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Subject :- Some Problems in the Theory of the Partial Differential Equations of Mathematical Physics.

Supplement :- Some Applications of Hölder's Inequality.

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This thesis consists only of my own unaided work.

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List of Published Papers.

(The papers marked $^{\times}$ are included in the present thesis; those marked $^{+}$ are contained in the short supplement.)

1. On a Linear Partial Differential Equation of Hyperbolic Type,
Proc. Edin. Math. Soc. (1), **41**, (1922-23), 76-81.

×2. The Conservation Theorems of a Damped Dynamical System,
ibid. (1), **42**, (1923-24), 61-68.

×3. On Self-Adjoint Partial Differential Equations of the Second Order,
ibid. (1), **43**, (1924-25), 35-38.

4. On Binet's Inverse Factorial Series for $\mu(x)$,
ibid. (1), **43**, (1924-25), 103-105.

×5. Partial Differential Equations and the Calculus of Variations,
Proc. Roy. Soc. Edin., **46**, (1925-26), 126-135.

×6. Note on Whittaker's Solution of Laplace's Equation,
Proc. Edin. Math. Soc., (1), **44**, (1925-26), 22-25.

×7. On the Integral Equations for the Lamé Functions,
ibid, (2), **1** (1927), 62-64.

†8. Note on Series of Positive Terms,
Journal Lond. Math Soc., **2**, (1927), 9-12

†9. Note on Series of Positive Terms,
ibid., **3**, (1928), 49-51.

†10. On Fourier Constants,
Proc. Roy. Soc. Edin., **48**, (1927-28), 15-19.

†11. On Hardy's Theory of m-Functions,
Proc. Edin. Math. Soc., (2), **1**, (1928), 129-134.

×12. On Electrostatics in a Gravitational Field,
Proc. Roy. Soc. (A), **118**, (1928), 184-194.

Some Problems in the Theory of the Partial Differential Equations of Mathematical Physics.

During 1923 and the first part of 1924, Professor Whittaker gave, in the Mathematical Institute of Edinburgh University, a course of lectures on linear partial differential equations of the second order and their application to the problems of mathematical physics. My knowledge of this subject is primarily derived from this course of lectures, and the problems discussed in this thesis were suggested by them and by the reading I did in connection with them. Accordingly I should like to take this opportunity of thanking Professor Whittaker for allowing me to attend his lectures and for his kind interest, encouragement and advice to me in my work.

Chapter I.

I The Derivation of Linear Partial Differential Equations from a Calculus of Variations Problem.

§1. Introduction.

Let us consider first, as an illustration, the transverse vibrations of a violin string. If it is of uniform line density ρ, has tension ρc^2 and has its ends fixed at $x = 0$ and $x = l$, then the kinetic and potential energies at time t are $\frac{1}{2}\rho \int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$ and $\frac{1}{2}\rho c^2 \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$, correct to the second order in u, the small transverse displacement at the point x at time t. It follows, from Hamilton's principle, that the vibrations of the string are such that the first variation, calculated according to the methods of the Calculus of Variations, of the double integral

$$\frac{1}{2}\rho c^2 \int_{t_0}^{t} \int_0^l \left[\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2\right] dx\, dt$$

vanishes. This means that

$$\rho c^2 \int_{t_0}^{t} \int_0^l \left[\frac{1}{c^2} \frac{\partial u}{\partial t} \cdot \frac{\partial \delta u}{\partial t} - \frac{\partial u}{\partial x} \cdot \frac{\partial \delta u}{\partial x}\right] dx\, dt = 0$$

and hence, after integration by parts, that

$$\rho c^2 \int_{t_0}^{t} \int_0^l \delta u \left[\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right] dx\, dt = 0.$$

The fact that δu is arbitrary implies that u satisfies the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

It is well known that a P.D.E.[x] derived from a Calculus of Variations problem is self-adjoint.[†] The first fact I noticed in this connection was that the equation[‡]

$$\frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} + \gamma u$$

where α, β, γ are constants is not self-adjoint, and yet may be derived by annulling the variation of the double integral

$$\iint e^{\frac{\beta}{\alpha}t} \left[\left(\frac{\partial u}{\partial x}\right)^2 - \alpha\left(\frac{\partial u}{\partial t}\right)^2 + \gamma u^2 \right] dx . dt .$$

The point is that the equation becomes self-adjoint if we multiply through by the factor $e^{\frac{\beta}{\alpha}t}$; for

$$e^{\beta t/\alpha} \frac{\partial^2 u}{\partial x^2} - \alpha e^{\beta t/\alpha} \frac{\partial^2 u}{\partial t^2} - \beta e^{\beta t/\alpha} \frac{\partial u}{\partial t} - \gamma e^{\beta t/\alpha} u$$

$$\equiv \frac{\partial^2}{\partial x^2}\left(e^{\beta t/\alpha} u\right) - \frac{\partial^2}{\partial t^2}\left(\alpha e^{\beta t/\alpha} u\right) + \frac{\partial}{\partial t}\left(\beta e^{\beta t/\alpha} u\right) - \gamma e^{\beta t/\alpha} u$$

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x In future, the abbreviation 'P.D.E.' for 'partial differential equation' will be used.

† If $F(u) = \sum_{ik} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i B_i \frac{\partial u}{\partial x_i} + Cu$

$$G(u) = \sum_{ik} \frac{\partial^2}{\partial x_i \partial x_k}(A_{ik} u) - \sum_i \frac{\partial}{\partial x_i}(B_i u) + Cu ,$$

$F(u)$ and $G(u)$ are called adjoint differential expressions. If $F(u) \equiv G(u)$, $F(u)$ is said to be self-adjoint. See Goursat, Cours d'Analyse, (1923), 3, pp. 146-147.

‡ This equation occurs in the theory of telegraphy when the cable possesses capacity, self-inductance, resistance, and leakage. See Rayleigh's Sound, 1, p. 467.

The question then arises "How general an equation may be derived by annulling the variation of an integral?" Alternatively "How general a P.D.E. may be made self-adjoint by multiplication by a suitable factor?" An answer is obtained in the remainder of Chapter I of this thesis.

It is not claimed that the derivation of an equation from a Calculus of Variations problem has any physical meaning. Nevertheless such derivation of an equation is not merely a curious result, but has important applications.

The first possible application is in connection with the so-called "direct" method for solving the boundary-value problems of mathematical physics. This method, developed largely by Courant,[x] is a reversal of the ordinary process. Instead of trying to find a solution of the P.D.E. satisfying the

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[x] A slight account of the method is given in Courant and Hilbert, Methoden der mathematischen Physik, (1924), 1, p. 155 - p. 165. A full account is promised in the second volume, not yet published

given boundary conditions, in the direct method we try to find, out of the whole class of functions satisfying the given boundary conditions, that one which makes a minimum the integral which gives rise to the P.D.E. in question.

The second possible application is due to E. Noether[x], who shewed that the knowledge of a group of transformations "admitted" by an integral leads to certain "conservation" theorems relating to the solutions of the P.D.E. obtained by annulling the variation of the integral in question. These conservation theorems play the same part as the integrals of energy, momentum and angular momentum in dynamics. These ideas are applied to a particular problem in Chapter II of this thesis.

That there was need for an investigation into the derivation of a P.D.E. from a Calculus of Variations problem is shewn by a reference to the book by Courant and Hilbert already quoted. On page 205 of volume 1, they purport to obtain

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[x] Gött. Nach., (1918), pp. 238 - 257

an equation of parabolic type by annulling the variation of an integral. Carrying out their working, the equation obtained is

$$c''' \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial c'''}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} + \frac{1}{2}\left(\frac{\partial d'''}{\partial \xi} + \frac{\partial e'''}{\partial \eta} - 2f''' \right) u = 0,$$

which is an ordinary differential equation, not a P.D.E. It will be shewn below that a parabolic equation cannot be obtained from a Calculus of Variations problem.

This section of the thesis has already been published, in a shortened form, in two papers, viz <u>Proc. Edin. Math. Soc.</u>, (1), 43, (1924-25), pp. 35-38, <u>Proc. Roy. Soc. Edin.</u>, 46, (1925-26), pp. 126 – 135. The former deals with the case of constant coefficients, the latter with the more general case; we give below the second of these papers.

§ 2. Notation.†

The independent variables are denoted by $x^1, x^2, \ldots x^n$; the dependent variable is u. The operator $\frac{\partial}{\partial x^\mu}$ is written as D_μ. The equation of the type we are considering may then be written

$$\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} p^{\beta\gamma} D_\beta D_\gamma u + \sum_{\beta=1}^{n} q^\beta D_\beta u + ru = 0 \qquad (2.1)$$

† It is, of course, possible to adopt in this work the Summation Convention of the Tensor Calculus, that when a Greek letter occurs twice in a term, once as index and once as suffix, then the terms obtained by giving all possible values to this Greek letter are to be summed. For the sake of clearness and simplicity this has, however, not been done.

Here $p^{\beta\gamma}$, q^{β}, and r denote functions of the independent variables only, and $p^{\beta\gamma}=p^{\gamma\beta}$; the functions $p^{\beta\gamma}$, q^{β}, and r have no common factor.

Later we discuss systems of equations involving m dependent variables, $u^1, u^2, \ldots u^m$. In order that such a system may be derivable from a problem in the Calculus of Variations it is necessary (but not sufficient) that the system consists of m linearly independent equations. We write this system in the form

$$\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} p^{\beta\gamma}_{\delta\alpha} D_\beta D_\gamma u^\alpha + \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} q^{\beta}_{\delta\alpha} D_\beta u^\alpha + \sum_{\alpha=1}^{m} r_{\delta\alpha} u^\alpha = 0 \quad . \quad . \quad (2.2)$$

where the suffix δ takes the values $1, 2, \ldots m$. Here the coefficients $p^{\beta\gamma}_{\delta\alpha}$, $q^{\beta}_{\delta\alpha}$, $r_{\delta\alpha}$ are functions of the independent variables $x^1, x^2, \ldots x^n$ only.

We also use a symbolism* for denoting the symmetrical and antisymmetrical parts of any quantity with regard to a pair of suffixes. We write

$$r_{(\delta\alpha)} = \tfrac{1}{2}(r_{\delta\alpha} + r_{\alpha\delta}) \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

and

$$r_{[\delta\alpha]} = \tfrac{1}{2}(r_{\delta\alpha} - r_{\alpha\delta}) \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

These are the symmetrical and antisymmetrical parts respectively.

§ 3. The Self-Adjoint Property and its Generalisation.

In order to obtain an equation of the type (2.1) it is necessary to consider the problem of annulling the variation of the integral

$$\int V \,.\, exp\phi \,.\, dx^1 dx^2 \ldots dx^n \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

where ϕ is a function of the independent variables only, whilst V is a quadratic homogeneous expression in u and its first derivatives, *i.e.*

$$V = \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} a^{\beta\gamma} D_\beta u \,.\, D_\gamma u + 2\sum_{\beta=1}^{n} b^\beta u \,.\, D_\beta u + cu^2 \quad . \quad . \quad . \quad (3.2)$$

where $a^{\beta\gamma}$, b^β, c are functions of $x^1, x^2, \ldots x^n$ only, and $a^{\beta\gamma}=a^{\gamma\beta}$.

The first variation of (3.1) has the form

$$-2\int \delta u \,.\, exp\phi \Big\{ \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} a^{\beta\gamma} D_\beta D_\gamma u + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} (a^{\beta\gamma} D_\gamma \phi + D_\gamma a^{\beta\gamma}) D_\beta u$$
$$+ \Big(\sum_{\beta=1}^{n}[D_\beta b^\beta + b^\beta D_\beta\phi] - c\Big)u \Big\} dx' \ldots dx^n$$

where δu denotes the variation of u. We have integrated by parts and omitted terms which can be made to vanish by suitably assigning the boundary conditions. Equating to zero the coefficient of δu in the

* See Schouten, *Der Ricci-Kalkül*, p. 25.

integrand, we have as the condition for a stationary value of the integral (3.1) the equation

$$exp\phi \,.\left\{\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}a^{\beta\gamma}\mathrm{D}_\beta\mathrm{D}_\gamma u+\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}(a^{\beta\gamma}\mathrm{D}_\gamma\phi+\mathrm{D}_\gamma a^{\beta\gamma})\mathrm{D}_\beta u + \left(\sum_{\beta=1}^{n}[\mathrm{D}_\beta b^\beta+b^\beta\mathrm{D}_\beta\phi]-c\right)u\right\}=0 \quad . \quad (3.31)$$

The common factor $exp\phi$ is not zero, and may be cancelled out. The resulting equation is

$$\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}a^{\beta\gamma}\mathrm{D}_\beta\mathrm{D}_\gamma u+\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}(a^{\beta\gamma}\mathrm{D}_\gamma\phi+\mathrm{D}_\gamma a^{\beta\gamma})\mathrm{D}_\beta u+\left(\sum_{\beta=1}^{n}[\mathrm{D}_\beta b^\beta+b^\beta\mathrm{D}_\beta\phi]-c\right)u\Big\}=0 \quad . \quad (3.32)$$

The equation (3.31) is self-adjoint, whilst (3.32) is not identical with its adjoint equation.*

It follows then that an equation of the form (2.1) which is obtained by annulling the variation of an integral (3.1) must either be self-adjoint or be rendered self-adjoint after multiplication by a suitable function of the independent variables.

In order that the equation (2.1) may be derivable from a Calculus of Variations problem, it is necessary and sufficient that $a^{\beta\gamma}$, b^β, c, and ϕ can be determined so that

$$\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}p^{\beta\gamma}\mathrm{D}_\beta\mathrm{D}_\gamma u+\sum_{\beta=1}^{n}q^\beta\mathrm{D}_\beta u+ru\equiv$$

$$\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}a^{\beta\gamma}\mathrm{D}_\beta\mathrm{D}_\gamma u+\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}(a^{\beta\gamma}\mathrm{D}_\gamma\phi+\mathrm{D}_\gamma a^{\beta\gamma})\mathrm{D}_\beta\phi+\left(\sum_{\beta=1}^{n}[\mathrm{D}_\beta b^\beta+b^\beta\mathrm{D}_\beta\phi]-c\right)u.$$

On account of the symmetry of $p^{\beta\gamma}$ and $a^{\beta\gamma}$ this condition reduces to the three sets of equations

$$p^{\beta\gamma}=a^{\beta\gamma} \quad . \quad (3.41)$$

$$q^\beta=\sum_{\gamma=1}^{n}(a^{\beta\gamma}\mathrm{D}_\gamma\phi+\mathrm{D}_\gamma a^{\beta\gamma}) \quad . \quad (3.42)$$

$$r=\sum_{\beta=1}^{n}(\mathrm{D}_\beta b^\beta+b^\beta\mathrm{D}_\beta\phi)-c \quad . \quad (3.43)$$

to determine $a^{\beta\gamma}$, b^β, c, and ϕ in terms of $p^{\beta\gamma}$, q^β, and r.

Since equation (3.43) alone involves the quantities b^β and c, it is comparatively unimportant. After the quantities b^β are arbitrarily assigned, and after $a^{\beta\gamma}$ and ϕ are determined from (3.41) and (3.42) the relation (3.43) gives the appropriate value of c.

* See Goursat, *Cours d'Analyse* (third edition) (1923), 3, p. 147, on adjoint equations.

The function ϕ must satisfy the system of equations

$$q^{\beta}-\sum_{\gamma=1}^{n}D_{\gamma}p^{\beta\gamma}=\sum_{\gamma=1}^{n}p^{\beta\gamma}D_{\gamma}\phi \quad . \quad . \quad . \quad . \quad (3.5)$$

where β takes the values 1, 2, . . . n. This system must determine ϕ uniquely, and so must form "a complete system." * As the equations are not in the usual form, we make an obvious transformation. Let us suppose ϕ is determined by a relation of the form

$$f(x^1, x^2, \ldots x^n, \phi)=0.$$

If we write D_0 for $\dfrac{\partial}{\partial\phi}$, the system (3.5) may be written

$$\sum_{\gamma=1}^{n}p^{\beta\gamma}D_{\gamma}f+(q^{\beta}-\sum_{\gamma=1}^{n}D_{\gamma}p^{\beta\gamma})D_0f=0 \quad . \quad . \quad . \quad (3.51)$$

Now let the determinant of the coefficients $p^{\beta\gamma}$ be P, and let the co-factor of $p^{\beta\gamma}$ in P be $P_{\beta\gamma}$. The system of equations (3.51) can be solved for $D_{\alpha}f$ (provided P is not zero); we find that

$$D_{\alpha}f+\frac{1}{P}\sum_{\beta=1}^{n}P_{\alpha\beta}\Big(q^{\beta}-\sum_{\gamma=1}^{n}D_{\gamma}p^{\beta\gamma}\Big)D_0f=0 \quad . \quad . \quad . \quad (3.52)$$

If the system of equations (3.52) forms "a complete Jacobian system," then ϕ is uniquely determined. The condition for the existence of a unique function ϕ is then that the relations

$$D_{\alpha}\Big[\sum_{\gamma=1}^{n}P_{\beta\gamma}\Big(q^{\gamma}-\sum_{\epsilon=1}^{n}D_{\epsilon}p^{\gamma\epsilon}\Big)\Big/P\Big]=D_{\beta}\Big[\sum_{\gamma=1}^{n}P_{\alpha\gamma}\Big(q^{\gamma}-\sum_{\epsilon=1}^{n}D_{\epsilon}p^{\gamma\epsilon}\Big)\Big/P\Big] \quad . \quad (3.6)$$

hold for all pairs of values of α and β from 1 to n.

This may be transformed into

$$\sum_{\lambda=1}^{n}p^{\lambda\alpha}D_{\lambda}\Big(q^{\beta}-\sum_{\mu=1}^{n}D_{\mu}p^{\mu\beta}\Big)-\sum_{\lambda=1}^{n}p^{\lambda\beta}D_{\lambda}\Big(q^{\alpha}-\sum_{\mu=1}^{n}D_{\mu}p^{\mu\alpha}\Big)$$
$$=\frac{1}{P}\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}P_{\mu\nu}\Big(q^{\nu}-\sum_{\pi=1}^{n}D_{\pi}p^{\nu\pi}\Big)\Big(p^{\lambda\alpha}D_{\lambda}p^{\mu\beta}-p^{\lambda\beta}D_{\lambda}p^{\mu\alpha}\Big) \quad . \quad (3.7)$$

for all pairs of values of α and β.

This system of relations (3.7) can be satisfied in various ways, of which the following are particular cases:

(i) $$D_{\gamma}p^{\alpha\beta}=0, \qquad D_{\gamma}q^{\beta}=0 \quad . \quad . \quad . \quad . \quad (3.71)$$

This case, which has already been discussed elsewhere,† is that in which the coefficients of the first and second derivatives are mere constants. It

* Goursat, *Cours d'Analyse*, (third edition), (1918), 2, p. 620, *et seq.*

† E. T. Copson, *Proc. Edin. Math. Soc.*, 43, (1924–25), pp. 35–38.

can be shown easily that, in this case, ϕ is linear in the independent variables $x^1, x^2, \ldots x^n$.

(ii)
$$q^\nu = \sum_{\pi=1}^{n} D_\pi p^{\nu\pi} \quad . \quad . \quad . \quad . \quad . \quad (3.72)$$

in which case the equation (2.1) is self-adjoint; the function ϕ is now a mere constant.

(iii)
$$\sum_{\lambda=1}^{n} p^{\lambda\alpha} D_\lambda p^{\mu\beta} = \sum_{\lambda=1}^{n} p^{\lambda\beta} D_\lambda p^{\mu\alpha} \quad . \quad . \quad . \quad . \quad . \quad (3.731)$$

$$\sum_{\lambda=1}^{n} p^{\lambda\alpha} D_\lambda q^{\beta} = \sum_{\lambda=1}^{n} p^{\lambda\beta} D_\lambda q^{\alpha} \quad . \quad . \quad . \quad . \quad . \quad (3.732)$$

In this case the equations of the system (3.5) are complete *identically* and not in virtue of the equations of the system.

It should be noticed that self-adjoint equations are merely a particular subclass of the much wider class of equations whose coefficients satisfy the relations (3.6).

§ 4. The Parabolic Case.

We have now to consider the equation

$$\sum_{\beta=1}^{n} \sum_{\gamma=1}^{n} p^{\beta\gamma} D_\beta D_\gamma u + \sum_{\beta=1}^{n} q^\beta D_\beta u + ru = 0 \quad . \quad . \quad . \quad . \quad (2.1)$$

when the determinant P of the coefficients $p^{\beta\gamma}$ vanishes identically; in this case, which was omitted in § 3, the equation is of parabolic type.

Here the system of equations (3.5) is not a linearly independent system. Let us suppose * that the co-factors of the elements $p^{1n}, p^{2n}, \ldots p^{nn}$ of the vanishing determinant P do not all vanish. The system (3.5) will reduce to a consistent linearly independent system of $(n-1)$ equations if the determinant

$$\begin{vmatrix} p^{1,1} & , p^{2,1} & , \ldots, p^{n,1} \\ p^{1,2} & , p^{2,2} & , \ldots, p^{n,2} \\ \vdots & \vdots & \vdots \\ p^{1,n-1} & , p^{2,n-1} & , \ldots, p^{n,n-1} \\ q^1 - \sum_{\gamma=1}^{n} D_\gamma p^{1\gamma}, & q^2 - \sum_{\gamma=1}^{n} D_\gamma p^{2\gamma}, & \ldots, q^n - \sum_{\gamma=1}^{n} D_\gamma p^{n,\gamma} \end{vmatrix} \quad . \quad . \quad (4.1)$$

vanishes. We have then to investigate the meaning of the vanishing of this determinant and of P. It is convenient to do this by performing a change of independent variable.

* There is no loss in generality in this supposition, except in the case when all the first minors of the vanishing determinant P vanish. This case is of no very great interest, however.

Let us introduce a new system of independent variables $y^1, y^2, \ldots y^n$, which are functions, as yet arbitrary, of the valuables $x^1, x^2, \ldots x^n$. Let the operator $\frac{\partial}{\partial y^a}$ be written as Δ_a. Then it is easily seen that

$$D_\beta u = \sum_{\epsilon=1}^{n} D_\beta y^\epsilon \,.\, \Delta_\epsilon u.$$

If J^{-1} denote the determinant of the quantities $D_\beta y^\epsilon$, then J is the Jacobian of the variables x^a with respect to the y^β.

The equation (2.1) is then derivable by annulling the variation of the integral

$$\int VJ\, exp\phi \,.\, dy^1 dy^2 \ldots dy^n \quad . \quad . \quad . \quad (4.21)$$

where

$$V = \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} a^{\beta\gamma} D_\beta u \,.\, D_\gamma u + 2\sum_{\beta=1}^{n} b^\beta u \,.\, D_\beta u + cu^2 \quad . \quad . \quad . \quad (3.2)$$

From the vanishing of P and the relation (3.41) it follows that the determinant of $a^{\beta\gamma}$ vanishes.

In terms of the new co-ordinates, we have

$$V = \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n}\sum_{\epsilon=1}^{n}\sum_{\eta=1}^{n} a^{\beta\gamma} D_\beta y^\epsilon \,.\, D_\gamma y^\eta \,.\, \Delta_\epsilon u \,.\, \Delta_\eta u + 2\sum_{\beta=1}^{n}\sum_{\epsilon=1}^{n} b^\beta u D_\beta y^\epsilon \,.\, \Delta_\epsilon u + cu^2 \quad . \quad (4.22)$$

We now try to determine the function y^n so that the equations

$$\sum_{\beta=1}^{n} a^{\beta\gamma} D_\beta y^n = 0$$

may hold for $\gamma = 1, 2, \ldots n$. Since the determinant of $a^{\beta\gamma}$ vanishes whilst the co-factors of $a^{1,n}, a^{2,n}, \ldots a^{n,n}$ are not all zero, we can take the system of $(n-1)$ equations

$$\sum_{\beta=1}^{n} a^{\beta\gamma} D_\beta y^n = 0 \qquad (\gamma = 1, 2, \ldots, n-1) \quad . \quad . \quad . \quad (4.31)$$

which are linearly independent, to define the function y^n.

The system of equations (4.31) determines y^n uniquely if the system is a complete system. The condition for this is that

$$F_a D_\beta F_n - F_\beta D_a F_n = F_n (D_\beta F_a - D_a F_\beta) \quad . \quad . \quad . \quad (4.32)$$

for all pairs of values of a and β from 1 to $n-1$, where F_a is defined by the relation *

$$P_{a\beta} = F_a F_\beta.$$

* That $P_{a\beta}$ has this form follows from Bellavitis' Theorem on vanishing determinants. See *Mem. Istituto, Venezia*, 7 (1857).

If the conditions (4.32) are satisfied, then we may write

$$\mathrm{VJ}\; exp\phi = \sum_{\beta=1}^{n-1}\sum_{\gamma=1}^{n-1} f^{\beta\gamma}\Delta_\beta u\,.\,\Delta_\gamma u + 2\sum_{\beta=1}^{n} g^\beta \Delta_\beta u + hu^2$$

where the coefficients $f^{\beta\gamma}$, g^β, and h are functions of $y^1, y^2, \ldots y^n$. The equation obtained by annulling the variation of the integral (4.21) is

$$\sum_{\beta=1}^{n-1}\sum_{\gamma=1}^{n-1} f^{\beta\gamma}\Delta_\beta\Delta_\gamma u + \sum_{\beta=1}^{n-1}\sum_{\gamma=1}^{n-1}\Delta_\gamma f^{\beta\gamma}\,.\,\Delta_\beta u + \Big(\sum_{\beta=1}^{n}\Delta_\beta g^\beta - h\Big)u = 0.$$

But this is not a partial differential equation of parabolic type in n independent variables; it is a partial differential equation in $(n-1)$ variables, whilst y^n is a mere parameter.

We have then the result :—

If the equation (2.1) *is such that the determinant* P *and the determinant* (4.1) *both vanish and such that the condition* (4.32) *holds, then the equation* (2.1) *involves fewer than* n *independent variables.*

In the case of two independent variables, the condition (4.32) always holds. We can then deduce, by a method of descent from s variables to $s-1$ variables, that *

No differential equation

$$\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} p^{\beta\gamma}\mathrm{D}_\beta\mathrm{D}_\gamma u + \sum_{\beta=1}^{n} q^\beta \mathrm{D}_\beta u + ru = 0$$

of parabolic type can be derived from a Calculus of Variations Problem.

In other words

There is no self-adjoint equation of parabolic type.

§ 5. Systems of Equations.

We are now going to determine the conditions under which the system of m equations

$$\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} p_{\delta\alpha}^{\beta\gamma}\mathrm{D}_\beta\mathrm{D}_\gamma u^\alpha + \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} q_{\delta\alpha}^{\beta}\mathrm{D}_\beta u^\alpha + \sum_{\alpha=1}^{m} r_{\delta\alpha}u^\alpha = 0 \quad . \quad . \quad (2.2)$$

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* *Cf.* Courant u. Hilbert, *Methoden der Mathematischen Physik*, i, p. 205, where we find the statement: "Wir erkennen also, dass elliptische Differentialgleichungen aus einem definiten, hyperbolische aus einem indefiniten und parabolische aus einem semidefiniten Integranden durch Variation entspringen."

This statement is true only as regards the elliptic and hyperbolic cases and follows from the relation $a^{\beta\gamma} = p^{\beta\gamma}$. The semidefinite integrand leads to an equation apparently in n variables, but actually in fewer than n.

where δ takes the values 1, 2, . . . m, may be derived from annulling the variation of an integral

$$\int exp\phi . \mathrm{V} dx^1 dx^2 \ldots dx^n \quad . \quad . \quad . \quad . \quad (5.11)$$

where ϕ is a function of the independent variables only, and where

$$\mathrm{V} = \sum_{\alpha=1}^{m}\sum_{\delta=1}^{m}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} a^{\beta\gamma}_{\alpha\delta}\mathrm{D}_\beta u^\alpha . \mathrm{D}_\gamma u^\delta + 2\sum_{\alpha=1}^{m}\sum_{\delta=1}^{m}\sum_{\beta=1}^{n} b^{\beta}_{\alpha\delta} u^\alpha \mathrm{D}_\beta u^\delta + \sum_{\alpha=1}^{m}\sum_{\delta=1}^{m} c_{\alpha\delta}u^\alpha u^\delta \quad . \quad (5.12)$$

There is obviously no loss in generality in supposing that

$$p^{\beta\gamma}_{\delta\alpha} = p^{\gamma\beta}_{\alpha\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.21)$$

$$a^{\beta\gamma}_{\alpha\delta} = a^{\gamma\beta}_{\delta\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.22)$$

$$c_{\alpha\delta} = c_{\delta\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5{\cdot}23)$$

for any antisymmetrical parts would go out on performing the summations involved in (2.2) and (5.12).

It is easily shown that, after we integrate by parts and neglect a hypersurface integral which may be made to vanish by a suitable choice of boundary conditions, the first variation of (5.11) is of the form

$$-2\int exp\phi\Big[\sum_{\alpha=1}^{m}\sum_{\epsilon=1}^{m}\delta u^\epsilon\Big\{\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} a^{\beta\gamma}_{\alpha\epsilon}\mathrm{D}_\beta\mathrm{D}_\gamma u^\alpha + \sum_{\beta=1}^{n}\Big(\sum_{\gamma=1}^{n}\mathrm{D}_\gamma a^{\beta\gamma}_{\alpha\epsilon} + \sum_{\gamma=1}^{n} a^{\beta\gamma}_{\alpha\epsilon}\mathrm{D}_\gamma\phi + b^{\beta}_{\alpha\epsilon} - b^{\beta}_{\epsilon\alpha}\Big)\mathrm{D}_\beta u^\alpha$$

$$+\Big(\sum_{\beta=1}^{n}\mathrm{D}_\beta b^{\beta}_{\alpha\epsilon} + \sum_{\beta=1}^{n} b^{\beta}_{\alpha\epsilon}\mathrm{D}_\beta\phi - c_{\alpha\epsilon}\Big)u^\alpha\Big\}\Big] dx' \ldots dx^n$$

where δu^ϵ denotes the variation of u^ϵ.

By annulling the variation of the integral (5.11) we obtain the system of equations

$$\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} a^{\beta\gamma}_{\alpha\delta}\mathrm{D}_\beta\mathrm{D}_\gamma u^\alpha + \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n}\Big(\sum_{\gamma=1}^{n}\mathrm{D}_\gamma a^{\beta\gamma}_{\alpha\delta} + \sum_{\gamma=1}^{n} a^{\beta\gamma}_{\alpha\delta}\mathrm{D}_\gamma\phi + b^{\beta}_{\alpha\delta} \cdot b^{\beta}_{\delta\alpha}\Big)\mathrm{D}_\beta u^\alpha$$

$$+\Big(\sum_{\beta=1}^{n}\mathrm{D}_\beta b^{\beta}_{\alpha\delta} + \sum_{\beta=1}^{n} b^{\beta}_{\alpha\delta}\mathrm{D}_\beta\phi - c_{\alpha\delta}\Big)u^\alpha = 0 \quad . \quad . \quad (5.3)$$

where δ takes the values 1, 2, . . . m.

In order that the systems (5.3) and (2,2) may be identical, it is first of all necessary that the system (2.2) may be reducible, by multiplying by suitable factors, to a form in which

$$p^{\beta\gamma}_{\delta\alpha} = p^{\beta\gamma}_{\alpha\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.31)$$

Remembering the notation introduced in § 2 and using the relation

(5.22), we find that the quantities $a^{\beta\gamma}_{\delta\alpha}$, $b^{\beta}_{\alpha\delta}$, $c_{\alpha\delta}$, and ϕ are to be determined by the relations

$$p^{\beta\gamma}_{\delta\alpha} = a^{\beta\gamma}_{(\delta\alpha)} \quad . \quad . \quad . \quad . \quad . \quad (5.32)$$

$$q^{\beta}_{\delta\alpha} = \sum_{\gamma=1}^{n}\left(D_\gamma a^{\beta\gamma}_{\alpha\delta} + a^{\beta\gamma}_{\alpha\delta} D_\gamma \phi\right) + 2b^{\beta}_{[\alpha\delta]} \quad . \quad . \quad . \quad (5.33)$$

$$r_{\delta\alpha} = \sum_{\beta=1}^{n}\left(D_\beta b^{\beta}_{\alpha\delta} + b^{\beta}_{\alpha\delta} D_\beta \phi\right) - c_{\alpha\delta} \quad . \quad . \quad . \quad . \quad (5.34)$$

We have these three systems of equations (5.32), (5.33), and (5.34) to determine the quantities $a^{\beta\gamma}_{\alpha\delta}$, $b^{\beta}_{\alpha\delta}$, $c_{\alpha\delta}$, and ϕ in terms of $p^{\beta\gamma}_{\delta\alpha}$, $q^{\beta}_{\delta\alpha}$, $r_{\delta\alpha}$.

Eliminating $a^{\beta\gamma}_{\alpha\delta}$ between (5.32) and (5.33) gives the equations

$$q^{\beta}_{(\delta\alpha)} - \sum_{\gamma=1}^{n} D_\gamma p^{\beta\gamma}_{\delta\alpha} = \sum_{\gamma=1}^{n} p^{\beta\gamma}_{\delta\alpha} D_\gamma \phi \quad . \quad . \quad . \quad . \quad (5.41)$$

It can further be shown that

$$2r_{[\delta\alpha]} - \sum_{\beta=1}^{n} D_\beta q^{\beta}_{[\delta\alpha]} = \sum_{\beta=1}^{n} q^{\beta}_{[\delta\alpha]} D_\beta \phi \,. \quad . \quad . \quad . \quad . \quad (5.42)$$

The two sets of relations (5.41) and (5.42) must determine ϕ uniquely. Now the number of equations is $\frac{1}{2}m(mn+m+n-1)$; these must only involve n linearly independent relations. In order that this may be the case, it is necessary that *every determinant which can be formed out of the array*

$$\left\| \begin{matrix} \cdots\, p^{\beta 1}_{\delta\alpha} & , \;\ldots, & q^{1}_{[\delta\alpha]} & , \;\ldots \\ \cdots\, p^{\beta 2}_{\delta\alpha} & , \;\ldots, & q^{2}_{[\delta\alpha]} & , \;\ldots \\ \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot \\ \cdots\, p^{\beta n}_{\delta\alpha} & , \;\ldots, & q^{n}_{[\delta\alpha]} & , \;\ldots \\ \cdots\, q^{\beta}_{(\delta\alpha)} - \sum_{\gamma=1}^{n} D_\gamma p^{\beta\gamma}_{\delta\alpha} & , \;\ldots, & 2r_{[\delta\alpha]} - \sum_{\gamma=1}^{n} D_\gamma q^{\gamma}_{[\delta\alpha]} & , \;\ldots \end{matrix} \right\| \quad . \quad . \quad . \quad . \quad (5.5)$$

must vanish.

If, however, the condition (5.5) does not hold, the equations (5.41) and (5.42) to determine ϕ are inconsistent; the system (2.2) of partial differential equations cannot then be derived from a Calculus of Variations problem.

If the condition (5.5) holds, we can choose out of the systems (5.41) and (5.42) n linearly independent equations to determine ϕ. In order that ϕ may be uniquely determined it is necessary and sufficient that these n equations may form a complete system.

The conditions for completeness can be obtained in a manner similar to that adopted in § 3. If, for example, the determinant P_{11} of the coefficients $p_{11}^{\beta\gamma}$ does not vanish, we may take the linearly independent system

$$q_{11}^{\beta} - \sum_{\gamma=1}^{n} D_{\gamma} p_{11}^{\beta\gamma} = \sum_{\gamma=1}^{n} p_{11}^{\beta\gamma} D_{\gamma} \phi \qquad (\beta = 1, 2, \ldots n)$$

to determine ϕ. This system is complete if

$$\sum^{n} p_{11}^{\lambda\alpha} D_{\lambda}\Big(q_{11}^{\beta} - \sum_{\mu=1}^{n} D_{\mu} p_{11}^{\mu\beta}\Big) - \sum_{\lambda=1}^{n} p_{11}^{\lambda\beta} D_{\lambda}\Big(q_{11}^{\alpha} - \sum_{\mu=1}^{n} D_{\mu} p_{11}^{\mu\alpha}\Big)$$
$$= \frac{1}{P_{11}} \sum_{\lambda=1}^{n} \sum_{\mu=1}^{n} \sum_{\nu=1}^{n} P_{11,\,\mu\nu} \Big(q_{11}^{\nu} - \sum_{\pi=1}^{n} D_{\pi} p_{11}^{\nu\pi}\Big)\Big(p_{11}^{\lambda\alpha} D_{\lambda} p_{11}^{\mu\beta} - p_{11}^{\lambda\beta} D_{\lambda} p_{11}^{\mu\alpha}\Big)$$

holds for all pairs of values of α and β; here $P_{11,\,\mu\nu}$ denotes the co-factor of $p_{11}^{\mu\nu}$ in P_{11}.

§ 6. Particular Cases of § 5.

If P_{11} does not vanish, the conditions for "completeness" may be satisfied by supposing that

$$q_{11}^{\beta} - \sum_{\mu=1}^{n} D_{\mu} p_{11}^{\mu\beta} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6.1)$$

But in this case ϕ is a constant. Hence we have from (5.41) and (5.42)

$$q_{(\delta\alpha)}^{\beta} - \sum_{\gamma=1}^{n} D_{\gamma} p_{\delta\alpha}^{\beta\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6.11)$$

$$2r_{[\delta\alpha]} - \sum_{\beta=1}^{n} D_{\beta} q_{[\delta\alpha]}^{\beta} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6.12)$$

In this case we say that the system (2.2) of partial differential equations forms a self-adjoint system.

In the more general cases it seems to be preferable not to attempt a detailed classification of systems of equations, but to treat each system, as it arises in practical problems, on its own merits by an application of the methods of § 5.

(Issued separately April 6, 1926.)

Chapter II.

II Conservation Theorems

§1. E. Noether's Theorem[×] on Invariant Calculus of Variations Problems.

Suppose that

$$\iint f\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) dx\, dt.$$

"admits" (i.e. is invariant with respect to) the continuous group of transformations whose infinitesimal transformation is

$$X = x + \Delta x, \qquad T = t + \Delta t, \qquad U = u + \Delta u.$$

Write

$$\bar{\delta} u = \Delta u - \frac{\partial u}{\partial x}\Delta x - \frac{\partial u}{\partial t}\Delta t,$$

$$\psi = \frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial(\partial u/\partial x)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial(\partial u/\partial t)}\right);$$

then

$$\psi\, \bar{\delta} u = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial t}.$$

where

$$B_1 = -f\Delta x - \frac{\partial f}{\partial(\partial u/\partial x)}\bar{\delta} u$$

$$B_2 = -f\Delta t - \frac{\partial f}{\partial(\partial u/\partial t)}\bar{\delta} u.$$

Now $\psi = 0$ is the equation obtained by annulling the

× Gött. Nach., (1918), pp. 235-257. A short account of this is given (in a different notation) by Courant and Hilbert, loc. cit., pp. 216-219.

the variation of the given integral.

Hence when u is a solution of $\psi = 0$, it satisfies the divergence relation

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial t} = 0$$

identically; by integrating this divergence relation, we obtain just as many conservation theorems as there are parameters in the group admitted by the given integral.

This is the particular case of E. Noether's theorem, which we are about to apply. The theorem is a generalisation of well known results in dynamics. In the problem of n bodies moving freely under their mutual attractions according to the inverse square law, the integral

$$\int L \, dt,$$

where L is the Lagrangian function, admits the transformations of the ten-parameter Galileo-Newton group. From this, ten first integrals of the equations of motion

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

may be obtained. For example, from the subgroup of rotations, the angular momentum integrals are derived.

§2. Determination of the Infinitesimal Transformation in a Particular Problem.[x]

We have already seen that the equation

$$\frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial^2 u}{\partial t^2} + \gamma\alpha \frac{\partial u}{\partial t} + \beta u$$

(usually known as the equation of telegraphy) where α, β, γ are constants, may be derived by annulling, according to the methods of the Calculus of Variations, the first variation of

$$\iint \left[\left(\frac{\partial u}{\partial x}\right)^2 - \alpha \left(\frac{\partial u}{\partial t}\right)^2 + \beta u^2 \right] e^{\gamma t} \, dx \, dt .$$

To apply E. Noether's theorem we must first find the infinitesimal transformation of the continuous group of transformations which leaves this integral invariant. The appropriate form for this infinitesimal transformation is

$$X = x + \varepsilon \xi$$
$$T = t + \varepsilon \eta$$
$$U = u + \varepsilon u \zeta$$

where ε is a constant whose square is to be neglected, whilst ξ, η, ζ are functions of x and t only. That the

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[x] This work I have already published in the Proc. Edin. Math. Soc. (1) 42, (1923-24), pp. 61-68. As several errors were made in the interpretation of the conservation theorems, the printed form of this work is not given here.

infinitesimal transformation must have this form follows from some work of Lie[x]; with any other form of infinitesimal transformation the resulting P.D.E. would not be linear.

We easily find that the invariance of the given integral implies that

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 - \alpha\left(\frac{\partial u}{\partial t}\right)^2 + \beta u^2\right] e^{\gamma t} =$$

$$\left[\begin{array}{l} \left(\frac{\partial u}{\partial x}\right)^2 - 2\varepsilon\left\{\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial \eta}{\partial x} - \left(\frac{\partial u}{\partial x}\right)^2\zeta - u\frac{\partial u}{\partial x}\frac{\partial \zeta}{\partial x}\right\} \\ - \alpha\left(\frac{\partial u}{\partial t}\right)^2 + 2\alpha\varepsilon\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial \xi}{\partial t} + \left(\frac{\partial u}{\partial t}\right)^2\frac{\partial \eta}{\partial t} - \left(\frac{\partial u}{\partial t}\right)^2\zeta - u\frac{\partial u}{\partial t}\cdot\frac{\partial \zeta}{\partial t}\right\} \\ + \beta u^2 + 2\varepsilon\beta u^2 \end{array}\right].$$

$$\cdot e^{\gamma t}\cdot\left[1 + \varepsilon\left(\gamma\eta + \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial t}\right)\right].$$

Equating the various powers and products of u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial t}$, we find that ξ, η, ζ are given by

$$\begin{aligned} \frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial t} - \gamma\eta - 2\zeta &= 0 \\ \frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial t} + \gamma\eta + 2\zeta &= 0 \\ \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial t} + \gamma\eta + 2\zeta &= 0 \\ \alpha\frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial x} \qquad &= 0 \\ \frac{\partial \zeta}{\partial x} &= 0 \\ \frac{\partial \zeta}{\partial y} &= 0. \end{aligned}$$

Hence ξ, η, ζ are constants such that $\gamma\eta + 2\zeta$ is zero.

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[x] Lie, Leipziger Berichte, (1894-95), p. 322.

We have thus shewn that

$$\iint f \, dx \, dt,$$

where $f = \left[\left(\frac{\partial u}{\partial x}\right)^2 - \alpha \left(\frac{\partial u}{\partial t}\right)^2 + \beta u^2 \right] e^{\gamma t}$, is invariant with respect to the two parameter continuous group of transformations whose infinitesimal transformation is

$$X = x + \varepsilon_1$$
$$T = t + \varepsilon_2$$
$$U = u - \tfrac{1}{2} \gamma \varepsilon_2 u$$

where ε_1 and ε_2 are two constants whose squares and products are to be neglected.

§3. The Conservation Theorems.

In our particular problem

$$f = \left[\left(\frac{\partial u}{\partial x}\right)^2 - \alpha \left(\frac{\partial u}{\partial t}\right)^2 + \beta u^2 \right] e^{\gamma t}$$

$$\Delta x = \varepsilon_1$$

$$\Delta t = \varepsilon_2$$

$$\Delta u = -\tfrac{1}{2} \gamma \varepsilon_2 u$$

$$\bar{\delta} u = -\varepsilon_1 \frac{\partial u}{\partial x} - \varepsilon_2 \left(\frac{\partial u}{\partial t} + \tfrac{1}{2} \gamma u \right)$$

$$\psi = 2 e^{\gamma t} \left[\alpha \frac{\partial^2 u}{\partial t^2} + \gamma \alpha \frac{\partial u}{\partial t} + \beta u - \frac{\partial^2 u}{\partial x^2} \right].$$

Consequently

$$\psi \bar{\delta} u = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial t}$$

where

$$B_1 = \varepsilon_1 e^{\gamma t} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \alpha \left(\frac{\partial u}{\partial t}\right)^2 - \beta u^2 \right] + 2 \varepsilon_2 e^{\gamma t} \frac{\partial u}{\partial x} \left[\frac{\partial u}{\partial t} + \tfrac{1}{2} \gamma u \right]$$

and

$$B_2 = -2\varepsilon_1 e^{\gamma t}\alpha \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t} - \varepsilon_2 e^{\gamma t}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \alpha\left(\frac{\partial u}{\partial t}\right)^2 + \beta u^2 + \gamma\alpha u \frac{\partial u}{\partial t}\right].$$

Since ε_1 and ε_2 are independent, we see that, if u is a solution of

$$\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^2 u}{\partial t^2} + \gamma\alpha\frac{\partial u}{\partial t} + \beta u,$$

then there are two divergence relations

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \alpha\left(\frac{\partial u}{\partial t}\right)^2 - \beta u^2\right] = 2\alpha\left[\frac{\partial}{\partial t} + \gamma\right]\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t},$$

$$\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t} + \gamma u\frac{\partial u}{\partial x}\right] = \left[\frac{\partial}{\partial t} + \gamma\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \alpha\left(\frac{\partial u}{\partial t}\right)^2 + \beta u^2 + \gamma\alpha u\frac{\partial u}{\partial t}\right]$$

From these relations we can obtain by integration two conservation theorems.

§4. The Conservation Theorems of a Parabolic Equation.

If in the equation

$$\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^2 u}{\partial t^2} + \gamma\alpha\frac{\partial u}{\partial t} + \beta u$$

we put $\gamma\alpha = \kappa$, and then let $\alpha \to 0$, $\gamma \to \infty$, whilst κ remains fixed, we obtain the equation of parabolic type

$$\frac{\partial^2 u}{\partial x^2} = \kappa\frac{\partial u}{\partial t} + \beta u.$$

This equation, as we have seen, cannot be derived from a Calculus of Variations problem. It will however possess conservation theorems which may be deduced from those of the previous section by the same limiting process. We find that if u is a solution of

$$\frac{\partial^2 u}{\partial x^2} = \kappa \frac{\partial u}{\partial t} + \beta u$$

then there are two divergence relations

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^2 - \beta u^2\right] = 2\kappa \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}$$

$$\frac{\partial}{\partial x}\left[\kappa u \frac{\partial u}{\partial x}\right] = \kappa\left[\left(\frac{\partial u}{\partial x}\right)^2 + \beta u^2 + \kappa u \frac{\partial u}{\partial t}\right].$$

From these relations we can obtain by integration two conservation theorems.

This rather trivial example shews how conservation theorems may be obtained for equations of parabolic type. In general such equations are limiting forms of equations of non-parabolic type which may be obtained by annulling, according to the methods of the Calculus of Variations, the first variation of an integral. From the divergence relations of the non-parabolic equation those for the parabolic equation may be obtained by a limiting process.

§ 5. The Damped Vibrations of a Violin String.

Consider a violin string whose equilibrium position is along the x axis and whose ends are fixed at $x = 0$ and $x = l$. Using the notation

ρ = density of string

ρc^2 = tension of string

u = small transverse displacement at point x at time t,

we can easily shew that the equation, expressing the rate of change of momentum perpendicular to Ox, when the string vibrates and is damped by a force $\rho\gamma \frac{\partial u}{\partial t}$ (i.e. by a force proportional to the velocity), is, neglecting cubes of u and its derivates,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{c^2}\frac{\partial u}{\partial t}.$$

The divergence relations obtained from this equation are

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2\right] = \frac{2}{c^2}\left[\frac{\partial}{\partial t} + \gamma\right]\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}$$

$$\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t} + \gamma u\frac{\partial u}{\partial x}\right] = \left[\frac{\partial}{\partial t} + \gamma\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{\gamma}{c^2}u\frac{\partial u}{\partial t}\right].$$

Since the only quantities which are of interest are those which persist for all time, we integrate these identities with respect to x between the limits 0 and l, remembering that u and $\frac{\partial u}{\partial t}$ vanish at the ends of the string.

In this way we obtain two conservation theorems which we must interpret physically; these are

$$\left[\tfrac{1}{2}\rho c^2\left(\frac{\partial u}{\partial x}\right)^2\right]_0^l = \left(\frac{d}{dt}+\gamma\right)\int_0^l \rho\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\cdot dx$$

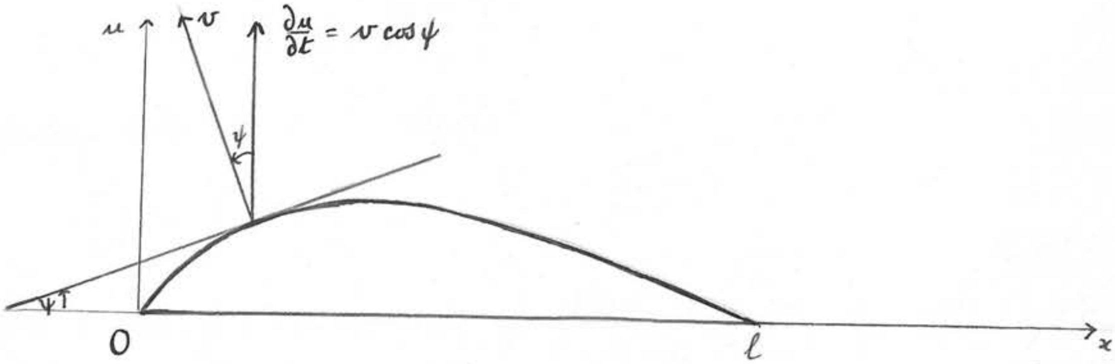
$$0 = \left(\frac{d}{dt}+\gamma\right)\int_0^l \tfrac{1}{2}\rho c^2\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{\gamma}{c^2}u\frac{\partial u}{\partial t}\right]dx.$$

The physical interpretation of the first of these conservation theorems is by no means obvious. But since it corresponds to the infinitesimal transformation

$$\Delta x = \varepsilon_1, \quad \Delta t = 0, \quad \Delta u = 0,$$

it must, from general principles, express in some way the conservation of momentum parallel to Ox.

Now since the string is not capable of executing longitudinal vibrations any point of it is moving in the instantaneous direction of the normal to the curve. (See figure below, which exaggerates the actual displacement.) Let v be the velocity along the



normal. If ψ is the angle the tangent makes with Ox, then we see that

$$\frac{\partial u}{\partial t} = v \cos \psi .$$

Hence the x component of velocity is

$$- v \sin \psi = - \frac{\partial u}{\partial t} \tan \psi = - \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} .$$

Similarly the damping force is actually along the normal and has components $\rho \gamma \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x}$ parallel to Ox and $\rho \gamma \frac{\partial u}{\partial t}$ parallel to Ou. Consequently

$$\left(\frac{d}{dt} + \gamma \right) \int_0^l \rho \, \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial t} \, dx =$$

− rate of change of x-momentum

+ total x-component of damping force.

Again the peg at $x = l$ exerts on the string a force whose x component is

$$\rho c^2 \cos \psi_l = \rho c^2 \left[1 + \left(\frac{\partial u}{\partial x} \right)_l^2 \right]^{-\frac{1}{2}} = \rho c^2 \left[1 - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)_l^2 \right]$$

correct to the second order in the small quantity u. Similarly the peg at $x = 0$ exerts a force on the string which has x component

$$- \rho c^2 \left[1 - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)_0^2 \right]$$

neglecting cubes of u. Hence

$$\left[\frac{1}{2} \rho c^2 \left(\frac{\partial u}{\partial x} \right)^2 \right]_0^l =$$

− (sum of x-components of forces exerted by the pegs).

The first conservation theorem is then the equation of conservation of x-momentum; in words it is

$-$ [x component of forces exerted by pegs]

$= -$ [rate of change of x-momentum]

$+$ [total x-component of damping forces].

The second conservation theorem states that, if

$$E = \int_0^l \tfrac{1}{2}\rho\left[c^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \gamma u \frac{\partial u}{\partial t}\right] dx ,$$

then $E = E_0 e^{-\gamma t}$, where E_0 is the (constant) value of E when $t = 0$. When there is no damping force (i.e. $\gamma = 0$) this shews that E is constant, i.e. the total energy of the system is conserved. The second conservation theorem is the generalisation of the energy integral for this damped dynamical system.

The second conservation theorem may be easily transformed into the more usual form involving Rayleigh's Dissipation Function F.[x] Denote by H the total energy of the system, so that

$$H = \int_0^l \tfrac{1}{2}\rho\left[c^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right] dx .$$

Then our theorem states that

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[x] Proc. Lond. Math. Soc., (1), 4, (1873),

$$-\frac{dH}{dt} = \gamma E + \frac{d}{dt}\int_0^l \tfrac{1}{2}\gamma\rho u \frac{\partial u}{\partial t}\,dx$$

$$= \gamma E + \int_0^l \tfrac{1}{2}\gamma\rho\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^l \tfrac{1}{2}\gamma\rho\left[c^2\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial u}{\partial t}\right] u\,dx$$

$$= \int_0^l \rho\gamma\left(\frac{\partial u}{\partial t}\right)^2 dx$$

on integrating by parts. Hence the total energy of the system is being dissipated at the rate $\int_0^l \rho\gamma\left(\frac{\partial u}{\partial t}\right)^2 dx$,

or
$$\frac{dH}{dt} = -2F$$

where F is Rayleigh's Dissipation function which has here the value $\int_0^l \frac{1}{2}\rho\gamma\left(\frac{\partial u}{\partial t}\right)^2 dx$.

We see then that from the equation which expresses essentially the equation of rate of change of momentum perpendicular to Ox we deduce by means of the theory of groups of transformations the equation of energy and the equation of rate of change of x-momentum.

§6. The Equation of Telegraphy

Consider the propagation of signals in a telegraph cable,[x] which has capacity C, self-inductance L, resistance R, and leakage A, per unit length, where C, L, R and A are constants. If the potential and current at time t at a distance x from one end of the cable are V and j respectively, then the equation of conservation of electric charge is

$$\frac{\partial j}{\partial x} + AV + C\frac{\partial V}{\partial t} = 0 \qquad \text{------} \qquad 6{\cdot}1$$

whilst Neumann's Law of Electromagnetic Induction takes the form

$$L\frac{\partial j}{\partial t} + Rj + \frac{\partial V}{\partial x} = 0 \qquad \text{-----} \qquad 6{\cdot}2.$$

From these two equations we see that

$$V = \frac{\partial \theta}{\partial x} \qquad \text{-----} \qquad 6{\cdot}3$$

$$j = -A\theta - C\frac{\partial \theta}{\partial t} \qquad \text{-----} \qquad 6{\cdot}4$$

where θ is a solution of the equation of telegraphy

$$\frac{\partial^2 \theta}{\partial x^2} = LC\frac{\partial^2 \theta}{\partial t^2} + (LA + RC)\frac{\partial \theta}{\partial t} + RA\theta \qquad \text{----} \qquad 6{\cdot}5.$$

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[x] See Rayleigh's Sound, 1, p. 467.

By integrating the second divergence relation of §3, with the appropriate values of the constants which occur, with respect to x between 0 and ℓ, we obtain

$$\left[2LC\,\frac{\partial\theta}{\partial x}\cdot\frac{\partial\theta}{\partial t} + (LA+RC)\,\theta\,\frac{\partial\theta}{\partial x}\right]_0^{\ell}$$

$$= \left[LC\,\frac{d}{dt} + LA+RC\right]\int_0^{\ell}\left[\left(\frac{\partial\theta}{\partial x}\right)^2 + LC\left(\frac{\partial\theta}{\partial t}\right)^2 + RA\,\theta^2 + (LA+RC)\,\theta\,\frac{\partial\theta}{\partial t}\right]$$

On general principles we should expect this to be the equation of energy, as is indeed the case.

In the simplest case when R and A are zero, this equation becomes, if we use 6·3 and 6·4,

$$-\left[Vj\right]_0^{\ell} = \frac{d}{dt}\int_0^{\ell}\tfrac{1}{2}\left(Lj^2 + CV^2\right)dx.$$

The right hand side represents the rate of change of the total energy, both electrical and magnetic, in the cable, whilst the left hand side represents the rate at which energy enters the cable from the external source of current.

This result also holds in the general case when R and A do not vanish, though the transformation of the integrals becomes rather more complicated.

In the general case, the conservation theorem is

$$\left[-2Vj + (RC - \lambda A)L^{-1}V\theta\right]_0^l$$

$$= \left[C\frac{d}{dt} + (\lambda A + RC)L^{-1}\right]\int_0^l \left[V^2 + \frac{L}{C}j^2 + \frac{\lambda A - RC}{C}\theta j\right] dx$$

The right hand side is

$$\frac{d}{dt}\int_0^l (CV^2 + Lj^2)\,dx + \int_0^l \left[\frac{LA+RC}{L}V^2 + \frac{\lambda A+RC}{C}j^2 + \frac{L^2A^2 - R^2C^2}{\lambda C}\theta j\right]$$

$$+ \int_0^l (\lambda A - RC)\left(\frac{\partial\theta}{\partial t}j + \theta\frac{\partial j}{\partial t}\right) dx$$

$$= \frac{d}{dt}\int_0^l (CV^2 + Lj^2)\,dx +$$

$$\int_0^l \left[\begin{array}{l} \dfrac{LA+RC}{L}V^2 + \dfrac{LA+RC}{C}j^2 + \dfrac{L^2A^2 - R^2C^2}{LC}\theta j - \dfrac{\lambda A - RC}{C}(j^2 + jA\theta) \\ - \dfrac{(\lambda A - RC)}{L}\left(\theta Rj + \theta\dfrac{\partial V}{\partial x}\right)\end{array}\right] dx$$

$$= \frac{d}{dt}\int_0^l (CV^2 + Lj^2)\,dx + 2\int_0^l (AV^2 + Rj^2)\,dx - \left[(\lambda A - RC)L^{-1}\theta V\right]_0^l$$

The conservation theorem is then

$$\left[-Vj\right]_0^l = \int_0^l AV^2\,dx + \int_0^l Rj^2\,dx + \frac{d}{dt}\int_0^l \tfrac{1}{2}(Lj^2 + CV^2)\,dx\,.$$

The left hand side is equal to the rate at which energy enters the cable at the ends from the source of current; the right hand side consists of three terms, of which the first represents the rate at which energy is being lost because of the leakage due to faulty insulation, the second represents the rate at which electrical energy is turned into heat, whilst the last represents the rate of change of the energy resident in the cable. The equation gives a complete account of how the energy put into the cable is used up; it is, in fact, the energy equation.

The other divergence relation does not give a conservation theorem of any physical importance.

§7. The Laminar Motion of a Viscous Fluid.

The laminar motion of a viscous incompressible fluid under no external forces is governed by a parabolic P.D.E. of the type considered in §4.[x]

Suppose that the fluid moves with velocity u parallel to the axis of y, and has no motion parallel to the other axes; u is a function of x and t only. All the stress components vanish except $p_{xy} = \rho\nu \frac{\partial u}{\partial x}$, where ρ is the density of the fluid and ν is Maxwell's kinematic coefficient of viscosity. Then u satisfies the P.D.E.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\nu}\frac{\partial u}{\partial t}.$$

From §4, we obtain the two divergence relations

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x}\right)^2\right] = \frac{2}{\nu}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t},$$

$$\frac{\partial}{\partial x}\left[\frac{1}{\nu}\, u\,\frac{\partial u}{\partial x}\right] = \frac{1}{\nu}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{\nu}\, u\,\frac{\partial u}{\partial t}\right].$$

If we integrate the latter with respect to x between l and l', we obtain

$$\left[u\frac{\partial u}{\partial x}\right]_l^{l'} = \int_l^{l'}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{\nu}\, u\,\frac{\partial u}{\partial t}\right\}dx.$$

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[x] See Lamb's <u>Hydrodynamics</u> (4th Edition), p. 609.

Consequently we have

$$\left[u p_{xy} \right]_{l}^{l'} = \int_{l}^{l'} \nu \rho \left(\frac{\partial u}{\partial x} \right)^2 dx + \frac{d}{dt} \int_{l}^{l'} \tfrac{1}{2} \rho u^2 \, dx .$$

This conservation theorem is the equation of energy. The left-hand-side is the rate at which energy is added to the fluid between the planes $x = l$ and $x = l'$. The second term on the right-hand-side is the rate at which the kinetic energy of the fluid increases, whilst the first term on that side is the rate at which energy is dissipated in the form of heat.[x]

The other divergence relation does not lead to a conservation theorem of any physical importance, just as happened in §6. E. Noether's Theorem gives all the conservation-theorems of a physical system which are of physical importance, but it may, as here, give others of no physical interest. For example, the fact that the calculus of variations problem giving Maxwell's equations admits the fifteen parameter group of conformal transformations of the space-time of special relativity, should give

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[x] See Lamb, <u>loc. cit.</u>, p. 575.

fifteen conservation theorems. Of these only seven are of any physical importance, viz. the equations of conservation of energy, momentum and angular momentum of the electromagnetic field.[×]

[×] See E. Bessel-Hagen, Math. Ann., 84, (1921), pp. 258-276.

Chapter III.

III The General Solution of Laplace's Equation

§1. One of the most important P.D.E.'s of physics is the equation of Laplace,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \cdots\cdots (1\cdot1),$$

which occurs frequently in the theory of Newtonian gravitational attraction, in the theory of the irrotational motion of a perfect fluid, etc. A particular case of this equation, namely

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

possesses as its most general solution

$$f(x+iy) + g(x-iy)$$

where f and g denote arbitrary functions. The question then arises, What is the most general solution of Laplace's equation?

In the year 1902, Professor Whittaker[×] shewed that

$$V = \int_{-\pi}^{\pi} f(z + ix\cos u + iy\sin u,\ u)\, du \quad \cdots\cdots (1\cdot2)$$

is a general solution of Laplace's equation. By a cyclic interchange of x, y, z, the following other

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× Math. Ann., 57, p. 333. The proof occurs also in Modern Analysis, §18.3.

general solutions

$$V = \int_{-\pi}^{\pi} g(x + iy\cos u + iz\sin u, u)\,du \quad ----- \quad (1\cdot3)$$

$$V = \int_{-\pi}^{\pi} h(y + iz\cos u + ix\sin u, u)\,du \quad ---- \quad (1\cdot4)$$

are obtained. Here f, g, h denote arbitrary functions.

Three questions arise in connexion with these three forms of solution :-

(i) Can every solution of Laplace's equation be represented in each of these forms?

(ii) If there exist more than one representation of a given solution, what is the connexion between them?

(iii) If every solution cannot be represented in one of these forms, is there a more general solution than Whittaker's?

By consideration of particular examples, we find that the answer to (i) is "No," & that (ii) if a solution possesses several Whittaker representations, they have different regions of validity and cannot be transformed into each other.

A closer discussion of Whittaker's original proof shews, in answer to (iii), that while it is not possible always to represent a solution of Laplace's equation in the form (1·2), it is always possible to represent it in the form

$$\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f_n(z + ix\cos u + iy\sin u,\ u)\, du,$$

where it may or may not be legitimate to invert the order of integration and summation.

§2. Connexion with the Equation of Wave Motions.

It might be suggested that the solution (1·2) could be generalised by using a contour integral instead of a real integral from $-\pi$ to π. There are however physical reasons why this should not be necessary.

To every solution of the equation of Laplace corresponds a solution of the two-dimensional equation of wave motions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \text{------} \quad (2\cdot1)$$

obtained by putting ict for z. The Whittaker solution of (2·1) is now

$$V = \int_{-\pi}^{\pi} f(ct + x\cos u + y\sin u, u)\, du \quad \ldots\ldots \quad (2\cdot2)$$

But $f(ct + x\cos u + y\sin u, u)$ represents a wave, travelling to the origin, in which the wave front makes an angle u with Oy. Every solution of $(2\cdot1)$ should then, on physical grounds, be representable in the form $(2\cdot2)$ or as the sum of an infinite series of terms of this form. If u be allowed to have complex values, this representation would have no physical meaning.

§3. The fundamental solution $[x^2+y^2+z^2]^{-\frac{1}{2}}$.

This simplest solution of Laplace's equation is $[x^2+y^2+z^2]^{-\frac{1}{2}}$; from it, as Maxwell shewed, all the spherical harmonics of negative degree, can be obtained by differentiations. It seems then to be the most suitable solution for us to discuss first in relation to Whittaker's general solutions.

It can be easily shewn that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [z + ix\cos u + iy\sin u]^{-1}\, du = [x^2+y^2+z^2]^{-\frac{1}{2}} \quad \text{if } z > 0$$

$$= -[x^2+y^2+z^2]^{-\frac{1}{2}} \quad \text{if } z < 0.$$

The solution $[x^2+y^2+z^2]^{-\frac{1}{2}}$ has then two representations in of the form (1·2), each valid in a different half-space. The expression

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{du}{z+ix\cos u+iy\sin u} \qquad \text{- - - -} \qquad (3\cdot 1)$$

is divergent on the place $z=0$ separating the two half-spaces $z>0$ and $z<0$. The two representations are essentially distinct and cannot be transformed into one another. On the plane $z=0$, except at the singular point $(0,0,0)$ of $[x^2+y^2+z^2]^{-\frac{1}{2}}$, the expression (3·1) does, however, exist as a Cauchy principal value; but as this principal value is zero, the integral, regarded as a Cauchy principal value, ceases to represent $[x^2+y^2+z^2]^{-\frac{1}{2}}$.

Hence we see that <u>the solution $[x^2+y^2+z^2]^{-\frac{1}{2}}$ of Laplace's equation possesses six Whittaker representations, valid respectively for $x>0$, $x<0$, $y>0$, $y<0$, $z>0$, $z<0$; these are essentially distinct and cannot be transformed into one another.</u>

§4 The Second Solution of order -1.†

Laplace's equation possesses, besides the solution $[x^2+y^2+z^2]^{-1}$, another solution of degree -1, which becomes infinite at the origin. In terms of spherical-polar coordinates defined by

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta\,,$$

this solution is $r^{-1}Q_0(\cos\theta)$, where $Q_0(\cos\theta)$ denotes the Legendre function of the second kind. This solution has as singular line $x=y=0$, or $\theta=0$. No attempt appears to have been made to represent this solution in Whittaker's form; in this section, the possibility of such representation is investigated.

Instead of this solution, we take the solution which has $\theta=\alpha$, $\phi=0$, as singular line, and then intend to examine what happens as α tends to zero. The solution is

$$r^{-1}Q_0\,(\cos\varpi)$$

where

$$\cos\varpi = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\phi.$$

Using the addition theorem for Legendre functions,* we have
$r^{-1}Q_0\,(\cos\varpi)$

$$= r^{-1}P_0(\cos\theta)\,Q_0(\cos\alpha) + 2\sum_{m=1}^{\infty} r^{-1}P_0^{-m}(\cos\theta)\cos m\phi\, Q_0^{m}(\cos\alpha) \ldots (4\cdot1)$$

In this way, we have expressed our solution as an infinite series of terms which are expressible in the form (1·2). But this representation is only valid in a region for which

$$\theta < \alpha \leqq \pi/2,$$

i.e. inside a circular cone with Oz as axis and the singular line $\theta=\alpha$, $\phi=0$ as generator.

† I have already published the substance of this section in my note, Proc. Edin. Math Soc. (1), 44, (1925-26), pp. 22-25.

* See Whittaker and Watson: *Modern Analysis* (3rd Edn.) 329. We are using Hobson's definition (*Phil. Trans.* A **187** (1896)) of the associated functions.

Now Hobson * has shewn that

$$2\pi P_0^{-m}(\cos\theta)(-1)^m \cos m\phi = \frac{1}{m!}\int_0^{2\pi} \frac{\cos mu\, du}{\cos\theta + i\sin\theta\cos(u-\phi)} \ldots (4{\cdot}2)$$

Hence, if $z > 0$, and if $\theta < \alpha \leq \pi/2$, we have

$$2\pi r^{-1} Q_0(\cos\varpi) = Q_0(\cos\alpha)\int_0^{2\pi} \frac{du}{z + ix\cos u + iy\sin u}$$
$$+ 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} Q_0^m(\cos\alpha)\int_0^{2\pi} \frac{\cos mu\, du}{z + ix\cos u + iy\sin u} \ldots (4{\cdot}3).$$

Before we invert the order of integration and summation, it is necessary to examine the uniformity of convergence of the series

$$Q_0(\cos\alpha) + 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} Q_0^m(\cos\alpha)\cos mu \quad \ldots\ldots\ldots\ldots\ldots (4{\cdot}4)$$

It follows from some recent work† on the asymptotic expansions of the Hypergeometric Function that

$$2\,\frac{(-1)^m}{m!}\,Q_0^m(\cos\alpha) \sim \frac{i^{-m}}{m}\left[\cot^m\frac{\alpha}{2} - (-1)^m\tan^m\frac{\alpha}{2} + O\left(\frac{1}{m}\right)\right]$$

for large values of m. The coefficients of the trigonometric series (4·4) are then not bounded as $m \to \infty$, unless $\alpha = \pi/2$. If $\alpha = \pi/2$, the series (4·4) converges uniformly in the interval $(0, 2\pi)$ except at $u = \pi/2$ and $u = 3\pi/2$, where it has finite discontinuities.

It is then legitimate to invert the order of integration and summation in (4·3), only if $\alpha = \pi/2$. We have then

$$2\pi r^{-1} Q_0(\sin\theta\cos\phi) = \int_0^{2\pi} \frac{f(u)\, du}{z + ix\cos u + iy\sin u}$$

where $f(u)$ is the sum of the series

$$Q_0(0) + 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} Q_0^m(0)\cos mu,$$

provided only that $z > 0$.

We see thus that it is impossible to construct a definite integral representation in finite terms of the form (1·2) of the solution $r^{-1}Q_0(\cos\theta)$.

Representations of the forms (1·3) and (1·4) do, however, exist. For by interchanging z and x in

* loc. cit., p. 499.

† G. N. Watson, Cambridge Phil. Trans., 22, (1918), pp. 277-308.

the above investigation, we find that

$$r^{-1} Q_0(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(u)\,du}{x + iz\cos u + iy\sin u}$$

which is valid for $x > 0$, and similarly that

$$r^{-1} Q_0(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(u)\,du}{y + iz\cos u + ix\sin u}$$

valid for $y > 0$; $f(u)$ denotes, as before, the sum of the series (4·4).

The first of these integrals represents for $x < 0$ the function $-r^{-1}Q_0(\cos\theta)$; on $x = 0$, the integral diverges, but has the Cauchy principal value zero except on the singular line $x = y = 0$. Similar results hold for the second integral.

We see then that <u>a solution of Laplace's equation need not be representable in all of the forms given by Whittaker</u>.

§ 6 Solutions involving Bessel's Functions.

If we write

$$x = \rho\cos\varphi \qquad y = \rho\sin\varphi\,,$$

we know that Laplace's equation possesses solutions $e^{kz}\cdot\genfrac{}{}{0pt}{}{\cos}{\sin} m\varphi\,.\,J_m(k\rho)$ which possess no singularity in the finite part of space, and the corresponding second solutions $e^{kz}\cdot\genfrac{}{}{0pt}{}{\cos}{\sin} m\varphi\,.\,Y_m(k\rho)$, which have a logarithmic singularity $\rho = 0$, i.e. a singular line Oz.

The first solutions may be represented in Whittaker's form thus:-[x]

$$e^{kz}\cdot\genfrac{}{}{0pt}{}{\cos}{\sin} m\varphi\,.\,J_m(k\rho) = \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{k(z + ix\cos u + iy\sin u)}\genfrac{}{}{0pt}{}{\cos}{\sin} mu\,.\,du$$

valid over the finite part of space. It does not appear to be possible to represent this solution, which is specially related to the axis of z, in either of the forms involving $x + iy\cos u + iz\sin u$, $y + iz\cos u + ix\sin u$. The second solutions involving the functions Y_m have apparently never been expressed in Whittaker's form; this is the problem which will be discussed in the remainder of this section.

For simplicity, we shall discuss the case $m = 0$, $k = 1$, the working of which will be typical of the general case. The mode of attack will be to express $e^z\,Y_0(\rho)$ as an infinite series of terms expressible in Whittaker's form, and then try to invert the order of integration and summation,

[x] Whittaker and Watson, Modern Analysis (3rd. Edition, 1920), § 18.5

which should be possible if a representation of the required form exists.

Now put

$$x = -a + \rho'\cos\varphi' \quad , \quad y = \rho'\sin\varphi', \qquad a > 0 .$$

Then

$$e^{z} Y_0(\rho) = e^{z} Y_0\left(\sqrt{\rho'^2 + a^2 - 2a\rho'\cos\varphi'}\right)$$

$$= e^{z} J_0(a)\, Y_0(\rho') + 2\sum_{1}^{\infty} e^{z} J_n(a)\, Y_n(\rho') \cos n\varphi'$$

$$(\alpha) \qquad = e^{z} J_0(\rho')\, Y_0(a) + 2\sum_{1}^{\infty} e^{z} J_n(\rho')\, Y_n(a) \cos n\varphi',$$

by K. Neumann's Addition Theorem;[x] the first of these series is valid for $\rho' > a > 0$, the second for $a > \rho' > 0$. By the use of the known expression

$$e^{z}\cos n\varphi'\, J_n(\rho') = \frac{1}{2\pi i^n}\int_{-\pi}^{\pi} e^{z + ix\cos u + iy\sin u} \cdot e^{ia\cos u} \cos nu\, d$$

we see that, if $0 < \rho' < a$, then

$$e^{z} Y_0(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z + ix\cos u + iy\sin u}\, e^{ia\cos u}\, Y_0(a)\, du$$

$$+ \frac{1}{\pi}\sum_{1}^{\infty}\int_{-\pi}^{\pi} e^{z + ix\cos u + iy\sin u} \cdot e^{ia\cos u} \cos nu\, Y_n(a)\, i^{-n} du .$$

Consequently, within the cylinder $x^2 + y^2 = 2ax$ which passes through the singular line $x = y = 0$ of $e^{z} Y_0(\rho)$, this solution of Laplace's equation can be represented as an infinite series of terms of Whittaker's type.
If it is possible to invert the order of integration and summation the problem is solved.

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[x] See Watson's <u>Bessel Functions</u>, § 11.3. This mode of attacking the definite integral representation of $e^{z} Y_0(\rho)$ was suggested to me by Prof. Whittaker.

Now for large values of n the general term of the series $Y_0(a) + 2\sum_1^\infty i^{-n} Y_n(a)\cos nu$ is such that [x]

$$2i^{-n}Y_n(a)\cos nu \sim -\frac{2^{n+1}(n-1)!}{\pi a^n i^n}\cos nu$$

and hence this series does not converge. The inversion of the order of integration and summation is not then valid.

Consequently the solution $e^z Y_0(\sqrt{x^2+y^2})$ may be represented, within the cylinder $x^2+y^2 = 2ax$, as the sum of a convergent series of terms of the form

$$\int_{-\pi}^{\pi} f(z + ix\cos u + iy\sin u,\ u)\,du,$$

but cannot be represented as a single term of this type since the order of integration and summation may not be inverted.

This suggests that the proof given by Whittaker of his theorem is not valid. An examination of this proof shews where the error lies; at one stage[†], the order of integration and summation in an infinite series is unjustifiably inverted. What Whittaker actually proved then is that

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[x] N. Nielsen, Handbuch der Theorie der Cylinderfunktionen, (Leipzig, 1904), p. 11.

† At the top of p. 390 of Modern Analysis.

Every solution of Laplace's equation may be represented as an infinite series of terms of the form

$$\int_{-\pi}^{\pi} f(z + ix\cos u + iy\sin u,\, u)\,du\,;$$

the series converges in a region in which the solution is regular, but, in general, it is not permissible to invert the order of integration and summation.

§6. On a P.D.E. discussed by Poisson.

Poisson's "Mémoire sur l'Intégration des Equations Linéaires aux Différences Partielles" × is of considerable interest in connection with the problem of representing in finite terms the general solution of Laplace's equation. In it, he discussed the solution of the equation

$$\frac{\partial^2 u}{\partial \rho^2} - \frac{mu}{\rho^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{----} \quad (6.1).$$

If, in Laplace's equation, we put $z = it$, $V = \rho^{-\frac{1}{2}} u(\rho, t)\, e^{n\varphi i}$, where $\rho^2 = x^2 + y^2$, $\varphi = \arctan(y/x)$ we easily find that u satisfies (6.1) with $m = n^2 - \frac{1}{4}$.

Poisson's general solution, valid if $m + \frac{1}{4}$ is not an integer or zero, is

$$u = \rho^k \int_0^\pi \Phi(\rho\cos\omega + t) \sin^{2k-1}\omega \, d\omega + \rho^{k'} \int_0^\pi \Psi(\rho\cos\omega + t) \sin^{2k'-1}\omega \, d\omega,$$

where Φ and Ψ are arbitrary functions and where k and k' are the roots of $k^2 - k - m = 0$. When $m + \frac{1}{4}$ is zero, this solution loses its generality, since the two terms are identical. In this case, Poisson gave the solution

$$u = \rho^{\frac{1}{2}} \int_0^\pi \left[\Phi(\rho\cos\omega + t) + \Psi(\rho\cos\omega + t) \log(\rho \sin^2\omega) \right] d\omega,$$

involving two arbitrary functions.

× *Journ. de l'Ecole R. Polyt.*, (1823), Tome $\underline{\text{XII}}$, Cahier 19, p. 215 *et seq.*

A particular case of this result of Poisson's is that

$$e^{z}\,Y_0(\rho) = \frac{1}{\pi^2}\int_{-\pi}^{\pi} e^{z+i\rho\cos\omega}\left[\gamma + \log(2\rho\sin^2\omega)\right]d\omega,$$

where γ is Euler's constant. This certainly represents $e^{z}Y_0(\rho)$ in a finite form somewhat resembling Whittaker's general solution (1·2). For, by a few trivial transformations, it becomes

$$e^{z}\,Y_0(\rho) = \frac{1}{\pi^2}\int_{-\pi}^{\pi} e^{z+ix\cos u+iy\sin u}\left[\gamma + \log(2\rho\sin^2(u-\varphi))\right]du;$$

the integrand cannot, however, be put in the form

$$f(z+ix\cos u+iy\sin u,\, u).$$

In his discussion of the case when $m+\frac{1}{4}$ is the square of an integer, Poisson arrives at an incorrect result through an error in his analysis.[×] The correct result is, if $m+\frac{1}{4}=n^2$ where n is an integer,

$$u = \rho^{-n+\frac12}\left[\varphi(\rho+t) + a_1\rho\,\varphi_1(\rho+t) + \cdots + a_{2n-1}\rho^{2n-1}\varphi_{2n-1}(\rho+t)\right]$$

$$+\ \rho^{n+\frac12}\int_0^{\pi}\sin^{2n}\omega\left[\chi(\rho\cos\omega+t) - c\,\psi(\rho\cos\omega+t)\left\{\log(\rho\sin^2\omega)+4nq'\right\}\right]d$$

× This error does not appear to have been pointed out previously, although the paper is often quoted.

where φ and χ are arbitrary functions,

$$\varphi_s(t) = \frac{d^s \varphi(t)}{dt^s}, \qquad \psi(t) = \varphi_{2n}(t),$$

$$a_s = \frac{(-1)^s}{2^s} \cdot \frac{\Gamma(2n-s) \cdot \Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{1}{2}-s)\,\Gamma(2n)},$$

$$c = (-1)^n / 2\pi \cdot \Gamma(2n),$$

$$q = \int_\pi^\omega \sin^{2n}\omega \, d\omega,$$

$$q' = \int q \sin^{-2n}\omega \, d\omega.$$

The analysis is rather long and tedious, and so has been omitted.

By means of this result, solutions of Laplace's equation of the form $f(\rho, z)\, e^{n\varphi i}$ may all be expressed in finite terms, but a general solution of Laplace's equation is not obtained. Further the formulae are exceedingly complicated, andlack the formal simplicity of Whittaker's representations of solutions of Laplace's equation.

Chapter IV.

IV Some Particular Solutions of Laplace's Equation.

§1. In the preceding chapter, the general solution of Laplace's equation has been discussed. When we deal with particular problems involving boundary conditions it is usually more convenient to build up solutions from elementary particular solutions. For example, in the boundary value problems associated with the sphere, solutions are formed as sums of spherical harmonics.

Two of the most interesting particular solutions of this type are those involving Lamé functions and Mathieu functions, which occur when the boundary is either an ellipsoid or an elliptic cylinder respectively.[x]

[x] The Mathieu and Lamé functions are fully discussed in Chapters XIX and XXIII of Whittaker and Watson's Modern Analysis. A brief account is also given in Pierre Humbert's tract, Fonctions de Lamé et Fonctions de Mathieu (Gauthier-Villars: Paris; 1926), which contains an excellent bibliography

§2. The Lamé Functions.

Consider the curvilinear coordinates α, β, γ defined by

$$x = (a^2-b^2)\,\mathrm{sn}\,\alpha\ \mathrm{sn}\,\beta\ \mathrm{sn}\,\gamma \,/\, \sqrt{a^2-c^2}$$

$$y = (b^2-a^2)\,\mathrm{cn}\,\alpha\ \mathrm{cn}\,\beta\ \mathrm{cn}\,\gamma \,/\, \sqrt{b^2-c^2}$$

$$z = (c^2-a^2)\,\mathrm{dn}\,\alpha\ \mathrm{dn}\,\beta\ \mathrm{dn}\,\gamma \,/\, \sqrt{c^2-b^2}$$

where $\underline{a}, \underline{b}, \underline{c}$ are constants, and where the modulus of the elliptic functions is given by

$$k^2 = (a^2-b^2)/(a^2-c^2), \qquad (a^2 > b^2 > c^2).$$

If we suppose that $\alpha = \xi + iK'$, $\beta = \eta$, $\gamma = K + i\zeta$, where ξ, η, ζ are real, then the surfaces $\alpha =$ constant, $\beta =$ constant, $\gamma =$ constant form a triply orthogonal system of ellipsoids, hyperboloids of two sheets, and hyperboloids of one sheet respectively. These quadrics are confocal with the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We can easily shew that

$$(dx)^2 + (dy)^2 + (dz)^2 = \frac{(a^2-b^2)^2}{(a^2-c^2)} \sum (\mathrm{sn}^2\alpha - \mathrm{sn}^2\beta)(\mathrm{sn}^2\alpha - \mathrm{sn}^2\gamma)(d\alpha)^2;$$

by the use of the well known formula for Beltrami's second differential parameter,[×] we deduce that, in terms of these curvilinear coordinates, Laplace's equation takes the form

$$\sum (\operatorname{sn}^2\beta - \operatorname{sn}^2\gamma) \frac{\partial^2 V}{\partial \alpha^2} = 0.$$

It follows that Laplace's equation possesses normal solutions of the form

$$E(\alpha).\ E(\beta).\ E(\gamma)$$

where $E(t)$ satisfies the equation

$$\frac{d^2E}{dt^2} = \{n(n+1)\,k^2 \operatorname{sn}^2 t + A\}\, E$$

where A and n are constants. This equation is known as Lamé's equation.

When $\underline{n}$ is an integer, it can be shewn that there are $2n+1$ values of the constant A, for which Lamé's equation has, as solutions, polynomials in the elliptic functions $\operatorname{sn} t$, $\operatorname{cn} t$ and $\operatorname{dn} t$. These are the $2n+1$ Lamé functions of degree n; they are usually denoted by[†] $E_n^m(t)$ $(m = 1, 2, \ldots 2n+1)$.

× See, for example, Levi-Civita's Absolute Differential Calculus, p. 154.

† See Whittaker and Watson's Modern Analysis, §§ 23·21 – § 23·24 and § 23·5.

§3. An Alternative Mode of Introducing the Lamé Functions.

In the preceding section we have given the classical method of introducing Lamé functions from the problem of ellipsoidal harmonics. We now discuss another transformation of Laplace's equation, whose solution involves the use of Lamé functions in ~~a~~ manner which emphasises their connection with the spherical harmonics much better than the classical method. This section contains work which is apparently new; if it is not new, it is remarkable that it should have been so neglected.

Consider the system of curvilinear coordinates defined by

$$x = kr \, sn\,\alpha \; sn\,\beta$$

$$y = r \, dn\,\alpha \; dn\,\beta / k'$$

$$z = ikr \, cn\,\alpha \; cn\,\beta / k'.$$

where k is the modulus of the elliptic functions. Here r denotes, as usual, the distance of (x, y, z) from the origin. To obtain real values of (x, y, z)

we must suppose that α is real and that $\beta = K + i\eta$, where η is real. The surfaces of constant parameter are then

(i) spheres $\qquad x^2+y^2+z^2 = r^2$

(ii) elliptic cones $\qquad \dfrac{k^2y^2}{dn^2\alpha} + \dfrac{z^2}{cn^2\alpha} = \dfrac{x^2}{sn^2\alpha}$

(iii) elliptic cones $\qquad k'^2x^2 + \dfrac{z^2}{sn^2(\eta, k')} = \dfrac{k^2y^2}{cn^2(\eta, k')}$.

These form a triply orthogonal system of surfaces.

Since

$$(dx)^2+(dy)^2+(dz)^2 = (dr)^2 - k^2r^2(sn^2\alpha - sn^2\beta)\left[(d\alpha)^2-(d\beta)^2\right],$$

we find that Laplace's equation becomes

$$\frac{\partial^2 V}{\partial\alpha^2} - \frac{\partial^2 V}{\partial\beta^2} = k^2(sn^2\alpha - sn^2\beta)\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right), \qquad \ldots\ldots (3\cdot$$

when expressed in terms of the curvilinear coordinates r, α, β.

The potential functions, which are homogeneous polynomials of degree $\underline{n}$ in $\underline{x}, \underline{y}, \underline{z}$, are then of the form $r^n u(\alpha,\beta)$, where $u(\alpha,\beta)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2} = n(n+1)k^2(sn^2\alpha - sn^2\beta)u \qquad \ldots\ldots (3\cdot2)$$

But from the definition of u, it must be a surface harmonic of degree n, say $S_n(\theta, \varphi)$ where

$$\cos\theta = k \,\mathrm{sn}\,\alpha \,\mathrm{sn}\,\beta$$

$$\sin\theta \cos\varphi = \mathrm{dn}\,\alpha \,\mathrm{dn}\,\beta / k'$$

$$\sin\theta \sin\varphi = ik \,\mathrm{cn}\,\alpha \,\mathrm{cn}\,\beta / k'.$$

From the theory of surface harmonics, it follows that the equation (3.2) has $(2n+1)$ linearly independent solutions which are polynomials in $\sin\theta$, $\cos\theta$, $\sin\varphi$ and $\cos\varphi$, and therefore are polynomials in $\mathrm{sn}\,\alpha$, $\mathrm{sn}\,\beta$, $\mathrm{cn}\,\alpha$, $\mathrm{cn}\,\beta$, $\mathrm{dn}\,\alpha$, $\mathrm{dn}\,\beta$; for they are $P_n(\cos\theta)$ and $P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\varphi$ $(m = 1, 2, \ldots n)$.

To determine these $(2n+1)$ solutions, we put in (3.2) u equal to $v(\alpha)\, w(\beta)$, and obtain

$$\frac{1}{v}\frac{d^2 v}{d\alpha^2} - n(n+1)\, k^2 \mathrm{sn}^2\alpha = \frac{1}{w}\frac{d^2 w}{d\alpha^2} - n(n+1)\, k^2 \mathrm{sn}^2 \beta.$$

Hence $v(\alpha) = E(\alpha)$, $w(\beta) = E(\beta)$, where $E(t)$ is a solution of

$$\frac{d^2 E}{dt^2} = \{ n(n+1)\, k^2 \mathrm{sn}^2 t + A \} E,$$

where the constant A is so chosen that $E(t)$ is a polynomial in $\mathrm{sn}\, t$, $\mathrm{cn}\, t$, $\mathrm{dn}\, t$. But this equation is the equation of Lamé.

From the existence of $(2n+1)$ linearly independent surface harmonics of degree n, it follows that the equation (3·2) has $(2n+1)$ linearly independent solutions of the form $E(\alpha)\,E(\beta)$ where $E(\alpha)$ denotes a Lamé function of degree n. In particular, it follows that there are $(2n+1)$ linearly independent Lamé functions of degree n. Further every solution of the equation (3·2) which is a polynomial in the elliptic functions of arguments α and β must be of the form

$$\sum_{1}^{2n+1} k_m\, E_n^{m}(\alpha)\, E_n^{m}(\beta)$$

where the k_m are constants.

§4 The Integral Equations for the Lamé Functions.

The method of introducing the Lamé functions discussed in the previous section is preferable to the classical method, not only because it follows closely the ordinary methods of spherical harmonics, but also because it enables us to obtain immediately, in a very obvious manner, the integral equations satisfied by the Lamé functions.

In 1914 and 1915, Professor Whittaker[x] gave a number of homogeneous Fredholm integral equations of the form

$$u(t) = \lambda \int_{-2K}^{2K} N(t, s)\, u(s)\, ds \qquad \ldots\ldots \quad (4\cdot1)$$

which are satisfied by the whole set of Lamé functions of degree $\underline{n}$, or by some particular sub-class of them. The method of proof consisted in shewing that $N(t,s)$ which is a polynomial in $\mathrm{sn}\, t$, $\mathrm{sn}\, s$, $\mathrm{cn}\, t$, $\mathrm{cn}\, s$,

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[x] <u>Proc. Lond. Math. Soc.</u>, (2), <u>14</u>, (1915), 260.
<u>Proc. Roy. Soc. Edin.</u>, <u>35</u>, (1914-15), 70.
See also Whittaker and Watson, <u>Modern Analysis</u>, §§ 23·6, 23·61.

dn t and dn s, is a solution of

$$\frac{\partial^2 N}{\partial t^2} - \frac{\partial^2 N}{\partial s^2} = n(n+1) k^2 (\mathrm{sn}^2 t - \mathrm{sn}^2 s) N; \quad \cdots\cdots \quad (4\cdot2)$$

from this, the fact that the Lamé functions of degree $\underline{n}$ satisfy the integral equation (4·1) easily follows.

Prof. Whittaker gave various forms for the nucleus $N(t, s)$, which he verified were solutions of (4·2) of the required type; but he gave no general formula for the nucleus, and no indication of how he arrived at his results.

A comparison of equation (4·2) with equation (3·2) shews at once that

$$N(t, s) = S_n'(\theta, \varphi)$$

where $S_n(\theta, \varphi)$ is a surface harmonic of degree n,[×] and where

$$\cos\theta = k \,\mathrm{sn}\, t \,\mathrm{sn}\, s$$

$$\sin\theta \cos\varphi = \mathrm{dn}\, t \,\mathrm{dn}\, s / k'$$

$$\sin\theta \sin\varphi = ik \,\mathrm{cn}\, t \,\mathrm{cn}\, s / k'.$$

× I have already published, in a shortened form, this method of arriving at the integral equations for the Lamé functions; Proc. Edin. Math. Soc., (2), $\underline{\underline{1}}$, (1927), 62-64.

One of the most simple spherical harmonics of degree n is

$$P_n(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\overline{\varphi-\varphi_1})$$

where θ_1 and φ_1 are constants. From this we obtain the nucleus (4·3)

$$P_n\left(k\,\mathrm{sn}\,t\,\mathrm{sn}\,s\cos\theta_1 + \frac{1}{k'}\mathrm{dn}\,t\,\mathrm{dn}\,s\sin\theta_1\cos\varphi_1 + \frac{ik}{k'}\mathrm{cn}\,t\,\mathrm{cn}\,s\sin\theta_1\sin\varphi\right.$$

or, if t_1 and s_1 are the constant values of t and s corresponding to $\theta=\theta_1$, $\varphi=\varphi_1$, [x]

$$P_n\left(k^2\,\mathrm{sn}\,t\,\mathrm{sn}\,s\,\mathrm{sn}\,t_1\,\mathrm{sn}\,s_1 + \frac{1}{k'^2}\mathrm{dn}\,t\,\mathrm{dn}\,s\,\mathrm{dn}\,t_1\,\mathrm{dn}\,s_1 - \frac{k^2}{k'^2}\mathrm{cn}\,t\,\mathrm{cn}\,s\,\mathrm{cn}\,t_1\,\mathrm{cn}\,s\right.$$

The homogeneous Fredholm integral equation (4·1), with this nucleus, has, for suitable values of θ_1 and φ_1, as solutions all the $2n+1$ Lamé functions of degree n. By specialising θ_1 and φ_1 we obtain simpler forms of nucleus; for these simpler forms the solutions consist generally of some subclass of Lamé functions of degree n.

By writing in (4·3), the following sets of values of θ_1 and φ_1,

(i) $\theta_1 = 0$, $\quad\varphi_1$ real but arbitrary

(ii) $\theta_1 = \frac{1}{2}\pi$, $\quad\varphi_1 = \frac{1}{2}\pi$

(iii) $\theta_1 = \frac{1}{2}\pi$, $\quad\varphi_1 = 0$

we obtain the following three types of nucleus

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[x] A similar result is given by Whittaker in §5 of his London Mathematical Society paper already quoted.

$$P_n(k \operatorname{sn} t \operatorname{sn} s), \qquad \cdots\cdots \quad (4{\cdot}31)$$

$$P_n\left(\frac{ik}{k'} \operatorname{cn} t \operatorname{cn} s\right), \qquad \cdots\cdots \quad (4{\cdot}32)$$

$$P_n\left(\frac{1}{k'} \operatorname{dn} t \operatorname{dn} s\right), \qquad \cdots\cdots \quad (4{\cdot}33)$$

which were given by Prof. Whittaker in the London Mathematical Society paper already quoted.

A fourth nucleus, given by Prof. Whittaker in his R.S.E. paper, is a limiting form of (4·3). For we have, putting $s_1 = iK' + \varepsilon$,

$$\operatorname{sn}^n \varepsilon \, P_n\left(k^2 \operatorname{sn} t \operatorname{sn} s \operatorname{sn} t_1 \operatorname{sn}(iK'+\varepsilon) + \cdots\cdots \qquad \right)$$

$$= \operatorname{sn}^n \varepsilon \, P_n\left(k \operatorname{sn} t \operatorname{sn} s \operatorname{sn} t_1 \operatorname{ns} \varepsilon - \frac{i}{k'^2} \operatorname{dn} t \operatorname{dn} s \operatorname{dn} t_1 \operatorname{ds} \varepsilon + \frac{ik}{k'^2} \operatorname{cn} t \operatorname{cn} s \operatorname{cn} t_1 \operatorname{ds} \varepsilon\right.$$

$$\to \frac{(2n)!}{2^n (n!)^2} \left(k \operatorname{sn} t \operatorname{sn} s \operatorname{sn} t_1 - \frac{i}{k'^2} \operatorname{dn} t \operatorname{dn} s \operatorname{dn} t_1 + \frac{ik}{k'^2} \operatorname{cn} t \operatorname{cn} s \operatorname{cn} t_1\right)^n$$

as $\varepsilon \to 0$

$$= \frac{(2n)!}{2^n (n!)^2} \left(-\frac{i \operatorname{dn} t_1}{k'^2}\right)^n \left(\operatorname{dn} t \operatorname{dn} s - k \operatorname{cn} t \operatorname{cn} s \operatorname{cd} t_1 + ikk'^2 \operatorname{sn} t \operatorname{sn} s \operatorname{sd} t_1\right)$$

Consequently

$$(\operatorname{dn} t \operatorname{dn} s - k \operatorname{cn} t \operatorname{cn} s \operatorname{cd} t_1 + ikk'^2 \operatorname{sn} t \operatorname{sn} s \operatorname{sd} t_1)^n$$

is a possible form of nucleus for the integral equation of the Lamé functions. If we replace t_1 by the constant γ defined by the two consistent equations

$$\operatorname{cd} t_1 = -\cosh \gamma, \qquad \operatorname{sd} t_1 = -\frac{i}{k'} \sinh \gamma,$$

this nucleus takes the form given by Prof. Whittaker, viz.

$$(\mathrm{dn}\, t\, \mathrm{dn}\, s + k \cosh \gamma\, \mathrm{cn}\, t\, \mathrm{cn}\, s + kk' \sinh \gamma\, \mathrm{sn}\, t\, \mathrm{sn}\, s)^n \quad \text{......} \quad (4\cdot34)$$

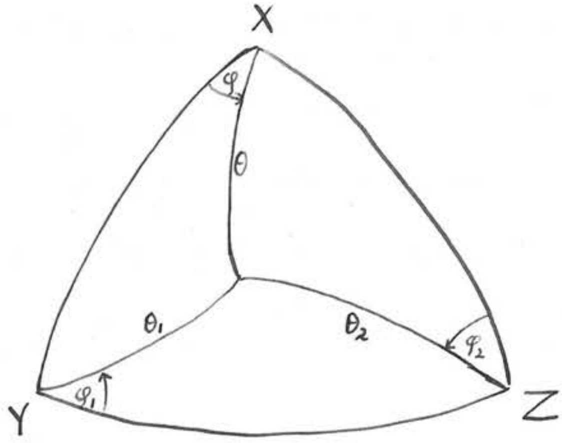
Another simple surface harmonic of degree n is $\frac{1}{2} P_n^2(\cos\theta) \sin 2\varphi$, where $P_n^2(\cos\theta)$ denotes the associated Legendre function. Writing $P_n''(\mu)$ for $d^2 P_n(\mu)/d\mu^2$, this surface harmonic may be written as $P_n''(\cos\theta) . \sin^2\theta . \sin\varphi . \cos\varphi$. Here (r, θ, φ) are the polar coordinates of (x, y, z) referred to OX as polar axis; let ~~these~~ the polar coordinates with OY and OZ as polar axes be (r, θ_1, φ_1), (r, θ_2, φ_2) respectively. Then

$$P_n''(\cos\theta_1) . \sin^2\theta_1 . \sin\varphi_1 . \cos\varphi_1$$

and

$$P_n''(\cos\theta_2) . \sin^2\theta_2 . \sin\varphi_2 . \cos\varphi_2$$

are also surface harmonics of degree n.



By applying the ordinary formulae of spherical trigonometry in the above figure, the following relations connecting $\theta, \theta_1, \theta_2, \varphi, \varphi_1, \varphi_2$ are easily obtained:—

$$\cos\theta_1 = \sin\theta\cos\varphi = \sin\theta_2\sin\varphi_2$$

$$\sin\theta_1\sin\varphi_1 = \cos\theta = \sin\theta_2\cos\varphi_2$$

$$\sin\theta_1\cos\varphi_1 = \sin\theta\sin\varphi = \cos\theta_2.$$

We thus obtain three spherical harmonics involving $P_n''(\mu)$ viz.

$$P_n''(\cos\theta).\ \sin^2\theta.\ \sin\varphi.\cos\varphi,$$

$$P_n''(\sin\theta\cos\varphi).\ \cos\theta.\ \sin\theta.\sin\varphi,$$

$$P_n''(\sin\theta\sin\varphi).\ \cos\theta.\ \sin\theta\cos\varphi.$$

From these, ~~the~~ three forms of nucleus of the integral equation for the Lamé functions are obtained, viz.

$$P_n''(k\,\mathrm{sn}\,t.\ \mathrm{sn}\,s)\ \mathrm{cn}\,t.\ \mathrm{dn}\,t.\ \mathrm{cn}\,s.\ \mathrm{dn}\,s. \quad \text{------} \quad (4.41)$$

$$P_n''(\mathrm{dn}\,t.\ \mathrm{dn}\,s./k')\ \mathrm{sn}\,t.\ \mathrm{cn}\,t.\ \mathrm{sn}\,s.\ \mathrm{cn}\,s. \quad \text{----} \quad (4.42)$$

$$P_n''(ik\,\mathrm{cn}\,t.\mathrm{cn}\,s./k')\ \mathrm{sn}\,t.\ \mathrm{dn}\,t.\ \mathrm{sn}\,s.\ \mathrm{dn}\,s. \quad \text{-----} \quad (4.43)$$

These were given by Whittaker and Watson when the third Edition of Modern Analysis was published. (See § 23.61). It is possible to write down a nucleus of this type involving two arbitrary constants (cf. nucleus (4·3)), by choosing an arbitrary polar axis instead of one of the axes of coordinates. But this is of no particular interest.

§5 The Connexion between the Lamé and the Mathieu Functions.

The structure of spheres, cones and planes associated with polar coordinates approximates, in a region of space, near the polar axis, but at a great distance from the origin, to the structure of planes, circular cylinders and planes associated with cylindrical coordinates. Corresponding to this, it can be shewn that the normal solution associated with polar coordinates, viz.

$$r^n P_n^m(\cos\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi,$$

approximates to the normal solution associated with cylindrical coordinates, viz.

$$e^{\pm kz} J_m(k\rho) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\varphi.$$

A discussion of this limiting process will be found in Watson's Bessel Functions, § 5.71.

A similar limiting process may be applied to the elliptical-polar coordinates of § 3 of this chapter. Suppose that $k \to 0$; then in the neighbourhood of the point at infinity on the y-axis, the structure of spheres and cones associated with the elliptical-polar coordinates approximates to the structure

(i) planes $y = \text{constant}$

(ii) hyperbolic cylinders $\dfrac{x^2}{\sin^2\alpha} - \dfrac{z^2}{\cos^2\alpha} = h^2$

(iii) elliptic cylinders $\dfrac{x^2}{\cosh^2\eta} + \dfrac{z^2}{\sinh^2\eta} = h^2$,*

which is that associated with the elliptical-cylindrical coordinate system. For the two families (ii) and (iii) are confocal, the foci being $x = \pm h$, $z = 0$. This suggests that the normal solution of Laplace's equation

$$r^n E_n^m(\alpha)\, E_n^m(\beta)$$

associated with the elliptical-polar coordinates should go over by a limiting process into the normal solution, involving Mathieu functions, of Laplace's equation in elliptical-cylindrical coordinates.

The limiting process just mentioned may be performed analytically as follows :- first change the origin to the point $\left(0, \frac{h}{k}, 0\right)$; then the equations†

$$x = kr \operatorname{sn}\alpha \operatorname{sn}\beta$$
$$y = y$$
$$z = ikr \operatorname{cn}\alpha \operatorname{cn}\beta$$

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* I have used here the formulae

$$\operatorname{sn}(\eta, 1) = \tanh\eta$$
$$\operatorname{cn}(\eta, 1) = \operatorname{sech}\eta$$

which may be easily proved from the integral definition of $\operatorname{sn}(\eta, 1)$.

† y is written for what was previously called $y - \frac{h}{k}$.

where

$$r^2 = x^2+(y+\frac{h}{k})^2+z^2 = \frac{h^2}{k^2}\left\{1+2\frac{k}{h}y+\frac{k^2}{h^2}(x^2+y^2+z^2)\right\},$$

express the coordinates of a point in terms of three parameters y, α, β, and are essentially equivalent to the elliptical-polar coordinate system. As $k \to 0$, these equations become

$$x = h\cos\xi\cos\eta$$

$$y = y$$

$$z = ih\sin\xi\sin\eta$$

where $\xi = \frac{\pi}{2}-\beta$, $\eta = \frac{\pi}{2}-\alpha$. These are the ordinary equations expressing (x, y, z) in elliptical-cylindrical coordinates; it is usual to suppose that ξ is purely imaginary, and that η is real.

If $u(\alpha,\beta)$ is a solution of the equation (3·2) which is a polynomial in the elliptic functions of argument α and β, then

$$\left(\frac{kr}{h}\right)^n u(\alpha,\beta)$$

is a solution of Laplace's equation. It may be written

$$\left\{1+\frac{2k}{h}y+\frac{k^2}{h^2}(x^2+y^2+z^2)\right\}^{\frac{n}{2}} u(\alpha,\beta).$$

In this put $k = imh/n$, and let $n \to \infty$, m remaining fixed, so that $k \to 0$. This solution of Laplace's equation tends to

$$e^{imy}\, v(\xi,\eta)$$

where ξ and η are as just defined. This is the normal solution of Laplace's equation (which is unaltered in its Cartesian form by this limiting process), appropriate to problems involving elliptical cylinders as boundaries.

The function $v(\xi, \eta)$ satisfies the P.D.E.

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \eta^2} + m^2 h^2 (\cos^2 \xi - \cos^2 \eta) v = 0 \qquad \text{---} \quad (5 \cdot 1)$$

and is periodic in ξ and in η, of period 2π; for $(5 \cdot 1)$ is the limiting form of $(3 \cdot 2)$. Solving $(5 \cdot 1)$ by the method of separation of independent variables, we find that

$$v(\xi, \eta) = \sum a\, G(\xi)\, G(\eta)$$

where $\underline{a}$ is a constant, and where $G(t)$ is a periodic solution, of period 2π, of Mathieu's equation

$$\frac{d^2 G}{dt^2} + (A + m^2 h^2 \cos^2 t)\, G = 0 \qquad \text{----} \quad (5 \cdot 2);$$

the constant A is so chosen that a periodic solution exists. The corresponding functions $G(t)$ are called the Mathieu functions.[×]

[×] See Whittaker and Watson, Modern Analysis, Chapter XIX and also Humbert's tract already quoted.

We have thus sketched a proof of the following result:—

The solution $r^n E_n^h(\alpha) E_n^h(\beta)$, of Laplace's equation in elliptical-polar coordinates, involving Lamé functions of degree n, degenerates, when the elliptical-polar coordinates degenerate into elliptical-cylindrical ones, into the solution $\sum e^{im\gamma} G(\xi) G(\eta)$, where $G(t)$ denotes a Mathieu function.

It does not appear at present possible to give a rigorous proof of the above theorem. First of all, there is no satisfactory notation for the Lamé functions, and also no simple mode of expressing them, corresponding to the expression of a Legendre function as a hypergeometric function. The Lamé functions are extremely difficult to handle analytically. Also very little is known about the series representations of the Mathieu functions, for the recurrence relations giving the coefficients are extremely complicated.

§6. The Integral Equations for the Mathieu Functions.

We have seen in §4 that the Lamé functions are the solutions of the integral equation

$$u(t) = \lambda \int_{-2K}^{2K} N(t, s)\, u(s)\, ds\,, \qquad \text{- - - -} \quad (4\cdot 1)$$

and in §5 that the Mathieu functions are limiting forms of the Lamé functions. It seems probable then that the Mathieu functions will satisfy integral equations

$$G(\xi) = \lambda \int_{-\pi}^{\pi} K(\xi, \eta)\, G(\eta)\, d\eta \qquad \text{- - - -} \quad (6\cdot 1)$$

where $K(\xi, \eta)$ is the limiting form of $N(t, s)$ when $k = imh/n$, and $n \to \infty$, and where $\xi = \frac{\pi}{2} - t$, $\eta = \frac{\pi}{2} - s$.

Various integral equations for the Mathieu functions have been given.[×] The simplest method of obtaining the form of the nucleus seems to be that used by Poole, who deduces each nucleus from a particular solution of the equation of wave-motions in two-dimensions.

[×] Whittaker, Proc. Int. Congress of Math., 1912.
Whittaker and Watson, Modern Analysis, (1920), §19·21, and p. 426, Exx. 1, 2.
Poole, Proc. Lond. Math. Soc., (2), 20, (1921), p. 374.
S. C. Dhar, Journ. of Dept. of Sc., Calcutta University, III (1922).

The method is, in fact, similar to that used in § 4 for obtaining integral equations for the Lamé functions. It is shewn below that, by the limiting process used in § 5, the integral equations for the Mathieu functions can be obtained from those for the Lamé functions.

To determine these limiting forms, the following results are required :—

As $n \to \infty$ by positive integral values,

$$(-1)^n \sqrt{n\pi}\; P_{2n}\left(\frac{z}{2n}\right) \to \cos z$$

$$(-1)^n \sqrt{n\pi}\; P_{2n+1}\left(\frac{z}{2n+1}\right) \to \sin z$$

$$(-1)^{n+1} \sqrt{n\pi}\; P''_{2n}\left(\frac{z}{2n}\right)/4n^2 \to \cos z$$

$$(-1)^{n+1} \sqrt{n\pi}\; P''_{2n+1}\left(\frac{z}{2n+1}\right)/4n^2 \to \sin z,$$

uniformly in z.

For we have

$$(-1)^n \sqrt{n\pi}\; P_{2n}\left(\frac{z}{2n}\right) = \sqrt{n\pi} \sum_{r=0}^{n} (-1)^r \frac{(2n+2r)!}{2^{2n}.(2r)!\,(n-r)!\,(n+r)!} \left(\frac{z}{2n}\right)^{2r}$$

For fixed r, we see that

$$(-1)^n \sqrt{n\pi}\; \frac{(2n+2r)!}{2^{2n}.(2r)!\,(n-r)!\,(n+r)!} \left(\frac{z}{2n}\right)^{2r} \to (-1)^r \frac{z^{2r}}{(2r)!}$$

as $n \to \infty$; this follows from Stirling's asymptotic formula for the Gamma function. It may also be shewn that there exists a constant C independent of r and n such that

$$\frac{\sqrt{n\pi}\;(2n+2r)!}{2^{2n}\,(n-r)!\,(n+r)!\,(2n)^{2r}} \leq C$$

for all n and for all $r \leq n$. Consequently we have

$$\left| \sqrt{n\pi}\,(-1)^r \frac{(2n+2r)!}{2^{2n}\,(n-r)!\,(n+r)!} \left(\frac{z}{2n}\right)^{2r} \right| \leq C \frac{|z|^{2r}}{(2r)!}$$

and the series $\sum \frac{|z|^{2r}}{(2r)!}$ is uniformly convergent with respect to z. It follows that, by Tannery's theorem,[×]

$$(-1)^n\sqrt{n\pi}\; P_{2n}\left(\frac{z}{2n}\right) \rightarrow \sum_0^\infty (-1)^r \frac{z^{2r}}{(2r)!} \qquad \text{as } n \rightarrow \infty,$$

$$\text{i.e.} \rightarrow \cos z.$$

The other three results may be proved in a similar manner.

The four forms of nucleus (4·31), (4·32), (4·41), (4·43) give rise to eight forms of nucleus for the integral equation satisfied by the Mathieu functions. For example,

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× Bromwich, Infinite Series, (2nd. Edn., 1926), §49.

when we put $k = imh/n$ and let $n \to \infty$, we obtain from (4·31) two limiting forms

$$\text{viz. } \lim_{p\to\infty} (-1)^p \sqrt{p\pi}\, P_{2p}\left(\frac{imh}{2p} \operatorname{sn} t \operatorname{sn} s\right)$$

$$\text{and } \lim_{p\to\infty} (-1)^p \sqrt{p\pi}\, P_{2p+1}\left(\frac{imh}{2p+1} \operatorname{sn} t \operatorname{sn} s\right);$$

these are

$$\cosh(mh \cos\xi \cos\eta) \quad \text{---------} \quad (6{\cdot}21)$$

and

$$\sinh(mh \cos\xi \cos\eta), \quad \text{---------} \quad (6{\cdot}22)$$

where the usual transformation $t = \frac{\pi}{2} - \xi$, $s = \frac{\pi}{2} - \eta$ has been made and an unimportant multiplier i in the second nucleus has been neglected. Similarly from (4·32), (4·41), (4·43) the following six forms of nucleus for the Mathieu functions are obtained:—

$$\cos(mh \sin\xi \sin\eta) \quad \text{-----} \quad (6{\cdot}23)$$

$$\sin(mh \sin\xi \sin\eta) \quad \text{-----} \quad (6{\cdot}24)$$

$$\cosh(mh \cos\xi \cos\eta) \sin\xi \sin\eta \quad \text{----} \quad (6{\cdot}25)$$

$$\sinh(mh \cos\xi \cos\eta) \sin\xi \sin\eta \quad \text{----} \quad (6{\cdot}26)$$

$$\cos(mh \sin\xi \sin\eta) \cos\xi \cos\eta \quad \text{---} \quad (6{\cdot}27)$$

$$\sin(mh \sin\xi \sin\eta) \cos\xi \cos\eta. \quad \text{---} \quad (6{\cdot}28)$$

Of these (6·21), (6·22), (6·23) and (6·24) are due to Whittaker, the rest to Poole.

If we use the relation, due to Heine,[×]

$$\lim_{n\to\infty}\left[n^{-m}P_n^{\,m}\left(1-\frac{z^2}{2n^2}\right)\right] = J_m(z),$$

we obtain, if we put $k = imh/n$ in (4·3) and let $n\to\infty$, the nucleus (6·3)

$$J_0\left[imh\left(\cos^2\xi+\cos^2\eta+\cos^2\xi_1+\cos^2\eta_1-2-2\cos\xi\cos\eta\cos\xi_1\cos\eta_1+2\sin\xi\sin\eta\sin\xi_1\sin\eta_1\right)^{\frac{1}{2}}\right]$$

involving two arbitrary constants ξ_1, η_1. This is a new nucleus for the Mathieu functions; a direct proof of this result may be easily obtained by either Whittaker's or Poole's method. A particular case of the nucleus (6·3) is given by Whittaker and Watson[‡]; it is that obtained by putting $\xi_1 = 0$, $\eta_1 = \pi$, viz.

$$J_0\left[imh(\cos\xi+\cos\eta)\right] \qquad \text{----} \qquad (6·31)$$

~~This nucleus (6·31) is the limiting form of (4·33).~~

If, in (6·3), we put $\xi_1 = 0$, $\eta_1 = \frac{\pi}{2}$, we obtain

$$J_0\left[imh\left(\cos^2\xi+\cos^2\eta-1\right)^{\frac{1}{2}}\right] \qquad \text{-----} \qquad (6·32)$$

as a possible nucleus for the integral equations for the Mathieu functions. This is the limiting form of (4·33); again a direct proof may be easily constructed.

× See Modern Analysis, § 17·4

‡ ibid., p. 426, Ex 2.

By the use of the same formula of Heine, we easily obtain as the limiting form of (3·8)

$$\frac{J_2\left[imh\,(\cos^2\zeta+\cos^2\eta-1)^{\frac{1}{2}}\right]}{\cos^2\zeta+\cos^2\eta-1}\sin\zeta\sin\eta\cos\zeta\cos\eta \quad \cdots \quad (6\cdot4),$$

a new nucleus for the Mathieu functions.

Lastly, as Professor Whittaker has shewn in his R.S.E. paper already quoted, the exponential limit

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

gives as the limiting form of (4·34) the nucleus for the Mathieu functions

$$e^{imh(\cosh\gamma\sin\zeta\sin\eta+\sinh\gamma\cos\zeta\cos\eta)} \quad \cdots\cdots \quad (6\cdot5);$$

the particular cases of (6·5) corresponding to $\gamma=0$ and to $\gamma=-\frac{\pi i}{2}$, viz.

$$e^{imh\sin\zeta\sin\eta}.$$

and

$$e^{mh\cos\zeta\cos\eta}$$

are quite well known.

Chapter V.

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Chapter V

On Electrostatics in a Gravitational Field.

By E. T. COPSON.

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On Electrostatics in a Gravitational Field.

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§ 1. *Introduction.*

In a recent paper, Prof. Whittaker* has discussed the effect, according to the general theory of relativity, of gravitation on electromagnetic phenomena. In particular, he has discussed electrostatics in gravitational fields of two kinds, namely (i) the field due to a single gravitating mass, in which case space-time has the metric discovered by Schwarzschild, and (ii) a limiting case of this, called a *quasi-uniform* field, in which the gravitational force is, in the neighbourhood of the origin, uniform.

Whittaker's general method, so far as electrostatical problems were concerned, was to solve the partial differential equation satisfied by the electrostatic potential in terms of generalised harmonic functions, and then, from these, to build up other solutions. In this way, he succeeded in finding an algebraic expression which represents the potential of a single electron in the quasi-uniform field; he did not, however, obtain a corresponding algebraic expression for the potential of an electron in the Schwarzschild field.

The chief result of the present paper is the solution of the problem which is thus presented, namely, to determine the potential of an electron in the Schwarzschild field in an algebraic form. In order to obtain it, I have departed altogether from Whittaker's method of investigation and have relied instead on Hadamard's theory of " elementary solutions " of partial differential equations. I show first, in § 2, how the solution obtained by Whittaker for the quasi-uniform field may be obtained by the aid of Hadamard's theory, and then show, in § 3, that the same methods yield the solution in an algebraic form (3.6) for the Schwartzschild field. In the last section, the new solution is expanded in terms of generalised harmonic functions; some of the relations obtained in this section are believed to be new properties of the Legendre functions.

The following brief résumé of Hadamard's theory of elementary solutions is given to make the rest of the paper intelligible.

The solution $[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}$ of Laplace's equation in three dimensions is distinguished from all other solutions by the following properties :—

* 'Roy. Soc. Proc.,' A, vol. 116, p. 720 (1927).

(i) it is continuous and differentiable as often as we please in any finite region of space, except in the neighbourhood of the singular point (a, b, c);

(ii) it becomes infinite, to as low an order as possible,* at the singular point and on all the isotropic lines through it.

But the isotropic lines are the "bicharacteristics"† of Laplace's equation. This suggests at once the appropriate generalisation.

Let $g^{\alpha\beta}$, h^{α}, k be holomorphic functions in a certain domain of real values of the variables $(x^1, x^2, \dots x^m)$. Then Hadamard‡ has shown that the partial differential equation

$$\sum_{\alpha, \beta=1}^{m} g^{\alpha\beta} \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} + \sum_{\alpha=1}^{m} h^\alpha \frac{\partial u}{\partial x^\alpha} + ku = 0,$$

possesses, when m is odd, a unique solution which is continuous and differentiable as often as we please in the domain where $g^{\alpha\beta}$, h^{α}, k are holomorphic, except in the neighbourhood of one singular point $(a^1, a^2, \dots a^m)$ and the bicharacteristics through it; in the neighbourhood of the singular point, this solution becomes infinite to as low an order as possible, and may be expanded in the form

$$\Gamma^{-\frac{m-2}{2}} (\mathrm{U}_0 + \mathrm{U}_1\Gamma + \mathrm{U}_2\Gamma^2 + \dots).$$

Γ here denotes the square of the geodesic distance from the singular point to $(x^1, x^2, \dots x^m)$ in the space whose metric is

$$ds^2 = \sum_{\alpha,\beta} g_{\alpha\beta}\, dx^\alpha\, dx^\beta\,;$$

the functions U_0, U_1, U_2, ... are holomorphic in the given domain. This solution is called the "elementary solution";

$$[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-\frac{1}{2}}$$

is obviously the elementary solution of Laplace's equation. Accordingly we shall assume that *the potential of an electron in a gravitational field is the elementary solution of the partial differential equation of electrostatic potential.*

* It is this property which distinguishes $[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-\frac{1}{2}}$ from the solution $(x-a)\,[(x-a)^2 + (y-b)^2 + (z-c)^2]^{-\frac{3}{2}}$.

† On "bicharacteristics," see Hadamard, "Lectures on Cauchy's Problem in Linear Partial Differential Equations" (Yale University Press, 1923), p. 75, *et seq.*

‡ *Loc. cit.*, Book II, chap. 3. We make use here of the tensor notation. Thus $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are corresponding elements in reciprocal determinants. We shall later write $g(x)$ for the value of the determinant $|g_{\alpha\beta}|$ at $(x^1, x^2, \dots, x^m)$.

In the course of his proof of the existence of the elementary solution, Hadamard shows how to construct it; thus the equation

$$U_0 = \sqrt{g(a)}\,\exp.\left[-\int_0^s\left(\sum_{\alpha,\beta} g^{\alpha\beta}\frac{\partial^2\Gamma}{\partial x^\alpha\,\partial x^\beta} + \sum_\alpha h^\alpha\frac{\partial\Gamma}{\partial x^\alpha} - 2m\right)\frac{ds}{4s}\right]$$

determines U_0. Here integration is along the geodesic from the singularity to $(x^1, x^2, \ldots, x^m)$, with respect to the arc s of the geodesic. The other functions U_n are given by recurrence relations.

§ 2. *Electrostatics in the Quasi-Uniform Gravitational Field.*

The electrostatic potential in the quasi-uniform gravitational field satisfies the equation*

$$\left(1+\frac{2gx}{c^2}\right)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{2.1}$$

If we apply the transformation

$$1+\frac{2gx}{c^2} = (1+\xi)^2, \qquad \frac{gy}{c^2} = \eta, \qquad \frac{gz}{c^2} = \zeta, \tag{2.2}$$

this equation becomes

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2} - \frac{1}{1+1\,\xi}\frac{\partial u}{\partial \xi} = 0.\dagger \tag{2.3}$$

In terms of these co-ordinates, the metric of space time is

$$ds^2 = (1+\xi)^2\,dt^2 - \frac{c^2}{g^2}(d\xi^2 + d\eta^2 + d\zeta^2),$$

so that ξ, η, ζ represent actual distance.

It is a consequence of Hadamard's theory, that the elementary solution is, if $1+\xi$ is positive,

$$\Gamma^{-\frac{1}{2}}[U_0 + U_1\Gamma + U_2\Gamma^2 + \ldots]$$

where

$$\Gamma = (\xi-\alpha)^2 + (\eta-\beta)^2 + (\zeta-\gamma)^2.$$

The formula for U_0‡ is

$$\begin{aligned} U_0 &= \exp.\left[-\int_\alpha^\xi\left\{\frac{\partial^2\Gamma}{\partial\xi^2} + \frac{\partial^2\Gamma}{\partial\eta^2} + \frac{\partial^2\Gamma}{\partial\zeta^2} - \frac{1}{\xi+1}\frac{\partial\Gamma}{\partial\xi} - 6\right\}\frac{d\xi}{4(\xi-\alpha)}\right] \\ &= \exp.\int_\alpha^\xi \frac{d\xi}{2(\xi+1)} \\ &= \left(\frac{1+\xi}{1+\alpha}\right)^{\frac{1}{2}}. \end{aligned}$$

* Whittaker, *loc. cit.*, p. 723 (13).

† This equation would also occur, in classical electrostatics, as the equation of potential when the specific inductive capacity is $(1+\xi)^{-1}$.

‡ Hadamard, *loc. cit.*, p. 94 (41).

The recurrence relations, from which the functions U_1, U_2, ... are to be determined, are*

$$U_n = -\frac{U_0}{2(2n-1)(\xi-\alpha)^n}\int_\alpha^\xi \frac{(\xi-\alpha)^{n-1}}{U_0}\left[\frac{\partial^2 U_{n-1}}{\partial\xi^2}+\frac{\partial^2 U_{n-1}}{\partial\eta^2}+\frac{\partial^2 U_{n-1}}{\partial\zeta^2}-\frac{1}{\xi+1}\frac{\partial U_{n-1}}{\partial\xi}\right]d\xi,$$

where integration is along the straight line from (α, β, γ) to (ξ, η, ζ). We easily find that

$$U_1 = \tfrac{3}{8}U_0(1+\xi)^{-1}(1+\alpha)^{-1}$$

$$U_2 = -\tfrac{5}{128}U_0(1+\xi)^{-2}(1+\alpha)^{-2},$$

and, generally, that

$$U_n = k_n U_0 (1+\xi)^{-n}(1+\alpha)^{-n}$$

where

$$\frac{k_n}{k_{n-1}} = -\frac{(2n+1)(2n-3)}{4\,.\,2n(2n-1)}\,.$$

The constants k_n are therefore the coefficients in the expansion of

$$(1+\tfrac{1}{2}x)(1+\tfrac{1}{4}x)^{-\frac{1}{2}}.$$

Hence the elementary solution, having singularity at (α, β, γ), is equal to

$$\left.\begin{aligned}
&\frac{U_0}{\Gamma^{\frac{1}{2}}}\left[1+k_1\frac{\Gamma}{(1+\xi)(1+\alpha)}+k_2\frac{\Gamma^2}{(1+\xi)^2(1+\alpha)^2}+\dots\right]\\
&=\frac{U_0}{\Gamma^{\frac{1}{2}}}\left(1+\frac{\Gamma}{2(1+\xi)(1+\alpha)}\right)\left(1+\frac{\Gamma}{4(1+\xi)(1+\alpha)}\right)^{-\frac{1}{2}}\\
&=\frac{[(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2+2(1+\xi)(1+\alpha)]}{[(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2+4(1+\xi)(1+\alpha)]^{\frac{1}{2}}[(\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2]^{\frac{1}{2}}(1+\alpha)}.\dagger
\end{aligned}\right\}. \quad (2.4)$$

If (a, b, c) is the point in the original co-ordinate system corresponding to (α, β, γ) the elementary solution becomes, in terms of x, y, z,

$$\frac{\left[2+2\frac{g}{c^2}(x+a)+\frac{g^2}{c^4}\{(y-b)^2+(z-c)^2\}\right]}{\left\{\left[2+2\frac{g}{c^2}(x+a)+\frac{g^2}{c^4}\{(y-b)^2+(z-c)^2\}\right]^2-4\left(1+\frac{2gx}{c^2}\right)\left(1+\frac{2ga}{c^2}\right)\right\}^{\frac{1}{2}}\sqrt{1+\frac{2ga}{c^2}}}$$

$$=\frac{1+\frac{g}{c^2}(x+a)+\frac{g^2}{2c^4}\{(y-b)^2+(z-c)^2\}}{\frac{g}{c^2}\sqrt{1+\frac{2ga}{c^2}}\left[(x-a)^2+(y-b)^2+(z-c)^2+\frac{g}{c^2}(x+a)\{(y-b)^2+(z-c)^2\}+\frac{g^2}{4c^4}\{(y-b)^2+(z-c)^2\}^2\right.}$$

* *Ibid.*, p. 95 (44′).

† It will be observed that this has a singularity at the image point of (α, β, γ) in $\xi = -1$. This is not of importance, since the region $\xi \leq -1$ is inaccessible in the relativity case. It is of interest when the P.D.E. is regarded as the classical potential equation with $K = (1+\xi)^{-1}$; K becomes infinite for $\xi = -1$.

This is precisely the formula given by Whittaker* for the potential of an electron in the quasi-uniform gravitational field.

§ 3. *Electrostatics in the Field due to a Single Gravitating Centre.*

The metric of space-time about a single gravitating centre is specified by Schwartzschild's formula

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{1}{c^2}\left(\frac{dR^2}{1 - \frac{\alpha}{R}} + R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2\right);$$

in this field, the equation of electrostatic potential† is

$$\left(1 - \frac{\alpha}{R}\right)\frac{\partial}{\partial R}\left(R^2 \frac{\partial u}{\partial R}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{3.1}$$

If we transform to the isotropic co-ordinates specified by

$$r = \frac{2R}{\alpha} - 1 + \frac{2}{\alpha}\sqrt{R^2 - \alpha R}, \tag{3.2}$$

the partial differential equation becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{2 - 4r}{r^2(r^2 - 1)}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 0‡ \tag{3.3}$$

where

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

In terms of the isotropic co-ordinates, the metric of space-time is§

$$ds^2 = \left(\frac{r-1}{r+1}\right)^2 dt^2 - \frac{\alpha^2}{16c^2}\left(1 + \frac{1}{r}\right)^4 (dx^2 + dy^2 + dz^2);$$

the transformation conformally represents the spatial part ($R > \alpha$) of Schwartzschild's metric on to the part ($r > 1$) of Euclidean space. In these co-ordinates, $r = 1$ is an inaccessible boundary.

The potential of an electron at $(0, 0, a)$, being the elementary solution of equation (3.3), must have the form

$$\Gamma^{-\frac{1}{2}}\left[U_0 + U_1\Gamma + U_2\Gamma^2 + \ldots\right] \tag{3.4}$$

where $\Gamma = x^2 + y^2 + (z - a)^2$ and where U_0, U_1, U_2, ... are holomorphic for $r > 1$.

* *Loc. cit.*, p. 727 (19).

† *Ibid.*, p. 729 (25).

‡ This equation would occur in classical electrostatics if the specific inductive capacity were assumed to be $(r + 1)^3/r^2(r - 1)$.

§ *Cf.* de Sitter, 'Amsterdam Akad. Verslagen,' vol. 25, p. 232 (1916), or Eddington, "Mathematical Theory of Relativity," p. 93 (1924).

The Hadamard formula for U_0 is

$$U_0 = \exp\left[-\int_0^s \frac{(1-2r)}{r^2(r^2-1)}(r^2 - az)\frac{ds}{s}\right]$$

where integration is along the straight line specified by

$$x = s \sin\beta \cos\gamma, \qquad y = s \sin\beta \sin\gamma, \qquad z = a + s\cos\beta,$$

where β and γ are constants. But evidently we have

$$(r^2 - az)\frac{ds}{s} = (s + a\cos\beta)\, ds = r\, dr\,;$$

hence

$$U_0 = \exp\left[\int_a^r \frac{2r-1}{r(r^2-1)}\, dr\right] = \frac{r(r-1)^{1/2}(a+1)^{3/2}}{a(a-1)^{1/2}(r+1)^{3/2}}.$$

Instead of using Hadamard's recurrence relations connecting U_0, U_1, U_2, . . ., we may apply the process by which they were obtained, that is, we may substitute (3.4) in equation (3.3), regarding U_1, U_2, . . . as holomorphic functions of r (> 1). If we do this, we obtain

$$U_1 = \frac{3}{2}\frac{U_0}{(r^2-1)(a^2-1)},$$

$$U_2 = -\frac{5}{8}\frac{U_0}{(r^2-1)^2(a^2-1)^2},$$

and so on. Hence the elementary solution is of the form

$$\frac{r(a+1)^2}{a(r+1)^2}\left[\frac{(r^2-1)}{(a^2-1)\Gamma}\right]^{\frac{1}{2}}\left[1 + \frac{3}{2}\frac{\Gamma}{(r^2-1)(a^2-1)} - \frac{5}{8}\frac{\Gamma^2}{(r^2-1)^2(a^2-1)^2} + \ldots\right].$$

The form of the first three terms in the elementary solution suggests that, instead of determining successively the remaining U_n, we should substitute, in the equation (3.3),

$$u = \frac{r}{(r+1)^2}F(\gamma),$$

where

$$\gamma = \Gamma/(r^2-1).$$

If we do this, we find that $F(\gamma)$ is a solution of

$$2\{\gamma^2 + (a^2-1)\gamma\}\frac{d^2F}{d\gamma^2} + 3\{2\gamma + a^2 - 1\}\frac{dF}{d\gamma} = 0,$$

and hence that

$$F(\gamma) = A + B\frac{2\gamma + (a^2-1)}{\{\gamma^2 + \gamma(a^2-1)\}^{\frac{1}{2}}}$$

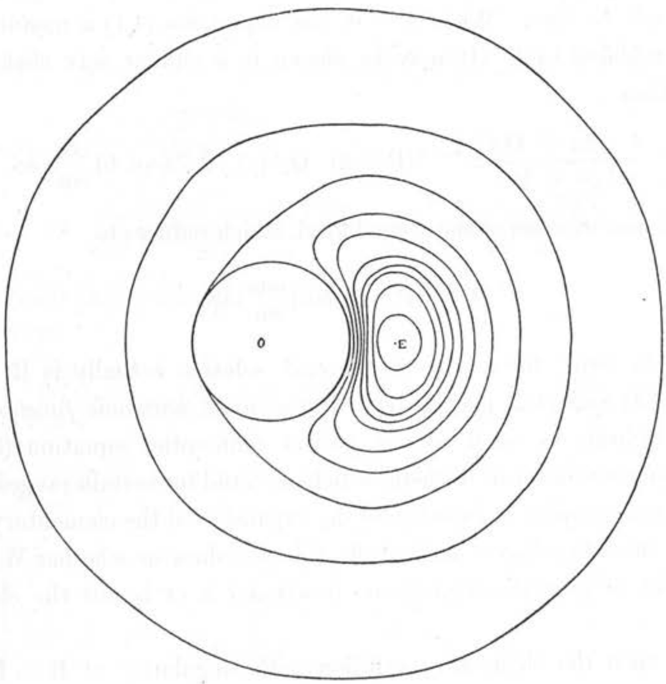
where A and B are arbitrary constants.

The elementary solution of equation (3.3) with singularity at $r = a$, $\theta = 0$, is thus seen to be a constant multiple of

$$\frac{r}{(r+1)^2}[2\Gamma + (a^2-1)(r^2-1)][\Gamma^2 + \Gamma(a^2-1)(r^2-1)]^{-\frac{1}{2}}* \tag{3.5}$$

$$= \frac{r}{(r+1)^2}\left[\left(\frac{a^2r^2+1-2ar\cos\theta}{r^2+a^2-2ar\cos\theta}\right)^{\frac{1}{2}} + \left(\frac{r^2+a^2-2ar\cos\theta}{a^2r^2+1-2ar\cos\theta}\right)^{\frac{1}{2}}\right].\dagger \tag{3.6}$$

When we transform back to the co-ordinates (R, θ, ϕ), this expression for the potential becomes rather complicated. In any practical application it seems preferable to use the isotropic co-ordinate system. The figure below shows the equipotential surfaces due to an electron E at $\mathrm{R} = 1{\cdot}56\alpha$, $\theta = 0$, corresponding to $a = 4$. With the exception of the two surfaces nearest to the electron, the difference of potential between consecutive surfaces is constant. It will be observed that $\mathrm{R} = \alpha$ is always an equipotential surface.



* A comparison of this formula with (2.4) shows that (2.4) is the limiting form of (3.5) when the Schwartzschild metric degenerates into the quasi-uniform metric. This is, of course, what one would expect.

† This shows that there is a singularity at $r = a$, $\theta = 0$, and at its image point in the sphere $r = 1$. In the relativity case, this latter singularity does not concern us; but when we regard (3.3) as the classical electrostatic potential equation with $\mathrm{K} = (r+1)^3/r^2(r-1)$ it is of some interest. *Cf.* footnote †, p. 187.

If we put $a = 1$ in the potential (3.6), we find that it reduces to a constant multiple of R^{-1}; in other words, the potential of an electron on the boundary sphere $R = \alpha$ is independent of its position on the sphere, a rather curious result.

§ 4. *The Expansion of the Potential in Terms of Generalised Harmonic Functions.*

It has been shown* that the equation (3.1) possesses the particular solution

$$\frac{2 \,.\, (n-1)\,!\, n\,!}{(2n)\,!} \alpha^{n-1} (R - \alpha)\, P_n'(\rho)\, P_n^m (\cos \theta) \frac{\cos}{\sin} m\phi \tag{4.1}$$

if $n = 1, 2, 3, \ldots$, where $\rho\alpha = 2R - \alpha$; this solution reduces to

$$R^n P_n^m (\cos \theta) \frac{\cos}{\sin} m\phi,$$

when α tends to zero. When $n = 0$, the expression (4.1) is meaningless and should be replaced by 1. It may be shown in a similar way that, if $n = 0, 1, 2, \ldots$, then

$$-\frac{4 \,.\, (2n+1)\,!}{n\,!\,(n+1)\,!} \alpha^{-n-2} (R - \alpha) \,.\, Q_n'(\rho) \,.\, P_n^m (\cos \theta) \frac{\cos}{\sin} m\phi \tag{4.2}$$

is the particular solution of the second kind, which reduces to

$$R^{-n-1} P_n^m (\cos \theta) \frac{\cos}{\sin} m\phi,$$

as α tends to zero; for $n = 0$, the second solution actually is R^{-1}. These functions (4.1) and (4.2) may be called *generalised harmonic functions*. Any solution, algebraic in $\cos \theta$, of the partial differential equation (3.1) must possess expansions in terms of them, which are valid for certain ranges of values of R. We now propose to investigate the expansion of the elementary solution in terms of these functions; incidentally this will show us whether Whittaker's infinite series of generalised harmonic functions† is or is not the elementary solution.

Consider then the elementary solution with singularity at $R = B$, $\theta = 0$, corresponding to $r = b$. If we choose the constant multiplier so that the potential is symmetrical in r and b, it has the form

$$V = \frac{4br}{\alpha\, (b+1)^2\, (r+1)^2} \left[\left(\frac{r^2 - 2br \cos \theta + b^2}{b^2 r^2 - 2br \cos \theta + 1} \right)^{\frac{1}{2}} + \left(\frac{b^2 r^2 - 2br \cos \theta + 1}{r^2 - 2br \cos \theta + b^2} \right)^{\frac{1}{2}} \right]. \tag{4.3}$$

* Whittaker, *loc. cit.*, p. 730 (31).

† *Ibid.*, p. 730 (33).

If $r > b > 1$, the value of this potential on $\theta = 0$ is

$$\mathrm{V}_0 = \frac{4br}{\alpha(b+1)^2(r+1)^2}\left[\frac{r-b}{br-1} + \frac{br-1}{r-b}\right]$$

$$= \frac{2(\mathrm{R}-\alpha)}{\alpha\mathrm{B}\sqrt{\rho^2-1}} \cdot \frac{r}{r^2-1}\left[\frac{1}{b}\left(1-\frac{b}{r}\right)\left(1-\frac{1}{br}\right)^{-1} + b\left(1-\frac{1}{br}\right)\left(1-\frac{b}{r}\right)^{-1}\right]$$

$$= \frac{2(\mathrm{R}-\alpha)}{\alpha\mathrm{B}\sqrt{\rho^2-1}}\frac{1}{r}\left[1+\frac{1}{r^2}+\frac{1}{r^4}+\dots\right]\left[\left(b+\frac{1}{b}\right)+\left(b^2-2+\frac{1}{b^2}\right)\frac{1}{r}+\dots\right].$$

$$= \frac{2(\mathrm{R}-\alpha)}{\alpha\mathrm{B}\sqrt{\rho^2-1}}\left[\left(b+\frac{1}{b}\right)\frac{1}{r}+\left(b^2-2+\frac{1}{b^2}\right)\frac{1}{r^2}+\left(b^3+\frac{1}{b^3}\right)\frac{1}{r^3} + \left(b^4-2+\frac{1}{b^4}\right)\frac{1}{r^4}+\dots\right].$$

Now Schläfli* has shown that

$$\{\rho+\sqrt{\rho^2-1}\}^{-n} = -n\sum_{m=0}^{\infty}\frac{2m+n-\frac{1}{2}}{2\pi}\frac{\Gamma(m-\frac{1}{2})\,\Gamma(m+n-\frac{1}{2})}{m!\,(m+n)!}\mathrm{Q}_{2m+n-1}(\rho).$$

Term-by-term differentiation may be easily shown to be valid here; it gives

$$\frac{r^{-n}}{\sqrt{\rho^2-1}} = \frac{\{\rho+\sqrt{\rho^2-1}\}^{-n}}{\sqrt{\rho^2-1}}$$

$$= \sum_{m=0}^{\infty}\frac{(2m+n-\frac{1}{2})}{2\pi}\frac{\Gamma(m-\frac{1}{2})\,\Gamma(m+n-\frac{1}{2})}{m!\,(m+n)!}\mathrm{Q}'_{2m+n-1}(\rho).$$

If we substitute this series for the various inverse powers of r in the expansion of V_0, we have

$$\mathrm{V}_0 = \frac{2(\mathrm{R}-\alpha)}{\mathrm{B}\alpha}\Bigg[\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\left(b^{2m+1}+\frac{1}{b^{2m+1}}\right)\cdot\frac{2p+2m+\frac{1}{2}}{2\pi}\cdot\frac{\Gamma(p-\frac{1}{2})\Gamma(2m+p+\frac{1}{2})}{p!\,(2m+p+1)!}\mathrm{Q}'_{2p+2m}(\rho)$$

$$+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\left(b^{2m+2}-2+\frac{1}{b^{2m+2}}\right)\cdot\frac{2p+2m+\frac{3}{2}}{2\pi}\cdot\frac{\Gamma(p-\frac{1}{2})\Gamma(2m+p+\frac{3}{2})}{p!\,(2m+p+1)!}\mathrm{Q}'_{2p+2m+1}(\rho)\Bigg]$$

$$= \frac{2(\mathrm{R}-\alpha)}{\mathrm{B}\alpha}\Bigg[\sum_{n=0}^{\infty}\sum_{t=0}^{n}\left\{b^{2(n-t+1)}-2+\frac{1}{b^{2(n-t+1)}}\right\}\frac{2n+\frac{3}{2}}{2\pi}\frac{\Gamma(t-\frac{1}{2})\,\Gamma(2n-t+\frac{3}{2})}{t!\,(2n-t+2)!}\mathrm{Q}'_{2n+1}(\rho)$$

$$+\sum_{n=1}^{\infty}\sum_{t=0}^{n}\left\{b^{2(n-t)+1}+\frac{1}{b^{2(n-t)+1}}\right\}\frac{2n+\frac{1}{2}}{2\pi}\frac{\Gamma(t-\frac{1}{2})\,\Gamma(2n-t+\frac{1}{2})}{t!\,(2n-t+1)!}\mathrm{Q}'_{2n}(\rho)$$

$$-\tfrac{1}{2}\left(b+\frac{1}{b}\right)\mathrm{Q}'_0(\rho)\Bigg].$$

Now by the use of the expansion

$$(1-hb)^{-3/2}(1-hb^{-1})^{-3/2} = \sum_{0}^{\infty}h^n\mathrm{P}'_{n+1}\left(\tfrac{1}{2}\left(b+\frac{1}{b}\right)\right),$$

* See Whittaker and Watson, 'Modern Analysis' (1920), p. 334, ex. 31.

we can easily show that

$$\left(b^2-2+\frac{1}{b^2}\right)\mathrm{P}'_{2n}\left(\tfrac{1}{2}\left(b+\frac{1}{b}\right)\right)$$

$$=-\frac{2n\,(2n+1)}{\pi}\sum_{t=0}^{\infty}\left\{b^{2(n-t)+1}+\frac{1}{b^{2(n-t)+1}}\right\}\frac{\Gamma\,(t-\frac{1}{2})\,\Gamma\,(2n-t+\frac{1}{2})}{t\,!\,(2n-t+1)\,!},$$

$$\left(b^2-2+\frac{1}{b^2}\right)\mathrm{P}'_{2n+1}\left(\tfrac{1}{2}\left(b+\frac{1}{b}\right)\right)$$

$$=-\frac{(2n+1)\,(2n+2)}{\pi}\sum_{t=0}^{\infty}\left\{b^{2(n-t+1)}-2+\frac{1}{b^{2(n-t+1)}}\right\}\frac{\Gamma\,(t-\frac{1}{2})\,\Gamma\,(2n-t+}{t\,!\,(2n-t+2)\,!}$$

If we use these expansions and write β for $\frac{1}{2}\left(b+\dfrac{1}{b}\right)$, we find that

$$\mathrm{V}_0=-\frac{(\mathrm{R}-\alpha)}{\mathrm{B}\alpha}\left[\left(b+\frac{1}{b}\right)\mathrm{Q}'_0\,(\rho)+\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{(2n+1)}{n\,(n+1)}\left(b^2-2+\frac{1}{b^2}\right)\mathrm{P}'_n\,(\beta)\,\mathrm{Q}'_n(\rho)\right]$$

$$=\frac{2\,(\mathrm{R}-\alpha)\,\mathrm{Q}'_0\,(\rho)}{\mathrm{B}\alpha}-\left[\frac{4}{\alpha^2}(\mathrm{R}-\alpha)\,\mathrm{Q}'_0\,(\rho)\right.$$

$$\left.+\frac{8}{\alpha^3}\sum_{n=1}^{\infty}\frac{(2n+1)}{n\,(n+1)}\,(\mathrm{B}-\alpha)\,\mathrm{P}'_n\,(\beta)\,(\mathrm{R}-\alpha)\,\mathrm{Q}'_n\,(\rho)\right]$$

$$=-\frac{\alpha}{2\mathrm{BR}}+\sum_{n=0}^{\infty}\mathrm{E}_n\,(\mathrm{B},\,0)\,.\,\mathrm{F}_n\,(\mathrm{R},\,0), \tag{4.4}$$

where

$$\mathrm{E}_n\,(\mathrm{R},\,\theta)=\frac{2\,.\,(n-1)\,!\,n\,!}{(2n)\,!}\,\alpha^{n-1}\,(\mathrm{R}-\alpha)\,\mathrm{P}'_n\,(\rho)\,\mathrm{P}_n\,(\cos\theta),$$

$$\mathrm{F}_n\,(\mathrm{R},\,\theta)=-\frac{4\,.\,(2n+1)\,!}{n\,!\,(n+1)\,!}\,\alpha^{-n-2}\,(\mathrm{R}-\alpha)\,\mathrm{Q}'_n\,(\rho)\,\mathrm{P}_n\,(\cos\theta).$$

Now V, which is given by (4.3), is a solution of the partial differential equation (3.1), is algebraic in $\cos\theta$, and does not involve ϕ. Accordingly, for a certain range of values of R, it is expansible in the form

$$\sum_{n=0}^{\infty}a_n\,\mathrm{E}_n\,(\mathrm{R},\,\theta)+\sum_{n=0}^{\infty}b_n\,\mathrm{F}_n\,(\mathrm{R},\,\theta),$$

where a_n, b_n do not involve R and θ. But, on $\theta=0$, V takes the value given by (4.4), if $\mathrm{R}>\mathrm{B}>\alpha$. Hence, if $\mathrm{R}>\mathrm{B}>\alpha$,

$$\mathrm{V}=-\frac{\alpha}{2\mathrm{BR}}+\sum_{n=0}^{\infty}\mathrm{E}_n\,(\mathrm{B},\,0)\,.\,\mathrm{F}_n\,(\mathrm{R},\,\theta). \tag{4.5}$$

Similarly, if $\mathrm{R}<\mathrm{B}$, we can show that

$$\mathrm{V}=-\frac{\alpha}{2\mathrm{BR}}+\sum_{n=0}^{\infty}\mathrm{F}_n\,(\mathrm{B},\,0)\,.\,\mathrm{E}_n\,(\mathrm{R};\,\theta). \tag{4.6}$$

We have thus shown that

The electric potential at the point (R, θ, φ) *due to a unit point-charge at rest at the point* (R = B, θ = 0) *in the gravitational field due to a mass at the origin is*

$$\frac{4br}{\alpha(b+1)^2(r+1)^2}\left[\left(\frac{r^2-2br\cos\theta+b^2}{b^2r^2-2br\cos\theta+1}\right)^{\frac{1}{2}}+\left(\frac{b^2r^2-2br\cos\theta+1}{r^2-2br\cos\theta+b^2}\right)^{\frac{1}{2}}\right],$$

where

$$\frac{4\mathrm{R}}{\alpha}=\frac{(r+1)^2}{r},\quad \frac{4\mathrm{B}}{\alpha}=\frac{(b+1)^2}{b}.$$

This potential may be expanded in the form

$$-\frac{\alpha}{2\mathrm{BR}}-\frac{4}{\alpha^2}(\mathrm{R}-\alpha)\,\mathrm{Q}_0'\left(\frac{2\mathrm{R}}{\alpha}-1\right)$$

$$-\frac{8}{\alpha^3}\sum_{n=1}^{\infty}\frac{(2n+1)}{n(n+1)}(\mathrm{B}-\alpha)\,.\,\mathrm{P}'_n\left(\frac{2\mathrm{B}}{\alpha}-1\right).\,(\mathrm{R}-\alpha)\,.\,\mathrm{Q}'_n\left(\frac{2\mathrm{R}}{\alpha}-1\right).\,\mathrm{P}_n(\cos\theta)$$

if $\mathrm{R} > \mathrm{B} > \alpha$.

When α tends to zero, the series (4.5) and (4.6) reduce to the classical formulæ

$$\sum_{n=0}^{\infty}\mathrm{B}^n\mathrm{R}^{-n-1}\mathrm{P}_n(\cos\theta),\qquad \sum_{n=0}^{\infty}\mathrm{B}^{-n-1}\mathrm{R}^n\mathrm{P}_n(\cos\theta),$$

which hold when the mass of the gravitating centre vanishes. Whittaker's infinite series is similar to the series (4.6), and both have the same limit when α tends to zero; they differ in that the constant coefficients are not the same and in the presence of the term $-\alpha/2\mathrm{BR}$. It seems unlikely that Whittaker's series can be expressed in terms of algebraic functions.

In conclusion I should like to thank Prof. Whittaker for his kind interest and encouragement during the progress of this work, and for his advice during its preparation for publication.

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Supplement to Thesis.

Supplement:– Some Applications of Hölder's Inequality.

I. Note on Series of Positive Terms.

II. Note on Series of Positive Terms.

III. On Fourier Constants.

IV. On Hardy's Theory of m-Functions.

I

NOTE ON SERIES OF POSITIVE TERMS

E. T. Copson*.

[*Extracted from the Journal of the London Mathematical Society, Vol. 2, Part 1.*]

Prof. E. B. Elliott† has recently given a very simple proof of a theorem, due to Hardy, concerning series of positive terms. His method is here applied to prove certain other similar theorems.

1. Theorem A‡. *Suppose that*

$$\kappa > 1, \quad \lambda_n > 0, \quad a_n > 0 \quad (n = 1, 2, \ldots),$$

that $\qquad A_n = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n, \quad \Lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n,$

and that $\Sigma \lambda_n a_n^\kappa$ *converges. Then*

$$\Sigma \lambda_n \left(\frac{A_n}{\Lambda_n}\right)^\kappa \leqslant \frac{\kappa}{\kappa-1} \Sigma \lambda_n a_n \left(\frac{A_n}{\Lambda_n}\right)^{\kappa-1} \leqslant \left(\frac{\kappa}{\kappa-1}\right)^\kappa \Sigma \lambda_n a_n^\kappa.$$

If further $\kappa \geqslant s \geqslant 1$, *then* $\kappa^s (\kappa-1)^{-s} \Sigma \lambda_n a_n^s (A_n/\Lambda_n)^{\kappa-s}$ *is an increasing function of* s.

For simplicity, write $\alpha_n = A_n \Lambda_n^{-1}$; then we have

$$\begin{aligned} \lambda_n \alpha_n^\kappa - \frac{\kappa}{\kappa-1} \lambda_n a_n \alpha_n^{\kappa-1} &= \left(\lambda_n - \Lambda_n \frac{\kappa}{\kappa-1}\right) \alpha_n^\kappa + \frac{\kappa}{\kappa-1} \Lambda_{n-1} \alpha_{n-1} \alpha_n^{\kappa-1} \\ &\leqslant \left(\lambda_n - \Lambda_n \frac{\kappa}{\kappa-1}\right) \alpha_n^\kappa + \Lambda_{n-1} \alpha_n^\kappa + \frac{1}{\kappa-1} \Lambda_{n-1} \alpha_{n-1}^\kappa \\ &= \frac{1}{\kappa-1} (\Lambda_{n-1} \alpha_{n-1}^\kappa - \Lambda_n \alpha_n^\kappa); \end{aligned}$$

this holds for $n = 1$ if we write $\Lambda_0 = 0$. By adding the inequalities for $n = 1, 2, 3, \ldots, N$, we find that

$$\sum_1^N \lambda_n \alpha_n^\kappa - \frac{\kappa}{\kappa-1} \sum_1^N \lambda_n a_n \alpha_n^{\kappa-1} \leqslant -\frac{1}{\kappa-1} \Lambda_N \alpha_N^\kappa \leqslant 0.$$

Now, by the inequality of Hölder, we know that

$$\left(\sum_1^N \lambda_n a_n \alpha_n^{\kappa-1}\right)^\kappa \leqslant \left(\sum_1^N \lambda_n a_n^\kappa\right) \left(\sum_1^N \lambda_n \alpha_n^\kappa\right)^{\kappa-1},$$

and so, for all integral values of N,

$$\sum_1^N \lambda_n \alpha_n^\kappa \leqslant \frac{\kappa}{\kappa-1} \sum_1^N \lambda_n a_n \alpha_n^{\kappa-1} \leqslant \left(\frac{\kappa}{\kappa-1}\right)^\kappa \sum_1^N \lambda_n a_n^\kappa.$$

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* Received and read 10 June, 1926.

† *Journal London Math. Soc.*, 1 (1926), 93–96.

‡ Hardy, *Messenger of Math.*, 54 (1925), 150–156, Theorem B.

Since the sequences $\sum\limits_1^N \lambda_n a_n^\kappa$, $\sum\limits_1^N \lambda_n a_n \alpha_n^{\kappa-1}$ are positive ascending, and the series $\Sigma\lambda_n a_n^\kappa$ converges, it follows that $\Sigma\lambda_n \alpha_n^\kappa$ and $\Sigma\lambda_n a_n \alpha_n^{\kappa-1}$ converge also, in such a way that

$$\Sigma\lambda_n \alpha_n^\kappa \leqslant \frac{\kappa}{\kappa-1}\Sigma\lambda_n a_n \alpha_n^{\kappa-1} \leqslant \left(\frac{\kappa}{\kappa-1}\right)^\kappa \Sigma\lambda_n a_n^\kappa.$$

This is the first result of Theorem A; it is known that the constants in these inequalities are the best possible.

By a further application of Hölder's inequality, we can show that $\Sigma\lambda_n a_n^s \alpha_n^{\kappa-s}$ converges when $\kappa \geqslant s \geqslant 0$. Let us write c_s for

$$\kappa^s(\kappa-1)^{-s}\Sigma\lambda_n a_n^s \alpha_n^{\kappa-s}.$$

From a theorem due to Jensen*, it follows that $\left(\frac{c_s}{c_{s_0}}\right)^{1/(s-s_0)}$, where s_0 is any real constant, is monotonic non-diminishing for $s > s_0$. Hence we have

$$\left(\frac{c_\beta}{c_\alpha}\right)^{1/(\alpha-\beta)} \leqslant \left(\frac{c_\gamma}{c_\beta}\right)^{1/(\beta-\gamma)}, \tag{1}$$

if $\alpha \geqslant \beta \geqslant \gamma$, and therefore

$$\left(\frac{c_\beta}{c_\alpha}\right)^{1/(\alpha-\beta)} \leqslant \left(\frac{c_1}{c_\beta}\right)^{1/(\beta-1)} \leqslant \frac{c_0}{c_1} \leqslant 1,$$

i.e. $c_\alpha \geqslant c_\beta$, if $\alpha \geqslant \beta \geqslant 1$; this proves the second part of Theorem A.

The behaviour of c_s in the interval $(0, 1)$ of values of s depends on the particular choice of λ_n and a_n. For example, if $a_n = 1$, we find that

$$c_s = \kappa^s(\kappa-1)^{-s}\Sigma\lambda_n,$$

so that c_s is monotonic increasing. But if $\lambda_n = 1$, $a_n = 1/n!$, $\kappa = 2$, c_s possesses a minimum in the interval $(0, 1)$ and is not monotonic†.

2. Theorem B‡. *Suppose that*

$$\kappa > 1, \quad \lambda_n > 0, \quad a_n > 0 \quad (n = 1, 2, 3, \ldots),$$

that $\quad \Lambda_n = \lambda_1+\lambda_2+\ldots+\lambda_n, \quad A_n = \frac{\lambda_n a_n}{\Lambda_n}+\frac{\lambda_{n+1} a_{n+1}}{\Lambda_{n+1}}+\ldots,$

* J. L. W. V. Jensen, *Acta Math.*, 30 (1906), 184.

† It follows from (1) that c_s is convex for $s \geqslant 0$, and so has no maximum and at most one minimum. As we have seen, both cases may occur. It may also be shown that equality between two values of c_s occurs only in the case in which the graph of c_s renders such equality intuitive. I am indebted to Mr. A. Oppenheim for these remarks.

‡ The particular case when $\kappa = 2$, $\lambda_n = 1$, was stated in a slightly different notation by Hardy, *Messenger of Math.*, 48 (1919), 107–112 (111).

and that $\Sigma\lambda_n a_n^\kappa$ *converges; then*

$$\Sigma\lambda_n A_n^\kappa \leqslant \kappa\Sigma\lambda_n a_n A_n^{\kappa-1} \leqslant \kappa^\kappa \Sigma\lambda_n a_n^\kappa,$$

the constants being the best possible. If further $\kappa \geqslant s \geqslant 1$, *then* $\kappa^s \Sigma\lambda_n a_n^s A_n^{\kappa-s}$ *is an increasing function of* s.

We obviously have from Hölder's inequality

$$A_n^\kappa \leqslant (\lambda_n a_n^\kappa + \lambda_{n+1} a_{n+1}^\kappa + \ldots)(\lambda_n \Lambda_n^{-\kappa/(\kappa-1)} + \lambda_{n+1}\Lambda_{n+1}^{-\kappa/(\kappa-1)} + \ldots)^{\kappa-1}.$$

Also

$$\lambda_n \Lambda_n^{-\kappa/(\kappa-1)} + \lambda_{n+1}\Lambda_{n+1}^{-\kappa/(\kappa-1)} + \ldots < \int_{\Lambda_{n-1}}^\infty n^{-\kappa/(\kappa-1)} dn = (\kappa-1)\Lambda_{n-1}^{-1/(\kappa-1)}$$

provided that $n > 1$. Hence, if $n > 1$, we have

$$A_n^\kappa < (\kappa-1)^{\kappa-1} R_{n-1} \Lambda_{n-1}^{-1},$$

where R_{n-1} denotes the remainder after $n-1$ terms of the series $\Sigma\lambda_n a_n^\kappa$. We have also

$$\lambda_n A_n^\kappa - \kappa\lambda_n a_n A_n^{\kappa-1} = (\lambda_n - \kappa\Lambda_n) A_n^\kappa + \kappa\Lambda_n A_n^{\kappa-1} A_{n+1}$$

$$\leqslant (\lambda_n - \kappa\Lambda_n) A_n^\kappa + (\kappa-1)\Lambda_n A_n^\kappa + \Lambda_n A_{n+1}^\kappa = \Lambda_n A_{n+1}^\kappa - \Lambda_{n-1} A_n^\kappa,$$

when $n > 1$; this relation also holds for $n = 1$ if we write $\Lambda_0 = 0$. By writing down the inequalities for $n = 1, 2, \ldots, N$, and adding, we see that

$$\sum_1^N \lambda_n A_n^\kappa - \kappa \sum_1^N \lambda_n a_n A_n^{\kappa-1} \leqslant \Lambda_N A_{N+1}^\kappa < (\kappa-1)^{\kappa-1} R_N.$$

Hence we have $$\sum_1^N \lambda_n A_n^\kappa < \kappa\left(\sum_1^N \lambda_n a_n A_n^{\kappa-1} + \epsilon_N\right),$$

where ϵ_N tends to zero when $N \to \infty$, because of the convergence of $\Sigma\lambda_n a_n^\kappa$. This inequality, together with

$$\left(\sum_1^N \lambda_n a_n A_n^{\kappa-1}\right)^\kappa \leqslant \sum_1^N \lambda_n a_n^\kappa \left(\sum_1^N \lambda_n A_n^\kappa\right)^{\kappa-1},$$

gives us $$\sum_1^N \lambda_n a_n A_n^{\kappa-1} < \kappa^{\kappa-1} \sum_1^N \lambda_n a_n^\kappa \left(1 + \epsilon_N \Big/ \sum_1^N \lambda_n a_n A_n^{\kappa-1}\right)^{\kappa-1}.$$

But the series $\Sigma\lambda_n a_n A_n^{\kappa-1}$ is a series of positive terms, and hence either converges or tends to $+\infty$; in either case we have

$$\sum_1^N \lambda_n a_n A_n^{\kappa-1} < \kappa^{\kappa-1} \sum_1^N \lambda_n a_n^\kappa (1 + \eta_N),$$

where $$\eta_N = \left(1 + \frac{\epsilon_N}{\sum_1^N \lambda_n a_n A_n^{\kappa-1}}\right)^{\kappa-1} - 1$$

tends to zero. Hence we see that

$$\sum_1^N \lambda_n A_n^\kappa < \kappa \sum_1^N \lambda_n a_n A_n^{\kappa-1} + \kappa\epsilon_N < \kappa^\kappa \sum_1^N \lambda_n a_n^\kappa + \kappa\epsilon_N + \kappa^\kappa \eta_N,$$

from which the first part of Theorem B follows immediately, whilst the third part may be proved in a manner similar to that employed in Theorem A.

We have now to show that the constant κ^κ is the best possible constant. To do this, we consider the particular case when

$$\lambda_n = 1, \quad \Lambda_n = n, \quad a_n = n^{-\mu-\epsilon} \quad (\mu = \kappa^{-1}, \epsilon > 0).$$

Then we have

$$A_n = \sum_n^\infty \nu^{-(1+\mu+\epsilon)} > \int_n^\infty \nu^{-(1+\mu+\epsilon)}\, d\nu = \frac{n^{-\mu-\epsilon}}{\mu+\epsilon},$$

and hence

$$\Sigma \lambda_n A_n^\kappa > (\mu+\epsilon)^{-\kappa} \Sigma n^{-(1+\kappa\epsilon)} = \frac{\kappa^\kappa}{(1+\kappa\epsilon)^\kappa} \Sigma \lambda_n a_n^\kappa.$$

It follows, since ϵ is arbitrarily small, that κ^κ is the best possible constant.

3. Prof. Hardy* has proved part of Theorem A as a deduction from a similar integral theorem. Theorem C (enunciated below) bears the same relation to Hardy's integral theorem as Theorem B does to Theorem A; in fact, Theorem B may be deduced from Theorem C.

THEOREM C†. *Suppose that $f(x) \geqslant 0$, $\kappa > 1$, and that $|f(x)|^\kappa$ is integrable (in the sense of Lebesgue) over $(0, \infty)$. Then $\int_x^\infty f(t) \frac{dt}{t}$ converges if $x > 0$, and defines a function $\phi(x)$ such that*

$$\int_0^\infty |\phi(x)|^\kappa dx \leqslant \kappa^\kappa \int_0^\infty |f(x)|^\kappa dx,$$

the constant being the best possible.

This theorem may be proved in a manner similar to that employed by Hardy for the integral equivalent of Theorem A, in §§ 1, 3 of the paper quoted.

* *Messenger of Math.*, 54 (1925), 150–156.

† The particular case of Theorem C when $\kappa = 2$ was given by Hardy, *l.c.* (f.n. ‡), 107.

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II

NOTE ON SERIES OF POSITIVE TERMS

E. T. Copson*.

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The theorems of which an account is here given are extensions of two convergence theorems recently discussed by Hardy and Littlewood†, and also of two more elementary theorems which I considered in a previous note‡.

1. Notation. Suppose that

$$\lambda_n > 0, \quad a_n > 0 \quad (n = 1, 2, \ldots),$$

that $\quad A_n = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n, \quad \Lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n,$

and that $\Sigma \lambda_n a_n$ is divergent.

Theorem 1.1. *If* $\kappa \geqslant c > 1$, *then*

$$\Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa \leqslant \left(\frac{\kappa}{c-1}\right)^\kappa \Sigma \lambda_n \Lambda_n^{\kappa-c} a_n^\kappa.$$

Theorem 1.2. *If* $c > \kappa > 1$, $\lambda_n/\lambda_{n+1} \leqslant U$, $\Lambda_n/\Lambda_{n+1} \geqslant M^{(\kappa-1)/(c-1)}$, *then*

$$\Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa \leqslant \left(\frac{\kappa U}{(c-1) M}\right)^\kappa \Sigma \lambda_n \Lambda_n^{\kappa-c} a_n^\kappa.$$

Theorem 1.3. *If* $c > 1 > \kappa > 0$, $0 < L \leqslant \lambda_n/\lambda_{n+1}$, *then*

$$\Sigma \lambda_n \Lambda_n^{\kappa-c} a_n^\kappa \leqslant \left(\frac{c-1}{\kappa L}\right)^\kappa \Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa.$$

2. Notation. Suppose that

$$\lambda_n > 0, \quad a_n > 0 \quad (n = 1, 2, \ldots),$$

that $\Sigma \lambda_n a_n$ is convergent, and that

$$A_n = \lambda_n a_n + \lambda_{n+1} a_{n+1} + \ldots, \quad \Lambda_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

Theorem 2.1. *If* $\kappa > 1 > c \geqslant 0$, *then*

$$\Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa \leqslant \left(\frac{\kappa}{1-c}\right)^\kappa \Sigma \lambda_n \Lambda_n^{\kappa-c} a_n^\kappa.$$

Theorem 2.2. *If* $\kappa > 1 > 0 > c$, $\lambda_n/\lambda_{n+1} \leqslant U$, $\Lambda_n/\Lambda_{n+1} \geqslant M^{1/(1-c)}$, *then*

$$\Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa \leqslant \left(\frac{\kappa U}{(1-c) M}\right)^\kappa \Sigma \lambda_n \Lambda_n^{\kappa-c} a_n^\kappa.$$

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* Received and read 10 November, 1927.

† *Journal für Math.*, 157 (1927), 141–158, in particular 143–145.

‡ *Journal London Math Soc.*, 2 (1927), 9–12.

THEOREM 2.3. *If* $1 > \kappa > 0$, $c < 1$, *then*

$$\Sigma \lambda_n \Lambda_n^{\kappa - c} a_n^\kappa \leqslant \left(\frac{N}{\kappa}\right)^\kappa \Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa,$$

where $N = 1$ *or* $1 - c$ *according as* $c \geqslant 0$ *or* $c < 0$.

3. In each of the six theorems, the series occurring on the right-hand side of the inequality is supposed convergent.

Theorems 1.1, 1.2 may be simply deduced from Theorem A of my previous note. For instance, if, in Theorem A, which may be written in the form

$$\Sigma \mu_n M_n^{-\kappa} B_n^\kappa \leqslant \left(\frac{\kappa}{\kappa - 1}\right)^\kappa \Sigma \mu_n b_n^\kappa,$$

where $\mu_n > 0, \quad b_n > 0,$

$$B_n = \mu_1 b_1 + \mu_2 b_2 + \ldots + \mu_n b_n, \quad M_n = \mu_1 + \mu_2 + \ldots + \mu_n,$$

we substitute $\mu_n = \lambda_n \Lambda_n^{(\kappa - c)/(1-\kappa)}, \quad \mu_n b_n = \lambda_n a_n,$

the result of Theorem 1.1 follows immediately.

Similarly Theorems 2.1 and 2.2 may be deduced from Theorem B of my previous note.

4. *Proof of Theorem* 1.3. We have

$$\begin{aligned}\kappa \lambda_n \Lambda_n^{1-c} A_n^{\kappa-1} a_n &= \kappa \Lambda_n^{1-c} A_n^\kappa - \kappa \Lambda_n^{1-c} A_n^{\kappa - 1} A_{n-1} \\ &\leqslant \kappa \Lambda_n^{1-c} A_n^\kappa - (\kappa - 1) \Lambda_n^{1-c} A_n^\kappa - \Lambda_n^{1-c} A_{n-1}^\kappa {}^* \\ &= \Lambda_n^{1-c} A_n^\kappa - \Lambda_n^{1-c} A_{n-1}^\kappa,\end{aligned}$$

so that

$$\begin{aligned}\Lambda_n^{1-c} A_{n-1}^\kappa - \Lambda_{n+1}^{1-c} A_n^\kappa &\leqslant \Lambda_n^{1-c} A_n^\kappa \left\{1 - \left(\frac{\Lambda_n}{\Lambda_{n+1}}\right)^{c-1}\right\} - \kappa \lambda_n \Lambda_n^{1-c} A_n^{\kappa-1} a_n \\ &< (c-1) L^{-1} \lambda_n \Lambda_n^{-c} A_n^\kappa - \kappa \lambda_n \Lambda_n^{1-c} A_n^{\kappa - 1} a_n \dagger.\end{aligned}$$

This inequality holds for $n = 1, 2, 3, \ldots, N$, if we write $A_0 = 0$; addition of these inequalities gives

$$-\Lambda_{N+1}^{1-c} A_N^\kappa < (c-1) L^{-1} \sum_1^N \lambda_n \Lambda_n^{-c} A_n^\kappa - \kappa \sum_1^N \lambda_n \Lambda_n^{1-c} A_n^{\kappa-1} a_n.$$

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* By the inequality $\kappa \alpha^{\kappa-1} \beta \geqslant (\kappa - 1) \alpha^\kappa + \beta^\kappa$, valid if $0 < \kappa < 1$.

† We have here used the inequality

$$\frac{\Lambda_n}{\lambda_n} \left\{1 - \left(\frac{\Lambda_n}{\Lambda_{n+1}}\right)^{c-1}\right\} < (c-1) L^{-1},$$

which may easily be proved.

By a further use of the inequality given in the last foot-note, we see that

$$\sum_N^\infty \lambda_n \Lambda_n^{-c} A_n^\kappa \geqslant A_N^\kappa \sum_N^\infty \lambda_n \Lambda_n^{-c} > L(c-1)^{-1} A_N^\kappa \sum_N^\infty (\Lambda_n^{1-c} - \Lambda_{n+1}^{1-c})$$
$$= L(c-1)^{-1} \Lambda_N^{1-c} A_N^\kappa \geqslant L(c-1)^{-1} \Lambda_{N+1}^{1-c} A_N^\kappa.$$

Since the series $\Sigma \lambda_n \Lambda_n^{-c} A_n^\kappa$ is convergent, it follows that $\Lambda_{N+1}^{1-c} A_N^\kappa \to 0$ as $N \to \infty$.

We have therefore proved that

$$(c-1) L^{-1} \sum_1^N \lambda_n \Lambda_n^{-c} A_n^\kappa + \epsilon_N > \kappa \sum_1^N \lambda_n \Lambda_n^{1-c} A_n^{\kappa-1} a_n$$
$$\geqslant \kappa \left(\sum_1^N \lambda_n \Lambda_n^{-c} A_n^\kappa \right)^{(\kappa-1)/\kappa} \left(\sum_1^N \lambda_n \Lambda_n^{\kappa-c} a_n^\kappa \right)^{1/\kappa} *,$$

where ϵ_N tends to zero as $N \to \infty$. This leads at once to the required result.

The proof of Theorem 2.3 follows exactly similar lines.

5. It is known that, when $\lambda_n = 1$, the constants in 1.1, 1.3, 2.1, 2.3 ($c \leqslant 0$) are the best possible, whilst those in 1.2, 2.2, 2.3 ($c > 0$) are not. A referee has pointed out that the constant in 1.1 is the best possible for any particular set of λ_n such that

$$\Lambda_n = \lambda_1 + \ldots + \lambda_n \to \infty$$

and
$$\lambda_n / \Lambda_n \to 0,$$

e.g. if $\lambda_n = 1/n$ (the most interesting case after $\lambda_n = 1$).

* By Hölder's inequality, with $0 < \kappa < 1$.

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III

On Fourier Constants.

By E. T. Copson.

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III.—On Fourier Constants. By E. T. Copson.

(MS. received October 17, 1927. Read November 7, 1927.)

§ 1. *Notation.*—Suppose that $a_0, a_1, a_2, \ldots$ are real constants, such that, for some fixed p ($\geqslant 1$), the series $\Sigma|a_n|^{1+\frac{1}{p}}$ is convergent. It is a consequence of the generalised Riesz-Fischer Theorem * that there exists a function $f(t)$, L^{1+p} over $(-1, 1)$, of which

$$\tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos n\pi t$$

is the Fourier Series.

If $0 < l \leqslant 1$, we shall write †

$$a_n(l) = \frac{2}{l}\int_0^l f(t) \cos \frac{n\pi t}{l} dt,$$

$$b_n(l) = \frac{2}{l}\int_0^l f(t) \sin \frac{n\pi t}{l} dt.$$

Then we have in $(-l, l)$

$$f(t) \sim \tfrac{1}{2}a_0(l) + \sum_1^\infty a_n(l) \cos \frac{n\pi t}{l},$$

whilst in $(0, l)$

$$f(t) \sim \sum_1^\infty b_n(l) \sin \frac{n\pi t}{l}.$$

§ 2. In the case $p=1$, the convergence of $\frac{1}{2}a_0^2 + \sum_1^\infty a_n^2$ implies the convergence of $\frac{1}{2}a_0^2(l) + \sum_1^\infty a_n^2(l)$ and of $\sum_1^\infty b_n^2(l)$; for, by Parseval's Theorem,‡ we have

$$\begin{aligned}\tfrac{1}{2}a_0^2(l) + \sum_1^\infty a_n^2(l) = \sum_1^\infty b_n^2(l) &= \frac{2}{l}\int_0^l \{f(t)\}^2 dt \\ &\leqslant \frac{2}{l}\int_0^1 \{f(t)\}^2 dt \\ &= \frac{1}{l}\left[\tfrac{1}{2}a_0^2 + \sum_1^\infty a_n^2\right].\end{aligned}$$

In this note an attempt is made to generalise this result; it is, in fact,

* Hobson, *Functions of a Real Variable* (2nd ed.), 2, 599, Theorem II.

† That $a_n(l)$ and $b_n(l)$ exist follows from the use of Hölder's inequality for integrals, and the fact that $f(t)$ is L^{1+p} over $(0, 1)$ and therefore over $(0, l)$.

‡ Hobson, *loc. cit.*, 575.

shown that in certain cases the convergence of $\Sigma|a_n|^{1+\frac{1}{p}}$ $(p\geqslant 1)$ does imply the convergence of $\Sigma|a_n(l)|^{1+\frac{1}{p}}$ and of $\Sigma|b_n(l)|^{1+\frac{1}{p}}$.

The generalisations of the Parseval and Riesz-Fischer Theorems * do not provide the required result, but merely demonstrate the convergence of $\Sigma|a_n(l)|^{1+p}$. For they give

$$\begin{aligned}\left|\frac{a_0(l)}{\sqrt{2}}\right|^{1+p}+\sum_1^\infty|a_n(l)|^{1+p}&\leqslant\frac{2^p}{l^p}\left\{\int_0^l|f(t)|^{1+\frac{1}{p}}dt\right\}\\&\leqslant\frac{2^p}{l}\int_0^l|f(t)|^{1+p}dt\\&\leqslant\frac{2^p}{l}\int_0^1|f(t)|^{1+p}dt\\&\leqslant\frac{2^{p-1}}{l}\left\{\left|\frac{a_0}{\sqrt{2}}\right|^{1+\frac{1}{p}}+\sum_1^\infty|a_n|^{1+\frac{1}{p}}\right\}^p;\end{aligned}$$

similarly for $\sum\limits_1^\infty|b_n(l)|^{1+p}$. Further generalisations of the Parseval and Riesz-Fischer Theorems have recently been obtained by Hardy and Littlewood,† but these also do not yield the required result.

We shall make use of the following theorem, due to Titchmarsh.‡ *If* $\sum\limits_{-\infty}^\infty|a_n|^p$ *is convergent* $(p>1)$, *and if*

$$\beta_m=\frac{\sin\pi\lambda}{\pi}\sum_{n=-\infty}^\infty\frac{a_n}{m+n+\lambda}$$

where λ *is real and not an integer, then* $\sum\limits_{-\infty}^\infty|\beta_n|^p$ *is convergent.*

§ 3. *Theorem I.—The convergence of* $\left|\frac{a_0}{\sqrt{2}}\right|^{1+\frac{1}{p}}+\sum\limits_1^\infty|a_n|^{1+\frac{1}{p}}$, *where* $p\geqslant 1$, *implies the convergence of* $\sum\limits_1^\infty|b_n|^{1+\frac{1}{p}}$, *and conversely.*

We are using the notation of § 1, with the exception of b_n for $b_n(1)$. Then

$$b_n=2\int_0^1 f(t)\sin n\pi t\,dt,$$

where $f(t)$ is L^{1+p} over $(0, 1)$ and has the cosine Fourier Series

$$\tfrac{1}{2}a_0+\sum_1^\infty a_n\cos n\pi t.$$

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* Hobson, *loc. cit.*, 599.

† *Math. Ann.*, **97** (1926), 159–209.

‡ *Math. Zeit.*, **25** (1926), 321–347.

By Parseval's Theorem for the product of two functions * we can substitute, in the integral for b_n, this Fourier Series and integrate term-by-term. We thus obtain

$$b_n = a_0 \frac{1-(-1)^n}{n\pi} + 2\sum_{r=1}^{\infty} a_r \frac{1-(-1)^{n+r}}{(n+r)(n-r)} \cdot \frac{n}{\pi},$$

and therefore

$$b_{2n} = \frac{1}{\pi}\sum_{-\infty}^{\infty} \frac{a_s}{n+s+\frac{1}{2}},$$

$$b_{2n+1} = \frac{1}{\pi}\sum_{-\infty}^{\infty} \frac{a_s'}{n+s+\frac{1}{2}},$$

where

$$a_s = a_{2s+1}(s \geqslant 0), \quad a_s = a_{-2s-1}(s<0), \quad a_s' = a_{-s}' = a_{2s}(s \geqslant 0).$$

It is a consequence of the data of the Theorem that both of the series $\sum_{-\infty}^{\infty} |a_s|^{1+\frac{1}{p}}$ and $\sum_{-\infty}^{\infty} |a_s'|^{1+\frac{1}{p}}$ converge. Hence, by the particular case $\lambda = \frac{1}{2}$ of Titchmarsh's Theorem, both of the series $\sum_{-\infty}^{\infty} |b_{2n}|^{1+\frac{1}{p}}$ and $\sum_{-\infty}^{\infty} |b_{2n+1}|^{1+\frac{1}{p}}$ converge; this leads at once to the required result.

§ 4. *Theorem II.—The convergence of* $\left|\frac{a_0}{\sqrt{2}}\right|^{1+\frac{1}{p}} + \sum_{1}^{\infty} |a_n|^{1+\frac{1}{p}}$, *where* $p \geqslant 1$, *implies the convergence of* $\left|\frac{a_0(l)}{\sqrt{2}}\right|^{1+\frac{1}{p}} + \sum_{1}^{\infty} |a_n(l)|^{1+\frac{1}{p}}$ *and* $\sum_{1}^{\infty} |b_n(l)|^{1+\frac{1}{p}}$, *if* l *is a positive rational number less than unity.*

In virtue of Theorem I, it suffices to prove the convergence of $\left|\frac{a_0(l)}{\sqrt{2}}\right|^{1+\frac{1}{p}} + \sum_{1}^{\infty} |a_n(l)|^{1+\frac{1}{p}}$. In the formula

$$a_n(l) = \frac{2}{l}\int_0^l f(t) \cos \frac{n\pi t}{l} dt$$

it is permissible, as before, to substitute the Fourier Series for $f(t)$ and integrate term-by-term. In this way we obtain

$$a_n(l) = \frac{(-1)^n}{\pi} \sum_{-\infty}^{\infty} a_r \frac{\sin rl\pi}{n+rl},$$

where $a_r = a_{-r}$ defines a_r for negative values of r; further, we define $a_n(l)$ by this relation when n is negative, so that $a_n(l) = a_{-n}(l)$. By the use of

* Hobson, *loc. cit.*, 614 (§ 399).

Hölder's inequality it is obvious that the series for $a_n(l)$ is absolutely convergent, and may therefore be deranged without altering its sum.

Since l is rational, it is equal to s/q where s and q are positive integers $(s<q)$. When r is positive, $rl=m+k/q$ where m is an integer and k is one of the numbers $0, 1, 2, \ldots q-1$; in this case we write

$$a_r = \mathrm{A}_{m,k}, \quad a_{-r} = \mathrm{A}_{-m,k}.$$

When m and k are integers such that $m+k/q$ is not an integral multiple of l, $\mathrm{A}_{\pm m,k}$ is defined to be zero. We have then

$$a_n(s/q) = \sum_{k=0}^{q-1} \gamma_{n,k},$$

where

$$\gamma_{n,k} = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} (-1)^{m+n} \frac{\mathrm{A}_{m,k} \sin \frac{k\pi}{q}}{m+n+\frac{k}{q}} \quad (k=1, 2, \ldots q-1)$$

$$\gamma_{n,0} = \mathrm{A}_{n,0};$$

obviously $\mathrm{A}_{n,0}$ vanishes except where n is an integral multiple of s, when we have $\mathrm{A}_{ts,0} = a_{tq}$. Titchmarsh's Theorem shows that

$$\sum_{n=-\infty}^{\infty} |\gamma_{n,k}|^{1+\frac{1}{p}} \leqslant \mathrm{K}(k/q) \sum_{m=-\infty}^{\infty} |\mathrm{A}_{m,k}|^{1+\frac{1}{p}} \quad (k=1, 2, \ldots q-1),$$

where $\mathrm{K}(k/q)$ denotes a constant depending only on k/q and p. This last inequality also holds for $k=0$, if we take $\mathrm{K}(0)=1$.

We now have

$$\begin{aligned}
\left(\sum_{n=-\infty}^{\infty} |a_n(s/q)|^{1+\frac{1}{p}}\right)^{\frac{p}{p+1}} &\leqslant \left\{\sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{q-1} |\gamma_{n,k}|\right)^{1+\frac{1}{p}}\right\}^{\frac{p}{1+p}} \\
&\leqslant \sum_{k=0}^{q-1} \left(\sum_{n=-\infty}^{\infty} |\gamma_{n,k}|^{1+\frac{1}{p}}\right)^{\frac{p}{1+p}} {}^{*} \\
&\leqslant \sum_{k=0}^{q-1} \left\{\mathrm{K}(k/q)^{\frac{p}{1+p}} \left(\sum_{n=-\infty}^{\infty} |\mathrm{A}_{n,k}|^{1+\frac{1}{p}}\right)^{\frac{p}{1+p}}\right\} \\
&\leqslant \left\{\sum_{k=0}^{q-1} \mathrm{K}(k/q)^{p}\right\}^{\frac{1}{1+p}} \left\{\sum_{k=0}^{q-1} \sum_{n=-\infty}^{\infty} |\mathrm{A}_{n,k}|^{1+\frac{1}{p}}\right\}^{\frac{p}{1+p}} {}^{\dagger} \\
&= \left\{\sum_{k=0}^{q-1} \mathrm{K}(k/q)^{p}\right\}^{\frac{1}{1+p}} \left\{\sum_{n=-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}\right\}^{\frac{p}{1+p}},
\end{aligned}$$

which proves the theorem.

* By Minkowski's inequality.

† By Hölder's inequality.

§ 5. The restriction that l be rational was introduced in order that Titchmarsh's Theorem might be applied; it seems likely that Theorem II is true without this restriction.

It might be thought that the restriction could be removed by regarding an irrational l as the limit of a sequence of rational numbers, s_m/q_m, where s_m and q_m tend to infinity with m. The proof would be complete if we could show that

$$\sum_{k=0}^{q-1} \mathrm{K}(k/q)^p \to \text{finite limit as } q \to \infty;$$

but this would imply that

$$\frac{1}{q} \sum_{k=0}^{q-1} \mathrm{K}(k/q)^p \to 0 \text{ as } q \to \infty,$$

and therefore that

$$\int_0^1 \mathrm{K}(x)^p dx = 0,$$

i.e. that $\mathrm{K}(x)$ is almost everywhere zero, and this is certainly not the case.

The restriction cannot be removed by this means; if it is unnecessary, some entirely different type of proof is required.

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UNIVERSITY OF EDINBURGH.

Mathematical Department.

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IV On Hardy's Theory of *m*-Functions.

By E. T. COPSON.

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On Hardy's Theory of m-Functions.

By E. T. Copson.

(*Received* 16*th October* 1927. *Read* 4*th November* 1927.)

§ 1. The Cardinal Function of Interpolation Theory[1] is the function

$$C(x) = \sum_{-\infty}^{\infty} a_n \frac{\sin \pi (x-n)}{\pi (x-n)}$$

which takes the values a_n at the points $x = n$. Ferrar[2] has recently proved

Theorem 1. *If* $\sum_{1}^{\infty} | a_n \log n | / n$ *and* $\sum_{1}^{\infty} | a_{-n} \log n | / n$ *are convergent,* $C(x)$ *is an m-function*[3] *for* $m \geqslant \pi$.

This means that $C(x)$ is a solution of the integral equation

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin m (x-t)}{x-t} f(t)\, dt \qquad \ldots\ldots\ldots\ldots(1)$$

Ferrar's proof deals with functions of a real variable and involves some rather difficult double limit considerations. In § 2 of the present paper is given a complex variable treatment, which provides a much more direct proof of the property in question.

In the concluding sections,[4] we show that this m-function property of the Cardinal Function is closely allied to the fact that it can be represented, under certain circumstances, by an integral of the form

$$C(x) = \int_0^1 [\phi(t) \cos \pi x t + \psi(t) \sin \pi x t]\, dt.$$

---

[1] This function was introduced by Prof. Whittaker, *Proc. Roy. Soc. Edin.*, **35** (1915), 181-194.

[2] *ibid.*, **46** (1926), 323-333 ; in particular 330-333.

[3] The theory of m-functions is due to Prof. Hardy, *Proc. Lond. Math. Soc.* (2), **7** (1909), 445-472.

[4] §§ 3, 4 have been rewritten in accordance with the valuable suggestions of Mr W. L. Ferrar, who kindly read the paper in manuscript.

§ 2. *Proof of Theorem* 1.

Under the conditions of Theorem 1, the Cardinal Function Series is uniformly and absolutely convergent in any finite part of the z-plane, and represents an integral function. Consider now

$$\int_{\Gamma} \frac{e^{im(z-x)}}{z-x} C(z)\, dz$$

where Γ is the contour formed of the segment of the real axis from $-R$ to R, indented at x, and the semicircle in the upper half plane on this segment as diameter; we suppose that $R = N + \delta$, where N is an integer, and $0 < \delta < 1$. This contour integral vanishes, since the integrand is analytic inside and on the contour. The evaluation of the contour integral gives at once

$$P\int_{-R}^{R} \frac{e^{im(t-x)}}{t-x} C(t)\, dt = \pi i\, C(x) - I(R)$$

where the integral on the left-hand-side is a Cauchy principal value, and where $I(R)$ is the integral round the semicircle.

Now we easily see that, if $0 \leqslant \theta \leqslant \pi$, $|C(Re^{\theta i})|$ is less than

$$\frac{1}{\pi} e^{\pi R \sin\theta} \left[\frac{|a_0|}{R} + \sum_1^\infty \frac{|a_n|}{\{R^2+n^2-2Rn\cos\theta\}^{\frac{1}{2}}} + \sum_1^\infty \frac{|a_{-n}|}{\{R^2+n^2+2Rn\cos\theta\}^{\frac{1}{2}}} \right]$$

the series on the right-hand-side being uniformly convergent with respect to θ. Since

$$I(R) = \int_0^\pi \frac{e^{im(R\cos\theta - x) - mR\sin\theta}}{Re^{\theta i} - x} C(Re^{\theta i})\, iRe^{\theta i}\, d\theta,$$

$\pi(R - |x|)\,|I(R)|\,/\,R$ is less than

$$\int_0^\pi e^{-(m-\pi)R\sin\theta} \left[\frac{|a_0|}{R} + \sum_1^\infty \frac{|a_n|}{\{R^2+n^2-2Rn\cos\theta\}^{\frac{1}{2}}} + \sum_1^\infty \frac{|a_{-n}|}{\{R^2+n^2+2Rn\cos\theta\}} \right] d\theta$$

$$\leqslant \int_0^\pi \left[\frac{|a_0|}{R} + \sum_1^\infty \frac{|a_n|}{\{R^2+n^2-2Rn\cos\theta\}^{\frac{1}{2}}} + \sum_1^\infty \frac{|a_{-n}|}{\{R^2+n^2+2Rn\cos\theta\}^{\frac{1}{2}}} \right] d\theta$$

$$\leqslant \pi \left\{ \frac{|a_0|}{R} + \frac{2}{\pi} \sum_1^\infty \frac{|a_n| + |a_{-n}|}{R+n} K\left[\frac{2\sqrt{Rn}}{R+n} \right] \right\}$$

where $K(k)$ denotes the complete elliptic integral of the first kind, modulus k. We have here used the condition that $m \geqslant \pi$, and have integrated term-by-term, which is obviously valid in this case.

This inequality may be written in the form

$$\frac{R-|x|}{R}\,|I(R)| \leqslant \frac{|a_0|}{R} + \frac{2}{\pi}\left(\sum_1^N + \sum_{N+1}^{\infty}\right)\frac{|a_n|+|a_{-n}|}{R+n}\,K\left[\frac{2\sqrt{Rn}}{R+n}\right]$$

where $R = N + \delta$. We shall shew from this that $I(R)$ tends to zero as N tends to infinity, δ being fixed.

Now when $n \geqslant N+1$, we have

$$\frac{2\sqrt{Rn}}{R+n} \leqslant \frac{2\sqrt{(N+\delta)(N+1)}}{2N+1+\delta} < 1 - \frac{C_1}{N^2}$$

where C_1 is a positive constant depending only on δ; since $K(k)$ is a monotone increasing function of K, if $0 \leqslant k \leqslant 1$, we see that

$$\sum_{N+1}^{\infty}\frac{|a_n|+|a_{-n}|}{R+\delta}\,K\left[\frac{2\sqrt{Rn}}{R+n}\right] \leqslant \sum_{N+1}^{\infty}\frac{|a_n|+|a_{-n}|}{N+\delta+n}\,K\left[1-\frac{C_1}{N^2}\right]$$

Now it can be easily shewn[1] that

$$K\left[1-\frac{C_1}{N^2}\right] / \log N$$

is positive and finite for all $N\ (>1)$ and tends to unity as $N \to \infty$. Consequently

$$\begin{aligned}\sum_{N+1}^{\infty}\frac{|a_n|+|a_{-n}|}{R+n}\,K\left[\frac{2\sqrt{Rn}}{R+n}\right] &\leqslant C_2\sum_{N+1}^{\infty}\frac{|a_n|+|a_{-n}|}{N+\delta+n}\log N\\ &\leqslant C_2\sum_{N+1}^{\infty}\frac{|a_n|+|a_{-n}|}{n}\log n\\ &\to 0 \text{ as } N\to\infty,\end{aligned}$$

since the two series $\overset{\infty}{\Sigma}\,|a_n \log n|/n$ and $\overset{\infty}{\Sigma}\,|a_{-n}\log n|/n$ are convergent.

It is a consequence of Tannery's Theorem[2] that

$$\frac{|a_0|}{R} + \frac{2}{\pi}\sum_1^N \frac{|a_n|+|a_{-n}|}{R+n}\,K\left[\frac{2\sqrt{Rn}}{R+n}\right] \to 0 \qquad \text{as } N\to\infty,$$

---

[1] It is an elementary consequence of the result (given in Whittaker and Watson, *Modern Analysis* (1920), § 22.737), $\lim_{k\to 0}\{K' - \log(4/k)\} = 0$.

[2] See Bromwich, *Infinite Series* (1926). § 49.

if we can shew that

$$\frac{|a_n| + |a_{-n}|}{R+n} K\left[\frac{2\sqrt{Rn}}{R+n}\right] \leqslant M_n$$

where M_n is independent of R, and ΣM_n is convergent. But, as above, we may shew that

$$\frac{|a_n| + |a_{-n}|}{R+n} K\left[\frac{2\sqrt{Rn}}{R+n}\right] \leqslant C_3 \frac{|a_n| + |a_{-n}|}{n} \log n,$$

which is sufficient for our purpose.

We have thus shewn that, under the conditions of Theorem 1, $I(R) \to 0$ as $N \to \infty$, and hence that

$$P\int_{-\infty}^{\infty} \frac{e^{im(t-x)}}{t-x} C(t)\, dt = \pi i\, C(x)$$

where the integral on the left-hand-side is a principal value, both at $t = x$, and $t = \infty$. Equating imaginary parts, we have at once, if $m \geqslant \pi$,

$$\int_{-\infty}^{\infty} \frac{\sin m(t-x)}{t-x} C(t)\, dt = \pi\, C(x);$$

the principal value sign has been omitted because $t = x$ is a removable singularity, and because $C(t)$ is an integral function finite on the real axis and

$$\int_{-\infty}^{\infty} \frac{\sin m(t-x)}{t-x}\, dt$$

exists. This completes the proof of Theorem 1.

It may be pointed out that the proof that $I(R) \to 0$ may be considerably shortened in the case $m > \pi$, by the use of the inequality

$$|C(Re^{\theta i})| \leqslant K e^{\pi R \sin\theta} R / \log R$$

if $0 \leqslant \theta \leqslant \pi$. But the proof by the use of this inequality fails in the case $m = \pi$.

§ 3. We have just seen that the fact that the Cardinal Function is a solution of the equation (1) depends chiefly upon the result that $I(R) \to 0$ as $R \to \infty$, the other parts of the proof being straightforward deductions from Cauchy's Integral Theorem.

Ferrar[1] has recently shown that, if $\sum\limits_{-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}$ is convergent $(p \geqslant 1)$, then the Cardinal Function has the definite integral representation

$$C(x) = \int_0^1 [\phi(t) \cos \pi x t + \psi(t) \sin \pi x t]\, dt,$$

where ϕ and ψ are each L^{1+p} over $(0, 1)$. From this result we are able to prove, with very little trouble, that $I(R) \to 0$ as $R \to \infty$, and thus to bring out the connection between the two properties of the Cardinal Function which we have noted in § 1.

For, considered as a function of the complex variable z, $C(z)$ is an integral function which remains finite as z tends to infinity in either direction along the real axis. In the upper half plane, we have

$$z = re^{\theta i} \qquad (0 \leqslant \theta \leqslant \pi)$$

$$\left| \begin{matrix} \cos \\ \sin \end{matrix} \pi z t \right| \leqslant e^{\pi r \sin \theta . t} \quad (t \geqslant 0).$$

Hence, by the use of Hölder's inequality for integrals, we have

$$|C(re^{\theta i})| \leqslant \int_0^1 |\phi| \, |\cos \pi r e^{\theta i} t| \, dt + \int_0^1 |\psi| \, |\sin \pi r e^{\theta i} t| \, dt$$

$$\leqslant \left\{ \left(\int_0^1 |\phi|^{1+p} dt \right)^{\frac{1}{1+p}} + \left(\int_0^1 |\psi|^{1+p} dt \right)^{\frac{1}{1+p}} \right\} \left\{ \int_0^1 e^{\pi \left(1 + \frac{1}{p}\right) r \sin \theta . t} dt \right\}^{\frac{p}{1+p}}$$

$$< K e^{\pi r \sin \theta} (r \sin \theta)^{-\frac{p}{1+p}}$$

where K is a finite constant, since ϕ and ψ are each L^{1+p} over $(0, 1)$.

We now have

$$|I(R)| \leqslant \frac{RK}{R - |x|} \int_0^\pi e^{-(m-\pi) R \sin \theta} R^{-\frac{p}{1+p}} \sin^{-\frac{p}{1+p}} \theta \, d\theta$$

$$\leqslant \frac{2RK}{R - |x|} \int_0^{\frac{1}{2}\pi} e^{-2(m-\pi) R\theta / \pi} \left(\frac{2R\theta}{\pi} \right)^{-\frac{p}{1+p}} d\theta$$

$$\to 0, \text{ as } R \to \infty, \text{ if } m \geqslant \pi.$$

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[1] *Proc. Roy. Soc. Edin.*, **47** (1927), 230–242. The particular case $p=1$ was previously discussed by J. M. Whittaker, *Proc. Edin. Math. Soc.* (2), **1** (1927), 41–46.

We can now easily complete the proof, exactly as in § 2, of the following theorem:—

Theorem 1.* *If* $\sum\limits_{-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}$ $(p \geqslant 1)$ *is convergent, then* $C(x)$ *possesses the definite integral representation*

$$C(x) = \int_0^1 [\phi(t) \cos \pi x t + \psi(t) \sin \pi x t]\, dt$$

where ϕ *and* ψ *are each* L^{1+p} *over* $(0, 1)$, *and is an m-function for* $m \geqslant \pi$.

Theorem 1* is, of course, included in Theorem 1; for, by Hölder's inequality, the convergence of $\sum\limits_{-\infty}^{\infty} |a_n|^{1+\frac{1}{p}}$ $(p \geqslant 1)$ implies the convergence of $\sum\limits_{1}^{\infty} |a_n \log n| / n$ and $\sum\limits_{1}^{\infty} |a_{-n} \log n| / n$, but not conversely.

§ 4. Finally, the use of functions of class L^p enables us to prove, by the same direct method, Theorems[1] 2 and 3 below.

Theorem 2. *The integrals*

$$f(x) = \int_a^A \phi(w) \begin{matrix}\cos \\ \sin\end{matrix}\, wx\, dw$$

represent m-functions, if $-m \leqslant a < A \leqslant m$, *provided only that* $\phi(w)$ *is* L^p $(p > 1)$ *over* (a, A).

Theorem 3. *The integral*

$$f(x) = \int_{-\infty}^{\infty} \frac{\sin \mu (w - x)}{w - x} \phi(w)\, dw$$

represents an m-function, if $m \geqslant \mu > 0$, *provided only that* $\phi(w)$ *is* L^p $(p > 1)$ *over* $(-\infty, \infty)$.

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[1] Compare the rather similar theorems given by Hardy, *loc. cit.*, 457, 459.