$ . In particular, we see that  $C \not\subseteq \text{Nklt}(X, \frac{\mu}{2}(S_x + S_y + S_z))$ . Thus, using [ACCFKGSSV, Lemma A.4.12], we may assume that  $S_x, S_y, S_z$  are not contained in  $\text{Supp}(D)$ .

If  $s = 0$  and  $C$  is the curve (5.6), then  $1 = D \cdot \ell \geq \text{mult}_C(D)$ , where  $\ell$  is a general fiber of the projection  $E_{x,y} \rightarrow \pi_{x,y}(E_{x,y})$ . But  $\text{mult}_C(D) > 1$ . Therefore, we see that  $s \neq 0$  or  $C$  is not the curve (5.6). Then  $\eta_x(C) \neq \Delta_x, \eta_y(C) \neq \Delta_y$  and  $\eta_z(C) \neq \Delta_z$ .

Let  $\ell$  be a general fiber of the morphism  $\eta_x|_{S_x}: S_x \rightarrow \eta_x(C)$ . Then  $\ell$  is not contained in the support of the divisor  $D$ , since  $S_x$  is not contained in its support. On the other hand, the curve  $\ell$  meets the curve  $C$ , so that  $2 = D \cdot \ell \geq \text{mult}_C(D)$ , which gives  $\text{mult}_C(D) \leq 2$ .

Let  $\eta: \widehat{X} \rightarrow X$  be the blow up of the curve  $C$ , and let  $F$  be the  $\eta$ -exceptional surface. Then the  $G$ -action lifts to  $\widehat{X}$ , and it follows from [ACCFKGSSV, Lemma A.4.3] that  $F$  contains a smooth irreducible  $G$ -invariant curve  $\mathcal{C}$  such that  $\mathcal{C}$  is a section of the natural projection  $F \rightarrow C$ . Let us show that such curve does not exist.

Let  $\widehat{S}_x, \widehat{S}_y, \widehat{S}_z$  be the proper transforms on  $\widehat{X}$  of the surfaces  $S_x, S_y, S_z$ , respectively. Then each intersection among  $\widehat{S}_x \cap F, \widehat{S}_y \cap F, \widehat{S}_z \cap F$  contains a unique component that is a section of the projection  $F \rightarrow C$ . Denote these sections by  $\mathcal{C}_x, \mathcal{C}_y, \mathcal{C}_z$ , respectively. Then

- $\mathcal{C}_x, \mathcal{C}_y, \mathcal{C}_z$  are distinct curves,
- $\mathcal{C}_x, \mathcal{C}_y, \mathcal{C}_z$  are  $\Gamma$ -invariant, and  $\Gamma$  acts faithfully on each of these curves,
- the whole group  $G$  permutes the curves  $\mathcal{C}_x, \mathcal{C}_y, \mathcal{C}_z$  two-transitively.

Thus, using [ACCFKGSSV, Corollary A.6.9], we conclude that  $F = \mathbb{P}^1 \times \mathbb{P}^1$ . Then, using [ACCFKGSSV, Lemma A.6.6], we conclude that the  $G$ -action on  $F$  is given by [ACCFKGSSV, Equation A.6.7] for some integers  $a > 0$  and  $b$ , which implies that  $F$  does not contain  $G$ -invariant sections of the projection  $F \rightarrow C$ , which contradicts the existence of the curve  $\mathcal{C}$ .  $\square$

Now, Proposition 5.24 follows from Theorem 2.24 and [ACCFKGSSV, Lemma 4.2.5].

### 5.2.5 Family №3.15

Let  $Q$  be the quadric  $\{x_0^2 + 2x_1x_2 + 2x_1x_4 + 2x_2x_3 = 0\} \subset \mathbb{P}^4$ , where  $x_0, x_1, x_2, x_3, x_4$  are homogeneous coordinates on  $\mathbb{P}^4$ . Then the quadric  $Q$  is smooth. Let  $L$  be the line  $\{x_0 = x_1 = x_2 = 0\}$ , let  $\Pi$  be the plane  $\{x_3 = x_4 = 0\}$ , and let  $C = Q \cap \Pi$ . Then  $L \subset Q, L \cap \Pi = \emptyset$ , and  $C$  is a smooth conic. Let  $\pi: X \rightarrow Q$  be the blow up along the union  $L \cup C$ . Then  $X$  is a smooth Fano threefold from the deformation family № 3.15. By [CSP19, Lemma 5.10], the threefold  $X$  is the unique smooth member of this family.

**Proposition 5.28.** *The threefold  $X$  is  $K$ -polystable.*

Let  $G$  the subgroup in  $\text{Aut}(Q)$  generated by the involution  $\iota$  given by

$$[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_0 : x_2 : x_1 : x_4 : x_3]$$

and the transformations  $[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [\lambda x_0 : \lambda^2 x_1 : x_2 : \lambda^2 x_3 : x_4]$  for  $\lambda \in \mathbb{C}^*$ . Then  $G \cong \mathbb{C}^* \rtimes \mu_2$ . Since  $L$  and  $C$  are  $G$ -invariant, the action of the group  $G$  lifts to  $X$ . To prove Proposition 5.28, we will apply Theorem 2.19 to  $X$  equipped with  $G$ -action. But first, let us describe  $G$ -equivariant geometry of the threefold  $X$ .

Let  $\overline{R}$  be the surface  $\{x_2x_3 + x_1x_4 = 0\} \cap Q$ , and let  $R$  be its proper transform on  $X$ . Then the surface  $\overline{R}$  is irreducible, it is singular along  $L$ , and it contains both  $L$  and  $C$ , but  $R$  is smooth, and there is a  $G$ -equivariant birational morphism  $\eta: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$  that contracts  $R$  to a curve. Thus, we have the following  $G$ -equivariant commutative diagram:

$$\begin{array}{ccccc}
& & \mathbb{P}^1 \times \mathbb{P}^2 & & \\
& \swarrow \text{pr}_1 & \uparrow \eta & \searrow \text{pr}_2 & \\
\mathbb{P}^1 & & X & & \mathbb{P}^2 \\
\uparrow v & \swarrow \theta & & \searrow \phi & \uparrow \nu \\
Y & & & & V \\
& \searrow \varphi & \downarrow \pi & \swarrow \vartheta & \\
& & Q & & 
\end{array}$$

where  $\vartheta$  is the blow up of the line  $L$ ,  $\varphi$  is the blow up of the conic  $C$ ,  $v$  is a fibration into quadric surfaces,  $\nu$  is a  $\mathbb{P}^1$ -bundle,  $\text{pr}_1$  and  $\text{pr}_2$  are projections to the first and the second factors, respectively,  $\theta$  and  $\phi$  are blow ups of the preimages of  $L$  and  $C$ , respectively.

Let  $E_L$  and  $E_C$  be the exceptional surfaces of the morphisms  $\theta$  and  $\phi$ , respectively. let  $H_Q = \pi^*(\mathcal{O}_{\mathbb{P}^4}(1)|_Q)$ , let  $H_1 = (\text{pr}_1 \circ \eta)^*(\mathcal{O}_{\mathbb{P}^1}(1))$  and let  $H_2 = (\text{pr}_2 \circ \eta)^*(\mathcal{O}_{\mathbb{P}^2}(1))$ . Then

$$\begin{aligned}
\text{Pic}(X) &= \mathbb{Z}[H_Q] \oplus \mathbb{Z}[E_L] \oplus \mathbb{Z}[E_C], \\
\text{Nef}(X) &= \mathbb{R}_{\geq 0}[H_Q] + \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2], \\
\overline{\text{Eff}}(X) &= \mathbb{R}_{\geq 0}[E_L] + \mathbb{R}_{\geq 0}[E_C] + \mathbb{R}_{\geq 0}[R] + \mathbb{R}_{\geq 0}[H_1].
\end{aligned}$$

Note that  $H_2 \sim H_Q - E_L$ ,  $H_1 \sim H_Q - E_C$ ,  $R \sim 2H_Q - 2E_L - E_C$ , and

$$-K_X \sim 3H_Q - E_L - E_C \sim H_Q + H_1 + H_2 \sim_{\mathbb{Q}} 2E_L + \frac{1}{2}E_C + \frac{3}{2}R, \quad (5.9)$$

so that  $\alpha_G(X) = \frac{1}{2}$  by [CS08, Lemma 8.15]. One can show that  $\text{Aut}(X) = G$ .

Let  $L'$  be the line  $\{x_0 = x_1 + 2x_3 = x_2 + 2x_4 = 0\} \subset Q$ . Then the line  $L'$  is  $G$ -invariant. Similarly, for every non-zero  $t \in \mathbb{C}$ , let  $C_t = \{(1-t)x_1 - 2tx_3 = (1-t)x_2 - 2tx_4 = 0\} \cap Q$ . Then  $C_t$  is an irreducible  $G$ -invariant conic for every non-zero  $t \in \mathbb{C}$ . Note that  $C = C_1$ . Note also that  $L \cap L' = \emptyset$ ,  $L \cap C_t = \emptyset$  and  $L' \cap C_t = \emptyset$  for every  $t \neq 0$ . Finally, observe that the conics  $C_{t_1}$  and  $C_{t_2}$  are also disjoint for  $t_1 \neq t_2$ .

**Lemma 5.29.** *Let  $Z$  be an irreducible  $G$ -invariant curve in the quadric hypersurface  $Q$ . Then either  $Z = L$ , or  $Z = L'$ , or  $Z = C_t$  for some non-zero  $t \in \mathbb{C}$ .*

*Proof.* Observe that the curve  $Z$  is rational, so that it contains a  $\iota$ -fixed point  $P$  such that the curve  $Z$  is the closure of the  $\mathbb{G}_m$ -orbit of this point. Thus, looking

at the  $\iota$ -fixed points in  $Q$ , we conclude that either  $P = [0 : 0 : 0 : 1 : -1]$ , or  $P = [0 : 2 : -2 : -1 : 1]$ , or

$$P = [4s : 4s^2 : 4s^2 : -2s^2 - 1 : -2s^2 - 1]$$

for some non-zero  $s \in \mathbb{C}$ . Then either  $Z = L$ , or  $Z = L'$ , or  $Z = C_t$  for  $t = -2s^2$ .  $\square$

In what follows, we will apply results from Section 2.2.3.2 to prove Proposition 5.28. We will use notations of this section. Let  $Z$  be an irreducible  $G$ -invariant curve in  $X$ .

**Lemma 5.30.** *Suppose that  $Z \subset E_C$ . Then  $S(W_{\bullet, \bullet}^{E_C}; Z) \leq \frac{51}{64}$ .*

*Proof.* One has  $E_C \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $s$  a section of the projection  $E_C \rightarrow C$  such that  $s^2 = 0$ , and let  $f$  a fiber of this projection. Take  $u \in \mathbb{R}_{\geq 0}$ . Then

$$-K_X - uE_C \sim_{\mathbb{R}} 2H_1 + \frac{1}{2}R + \left(\frac{3}{2} - u\right)E_C,$$

so that  $-K_X - uE_C$  is pseudo-effective if and only if  $u \leq \frac{3}{2}$ . Moreover, if  $u \leq \frac{3}{2}$ , then

$$P(-K_X - uE_C) = \begin{cases} -K_X - uE_C & \text{if } 0 \leq u \leq 1, \\ 2H_1 + (3 - 2u)H_2 & \text{if } 1 \leq u \leq \frac{3}{2}, \end{cases}$$

and we have

$$N(-K_X - uE_C) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u - 1)R & \text{if } 1 \leq u \leq \frac{3}{2}. \end{cases}$$

If  $u \leq 1$ , then we have  $P(-K_X - uE_C)|_{E_C} \sim (1 + u)s + (4 - 2u)f$ . Similarly, if  $1 \leq u \leq \frac{3}{2}$ , then  $P(-K_X - uE_C)|_{E_C} \sim 2s + (6 - 4u)f$ . Note that  $R|_{E_C}$  is a smooth curve in  $|s + 2f|$ . Thus, if  $Z = R|_{E_C}$ , then Corollary 2.32 gives

$$\begin{aligned} S(W_{\bullet, \bullet}^{E_C}; Z) &= \frac{3}{32} \int_0^1 \int_0^\infty \text{vol}((1 + u - v)s + (4 - 2u - 2v)f) dv du \\ &+ \frac{3}{32} \int_1^{\frac{3}{2}} 4(u - 1)(6 - 4u) du + \frac{3}{32} \int_1^{\frac{3}{2}} \int_0^\infty \text{vol}((2 - v)f + (6 - 4u - 2v)s) dv du \\ &= \frac{3}{32} \int_0^{\frac{1}{2}} \int_0^{1+u} 2(4 - 2u - 2v)(1 + u - v) dv du + \frac{3}{32} \int_{\frac{1}{2}}^1 \int_0^{2-u} 2(4 - 2u - 2v)(1 + u - v) dv du \\ &+ \frac{3}{32} \int_1^{\frac{3}{2}} 4(u - 1)(6 - 4u) du + \frac{3}{32} \int_1^{\frac{3}{2}} \int_0^{3-2u} 2(6 - 4u - 2v)(2 - v) dv du = \frac{15}{32} < \frac{51}{64}. \end{aligned}$$

If  $Z \neq R|_{E_C}$ , then we have  $S(W_{\bullet, \bullet}^{E_C}; Z) \leq S(W_{\bullet, \bullet}^{E_C}; s)$ , because  $|Z - s| \neq \emptyset$ , since

$Z \not\sim f$  as the conic  $C$  does not have  $G$ -fixed points. Therefore, if  $Z \neq R|_{E_C}$ , then

$$\begin{aligned} S(W_{\bullet, \bullet}^{E_C}; Z) &\leq S(W_{\bullet, \bullet}^{E_C}; s) = \frac{3}{32} \int_0^1 \int_0^\infty \text{vol}((1+u-v)s + (4-2u)f) dvdu \\ &+ \frac{3}{32} \int_1^{\frac{3}{2}} \int_0^\infty \text{vol}((2-v)f + (6-4u)s) dvdu = \frac{3}{32} \int_0^1 \int_0^{1+u} 2(4-2u)(1+u-v) dvdu \\ &\quad + \frac{3}{32} \int_1^{\frac{3}{2}} \int_0^2 2(6-4u)(2-v) dvdu = \frac{51}{64} \end{aligned}$$

by Corollary 2.32.  $\square$

**Lemma 5.31.** *Suppose that  $Z \subset E_L$ . Then  $S(W_{\bullet, \bullet}^{E_L}; Z) \leq \frac{29}{32}$ .*

*Proof.* First, we observe that  $E_L \cong \mathbb{F}_1$ . Let  $f$  be a fiber of the natural projection  $E_L \rightarrow L$ , and let  $s$  the  $(-1)$ -curve in  $E_L$ . Then  $R|_{E_L}$  is a smooth curve in  $|2s+2f|$ .

Take  $u \in \mathbb{R}_{\geq 0}$ . Using (5.9), we see that  $-K_X - uE_L$  is pseudo-effective  $\iff u \leq 2$ . Moreover, if  $u \leq 2$ , then

$$P(-K_X - uE_L) = \begin{cases} -K_X - uE_L & \text{if } 0 \leq u \leq 1, \\ (2-u)H_1 + (3-u)H_2 & \text{if } 1 \leq u \leq 2, \end{cases}$$

and we have

$$N(-K_X - uE_L) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u-1)R & \text{if } 1 \leq u \leq 2. \end{cases}$$

If  $u \leq 1$ , then we have  $P(-K_X - uE_L)|_{E_L} \sim (1+u)s + 3f$ . Similarly, if  $1 \leq u \leq 2$ , then we have  $P(-K_X - uE_L)|_{E_L} \sim (3-u)s + (5-2u)f$ . Thus, if  $Z = R|_{E_L}$ , then

$$\begin{aligned} S(W_{\bullet, \bullet}^{E_L}; Z) &= \frac{3}{32} \int_0^1 \int_0^\infty \text{vol}((1+u-2v)s + (3-2v)f) dvdu \\ &+ \frac{3}{32} \int_1^2 (u-1)(3-u)(7-3u) du + \frac{3}{32} \int_1^2 \int_0^\infty \text{vol}((3-u-2v)s + (5-2u-2v)f) dvdu \\ &= \frac{3}{32} \int_0^1 \int_0^{\frac{1+u}{2}} (u+1-2v)(5-u-2v) dvdu + \frac{17}{128} + \frac{3}{32} \int_1^2 \int_0^{\frac{3-u}{2}} (3-u-2v)(7-3u+2v) dvdu \end{aligned}$$

by Corollary 2.32, so that  $S(W_{\bullet, \bullet}^{E_L}; Z) = \frac{15}{32} < \frac{29}{32}$ .

If  $Z \neq R|_{E_L}$ , then  $S(W_{\bullet, \bullet}^{E_L}; Z) \leq S(W_{\bullet, \bullet}^{E_L}; s)$ , because  $|Z - s| \neq \emptyset$ , since  $Z \not\sim f$  as the line  $L$  does not have  $G$ -fixed points. Hence, if  $Z \neq R|_{E_L}$ , then Corollary 2.32 gives

$$\begin{aligned} S(W_{\bullet, \bullet}^{E_L}; Z) &\leq S(W_{\bullet, \bullet}^{E_L}; s) = \frac{3}{32} \int_0^1 \int_0^\infty \text{vol}((1+u-v)s + 3f) dvdu \\ &+ \frac{3}{32} \int_1^2 \int_0^\infty \text{vol}((3-u-v)s + (5-2u)f) dvdu = \frac{3}{32} \int_0^1 \int_0^{1+u} (1+u-v)(5-u+v) dvdu \\ &\quad + \frac{3}{32} \int_1^2 \int_0^{3-u} (3-u-v)(7-3u+v) dvdu = \frac{29}{32} \end{aligned}$$

as required.  $\square$

Let  $\bar{S}$  be the surface  $Q \cap \{x_1x_4 = x_2x_3\}$ . Then  $\bar{S}$  is a del Pezzo surface of degree 4 that has four ordinary nodes. It is well-known that  $\bar{S}$  is toric, and it contains four lines [CT88, CP21]. Two of them are the lines  $L$  and  $L'$  described above, and the remaining two lines in  $\bar{S}$  are the disjoint lines  $\ell = \{x_0 = x_1 = x_3 = 0\}$  and  $\ell' = \{x_0 = x_2 = x_4 = 0\}$ . Then

$$\begin{aligned} L \cap \ell &= [0 : 0 : 0 : 0 : 1], \\ L' \cap \ell &= [0 : 0 : 2 : 0 : -1] \\ L \cap \ell' &= [0 : 0 : 0 : 1 : 0], \\ L' \cap \ell' &= [0 : 2 : 0 : -1 : 0]. \end{aligned}$$

These are the singular points of  $\bar{S}$ . By [CP21, Lemma 2.9], the lines  $L, L', \ell, \ell'$  generate  $\text{Cl}(\bar{S})$ , which has rank 2. On the surface  $\bar{S}$ , we have  $2L \sim 2L', 2\ell \sim 2\ell'$  and

$$-K_{\bar{S}} \sim L + L' + \ell + \ell' \sim 2(L + \ell).$$

The surface  $\bar{S}$  also contains all conics  $C_t$  for  $t \in \mathbb{C}^*$  including the conic  $C = C_1$ , each conic  $C_t$  is contained in the smooth locus of the surface  $\bar{S}$ , and  $C_t \sim 2L$  for every  $t \in \mathbb{C}^*$ .

**Lemma 5.32.** *Suppose that  $\pi(Z) = C_t$  for  $t \in \mathbb{C} \setminus \{0, 1\}$ . Then  $S(W_{\bullet, \bullet}^S; Z) = \frac{79}{128}$ .*

*Proof.* Take  $u \in \mathbb{R}_{\geq 0}$ . Observe that

$$-K_X - uS \sim_{\mathbb{R}} \left(\frac{3}{2} - u\right)S + \frac{1}{2}E_L + \frac{1}{3}E_L,$$

which implies that  $-K_X - uS$  is pseudo-effective  $\iff u \leq \frac{3}{2}$ . Moreover, if  $u \leq \frac{3}{2}$ , then

$$P(-K_X - uS) = \begin{cases} -K_X - uS & \text{if } 0 \leq u \leq 1, \\ (3 - 2u)H_Q & \text{if } 1 \leq u \leq \frac{3}{2}, \end{cases}$$

and

$$N(-K_X - uS) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ (u - 1)(E_L + E_C) & \text{if } 1 \leq u \leq \frac{3}{2}. \end{cases}$$

In particular, we see that  $Z$  is not contained in the supports of the divisor  $N(-K_X - uS)|_S$ . Therefore, using Corollary 2.32, we obtain

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &= \frac{3}{32} \int_0^1 \int_0^\infty \text{vol}((-K_X - uS)|_S - vZ) dudv + \\ &\quad + \frac{3}{32} \int_1^{\frac{3}{2}} \int_0^\infty \text{vol}((3 - 2u)H_Q|_S - vZ) dudv. \end{aligned}$$

To compute these integrals, let us say few words about geometry of the surface  $S$ .

The morphism  $\pi$  induces a birational morphism  $\varpi: S \rightarrow \bar{S}$ , which is the minimal resolution of the two singular points  $[0 : 0 : 0 : 1 : 0]$  and  $[0 : 0 : 0 : 0 : 1]$  of

the surface  $\bar{S}$ . In particular, the surface  $S$  has exactly two singular points, and they are ordinary nodes. Denote the proper transforms on  $S$  of the curves  $L, L', \ell, \ell'$  and  $C_t$  by the same symbols, and denote by  $\mathbf{e}$  and  $\mathbf{e}'$  the two  $\varpi$ -exceptional curves such that  $\mathbf{e} \cap \ell \neq \emptyset$  and  $\mathbf{e}' \cap \ell' \neq \emptyset$ . Note that the Mori cone  $\overline{\text{NE}}(S)$  is generated by the curves  $L, \ell, \ell', \mathbf{e}, \mathbf{e}'$ .

On the surface  $S$ , we have  $C_t \sim 2L'$ ,  $2L + \mathbf{e} + \mathbf{e}' \sim 2L'$  and  $2\ell + \mathbf{e} \sim 2\ell' + \mathbf{e}'$ , and the intersections of the curves  $L, L', \ell, \ell', \mathbf{e}$  and  $\mathbf{e}'$  are given in the following table:

	$L$	$L'$	$\ell$	$\ell'$	$\mathbf{e}$	$\mathbf{e}'$
$L$	-1	0	0	0	1	1
$L'$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\ell$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	0
$\ell'$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1
$\mathbf{e}$	1	0	1	0	-2	0
$\mathbf{e}'$	1	0	0	1	0	-2

Let  $v$  be a non-negative real number. If  $u \leq 1$ , then

$$P(-K_X - uS)|_S - vZ \sim_{\mathbb{R}} \left( \frac{3-u}{2} - v \right) Z + (3-2u)(\ell + \ell') + \frac{2-u}{2}(\mathbf{e} + \mathbf{e}'),$$

so that the divisor  $P(-K_X - uS)|_S - vZ$  is pseudo-effective if and only if  $v \leq \frac{3-u}{2}$ . Moreover, if  $u \leq 1$  and  $v \leq \frac{3-u}{2}$ , its Zariski decomposition can be described as follows:

- if  $0 \leq v \leq 1$ , then  $P(-K_X - uS)|_S - vZ$  is nef,
- if  $1 \leq v \leq \frac{3-u}{2}$ , then the positive part of the Zariski decomposition is

$$\left( \frac{3-u}{2} - v \right) Z + (5-2u-2v)(\ell + \ell') + \frac{2-u}{2}(\mathbf{e} + \mathbf{e}'),$$

and the negative part is  $2(v-1)(\ell + \ell')$ .

Similarly, if  $1 \leq u \leq \frac{3}{2}$ , then

$$P(-K_X - uS)|_S - vZ \sim_{\mathbb{R}} (3-2u-v)Z + (3-2u)(\ell + \ell') + \left( \frac{3}{2} - u \right) (\mathbf{e} + \mathbf{e}').$$

so that this divisor is pseudo-effective  $\iff$  it is nef  $\iff v \leq 3-2u$ . Hence, we obtain

$$\begin{aligned} S(W_{\bullet, \bullet}^S; Z) &= \frac{3}{32} \int_0^1 \int_0^1 (3u^2 + 8uv - 16u - 12v + 17) dudv \\ &+ \frac{3}{32} \int_0^1 \int_1^{\frac{3-u}{2}} (3-u-2v)(7-3u-2v) dudv + \frac{3}{32} \int_1^{\frac{3}{2}} \int_0^{3-2u} 4(3-2u-v)(3-2u) dudv \\ &= \frac{79}{128}. \end{aligned}$$

as claimed.  $\square$

Now, let  $\overline{H}$  be the hyperplane section of the quadric threefold  $Q$  given by  $x_0 = 0$ , and let  $H$  be its proper transform on the threefold  $X$ . Then  $\overline{H}$  is a smooth quadric surface that contains the lines  $L$  and  $L'$ , and  $H$  is a smooth del Pezzo surface of degree six.

**Lemma 5.33.** *Suppose that  $\pi(Z) = L'$ . Then  $S(W_{\bullet, \bullet}^H; Z) = \frac{49}{64}$ .*

*Proof.* Take  $u \in \mathbb{R}_{\geq 0}$ . Note that  $-K_X - uH \sim_{\mathbb{R}} (2 - u)H + H_1 + E_L$ , which implies that the divisor  $-K_X - uH$  is pseudo-effective  $\iff u \leq 2$ . Moreover, if  $u \leq 2$ , then

$$P(-K_X - uS) = \begin{cases} -K_X - uH & \text{if } 0 \leq u \leq 1, \\ H_1 + (2 - u)H_Q & \text{if } 1 \leq u \leq 2, \end{cases}$$

and  $N(-K_X - uS) = (u - 1)E_L$  for  $u \in [1, 2]$ . Then  $Z \notin \text{Supp}(N(-K_X - uH)|_H)$ , so that

$$\begin{aligned} S(W_{\bullet, \bullet}^H; Z) &= \frac{3}{32} \int_0^1 \int_0^\infty \text{vol}((-K_X - uH)|_H - vZ) dudv + \\ &\quad + \frac{3}{32} \int_1^2 \int_0^\infty \text{vol}((H_1 + (2 - u)H_Q)|_H - vZ) dudv \end{aligned}$$

by Corollary 2.32.

The conic  $C$  intersects  $\overline{H}$  transversally at  $P_1 = [0 : 1 : 0 : 0 : 0]$  and  $P_2 = [0 : 0 : 1 : 0 : 0]$ , which are not contained in the lines  $L$  and  $L'$ . Thus, the morphism  $\pi$  induces a birational morphism  $\varpi: H \rightarrow \overline{H}$  that blows up  $P_1$  and  $P_2$ . Let  $\mathbf{e}_1$  and  $\mathbf{e}_2$  be the  $\varpi$ -exceptional curves that are contracted to  $P_1$  and  $P_2$ , respectively, let  $\mathbf{s}_1$  and  $\mathbf{f}_1$  be the proper transform on the surface  $H$  of the two rulings of the surface  $\overline{H} \cong \mathbb{P}^1 \times \mathbb{P}^1$  that pass through the point  $P_1$ , and let  $\mathbf{s}_2$  and  $\mathbf{f}_2$  be the proper transform on  $H$  of the two rulings that pass through  $P_2$ . We may assume that  $Z \sim \mathbf{s}_1 + \mathbf{e}_1 \sim \mathbf{s}_2 + \mathbf{e}_2$ , so that  $\mathbf{f}_1 + \mathbf{e}_1 \sim \mathbf{f}_2 + \mathbf{e}_2$  and  $\mathbf{f}_1 + \mathbf{s}_2 \sim \mathbf{f}_2 + \mathbf{s}_1$ . Observe that  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{s}_1, \mathbf{s}_2, \mathbf{f}_1, \mathbf{f}_2$  are all  $(-1)$ -curves in  $H$ .

Note that  $E_L|_H \sim \mathbf{s}_1 + \mathbf{e}_1$ ,  $H_Q|_H \sim \mathbf{f}_1 + \mathbf{s}_1 + 2\mathbf{e}_1$ ,  $H_1|_H \sim \mathbf{f}_1 + \mathbf{s}_2$  and  $H|_H \sim \mathbf{f}_1 + \mathbf{e}_1$ .

Let  $v$  be a non-negative real number. If  $u \leq 1$ , then

$$P(-K_X - uH)|_H - vZ \sim_{\mathbb{R}} (2 - u)\mathbf{f}_1 + \mathbf{f}_2 + (2 - v)\mathbf{s}_1 + (3 - u - v)\mathbf{e}_1$$

so that this divisor is pseudo-effective if and only if  $v \leq 2$ . Moreover, it is nef for  $v \in [0, 1]$ , and its Zariski decomposition for  $v \in [1, 2]$  is

$$\underbrace{(3 - u - v)(\mathbf{f}_1 + \mathbf{e}_1) + (2 - v)(\mathbf{s}_1 + \mathbf{f}_2)}_{\text{positive part}} + \underbrace{(v - 1)(\mathbf{f}_1 + \mathbf{f}_2)}_{\text{negative part}},$$

Similarly, if  $1 \leq u \leq 2$ , then

$$P(-K_X - uH)|_H - vZ \sim_{\mathbb{R}} (2 - u)\mathbf{f}_1 + \mathbf{f}_2 + (3 - u - v)\mathbf{s}_1 + (4 - 2u - v)\mathbf{e}_1$$

so that this divisor is pseudo-effective if and only if  $v \leq 4 - 2u - v$ . Moreover, it is nef for  $v \leq 2 - u$ , and its Zariski decomposition for  $v \geq 2 - u$  is

$$\underbrace{(4 - 2u - v)(\mathbf{f}_1 + \mathbf{e}_1) + (3 - u - v)(\mathbf{s}_1 + \mathbf{f}_2)}_{\text{positive part}} + \underbrace{(v - 2 + u)(\mathbf{f}_1 + \mathbf{f}_2)}_{\text{negative part}}.$$

Hence, using Corollary 2.32, we obtain

$$\begin{aligned} S(W_{\bullet, \bullet}^H; Z) &= \frac{3}{32} \int_0^1 \int_0^1 (2uv - 4u - 6v + 10) dv du + \frac{3}{32} \int_0^1 \int_1^2 2(2-v)(3-u-v) dv du \\ &+ \frac{3}{32} \int_1^2 \int_0^{2-u} (2u^2 + 2uv - 12u - 6v + 16) dv du + \frac{3}{32} \int_1^2 \int_{2-u}^{4-2u} 2(3-u-v)(4-2u-v) dv du \\ &= \frac{49}{64} \end{aligned}$$

as required.  $\square$

Now, we are ready to prove that  $X$  is  $K$ -polystable. Suppose that  $X$  is not  $K$ -polystable. Then, by Theorem 2.19, there is a  $G$ -invariant prime divisor  $F$  over  $X$  such that  $\beta(F) \leq 0$ . Let  $\mathcal{Z} = C_X(F)$ . Then  $Z$  is not a surface by Theorem 5.6, so that  $\mathcal{Z}$  and  $\pi(\mathcal{Z})$  are curves, since  $Q$  has no  $G$ -fixed points. Now, applying Lemmas 5.29, 5.30, 5.31, 5.32, 5.33, we get a contradiction with Corollary 2.32, since  $S_X(E_C) < 1$ ,  $S_X(E_L) < 1$ ,  $S_X(S) < 1$  and  $S_X(H) < 1$  by Theorem 5.6. Therefore, the threefold  $\mathbb{N}^\circ 3.15$  is  $K$ -polystable.

### 5.2.6 Family $\mathbb{N}^\circ 4.13$

Let  $X$  be a smooth Fano threefold  $\mathbb{N}^\circ 4.13$ . Then there is a birational morphism  $\pi: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  that is a blow up of a smooth curve  $C$  of degree  $(1, 1, 3)$ . Moreover, one can choose coordinates  $([x_0 : x_1], [y_0 : y_1], [z_0 : z_1])$  on  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  such that the curve  $C$  is given by one of the following two equations:

$$x_0 y_1 - x_1 y_0 = x_0^3 z_0 + x_1^3 z_1 + \lambda(x_0 x_1^2 z_0 + x_0^2 x_1 z_1) = 0 \quad (5.10)$$

for some  $\lambda \in \mathbb{C} \setminus \{\pm 1, \pm 3\}$ , or

$$x_0 y_1 - x_1 y_0 = x_0^3 z_0 + x_1^3 z_1 + x_0 x_1^2 z_0 = 0. \quad (5.11)$$

Note that for any  $\lambda \in \mathbb{C} \setminus \{0, \pm 1, \pm 3\}$ ,  $X_\lambda$  obtained as a blow up of  $C$  given by (5.10), are isomorphic. When  $\lambda = 0$ , the 3-fold  $X_0$ , obtained by blowing up  $C$  given by (5.11), is non-isomorphic to  $X_\lambda$  for  $\lambda \in \mathbb{C} \setminus \{0, \pm 1, \pm 3\}$ . Therefore, we will prove that  $X$  is  $K$ -polystable if  $C$  is given by (5.10) and this would in turn imply

**Corollary 5.34.** *Suppose that  $C$  is given by (5.11). Then  $X$  is strictly  $K$ -semistable.*

*Proof.* Arguing as in the proof of Corollary 5.13, we construct a test configuration for  $X$ , whose special fiber is the threefold  $X_0$ , which is the Fano threefold  $\mathbb{N}^{\circ}4.13$  that is a blow up of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  at the smooth curve given by (5.10) with  $\lambda = 0$ . Assuming that  $X_0$  is K-polystable, we see that  $X$  is strictly K-semistable by Corollary 5.4.  $\square$

From now on, we suppose that the curve  $C$  is given by (5.10). Let  $\overline{R} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be the surface given by the equation  $x_0y_1 - x_1y_0 = 0$  and let  $\text{pr}_3 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be the projection to the third factor. Then  $\overline{R} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , the surface  $\overline{R}$  contains the curve  $C$  and the curve  $C$  is of degree  $(3, 1)$  on the surface  $\overline{R}$ . Moreover, the projection  $\text{pr}_3$  induces a triple cover  $C \rightarrow \mathbb{P}^1$ . If  $\lambda = 0$ , this triple cover is ramified at exactly 2 points, which implies that  $\text{Aut}^0(X) \cong \mathbb{G}_m$  by [CSP19, Corollary 2.7], so that  $X$  is the unique smooth Fano threefold in the family  $\mathbb{N}^{\circ}4.13$  that has an infinite automorphism group [CSP19]. On the other hand, if  $\lambda \neq 0$ , then the triple cover is ramified at 4 distinct points. Now, arguing as in the proof of [CSP19, Corollary 8.12], we see that  $\text{Aut}(X)$  is a finite group provided that  $\lambda \neq 0$ .

Observe that the group  $\text{Aut}(X)$  is actually not trivial for every  $\lambda \in \mathbb{C} \setminus \{\pm 1, \pm 3\}$ . Namely, let  $A_1, A_2$  and  $A_3$  be the automorphisms of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  defined as follows:

$$\begin{aligned} A_1 &: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \mapsto ([x_0 : -x_1], [y_0 : -y_1], [z_0 : -z_1]), \\ A_2 &: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \mapsto ([x_1 : x_0], [y_1 : y_0], [z_1 : z_0]), \\ A_3 &: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) \mapsto ([y_0 : y_1], [x_0 : x_1], [z_0 : z_1]). \end{aligned}$$

Let  $G$  be the subgroup of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$  generated by  $A_1, A_2$  and  $A_3$ . Then  $|G| = 8$ , and the curve  $C$  is  $G$ -invariant, so that the action of the group  $G$  lifts to the threefold  $X$ . Thus, we can identify  $G$  with a subgroup of the group  $\text{Aut}(X)$ .

Let us show that  $X$  is K-polystable, so that  $X$  is K-stable for  $\lambda \neq 0$  by Corollary 2.15.

**Lemma 5.35.** *The following assertions hold:*

1.  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  does not contain  $G$ -fixed points.
2.  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  does not contain  $G$ -invariant irreducible curves of degree  $(d_1, d_2, d_3)$  such that one of the non-negative integers  $d_1, d_2$  or  $d_3$  is zero.
3.  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  contains sixteen  $G$ -invariant irreducible curves of degree  $(1, 1, 1)$ . Four of them lie on  $\overline{R}$ , and the remaining curves intersect  $\overline{R}$  in 2 points.
4. Let  $\Gamma$  be an irreducible curve of degree  $(1, 1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  such that  $\Gamma \not\subset \overline{R}$ . Then either  $\Gamma \cap C = \emptyset$  or  $\Gamma \cap C = \Gamma \cap \overline{R}$ .

*Proof.* Assertions (1) and (2) are obvious. To prove (3) and (4), let  $x = \frac{x_1}{x_0}$ ,  $y = \frac{y_1}{y_0}$ ,  $z = \frac{z_1}{z_0}$  be the non-homogeneous coordinates on each factor of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . There are precisely four irreducible curves of degree  $(1, 1)$  on  $\mathbb{P}_x^1 \times \mathbb{P}_y^1$ , which are invariant under the induced action of the group  $\langle A_1, A_2 \rangle$ . These are the curves given by  $y = \pm x^{\pm 1}$ . Similarly, there are also 4 irreducible curves of degree  $(1, 1)$

on  $\mathbb{P}_x^1 \times \mathbb{P}_z^1$  invariant under the induced action of the group  $\langle A_1, A_3 \rangle$ . These are the curves that are given by  $z = \pm x^{\pm 1}$ . This gives us 16 possibilities for a  $G$ -invariant curve in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  of degree  $(1, 1, 1)$ . These are the curves  $l_{\pm x^{\pm 1}, \pm x^{\pm 1}}$  that are given by  $(y, z) = (\pm x^{\pm 1}, \pm x^{\pm 1})$ , respectively. Four of these curves are contained in the surface  $\overline{R}$ , which is given by  $y = x$ . On the other hand, each of the remaining twelve curves meets  $\overline{R}$  in precisely 2 points. The assertion on the intersection with  $C$  is immediate to check.  $\square$

Now, let us recall from [F16] the descriptions of the Mori cone  $\overline{\text{NE}}(X)$ , the nef cone and the cone of effective divisors of the Fano threefold  $X$ . Let  $l_1, l_2, l_3$  be the proper transforms of curves of degree  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  that meet  $C$ . Denote by  $l_4$  the proper transform of a curve of degree  $(1, 1, 0)$  that is contained in  $\overline{R}$ , and denote by  $l_5$  a curve that contracted by  $\pi$  to a point. Then the cone  $\text{NE}(X)$  is generated by the curves  $l_1, l_2, l_3, l_4$  and  $l_5$ . Let  $H_1, H_2$  and  $H_3$  be general fibers of the del Pezzo fibrations  $\text{pr}_1 \circ \pi, \text{pr}_2 \circ \pi$  and  $\text{pr}_3 \circ \pi$ , where  $\text{pr}_1$  and  $\text{pr}_2$  are projections to the first and the second factors of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , respectively. Denote by  $E_1, E_2$  and  $E_3$  the exceptional divisors of the contractions of the extremal rays generated by  $l_1, l_2$  and  $l_3$ , respectively. Let  $R$  be the proper transform of  $\overline{R}$  in  $X$ . Then  $E_1 \sim 3H_2 + H_3 - E$ ,  $E_2 \sim 3H_1 + H_3 - E$ ,  $R = E_3 \sim H_1 + H_2 - E$ ,

$$\begin{aligned} \text{Nef}(X) = & \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2] + \mathbb{R}_{\geq 0}[H_3] + \\ & + \mathbb{R}_{\geq 0}[2H_1 + H_2 + H_3 - E] + \mathbb{R}_{\geq 0}[H_1 + 2H_2 + H_3 - E] \end{aligned}$$

and

$$\begin{aligned} \text{Eff}(X) = & \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2] + \mathbb{R}_{\geq 0}[H_3] + \mathbb{R}_{\geq 0}[H_1 + H_2 - E] + \\ & + \mathbb{R}_{\geq 0}[3H_1 + H_3 - E] + \mathbb{R}_{\geq 0}[3H_2 + H_3 - E] + \mathbb{R}_{\geq 0}[E]. \end{aligned}$$

**Lemma 5.36.** *Let  $D \neq 0$  be an effective  $G$ -invariant  $\mathbb{Z}$ -divisor on the threefold  $X$ . Suppose that  $-K_X - D$  is big. Then  $D = R$ .*

*Proof.* Since  $-K_X \sim 2R + E + 2H_3$ , the divisor  $D$  must be linearly equivalent to one of the following divisors:  $H_1, H_2, H_3, H_1 + H_3, H_2 + H_3, H_1 + H_2 - E$  or  $H_1 + H_2 + H_3 - E$ . But the linear systems  $|H_1|, |H_2|, |H_3|, |H_1 + H_3|, |H_2 + H_3|, |H_1 + H_2 + H_3 - E|$  do not contain  $G$ -invariant divisors. Thus, we see that  $D \sim H_1 + H_2 - E$ , so that  $D = R$ .  $\square$

In the following result and its proof, we use the notations introduced in Section 2.2.3.2.

**Lemma 5.37.** *Let  $Z$  be a  $G$ -invariant irreducible curve in  $R$ . Then  $S(W_{\bullet, \bullet}^R; Z) \leq \frac{27}{52}$ .*

*Proof.* Fix  $x \in \mathbb{R}_{\geq 0}$ . Then the divisor  $-K_X - xR$  is pseudo-effective if and only if  $x \leq 2$ . Let  $P(x) = P(-K_X - xR)$  and  $N(x) = N(-K_X - xR)$ . Then

$$P(x) = \begin{cases} -K_X - xR & \text{if } 0 \leq x \leq 1, \\ (2-x)(H_1 + H_2) + 2H_3 & \text{if } 1 \leq x \leq 2, \end{cases}$$

and  $N(x) = (x - 1)E$  if  $1 \leq x \leq 2$ .

Recall that  $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\ell_1$  and  $\ell_2$  be the rulings of the surface  $R$  such that both  $\text{pr}_1 \circ \pi$  and  $\text{pr}_2 \circ \pi$  contracts  $\ell_1$ , and  $\text{pr}_3 \circ \pi$  contracts  $\ell_2$ . Then  $-K_X|_R \sim -R|_R \sim \ell_1 + \ell_2$ . Let  $\mathcal{C} = R \cap E$ . Then  $\mathcal{C} \sim 3\ell_1 + \ell_2$ . If  $0 \leq x \leq 1$ , then  $P(x)|_R \sim (1 + x)(\ell_1 + \ell_2)$ . Likewise, if  $1 \leq x \leq 2$ , then  $P(x)|_R \sim (4 - 2x)\ell_1 + 2\ell_2$  and  $N(x)|_R = (x - 1)\mathcal{C}$ . Thus, if  $Z = \mathcal{C}$ , then

$$\begin{aligned} S(W_{\bullet, \bullet}^R; Z) &= \frac{3}{26} \int_0^1 \int_0^\infty \text{vol}((1 + x - 3y)\ell_1 + (1 + x - y)\ell_2) dy dx + \\ &+ \frac{3}{26} \int_1^2 \left( (x-1)((4-2x)\ell_1 + 2\ell_2)^2 + \int_0^\infty \text{vol}((4-2x-3y)\ell_1 + (2-y)\ell_2) dy \right) dx = \frac{44}{117} < \frac{27}{52} \end{aligned}$$

by Corollary 2.32. Thus, to complete the proof, we may assume that  $Z \neq \mathcal{C}$ .

Since linear systems  $|\ell_1|$  and  $|\ell_2|$  do not contain  $G$ -invariant curves by Lemma 5.35, we have  $Z \sim b_1\ell_1 + b_2\ell_2$  for some positive integers  $b_1$  and  $b_2$ . By Corollary 2.32, we get

$$\begin{aligned} S(W_{\bullet, \bullet}^R; Z) &= \frac{3}{26} \int_0^1 \int_0^\infty \text{vol}((1 + x - b_1y)\ell_1 + (1 + x - b_2y)\ell_2) dy dx \\ &\quad + \frac{3}{26} \int_1^2 \int_0^\infty \text{vol}((4 - 2x - b_1y)\ell_1 + (2 - b_2y)\ell_2) dy dx \\ &\leq \frac{3}{26} \int_0^1 \int_0^\infty \text{vol}((1+x-y)(\ell_1+\ell_2)) dy dx + \frac{3}{26} \int_1^2 \int_0^\infty \text{vol}((4-2x-y)\ell_1+(2-y)\ell_2) dy dx \\ &= \frac{3}{26} \int_0^1 \int_0^{1+x} 2(1+x-y)^2 dy dx + \frac{3}{26} \int_1^2 \int_0^{4-2x} 2(4-2x-y)(2-y) dy dx = \frac{27}{52}, \end{aligned}$$

which is exactly what we want.  $\square$

Now we are ready to prove

**Theorem 5.38.** *The threefold  $X$  is  $K$ -polystable.*

*Proof.* Suppose that  $X$  is not  $K$ -polystable. By Theorem 2.19, there is  $G$ -invariant prime divisor  $F$  over  $X$  such that  $\beta(F) \leq 0$ . Let  $Z = C_X(F)$ . Then  $Z$  is not a surface by Theorem 5.6. Thus, since  $X$  does not have  $G$ -fixed points by Lemma 5.35, we see that  $Z$  is a  $G$ -invariant irreducible curve. Now, using Lemma 2.22, we get  $\alpha_{G,Z}(X) < \frac{3}{4}$ . By Lemma 2.20, there is a  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $X$  and a positive rational number  $\lambda < \frac{3}{4}$  such that  $D \sim_{\mathbb{Q}} -K_X$ ,  $Z \subseteq \text{Nklt}(X, \lambda D)$ , and  $(X, \lambda D)$  is strictly log canonical at general point of the curve  $Z$ . Then  $\text{Nklt}(X, \lambda D)$  contains no surfaces except possibly the surface  $R$  by Lemma 5.36.

Using Corollary 2.32, Lemma 5.37 and Theorem 5.6, we see that  $Z \not\subset R$ . Hence, using Lemma 5.35 and applying [ACCFKGSSV, Corollary A.1.15] to  $(X, \lambda D)$  and the morphisms  $\text{pr}_1 \circ \pi$ ,  $\text{pr}_2 \circ \pi$  and  $\text{pr}_3 \circ \pi$ , we see that  $\pi(Z)$  is a curve of degree  $(1, 1, 1)$ . Then  $\pi(Z)$  is one of the twelve  $G$ -invariant curves described in Lemma 5.35.

Let  $\varphi: X \rightarrow X'$  be a birational morphism that contracts  $R$  to an ordinary double point, let  $D'$  be the proper transform of the divisor  $D$  on the threefold

$X'$ , and let  $Z' = \varphi(Z)$ . Then  $X'$  is a Fano threefold with terminal Gorenstein singularities, and  $D' \sim_{\mathbb{Q}} -K_{X'}$ . Moreover, the log pair  $(X', \lambda D')$  is strictly log canonical at general point of the curve  $Z'$ , and the locus  $\text{Nklt}(X', \lambda D')$  is one-dimensional. Then  $Z'$  is smooth by [ACCFKGSSV, Corollary A.1.17]. Thus, using Lemma 5.35, we deduce that  $\pi(Z) \cap C$  consists of two points.

Let  $Y$  be the unique surface in  $|H_1 + H_2|$  that contains  $Z$ , let  $\bar{Y}$  be its proper transform on  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , and let  $\varphi: Y \rightarrow \bar{Y}$  be the birational morphism that is induced by  $\pi$ . Then  $\varphi$  is the blow up of the intersection  $C \cap \bar{Y}$ , which consists of two points that are not contained in one ruling of the surface  $\bar{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Then  $Y$  is a sextic del Pezzo surface.

Let us apply results proved in Section 2.2.3.2 to  $Y$  and  $Z$  to derive a contradiction.

Fix a non-negative number  $x$ . Let  $P(x) = P(-K_X - xY)$  and  $N(x) = N(-K_X - xY)$ . Then  $-K_X - xY$  is nef  $\iff x \leq \frac{1}{2}$ , and  $-K_X - xY$  is pseudo-effective  $\iff x \leq 2$ . Using the description of the effective and nef cones above, we have

$$P(x) = \begin{cases} (2-x)(H_1 + H_2) + 2H_3 - E & \text{if } 0 \leq x \leq \frac{1}{2}, \\ (3-x)(H_1 + H_2) + 2H_3 + (2x-2)E & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

and  $N(x) = (2x-1)R$  if  $\frac{1}{2} \leq x \leq 1$ . Using Corollary 2.32, we get  $S(W_{\bullet, \bullet}^Y; Z) \geq 1$ , since we have  $S_X(Y) < 1$  by Theorem 5.6. Let us compute  $S(W_{\bullet, \bullet}^Y; Z)$ .

Let  $e_1$  and  $e_2$  are exceptional curves of the morphism  $\varphi$ , let  $f_1$  and  $f_2$  be the proper transform on  $Y$  of the rulings of the surface  $\bar{Y}$  that are contracted by  $\text{pr}_1$  and  $\text{pr}_2$  and pass through the points  $\varphi(e_1)$  and  $\varphi(e_2)$ , respectively. Then, on the surface  $Y$ , we have  $E|_Y = e_1 + e_2$ ,  $R|_Y = f_1 + f_2$ ,  $H_1|_Y \sim H_2|_Y \sim f_1 + e_1 \sim f_2 + e_2$ . Let  $h_1$  and  $h_2$  be the proper transform on  $Y$  of the rulings of the surface  $\bar{Y}$  that are contracted by the projection  $\text{pr}_3$  and pass through  $\varphi(e_1)$  and  $\varphi(e_2)$ , respectively. Then  $H_3|_Y \sim h_1 + e_1 \sim h_2 + e_2$  and  $Z \sim f_1 + h_2 \sim f_2 + h_1$ . Therefore, if  $0 \leq x \leq \frac{1}{2}$ , then we have  $P(x)|_Y \sim_{\mathbb{R}} (2-2x)f_1 + 2f_2 + (3-2x)e_1 + e_2 + 2h_1$ . Similarly, if  $\frac{1}{2} \leq x \leq 1$ , then  $P(x)|_Y \sim_{\mathbb{R}} (3-4x)f_1 + (3-2x)f_2 + (3-2x)e_1 + e_2 + 2h_1$  and  $N(x)|_Y = (2x-1)(f_1 + f_2)$ . Take  $y \in \mathbb{R}_{\geq 0}$ . Then Corollary 2.32 gives

$$\begin{aligned} S(W_{\bullet, \bullet}^Y; B) &= \frac{3}{26} \int_0^{\frac{1}{2}} \int_0^{\infty} \text{vol}((2-2x)f_1 + 2f_2 + (3-2x)e_1 + e_2 + 2h_1 - yZ) dy dx \\ &+ \frac{3}{26} \int_{\frac{1}{2}}^1 \int_0^{\infty} \text{vol}((3-4x)f_1 + (3-2x)f_2 + (3-2x)e_1 + e_2 + 2h_1 - yZ) dy dx, \end{aligned}$$

where  $e_1, e_2, f_1, f_2, h_1, h_2$  are  $(-1)$ -curves on the surface  $Y$ , and  $Z \sim f_1 + h_2 \sim f_2 + h_1$ . If  $x \leq \frac{1}{2}$  and  $y \leq 1$ , then  $(2-2x)f_1 + 2f_2 + (3-2x)e_1 + e_2 + 2h_1 - yZ$  is nef, so that

$$\text{vol}((2-2x)f_1 + 2f_2 + (3-2x)e_1 + e_2 + 2h_1 - yZ) = 4xy - 8x - 8y + 14.$$

If  $x \leq \frac{1}{2}$  and  $1 \leq y \leq 2$ , then the Zariski decompositions of this divisor is

$$\underbrace{(4-2x-y)(f_1 + e_1) + (2-y)(h_1 + e_1)}_{\text{positive part}} + \underbrace{(y-1)(e_1 + e_2)}_{\text{negative part}},$$

so that its volume is  $2(4 - 2x - y)(2 - y)$ . For  $y > 2$ , this divisor is not pseudo-effective. Similarly, if  $\frac{1}{2} \leq x \leq 1$  and  $0 \leq y \leq 2 - 2x$ , then

$$\text{vol}((3 - 4x)f_1 + (3 - 2x)f_2 + (3 - 2x)e_1 + e_2 + 2h_1) = 4xy - 8x^2 - 8x - 8y + 16.$$

If  $2 - 2x \leq y \leq \min\{2, 6 - 6x\}$ , then the volume of this divisor is  $2(6 - 6x - y)(2 - y)$ . For  $y > \min\{2, 6 - 6x\}$ , this divisor is not pseudo-effective. Now, using Corollary 2.32 and integrating, we get  $S(W_{\bullet, \bullet}^Y; Z) = \frac{257}{312} < 1$ . This shows that  $X$  is K-polystable.  $\square$

Therefore, if  $\lambda \neq 0$ , then  $X$  is K-stable by Corollary 2.15.

### 5.2.7 Family №4.3

Let  $C$  be the curve in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  of degree  $(1, 1, 2)$  given by

$$\begin{cases} x_0y_1 - x_1y_0 = 0, \\ x_0z_1^2 + x_1z_0^2 = 0, \end{cases}$$

where  $[x_0 : x_1]$ ,  $[y_0 : y_1]$  and  $[z_0 : z_1]$  are coordinates on the first, the second and the third factors of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , respectively. Observe that the curve  $C$  is smooth and irreducible. Let  $\pi: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be the blow up of  $C$ . Then  $X$  is the smooth Fano threefold № 4.3.

Let  $G$  be the subgroup of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$  generated by the following transformations:

$$\begin{aligned} \alpha: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) &\mapsto ([x_1 : x_0], [y_1 : y_0], [z_1 : z_0]), \\ \beta: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) &\mapsto ([y_0 : y_1], [x_0 : x_1], [z_0 : z_1]), \\ \gamma_\lambda: ([x_0 : x_1], [y_0 : y_1], [z_0 : z_1]) &\mapsto ([x_0 : \epsilon^2 x_1], [y_0 : \epsilon^2 y_1], [z_0 : \epsilon z_1]), \end{aligned}$$

where  $\epsilon \in \mathbb{C}^*$ . Then  $G \cong (\mathbb{G}_m \times \mu_2) \times \mu_2$ , and  $C$  is  $G$ -invariant, so that the  $G$ -action lifts to the threefold  $X$ .

Let  $R_C$  be the  $G$ -invariant surface  $\{x_0y_1 - x_1y_0 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ , let  $R$  be its proper transform via  $\pi$  on the threefold  $X$ , let  $E$  be the  $\pi$ -exceptional surface, and let  $H_i = (\text{pr}_i \circ \pi)^*(\mathcal{O}_{\mathbb{P}^1}(1))$ , where  $\text{pr}_i: \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is the  $i$ th-projection. Then

$$-K_X \sim 2H_1 + 2H_2 + 2H_3 - E,$$

and  $R \sim H_1 + H_2 - E$ , because  $C \subset R_C$ . Moreover, we have

**Lemma 5.39.** *The following assertions hold:*

1. both  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  and  $X$  do not contain  $G$ -fixed points,
2. if  $Z$  is a  $G$ -invariant curve in  $X$ , then  $H_i \cdot Z \geq 2$  for every  $i \in \{1, 2, 3\}$ ,
3. the linear system  $|H_1 + H_2 + H_3|$  contains no  $G$ -invariant surfaces,
4. if  $D$  is a non-zero effective  $G$ -invariant  $\mathbb{Z}$ -divisor on  $X$  such that  $-K_X - D$  is big, then  $D = R$ .

*Proof.* The first three assertions follow from the study of the  $G$ -action on  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . The remaining assertion immediately follows from the description of the cone of effective divisors of  $X$ , which is given in [F16].  $\square$

In the remaining part of the section, we will prove that  $X$  is  $K$ -polystable using results from Section 2.2.3.2. As usual, we will use notations introduced in this section. We start with

**Lemma 5.40.** *Let  $Z$  be a  $G$ -invariant irreducible curve in  $R$ . Then  $S(W_{\bullet, \bullet}^R; Z) < 1$ .*

*Proof.* Let us determine the (divisorial) Zariski decomposition of the divisor  $-K_X - xR$ , where  $x \in \mathbb{R}_{\geq 0}$  using the description of the  $\text{Nef}(X)$  and  $\text{Eff}(X)$  given in [F16]. Consider the divisor

$$-K_X - xR \sim_{\mathbb{R}} (2-x)H_1 + (2-x)H_2 + 2H_3 + (x-1)E.$$

If  $0 \leq x \leq 1$ , then  $-K_X - xR$  is nef. For  $x > 2$ ,  $-K_X - xR$  is not pseudoeffective. Finally, if  $1 \leq x \leq 2$ , then

$$P(-K_X - xR) = (2-x)H_1 + (2-x)H_2 + 2H_3; \quad N(-K_X - xR) = (x-1)E;$$

where we use notations introduced in Section 2.2.3.2.

Let  $\ell_1$  and  $\ell_2$  be the rulings of the surface  $R \cong \mathbb{P}^1 \times \mathbb{P}^1$  such that  $\ell_2$  is contracted by  $\text{pr}_3 \circ \pi$ , and  $\ell_1$  is contracted by both  $\text{pr}_1 \circ \pi$  and  $\text{pr}_2 \circ \pi$ . Then

$$(-K_X - xR)|_R \sim_{\mathbb{R}} 2\ell_1 + (x+1)\ell_2.$$

Let  $\mathcal{C} = R \cap E$ . Then  $\mathcal{C} \sim 2\ell_1 + \ell_2$ . If  $1 \leq x \leq 2$ , then

$$P(-K_X - xR)|_R \sim_{\mathbb{R}} (4-2x)\ell_1 + 2\ell_2; \quad N(-K_X - xR)|_R = (x-1)\mathcal{C}.$$

Thus, if  $Z = \mathcal{C}$ , then Corollary 2.32 gives

$$\begin{aligned} S(W_{\bullet, \bullet}^R; Z) &= \frac{1}{10} \int_1^2 ((4-2x)\ell_1 + 2\ell_2)^2 (x-1) dx + \\ &\frac{1}{10} \int_0^1 \int_0^\infty \text{vol}(2\ell_1 + (x+1)\ell_2 - yZ) dy dx + \frac{1}{10} \int_1^2 \int_0^\infty \text{vol}((4-2x)\ell_1 + 2\ell_2 - yZ) dy dx = \\ &= \frac{1}{10} \int_1^2 4(4-2x)(x-1) dx + \frac{1}{10} \int_0^1 \int_0^1 2(2-2y)(x+1-y) dy dx + \\ &\quad + \frac{1}{10} \int_1^2 \int_0^{2-x} 2(4-2x-2y)(2-y) dy dx = \frac{29}{60} < 1. \end{aligned}$$

Therefore, to complete the proof, we may assume that  $Z \neq \mathcal{C}$ . Then

$$\begin{aligned} S(W_{\bullet, \bullet}^R; Z) &= \frac{1}{10} \int_0^1 \int_0^\infty \text{vol}(2\ell_1 + (x+1)\ell_2 - yZ) dy dx \\ &+ \frac{1}{10} \int_1^2 \int_0^\infty \text{vol}((4-2x)\ell_1 + 2\ell_2 - yZ) dy dx \leq \frac{1}{10} \int_0^1 \int_0^\infty \text{vol}(2\ell_1 + (x+1)\ell_2 - y(\ell_1 + \ell_2)) dy dx \\ &\quad + \frac{1}{10} \int_1^2 \int_0^\infty \text{vol}((4-2x)\ell_1 + 2\ell_2 - y(\ell_1 + \ell_2)) dy dx \\ &= \frac{1}{10} \int_0^1 \int_0^{x+1} 2(2-y)(x+1-y) dy dx + \int_1^2 \int_0^{4-2x} 2(4-2x-y)(2-y) dy dx = \frac{13}{24} < 1 \end{aligned}$$

by Corollary 2.32. □

Now, we are ready to prove that  $X$  is K-polystable. Suppose that  $X$  is not K-polystable. Then, by Theorem 2.19, there is a  $G$ -equivariant birational morphism  $f: \tilde{X} \rightarrow X$  and a  $G$ -invariant prime divisor  $F \subset \tilde{X}$  such that  $\beta(F) = A_X(F) - S_X(F) \leq 0$ . Let  $Z = f(F)$ . Then  $Z$  is not a surface by Theorem 5.6, so that  $Z$  is a  $G$ -invariant irreducible curve, because  $X$  does not have  $G$ -invariant points by Lemma 5.39. Now, using Corollary 2.32 and Lemma 5.40, we see that  $Z \subset R$ , because  $S_X(R) < 1$  by Theorem 5.6.

Using Lemma 2.22, we get  $\alpha_{G,Z}(X) < \frac{3}{4}$ . By Lemma 2.20, there exists a  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $X$  such that  $D \sim_{\mathbb{Q}} -K_X$  and  $Z \subset \text{Nklt}(X, \lambda D)$  for a positive rational number  $\lambda < \frac{3}{4}$ . By Lemma 5.39, the only possible two-dimensional component of  $\text{Nklt}(X, \lambda D)$  is  $R$ . Since  $Z \not\subset R$ , we conclude that  $Z$  is an irreducible component of the locus  $\text{Nklt}(X, \lambda D)$ . Applying [ACCFKGSSV, Corollary A.1.15] to  $\text{pr}_1 \circ \pi$ ,  $\text{pr}_2 \circ \pi$ ,  $\text{pr}_3 \circ \pi$ , we get  $H_1 \cdot Z \leq 1$ ,  $H_2 \cdot Z \leq 1$ ,  $H_3 \cdot Z \leq 1$ . But this is impossible by Lemma 5.39. The obtained contradiction shows that  $X$  is K-polystable.

### 5.2.8 Family №5.1

This family contains unique smooth threefold. It is K-polystable. To prove this, we have to describe this threefold explicitly and compute its automorphism group. To start with, let  $Q$  be a smooth quadric  $\{x_1x_2 + x_2x_3 + x_3x_1 + yz = 0\} \subset \mathbb{P}^4$ , where  $x_1, x_2, x_3, y$  and  $z$  are homogeneous coordinates on  $\mathbb{P}^4$ . Let  $C$  be the smooth conic in the quadric  $Q$  that is cut out by  $y = z = 0$ , and let  $P_1 = [1 : 0 : 0 : 0 : 0]$ ,  $P_2 = [0 : 1 : 0 : 0 : 0]$ ,  $P_3 = [0 : 0 : 1 : 0 : 0]$ . Then  $C$  contains the points  $P_1, P_2, P_3$ . Let  $\theta: Y \rightarrow Q$  be the blow up of the points  $P_1, P_2, P_3$ , let  $\mathcal{C}$  be the strict transform on  $Y$  of the conic  $C$ , and let  $\eta: X \rightarrow Y$  be the blow up of the curve  $\mathcal{C}$ . Then  $X$  is the unique smooth Fano threefold №5.1.

Now, let us describe  $\text{Aut}(X)$ . Let  $G$  be a subgroup in  $\text{Aut}(Q)$  that is described as

$$G = \left\{ g \in \text{Aut}(Q) \mid g(C) = C \text{ and } g(\{P_1, P_2, P_3\}) = \{P_1, P_2, P_3\} \right\}.$$

Observe that the action of the group  $G$  lifts faithfully on the Fano threefold  $X$ , so that we can identify  $G$  with a subgroup of the automorphism group  $\text{Aut}(X)$ . Moreover, using the description of the Mori cone  $\text{NE}(X)$  given in [F16], we conclude that  $\text{Aut}(X) = G$ . Furthermore, we have  $G \cong \mathfrak{S}_3 \times (\mathbb{G}_m \rtimes \boldsymbol{\mu}_2)$  and  $G$  acts on  $Q$  as follows:

- if  $\sigma \in \mathfrak{S}_3$ , then  $\sigma$  acts by  $[x_1 : x_2 : x_3 : y : z] \mapsto [x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)} : y : z]$ ,
- if  $\lambda \in \mathbb{G}_m$ , then  $\lambda$  acts by  $[x_1 : x_2 : x_3 : y : z] \mapsto [\lambda x_1 : \lambda x_2 : \lambda x_3 : \lambda^2 y : z]$ ,
- if  $\iota \in \boldsymbol{\mu}_2$ , then  $\iota$  acts by  $[x_1 : x_2 : x_3 : y : z] \mapsto [x_1 : x_2 : x_3 : z : y]$ .

Then  $Q$  does not contain  $G$ -invariant points. Let  $Z$  be the smooth conic in  $Q$  that is cut out by  $x_1 - x_3 = x_2 - x_3 = 0$ . Then  $C \cap Z = \emptyset$ .

**Lemma 5.41.** *The curves  $C$  and  $Z$  are the only irreducible  $G$ -invariant curves in  $Q$ .*

*Proof.* Let  $\mathcal{C}$  be a  $G$ -invariant irreducible curve in  $Q$  that is different from  $C$ . Let us show that  $\mathcal{C} = Z$ . Since  $\mathcal{C} \neq C$ , it contains a point  $P = [x_1 : x_2 : x_3 : y : 1]$  with  $y \neq 0$ , which implies that  $\mathcal{C} = \overline{\mathbb{G}_m \cdot P}$ . In particular, for every  $\sigma \in \mathfrak{S}_3$ , there is  $\lambda \in \mathbb{C}^*$  such that

$$[x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)} : y : 1] = \left[ x_1 : x_2 : x_3 : \lambda y : \frac{1}{\lambda} \right] = [\lambda x_1 : \lambda x_2 : \lambda x_3 : \lambda^2 y : 1],$$

so that  $\lambda^2 = 1$ . Now, using  $\sigma = (1, 2)$  and  $\sigma = (2, 3)$ , we see that  $x_1 = x_2 = x_3 \neq 0$ , so that  $\mathcal{C} = Z$ .  $\square$

Let  $\phi_C: Y_C \rightarrow Q$  and  $\phi_Z: Y_Z \rightarrow Q$  be the blow up of the conics  $C$  and  $Z$ , respectively. Denote by  $F_C$  and  $F_Z$  the exceptional surfaces of the blow ups  $\phi_C$  and  $\phi_Z$ , respectively. Observe that the action of the group  $G$  on the quadric  $Q$  lifts to its actions on  $Y_C$  and  $Y_Z$ , and the surfaces  $F_C$  and  $F_Z$  are exceptional  $G$ -invariant prime divisors over  $Q$ .

**Lemma 5.42.** *The only exceptional  $G$ -invariant prime divisors over  $Q$  are  $F_C$  and  $F_Z$ .*

*Proof.* Recall that the center on  $Q$  of a  $G$ -invariant prime divisor over  $Q$  is a  $G$ -invariant irreducible subvariety in  $Q$ . Therefore, by Lemma 5.41, it is enough to show that the surfaces  $F_C$  and  $F_Z$  do not contain proper  $G$ -invariant irreducible subvarieties.

We start with  $F_C$ . Let  $\psi_C: U_C \rightarrow \mathbb{P}^4$  be the blow up of the linear span of the conic  $C$ , i.e. the blow up of the plane  $y = z = 0$ . We have the following  $G$ -equivariant diagram:

$$\begin{array}{ccc} Y_C & \hookrightarrow & U_C \\ \phi_C \downarrow & & \downarrow \psi_C \\ Q & \hookrightarrow & \mathbb{P}^4 \end{array}$$

Let us describe the  $G$ -action on  $U_C$ . The fourfold  $U_C$  can be covered by two charts. The first one is given in  $\mathbb{P}^4 \times \mathbb{A}_y^1$  by  $y = y'z$ , and the second is given  $\mathbb{P}^4 \times \mathbb{A}_z^1$  by  $z = z'y$ . Using these charts, the action of the group  $G$  can be described as follows:

- if  $\sigma \in \mathfrak{S}_3$ , then  $\sigma$  acts by  $([x_1 : x_2 : x_3 : y : z], y') \mapsto ([x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)} : y : z], y')$ ;
- if  $\lambda \in \mathbb{G}_m$ , then  $\lambda$  acts by

$$\left( [x_1 : x_2 : x_3 : y : z], y' \right) \mapsto \left( [x_1 : x_2 : x_3 : \lambda y : \frac{z}{\lambda}], \lambda^2 y' \right);$$

- if  $\iota \in \mu_2$ , then  $\iota$  acts by

$$\left( [x_1 : x_2 : x_3 : y : z], y' \right) \mapsto \left( [x_1 : x_2 : x_3 : z : y], \frac{1}{y'} \right).$$

Let  $E_C$  be the  $\psi_C$ -exceptional divisor. Then  $E_C$  can be identified with  $\mathbb{P}_{x_1, x_2, x_3}^2 \times \mathbb{P}_{\mathbf{y}, \mathbf{z}}^1$ , and  $F_C$  can be identified with its subvariety that is given by  $x_1x_2 + x_2x_3 + x_3x_1 = 0$ . Moreover, the action of the group  $G$  on the threefold  $E_C$  can be described as follows:

- if  $\sigma \in \mathfrak{S}_3$ , then  $\sigma$  acts by

$$\left( [x_1 : x_2 : x_3], [\mathbf{y} : \mathbf{z}] \right) \mapsto \left( [x_{\sigma(1)} : x_{\sigma(2)} : x_{\sigma(3)}], [\mathbf{y} : \mathbf{z}] \right);$$

- if  $\lambda \in \mathbb{G}_m$ , then  $\lambda$  acts by

$$\left( [x_1 : x_2 : x_3], [\mathbf{y} : \mathbf{z}] \right) \mapsto \left( [x_1 : x_2 : x_3], [\lambda \mathbf{y} : \frac{\mathbf{z}}{\lambda}] \right);$$

- if  $\iota \in \mu_2$ , then  $\iota$  acts by

$$\left( [x_1 : x_2 : x_3], [\mathbf{y} : \mathbf{z}] \right) \mapsto \left( [x_1 : x_2 : x_3], [\mathbf{z} : \mathbf{y}] \right).$$

This easily implies that the surface  $F_C$  does not contain irreducible  $G$ -invariant curves, because  $C$  does not have  $\mathfrak{S}_3$ -invariant points. Since  $F_C$  does not contain  $G$ -invariant points, we see that  $F_C$  does not contain proper  $G$ -invariant irreducible subvarieties.

Similarly, we see that  $F_Z$  does not contain proper  $G$ -invariant subvarieties.  $\square$

Now we are ready to prove

**Theorem 5.43.** *The threefold  $X$  is  $K$ -polystable.*

*Proof.* Let  $F$  be a  $G$ -invariant prime divisor over  $X$ . By Theorem 2.19, it is enough to prove that  $\beta(F) > 0$ . If  $F$  is a prime divisor on  $X$ , then  $\beta(F) > 0$  by Theorem 5.6. Therefore, we may assume that  $F$  is exceptional over  $X$ . Let  $\mathcal{Z}$  be the proper transform on  $X$  of the curve  $Z$ , and let  $\sigma: \tilde{X} \rightarrow X$  be the blow-up of the curve  $\mathcal{Z}$ . Then  $F$  is the  $\sigma$ -exceptional surface by Lemma 5.42.

We claim that  $\sigma^*(-K_X) - 2F$  is not big. To prove this fact, observe that there exists the following commutative diagram:

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\sigma} & X & \xrightarrow{\eta} & Y \\ \tilde{\vartheta} \downarrow & & \downarrow \vartheta & & \downarrow \theta \\ \tilde{Y}_C & \xrightarrow{\varsigma} & Y_C & \xrightarrow{\phi_C} & Q \end{array}$$

where  $\vartheta$  is the blow up of the fibers of the projection  $F_C \rightarrow C$  over the points  $P_1, P_2, P_3$ , i.e. the blow up of the preimages of these points via  $\phi_C$ ,  $\varsigma$  is the blow up of the proper transform of the curve  $Z$ , and  $\tilde{\vartheta}$  is the blow up of the preimages of  $P_1, P_2, P_3$  via  $\phi_C \circ \varsigma$ . Thus, if  $\sigma^*(-K_X) - 2F$  is big, then  $\varsigma^*(-K_{Y_C}) - 2\tilde{F}$  is big, where  $\tilde{F}$  is the  $\varsigma$ -exceptional surface. But the pseudoeffective cone of the threefold  $\tilde{Y}_C$  is described in [F16, Section 10]. Note that  $\tilde{Y}_C$  is a smooth Fano threefold  $\mathbb{N}^3.10$ . Now, using [F16, Section 10], we conclude that  $\varsigma^*(-K_{Y_C}) - 2\tilde{F}$  is not big, so that  $\sigma^*(-K_X) - 2F$  is not big either.

We see that the pseudo-effective threshold  $\tau(F) \leq 2$  (see Section 2.2.1). Thus, it follows from [Fu19:2, Lemma 2.1] that  $S_X(F) \leq \frac{3}{4}\tau(F) \leq \frac{3}{2} < 2 = A_X(F)$ , so that  $\beta(F) > 0$ . Hence, the threefold  $X$  is  $K$ -polystable.  $\square$

# Chapter 6

## Higher dimensions

In this chapter, we first study rational homogeneous spaces, whose polarised minimal families of rational curves have been studied in [LM03] and this in turn helps us provide new examples of 2-Fano manifolds. This is joint work with Carolina Araujo, Roya Beheshti, Ana-Maria Castravet, Kelly Jabbusch, Svetlana Makarova, Enrica Mazzon and Libby Taylor (See [?]).

We get the following classification of rational homogeneous spaces of Picard rank 1 that satisfy the  $\mathfrak{F}_2$  condition.

**Theorem 6.1.** *(Also Main Result 6) The following is the complete list of rational homogeneous spaces of Picard rank 1 satisfying the condition  $\mathfrak{F}_2$ :*

- $A_n/P^k$ , for  $k = 1, n$  and for  $n = 2k - 1, 2k$  when  $2 \leq k \leq \frac{n+1}{2}$ ;
- $B_n/P^k$ , for  $k = 1, n$  and for  $2n = 3k + 1$  when  $2 \leq k \leq n - 1$ ;
- $C_n/P^k$ , for  $k = 1, n$  and for  $2n = 3k - 2$  when  $2 \leq k \leq n - 1$ ;
- $D_n/P^k$ , for  $k = 1, n - 1, n$  and for  $2n = 3k + 2$  when  $2 \leq k < n - 1$ ;
- $E_n/P^k$ , for  $n = 6, 7, 8$  and  $k = 1, 2, n$ ;
- $F_4/P^4$ ;
- $G_2/P^k$ , for  $k = 1, 2$ .

However, we get no new examples of Fano manifolds satisfying  $\mathfrak{F}_3$ .

**Theorem 6.2.** *(Also Main Result 7) The only rational homogeneous spaces of Picard rank 1 satisfying  $\mathfrak{F}_3$ , are projective spaces  $\mathbb{P}^n$ ,  $n \geq 3$ , and quadric hypersurfaces  $Q^n \subset \mathbb{P}^{n+1}$ ,  $n \geq 7$ .*

We also go through the list of 2-Fano manifolds with large index in [AC13] and check the  $\mathfrak{F}_3$  condition for those. We obtain the following classification.

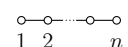
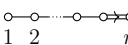
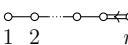
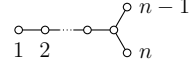
**Theorem 6.3.** *(Also Main Result 8) Let  $X$  be a Fano manifold of dimension  $n \geq 3$  and index  $i_X \geq n - 2$ . If  $X$  satisfies  $\mathfrak{F}_3$ , then  $X$  is isomorphic to one of the following.*

- $\mathbb{P}^n$ .
- *Complete intersections in projective spaces:*
  - *Quadric hypersurfaces  $Q^n \subset \mathbb{P}^{n+1}$  with  $n > 6$ ;*
  - *Complete intersections of quadrics  $X_{2,2} \subset \mathbb{P}^{n+2}$  with  $n > 13$ ;*
  - *Cubic hypersurfaces  $X_3 \subset \mathbb{P}^{n+1}$  with  $n > 25$ ;*
  - *Quartic hypersurfaces in  $\mathbb{P}^{n+1}$  with  $n > 62$ ;*
  - *Complete intersections  $X_{2,3} \subset \mathbb{P}^{n+2}$  with  $n > 32$ ;*
  - *Complete intersections  $X_{2,2,2} \subset \mathbb{P}^{n+3}$  with  $n > 20$ .*
- *Complete intersections in weighted projective spaces:*
  - *Degree 4 hypersurfaces in  $\mathbb{P}(2, 1, \dots, 1)$  with  $n > 55$ ;*
  - *Degree 6 hypersurfaces in  $\mathbb{P}(3, 2, 1, \dots, 1)$  with  $n > 181$ ;*
  - *Degree 6 hypersurfaces in  $\mathbb{P}(3, 1, \dots, 1)$  with  $n > 188$ ;*
  - *Complete intersections of two quadrics in  $\mathbb{P}(2, 1, \dots, 1)$  with  $n > 6$ .*

In Section 2, we provide the necessary background on Chern class computations (Section 2.3), polarised family of minimal rational curves (Section 2.4), Schubert Calculus (Section 2.5), and rational homogeneous spaces (Section 2.6). The proofs of Theorem 6.1 and Theorem 6.2 are given by the various results in Sections 6.1 and 6.2 and the proof of Theorem 6.3 is given in Section ??.

## 6.1 Rational Homogeneous Variety of Classical Type

In this section we consider rational homogeneous spaces constructed from Dynkin diagrams of type  $A, B, C$  and  $D$  as quotients by a maximal parabolic subgroup  $P^k$ . These give rise to the following varieties, see [Man20, §1.1] for a reference:

$A_n$		$A_n/P^k = \text{Gr}(k, n+1)$
$B_n$		for $k < n$ : $B_n/P^k = \text{OG}(k, 2n+1)$ for $k = n$ : $B_n/P^n = \text{OG}_+(n+1, 2(n+1))$
$C_n$		$C_n/P^k = \text{SG}(k, 2n)$
$D_n$		for $k < n-1$ : $D_n/P^k = \text{OG}(k, 2n)$ for $k \in \{n-1, n\}$ : $D_n/P^{n-1} \cong D_n/P^n = \text{OG}_+(n, 2n)$

We recall in a table, results from [AC12, §5] and [AC13, §6.2] on the condition  $\mathfrak{F}_2$  for homogeneous spaces of classical type. Note that we have  $\text{Gr}(1, n) \cong \mathbb{P}^{n-1}$ ,  $\text{SG}(1, n) \cong \mathbb{P}^{n-1}$  ( $n$  even) and  $\text{OG}(1, n) = Q^{n-2}$ .

		dim	$\mathfrak{F}_2$ is satisfied $\iff$
for $2 \leq k \leq \frac{n}{2}$	$\text{Gr}(k, n)$	$k(n - k)$	$n \in \{2k, 2k + 1\}$
for $2 \leq k < \frac{n}{2} - 1$	$\text{OG}(k, n)$	$\frac{k(2n-3k-1)}{2}$	$n = 3k + 2$
	$\text{OG}_+(k, 2k)$	$\frac{k(k-1)}{2}$	$\forall k$
for $2 \leq k \leq \frac{n}{2}$	$\text{SG}(k, n)$	$\frac{k(2n-3k+1)}{2}$	$n = 3k - 2$
	$\text{SG}(k, 2k)$	$\frac{k(k+1)}{2}$	$\forall k$

We continue the previous table, collecting the results on  $(H_x, L_x)$  from [AC12, §5] and Theorem 2.37:

		$(H_x, L_x)$
for $2 \leq k \leq \frac{n}{2}$	$\text{Gr}(k, n)$	$\left( \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}, p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1) \right)$
for $2 \leq k < \frac{n}{2} - 1$	$\text{OG}(k, n)$	$\left( \mathbb{P}^{k-1} \times \mathbb{Q}^{n-2k-2}, p_1^* \mathcal{O}(1) \otimes p_2^* \mathcal{O}(1) \right)$
	$\text{OG}_+(k, 2k)$	$(\text{Gr}(2, k), H)$
for $2 \leq k \leq \frac{n}{2}$	$\text{SG}(k, n)$	$\left( \mathbb{P}_{\mathbb{P}^{k-1}} \left( \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2k} \right), \mathcal{O}_{\mathbb{P}}(1) \right)$
	$\text{SG}(k, 2k)$	$(\mathbb{P}^{k-1}, \mathcal{O}(2))$

### 6.1.1 Type A: Grassmannians

**Proposition 6.4.** *For  $2 \leq k \leq n - 2$ ,  $\text{Gr}(k, n)$  does not satisfy the condition  $\mathfrak{F}_3$ .*

*Proof.* Let  $X = \text{Gr}(k, n)$ . As  $H_x \cong \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$  is a product, by [dJS06a, §3.3] it does not satisfy  $\mathfrak{F}_2$ . By Theorem 2.33 it follows that  $X$  does not satisfy  $\mathfrak{F}_3$ .  $\square$

### 6.1.2 Types B and D: orthogonal Grassmannians

**Proposition 6.5.** *For  $2 \leq k < \frac{n}{2} - 1$ ,  $\text{OG}(k, n)$  does not satisfy the condition  $\mathfrak{F}_3$ .*

*Proof.* Let  $X = \text{OG}(k, n)$ . As  $H_x$  is a product, by [dJS06a, §3.3] it does not satisfy  $\mathfrak{F}_2$ . By Theorem 2.33 it follows that  $X$  does not satisfy  $\mathfrak{F}_3$ .  $\square$

Now we consider  $\text{OG}_+(k, 2k)$ , whose family of minimal rational curves through a general point is  $H_x \cong \text{Gr}(2, k)$ , by Theorem 2.37. We have that  $\text{Gr}(2, k)$  satisfies  $\mathfrak{F}_2$  if and only if  $4 \leq k \leq 5$  by [AC12, §5.2]. Thus, by Theorem 2.33 the condition  $k \in \{4, 5\}$  is necessary for  $\text{OG}_+(k, 2k)$  to satisfy the condition  $\mathfrak{F}_3$ . However, we show that  $\text{OG}_+(k, 2k)$  does not satisfy  $\mathfrak{F}_3$  by considering a 3-cycle whose intersection with  $\text{ch}_3(\text{OG}_+(k, 2k))$  is non-positive. To this end, we start by recalling a result by Coskun about restriction of the Schubert cycles from  $\text{Gr}(k, n)$  to  $\text{OG}_+(k, n)$ .

**Proposition 6.6** ([Cos11, Proposition 6.2]). *Let  $j : \text{OG}(k, n) \hookrightarrow \text{Gr}(k, n)$  be the natural inclusion, and  $\sigma_{\lambda_1, \dots, \lambda_k}$  a Schubert cycle in  $\text{Gr}(k, n)$ . Then*

- (1)  $j^* \sigma_{\lambda_1, \dots, \lambda_k} = 0$  unless  $n - k - i \geq \lambda_i$  for all  $i$  with  $1 \leq i \leq k$ .

(2) Suppose that  $n - k - i > \lambda_i$  for all  $1 \leq i \leq k$ . Then  $j^* \sigma_{\lambda_1, \dots, \lambda_k}$  is effective and nonzero.

(3) Suppose that  $n = 2k$  and  $k - i = \lambda_i$  for all  $1 \leq i \leq k$ . Then  $j^* \sigma_{\lambda_1, \dots, \lambda_k}$  is  $2^{k-1}$  times the Poincaré dual of a point.

From (2.15), we have the following expression for the third Chern character in terms of restrictions of Schubert cycles from  $\text{Gr}(k, 2k)$ :

$$\text{ch}_3(\text{OG}_+(k, 2k)) = -\frac{k+7}{6} j^* \sigma_3 + \frac{k+4}{6} j^* \sigma_{2,1} - \frac{k+1}{6} j^* \sigma_{1,1,1}.$$

**Lemma 6.7.** For  $X = \text{OG}_+(k, 2k)$  with  $k \geq 3$ , we have  $j^* \sigma_3 = j^* \sigma_{1,1,1}$  and

$$\text{ch}_3(X) = \frac{k+4}{2} \left( \frac{1}{6} j^* \sigma_1^3 - j^* \sigma_3 \right).$$

*Proof.* By [AC13, Claim 33] the equality  $j^* \sigma_2 = j^* \sigma_{1,1} = \frac{1}{2} j^* \sigma_1^2$  holds on  $X$ . Applying Pieri's formula we have

$$\begin{aligned} \frac{1}{2} j^* \sigma_1^3 &= j^*(\sigma_1 \cdot \sigma_2) \stackrel{\text{Pieri}}{=} j^* \sigma_{2,1} + j^* \sigma_3 \\ \frac{1}{2} j^* \sigma_1^3 &= j^*(\sigma_1 \cdot \sigma_{1,1}) \stackrel{\text{Pieri}}{=} j^* \sigma_{2,1} + j^* \sigma_{1,1,1}. \end{aligned}$$

This implies that  $j^* \sigma_3 = j^* \sigma_{1,1,1}$  and

$$\text{ch}_3(X) = -\frac{k+7}{6} j^* \sigma_3 + \frac{k+4}{6} \left( \frac{1}{2} j^* \sigma_1^3 - j^* \sigma_3 \right) - \frac{k+1}{6} j^* \sigma_3 = \frac{k+4}{2} \left( \frac{1}{6} j^* \sigma_1^3 - j^* \sigma_3 \right).$$

□

**Proposition 6.8.** For  $k = 4$  or  $5$ ,  $\text{OG}_+(k, 2k)$  does not satisfy the condition  $\mathfrak{F}_3$ .

*Proof.* Consider the codimension 3 Schubert cycle  $\sigma_{2,1}$  on  $\text{Gr}(4, 8)$ , and respectively the codimension 7 Schubert cycle  $\sigma_{3,2,1,1}$  on  $\text{Gr}(5, 10)$ . We have

$$\begin{aligned} \frac{6}{8} \text{ch}_3(\text{OG}_+(4, 8)) \cdot j^* \sigma_{2,1} &\stackrel{\text{Lemma 6.7}}{=} j^* \left( \frac{1}{2} \sigma_1^3 \cdot \sigma_{2,1} - 3 \sigma_3 \cdot \sigma_{2,1} \right) \\ &\stackrel{\text{Pieri + Proposition 6.6}}{=} j^* \left( \frac{1}{2} 6 \sigma_{3,2,1} - 3 \sigma_{3,2,1} \right) = 0 \\ \frac{6}{9} \text{ch}_3(\text{OG}_+(5, 10)) \cdot j^* \sigma_{3,2,1,1} &\stackrel{\text{Lemma 6.7}}{=} j^* \left( \frac{1}{2} \sigma_1^3 \cdot \sigma_{3,2,1,1} - 3 \sigma_3 \cdot \sigma_{3,2,1,1} \right) \\ &\stackrel{\text{Pieri + Proposition 6.6}}{=} j^* \left( \frac{1}{2} 6 \sigma_{4,3,2,1} - 3 \sigma_{4,3,2,1} \right) = 0. \end{aligned}$$

By Proposition 6.6,  $j^* \sigma_{2,1}$  and  $j^* \sigma_{3,2,1,1}$  are nonzero effective 3-cycles which intersect non-positively with  $\text{ch}_3$ , giving that  $\text{OG}_+(k, 2k)$  does not satisfy  $\mathfrak{F}_3$  for  $k = 4, 5$ . □

### 6.1.3 Linear Sections in Orthogonal Grassmanians:

We consider a linear section  $X$  of  $\text{OG}_+(k, 2k)$  of codimension  $c$ , i.e.

$$X = \text{OG}_+(k, 2k) \cap H_1 \cap \dots \cap H_c,$$

where  $H_i \sim \frac{\sigma_1}{2}$  is a hyperplane section of the half-spinor embedding of  $\text{OG}_+(k, 2k)$ . By [AC13, Proposition 34],  $X$  satisfies  $\mathfrak{F}_2$  if and only if  $c < 4$ . By Theorem 2.37 the family of minimal rational curves on  $\text{OG}_+(k, 2k)$  is  $\text{Gr}(2, k)$ . Since these minimal rational curves are lines under the half-spinor embedding, the family of minimal rational curves on  $X$  is

$$H_x = \text{Gr}(2, k) \cap L^c$$

where  $L$  is a hyperplane section under the Plücker embedding.

**Proposition 6.9.** *For  $k = 5$  and  $c < 4$ ,  $\text{OG}_+(5, 10) \cap H^c$  does not satisfy the condition  $\mathfrak{F}_3$ .*

*Proof.* This follows from Theorem 2.33 as  $H_x = \text{Gr}(2, 5) \cap L^c$  does not satisfy  $\mathfrak{F}_2$  by [AC13, Proposition 32 (iv)].  $\square$

### 6.1.4 Type C: Symplectic Grassmanians

**Proposition 6.10.** *For  $2 \leq k < \frac{n}{2}$ ,  $\text{SG}(k, n)$  does not satisfy the condition  $\mathfrak{F}_3$ .*

*Proof.* Let  $X = \text{SG}(k, n)$ . By Theorem 2.33  $X$  does not satisfy  $\mathfrak{F}_3$ , as  $\rho(H_x) > 1$ .  $\square$

Now we consider  $\text{SG}(k, 2k)$  and show that in fact this does not satisfy the condition  $\mathfrak{F}_3$  as well. First, we consider the following result by Coskun.

**Proposition 6.11** ([Cos13, Corollary 3.38]). *Let  $i : \text{SG}(k, 2k) \hookrightarrow \text{Gr}(k, 2k)$  be the natural inclusion, and  $\sigma_{\lambda_1, \dots, \lambda_j}$  a Schubert cycle in  $\text{Gr}(k, 2k)$ . Then  $i^* \sigma_{\lambda_1, \dots, \lambda_j} = 0$  unless  $k + 1 - j \geq \lambda_j$  for all  $j$  with  $1 \leq j \leq k$ .*

**Corollary 6.12.**  $i^* \sigma_{k, k-1, \dots, 2, 1} \neq 0$ .

*Proof.* By Proposition 6.11, the condition on a Schubert cycle to be nonzero on  $\text{SG}(k, 2k)$  is that  $k \geq \lambda_1, k-1 \geq \lambda_2, \dots, 2 \geq \lambda_{k-1}, 1 \geq \lambda_k$ . Therefore the only codimension  $\frac{k(k+1)}{2}$  Schubert cycle whose restriction could be nonzero is  $\sigma_{k, k-1, \dots, 2, 1}$ . By non-degeneracy of the intersection pairing in cohomology on  $\text{Gr}(k, 2k)$ , we have necessarily that  $i^* \sigma_{k, k-1, \dots, 2, 1} \neq 0$ .  $\square$

From (2.16), we have the following expression for the 3-rd Chern character in terms of restrictions of Schubert cycles from  $\text{Gr}(k, 2k)$ :

$$\text{ch}_3(\text{SG}(k, 2k)) = \frac{-k+1}{6} i^* \sigma_3 - \frac{-k+4}{6} i^* \sigma_{2,1} + \frac{-k+7}{6} i^* \sigma_{1,1,1}.$$

**Lemma 6.13.** *For  $X = \text{SG}(k, 2k)$  with  $k \geq 3$ , we have  $i^* \sigma_3 = i^* \sigma_{1,1,1}$  and*

$$\text{ch}_3(X) = \frac{-k+4}{2} i^* \left( \sigma_3 - \frac{1}{6} \sigma_1^3 \right).$$

*Proof.* By [AC13, Claim 35] the equality  $i^*\sigma_2 = i^*\sigma_{1,1} = i^*\frac{1}{2}\sigma_1^2$  holds on  $X$ . Applying Pieri's formula we have

$$\begin{aligned}\frac{1}{2}i^*\sigma_1^3 &= i^*(\sigma_1 \cdot \sigma_2) \stackrel{\text{Pieri}}{=} i^*\sigma_{2,1} + i^*\sigma_3 \\ \frac{1}{2}i^*\sigma_1^3 &= i^*(\sigma_1 \cdot \sigma_{1,1}) \stackrel{\text{Pieri}}{=} i^*\sigma_{2,1} + i^*\sigma_{1,1,1}.\end{aligned}$$

This implies that  $i^*\sigma_3 = i^*\sigma_{1,1,1}$  and

$$\text{ch}_3(X) = \frac{-k+1}{6}i^*\sigma_3 - \frac{-k+4}{6}i^*\left(\frac{1}{2}\sigma_1^3 - \sigma_3\right) + \frac{-k+7}{6}i^*\sigma_3 = \frac{-k+4}{2}i^*\left(\sigma_3 - \frac{1}{6}\sigma_1^3\right).$$

□

**Proposition 6.14.**  $\text{SG}(k, 2k)$  does not satisfy the condition  $\mathfrak{F}_3$ .

*Proof.* We consider the cycle  $\rho := \sigma_{k-1, k-2, k-3, k-3, k-4, \dots, 2, 1}$  on  $\text{Gr}(k, 2k)$ : it has codimension  $\frac{k(k+1)}{2} - 3$  and its restriction  $i^*\rho$  to  $\text{SG}(k, 2k)$  is effective as non-negative linear combination of effective cycles on  $\text{SG}(k, 2k)$  by [Pra00, §1]. By the Pieri rule and Proposition 6.11 we have

$$i^*(\sigma_3 \cdot \rho) = i^*\sigma_{k, k-1, \dots, 2, 1}, \quad i^*(\sigma_1^3 \cdot \rho) = 6 \cdot i^*\sigma_{k, k-1, \dots, 2, 1}.$$

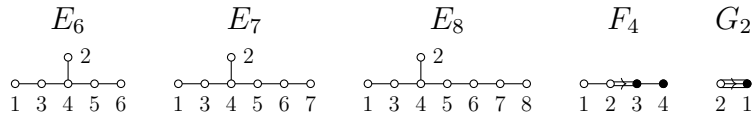
It follows that  $i^*\rho \neq 0$  as  $i^*\sigma_{k, k-1, \dots, 2, 1} \neq 0$  by Corollary 6.12, and

$$\begin{aligned}\text{ch}_3(\text{SG}(k, 2k)) \cdot i^*\rho &\stackrel{\text{Lemma 6.13}}{=} \frac{-k+4}{2} \left(\sigma_3 - \frac{1}{6}\sigma_1^3\right) \cdot i^*\rho \\ &= \frac{-k+4}{2} i^*\left(\sigma_{k, k-1, \dots, 2, 1} - \frac{1}{6}6 \cdot \sigma_{k, k-1, \dots, 2, 1}\right) = 0.\end{aligned}$$

Therefore  $\text{SG}(k, 2k)$  does not satisfy  $\mathfrak{F}_3$ . □

## 6.2 Rational homogeneous varieties of exceptional type

In this section, we show that none of the exceptional groups, when quotiented by a maximal parabolic subgroup, satisfies the condition  $\mathfrak{F}_3$ , and we study when the condition  $\mathfrak{F}_2$  is satisfied. We recall the Dynkin diagrams of exceptional type and mark in black the short roots, i.e. the roots such that there is an arrow in the diagram pointing in its direction.



**Proposition 6.15.**  $E_n/P^\alpha$  with  $\alpha \neq 1, 2, n$  and  $F_4/P^2$  do not satisfy  $\mathfrak{F}_2$ .

*Proof.* Let  $X$  be one of the homogeneous spaces in the statement. By Theorem 2.37,  $H_x$  is a product and thus has Picard rank  $> 1$ . To show that  $X$  does not satisfy  $\mathfrak{F}_2$ , we check that the polarized variety  $(H_x, L_x)$  is not isomorphic to any of the exceptional pairs (a) – (e) from the list in Theorem 2.34:

- For  $X = E_n/P^\alpha$  and  $\alpha = 3, 5$ , we have  $H_x \cong \text{Gr}(2, k) \times \mathbb{P}^l$ , for some  $k = 5, 6, 7$  and  $l = 1, 2, 3$ .
- For  $X = E_n/P^4$ , we have  $H_x \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^{n-4}$ .
- For  $X = E_n/P^6$ ,  $n = 7, 8$ , we have  $H_x \cong D_5/P^5 \times \mathbb{P}^{n-6} \cong \text{OG}_+(5, 10) \times \mathbb{P}^{n-6}$ .
- For  $X = E_8/P^7$ , we have  $H_x \cong E_6/P^6 \times \mathbb{P}^1$ .
- For  $X = F_4/P^2$ , we have  $H_x \cong \mathbb{P}^1 \times \mathbb{P}^2$ . By Theorem 2.37, the embedding of  $H_x$  in  $\mathbb{P}(T_x X)$  is not minimal, and thus  $L_x \neq \mathcal{O}(1, 1)$ .

Thus,  $X$  does not satisfy  $\mathfrak{F}_2$ . □

## 6.2.1 Type E

### 1. Parabolic groups $P^1, P^2$ :

**Proposition 6.16.**  $E_n/P^1$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$  for  $n = 6, 7, 8$ .

*Proof.* Let  $X = E_n/P^1$ . By Theorem 2.37 we have  $H_x = D_{n-1}/P^{n-1} = \text{OG}_+(n-1, 2(n-1))$  and  $L_x$  is a generator of  $\text{Pic}(H_x)$ . Write  $k = n-1$ . We have  $d = \dim(H_x) = \frac{k(k-1)}{2}$  and  $\frac{\sigma_1}{2} \sim L_x$ , as  $\text{Pic}(H_x) = \mathbb{Z}[\frac{\sigma_1}{2}]$ , where we denote by the same symbol the Schubert cycle  $\sigma_1$  in  $\text{Gr}(k, 2k)$  and its restriction to  $\text{OG}_+(k, 2k)$ .

As  $b_4(X) = 1$  by Lemma 2.36, we apply Theorem 2.34 and conclude that  $X$  satisfies  $\mathfrak{F}_2$  as

$$-2K_{H_x} - dL_x = 2 \cdot 2(k-1)L_x - \frac{k(k-1)}{2}L_x = \frac{(k-1)(8-k)}{2}L_x$$

is ample. We compute

$$\begin{aligned} T(\text{ch}_3(X)) &\stackrel{(2.19)}{=} \text{ch}_2(H_x) - \frac{1}{2} \left( c_1(H_x) - \frac{d}{2}c_1(L_x) \right) L_x - \frac{d-4}{12}L_x^2 \\ &= 2L_x^2 - \frac{1}{2} \left( 2(k-1) - \frac{k(k-1)}{4} \right) L_x^2 - \frac{k(k-1)-8}{24}L_x^2 \\ &= \frac{(k-5)(k-8)}{12}L_x^2. \end{aligned}$$

which implies that  $\text{ch}_3(X)$  is not positive, hence  $X$  does not satisfy  $\mathfrak{F}_3$ . □

**Remark 6.17.** As  $E_6/P^6 \cong E_6/P^1$ , Proposition 6.16 holds for  $E_6/P^6$  as well.

**Proposition 6.18.**  $E_n/P^2$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$  for  $n = 6, 7, 8$ .

*Proof.* Let  $X = E_n/P^2$ . By Theorem 2.37 we have  $H_x = A_{n-1}/P^3 = \text{Gr}(3, n)$  and  $L_x$  is a generator of  $\text{Pic}(H_x)$ , therefore  $d = \dim(H_x) = 3(n-3)$  and  $\sigma_1 \sim L_x$ . As  $b_4(X) = 1$  by Lemma 2.36, we apply Theorem 2.34 and conclude that  $X$  satisfies  $\mathfrak{F}_2$  as  $-2K_{H_x} - dL_x = 2nL_x - 3(n-3)L_x = (9-n)L_x$  is ample.

For  $n = 8$ ,  $H_x$  does not satisfy  $\mathfrak{F}_2$ , hence  $X$  does not satisfy  $\mathfrak{F}_3$  by Theorem 2.33. From now on we assume that  $n = 6, 7$ . We have

$$\begin{aligned} T(\text{ch}_3(X)) &\stackrel{(2.19)}{=} \text{ch}_2(H_x) - \frac{1}{2} \left( c_1(H_x) - \frac{d}{2} c_1(L_x) \right) L_x - \frac{d-4}{12} L_x^2 \\ &= \frac{n-4}{2} \sigma_2 - \frac{n-8}{2} \sigma_{1,1} - \frac{1}{2} \left( n - \frac{3(n-3)}{2} \right) \sigma_1^2 - \frac{3n-13}{12} \sigma_1^2 \\ &\stackrel{\text{Pieri}}{=} \left( \frac{n-4}{2} - \frac{n}{2} + \frac{3(n-3)}{4} - \frac{3n-13}{12} \right) \sigma_1^2 - \left( \frac{n-8}{2} + \frac{n-4}{2} \right) \sigma_{1,1} \\ &= \frac{-19+3n}{6} \sigma_1^2 - (n-6) \sigma_{1,1}. \end{aligned}$$

If  $n = 6$  we have  $T(\text{ch}_3(X)) < 0$ , and if  $n = 7$  we have  $T(\text{ch}_3(X)) \cdot \sigma_{4,3,3} = -\frac{2}{3} < 0$ . In both cases, this implies that  $\text{ch}_3(X)$  is not positive, hence  $X$  does not satisfy  $\mathfrak{F}_3$ .  $\square$

## 2. Freudenthal variety :

The homogeneous variety  $E_7/P^7$  is also known as the Freudenthal variety  $G(\mathbb{O}^3, \mathbb{O}^6)$ ; it has dimension 27 and index 18. We refer for instance to [CMP08, §2.1, §2.3] for more details on the geometry of  $E_7/P^7$ .

**Proposition 6.19.**  $E_7/P^7$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .

*Proof.* Let  $X = E_7/P^7$ . By Theorem 2.37 we have  $H_x = E_6/P^6$  and  $L_x$  is a generator of  $\text{Pic}(H_x)$ , therefore  $d = \dim(H_x) = 16$  and  $L_x \sim H$ , as the hyperplane section  $H$  in  $H_x$  is a generator of  $\text{Pic}(H_x)$  by [IM05, Proposition 5.1]. By Lemma 2.36,  $b_4(X) = 1$ , hence we apply Theorem 2.34 and obtain that  $X$  satisfies  $\mathfrak{F}_2$  as

$$2T(\text{ch}_2(H_x)) = -2K_{H_x} - dL_x = 24H - 16H = 8H$$

is ample. We apply Eq. (2.17) to compute:

$$\begin{aligned} T(\text{ch}_3(X)) &\stackrel{(2.19)}{=} \text{ch}_2(H_x) - \frac{1}{2} \left( c_1(H_x) - \frac{d}{2} c_1(L_x) \right) \cdot L_x - \frac{d-4}{12} L_x^2 \\ &\stackrel{\text{Lemma 6.20}}{=} 3H^2 - \frac{1}{2} 4H^2 - H^2 = 0. \end{aligned}$$

We conclude that  $X$  does not satisfy  $\mathfrak{F}_3$ .  $\square$

**Lemma 6.20.**  $\text{ch}_2(E_6/P^6) = 3H^3$  and  $\text{ch}_3(E_6/P^6) = 0$ .

*Proof.* Let  $X = E_6/P^1 = E_6/P^6$ ; as described for instance in [IM05],  $X$  admits an embedding in  $\mathbb{P}^{26}$ . Denote by  $\mathcal{N}$  the normal bundle of  $X$  in  $\mathbb{P}V \cong \mathbb{P}^{26}$ ; by [IM05, Proposition 7.1] we have  $c_1(\mathcal{N}) = 15H$ ,  $c_2(\mathcal{N}) = 102H^2$ , and  $c_3(\mathcal{N}) = 414H^3$ , where  $H$  denotes a hyperplane section. Then we obtain

$$\begin{aligned}\text{ch}_2(X) &= \text{ch}_2(\mathbb{P}^{26})|_X - \text{ch}_2(\mathcal{N}) = 3H^2, \\ \text{ch}_3(X) &= \text{ch}_3(\mathbb{P}^{26})|_X - \text{ch}_3(\mathcal{N}) = 0.\end{aligned}$$

□

**Proposition 6.21.**  $E_8/P^8$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .

*Proof.* Let  $X = E_8/P^8$ . By Theorem 2.37 we have  $H_1 := H_x = E_7/P^7$  the Freundenthal variety, whose corresponding minimal family of rational curves through a general point is  $H_2 = E_6/P^6$ . Thus, we have  $d = \dim(H_1) = 27$ ,  $c_1(H_1) = 18L_1$  and by [CMP08, §2.3]  $\text{Pic}(H_i) = \mathbb{Z}[h_i]$  where  $h_i$  denote a hyperplane class on  $H_i$ ; by Theorem 2.37  $h_i \sim L_i$ .

We have  $b_4(X) = 1$  by Lemma 2.36, hence we apply Theorem 2.34 and obtain that  $X$  satisfies  $\mathfrak{F}_2$  as  $-2K_{H_1} - dL_1 = 36L_1 - 27L_1 = 9L_1$  is ample. We compute

$$\begin{aligned}T(\text{ch}_3(X)) &\stackrel{(2.19)}{=} \text{ch}_2(H_1) - \frac{1}{2} \left( c_1(H_1) - \frac{d}{2}c_1(L_1) \right) L_1 - \frac{d-4}{12}L_1^2 \\ &= \text{ch}_2(H_1) - \frac{9}{4}L_1^2 - \frac{23}{12}L_1^2 = \text{ch}_2(H_1) - \frac{25}{6}L_1^2 \\ T \circ T(\text{ch}_3(X)) &= T(\text{ch}_2(H_1)) - \frac{25}{6}T(L_1^2) \stackrel{\text{Prop (6.19)}}{=} 4L_2 - \frac{25}{6}T(L_1^2) \\ &\leq 4L_2 - \frac{25}{6}L_2 = -\frac{1}{6}L_2\end{aligned}$$

where the last inequality holds by [AC12, Lemma 2.7 (1)]. We conclude that  $X$  does not satisfy  $\mathfrak{F}_3$ . □

## 6.2.2 Type F

**Proposition 6.22.**  $F_4/P^1$  does not satisfy the condition  $\mathfrak{F}_2$ .

*Proof.* Let  $X = F_4/P^1$ . By Theorem 2.37 we have  $H_x = C_3/P^3 = \text{SG}(3, 6)$ , and the embedding of  $H_x$  in  $\mathbb{P}(T_x X)$  is not minimal. Therefore  $\text{Pic}(H_x)$  is not generated by  $[L_x]$ . Since this pair  $(H_x, L_x)$  is not in the exceptional list of Theorem 2.34, we conclude that  $X$  does not satisfy  $\mathfrak{F}_2$ . □

**Proposition 6.23.**  $F_4/P^3$  does not satisfy the condition  $\mathfrak{F}_2$ .

*Proof.* Let  $X = F_4/P^3$ . By [LM03, Proposition 6.9]  $H_x$  is a nontrivial  $Q^4$ -bundle over  $\mathbb{P}^1$ , in particular  $\rho(H_x) \neq 1$ . Note that none of the exceptional pairs  $(H_x, L_x)$  in Theorem 2.34 admit a nontrivial  $Q^4$ -fibration. We conclude that  $X$  does not satisfy  $\mathfrak{F}_2$ .  $\square$

**Proposition 6.24.**  $F_4/P^4$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .

*Proof.* By [LM03, proof of Proposition 6.5]  $F_4/P^4$  is the generic hyperplane section of  $E_6/P^6$ . We write  $X = E_6/P^6 \cap H$ ,  $Y = E_6/P^6$  and recall that  $\text{Pic}(X) = \mathbb{Z}[H]$ . From Lemma 6.20 and (2.11), we obtain

$$\begin{aligned} \text{ch}_2(X) &= \left( \text{ch}_2(Y) - \frac{1}{2}H^2 \right)_{|Y} = \left( 3H^2 - \frac{1}{2}H^2 \right)_{|Y} = \frac{5}{2}H^2_{|X} \\ \text{ch}_3(X) &= \left( \text{ch}_3(Y) - \frac{1}{6}H^3 \right)_{|Y} = -\frac{1}{6}H^3_{|X}. \end{aligned}$$

This implies that  $X = F_4/P^4$  satisfies  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .  $\square$

### 6.2.3 Type G

**Proposition 6.25.**  $G_2/P^1$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .

*Proof.* By [LM03, §6.1], we have that  $G_2/P^1 = Q^5 \subset \mathbb{P}^6$ . Applying ?? and Eq. (2.11), we conclude that  $G_2/P^1$  satisfies  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .  $\square$

**Proposition 6.26.**  $G_2/P^2$  satisfies the condition  $\mathfrak{F}_2$  but not  $\mathfrak{F}_3$ .

*Proof.* Let  $X = G_2/P^2$ . Then  $X$  satisfies  $\mathfrak{F}_2$  by ?. Moreover, we have

$$\begin{aligned} 6 \cdot \text{ch}_3(X) \cdot c_1^2 &= (c_1^3 - 3c_1c_2 + 3c_3) \cdot c_1^2 = c_1^5 - 3c_1^3c_2 + 3c_3c_1^2 \\ &= 4374 - 3 \cdot 2106 + 3 \cdot 594 = -162 \end{aligned}$$

where the Chern numbers are computed in [KT20, Table 1]. As  $\frac{c_1^2}{27}$  is an integral class by [KT20, §2], we obtain  $\text{ch}_3(X) \cdot \frac{c_1^2}{27} = -1$ , so that  $X$  does not satisfy  $\mathfrak{F}_3$ .  $\square$

## 6.3 Higher Fano Manifolds with High Index

In this section, we will prove Theorem 6.3, which gives the complete classification of 2-Fano manifolds of high index that satisfy the  $\mathfrak{F}_3$  condition.

*Proof of Theorem 6.3:* In order to obtain the list in Theorem 6.3, we go through the list in [AC13, Theorem 3] and we check the  $\mathfrak{F}_3$  condition for each, one by one, using the formulas for the third Chern character given in Section 2.3. These give the various bounds on the dimension  $n$  of  $X$ , as in the list. The remaining homogeneous varieties do not satisfy the condition  $\mathfrak{F}_3$  by Propositions 6.4, 6.8, 6.9, 6.14 and 6.26.  $\square$

# Chapter 7

## Conclusion

The work detailed in this thesis began with the quest to simply understand singular curves and singular del Pezzo surfaces. This led to the work on Curves (Chapter 3), which was built on the work of Prof. Ivan Cheltsov in [C17].

Having done this, the most reasonable varieties to study in order to understand the notion of K-stability of Fano varieties are del Pezzo surfaces, since a lot has been done to determine K-stability of these surfaces. We have looked at understanding these objects using different techniques, namely, determining the applicability of Dervan's criterion, using the Valuative criterion and the stability threshold.

Building upon the work of Cheltsov and Martinez-Garcia in [CM20] on the applicability of Dervan's criterion for smooth del Pezzo surfaces, we have determined the same for del Pezzo surfaces with du Val singular points (Section 4.1.2). This is still work in progress. Just as in [CM20, Theorem 1.6], wherein partial results in the case of smooth cubic surfaces in  $\mathbb{P}^3$  were possible, there is still hope that more examples of K-stable singular del Pezzo surfaces with any given ample divisor, can be given.

On the other hand, in my joint work with Won and Kim, we have answered the Calabi problem (Section 1.2.1.2) for all quasi-smooth, well-formed hypersurfaces of index 2, using the techniques explained in Section 2.2. This problem is still open for such hypersurfaces of higher index and with the recent development of powerful methods, for instance [AZ20], I am certain that one can completely determine the K-stability of such hypersurfaces.

In order to further understand the geometry of these del Pezzo surfaces, we considered the problem of the existence of  $H$ -polar cylinders for any ample divisor  $H$  on a du Val del Pezzo surface, in my joint work with Belousov. In Section 4.3, we have explicitly constructed such cylinders for del Pezzo surfaces with at least one singular point of type  $E_n$ , for  $n = 6, 7$  and  $D_n$ , for  $n = 4, 5, 6, 7$ . In fact, we think that we can prove a much stronger result:

**Conjecture.** *Let  $S$  be a del Pezzo surface with du Val singularities. Suppose  $S$  has a  $(-K_S)$  polar cylinder. Then  $S$  has a  $H$ -polar cylinder for any ample divisor  $H$  on  $S$ .*

In the case of smooth Fano 3-folds, in my joint work with Araujo, Castravet, Cheltsov, Fujita, Kaloghiros, Martinez-Garcia, Shramov and Süß, the Calabi

problem is completely solved for 71 of the 105 deformation families. That is, we know which smooth Fano 3-folds are K-polystable and which are not, for 71 of the deformation families. For the remaining 34 families, Main Result 5 tells us that the general member is K-polystable. We conjecture that for 27 of the 34 remaining families, all members are K-stable.

**Conjecture.** *All smooth Fano threefolds in the deformation families*

*N<sup>o</sup>1.9, N<sup>o</sup>2.1, N<sup>o</sup>2.2, N<sup>o</sup>2.3, N<sup>o</sup>2.4, N<sup>o</sup>2.5, N<sup>o</sup>2.6, N<sup>o</sup>2.7, N<sup>o</sup>2.8,  
N<sup>o</sup>2.9, N<sup>o</sup>2.10, N<sup>o</sup>2.11, N<sup>o</sup>2.12, N<sup>o</sup>2.13, N<sup>o</sup>2.14, N<sup>o</sup>2.15, N<sup>o</sup>2.16, N<sup>o</sup>2.17,  
N<sup>o</sup>2.18, N<sup>o</sup>2.19, N<sup>o</sup>3.2, N<sup>o</sup>3.3, N<sup>o</sup>3.4, N<sup>o</sup>3.6, N<sup>o</sup>3.7, N<sup>o</sup>3.11, N<sup>o</sup>4.1*

*are K-stable and, in particular, they are K-polystable.*

All remaining seven families N<sup>o</sup>1.10, N<sup>o</sup>2.20, N<sup>o</sup>2.21, N<sup>o</sup>2.22, N<sup>o</sup>3.5, N<sup>o</sup>3.8, N<sup>o</sup>3.12 contain non-K-polystable smooth Fano threefolds, but their general members are K-polystable. Conjectural characterizations of their K-polystable members are in [ACCFKGSSV]. It would be very interesting to prove these conjectures and hence completely determine the K-stability of all members of all 105 deformation families.

In higher dimensions, there has been great effort in recent times to define a suitable higher analogue to the Fano condition. As described in Chapter 6, in my joint work with Araujo, Beheshti, Castravet, Jabbusch, Makarova, Mazzon and Taylor, we suggest a possible candidate for such an analogue (Definition 2). This is motivated by the definition of 2-Fano manifolds that was introduced by de Jong and Starr in [dJS06a]. But this is known to impose very severe restrictions on the polarized minimal family of rational curves  $(H_x, L_x)$  (See Section 2.4). In our work (See Chapter 6), we have described many new examples of higher Fano manifolds, including rational homogeneous spaces and have also given a complete classification of 2-Fano rational homogeneous spaces of Picard rank 1 (See Main Result 6). We also get no new examples of rational homogeneous spaces of Picard rank 1 satisfying  $\mathfrak{F}_3$ , other than Projective spaces and quadric hypersurfaces (See Main Result 7). In Main Result 8, we have also given a complete classification of Fano manifolds of large index that satisfy  $\mathfrak{F}_3$ . It would be interesting to next determine which rational homogeneous spaces of Picard rank bigger than 1 satisfy  $\mathfrak{F}_2$  and  $\mathfrak{F}_3$  condition.

Also, in our joint work, we propose the following problem.

**Problem.** *For fixed  $n$ , find the smallest integer  $k = k(n)$  such that the following holds. If  $X$  is an  $n$ -dimensional Fano manifold satisfying condition  $\mathfrak{F}_k$ , then  $X$  is a complete intersection in a weighted projective space. Can this integer  $k$  be chosen independently of  $n$ ?*

This is motivated by the following problem.

**Problem.** *Find examples of Fano manifolds satisfying condition  $\mathfrak{F}_3$  other than complete intersections in weighted projective spaces.*

These questions show that indeed studying such possible ‘analogues’ to the Fano condition in higher dimension, could provide interesting characterisations of projective spaces and complete intersections in terms of positivity of Chern characters.

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