THE INTRINSIC FORMALITY OF CERTAIN TYPES OF ALGEBRAS

Thesis Submitted By

Gregory Lupton

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Acknowledgements

I would like to take this traditional opportunity to thank my supervisor, Elmer Rees, for a number of things. For first suggesting to me that I work in rational homotopy theory, and then that I consider the problem which resulted in this thesis. For remaining patient with me and interested in my work, even through periods when my attitude towards my own work was, I suspect, somewhat more relaxed than his. Also for all the smaller things which, in the end, are probably more important. Thank you Elmer. I would also like to thank Jo Neisendorfer for a conversation he had with me which, though brief, was encouraging; and without which, (4.3.4.) and (A.2.1.) would now be more cumbersome.

It is indeed unfortunate that mathematics rarely contains any hint of the human support necessary for its existence, other than in the - once again, traditional - formal setting of an acknowledgement. This being said, however, I cannot pass up the opportunity of symbolising, in this fashion, the importance to me of one particular person, Suzannah Dunn. For contributing to the love and strength which lies behind creative endeavour; thank you, Suzannah.

Declaration

This thesis and the material presented in it is my own work, except where otherwise indicated.

DEDICATION

To all those people who, having the strength, courage and conviction, through their actions big or small, inwardly or outwardly directed, attempt to change the World into a better, happier place to be; this thesis is respectfully and humbly dedicated.

" Nobody made a greater mistake than he who did nothing because he could only do a little. "

Edmund Burke

" Do you know what counts? The detail. Only the detail counts. . ."

Arthur Koestler

" Many a mickle makes a muckle. "

Traditional

ABSTRACT

The central objectives of this thesis are two-fold. Firstly to consider the general problem: given a graded, 1-connected, rational algebra of finite type, is there a unique rational homotopy type having cohomology isomorphic to that algebra? If there is, then the algebra is said to be intrinsically formal. Secondly, to provide examples of intrinsically formal algebras. Chapter 2 is an introduction to the general problem, and the methods available for analysing it. A strategy for attacking this general problem is gathered together from the literature, and laid out in that chapter. Also, a comparison is made of the use of differential graded algebra methods and differential graded Lie algebra methods in the context of this problem.

In chapter 3, particular types of algebras are introduced, motivated by examples from algebraic geometry, and called Lefschetz algebras (3.1.2.). Some general properties of such algebras are established. In addition, a technical result is proved which gives useful linear independence criteria for certain elements in a free Lie algebra. These criteria are subsequently used at a number of places in the thesis.

The main results of the thesis are in chapters 4 and 5. Here it is proved that many of the algebras introduced in chapter 3 are intrinsically formal. Specifically, let H be a Lefschetz algebra of type H(n,k) such that $n \leqq 2k-1$ (c.f. 3.1); then H is intrinsically formal (5.1.9.). Further, let H be a Lefschetz algebra of type H(J;n,k) for any J and such that $n \leqq k+1$ (c.f. 4.3); then H is intrinsically formal (4.3.9).

In chapter 6, methods of constructing the algebras introduced in chapter 3 are given. These methods can be used to construct algebras of the type already shown to be intrinsically formal in chapters 4 and 5. This provides a large number of examples of intrinsically formal algebras. Finally, an example is given to contradict the suggestion that all algebras of the type introduced in chapter 3 are intrinsically formal. This example also demonstrates that the results of chapters 4 and 5 are best possible.

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CHAPTER 1

The concept of formality of spaces was introduced by Sullivan in his work on rational minimal models [Su_1], [D-G-M-Su], [Su_2]. The concept of formality can be explained as follows. Amongst other things, Sullivan constructed an equivalence between the rational homotopy category of nilpotent spaces having rational homology of finite type, and the homotopy category of associative, commutative, differential, graded, rational algebras having minimal models of finite type. So a rational homotopy type, represented by a space X, say, can be regarded as a homotopy equivalence class of differential graded algebras. A specific representative for this homotopy equivalence class was also constructed - the minimal model of X, written (M_X,d). Now the rational cohomology of X, $H^*(X;\mathbb{Q})$, can be regarded as a differential graded algebra with trivial differential, but in general it will not be in the same homotopy equivalence class as (M_X,d). If it is, then X is said to be formal (see (2.1.1.)). A wide range of formal spaces was given by Sullivan [Su_2 - sec.12].

The concept of intrinsic formality of algebras was also introduced in Sullivan's work [Su_2]. An algebra of finite type is intrinsically formal if there is a unique rational homotopy type having cohomology isomorphic to that algebra (see chapter 2 and (2.1.3.)). Apart from examples of a trivial nature, such as the rational cohomology of S^n, the first examples of intrinsically formal algebras appeared in [Su_2]. Let H be isomorphic to $\Lambda(x_1, x_2, \ldots\ldots, x_n)/(R_1, \ldots\ldots, R_m)$, where $(R_1, \ldots\ldots, R_m)$ is a regular sequence in $\Lambda(x_1, \ldots\ldots, x_n)$ - the notation Λ means the free algebra with polynomial generators in even degrees, and exterior generators in odd degrees; then H is intrinsically formal. Work by a number of authors ([B-L], [N_1], [N-M]), extending Quillen's results on rational homotopy [Q], meant that problems concerning minimal models, such as formality, became susceptible to attack by Lie algebra methods. In [N-M], differential graded Lie algebras were used to prove that all 1-connected Poincaré duality algebras of dimension at most 6 are intrinsically formal. This result was subsequently generalised in [M_2] to prove that all (k-1)-connected Poincaré duality algebras of dimension at most 4k-2 are intrinsically formal. In addition, several authors have proved, using either differential graded algebras [H-St], or differential graded Lie algebras [N-M], that all (k-1)-connected algebras of dimension at most 3k-2 are intrinsically formal. It has also been proven that the rational cohomology of an infinite stunted complex projective space, $H^*(CP^\infty/CP^n;Q)$, is intrinsically formal [T].

Formality and intrinsic formality are related as follows. If a space X is formal, then the rational cohomology of X, $H^*(X;Q)$, need not be intrinsically formal. However, if a space X is such that $H^*(X;Q)$ is intrinsically formal, then X must be a formal space (see chapter 2). Therefore, in order to prove a space X formal, it is sufficient to prove $H^*(X;Q)$ intrinsically formal; and indeed, the intrinsic formality problem has often been treated as a means of providing examples of formal spaces, rather than as a problem in its own right.

One of the main objectives of this thesis is to provide further examples of 1-connected algebras of finite type which are intrinsically formal. This is done in chapters 4 and 5. The choice of algebras used here was motivated by two considerations. Firstly, the knowledge that Kähler manifolds, the examples of prime geometric interest, are formal [D-G-M-Su]. Secondly, the question of how much of this latter fact could be derived from a knowledge of the algebra alone (c.f. 3.1.). In addition, the thesis serves to collect together, in a form convenient to a consideration of the intrinsic formality problem, results and techniques used by a number of authors employing differential graded Lie algebras.

Furthermore, although the strategy at (2.2.6.), which is the method used in this thesis, has been used before in order to prove intrinsic formality results, [M_2] [N–M]; the results of chapters 4 and 5 are the first examples where the execution of that strategy is non-trivial at all of its stages. This can be expressed more specifically as follows. Let H be a graded algebra; (2.2.6.) first requires the construction of the differential graded Lie algebra minimal model of H, written $L(s^{-1}H_*,\partial)$ – c.f. section 2.2. Now let $L(s^{-1}H_*,\partial+P)$ be a perturbation of this model – c.f. 2.2 . Then (2.2.6.) requires the construction of an isomorphism of differential graded Lie algebras

$$\Phi\colon L(s^{-1}H_*,\ \partial) \longrightarrow L(s^{-1}H_*,\ \partial+P)\ .$$

The results of 4.1 reduce this latter requirement to first constructing a linear map

$$\Phi\colon s^{-1}H_* \longrightarrow L(s^{-1}H_*)\ ,$$

between the vector space $s^{-1}H_*$, and the Lie algebra $L(s^{-1}H_*)$; and secondly checking that Φ defines a differential graded isomorphism between $L(s^{-1}H_*,\ \partial)$ and $L(s^{-1}H_*,\ \partial+P)$. Now, using this 'three-stage' method of attack, the result proving that all (k-1)-connected algebras of dimension at most 3k-2 are intrinsically formal, becomes a triviality; since all perturbations in this case must be zero, so the map Φ in the above can be taken to be the identity, and then Φ trivially induces the required isomorphism. The result of

[N-M (4.6)], generalised in [M_2], is more substantial to prove. These results necessitate taking into account the multiplicative structure of the algebra. More precisely, by a judicious choice of basis for H, motivated by Poincaré duality, the construction of the linear map $\Phi: s^{-1}H_* \longrightarrow L(s^{-1}H_*)$, as above, is achieved. This is where the bulk of the work in the results [N-M (4.6)] and [M_2] lies. However, in both these results it is a triviality to check that Φ, once defined, induces the necessary differential graded isomorphism referred to above. The results of chapters 4 and 5, then, also require the taking into account of the multiplicative structure of the algebras under consideration. Once more, this structure is used in order to make a judicious choice of basis for H (3.3.2.). Once this is done, however, further work is necessary simply to get to the stage of defining a linear map $\Phi: s^{-1}H_* \longrightarrow L(s^{-1}H_*)$. Indeed, in chapter 5, this stage requires a completely different construction to that used in [N-M (4.6)] and [M_2], depending on a multiplicative property - other than Poincaré duality - of the algebras under consideration (5.1.5.), and in addition requires the substantial calculations of the appendix. Once this map is defined, though, it is still non-trivial to check that it in turn defines a differential graded isomorphism Φ from $L(s^{-1}H_*, \partial)$ to $L(s^{-1}H_*, \partial+P)$. In (4.3.8.), this checking requires the preliminary lemma of (3.3.3.). Since, via the results of section 4.1, (2.2.6.) is separated, broadly, into three stages, as above; the results of chapters 4 and 5 go some way to indicating limits for the usefulness of (2.2.6.) - c.f. also the comments after (4.3.7.).

In chapter 6, methods of constructing examples of the types of algebras considered earlier in the thesis are given. Combining these with the results of chapters 4 and 5, one obtains a large number of examples of intrinsically formal algebras. A specific example is also constructed of a Lefschetz algebra which is not intrinsically formal. This example lies immediately outside the ranges covered by the results of chapters 4 and 5, and so demonstrates that those results are best possible.

This thesis uses differential graded Lie algebras in preference to differential graded algebras. A comparison of these two alternative approaches is given in chapter 2. Remark (2.3.4.) explains why the choice of differential graded Lie algebras was made in preference to differential graded algebras. It would seem that rational homotopy theory as it now stands, has reached the stage where it is capable of solving, and indeed posing, problems which are not amenable to solution by classical methods of algebraic topology - c.f. the remarks at the beginning of [Su-V]. In view of the problems faced when coming to terms with using differential graded Lie algebra methods - often perceived as more obscure and less 'natural' than Sullivan's differential graded algebras; it is interesting to speculate whether or not there are certain problems which, notwithstanding the equivalence between the two categories, are solvable by differential graded Lie algebra methods, and yet are intractable to the differential graded algebra approach. Certainly it is difficult to see how the results of, say, [M_2], [St], or chapter 5 of this thesis could have been achieved using differential graded algebras. Section 2.3 provides some specific examples for comparison.

Finally, some general notation to be used for the remainder of the thesis is fixed here. Unless otherwise specified, space will mean a 1-connected topological space with rational homology of finite type; algebra will mean an associative, commutative, graded, 1-connected rational algebra of finite type; coalgebra will mean an associative, commutative, graded, 1-connected rational coalgebra of finite type and Lie algebra will mean a graded Lie algebra over the rationals, which is zero in degree 0. Algebra differentials will be of degree +1, whilst coalgebra and Lie algebra differentials will be of degree −1, and the prefix DG will be used to signify 'differential graded'. A minimal DG algebra is a differential graded algebra (M,d) such that M is a free algebra and d is decomposable. A minimal DG coalgebra is a differential graded coalgebra (C,d) such that C is a symmetric coalgebra and d is zero on the primitive elements. A minimal DG Lie algebra is a differential graded Lie algebra (L,d) such that L is a free Lie algebra and d increases bracket length by at least one. If V is a graded vector space, then L(V) will denote the free Lie algebra on V; and Λ(V) the free commutative algebra on V – that is, polynomial on generators of even degree and exterior on generators of odd degree.

CHAPTER 2

In this chapter, and for the rest of the thesis, attention is focussed on a seminal problem of topology: How many homotopy types can share a given cohomology ring? In general, there could be more than one, but in suitable cases there might be one only. In rational homotopy theory, this latter phenomenon has a name:

DEFINITION A graded algebra H is said to be intrinsically formal if there is a unique rational homotopy class of spaces, such that $H^*(X;\mathbb{Q})$ is isomorphic to H for any representative, X, of the class.

[Remark – In rational homotopy theory there is no problem with 'realizing' algebras. For any algebra H, there always exists at least one rational homotopy type having cohomology ring isomorphic to H, [D-G-M-Su (3.2)] or [Su_2].]

This chapter will be concerned with considering the problem:

Given a graded algebra, H, is H intrinsically formal?

This problem is characterised as "the intrinsic formality problem", and the rest of the chapter is devoted to introducing basic ideas relevant to this problem. The chapter consists mostly of a convenient restatement of results to be found in the literature. In particular, see [N-M], [M_1], [M_2], and [H-St]. The main purpose of this chapter is to give a strategy for attacking the intrinsic formality problem (2.2.6.). When considering intrinsic formality, it is helpful to distinguish it carefully from the related concept of formality. There is an added difficulty when considering both these concepts in terms of DG Lie algebras rather than DG algebras – their original, and most natural, setting. The following two sections are intended to clarify these concepts, and the relationships between them.

2.1. FORMALITY; INTRINSIC FORMALITY; DG ALGEBRAS.

The concepts of formality and intrinsic formality were first introduced by Sullivan. Corresponding to the various categories in which it is now possible to do rational homotopy theory, there are equivalent formulations of these concepts [N-M]. This section is concerned with setting up these concepts in the DG algebra category – the one used by Sullivan.

Recall that the minimal model of a DG algebra (A,d) is a map of DG algebras

$$\rho: (M_A, d) \longrightarrow (A,d)$$

such that (M_A, d) is a minimal DG algebra and ρ^* is an isomorphism on cohomology. This always exists for any given DG algebra [Su_1], [D-G-M-Su], [Su_2].

Recall also that given a space X, there exists the DG algebra of PL forms on X, $(E^*(X), d)$, [Su_1]. This DG algebra captures the rational homotopy information of X, and the minimal model of the space X is defined as the minimal model of $(E^*(X), d)$:

$$\rho_X: (M_X, d) \longrightarrow (E^*(X), d) \ .$$

Regarding a graded algebra as a DG algebra with trivial differential, it is possible to make the:

2.1.1. Definition. [D-G-M-Su], [Su_2]

A minimal DG algebra (M,d) is formal if there exists a map of DG algebras

$$\psi: (M,d) \longrightarrow (H^*(M,d), 0)$$

which is an isomorphism on cohomology.

A DG algebra is formal if its minimal model is formal.

A space X is formal if $(E^*(X), d)$ is formal.

To see that there exist plenty of formal rational homotopy types, consider any graded algebra H to be a DG algebra having trivial differential. Regarded as such, it has a minimal model which can be written :

$$\rho_H: (M_H,d) \longrightarrow (H,0) \quad .$$

But notice that ρ_H fits the definition of formality given above; and so the minimal model of a graded algebra is formal. Thus, the minimal model of any graded algebra is a formal rational homotopy type, and has cohomology isomorphic to that algebra.

2.1.2. Remarks.

1. In general, for an algebra H, there will exist other rational homotopy types apart from (M_H,d), having cohomology isomorphic to H.

2. Given an algebra H, with minimal model

$$\rho_H: (M_H,d) \longrightarrow (H,0) \quad ,$$

consider a formal minimal DG algebra (N,d), such that $H^*(N,d)$ is also isomorphic to H. Then there exists a diagram:

$$\begin{array}{ccc} (M_H, d) & & (N,d) \\ \downarrow & & \downarrow \\ (H,0) & \xrightarrow{\cong} & H^*(N,d) \end{array}$$

and by a standard lifting type argument - c.f. [D-G-M-Su (1.2)] - it is possible to construct an isomorphism of DG algebras

$$\Phi: (M_H, d) \longrightarrow (N,d) ,$$

hence (M_H, d) and (N,d) represent the same rational homotopy type. Thus, given an algebra H, there is one and only one formal rational homotopy type which has cohomology isomorphic to H, and it is given by the minimal model of H. Any different rational homotopy types having cohomology isomorphic to H must necessarily be non-formal.

3. If an algebra H is intrinsically formal, then the rational homotopy type having cohomology isomorphic to H must be represented by the minimal model of H and hence must be formal.

So remarks 2 and 3 above give an equivalent definition of intrinsic formality:

2.1.3. Equivalent Definition.

A graded algebra H is intrinsically formal iff every rational homotopy type having cohomology isomorphic to H is formal.

It is now possible to outline a strategy for attacking the intrinsic formality problem: Given an algebra H, first take a minimal model to get a formal rational homotopy type having cohomology isomorphic to H. After this, search for other - necessarily non-formal - rational homotopy types having cohomology isomorphic to H. In fact, this strategy can be realised in the DG algebra category by the machinery of [H-St]. A brief summary of that machinery is now given, from the particular viewpoint relevant to the intrinsic formality problem.

2.1.4. **Summary of Machinery of Halperin and Stasheff.**

Given a 1-connected graded algebra H, in the paper [H-St] a 'canonical' way of constructing a minimal model for H is given which differs from the original construction used by Sullivan. This is called the bigraded model of the graded algebra H, and written

$$\rho: (\Lambda V, d) \longrightarrow (H, 0) \ .$$

Here the notation ΛV means the free graded commutative algebra on the (graded) vector space V. $(\Lambda V, d)$ is constructed in a similar fashion to the Tate resolution of an ungraded commutative algebra (c.f. also [Joz]). In fact, a vector space basis for V is constructed in such a way that it is bigraded; one grading - upper - is given by degree, and a second - lower - grading comes from the nature of the construction, which is performed in stages. The lower grading is determined by the stage at which elements are added. More properly, let H^+ be $\bigoplus_{i>0} H^i$. Then V_0 is defined to be $H^+/(H^+.H^+)$, retaining the same upper grading, and $\rho: \Lambda V_0 \longrightarrow H^+$ is defined by extending a splitting of $H^+ \longrightarrow V_0$; define d to be zero on V_0. Since H, in general, is not free, ker ρ, in general, is non-zero. So define V_1 to be $\ker\rho/(\ker\rho.\Lambda^+ V_0)$, with upper grading shifted down by one, and extend ρ to $\Lambda V_{(1)} = \Lambda(V_0 \oplus V_1)$ by defining ρ to be zero on V_1; define d on V_1 by requiring it to be a splitting of $\ker \rho \longrightarrow \ker \rho/(\ker\rho.\Lambda^+ V_0)$. After this, the remaining V_i's for $i \geq 2$ are constructed in such a fashion so as to kill off all higher cohomology - c.f. [H-St section 3]. Eventually, $V = \bigoplus_{i \geq 0} V_i$. This lower grading on a basis of V induces a lower grading on a vector space basis of the algebra ΛV. The differential d is of degree +1 with respect to the upper grading and of degree −1 with respect to the lower grading.

By putting ΛV_n equal to the vector space generated by all elements in ΛV of lower grading n, an increasing filtration of ΛV is defined:

$$F_n = \bigoplus_{i \leq n} \Lambda V_i \ .$$

By the comment after (2.1.1.) above, $(\Lambda V,d)$ must be a formal DG algebra since it is the minimal model of a graded algebra. Furthermore, $(\Lambda V,d)$ has cohomology isomorphic to H, by construction. The question arises then, which other rational homotopy types have cohomology isomorphic to H, and what is their relationship to $(\Lambda V,d)$?

If $(\Lambda V,d)$ is the bigraded model of a graded algebra H as above, and the filtration F_n is also as above, then a new differential D, say, on the same DG algebra ΛV, is said to be a perturbation of d if it satisfies the condition:

$$D-d: V_n \longrightarrow F_{n-2}$$

for all n. Now let (A,d) be some DG algebra such that $H^*(A,d)$ is isomorphic to H. Then

Theorem ([H-St] 4.4) With the above notation, there exists a perturbation D of d , and a map of DG algebras

$$\pi: (\Lambda V,D) \longrightarrow (A,d)$$

such that π^* is an isomorphism on cohomology - i.e. $(\Lambda V,D)$ represents the same rational homotopy type as (A,d) - and $(\Lambda V,D)$ and π are unique in an appropriate sense.

Remark. π: $(\Lambda V, D) \longrightarrow (A,d)$ is called the filtered model of the DG algebra (A,d).

This completes the summary of [H-St].

Now recall what the strategy for the intrinsic formality problem was, as outlined above (below 2.1.3). Using the above machinery, that strategy can be carried out as follows:-

2.1.5. Strategy.

Given a graded algebra H, construct the bigraded model of H. This gives the unique formal rational homotopy type having cohomology isomorphic to H. Consider all possible perturbations of the bigraded model; if these can be shown to be isomorphic to the bigraded model itself, then H must be intrinsically formal since by the theorem quoted above, all rational homotopy types having cohomology isomorphic to H are displayed in this way.

The advantages of using this machinery for the intrinsic formality problem are clear since after constructing the bigraded model, attention can be restricted to considering alterations of the differential alone, rather than the whole DG algebra. In section 3 of this chapter, some examples will be given which demonstrate the effectiveness of the above machinery in this context, and which also compare the use of DG algebras and DG Lie algebras in this context. However, attention is now turned to considering the intrinsic formality problem in the category which will be used for the bulk of the thesis - that of DG Lie algebras.

2.2. FORMALITY; INTRINSIC FORMALITY; DG LIE ALGEBRAS.

In this section, the concepts of formality and intrinsic formality are translated into DG Lie algebra terms. This is achieved essentially by using the Functors L and C introduced by Quillen [Q], [N_1], [N-M]. One problem in doing this, though, is that the resulting definitions look cumbersome and unnatural in comparison to their original forms in the DG algebra category. Good references for this section are [N-M], [N_1], [M_2], and [Q - appendix B].

Since, in the DG algebra category, formality is the property of a DG algebra having the same rational homotopy type as its cohomology; it would perhaps be tempting to lift this definition verbatim into the DG Lie algebra category. That is, it would perhaps be tempting to define a DG Lie algebra to be formal if it has the same rational homotopy type as its Lie algebra homology. This definition, however, is reserved for the property of coformality - see (2.2.7.) below. The definition of formality in DG Lie algebra terms is given below and followed by a discussion to make clear that this second definition of formality is precisely the original one transferred to DG Lie algebras using the Quillen functor. This latter fact appears as part of proposition 3.2 in [N-M].

Let (L,d) be a minimal DG Lie algebra. Minimality implies that the differential d must increase bracket length by at least one. If the differential increases bracket length by exactly one, then the differential is said to be quadratic.

2.2.1. Definition.

A minimal DG Lie algebra is formal if it is isomorphic to a minimal DG Lie algebra having a quadratic differential.

A DG Lie algebra is formal if its DG Lie algebra minimal model is formal.

A space is formal if its DG Lie algebra minimal model is formal.

2.2.2. Discussion.

Let H be a graded algebra of finite type, and let H_* be the dual coalgebra. H can be thought of as representing a formal - in the sense of (2.1.1.) - rational homotopy type, and indeed any formal rational homotopy type can be represented in this way. Recall how the Quillen functor L acts on such an object as H_* ([N_1 - Section 4] or [Q]). In general, for a DG coalgebra of finite type (C,d) say, a vector space is formed by taking the desuspension of C to get $s^{-1}C$. The free Lie algebra on this vector space is then formed, and a differential ∂ on $L(s^{-1}C)$ defined using both the coalgebra structure of H_* and the differential on C. In fact, the differential in $L(s^{-1}C,\partial)$ splits as the sum of two differentials,

$$\partial = \partial_d + \partial_\Delta$$

and for any element c in C, $\partial_d\,(s^{-1}c) = -\,s^{-1}(dc)$, so if the coalgebra differential, d, is zero, as in the cases of interest here, then the differential in $L(s^{-1}C,\partial)$ arises from the coalgebra comultiplication structure alone. Specifically, this is given as follows. Because C is commutative, it is possible to write the comultiplication Δ on an element c of C as

$$\Delta(c) = c \otimes 1 + 1 \otimes c + \sum_{i<j} c_i \otimes c_j + (-1)^{|c_i||c_j|} c_j \otimes c_i$$

$$+ \sum_i c_i \otimes c_i$$

where, in the first sum, if $i \neq j$, then $c_i \neq c_j$, and in the second sum, the degree of each c_i must be even. Then by doing this, it is possible to read off $\partial(s^{-1}c)$ from the formula for $\Delta(c)$ as

$$\partial(s^{-1}c) = -\sum_{i<j} (-1)^{|c_i|} [s^{-1}c_i , s^{-1}c_j] - 1/2 \sum_i [s^{-1}c_i , s^{-1}c_i] .$$

So, if the coalgebra has trivial differential - for example H_* - then the differential on the corresponding DG Lie algebra arises from the coalgebra structure alone, and in this case the resulting DG Lie algebra will have a quadratic differential.

Thus, since any formal - in the sense of (2.1.1.) - rational homotopy type X has a minimal model (M_X, d) for which there is a map

$$\psi: (M_X , d) \longrightarrow (H^*(X), 0)$$

such that ψ^* is an isomorphism; then under the equivalence

$$\text{DG ALGEBRAS} \equiv \text{DG LIE ALGEBRAS} ,$$

which takes weak equivalences to weak equivalences, the rational homotopy type of X is represented by $L(s^{-1}(M_X)_*, d)$ in DG LIE ALGEBRAS, and there will be a DG Lie algebra map

$$L(\psi_*): L(s^{-1}(M_X)_*, d) \longrightarrow L(s^{-1}H_*, \partial)$$

which is an isomorphism on (Lie algebra) homology. Now, the DG Lie algebra minimal model of X is by definition the DG Lie algebra minimal model of $L(s^{-1}(M_X)_*, d)$, and so there exists a map of DG Lie algebras

$$L(\psi_*).\rho: L_X,d \longrightarrow L(s^{-1}H_*,\partial) ,$$

which is an isomorphism on homology. But on minimal DG Lie algebras, weak isomorphisms are isomorphisms, hence the DG Lie algebra minimal model of X will be isomorphic to $L(s^{-1}H_*,\partial)$ which by the above has a quadratic differential. Thus , if a space X is formal according to the original definition (2.1.1.) , then it is formal according to definition (2.2.1.) above.

Conversely, assume a DG Lie algebra, (L, d), is formal according to definition (2.2.1.) above. Then since freeness is part of the definition of minimality, there is a DG map

$$\psi: (L,d) \longrightarrow L(V,\partial)$$

such that ψ_* is an isomorphism on (Lie algebra) homology, and $L(V,\partial)$ is a free Lie algebra on some vector space V with a quadratic differential, ∂. Thus the Quillen functor **C** [Q], [N_1], gives a map

$$\mathbf{C}(\psi): \mathbf{C}(L(V,\partial)) \longrightarrow \mathbf{C}(L,d)$$

which is an isomorphism on (coalgebra) homology.

Now in the special case that $L(V,\partial)$ is free with quadratic differential, it is possible to define a DG coalgebra (C,d) as follows:-

The coalgebra C, as a vector space, is sV - the suspension of the vector space V. The comultiplication on sV is defined by

$$\Delta(sv) = sv \otimes 1 + 1 \otimes sv + \sum (sv_i' \otimes sv_i'' + (-1)^{|sv_i'||sv_i''|} sv_i'' \otimes sv_i')$$

if $\partial(v) = \sum_i [v_i' , v_i'']$ in $L(V,\partial)$; and as a DG coalgebra, C has the trivial differential.

Then by construction, $\mathbf{L}(C,0) = \mathrm{L}(V,\partial)$. Thus, applying the functor $\mathbf{C}$ there is a map

$$\mathbf{C}(i)\colon\ \mathbf{C}(\mathbf{L}(C,0)) \longrightarrow \mathbf{C}(\mathrm{L}(V,\partial))$$

which is an isomorphism on homology, and composing this with the adjunction map $: C \longrightarrow \mathbf{CL}C$ which is also a weak isomorphism [N_1 – (4.1)], gives a map

$$\theta\colon\ (C,0) \longrightarrow \mathbf{C}(\mathrm{L}(V,\partial))$$

which is a weak isomorphism and composing this with $\mathbf{C}(\psi)$ from above, gives a weak isomorphism

$$\mathbf{C}(\psi)\bullet\theta\colon\ (C,0) \longrightarrow \mathbf{C}(L,d)\,.$$

Finally, dualising gives a map of DG algebras

$$\phi\colon\ (\mathbf{C}(L,d))^{*} \longrightarrow (C,\ 0)^{*}$$

which is an isomorphism on (DG algebra) homology. Hence, as $(C,0)^{*}$ is simply a graded algebra (trivial differential), $(\mathbf{C}(L,d))^{*}$ fits into the original definition of formality (2.1.1.) for DG algebras. But $(\mathbf{C}(L,d))^{*}$ is precisely the representative for the rational homotopy type of (L,d) when transferred from right to left in the equivalence

$$\text{DG ALGEBRAS} \quad \equiv \quad \text{DG LIE ALGEBRAS}\,.$$

Thus, if a DG Lie algebra is formal according to definition (2.2.1.) given above, then its rational homotopy type is represented by a DG algebra which is formal in the original sense of (2.1.1.).

This completes the discussion of the equivalence of the two definitions of formality.

At this point, an analogous set of remarks to that at (2.1.2) can be made. Here, it should be borne in mind that one of the main points of this thesis is to investigate the intrinsic formality problem, so the slant of these remarks is towards that objective.

2.2.3. Remarks.

1. Recall that given a graded algebra H of finite type , the DG algebra minimal model of H is a formal DG algebra having cohomology isomorphic to H. By the first part of the above discussion, the DG Lie algebra minimal model of H is a formal DG Lie algebra. By the second part of the discussion, this DG Lie algebra minimal model has homology coalgebra isomorphic to H_*. Hence, for every graded algebra H of finite type, there is a formal DG Lie algebra having homology coalgebra isomorphic to H_*.

2. Let H_* be a coalgebra of finite type. Let (K,d) and (L,d) be two formal minimal DG Lie algebras both having homology coalgebras isomorphic to $s^{-1}H_*$. Then it is possible, using a series of lifting type arguments (c.f. [D-G-M-Su (1.2)] and [B-L (1.4)]), to construct an isomorphism between (K,d) and (L,d). Hence, any two formal DG Lie algebras having isomorphic homology coalgebras will have isomorphic DG Lie algebra minimal models and thus represent the same rational homotopy type.

Since the two definitions of formality given so far correspond under the equivalences **L** and **C**, there is no need to alter the definition of intrinsic formality already given. Let H be a graded algebra of finite type. H is intrinsically formal if every rational homotopy type having cohomology isomorphic to H is formal. Now a rational homotopy type having cohomology isomorphic to H is represented in DG Lie algebra terms by a DG Lie algebra having homology coalgebra isomorphic to H_*. Thus

2.2.4. Definition.

A graded algebra H is intrinsically formal if every DG Lie algebra having homology coalgebra isomorphic to H_* is formal.

This definition clearly agrees with that given before.

So remarks 1 and 2 above give a strategy similar to that outlined at the end of section 1 for attacking the intrinsic formality problem; but this time in terms of DG Lie algebras. Given an algebra H of finite type, form the DG Lie algebra minimal model of H - denoted $L(s^{-1}H_*,\partial)$. By remarks 1 and 2 above, this is the unique formal rational homotopy type having homology coalgebra isomorphic to H_*. Any different rational homotopy types having homology coalgebra isomorphic to H_* must necessarily be non-formal. Fortunately, all rational homotopy types sharing a common homology coalgebra are related as follows:

Let H_* be a coalgebra,and form the DG Lie algebra minimal model of H_* to get $L(s^{-1}H_*,\partial)$ which has a purely quadratic differential by the discussion above. Any differential on $L(s^{-1}H_*)$ which retains the same quadratic part, but has an additional term which increases bracket length by at least 2 is called a perturbation of ∂ and is denoted by $L(s^{-1}H_*,\partial + P)$.

Then with this notation ;

2.2.5. Theorem [M_2] :

Given a coalgebra H_* of finite type, there is a bijection of sets:-

RATIONAL HOMOTOPY TYPES HAVING HOMOLOGY COALGEBRA H_*.	$\equiv$	ISOMORPHISM CLASSES OF DG LIE ALGEBRAS $L(s^{-1}H_*, \partial+P)$.

So although in general it is necessary to consider all rational homotopy types having homology coalgebra isomorphic to H_*, for the purposes of trying to solve the intrinsic formality problem, this task is made simpler by being able to restrict attention to those DG Lie algebras whose differentials are perturbations of the DG Lie algebra minimal model of H.

2.2.6. Summary.

To try and prove a graded algebra of finite type is intrinsically formal, it is possible to proceed as follows:-

First construct the DG Lie algebra minimal model of H, $L(s^{-1}H_*, \partial)$. Secondly, consider all DG Lie algebras of the form $L(s^{-1}H_*, \partial+P)$ where $\partial+P$ is a differential and P increases bracket length by at least two. If it is possible to define an isomorphism

$$\Phi : L(s^{-1}H_*, \partial) \longrightarrow L(s^{-1}H_*, \partial+P)$$

for a general such P, then H is intrinsically formal.

This is in fact the strategy that will be followed in this thesis. A specific example of this strategy will be offered in section 4 of this chapter, to illustrate some of the above discussion. One feature of this strategy is that since the DG Lie algebras on which it is necessary to construct maps are free, it is possible to work over a homogeneous bracket length at a time. This leads to the possibility of approaching the problem in the fashion of an obstruction theory. For more details see [St]. Use will be made of this feature in later results, c.f. (4.1.3.).

Finally, after explaining exactly what the definition of formality in terms of DG Lie algebras should be, and why; the definition of coformality is offered here by way of contrast:

2.2.7. Definition.

Let (L,d) be a minimal DG Lie algebra. (L,d) is coformal if there is a DG Lie algebra map

$$\psi: (L,d) \longrightarrow (\mathrm{H}(L,d),0)$$

such that ψ_* is an isomorphism on (Lie algebra) homology.

That this is not the same as formality can easily be seen from considering simple examples such as those in [N-M].

2.3. COMPARISON OF DG ALGEBRAS AND DG LIE ALGEBRAS.

When considering the intrinsic formality problem for a graded algebra of finite type there is a choice as to which approach to use. DG algebras can be used as in section 1 or DG Lie algebras can be used as in section 2. In both cases, the formal minimal model of a graded algebra must be constructed; then explicit isomorphisms must be constructed between it and any possible perturbations. The examples which follow, aside from consolidating the ideas of the above two sections, are intended to show that the choice of which category to work in can affect the amount of work to be done in order to prove intrinsic formality results.

2.3.1. Example 1.

Let H be a free algebra on n generators, say. This can be written

$$H = \Lambda(x_1 ,\ldots\ldots, x_n) .$$

The bigraded model of H is simply ($\Lambda(x_1, \ldots x_n)$, d=0). Because there are no relations to be introduced into the cohomology, it turns out that all elements of ΛV in this bigraded model have lower grading equal to 0. Since any perturbation of d must satisfy

$$D-d: V_n \longrightarrow F_{n-2}$$

then any perturbation must satisfy D = 0 = d, as F_{-2} =0. Hence there are no non-trivial perturbations of d and H is intrinsically formal.

However, the DG Lie algebra minimal model – written $L(s^{-1}H_*,\partial)$ – of H is in general quite difficult even to write down! No degrees for the algebra generators of H have as yet been given, and in any case $s^{-1}H_*$ will in general be an infinite dimensional vector space, let alone $L(s^{-1}H_*)$. So to write down explicit isomorphisms between this formal DG Lie algebra and a general perturbation in this context could be very awkward to say the least.

2.3.2. Example 2.

Let H be a graded algebra of finite type which is oddly graded. i.e.

$$H^{2i} = 0 \text{ , all i.}$$

Then all elements of $s^{-1}H_*$ are in even degrees, and hence all elements of $L(s^{-1}H_*)$ are of even degree also. The formal minimal DG Lie algebra minimal model of H has ∂ equal to 0 since in the coalgebra H_* all elements are primitive – c.f. section 2. Also, any perturbation of ∂ must be a degree –1 differential and because of the grading, it must also be trivial on elements of $s^{-1}H_*$. Hence there are no non-trivial perturbations of the DG Lie algebra minimal model of H, and so H is intrinsically formal.

However, to write down the bigraded model of such an algebra explicitly is quite hard, and to work with such models in general rapidly becomes complicated - c.f. [H-St (7.10)].

2.3.3. <u>Example 3.</u>

Let H be a Poincaré duality algebra. Then by choosing an appropriate basis for H, and hence by duality for H_*, it is possible to make the DG Lie algebra minimal model of H display Poincaré duality in an explicit fashion -c.f. the end of this chapter or [St] or [M_2]. This fact can be exploited in order to prove results about the rational homotopy types of such algebras, and indeed, in [M_2] is used to prove an intrinsic formality result along the same lines which will be used in this thesis.

As will be seen in later chapters of this thesis, it is possible, for certain types of algebras H, and with a wise choice of basis for H, to display multiplicative properties other than Poincaré duality in the DG Lie algebra minimal model of H - c.f. (3.2.3.). This can then be used to perform the kind of manipulations necessary in order to prove intrinsic formality in certain cases. Section 4 of this chapter is devoted to an explicit example intended to demonstrate this latter phenomenon.

However, if one chooses to work with DG algebras in these latter types of cases; then the rich structure of H is not necessarily displayed in a useful way, possibly making it more difficult to prove results.

2.3.4. Remark.

In the above examples, 1 and 2 were more to do with the difference in 'size' between the different types of minimal model. Example 3, however, gives some indication that, for the purposes of certain types of calculation, there is a very real qualitative difference between using DG algebras and DG Lie algebras. In particular, this has to do with the way in which DG isomorphisms can be suggested by the structure of the differential in the DG Lie algebra minimal model - c.f. the next section and chapters 4 and 5, or [St] and [M_2]. This is the reason why this thesis uses DG Lie algebras to analyse a concept which was first introduced in terms of DG algebras.

2.4. AN EXAMPLE.

The final section in this chapter is devoted to giving an explicit computation using the approach outlined in section 2. An additional feature of this example is that it uses a method of constructing maps similar to a method used subsequently and in much greater generality - c.f. chapters 4 and 5. This technique mimics ones used by [M_2] and [St] in the context of Poincaré duality algebras.

Let $X = (CP^2 \times S^2) \vee S^3$. Then $H^*(X;Q)$ has generators in the following dimensions:-

6	x^2a		
5			
4	x^2	xa	
3			b
2	x	a	
1			
0	1		

with relations $x^3=0$, $a^2=0$, $xb=0$, $ab=0$; where x, a and b are the obvious generators of $H^*(CP^2;Q)$, $H^*(S^2;Q)$ and $H^*(S^3;Q)$ respectively.

2.4.1. Proposition.

$H = H^*(X;Q)$ is intrinsically formal.

Proof. The DG Lie algebra minimal model of $H^*(X;Q)$ is first constructed, and then it is shown that any perturbation of this must be an isomorphic DG Lie algebra. In fact, an explicit isomorphism is constructed.

As a vector space, $s^{-1}H_*$ can be described by :-

5	$s^{-1}x^2a$		
4			
3	$s^{-1}x^2$	$s^{-1}xa$	
2			$s^{-1}b$
1	$s^{-1}x$	$s^{-1}a$	
0			

(abusing notation by using the same letters for dual elements) and according to the formula given in chapter 1, the differential on the free Lie algebra $L(s^{-1}H_*)$ is given by the formulae:-

$$\partial(s^{-1}x) = 0$$

$$\partial(s^{-1}a) = 0$$

$$\partial(s^{-1}b) = 0$$

$$\partial(s^{-1}xa) = -[s^{-1}x, s^{-1}a]$$

$$\partial(s^{-1}x^2) = -1/2\,[s^{-1}x, s^{-1}x]$$

$$\partial(s^{-1}x^2a) = -[s^{-1}x, s^{-1}xa] - [s^{-1}a, s^{-1}x^2].$$

This completes the definition of $L(s^{-1}H_*,\partial)$, the DG Lie algebra minimal model of H^*.

Now consider DG Lie algebras of the form $L(s^{-1}H_*,\partial+P)$ where P increases bracket length by at least 2. For dimensional reasons, P must be zero on all elements of $s^{-1}H_*$ apart from $s^{-1}x^2a$. Using the Jacobi identity it is possible to write the formula for P on this element as:

$$P(s^{-1}x^2a) = [s^{-1}x, R_2 + R_3] + [s^{-1}a, T_2 + T_3]$$

where R_2 , T_2 and R_3 , T_3 are terms of bracket length two and three respectively. There may not be a unique way of writing such an expression, but for the requirements of this proof, any choice of such an expression will suffice. So assume a choice has been made and fixed. It is trivial to check that $(\partial+P)^2 = 0$ for any such P and so any such expression defines a DG Lie algebra $L(s^{-1}H_*, \partial+P)$.

For P written as above, define an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*,\partial) \longrightarrow L(s^{-1}H_*,\partial+P)$$

as follows:

First, put Φ to be of the form $\Phi = 1 + \psi$ where ψ extends bracket length by greater than 1. Then comparing the expressions for P and ∂ on $s^{-1}x^2a$ from above, define ψ on $s^{-1}H_*$ by:-

$$\psi(s^{-1}xa) = -(R_2+R_3)$$

$$\psi(s^{-1}x^2) = -(T_2+T_3)$$

and $\psi=0$ elsewhere. Then extend Φ to a map of Lie algebras in the obvious way. That Φ is an isomorphism follows from the fact that it is of the form $1+\psi$, and to check that Φ is a chain map is trivial on all elements of $s^{-1}H_*$ apart from $s^{-1}x^2a$ where it is a chain map by construction. Thus, any DG Lie algebra of the form $L(s^{-1}H_*,\partial+P)$ is isomorphic to $L(s^{-1}H_*,\partial)$ and hence by the theorem at (2.2.5) above, $H^*(X;\mathbb{Q})$ is intrinsically formal. **QED.**

This example gives some indication of how the structure of the differential, in conjunction with writing the perturbation in a particular way, can suggest the construction of DG Lie algebra isomorphisms. Note that the above algebra is not strictly a Poincaré duality algebra, but possesses a similar kind of multiplicative structure.

CHAPTER 3

3.1. INTRODUCTION; BACKGROUND .

This section introduces a type of graded algebra which will occupy a central place in the rest of this thesis. Ultimately, it will be shown that this type of algebra provides a large number of intrinsically formal examples. First, however, some background information is provided.

In [D-G-M-Su], it was proved that compact Kähler manifolds are formal spaces. The proof relies heavily on a technical lemma - the so-called dd^c lemma - which is concerned with the differential geometry of the situation. In fact, formality over the real numbers was proved there, and subsequent 'theorems of descent' have had to be invoked to obtain formality over the rational numbers. It was shown by Lefschetz that compact Kähler manifolds have an interesting cohomology structure. This is described by the 'hard Lefschetz theorem', which in this context can be taken as reading:-

3.1.1. Hard Lefschetz Theorem.

Let X be a compact Kähler manifold of (real) dimension 2n. Then there exists a distinguished 2-dimensional class ω in $H^2(X;\mathbb{Q})$ such that the map

$$\omega^r: H^{n-r}(X;\mathbb{Q}) \longrightarrow H^{n+r}(X;\mathbb{Q}) ,$$

given by taking the cup product with ω r-times, is an isomorphism for all r; $0 \leqq r \leqq n$.

The question arises then; what, if anything, can be said about the rational homotopy of Kähler manifolds from a knowledge of the cohomology ring alone? For instance, if the cohomology ring could be shown to be intrinsically formal then any space having that ring for its cohomology would be formal - regardless of Kähler metrics and so on - and so its rational homotopy could be constructed from the cohomology alone. To this end, the appropriate algebraic properties are abstracted from the hard Lefschetz theorem to furnish the:

3.1.2. Definition.

Let H be a graded algebra of dimension 2n. H is a Lefschetz algebra if:-

1. H is a Poincaré duality algebra.

2. There exists ω in H^2 such that $\omega^r:H^{n-r} \longrightarrow H^{n+r}$ is an isomorphism for all $0 \leq r \leq n$.

3.1.3. Examples.

1. If X is a compact Kähler manifold then $H^*(X;\mathbb{Q})$ is a Lefschetz algebra.

2. Some intersection cohomology algebras provide examples of Lefschetz algebras [Mac].

3. Examples of a more artificial, but nonetheless interesting, kind can be constructed by forming spaces of type:

$$CP^n \# (S^n \times S^n)$$

and then building from this a space as:-

$$CP^n \times \{ CP^n \# (S^n \times S^n) \}$$

and so on. At each stage, $H^*(X;\mathbb{Q})$ gives a Lefschetz algebra - c.f. the beginning of chapter 6 where examples of this kind are given in a more formal way.

3.1.4. Lefschetz Decomposition.

If H is a Lefschetz algebra, there is a way of decomposing H - a kind of block decomposition - which is very useful for displaying H, as a vector space, in such a way that it helps calculations. Consider the isomorphisms:

$$\omega^r: H^{n-r} \longrightarrow H^{n+r}.$$

When r = n, this implies that ω^n is a non-zero multiple of the fundamental class of H, and thus none of the elements ω^i are zero, for $1 \leq i \leq n$. By assumption, H is 1-connected, although this is an unnecessary assumption for the purposes of the decomposition, so consider ker (ω^{n-1}) in deg 2. Denote this vector space by V_2. Then ω is not in V_2. Furthermore, as vector spaces,

$$H^2 \cong Q[\omega] \oplus V_2$$

and $\omega^r(Q[\omega] \oplus V_2)$ is of rank equal to rank H^2 for $0 \leq r \leq n-2$.

Now consider ker (ω^{n-2}) in degree 3. Denote this by V_3. Then $\omega^r(V_3)$ has rank equal to rank H^3 for all $0 \leq r \leq n-3$. The same procedure can be followed for ker (ω^{n-3}) in degree 4, and so on, until it is possible to draw the picture of H, displayed as:

2n	ω^n				
2n−1					
2n−2	ω^{n-1}	$\omega^{n-2}V_2$			
2n−3			$\omega^{n-3}V_3$		
2n−4	ω^{n-2}	$\omega^{n-3}V_2$		$\omega^{n-4}V_4$	
.	.	.	.	.	
.	.	.	.	.	
n	.	.	.	.	V_n
.	.	.	.	.	
.	.	.	.	.	
4	ω^2	ωV_2		V_4	
3			V_3		
2	ω	V_2			
1					
0	1				

Notice that all indecomposable elements of H must lie on the 'bottom edges' of the columns in such a display; in particular, there can be no indecomposable elements of H in degree greater than n. Of course, apart from multiplication by ω, such a decomposition does not say anything about the multiplicative structure of H. In certain cases, however, it does tell enough to help in proving the results desired later on. In particular, in section 2 of this chapter, the above decomposition is used to write down a particularly useful basis for the dual coalgebra to H; which in turn gives a basis for the DG Lie algebra minimal model of H, also known as the Quillen model of H.

This section is completed with the introduction of some notation:

3.1.5. Definition.

Let H be a Lefschetz algebra of dimension 2n, and assume a Lefschetz decomposition of H has been performed as above. Then if the vector spaces V_i are zero, for all $i \leq k-1$, H is said to be of type H(n,k).

3.2. MULTIPLICATION AND BASES FOR LEFSCHETZ ALGEBRAS.

Towards the end of the last chapter, allusions were made to a multiplicative property similar to Poincaré duality which Lefschetz algebras possess. Since any Lefschetz algebra H is also a Poincaré duality algebra, then for any class a in H, there exists some class x, of appropriate dimension – the Poincaré dual of a – such that

$$ax = \omega^n$$

where H is of dimension 2n and ω^n is the fundamental class of H. The idea is that the various ω^i for i less than n also act like fundamental classes for certain elements of H. Specifically:

3.2.1. Lemma.

Let H be a Lefschetz algebra of dimension 2n and type H(n,k), and let r be such that $|\omega^r| > 2n-k$. Then for any a such that $|a| \leq |\omega^r|/2$, there exists some x in H such that

$$ax = \omega^r.$$

Proof. Since H is of type H(n,k) then for $0 \leq i \leq k-1$, H^i is of rank 0 or 1 according as i is odd or even respectively. So by Poincaré duality, H^i is of rank 0 or 1 according as i is odd or even respectively, for $2n-k+1 \leq i \leq 2n$.

When $r = n$, Poincaré duality gives the desired result. So consider r in the range $2n-k+1 \leq 2r < 2n$, and let a be some element of degree $\leq r$; it is necessary to show that a 'inverts' in ω^r. Let a be an element of H^{r-j} for the appropriate $j \geq 0$; then by Poincaré duality there is some x such that

$$ax = \omega^n$$

and $|x| = 2n-(r-j) = n + (n-(r-j))$. But there is an isomorphism

$$\omega^{n-(r-j)}: H^{n-(n-(r-j))} \longrightarrow H^{n+(n-(r-j))}$$

and so $x = \omega^{n-(r-j)}y$ for some y, i.e.

$$x = \omega^{n-r}\,\omega^j\,y\,.$$

Now substitute this in the equation $ax = \omega^n$ to get

$$a\omega^{n-r}\omega^j y = \omega^n = \omega^{n-r}\omega^r.$$

And hence $|a\omega^j y| = |\omega^r|$, so because in this dimension rank H^i is 0 or 1, it is possible to write:

$$a\omega^j y = \lambda\omega^r.$$

But then $\omega^n = \omega^{n-r}a\omega^j y = \lambda\omega^n$ and so $\lambda \neq 0$. So define $x' = \omega^j y/\lambda$ and $ax' = \omega^r$ as desired. **QED.**

In the calculations which it is necessary to perform later on, the Quillen model of a Lefschetz algebra will be used. For this purpose, it is convenient to choose a particular type of basis for the dual coalgebra which displays both Poincaré duality and the 'Lefschetz' property. This basis is now constructed for a general Lefschetz algebra.

Recall that it is possible to perform a Lefschetz decomposition on a Lefschetz algebra H, as in section 1 of this chapter, and that this makes it possible to display H, as a vector space, as :-

2n	ω^n				
2n-2	ω^{n-1}	$\omega^{n-2}V_2$			
2n-3			$\omega^{n-3}V_3$		
2n-4	ω^{n-2}	$\omega^{n-3}V_2$		$\omega^{n-4}V_4$	
.	.	.	.	.	
.	.	.	.	.	
n	.	.	.	.	V_n
.	.	.	.	.	
.	.	.	.	.	
4	ω^2	ωV_2		V_4	
3			V_3		
2	ω	V_2			
0	1				

Thus, one possibility for a vector space basis of H would be to choose bases for the vector spaces V_i , and then let this choice determine a basis for the vector spaces ωV_i , $\omega^2 V_i$, and so on whenever these spaces are non-zero; then the elements ω^j would complete the basis. However, such a choice of basis would not reflect Poincaré duality in any particularly neat way. So a different basis is chosen in the following way.

With the above decomposition of H, consider the Poincaré duals of elements in the vector spaces V_i. These must all lie in H^{2n-i}, which is spanned by $\{\omega^{n-i}V_i, \omega^{n-i+1}V_{i-2}, \ldots\ldots, \omega^{n-i/2-1}V_2, \omega^{n-i/2}\}$, if i is even; and if i is odd, then the last two terms in this bracket must be replaced by the term $\omega^{n-i/2-3/2}V_3$. But ω^{n-i+j} annihilates V_i, for $j \geqq 1$, by construction, so Poincaré duality must define non-degenerate bilinear pairings:

$$:V_i \times \omega^{n-i}V_i \longrightarrow Q$$

for all i such that $2 \leqq i \leqq n$. Hence, given a choice of basis for each V_i, say

$$V_i = sp\{ v_{i1}, v_{i2}, \ldots\ldots, v_{i\nu(i)} \}$$

there is a corresponding dual basis of $\omega^{n-i}V_i$, written

$$\omega^{n-i}V_i = sp\{ v_{i1}^*, \ldots\ldots, v_{i\nu(i)}^* \}.$$

And furthermore, by construction, $v_{ij}v_{ik}^* = \delta_{jk}\omega^n$ for all i. Now, for $i < n-2$, consider the epimorphism

$$\omega: H^{2n-i-2} \longrightarrow H^{2n-i}.$$

This restricts to an isomorphism of vector spaces:

$$\omega: \omega^{n-i-1}V_i \longrightarrow \omega^{n-i}V_i .$$

Similarly, the map

$$\omega: V_i \longrightarrow \omega V_i$$

is a vector space isomorphism, and so with the choice of bases for V_i and $\omega^{n-i}V_i$ as above, each element v_{ij}^* in $\omega^{n-i}V_i$ is equal to ωu_{ij} for a unique u_{ij} in $\omega^{n-i-1}V_i$ and it is possible to define bases

$$\omega^{n-i-1}V_i = sp\{ u_{i1}, \ldots\ldots, u_{i\nu(i)} \}$$

and

$$\omega V_i = sp\{ \omega v_{i1}, \ldots\ldots, \omega v_{i\nu(i)} \}.$$

The Poincaré duality relations above imply that

$$v_{ij}(\omega u_{ik}) = \delta_{jk}\omega^n .$$

i.e.

$$(\omega v_{ij})u_{ik} = \delta_{jk}\omega^n$$

and so $\{ u_{ij} \}_j$ and $\{ \omega v_{ij} \}_j$ are bases dual to each other. Hence, writing $u_{ij} = (\omega v_{ij})^*$, this defines bases for V_i , ωV_i and respective dual bases for $\omega^{n-i}V_i$ and $\omega^{n-i-1}V_i$, with the dual bases connected via the formulae:

$$v_{ij}^* = \omega(\omega v_{ij})^*$$

for $2 \leq i \leq n-2$; and $1 \leq j \leq \nu(i)$.

It is clear that, for $i < n-3$, this construction can be repeated, and succesive repetitions of this, working down and up the various columns, will provide a basis for all of H apart from elements in degree n, and elements of the form ω^k for the various values of k.

3.2.2. Definition. (Poincaré – Lefschetz basis.)

Let H be a Lefschetz algebra of dimension 2n. A vector space basis for H will be called a Poincaré – Lefschetz basis if it is chosen in the following way:

First, perform a Lefschetz decomposition on H, and with the notation as in the preceding paragraphs, choose bases for the spaces V_i, $2 \leq i \leq n-1$. For $i < n-2$, let these bases determine bases for ωV_i in the obvious way; and in general, for $i + 2k < n$, let these bases determine bases for $\omega^k V_i$. This defines a basis for H in degrees less than n apart from those elements ω^k of degree less than n. For H in degrees greater than n, first form the 'dual bases' to those bases already chosen for the V_i's, as in the above, to give bases for $\omega^{n-i}V_i$ for $2 \leq i \leq n-1$. Then according to the procedure above, let these bases determine bases for $\omega^{n-i-k}V_i$ satisfying the relations

$$v_{ij}^* = \omega^k(\omega^k v_{ij})^*$$

for all k such that $2n - 2i - 2k + i > n$. This defines a basis for all elements of H apart from those elements in degree n and those elements of type ω^k. In degree n, let the bases of V_i for $2 \leq i \leq n$ determine bases in the obvious way, and let the elements ω^k for $1 \leq k \leq n$ be basis elements for the remainder. This completes the choice of a basis for H.

So for a Lefschetz algebra, the above gives a preferred choice of basis to use when working with the Quillen model of H. As indicated above, this basis reflects not only the 'Lefschetz' property of H but also the fact that H is a Poincaré duality algebra; and it does so in an explicit fashion. This is to our advantage because it allows us to make some remarks of a general nature about the differential in the Quillen model of H, which is the key tool for the manipulations of chapter 5.

3.2.3. <u>Remarks.</u>

Recall that in the definition of the Quillen model of an algebra H, the differential ∂ is 'read off' from the coalgebra structure of H_* – c.f. (2.2.2.). Let H be a Lefschetz algebra of dimension 2n, and assume that a Lefschetz – Poincaré basis has been chosen for H in the above fashion. Dualising to the coalgebra H_* and then forming $s^{-1}H_*$ – the desuspension of H_* – gives a vector space basis for $s^{-1}H_*$, which by abuse of notation can be written as:–

$$s^{-1}H_* = sp\{ s^{-1}\omega^k,\ s^{-1}\omega^p v_{ij}\ ,\ s^{-1}\omega^p v_{ij}{}^*,\ s^{-1}\omega^q v_{ij}\}$$

where $1 \leq k \leq n$; $2 \leq i \leq n$; $1 \leq j \leq \nu(i)$; p is such that $i \leq |\omega^p v_{ij}| < n$ and q is such that $|\omega^q v_{ij}| = n$. Then the formula for ∂, in the Quillen model of H, on some of these elements can be written

$$\partial(s^{-1}\omega^r v_{ij}) = - [s^{-1}\omega,\ s^{-1}\omega^{r-1}v_{ij}] - [s^{-1}\omega^2,\ s^{-1}\omega^{r-2}v_{ij}] + \ldots\ldots - [s^{-1}\omega^r,\ s^{-1}v_{ij}]$$

$$+ \{ \text{ brackets having no entries equal to } s^{-1}\omega^k \text{ for any } k \} .$$

and

$$\partial(s^{-1}\omega^p v_{ij}{}^*) =$$

$$- [s^{-1}\omega,\ s^{-1}\omega^{p+1}v_{ij}{}^*] - [s^{-1}\omega^2,\ s^{-1}\omega^{p+2}v_{ij}{}^*] - \ldots\ldots - [s^{-1}\omega^t,\ s^{-1}\omega^{p+t}v_{ij}{}^*]$$

$$+ \text{ brackets with entries of degree } \leq n-1$$

where, in this last expression, t is such that $|s^{-1}\omega^{p+t}v_{ij}{}^*| > n-1$.

Notice also, that there is no ambiguity about the notation $s^{-1}\omega v_{ij}{}^*$, as there is no such element as $(\omega)(v_{ij}{}^*)$ in the chosen basis of H, and so $s^{-1}\omega v_{ij}{}^*$ means $s^{-1}(\omega v_{ij})^*$.

3.2.4. Remark.

If H is an algebra, and $L(s^{-1}H_*,\partial)$ is the Quillen model of H; then those elements of $s^{-1}H_*$ in the kernel of ∂ are just $s^{-1}PH_*$ – the desuspensions of the primitive elements in the coalgebra H_*. Hence, if H is any Lefschetz algebra, then those elements of $s^{-1}H_*$ which are in the kernel of ∂ are contained in the set $sp\{ s^{-1}\omega, s^{-1}V_2 ,\ldots\ldots, s^{-1}V_n \}$. In the restricted case H is of type H(n,k) with $n \leq 2k-1$, then those elements of $s^{-1}H_*$ which are in the kernel of ∂ are exactly $sp\{ s^{-1}\omega, s^{-1}V_k ,\ldots\ldots, s^{-1}V_n \}$.

3.3. DG LIE ALGEBRAS AND UNIVERSAL ENVELOPING ALGEBRAS.

In later chapters, it will be necessary to check certain formulae and relations involving DG Lie algebras. Now it is well known that such checking can often be carried out in an easier fashion if the universal enveloping algebra is employed. This section, then, is devoted to a brief resumé of the salient properties of universal enveloping algebras; and in the last part, a result is proved concerning the linear independence of certain types of elements in a DG Lie algebra, and which will be invoked later on in the thesis. These facts can be found in the literature (eg. [Jac] with the appropriate adjustments to the graded case, [M-M], and [Q-appendix B]).

Let A be an associative algebra, and let A_L denote the Lie algebra of A. Let L be a (graded) Lie algebra.

3.3.1. Definition.

A pair $(U(L),i)$, where $U(L)$ is an associative algebra and i is a (graded) Lie algebra homomorphism

$$i: L \longrightarrow (U(L))_L ,$$

is a universal enveloping algebra of L if, for any (graded) Lie algebra map

$$\theta: L \longrightarrow A_L$$

there exists a unique map of associative algebras

$$\theta': U(L) \longrightarrow A$$

such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\;\theta\;} & A_L \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \theta'} & \\ & (U(L))_L & \end{array}$$

is a commutative diagram of (graded) Lie algebras.

The basic facts concerning universal enveloping algebras which will be needed here are as follows:-

1. For any Lie algebra L, there exists a universal enveloping algebra, unique up to isomorphism.

2. The map i is an inclusion of Lie algebras.

3. If (L, d) is a DG Lie algebra with d of degree -1, then there exists a degree -1 differential d on $U(L)$ such that i is a chain map.

4. There is a DG left inverse to i, r say, which is a 'shuffle' type map [Q]. That is; r of a single tensor element in the enveloping algebra, is a sum of brackets, all of which correspond to some permutation of the entries of the tensor.

5. If L is a free Lie algebra, say L(V) for some (graded) vector space V, then $U(L)$ can be taken to be T(V) – the tensor algebra on V.

In the case V is a graded vector space, and L is a free DG Lie algebra L(V, d), the above information can be summarised as:-

3.3.2. <u>Summary.</u>

The universal enveloping algebra of L(V, d) is

$$i\colon L(V, d) \longrightarrow (T(V),\ d)$$

such that i is an inclusion, and

1. On elements of V, $i(v) = v$.

2. On brackets, i is determined by

$$i([u, v]) = u \otimes v - (-1)^{|u||v|} v \otimes u.$$

3. d is determined by the fact that it is a derivation on T(V), together with the formula $d(iv) = i(d\,v)$ and 2 above.

4. A DG left inverse to i exists, r say, and r has the form $r(v_1 \otimes v_2 \otimes ..... \otimes v_n) =$

$$\sum \lambda_\sigma [v_{\sigma(1)} , [..... , [v_{\sigma(n-1)} , v_{\sigma(n)}]...]]$$

where the sum is over some set of permutations $\{\sigma\}$.

Finally, a result is given which gives linear independence criteria relevant to a number of situations of practical interest to this thesis. In particular, it is useful for checking formulae and relations defined on Quillen models - c.f. chapter 4 and subsequently.

3.3.3. Lemma.

Let V be a finite graded vector space, and let $\{v_1 ,......., v_n\}$ be a totally ordered basis for V. Let L(V) be the free Lie algebra on V, and x in L(V) an element of homogeneous bracket length r+1 of the form

$$x = \sum_i [v_i , X_i],$$

with

$$X_i = \sum_J \lambda_{(i,J)} [v_{j1}, [v_{j2}, [....., [v_{j(r-1)}, v_{jr}]]..]]$$

such that $\lambda_{(i,J)}$ is in Q,and $\lambda_{(i,J)} = 0$ unless Max{ v_i, v_{j1},......, $v_{j(r-2)}$ } < v_{jr} in each bracket, and $v_{j(r-1)} \leqq v_{jr}$ in each bracket, with respect to the given order. Then if x is zero in L(V), $\lambda_{(i,J)}$ is zero for each (i,J); and hence X_i is zero for each i.

Proof. By freeness, it is possible to consider the parts of x with $v_{j(r-1)} < v_{jr}$ and the parts with $v_{j(r-1)} = v_{jr}$ separately. First, the former. Consider the image of this component of x in the universal enveloping algebra T(V). This will be a sum of terms of tensor length r+1 of the form

$$\sum v_i \otimes i(X_i) \pm i(X_i) \otimes v_i$$

and all terms in x for which $v_{j(r-1)} < v_{jr}$ will make a unique contribution to i(x) with terms of configuration

$$\lambda_{(i,J)}\, v_i \otimes v_{j1} \otimes \ldots\ldots \otimes v_{j(r-1)} \otimes v_{jr} \,;$$

where v_{jr} is strictly maximal in each tensor. Thus if x, and hence i(x), is zero, then the component of i(x) in T(V) containing all such terms must sum to zero in T(V) independently. i.e.

$$\sum v_i \otimes \left(\sum \lambda_{(i,J)}\, v_{j1} \otimes \ldots\ldots \otimes v_{jr} \right) = 0 \,;$$

for all such terms. And hence, for each i, $\lambda_{(i,J)} = 0$ for all J such that $v_{j(r-1)} < v_{jr}$. This leaves only those brackets of x with $v_{j(r-1)} = v_{jr}$. Now in such brackets, $|v_{jr}|$ is odd, otherwise the bracket is zero, and hence under i,

$$i([v_{jr}, v_{jr}]) = 2\, v_{jr} \otimes v_{jr}.$$

Now, in a similar fashion to the above, i(x) contains unique contributions of the form

$$\lambda_{(i,J)} \; v_i \otimes v_{j1} \otimes \ldots\ldots \otimes v_{jr} \otimes v_{jr}$$

with $v_{j(r-1)} = v_{jr}$, and v_{jr} of maximal degree in each tensor, and so these terms must sum to zero independently of all other terms in i(x). Thus the formula

$$\sum v_i \left(\sum \lambda_{(i,J)} v_{j1} \otimes \ldots \otimes v_{jr} \right) = 0$$

holds for such terms, and hence once again $\lambda_{(i,J)} = 0$. **QED**.

CHAPTER 4

This chapter begins to bring together the general theory behind the intrinsic formality problem, which was introduced in chapter 2, and the information in chapter 3 concerning Lefschetz algebras. The end purpose is to prove intrinsic formality results in particular cases. The first section is taken up with introducing some technical tools which are used in later sections; and in sections 2 and 3, intrinsic formality results are proven for certain restricted types of Lefschetz algebras. The results of section 2 are generalised both in section 3 of this chapter, and in chapter 5. However, they are included to introduce the methods of proof, and techniques thereof, in context and in as smooth a way as possible.

4.1. SOME GENERAL RESULTS.

This section provides three results of a general nature which are useful tools when considering formality, perturbations, and so-on. Proofs may exist in the literature, but here the results are stated in a form convenient to this thesis, and so proofs are included for completeness.

Recall that L(V) denotes the free Lie algebra on the vector space V. Henceforth, ∂ denotes a quadratic differential on a Lie algebra, and $\partial + P$ a perturbation of the particular differential in question.

4.1.1. Proposition.

Let V be a (0-connected, graded) vector space, and $L(V,\partial)$ a minimal DG Lie algebra. Let $L(V,\partial + P)$ be a perturbation, and Φ a map of vector spaces:

$$\Phi: V \longrightarrow L(V)$$

of the form $\Phi = 1 + \psi$, where ψ extends bracket lengths by at least one, but not necessarily homogeneously. If ψ is extended to act on brackets of L(V) according to the formula

$$\psi([X,Y]) = [\psi X, Y] + [X, \psi Y] + [\psi X, \psi Y] ;$$

then there exists a new perturbation $L(V, \partial + Q)$ say, which satisfies

$$Q = P + \psi\partial + \psi P - \partial\psi - Q\psi ;$$

and in these circumstances,

$$\Phi: L(V, \partial + P) \longrightarrow L(V, \partial + Q)$$

is an isomorphism of DG Lie algebras.

Proof. Given such a Φ, the perturbation Q can be defined on elements of V using the above formula, inductively over degree. First define Q to be zero on elements of lowest degree, then extend to brackets of these elements, then go up a degree and using the formula, define Q on elements of V of this higher degree, and so-on. At all times, ψ must be required to act as above on brackets, and Q must act as a derivation. Having defined Q, it is necessary to check that $\partial + Q$ is a differential, and that Φ is a DG isomorphism.

$\Phi = 1 + \psi$ is a map of Lie algebras, since Φ is a homomorphism if, and only if ψ acts as specified on brackets. That Φ will be an isomorphism is trivial to check, it is a consequence of Φ being of the form $1 + \psi$.

For Φ to be a chain map, it is necessary to check that

$$\Phi(\partial + P) = (\partial + Q)\Phi .$$

Substituting $\Phi = 1 + \psi$ in this relation, it is necessary to check

$$\partial + P + \psi\partial + \psi P = \partial + \partial\psi + Q + Q\psi .$$

To check this relation, acting on elements of V, it is sufficient to check separately , all homogeneous bracket lengths. So the relation splits into two parts; one which extends bracket length by one, and this part reduces to $\partial = \partial$; and a second part which extends bracket length by greater than one, and this is simply a re-writing of the formula used to define Q. Thus, since Φ is a chain map on L(V) iff it is a chain map on elements of V, it only remains to check that $\partial + Q$ is a differential.

Again, for checking the relation $(\partial + Q)^2 = 0$ acting on elements of V, it is sufficient to check that

$$\partial Q + Q\partial + QQ = 0 ,$$

on all elements of V; since $\partial^2(v) = 0$ by assumption. On elements of degree 1, this is true trivially; so assume inductively that this is true on all elements of V of degree $\leq$ k, and all brackets of elements of degree $\leq$ k. Let v be of degree k+1 in V; then from the definition of Q,

$$Q(v) = P(v) + \psi\partial(v) + \psi P(v) - \partial\psi(v) - Q\psi(v)$$

and applying ∂ to this equation gives

$$\partial Q(v) = \partial P(v) + \partial\psi\partial(v) + \partial\psi P(v) - \partial Q\psi(v) ;$$

Applying the definition of Q to $\partial(v)$ gives

$$Q\partial(v) = P\partial(v) + \psi P\partial(v) - \partial\psi\partial(v) - Q\psi\partial(v)$$

and adding these latter two equations gives

$$\partial Q(v) + Q\partial(v) = \partial P(v) + P\partial(v) + \partial\psi\partial(v) - \partial\psi\partial(v) + \psi P\partial(v)$$

$$+ \partial\psi P(v) - \partial Q\psi(v) - Q\psi\partial(v)\,. \qquad (1)$$

But as $\partial + P$ is a differential, $(\partial P + P\partial + PP)(v) = 0$. So substituting this in (1) gives

$$\partial Q(v) + Q\partial(v) = -PP(v) + \psi P\partial(v) + \partial\psi P(v) - \partial Q\psi(v) - Q\psi\partial(v). \qquad (2)$$

Now, applying the definition of Q to P(v) gives the equation

$$QP(v) = PP(v) + \psi\partial P(v) + \psi PP(v) - \partial\psi P(v) - Q\psi P(v)$$

$$= PP(v) - \psi P\partial(v) - \partial\psi P(v) - Q\psi P(v)$$

with the last line following since $\partial P(v) + PP(v) + P\partial(v) = 0$. Using this last formula to substitute for $-PP(v)$ in (2) gives the following equation:

$$\partial Q(v) + Q\partial(v) = -QP(v) - \psi P\partial(v) - \partial\psi P(v) - Q\psi P(v)$$

$$+ \psi P\partial(v) + \partial\psi P(v) - \partial Q\psi(v) - Q\psi\partial(v)$$

$$= -QP(v) - Q\psi\partial(v) - Q\psi P(v) - \partial Q\psi(v)\,.$$

But $\psi(v)$ consists of a sum of brackets of length greater than one, and hence, a sum of brackets, all of whose entries are of degree less than k. $(\partial + Q)^2 = 0$ on $\psi(v)$, and so $-\partial Q(\psi(v)) = Q\partial(\psi(v)) + QQ(\psi(v))$, and finally, substituting this latter piece of information in the above gives

$$\partial Q(v) + Q\partial(v) = -Q\,(\,P(v) + \psi\partial(v) + \psi P(v) - \partial\psi(v) - Q\psi(v)\,)$$

$$= -QQ(v).$$

i.e.

$$(\partial Q + Q\partial + QQ)(v) = 0$$

(*)
for elements v of V in degree k+1. Therefore, by induction on k, $(\partial + Q)^2 = 0$ on L(V). **QED**

This technical result has, as a corollary, a result which is of great practical value when considering perturbations in general, and intrinsic formality in particular. The following result will be invoked at a number of places later in the thesis.

4.1.2. Corollary.

Let V be a (0-connected, graded) vector space. Let $L(V, \partial)$ be a minimal DG Lie algebra, and let $L(V, \partial + P)$ be any perturbation. Assume that all bracket length ≥ 3 terms in $H(L(V, \partial))$ are zero in degrees $\leq r$. Then there exists a new perturbation $L(V, \partial + Q)$ say, and an isomorphism of DG Lie algebras

$$\Phi: L(V, \partial + P) \longrightarrow L(V, \partial + Q)$$

such that $Q = 0$ on all elements of L(V) of degree $\leq r+1$.

Proof. A vector space map $\Phi = 1 + \psi$ will be constructed, and extended to act on brackets of L(V) according to the formula

$$\psi([X, Y]) = [\psi X, Y] + [X, \psi Y] + [\psi X, \psi Y] .$$

The perturbation Q will automatically follow from the above proposition. ψ is constructed inductively over degree as follows:

Define $\psi(v) = 0$ on all elements of degree 1 in V, and all brackets of

(*) Since $(\partial + Q)$ is a derivation, this implies that $(\partial + Q)^2 = 0$ on all brackets having entries of degree $\leq$ k+1.

elements of degree 1 in L(V), in the above fashion. Now inductively assume ψ has been defined on all elements of V in degrees $\leq k$ for some $k \leq r$, and extended to all brackets of such elements, such that the relation

$$\partial\psi = \psi\partial + P + \psi P \qquad (\dagger)$$

is true. Consider v in a chosen basis of V of degree k+1. $(\partial + P)(v)$ is a sum of brackets, all of whose entries are of degree less than k. Thus applying $(\dagger)$ to $(\partial + P)(v)$, gives the equation

$$\partial\psi(\partial + P)(v) = \psi\partial(\partial + P)(v) + P(\partial + P)(v) + \psi P(\partial + P)(v)$$

i.e.

$$\partial\psi\partial(v) + \partial\psi P(v) = \psi\partial\partial(v) + \psi\partial P(v) + P\partial(v) + PP(v) + \psi P\partial(v) + \psi PP(v)$$

so

$$\partial\psi\partial(v) + \partial\psi P(v) = P\partial(v) + PP(v) + \psi(\partial P + PP + P\partial)(v) .$$

But $(\partial P + PP + P\partial) = 0 = (\partial + P)^2$, since $\partial^2 = 0$.

Substituting this in the last equation gives

$$\partial\psi\partial(v) + \partial\psi P(v) = -\partial P(v)$$

i.e.

$$\partial(\psi\partial(v) + \psi P(v) + P(v)) = 0.$$

Hence, $(\psi\partial(v) + \psi P(v) + P(v))$ is a cycle of degree k, which is less than or equal to r by assumption. Thus, for some η in L(V) of bracket length ≥ 2,

$$\psi\partial(v) + \psi P(v) + P(v) = \partial\eta$$

by the assumptions on H(L(V, ∂)). Now define

$$\psi(v) = \eta$$

and by the above equation, (†) is satisfied when acting on v. This defines ψ on all elements v of degree k+1 such that (†) is true when acting on elements of V of degree k+1. Using the given formula to extend ψ to brackets of elements of degree $\leq$ k+1; it is easy to check that (†) is true on all such elements of L(V) also. By repeating the above steps inductively, define ψ on all elements of V of degree $\leq$ r+1. On any remaining elements of V of degree greater than r+1, define $\psi = 0$. This completes the definition of ψ, and defining $\Phi = 1 + \psi$ gives a vector space map of the form required to invoke the above proposition (4.1.1.).

Hence, by that proposition, the formula

$$Q = P + \psi\partial + \psi P - \partial\psi - Q\psi$$

inductively defines a perturbation L(V, ∂ + Q) such that

$$\Phi\colon L(V, \partial + P) \longrightarrow L(V, \partial + Q)$$

is an isomorphism of DG algebras. But Q is clearly zero on elements of V of degree one, and inductively assuming Q to be zero on elements of degree k for some k $\leq$ r, then the formula (†) above clearly implies that Q is zero on elements of degree k+1 also. Thus, by induction, Q is zero on all elements of degree $\leq$ r+1. **QED**

As mentioned previously, the above corollary will be used later in this thesis and in any case, it is of general use when considering perturbations and similar ideas. However, in certain situations, it is of interest to be able to make statements about perturbations in degrees above the (length three) connectedness degree of $L(V, \partial)$; and in such situations, (4.1.2.) may not be applicable. For instance, if it is required to show two non-trivial perturbations to be isomorphic, then the above result tends to require hypotheses which are too strict to be of practical use. This latter situation is not of concern to this thesis. However, it will certainly be necessary to have a general result along the lines of the above yet with weaker initial hypotheses. To this end, the following result is also included, which is a rewording of a result stated in [St].

If $L(V, \partial)$ is a free DG Lie algebra, and $L(V, \partial+P)$ is a perturbation, then in general, a (second) grading can be placed on a vector space basis of $L(V)$ simply by grading elements according to their bracket length. This, in turn, grades all elements of $L(V)$ of homogeneous bracket length. Now on all elements of V, P can be split up into its various homogeneous length parts, and hence in general - i.e. on all elements of $L(V)$ - can be written

$$P = P_3 + P_4 + P_5 + \ldots\ldots ,$$

where P_i extends bracket length by i-1. Similarly, with the same filtration on $L(V)$, any map of vector spaces

$$\Phi : V \longrightarrow L(V)$$

can be split into its homogeneous length parts, and written

$$\Phi = \phi_1 + \phi_2 + \phi_3 + \ldots\ldots ,$$

where ϕ_i extends bracket length by i-1. For the rest of the thesis, this will be regarded as standard notation.

Let V be a 0-connected, graded vector space, and $L(V,\partial)$ a minimal DG Lie algebra, with $L(V,\partial+P)$ a perturbation. As above, write $P = P_3 + \cdots$, and let Φ be a vector space map

$$\Phi = 1 + \Psi : V \longrightarrow L(V).$$

Writing Ψ as $\Psi_2 + \cdots : V \longrightarrow L(V)$, assume also the hypotheses that for any bracket $[X,Y]$ in $L(V)$ of degree less than the dimension of V; or for all brackets in $L(V)$ if V is infinite dimensional; $[\Psi_i X, \Psi_j Y] = 0$ for any i,j.

If $\Psi : L(V) \rightarrow L(V)$ and hence $\Phi : L(V) \rightarrow L(V)$ is the map obtained by extending the vector space map $\Psi : V \rightarrow L(V)$ as at (4.1.1.); then Ψ_i is defined on $L(V)$ as that part of Ψ which extends bracket lengths by i-1. With this notation, the formula of (4.1.1.) can be written

$$Q_n = P_n + \Sigma_{j \geq 2} (\Psi_j P_{n+1-j} - Q_{n+1-j} \Psi_j)$$

for all n, with the convention that P_2 and Q_2 both equal ∂ and $Q_i = 0 = P_i$ for $i \leq 1$. Then $Q = Q_3 + \cdots$ defines a perturbation, $L(V,\partial+Q)$ for which $\Phi : L(V,\partial+P) \longrightarrow L(V,\partial+Q)$ is a DG isomorphism.

The vector space map $\Psi = \Psi_2 + \cdots : V \rightarrow L(V)$ can also be used to define first maps $\Psi_i' : L(V) \rightarrow L(V)$, extended to brackets according to the rule

$$\Psi_i'([X,Y]) = [\Psi_i' X, Y] + [X, \Psi_i' Y];$$

and hence $\Psi' = \Psi_2' + \cdots : L(V) \rightarrow L(V)$. This gives a map $\Phi' = 1 + \Psi' : L(V) \rightarrow L(V)$, which, in general, will not be a Lie algebra map. However, under the given hypotheses, the maps Ψ and Ψ' coincide, at least on elements of V and on elements of $L(V)$ of degree less than the dimension of V. So the formula

$$Q_n = P_n + \Sigma_{j \geq 2} (\Psi_j' P_{n+1-j} - Q_{n+1-j} \Psi_j')$$

for all n, with the convention that Q_2 and P_2 both equal ∂, and $Q_i = 0 = P_i$ for $i \leq 1$ can be used to define a new perturbation, and in these circumstances,

$$\Phi : L(V,\partial+P) \longrightarrow L(V,\partial+Q)$$

is a DG isomorphism.

4.1.4. Remark.

It was stated above that the idea of this last result was that it provides a useful tool for the intrinsic formality problem in particular, and working with perturbations in general. However, the restriction on Φ looks, at first sight, like one which may not pertain in all but the most special of cases. In fact this turns out not to be the case, and indeed, in section 3 of this chapter, and in chapter 5, the results (4.1.2.) and (4.1.3.) are used together to great effect. Typically, a situation where the above result may be of use is the following: If H is a Poincaré duality algebra, or indeed any finite algebra; if it can be shown that all perturbations of the Quillen model of H are trivial up to 'the middle' degree - for instance with the help of (4.1.2.) - then maps being constructed between two such perturbations can be taken to be the identity on degrees 'below the middle', and so will fit the hypotheses of (4.1.3.).

4.2. THE CASES H(n,n).

In this section, a first intrinsic formality result is proven. It is generalised, separately, both in the next section and in chapter 5. If X is a complete intersection of complex dimension n, then $H^*(X;\mathbb{Q})$ is a Lefschetz algebra of type H(n,n). This case has been treated in [N_2], and indeed, the results of this section could effectively be read off from that paper; although no explicit mention of intrinsic formality is made there. The main result of this section, (4.2.3.), depends on the computational result (4.2.2.), which in turn is a special case of (A.2.2.). The result (4.2.2.) is also contained in the results of [N_2]; but it should be pointed out that the 'spectral sequence' method of calculating the homotopy Lie algebra of complete intersections used in [N_2], although yielding a complete solution there, does not extend beyond that particular situation, and in particular is not applicable to the algebras to be studied in the next section, and in chapter 5.

Recall from chapter 3 the notions of Lefschetz decomposition and a Lefschetz - Poincaré basis for a Lefschetz algebra. In the cases H is of type H(n,n) for some n, these two provide a picture of H as :

2n	ω^n	
2n−2	ω^{n-1}	
2n−4	ω^{n-2}	
.	.	
.	.	
n		V_n
.	.	
.	.	
4	ω^2	
2	ω	
0	1	

where V_n is some vector space. The rank of H in degree n will of course depend upon whether n is odd or even. Let V_n have rank m, say. Then the rank of H^n is m or m+1 according as n is odd or even. It is now possible to write a Lefschetz - Poincaré basis for $s^{-1}H_*$ according to the procedure given at (3.2.2.). In fact it simply reduces to the obvious choice. Let $\{ v_1 , v_2 , \ldots , v_m \}$ be a vector space basis for V_n. Then the Quillen model of H can be written

$$L(\{ s^{-1}\omega^i \}, \{ s^{-1}v_j \} ; \partial)$$

where $1 \leq i \leq n$, and $1 \leq j \leq m$.

4.2.1. A Filtration on $L(s^{-1}H_*)$.

In order to give a useful vocabulary to help with calculations, the following notation is introduced - c.f. section A.1., for greater generality. This has nothing to do with any other gradings or filtrations previously mentioned. Let H be a Lefschetz algebra of type H(n,n), and assume that a Lefschetz decomposition has been performed on H. Then using the previous notation for a vector space basis of $s^{-1}H_*$, put a grading on a vector space basis of $s^{-1}H_*$ as

$$s^{-1}\omega^i \text{ has grading } 0 \text{ for all } i\ ;$$

$$s^{-1}v_j \text{ has grading } -1 \text{ for all } j\ .$$

This induces a filtration on $L(s^{-1}H_*)$,

$$L(s^{-1}H_*) = F_0 \supset F_{-1} \supset F_{-2} \supset \ \ldots\ldots\ .$$

4.2.2. Lemma.

Let H be a Lefschetz algebra of type H(n,n). Let $L(s^{-1}H_*, \partial)$ be the Quillen model of H. Then all terms of degree $\leq 2n-2$ and of bracket length ≥ 3 in $H(\ L(s^{-1}H_*,\partial)\ )$ are zero.

Proof. The filtration chosen for $L(s^{-1}H_*)$ here, coincides with that assumed for lemma (A.2.2.). Thus, according to that lemma, any bracket length greater than or equal to three class in $H(L(s^{-1}H_*,\partial))$ of degree less than or equal to 2n-2, can be represented by an element of F_{-2}. But all elements of $s^{-1}H_*$ of grading -1, in the case H is of type H(n,n), must be of degree n-1; thus, any non-zero bracket length greater than or equal to three term in F_{-2} must be of degree greater than 2n-2. **QED**

So it is now possible to prove an intrinsic formality result.

4.2.3. Theorem.

Let H be a Lefschetz algebra of degree 2n and of type H(n,n). Then H is intrinsically formal.

Proof. Let $L(s^{-1}H_*, \partial)$ be the Quillen model of H, and let $L(s^{-1}H_*, \partial + P)$ be any perturbation. Then by the above, all bracket length ≥ 3 terms in $H(L(s^{-1}H_*, \partial))$ in degrees $\leq 2n-2$ are zero. Hence, by (4.1.2.), there is a new perturbation Q such that Q is zero on elements of degree $\leq 2n-1$ and an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q) .$$

But Q must clearly be the trivial perturbation, and so by the general theory of chapter 2, H is intrinsically formal. QED

4.3. THE CASES H(J;n,n-1).

In the previous section, Lefschetz algebras with rank $H^2 = 1$ were considered. The cases considered there can be thought of as 'simplest possible', subject to the condition that rank $H^2 = 1$. In this section, the latter condition is relaxed, and a result is proved which is slightly better than 'simplest possible', subject to this relaxed condition, analogous to the cases H(n,n-1).

In general, if H is a Lefschetz algebra with rank $H^2 = r$, say, then as usual there exists an inclusion of algebras

$$i_H: H^*(CP^n; Q) \longrightarrow H .$$

However, there is no reason to assume that there exists an inclusion

$$i: H^*(CP^{j_1} \times CP^{j_2} \times \ldots \times CP^{j_r} ; Q) \longrightarrow H$$

for any $J = (j_1, \ldots, j_r)$, with $r \geqq 2$. Nonetheless, there do exist algebras of this kind, and they will be the concern of this section.

4.3.1. Definition.

Let H be a Lefschetz algebra of degree 2n such that there exists an inclusion

$$i: H^*(CP^{j_1} \times \ldots \times CP^{j_r} ; Q) \longrightarrow H$$

where $n = j_1 + \ldots + j_r$. Write $J = (j_1, \ldots, j_r)$. If i is an isomorphism in degrees $\leqq k-1$, then say that H is of type H(J;n,k).

4.3.2. Example.

If H is a Lefschetz algebra of type H(n,k), then it is also of type H(n;n,k).

4.3.3. Lefschetz Decompositions.

Given a Lefschetz algebra of degree 2n and of type H(J;n,k), the algebra $H^*(CP^J;Q) = H^*(CP^{j_1} \times \ldots \times CP^{j_r};Q)$ is also a Lefschetz algebra, c.f. (6.1.1.), and so can be decomposed as such. It is easy to see that H can be decomposed in such a way that the map

$$i: H^*(CP^{j_1} \times \ldots CP^{j_r} ; Q) \longrightarrow H$$

will respect these decompositions. If a decomposition for $H^*(CP^J;Q)$ is written

2n	ω^n			
2n−1				
2n−2	ω^{n-1}	$\omega^{n-2}V_2$		
2n−3	$\vdots$	$\vdots$	$\omega^{n-3}V_3$	
.	$\vdots$		$\vdots$	
n		$\vdots$	$\vdots$	V_n
.	$\vdots$			
3	$\vdots$	$\vdots$	V_3	
2	ω	V_2		
1				
0	1			

Then consider $i(\omega)$ in H^2. Since $i(\omega^n)$ is non-zero in H, then $i(\omega)$ can be taken to be the Kähler class of H. Now consider $\ker(i(\omega)^{n-1})$ in H^2. This certainly contains $i(V_2)$, and so a basis for $\ker(i(\omega)^{n-1})$ can be chosen to extend this , to give a W_2 such that

$$H^2 = i(\,\omega \oplus V_2\,) \oplus W_2.$$

Continuing in this fashion will result in a Lefschetz decomposition for H which can be displayed as follows:

2n	ω^n		
2n−1			
2n−2	ω^{n-1}	$\omega^{n-2}V_2$; $\omega^{n-2}W_2$	
2n−3	$\vdots$	$\vdots$	$\omega^{n-3}V_3$; $\omega^{n-3}W_3$
.	$\vdots$	$\vdots$	$\vdots$
.			$\vdots$
.	$\vdots$	$\vdots$	$\vdots$
3	$\vdots$	$\vdots$	V_3 ; W_3
2	ω	V_2 ; W_2	
1			
0	1		

This gives a useful notation for proving a result analogous to (4.2.2.). Keeping the above notation, and writing the Quillen model of H as $L(s^{-1}H_*, \partial)$;

4.3.4. Lemma.

Let H be a Lefschetz algebra of degree 2n and of type H(J;n,k). Then all terms of $H(L(s^{-1}H_*, \partial))$ in degrees $\leq 2n-2$, and of bracket length ≥ 3 can be represented by elements in the (Lie) ideal W generated by $\{s^{-1}W_i\ ,\ s^{-1}\omega W_i\ ,\ \ldots\ ,\ s^{-1}\omega^{n-i}W_i\}$, where $2 \leq i \leq n$.

Proof. Consider the inclusion of algebras

$$i\colon H^*(CP^J;\mathbb{Q}) \longrightarrow H\,.$$

This induces a surjection of DG Lie algebras

$$p\colon L(s^{-1}H_*, \partial) \longrightarrow L(s^{-1}H_*(CP^J;\mathbb{Q}), \partial).$$

Now, as at (A.2.1.), this can be fitted into a short exact sequence of Lie algebras,

$$0 \longrightarrow \ker(p) \longrightarrow L(s^{-1}H_*,\partial) \longrightarrow L(s^{-1}H_*(CP^J;Q),\partial) \longrightarrow 0 ,$$

which is in fact a short exact sequence of DG Lie algebras. Thus there is an induced long exact sequence on (Lie) homology. Now, it is known that the right hand term only has non-zero terms in homology of bracket length 1 and 2; so for bracket lengths ≥ 3, in degrees $\leq 2n-2$, j_* is surjective. Furthermore, all elements in the image of j are clearly in the ideal W. **QED**

As an immediate corollary of the above, there is

4.3.5. Corollary.

Let H be a Lefschetz algebra of degree 2n and of type H(J;n,n−1) for some J. Then all terms in $H(L(s^{-1}H_*, \partial))$ of bracket length ≥ 3 and of degree $\leq n-1$ are zero.

Proof. Let x be such a class. Then by the above, it is possible to assume x is in the ideal generated by $\{s^{-1}W_{n-1}, s^{-1}W_n, s^{-1}\omega W_{n-1}\}$. But the lowest degree of any element in this set is n−2, and so any length ≥ 3 bracket containing such an entry must be of degree n or greater. So x must be zero. **QED**

Unfortunately, this homology result is not strong enough to imply intrinsic formality on its own, as was the case in the analogous situation of section 2. To prove intrinsic formality here, it is first necessary to prove some preliminary results. As usual, denote the Quillen model of an algebra H by $L(s^{-1}H_*, \partial)$.

4.3.6. Lemma.

Let H be a Lefschetz algebra of degree 2n and of type H(J;n,k). Let W be the (Lie) ideal in $L(s^{-1}H_*, \partial)$ generated, as in (4.3.4.), by the set $\{s^{-1}W_i, s^{-1}\omega W_i, \ldots, s^{-1}\omega^{n-i}W_i\}$, for $k \leq i \leq n$; and let $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H such that im(P) is contained in W for P acting on elements of $s^{-1}H_*$ of degree $\leq r-1$ for some $r \leq 2n-1$.

Then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$ and an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q)$$

where Q is such that im(Q) is contained in W for Q acting on elements of $L(s^{-1}H_*)$ of degree $\leq r$.

Proof. Let v be an element of degree r, and consider P(v). Since $(\partial + P)^2 = 0$, then $\partial P + PP + P\partial = 0$ also, and hence $\partial P(v)$ is contained in W. So writing $P(v) = \xi + \chi$ where ξ is not an element of W, and χ is an element of W; a vector space map $\Phi = 1 + \psi$ is constructed as follows.

Consider the short exact sequence

$$0 \longrightarrow \ker(p) \longrightarrow L(s^{-1}H_*, \partial) \longrightarrow L(s^{-1}H_*(CP^J, Q), \partial) \longrightarrow 0,$$

as in the proof of (4.3.4.). Since $\partial P(v)$ is contained in W, $p\partial P(v) = 0$; and hence, $\partial pP(v) = 0$ also. But since pP(v) is of bracket length ≥ 3, it is exact, since $L(s^{-1}H_*(CP^J;Q), \partial)$ has no non-zero bracket length ≥ 3 terms in homology. Say $pP(v) = \partial\zeta$. Since p is onto, $\zeta = p(\eta)$ for some η, and so $p(\partial\eta - P(v)) = 0$. So by exactness, $\partial\eta - P(v) = j(\theta)$ for some θ, and so it is possible, given any v in degree r, to write

$$P(v) = \partial\eta - j(\theta),$$

where $j(\theta)$ is in W. Given such a choice for all v in $s^{-1}H_*$ of degree r, define $\Phi = 1 + \psi$ by putting $\psi = 0$ on degrees $\leqq r-1$; on degree r, put $\psi(v) = \eta$; and in degrees $\geqq r+1$, put $\psi = 0$. This completes the definition of Φ.

This defines a vector space map

$$\Phi\colon s^{-1}H_* \longrightarrow L(s^{-1}H_*)$$

which by (4.1.1.) automatically defines a perturbation $L(s^{-1}H_*, \partial + Q)$ according to the formula

$$Q = P + \psi\partial + \psi P - \partial\psi - Q\psi.$$

But Q will clearly equal P in degrees $\leqq r-1$, and so by assumption will have image in W on elements of degree $\leqq r-1$, and on brackets of such elements, and by construction, Q will also have image in W on elements of $s^{-1}H_*$ of degree r. **QED**

4.3.7. Corollary.

Let H be a Lefschetz algebra of degree 2n and type H(J;n,k), and $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H. Let W be the ideal in $L(s^{-1}H_*)$ of (4.3.4.). Then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$, isomorphic to the original, and such that im(Q) is contained in W.

Proof. Apply the above result successively, at each stage raising the degree below which im(Q) is contained in W, and starting with Q = 0 in degree 1. Eventually, im(Q) will be contained in W on the whole of $s^{-1}H_*$. **QED**

With this last result, it is now possible to prove another intrinsic formality result. The nature of the proof is based on work in [St]. It is possible to follow through the first part of the proof of the following lemma in more general situations than the one dealt with here; however, the success of the approach taken here depends critically on the second part of the proof which is tantamount to a 'checking' of formulae. The above result, which enables the perturbation to be suitably re-arranged first, in conjunction with (3.3.3.) is what makes the adopted approach successful.

Recall the definition of a Poincaré-Lefschetz basis for a Lefschetz algebra H. It is convenient to alter the notation slightly for the purposes of the following proof. Let H be a Lefschetz algebra of degree 2n and of type H(J;n,n-1). Then as at (4.3.3.) above, it is possible to perform a Lefschetz decomposition of H in such a way that it respects a decomposition of $H^*(CP^J;Q)$. Assume this has been done for H, then H is displayed, as a vector space, by:-

$$H= sp\{\ i(H^*(CP^J;Q)),\ W_{n-1},\ W_n,\ \omega W_{n-1}\ \}\ .$$

Denoting those elements of H which are in the image of i by x^K for p-tuples K, if J is a p-tuple, and following the procedure at (3.2.2.) to choose basis elements, then a basis for $s^{-1}H_*$ which will be used in the following proof, can be taken as:

$$s^{-1}H_* = sp\{\ s^{-1}x^K,\ s^{-1}x^{K\,*},\ s^{-1}x^L,\ s^{-1}w_{n-1,\,j}\ ,\ s^{-1}w_{n,\,j}\ ,\ s^{-1}w_{n-1,\,j}^{\ *}\ \};$$

for $|s^{-1}x^L| = n-1$, and $|s^{-1}x^K| < n-1$.

4.3.8. Lemma.

Let H be a Lefschetz algebra of degree 2n and of type H(J;n,n−1), and $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H. Assume that P is zero on elements of degree $\leq$ n and that im(P) is contained in W, where W is the ideal in $L(s^{-1}H_*)$ generated by $\{s^{-1}W_{n-1}, s^{-1}W_n, s^{-1}W_n^*\}$ as at (4.3.4.).

If P is of the form $P = P_m + P_{m+1} + P_{m+2} + \ldots$, for some $m \geq 3$, then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$ and an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q)$$

such that Q is zero on all elements of degree $\leq$ n, im(Q) is contained in W and $Q = Q_{m+1} + Q_{m+2} + \ldots$.

Proof. Since P is zero on degrees $\leq$ n, it is possible to use the 'specific' version of (4.1.1.) - i.e. (4.1.3.). A map $\Phi = 1 + \psi$ will be constructed such that ψ is of the form $\psi = \psi_{m-1} + \psi_m + \ldots$, and satisfying $\psi = 0$ on all elements of $s^{-1}H_*$ of degree $\leq$ n−1. Hence $[\psi_i X, \psi_j Y]$ will equal zero for all brackets [X, Y] of $L(s^{-1}H_*)$ of degree $\leq$ 2n−1 and (4.1.3.) will apply.

The map of vector spaces $\Phi: s^{-1}H_* \longrightarrow L(s^{-1}H_*)$ is constructed as follows. Consider P on the desuspension of the fundamental class of H, which will be denoted $s^{-1}x^J$; Without loss of generality, by using the Jacobi identity, this can be written:

$$P(s^{-1}x^J) = \sum [s^{-1}x^{J\,r_1}, B_{r1}] + \sum \lambda_{(1,r1,R)}\ [s^{-1}x^{J\,r_1}, [s^{-1}w_{n-1,\,r2}, s^{-1}w_{n,\,r3}]]$$

$$+ \sum \lambda_{(2,r1,R)}\ [s^{-1}w_{n-1,\,r1}, [s^{-1}x^{J\,r_2}, s^{-1}w_{n,\,r3}]]$$

$$(\dagger)_1 \qquad + \sum \lambda_{(3,r1,R)}\ [s^{-1}x^{J\,r_1}, [s^{-1}x^{J\,r_2}, [s^{-1}w_{n-1,\,r3}, s^{-1}w_{n-1,\,r4}]]]$$

$$+ \sum \lambda_{(4,r1,R)}\ [s^{-1}x^{J\,r_1}, [s^{-1}w_{n-1,\,r2}, [s^{-1}x^{J\,r_3}, s^{-1}w_{n-1,\,r4}]]]$$

$$+ \sum \lambda_{(5,r1,R)}\ [s^{-1}w_{n-1,\,r1}, [s^{-1}x^{J\,r_2}, [s^{-1}x^{J\,r_3}, s^{-1}w_{n-1,\,r4}]]] \ ;$$

where B_{r1} is a sum of brackets each having exactly one entry from $\{s^{-1}W_{n-1}, s^{-1}W_n, s^{-1}W_{n-1}{}^*\}$. And furthermore, for dimensional reasons, the terms containing B_r's are the only ones which can be of length $\geqq 5$.

From now on, it is sufficient to consider homogeneous length terms, say of length t+1. Each homogeneous length part of B_{r1} can be written –again using Jacobi– as

$$B_{r1} = \sum \lambda_{(r1,R)}\ [s^{-1}x^{J\,r_2}, [s^{-1}x^{J\,r_3}, \ldots\ldots, [s^{-1}x^{J\,r_t}, s^{-1}w_{n-1,\,r(t+1)}]]\ldots]]$$

$$+ \sum \mu_{(r1,R)}\ [s^{-1}x^{J\,r_2}, [s^{-1}x^{J\,r_3}, \ldots\ldots, [s^{-1}x^{J\,r_t}, s^{-1}w_{n,\,r(t+1)}]]\ldots]]$$

$$+ \sum \nu_{(r1,R)}\ [s^{-1}x^{J\,r_2}, \ldots\ldots, [s^{-1}x^{J\,r_t}, s^{-1}w_{n,\,r(t+1)}{}^*]]\ldots.]] \ .$$

Now consider the formula for $\partial(s^{-1}x^J)$;

$$\partial(s^{-1}x^J) = \sum (-1)^{c_r}\ [s^{-1}x^{J\,r}, s^{-1}x^{J\,r\,*}] + \sum (-1)^{a_r}\ [s^{-1}w_{n-1,\,r}, s^{-1}w_{n-1,\,r}{}^*]$$

$$+ \text{ terms with entries of degree } n-1 \text{ only.} \qquad (\dagger)_2$$

Where $a_r = |s^{-1}w_{n-1,\,r}|$, and $c_r = |s^{-1}x^{J\,r}|$.

In the formula $(\dagger)_1$, denote the bracket length i part of each B_r by $(B_r)_{(i)}$. Define $\psi = 0$ on all elements of degree $\leqq n-1$, $\psi = 0$ on $s^{-1}x^J$, and using the expressions $(\dagger)_1$ and $(\dagger)_2$ in conjunction with each other, 'read off' the definition of ψ on elements of degree inbetween these latter two as:

On elements of type $s^{-1}x^{J}{}_{r}{}^{*}$; if m=3,

$$\psi_2(s^{-1}x^{J\ r}{}_1{}^*) = - (-1)^{cr_1} (B_{r1})_{(2)}$$

$$- \sum (-1)^{cr_1} \lambda_{(1,r1,R)} [s^{-1}w_{n-1,\ r2} , s^{-1}w_{n,\ r3}]$$

if $m \leqq 4$,

$$\psi_3(s^{-1}x^{J\ r}{}_1{}^*) = - (-1)^{cr_1} (B_{r1})_{(3)}$$

$$- \sum (-1)^{cr_1} \lambda_{(3,r1,R)} [s^{-1}x^{J\ r}{}_2 , [s^{-1}w_{n-1,\ r3} , s^{-1}w_{n-1,\ r4}]]$$

$$- \sum (-1)^{cr_1} \lambda_{(4,r1,R)} [s^{-1}w_{n-1,\ r2} , [s^{-1}x^{J\ r}{}_3 , s^{-1}w_{n-1,\ r4}]]$$

and, for $i \geqq 4$,

$$\psi_i(s^{-1}x^{J\ r}{}_1{}^*) = - (-1)^{cr_1} (B_{r1})_{(i)} \quad .$$

On elements of type $s^{-1}\omega_{n-1,r}{}^{*}$; if m = 3,

$$\psi_2(s^{-1}w_{n-1,\ r1}{}^{*}) = - \sum (-1)^{ar_1} \lambda_{(2,r1,J)} [s^{-1}x^{J\ r}{}_2 , s^{-1}w_{n,\ r3}]$$

if $m \leqq 4$,

$$\psi_3(s^{-1}w_{n-1,r1}{}^{*}) = - \sum (-1)^{ar_1} \lambda_{(5,r1,R)} [s^{-1}x^{J\ r}{}_2, [s^{-1}x^{J\ r}{}_3 , s^{-1}w_{n-1,\ r4}]]$$

and, for $i \geqq 4$,

$$\psi_i(s^{-1}w_{n-1,\ r1}{}^{*}) = 0 .$$

This completes the definition of Φ: $s^{-1}H_* \longrightarrow L(s^{-1}H_*)$.

Now by (4.1.3.), this map defines a new perturbation $L(s^{-1}H_*, \partial + Q)$ such that Q is of the form $Q = Q_3 + Q_4 + \dots$, and Q_n satisfies the formula

$$Q_n = P_n + \sum (\psi_j P_{n+1-j} - Q_{n+1-j}\psi_j).$$

Since P is zero on all elements of degree $\leq$ n, and ψ is by definition zero on degrees $\leq$ n−1, then Q = 0 on elements of degree $\leq$ n−1. Furthermore, as im(P) is contained in W, and also im(ψ), by construction, then clearly im(Q) is contained in W. It remains to show that $Q_i = 0$ for $i \leq m$, and that Q is zero on elements of degree n.

P was assumed to be of the form $P = P_m + P_{m+1} + \dots$ for some m. So by assumption, $P_i = 0$ for $i \leq m-1$. Furthermore, from the way ψ was constructed, this implies that $\psi_i = 0$ for $i \leq m-2$. So from the defining formula for Q_i above, it is clear that $Q_i = 0$ for $i \leq m-1$, and that the defining formula for Q_m reduces to:

$$Q_m = P_m + \psi_{m-1}\partial - \partial\psi_{m-1} .$$

<u>Claim</u>. Q_m is zero.

<u>Proof of claim.</u> It is sufficient to check on elements of $s^{-1}H_*$ of degree greater than n−1 alone. For elements of degree n; P is zero here, and as ψ is zero on elements of degree $\leq$ n−1 then $\psi\partial$ is zero here also. Thus on elements of degree n,

$$Q_m = \partial\psi_{m-1} .$$

Now ψ_{m-1} has image in W, and the only length $\geq$ 2 brackets of degree n in W are of form

$$[s^{-1}x^{Jp} , [s^{-1}x^{Jq}, s^{-1}w_{n-1, k}]] \quad \text{or} \quad [s^{-1}x^{J p} , s^{-1}w_{n, q}] ,$$

where $|s^{-1}x^{J}q| = j$, and $|s^{-1}x^{J\,p}| = 1$. Thus ∂ of such elements is zero, and hence $Q_m = 0$ on all elements of $s^{-1}H_*$ of degree n. It remains to check that $Q_m = 0$ on all elements of $s^{-1}H_*$ of degree $\geqq$ n+1,. All such elements are of the form $s^{-1}x^{J_r}$ for some index J_r .

On the fundamental class $s^{-1}x^{J}$, Q_m is clearly zero by construction. For other elements consider the following two formulae, valid on elements of degree $\leqq$ 2n-1:

$$\partial P_m + P_m \partial = 0 ; \qquad (1)$$

$$\psi_{m-1}\partial\partial = 0 . \qquad (2)$$

Where the first is the appropriate length term of $(\partial + P)^2 = 0$; and the second is just a homogeneous length part of $\psi\partial^2 = 0$. When applied to the desuspension of the fundamental class, these give, respectively:

$$0 = \sum [\partial s^{-1}x^{J\,r_1}, (B_{r1})_{(m-1)}] + \sum (-1)^{cr_1} [s^{-1}x^{J\,r_1}, \partial(B_{r1})_{(m-1)}]$$

$$+ \sum [s^{-1}x^{J\,r_1}, P_m(s^{-1}x^{J\,r_1 *})] \qquad (1);$$

and

$$0 = \sum (-1)^{cr_1} [\partial s^{-1}x^{J\,r_1}, \psi_{m-1}(s^{-1}x^{J\,r_1 *})]$$

$$+ \sum [s^{-1}x^{J\,r_1}, \psi_{m-1}\partial s^{-1}x^{J\,r_1 *}] \qquad (2).$$

Now, the first summand in (2) could potentially contribute terms which are not in the first summand of (1); however, where $\psi_{m-1}(s^{-1}x^{J\,r_1 *})$ does not equal $(B_{r1})_{(m-1)}$, then the corresponding (left-hand) term in each bracket, $s^{-1}x^{J\,r_1}$, must be of degree 1 or 2, and hence $\partial(s^{-1}x^{J\,r_1}) = 0$ here. Also, c_r is equal to one for all r; so in fact (2) reduces to

$$0 = \sum - [\partial s^{-1}x^{J\,r_1} , (B_{r1})_{(m-1)}] + \sum [s^{-1}x^{J\,r_1} , \psi_{m-1}\partial s^{-1}x^{J\,r_1{}^*}] \qquad (2).$$

And adding (1) and (2) gives:

$$0 = \sum [s^{-1}x^{J\,r_1} , (P_m - \partial\psi_{m-1} + \psi_{m-1}\partial) (s^{-1}x^{J\,r_1{}^*})] .$$

Now, since im(P_m) is contained in W, and so is im(ψ_{m-1}), and hence both im($\partial\psi_{m-1}$) and im($\psi_{m-1}\partial$) are, too; each bracket in this last sum has a unique entry from the set $\{s^{-1}W_{n-1}, s^{-1}W_n, s^{-1}W_{n-1}{}^* \}$, which must be of maximal degree for that bracket, and in fact all the brackets will have this entry of maximal degree as the 'right hand' entry. Thus it is possible to invoke (3.3.3.) and deduce that for each r_1 ,

$$(P_m - \partial\psi_{m-1} + \psi_{m-1}\partial)(s^{-1}x^{J\,r_1{}^*}) = 0 .$$

And hence that $Q_m = 0$. <u>End of proof of claim.</u>

That $Q = 0$ on elements of degree n follows as, in general, from the defining formula of Q, $Q = -\partial\psi$; and for dimensional reasons, all elements in the image of ψ in degree n must be closed, as in the above proof of claim.

By (4.1.3.), Φ is a DG isomorphism

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q) ,$$

and putting together the above, Q satisfies all the requirements of the lemma. QED

4.3.9. <u>Theorem.</u>

Let H be a Lefschetz algebra of degree 2n and of type H(J;n,n-1). Then H is intrinsically formal.

Proof. Let $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H. By

(4.3.5.) and (4.1.2.), it is possible to assume P is zero on all elements of degree $\leq$ n. And by (4.3.7.), it is possible to assume also that im(P) is contained in the ideal W. Hence the above result applies, and can be used repeatedly, at each repetition, increasing the number r for which P_i is zero for all i less than r. Since H is finite, this must at some point result in $L(s^{-1}H_*, \partial + P)$ being isomorphic to $L(s^{-1}H_*, \partial)$. Thus by the general theory of chapter 2, H is intrinsically formal. **QED**

4.3.10. Remark.

At the beginning of this section, it was stated that the above result was slightly better than 'simplest possible'. In fact, as the example in chapter six demonstrates, for Lefschetz algebras of type H(J;n,k) this is a best possible result.

CHAPTER 5

In this chapter, a result is proven which generalises that of 4.2. The cases under consideration for this chapter are restricted to the cases H is a Lefschetz algebra of degree 2n and of type H(n,k) for $n \leq 2k-1$.

5.1. THE CASES H(n,k) FOR $n \leq 2k-1$.

Let H be a Lefschetz algebra of degree 2n and of type H(n,k) where n and k are as in the title. Notice that the numbers n and k do not determine H as an algebra. In fact they do not even determine the Betti numbers in dimensions where the rank of H is not one. However, restricting to this range does make it possible to make some remarks of a general nature about such an H. In particular, it makes possible a crucial calculation of the homotopy Lie algebra of H in degrees less than the dimension of the desuspension of H. This calculation in turn makes it possible to prove the desired intrinsic formality result with a similar approach to that taken in chapter 4. The statement of the result of this calculation is as follows:

Recall from (3.2.2.) that for any Lefschetz algebra, it is possible to construct a Lefschetz-Poincaré basis; and that in such a basis, certain elements will typically be denoted $s^{-1}v_{ij}$ where i is the degree of v in H - i.e. before desuspension. If $n \leq 2k-1$ then these elements satisfy $\partial(s^{-1}v_{ij}) = 0$. Using this notation;

5.1.1. Proposition.

Let H be a Lefschetz algebra of type H(n,k) where $n \leqq 2k-1$, and let $L(s^{-1}H_*, \partial)$ be the Quillen model of H. Then in degrees less than $2n-1$, the bracket length greater than 2 terms of $H(L(s^{-1}H_*, \partial))$ are spanned, as a vector space, by the following elements :

1. Bracket length $\geqq 5$ - all terms are zero.

2. Bracket length 4

   a. if $n = 2k-1$, then in degree $2n-2$, types $[s^{-1}v_{kp}, [s^{-1}v_{kq}, [s^{-1}v_{kr}, s^{-1}v_{ks}]]]$.

   b. zero otherwise.

3. Bracket length 3

   a. brackets of type $[s^{-1}v_{ij}, [s^{-1}v_{pq}, s^{-1}v_{rs}]]$.

   b. elements of type $[s^{-1}v_{ij}, X_{ij}]$; where X_{ij} is ∂-closed and of degree $\geqq n-1$.

The proof of this proposition is relegated to the appendix. It takes the form of a somewhat painstaking elaboration on a basic method of calculation. The actual method of calculating, however, remains of interest; and indeed has already implicitly been used at (4.2.2.). As an immediate corollary of this result, there is

5.1.2. Corollary.

Let H be a Lefschetz algebra of type H(n,k) with $n \leqq 2k-1$. Then in degrees $\leqq n + k - 3$, all terms of bracket length $\geqq 3$ in $H(L(s^{-1}H_*,\partial))$ are zero.

Proof. In the above proposition, $|s^{-1}v_{kj}| = k-1$ for all j. Thus bracket length 4 terms are only non zero in degrees $\geqq 4k-4 \geqq n+k-3$, since k must be greater than 1; Bracket length 3 terms are only non zero in degrees $\geqq (k-1) + (n-1)$. All bracket length $\geqq 5$ terms are zero in these degrees. **QED**

This result in turn makes it possible to deduce:

5.1.3. Corollary.

Let H be a Lefschetz algebra of type H(n,k) where $n \leqq 2k-1$, and let $L(s^{-1}H_*,\partial)$ be the Quillen model of H. If $L(s^{-1}H_*, \partial + P)$ is any perturbation, then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$ and an isomorphism of DG Lie algebras

$$\Phi\colon L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q)$$

such that Q is zero on all elements of degree $\leqq n+k-2$.

Proof. Just combine the above corollary with (4.1.2.). **QED**

5.1.4. Remark.

Further to the remarks made at (3.2.3.) concerning the Quillen model of a general Lefschetz algebra, in the case H is of type H(n,k) with $n \leqq 2k-1$ it is possible to be more specific. Recall that a basis for $s^{-1}H_*$ can be written $\{ s^{-1}\omega^r, s^{-1}\omega^q v_{ij}, s^{-1}\omega^p v_{ij}, s^{-1}\omega^p v_{ij}^* \}$ for suitable r, p, and q; and when written as such, $|s^{-1}\omega^p v_{ij}| = 2p+i-1$, and $|s^{-1}\omega^p v_{ij}^*| = 2n-i-2p-1$.

Then the formula for ∂ in the Quillen model of H on some of these elements can be written:

$$\partial(s^{-1}\omega^p v_{ij}) = - [s^{-1}\omega, s^{-1}\omega^{p-1}v_{ij}] - [s^{-1}\omega^2, s^{-1}\omega^{p-2}v_{ij}] + ... + [s^{-1}\omega^p, s^{-1}v_{ij}] .$$

and

$$\partial(s^{-1}\omega^p v_{ij}^*) =$$

$$- [s^{-1}\omega, s^{-1}\omega^{p+1}v_{ij}^*] - [s^{-1}\omega^2, s^{-1}\omega^{p+2}v_{ij}^*] - ..... - [s^{-1}\omega^t, s^{-1}\omega^{p+t}v_{ij}^*]$$

$$+ \text{ brackets having entries of degree } \leq n-1 ;$$

where, in this last expression, t is the largest integer such that $| s^{-1}\omega^{p+t}v_{ij}^* | = 2n-i-2(p+t)-1$ is $\geq n$.

5.1.5. **Lemma.**

Let H be a Lefschetz algebra of type H(n,k) with $n \leq 2k-1$. Let $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H such that if P is written $P = P_3 + P_4 + ...$, then P_3 is zero on all elements of degree $\leq r-1$ for some r such that $n+k-2 \leq r-1 \leq 2n-2$. Then there exists a map of vector spaces Φ: $s^{-1}H_* \longrightarrow L(s^{-1}H_*)$ of the form $\Phi = 1 + \psi_2$ and such that $\psi_2 = 0$ on all elements of degree $\leq n-1$; and if ψ_2 is extended to act on brackets of $L(s^{-1}H_*)$, by the formula

$$\psi([X, Y]) = [\psi X, Y] + [X, \psi Y] + [\psi X, \psi Y] ,$$

then $P_3 + \psi_2\partial - \partial\psi_2 = 0$ on all elements of $L(s^{-1}H_*)$ of degree $\leq r$.

Proof. The proof uses a combination of some properties of the Quillen model of H made explicit with a Poincaré-Lefschetz basis, and the homology result of (5.1.1.) above. First, notice that since r is in the range for which the rank of $s^{-1}H_*$ is 1 or 0 according as the degree is odd or even respectively, then if r is even, Φ equal to the identity will satisfy all the requirements. So assume r is odd, $r = 2p-1$, say. Using a Poincaré-Lefschetz basis of H, $s^{-1}H_*$ can be pictured as follows:

$2n-1$	$s^{-1}\omega^n$			
⋮	⋮			
$2p-1$	$s^{-1}\omega^p$			
⋮	⋮			
$2n-k-1$	.	$s^{-1}V_k^*$		
$2n-k-2$	.		$s^{-1}V_{k+1}^*$	
$2n-k-3$	.	$s^{-1}\omega V_k^*$		$s^{-1}V_{k+2}^*$
$2n-k-4$	.		$s^{-1}\omega V_{k+1}^*$	
$2n-k-5$	.	$s^{-1}\omega^2 V_k^*$		$s^{-1}\omega V_{k+2}^*$
$2n-k-6$	.		$s^{-1}\omega^2 V_{k+1}^*$	
$2n-k-7$	.	⋮		$s^{-1}\omega^2 V_{k+2}^*$
⋮	⋮		⋮	⋮
$2p-k+3$	.	$s^{-1}\omega^{n-(p+2)}V_k^*$		
$2p-k+2$	.		$s^{-1}\omega^{n-(p+2)}V_{k+1}^*$	
$2p-k+1$	.	$s^{-1}\omega^{n-(p+1)}V_k^*$		$s^{-1}\omega^{n-(p+2)}V_{k+2}^*$
$2p-k$			$s^{-1}\omega^{n-(p+1)}V_{k+1}^*$	
$2p-k-1$	.	$s^{-1}\omega^{n-p}V_k^*$		$s^{-1}\omega^{n-(p+1)}V_{k+2}^*$
$2p-k-2$	.		$s^{-1}\omega^{n-p}V_{k+1}^*$	
$2p-k-3$	.			$s^{-1}\omega^{n-p}V_{k+2}^*$
⋮	⋮	⋮	⋮	⋮
$n-1$		⋮	⋮	$s^{-1}V_n$
⋮	⋮	⋮	⋮	⋮
$k+5$				$s^{-1}\omega^2 V_{k+2}$
$k+4$	.		$s^{-1}\omega^2 V_{k+1}$	
$k+3$		$s^{-1}\omega^2 V_k$		$s^{-1}\omega V_{k+2}$
$k+2$	.		$s^{-1}\omega V_{k+1}$	
$k+1$		$s^{-1}\omega V_k$		$s^{-1}V_{k+2}$
k	.		$s^{-1}V_{k+1}$	
$k-1$		$s^{-1}V_k$		
⋮	⋮			
1	$s^{-1}\omega$			
0				

Now consider P_3 on $s^{-1}\omega^p$; since P_3 is zero on elements of degree $\leq r-1$, then the equation $(\partial + P)^2 = 0$ implies that $\partial P_3 + P_3 \partial = 0$, and hence that $\partial P_3 = 0$; so $P_3(s^{-1}\omega^p)$ is a cycle of bracket length 3. Thus by (5.1.1.) above, it is possible to write

$$P_3(s^{-1}\omega^p) = \partial(\eta) + \sum_{ij \text{ in } IJ'} [s^{-1}v_{ij} , X_{ij}] \qquad (1)$$

for some index set IJ', where $\partial(X_{ij}) = 0$, and $|X_{ij}| \geqq n-1$. Using the Poincaré-Lefschetz basis, it is also possible to write

$$\begin{aligned} \partial(s^{-1}\omega^p) = & \sum_{ij \text{ in } IJ} (-1)^{c_{ij}} [s^{-1}v_{ij} , s^{-1}\omega^{n-p}v_{ij}^*] \\ & + \sum (-1)^{c_{ij}} [s^{-1}\omega v_{ij} , s^{-1}\omega^{n-p+1}v_{ij}^*] \qquad (2) \\ & + \ldots\ldots\ldots + \sum (-1)^{c_{ij}} [s^{-1}\omega^{q_i} v_{ij} , s^{-1}\omega^{n-p+q_i} v_{ij}^*] \\ & + \sum_{j \leqq [p/2]} [s^{-1}\omega^j , s^{-1}\omega^{p-j}] \\ & + \text{Brackets having entries of degree} \leqq n-1. \end{aligned}$$

Where the indexing set IJ for (2) contains the indexing set IJ' for (1), by (3.2.1.); q_i is the largest integer such that $|s^{-1}\omega^{n-p+q_i}v_{ij}^*| > n-1$, $[p/2]$ is the largest integer $\leqq p/2$, and $c_{ij} = |s^{-1}v_{ij}|$.

Using (1) and (2) in conjunction with each other, it is possible to begin the definition of Φ. First, take Φ to be of the form $1 + \psi_2$ as required. Define ψ_2 to be zero on elements $s^{-1}\omega^k$ for all $k \leqq p-1$. Using (1) and the first summand on the right hand side of (2), define

$$\psi_2(s^{-1}\omega^p) = \eta ,$$

$$\psi_2(s^{-1}\omega^{n-p}v_{ij}^*) = - (-1)^{c_{ij}} X_{ij} , \text{ if } ij \text{ in } IJ' ;$$

$$\psi_2(s^{-1}\omega^{n-p}v_{ij}^*) = 0 , \text{ if } ij \text{ not in } IJ' .$$

Since only terms from the first summand of (2) are used, there is a 'ψ_2 not equal to zero line' – not of a homogeneous degree! – running across $s^{-1}H_*$ and occupied by the elements $\{ s^{-1}\omega^{n-p}V_i^* \}_{i=k,\dots,n}$. This line never goes below degree n in $s^{-1}H_*$. Now define $\psi_2 = 0$ on all elements of $s^{-1}H_*$ lying below this line in their respective columns. In particular, $\psi_2 = 0$ on all elements of degree $\leqq n-1$.

Now, by construction, $P_3 + \psi_2\partial - \partial\psi_2 = 0$ on $s^{-1}\omega^p$. Also, since the X_{ij}'s are cycles, then $\partial\psi_2 = 0$ on all elements of degree $\leqq r-1$ for which ψ_2 has so far been defined. However, there are elements of type $s^{-1}\omega^m v_{ij}^*$ and of degree $\leqq r-1$, for which $\psi_2\partial$ need not equal zero; it is necessary, therefore, to extend the definition of ψ_2 over such elements.

Without loss of generality, it is possible to consider each 'column' of elements $\{ s^{-1}V_i , s^{-1}\omega V_i ,\dots, s^{-1}\omega V_i^* , \; s^{-1}V_i^* \}$, in $s^{-1}H_*$, separately for each i, at least for the purposes of defining ψ_2. So consider the part of a general such column which lies above the 'ψ_2 not equal to zero line', which is where ψ_2 must now be defined. This can be pictured, for the '$s^{-1}V_i$' column, as:

degree	elements
2n−i−1	$s^{-1}v_{i1}^*, \dots\dots, s^{-1}v_{i\nu(i)}^*$
2n−i−3	$s^{-1}\omega v_{i1}^*, \dots\dots, s^{-1}\omega v_{i\nu(i)}^*$
.	. .
.	. .
.	. .
2p−i+1	$s^{-1}\omega^{n-p-1}v_{i1}^*, \dots\dots, s^{-1}\omega^{n-p+1}v_{i\nu(i)}^*$
2p−i−1	$s^{-1}\omega^{n-p}v_{i1}^*, \dots\dots, s^{-1}\omega^{n-p}v_{i\nu(i)}^*$
2p−i−3	ψ_2 equal to zero on elements $\leqq$ this degree in this column.

In this column, ψ_2 has only been defined on elements of degree less than or equal to 2p-i-1 so far; and on elements of this degree, $\psi_2(s^{-1}\omega^{n-p}v_{ij})$ $= -(-1)^{c_{ij}} X_{ij}$ by definition, where $\partial X_{ij} = 0$; and below this degree, $\psi_2 = 0$. From the construction of the Poincaré-Lefschetz basis,

$$\partial(s^{-1}\omega^{n-p-1}v_{ij}^{*}) = - [s^{-1}\omega, s^{-1}\omega^{n-p}v_{ij}^{*}]$$

+ brackets whose entries are below the line;

and so $\psi_2\partial(s^{-1}\omega^{n-p-1}v_{ij}^{*}) = - [s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p}v_{ij}^{*})] = \pm [s^{-1}\omega, X_{ij}]$; which is ∂-closed. Now, $|s^{-1}\omega^{a}v_{bc}^{*}| \leq n+k-2$, for any (a,b,c); so $\psi_2\partial(s^{-1}\omega^{n-p-1}v_{ij}^{*})$ is of degree $\leq n+k-3$ and of bracket length ≥ 3, so by (5.1.2.) is exact. Say $\psi_2\partial(s^{-1}\omega^{n-p-1}v_{ij}^{*}) = \partial\xi$. Then define

$$\psi_2(s^{-1}\omega^{n-p-1}v_{ij}^{*}) = \xi \,;$$

and by definition, $\partial\psi_2(s^{-1}\omega^{n-p-1}v_{ij}^{*}) = - [s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p}v_{ij}^{*})]$.

Claim. On the elements $\{ s^{-1}\omega^{n-p-t}v_{ij}^{*} \}_{t=0,\dots,n-p}$, for each ij, it is possible to define ψ_2 inductively such that

$$\partial\psi_2(s^{-1}\omega^{n-p-t}v_{ij}^{*}) =$$

$$- [s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p-(t-1)}v_{ij}^{*})] - \dots - [s^{-1}\omega^{t}, \psi_2(s^{-1}\omega^{n-p}v_{ij}^{*})] .$$

Proof of claim. For the cases t=0 and t=1, this has already been done. So assume inductively that it has been done for t = 1, 2, ... q-1. Then consider the formula for $\partial(s^{-1}\omega^{n-p-q}v_{ij}^{*})$.

$$\partial(s^{-1}\omega^{n-p-q}v_{ij}^{*}) = - [s^{-1}\omega,\ s^{-1}\omega^{n-p-(q-1)}v_{ij}^{*}]$$

$$- [s^{-1}\omega^{2},\ s^{-1}\omega^{n-p-(q-2)}v_{ij}^{*}]$$

$$- \ldots\ldots\ldots\ldots - [s^{-1}\omega^{q},\ s^{-1}\omega^{n-p}v_{ij}^{*}]$$

+ terms on all of whose entries, $\psi_2 = 0$.

And so

$$\psi_2\partial(s^{-1}\omega^{n-p-q}v_{ij}^{*}) =$$

$$- [s^{-1}\omega,\ \psi_2(s^{-1}\omega^{n-p-(q-1)}v_{ij}^{*})]$$

$$- [s^{-1}\omega^{2},\ \psi_2(s^{-1}\omega^{n-p-(q-2)}v_{ij}^{*})]$$

$$- \ldots\ldots - [s^{-1}\omega^{q},\ \psi_2(s^{-1}\omega^{n-p}v_{ij}^{*})] .$$

Hence,

$$\partial\psi_2\partial(s^{-1}\omega^{n-p-q}v_{ij}^{*}) = [s^{-1}\omega,\ \partial\psi_2(s^{-1}\omega^{n-p-(q-1)}v_{ij}^{*})]$$

$$+ [s^{-1}\omega^{2},\ \partial\psi_2(s^{-1}\omega^{n-p-(q-2)}v_{ij}^{*})]$$

$$+ \ldots\ldots + [s^{-1}\omega^{q},\ \partial\psi_2(s^{-1}\omega^{n-p}v_{ij}^{*})]$$

$$- [\partial(s^{-1}\omega),\ \psi_2(s^{-1}\omega^{n-p-(q-1)}v_{ij}^{*})]$$

$$- [\partial(s^{-1}\omega^{2}),\ \psi_2(s^{-1}\omega^{n-p-(q-2)}v_{ij}^{*})]$$

$$- \ldots\ldots - [\partial(s^{-1}\omega^{q}),\ \psi_2(s^{-1}\omega^{n-p}v_{ij}^{*})] .$$

For the first set of terms in the above expression, it is possible to use the induction hypothesis to obtain expansions, and for the second set, since $q < k$, for dimensional reasons, then the formulae for ∂ on the various $s^{-1}\omega^{i}$

terms appearing here involve only brackets containing $s^{-1}\omega^j$ entries, for various values of j. Thus,

$$\partial\psi_2\partial(s^{-1}\omega^{n-p-q}v_{ij}^*) =$$

$$- [s^{-1}\omega, ([s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p-(q-2)}v_{ij}^*)] + [s^{-1}\omega^2, \psi_2(s^{-1}\omega^{n-p-(q-3)}v_{ij}^*)] + \dots + [s^{-1}\omega^{q-1}, \psi_2(s^{-1}\omega^{n-p}v_{ij}^*)])]$$

$$- [s^{-1}\omega^2, ([s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p-(q-3)}v_{ij}^*)] + [s^{-1}\omega^2, \psi_2(s^{-1}\omega^{n-p-(q-4)}v_{ij}^*)] + \dots + [s^{-1}\omega^{q-2}, \psi_2(s^{-1}\omega^{n-p}v_{ij}^*)])]$$

$$- \dots - [s^{-1}\omega^{q-1}, ([s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p}v_{ij}^*)])]$$

$$+ [1/2[s^{-1}\omega, s^{-1}\omega] , \psi_2(s^{-1}\omega^{n-p-(q-2)}v_{ij}^*)]$$

$$+ [[s^{-1}\omega, s^{-1}\omega^2] , \psi_2(s^{-1}\omega^{n-p-(q-3)}v_{ij}^*)]$$

$$+ [[s^{-1}\omega, s^{-1}\omega^3] + 1/2[s^{-1}\omega^2, s^{-1}\omega^2], \psi_2(s^{-1}\omega^{n-p-(q-4)}v_{ij}^*)]$$

$$+ \dots + [[s^{-1}\omega, s^{-1}\omega^{q-1}] + \dots +[s^{-1}\omega^{[q/2]}, s^{-1}\omega^{(q/2)}], \psi_2(s^{-1}\omega^{n-p}v_{ij}^*)] ;$$

where [q/2] is the largest integer less than or equal to q/2, and (q/2) is the smallest integer greater than or equal to q/2; and if these latter two are equal, then a 1/2 should be placed before the bracket. But for each triple $\{ s^{-1}\omega^b, s^{-1}\omega^c, \psi_2(s^{-1}\omega^{n-p-a}v_{ij}^*)\}$, for $0 \leq q$, and $1 \leq q-1$ with $a + b + c = q$, it is clear that the above formula contains copies of the — Jacobi identities:

for b not equal to c;

$$0 = - [\, s^{-1}\omega^{c} , [\, s^{-1}\omega^{b} , \psi_2(s^{-1}\omega^{n-p-a} v_{ij}^{*})]]$$

$$- [s^{-1}\omega^{b}, [s^{-1}\omega^{c}, \psi_2(s^{-1}\omega^{n-p-a} v_{ij}^{*})]]$$

$$+ [\, [s^{-1}\omega^{c}, s^{-1}\omega^{b}], \psi_2(s^{-1}\omega^{n-p-a} v_{ij}^{*})] \,;$$

and for b equal to c;

$$0 = - [s^{-1}\omega^{b}, [s^{-1}\omega^{b}, \psi_2(s^{-1}\omega^{n-p-a} v_{ij}^{*})]]$$

$$+ 1/2\, [\, [s^{-1}\omega^{b}, s^{-1}\omega^{b}], \psi_2(s^{-1}\omega^{n-p-a} v_{ij}^{*})] \,.$$

(In these latter formulae, both $s^{-1}\omega^{c}$ and $s^{-1}\omega^{b}$ have odd degree.) And furthermore, that these terms are in fact all that the formula for $\partial\psi_2\partial$ contains. Thus $\partial\psi_2\partial(s^{-1}\omega^{n-p-q} v_{ij}^{*}) = 0$, and so $\psi_2\partial(s^{-1}\omega^{n-p-q} v_{ij}^{*})$ is a length three cycle of degree $\leq$ n+k-3, and so by (5.1.2.) is exact; say it equals $\partial\eta$. Then defining

$$\psi_2(s^{-1}\omega^{n-p-q} v_{ij}^{*}) = \eta$$

gives that

$$\partial\psi_2(s^{-1}\omega^{n-p-q} v_{ij}^{*}) = \partial\eta = \psi_2\partial(s^{-1}\omega^{n-p-q} v_{ij}^{*}) =$$

$$- [s^{-1}\omega, \psi_2(s^{-1}\omega^{n-p-(q-1)} v_{ij}^{*})] - \ldots\ldots - [s^{-1}\omega^{q}, \psi_2(s^{-1}\omega^{n-p} v_{ij}^{*})]$$

as required. So the induction goes through and the claim is proved. <u>end of proof of claim.</u>

Now the above can clearly be done for sets of elements $\{ s^{-1}\omega^{n-p-t} v_{ij}^{*} \}_{t=0,1,\ldots,n-p}$ independently for each i and j, c.f. the formulae given at remark (5.1.4.) . And by so doing, the definition of ψ_2 is thus extended to all elements of $s^{-1}H_*$ of degree $\leq r$. On any elements of $s^{-1}H_*$ of degree $> r$, which elements must all be of type $s^{-1}\omega^m$ for some $m > p$, define $\psi_2 = 0$. This completes the definition of $\Phi = 1 + \psi_2$.

It now remains to check that $P_3 + \psi_2\partial - \partial\psi_2 = 0$ on all elements of degree $\leq r$. Since ψ_2 is zero on degrees $\leq n-1$, it is sufficient to check on elements of $s^{-1}H_*$ alone. On all elements of $s^{-1}H_*$ below the 'ψ_2 not equal to zero line', this is true trivially. On all elements which lie on this line, $\psi_2\partial$ is zero since ∂ of such an element is a sum of brackets all of whose entries are below the line. Furthermore, P_3 is zero on elements lying on the line, as elements here are of degree $< r$; also, ψ_2 of elements on the line is ∂-closed, since the X_{ij} are. Hence $P_3 + \psi_2\partial - \partial\psi_2 = 0$ on all elements lying on the line.

On the elements of $s^{-1}H_*$ of type $s^{-1}\omega^{n-p-q}v_{ij}^*$ lying above the 'ψ_2 not equal to zero line', and of degree $< r$ (as indeed all of them must be), P_3 is zero, and $\psi_2\partial - \partial\psi_2 = 0$ by construction. Therefore, $P_3 + \psi_2\partial - \partial\psi_2 = 0$ on these elements also.

On elements of type $s^{-1}\omega^i$ for $i < p$, P_3 is zero, ψ_2 is zero by definition, and for dimensional reasons, no elements of $s^{-1}H_*$ which lie on or above the 'ψ_2 not equal to zero line' can appear in the formula for $\partial(s^{-1}\omega^i)$; hence $\psi_2\partial$ will be zero here, and so $P_3 + \psi_2\partial - \partial\psi_2$ equals zero on all such elements.

Finally, on $s^{-1}\omega^p$, again no elements strictly above the 'ψ_2 not equal to zero line' can appear in the formula for $\partial(s^{-1}\omega^p)$, for dimensional reasons; and so $\psi_2\partial$ remains what it was constructed to be originally, since subsequent extensions of ψ_2 to other elements of $s^{-1}H_*$ only concerned elements above the line. And what it was constructed to be was precisely

$-P_3$. Furthermore, ψ_2 was defined to be η on $s^{-1}\omega^p$, and so $P_3 + \psi_2\partial - \partial\psi_2 = 0$ on $s^{-1}\omega^p$. But the required property has now been checked on all elements of $s^{-1}H_*$ of degree $\leq r$. **QED**

5.1.6. Lemma.

Let H be a Lefschetz algebra of type H(n,k) with $n \leq 2k-1$. Let $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H such that, if P is written $P = P_3 + P_4 + \dots$, then P_3 is zero on elements of $s^{-1}H_*$ of degree $\leq r-1$, for some $r-1 \geq n+k-2$. Then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$, and an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q)$$

such that Q_3 is zero on elements of degree $\leq r$.

Proof. Construct a map of vector spaces $\Phi: s^{-1}H_* \longrightarrow L(s^{-1}H_*)$ as in the previous Lemma. As ψ_2 is zero on all elements of degree $\leq n-1$, (4.1.3.) can be used. This defines a perturbation $Q = Q_3 + Q_4 + \dots$ such that

$$Q_3 = P_3 + \psi_2\partial - \partial\psi_2 .$$

Thus by the property which ψ_2 was constructed to have, Q_3 is zero on elements of degree $\leq r$. **QED**

5.1.7. Proposition.

Let H be a Lefschetz algebra of type H(n,k) where $n \leq 2k-1$. and let $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H. Then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$ and an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q)$$

such that if Q is written $Q = Q_3 + Q_4 + \ldots$, then $Q_3 = 0$.

Proof. By (5.1.3.) it is possible to assume $P = 0$ on elements of $s^{-1}H_*$ of degree $\leq n+k-2$. Then repeated application of (5.1.6.) above will eventually result in a perturbation Q, as required. **QED**

5.1.8. Proposition.

Let H be a Lefschetz algebra of type $H(n,k)$ with $n \leq 2k-1$, and let $L(s^{-1}H_*, \partial + P)$ be a perturbation of the Quillen model of H such that if P is written $P = P_3 + P_4 + \ldots$, then $P_3 = 0$ on all elements, and $P_4 = 0$ on elements of degree $\leq r-1$, for some $n-1 \leq r-1 \leq 4k-5$. Then there exists a new perturbation $L(s^{-1}H_*, \partial + Q)$ and an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial + Q)$$

such that $Q_3 = 0$ on all elements, and $Q_4 = 0$ on all elements of degree $\leq r$.

Proof. Consider $P_4(v)$ for any element v of $s^{-1}H_*$ of degree r. The 'length 5' part of the equation $(\partial + P)^2 = 0$, which is $\partial P_4 + P_3P_3 + P_4\partial = 0$, implies that $P_4(v)$ is a ∂-cycle of length 4, and of degree $\leq 4k-5$. So by (5.1.1.), it must be exact, say $P_4(v) = \partial\eta$. Then defining $\psi_3(v) = \eta$ on all elements of degree r, and $\psi_3 = 0$ on all other elements of $s^{-1}H_*$ defines a vector space map

$$\Phi = 1 + \psi_3: \quad s^{-1}H_* \longrightarrow L(s^{-1}H_*) .$$

This in turn defines a perturbation $L(s^{-1}H_*, \partial + Q)$, according to the formula at (4.1.3.). This perturbation will clearly have all the necessary properties. **QED**

5.1.9. Theorem.

Let H be a Lefschetz algebra of type H(n,k) with $n \leq 2k-1$. Then H is intrinsically formal.

Proof. Let $L(s^{-1}H_*; \partial + P)$ be any perturbation of the Quillen model of H. By (5.1.3.), it is possible to assume $P = 0$ on all elements of $s^{-1}H_*$ of degree $\leq n+k-2$, and furthermore, by (5.1.7.), to assume that if P is written $P = P_3 = P_4 + ...$, then $P_3 = 0$ also. Now by a repeated application of (5.1.8.) above, it is possible to assume also that P_4 is zero on all elements of degree $\leq 4k-4$.

Claim. It is possible further to assume that $P = 0$ on all elements of $s^{-1}H_*$ of degree $\leq 4k-4$.

Proof of claim. P_3 and P_4 have already been assumed to be zero on elements of degree $\leq 4k-4$, and P has been assumed to be zero on elements of degree $\leq n+k-2$. So say inductively, that P is zero on elements of $s^{-1}H_*$ of degree $\leq (r-1)$ for some $4k-5 \geq (r-1) \geq n+k-2$, and consider any element v of degree r in $s^{-1}H_*$. By considering that part of the equation $(\partial + P)^2 = 0$ which extends bracket length by greater than one, it is possible to write

$$\partial P + PP + P\partial = 0;$$

which, due to the assumptions already made concerning P, implies that $\partial P(v) = 0$. So P(v) is a ∂-cycle of degree $\leq 4k-5$ and, again because of the assumptions on P, of bracket length ≥ 5. Thus by (5.1.1.), P(v) must be exact, say

$$P(v) = \partial(\eta).$$

This argument can be used to define a map of vector spaces

$$\psi: s^{-1}H_* \longrightarrow L(s^{-1}H_*)$$

by putting $\psi = 0$ on all elements of $s^{-1}H_*$ of degree less than r and of degree greater than r, and on elements of degree equal to r, putting

$$\psi(v) = \eta.$$

Then according to (4.1.1.), $\Phi = 1 + \psi$ defines a new perturbation Q, isomorphic to $L(s^{-1}H_*, \partial+P)$, and satisfying the formula

$$Q = P + \psi\partial + \psi P - \partial\psi - Q\psi \ .$$

Clearly Q is zero on all elements of degree less than r, and on elements of degree r, Q is zero by construction. Thus by induction, which can be applied so long as $r \leq 4k-4$, $L(s^{-1}H_*, \partial+P)$ is isomorphic to a perturbation which is zero on all elements of degree $\leq 4k-4$. End of proof of claim.

If n is less than 2k-1, the result is proved already; so consider the case n = 2k-1. By the above, assume P is zero on all elements apart from on $s^{-1}\omega^n$, and that P_3 is zero also. Then by considering the equation $(\partial + P)^2 = 0$, $P(s^{-1}\omega^n)$ is a ∂-cycle of bracket length ≥ 4. So using again the result (5.1.1.), it is possible to write

$$P(s^{-1}\omega^n) = \partial\eta + \sum \lambda_J [s^{-1}v_{k\,j1} , [s^{-1}v_{k\,j2} , [s^{-1}v_{k\,j3} , s^{-1}v_{k\,j4}]]] \ .$$

Now using the part of the formula for $\partial(s^{-1}\omega^n)$ involving the terms $s^{-1}v_{kj}$; ie:

$$\partial(s^{-1}\omega^n) = - \sum (-1)^{c_{kj}} [s^{-1}v_{kj} , s^{-1}v_{kj}^*] \qquad + \text{other terms};$$

define a vector space map $\Phi = 1 + \psi: s^{-1}H_* \longrightarrow L(s^{-1}H_*)$ by

$$\psi_3(s^{-1}v_{kj}^{*}) = \sum_{J=(j,j2,j3,j4)}(-1)^{c_{k\,j1}} \lambda_J \, [s^{-1}v_{k\,j2} , [s^{-1}v_{k\,j3} , s^{-1}v_{k\,j4}]] ;$$

$$\psi(s^{-1}\omega^n) = \eta ;$$

and $\psi = 0$ on all other elements of $s^{-1}H_*$. Now since ψ is zero on elements of degree $\leq$ n−1, (4.1.3.) implies that this defines a new perturbation $L(s^{-1}H_*, \partial + Q)$ and an isomorphism between these two. But on all elements of $s^{-1}H_*$ apart from $s^{-1}\omega^n$, where ψ is not zero, $\partial\psi$ certainly is, and furthermore, by assumption P is also zero on such elements. Now for dimensional reasons, the elements $s^{-1}v_{kj}^{*}$ for any j cannot appear in the formula for $\partial(v)$ for any element of $s^{-1}H_*$ apart from $s^{-1}\omega^n$. Thus, according to the formula which is used to define Q, which, in general terms, is $Q = P + \psi\partial + \psi P - \partial\psi - Q\psi$, Q will clearly be zero on all elements other than $s^{-1}\omega^n$. But here, Q is zero by construction, and so Q is identically zero. Thus, Φ defines an isomorphism of DG Lie algebras

$$\Phi: L(s^{-1}H_*, \partial + P) \longrightarrow L(s^{-1}H_*, \partial);$$

and so by the general theory of chapter 2, H is intrinsically formal. **QED**

CHAPTER 6

In this final chapter, methods of constructing examples of Lefschetz algebras are given. These are then used to give an example of a Lefschetz algebra of type H(J;n,k) with k less than n−1 which is not intrinsically formal. Further, this example will also be of type H(n,k) with n greater than 2k−1. This will show that the results of chapters 4 and 5, within the terms set up there, are best possible.

6.1. CONSTRUCTING LEFSCHETZ ALGEBRAS.

Let H be a Lefschetz algebra of degree 2n. Recall from section (3.1.4.) that it is possible to perform a Lefschetz decomposition on H, and display H as:

$$H = sp\{ \omega^{j_1} , \omega^{j_2}V_2 , \ldots\ldots , \omega^{j_n}V_n \}$$

where $0 \leqq j_1 \leqq n$, $0 \leqq j_r \leqq n-r$, and V_r is defined as

$$\ker \omega^{n-r+1} : H^r \longrightarrow H^{2n-r+2} .$$

Here, ω is called the Kähler class of H. By the same procedure, given a Lefschetz algebra G of degree 2m, say, then it is possible to display G as:

$$G = sp\{ x^{i_1} , x^{i_2}U_2 , \ldots\ldots , x^{i_n}U_n \}$$

where $0 \leqq i_1 \leqq m$, and $0 \leqq i_r \leqq m-r$. For the following result, this will be regarded as fixed notation.

6.1.1. Proposition.

Let H be a Lefschetz algebra of degree 2n, with a Lefschetz decomposition having Kähler class ω, and let G be a Lefschetz algebra of degree 2m with a Lefschetz decomposition having Kähler class x. Then the tensor product of H and G, $H \otimes G$, is a Lefschetz algebra of degree 2(n+m) and ω + x can be chosen as a Kähler class of $H \otimes G$.

Proof. It is necessary to check two things; firstly that $H \otimes G$ is a Poincaré duality algebra, and secondly that $H \otimes G$ satisfies the 'Lefschetz' property. That $H \otimes G$ is Poincaré duality follows, since both H and G are, and the tensor product of two Poincaré duality algebras is itself a Poincaré duality algebra. Thus it remains to check that

$$(\omega + x)^k : (H \otimes G)^{n+m-k} \longrightarrow (H \otimes G)^{n+m+k}$$

is an isomorphism for all $0 \leqq k \leqq n+m$.

Well,

$$(H \otimes G)^r = \bigoplus_{s+t=r} H^s \otimes G^t = \bigoplus_{(p,q,i,j)} \omega^p V_i \otimes x^q U_j \ .$$

where the latter sum is over all p, q, i, and j (by convention, take $V_0 = 1$ and $U_0 = 1$), such that $0 \leqq p$, q, i and j; and $p \leqq n-i$, $q \leqq m-j$, $i \leqq n$, $j \leqq m$ and in addition, $2p + 2q + i + j = r$. Define the vector spaces

$$N_{r;i,j} = \bigoplus_{p,q} \omega^p V_i \otimes x^q U_j$$

for each triple (r,i,j), where the sum runs over all values of p and q such that $2p + 2q + i + j = r$, for $0 \leqq p \leqq n-i$, and $0 \leqq q \leqq m-j$. Then since multiplication by ω and x only send $\omega^p V_i$ to $\omega^{p+1} V_i$ and $x^q U_j$ to $x^{q+1} U_j$ respectively, clearly

$$(\omega+x)^k : N_{n+m-k;i,j} \longrightarrow N_{n+m+k;i,j}$$

for each pair i and j. And further, in order to prove the result, it is clearly sufficient to show that

$$(\omega + x)^k : N_{n+m-k;i,j} \longrightarrow N_{n+m+k;i,j}$$

is an isomorphism for all fixed i and j.

For each fixed pair, (i,j), the expression for $N_{n+m-k;i,j}$ sums over all p and q such that $2p + 2q = n + m - k - i - j$, so put $n + m - k - i - j = 2s$, so that $p + q = s$. Then

$$N_{n+m-k;i,j} = \bigoplus_{p+q=s} \omega^p V_i \otimes x^q U_j .$$

Writing this expression out gives

$$N_{n+m-k;i,j} = V_i \otimes x^s U_j \quad \oplus \quad ...... \quad \oplus \quad \omega^s V_i \otimes U_j ,$$

where some of these summands will be zero since $x^{m-j+1}U_j = 0$, and $\omega^{n-i+1}V_i = 0$. Similarly, since $n + m + k - i - j = 2s + 2k$, then

$$N_{n+m+k;i,j} = V_i \otimes x^{k+s} U_j \quad \oplus \quad .... \quad \oplus \quad \omega^{k+s} V_i \otimes U_j ;$$

where again, some of these summands will be zero for the same reasons as above. However, each summand $\omega^p V_i \otimes x^q U_j$ in each of these expressions is either zero, or isomorphic as a vector space to $V_i \otimes U_j$ under the obvious isomorphism

$$\omega^p \otimes x^q : V_i \otimes U_j \longrightarrow \omega^p V_i \otimes x^q U_j \;;$$

and so $(\omega + x)^k$ can be regarded as a vector space map

$$(\omega + x)^k : (V_i \otimes U_j)^{s+1 \text{ copies}} \longrightarrow (V_i \otimes U_j)^{k+s+1 \text{ copies}} .$$

Thus the map $(\omega + x)^k$ can be represented by a $(k+s+1) \times (s+1)$ matrix. Writing $N_{n+m-k;i,j}$ as

$$(\; V_i \otimes x^s U_j \quad \omega V_i \otimes x^{s-1} U_j \quad \ldots\ldots \quad \omega^s V_i \otimes U_j \;)^T ,$$

and similarly for $N_{n+m+k;i,j}$, the matrix representing $(\omega + x)^k$ can be written

$$\begin{pmatrix}
\binom{k}{0} & & & & \\
\binom{k}{1} & \binom{k}{0} & & O & \\
\vdots & \binom{k}{1} & \binom{k}{0} & & \\
\vdots & \vdots & \binom{k}{1} & \ddots & \\
\binom{k}{k} & \vdots & \vdots & \ddots & \\
 & \binom{k}{k} & \vdots & & \binom{k}{0} \\
 & & \binom{k}{k} & & \vdots \\
 & O & & \ddots & \binom{k}{k-1} \\
 & & & & \binom{k}{k}
\end{pmatrix}$$

So if it were possible to show this matrix to be a non-singular transformation from a rank $(s+1)$ space to a rank $(k+s+1)$ space, then $(\omega+x)^k$ is an isomorphism. However, it is a little more awkward than this, since it is not immediately clear which terms vanish, and which do not, in the expressions for $N_{n+m-k;i,j}$ and $N_{n+m+k;i,j}$.

Now, $n + m - k - i - j = 2s$; i.e.

$$(n - i) + (m - j) - k = 2s. \qquad (*)$$

So split the possible values for s into 3 cases, Case 1. $(n - i) \geqq (k + s)$, and hence by (*), $(m - j) \leqq s$.

Case 2. $(m - j) \geqq (k + s)$, and hence by (*), $(n - i) \leqq s$.

Case 3. $(k + s) > (n - i)$ and $(k + s) > (m - j)$, so by (*), $s < (n - i)$, and $s < (m - j)$.

These three cases are clearly exhaustive of all the possible values of s, and hence of all the possible values of p and q.

Consider case 1. Let $s = t + (m - j)$. Then the appropriate formulae for $N_{n+m-k;i,j}$ and $N_{n+m+k;i,j}$ are

$$N_{n+m-k;i,j} = \omega^t V_i \otimes x^{s-t} U_j \quad \oplus \quad \ldots\ldots \quad \oplus \quad \omega^s V_i \otimes U_j \ ,$$

and

$$N_{n+m+k;i,j} = \omega^{k+t} V_i \otimes x^{s-t} U_j \quad \oplus \quad \ldots\ldots \quad \oplus \quad \omega^{k+s} V_i \otimes U_j \ .$$

So both vector spaces have rank equal to $s - t + 1$ times that of $V_i \otimes U_j$, and the appropriate minor of the matrix of $(\omega+x)^k$ which gives the non-trivial part of the transformation is given by the bottom right hand square of the matrix written above; i.e.

$$\begin{pmatrix} \binom{k}{k} & \binom{k}{k-1} & \cdots & \binom{k}{k-s+t} \\ & \binom{k}{k} & & \vdots \\ & & \ddots & \vdots \\ \text{\Large 0} & & & \binom{k}{k} \end{pmatrix}$$

This is an upper triangular matrix with 1's on the diagonal, and so is non-singular; hence in this case, $(\omega + x)^k$ is an isomorphism.

Case 2 proceeds similarly. This time, the non-trivial part of the transformation is given by the top left square of the matrix; so $(\omega + x)^k$ is an isomorphism here also.

For case 3, put $(k + s) = b + (m - j)$, and so by $(*)$, $(k + s) = k - b + (n - i)$. Then here, the appropriate expressions for the N's are

$$N_{n+m-k;i,j} = V_i \otimes x^s U_j \quad \oplus \quad \ldots\ldots \quad \oplus \quad \omega^s V_i \otimes U_j \;;$$

and

$$N_{n+m+k;i,j} = \omega^b V_i \otimes x^{k+s-b} U_j \quad \oplus \quad \ldots\ldots \quad \oplus \quad \omega^{s+b} V_i \otimes x^{k-b} U_j \;.$$

These two both have rank equal to $s + 1$ times the rank of $V_i \otimes U_j$, and in this case, the appropriate minor of the matrix of $(\omega + x)^k$ which represents the non-trivial part of the transformation is given by the minor of the (s+1) columns, and the rows (b+1) to (b+s+1). This is the (s+1) by (s+1) matrix

$$\begin{pmatrix} \binom{k}{b} & \binom{k}{b-1} & \binom{k}{b-2} & \cdots & \cdots & \binom{k}{b-s} \\ \binom{k}{b+1} & \binom{k}{b} & & & & \binom{k}{b-s+1} \\ \binom{k}{b+2} & \vdots & & & & \vdots \\ \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & & & \vdots & \binom{k}{b-1} \\ \binom{k}{b+s} & \binom{k}{b+s-1} & \cdots & \cdots & \binom{k}{b+1} & \binom{k}{b} \end{pmatrix}$$

It is an exercise in the manipulation of determinants and binomial coefficients to show this matrix to be non-singular.

Thus, in all three cases, the map

$$(\omega + x)^k : N_{n+m-k;i,j} \longrightarrow N_{n+m+k;i,j}$$

for all k is an isomorphism, and so by the preceding remarks above,

$$(\omega + x)^k : (H \otimes G)^{n+m-k} \longrightarrow (H \otimes G)^{n+m+k}$$

is an isomorphism for all $0 \leq k \leq n+m$; and hence, $H \otimes G$ is a Lefschetz algebra of degree 2(n+m) with Kähler class $(\omega+x)$. **QED**

6.1.2. Proposition.

Let H be a Lefschetz algebra of degree 2n, having a Lefschetz decomposition with Kahler class ω. Let G be a Poincaré duality algebra of degree 2n which is (n−1)-connected. Then the connected sum of H and G, written H # G, is a Lefschetz algebra of degree 2n and ω can be taken to be a Kähler class for H # G .

Proof. Again, that H # G is a Poincaré duality algebra is immediate, since the connected sum of two Poincaré duality algebras is itself a Poincaré duality algebra. Thus it is necessary to check the 'Lefschetz' property. But this will clearly be satisfied, since the only degree in which H # G differs from H is n, by the connectedness assumption; and it is a triviality that

$$\omega^0 : (H \# G)^n \longrightarrow (H \# G)^n$$

is an isomorphism, since ω^0 is the identity. **QED**

6.1.3. Proposition.

Let H and G be Lefschetz algebras, both of degree 2n with Kähler classes ω and x respectively. Then it is possible to form a connected sum H # G, such that H # G, is a Lefschetz algebra of degree 2n, with Kähler class (ω + x).

Proof. H # G is certainly a Poincaré duality algebra, however the sum is formed, so it is sufficient to check the 'Lefschetz' property. From the construction of H # G, ωG and xH are both identically zero, and so

$$(\omega+x)^r : (H \# G)^{n-r} \longrightarrow (H \# G)^{n+r} ,$$

for $0 \leq r < n$, reduces to

$$\omega^r + x^r : H^{n-r} \oplus G^{n-r} \longrightarrow H^{n+r} \oplus G^{n+r} ,$$

with the map acting as the sum of two isomorphisms. Hence, $(\omega+x)^r$ is an isomorphism in this range. However, for $(\omega+x)^n$ to be an isomorphism, it is sufficient to check that this is non-zero, and this depends on how the sum is formed. Let μ and ν be the fundamental classes of H and G respectively. Assume that

$$\omega^n = \lambda_1 \mu,$$
$$x^n = \lambda_2 \nu,$$

where λ_1 and λ_2 are both non-zero. Then $(\omega+x)^n = \omega^n + x^n = \lambda_1\mu + \lambda_2\nu$. So if $(\lambda_1 + \lambda_2)$ is non-zero, form the sum H # G by introducing the relation $\mu - \nu = 0$, and if $\lambda_1 + \lambda_2 = 0$, form the sum H # G by introducing the relation $\mu + \nu = 0$. These two cases are mutually exclusive, and exhaustive of the possible values of λ_1 and λ_2, and in both cases $(\omega+x)^n$ is non-zero.
QED

The above results give means of constructing a plenty of Lefschetz algebras. By starting with the algebra $H = Q[\omega]/(\omega^{n+1})$, for any n, and for ω in degree 2, which algebras are the simplest kind of Lefschetz algebras; it is possible to use (6.1.2.) to attach elements in degree n, and then tensor the resulting algebra with either itself, or the algebra $Q[x]/(x^{m+1})$ for any m and for x in degree 2 also. By (6.1.1.), this will result in a Lefschetz algebra, and it is clear that this process can be repeated as many times as is desired, and combined with applications of (6.1.3.). Obviously, at each stage of such a process, the algebra is rapidly becoming more complicated, and also, for each application of (6.1.1.), increasing in dimension. The following, and final, section is devoted to constructing an example, making use of this process, of a Lefschetz algebra having the properties mentioned in the introduction to the chapter.

It should be noted that (6.1.1.), and (6.1.3.), cannot be used to construct examples of Lefschetz algebras of type H(n,k) with values of k greater than 2. This is clear, since with each application of these results, the rank in dimension 2 is increased by 1. The easiest way to construct examples of type H(n,k) with k greater than 2, then, is to consider simple examples of the following form.

Choose some pair n and k such that $2 \leqq k \leqq n$, and k is even. Then define an algebra H by

$$H = \mathbb{Q}[\ \omega, a\ ]/(\ a^2 - \omega^k\ ,\ \omega^{n-k+1}a\ )\ \ .$$

where ω is of degree 2, and a is of degree k. Then as a vector space, H has generators in degrees:

2n	ω^n	
2n−2	ω^{n-1}	
2n−4	ω^{n-2}	
.	.	
.	.	
.	.	
2n−k	.	$\omega^{n-k}a$
.	.	$\omega^{n-k-1}a$
.	.	.
n	.	.
.	.	.
.	.	ωa
k	.	a
.	.	
.	.	
.	.	
4	ω^2	
2	ω	
0	1	

It is trivial to check that H is a Lefschetz algebra of degree 2n and type

H(n,k). It is possible to increase the complexity of examples such as this one, thus generating a large number of examples in addition to those coming from (6.1.1.) and (6.1.3.).

6.2. AN EXAMPLE.

In this final section, a Lefschetz algebra of degree 8 and of type H(J;4,2), where J = (2,1,1), will be constructed using the methods of section 6.1. This algebra will also be of type H(4,2) and, as promised, will not be intrinsically formal.

Let $H_1 = \mathbb{Q}[\omega_1 , a, b]/(a^2, b^2, \omega_1 a, \omega_1 b, ab-\omega_1^2)$, where a, b and ω_1 are all of degree 2. Then as a vector space, H_1 has generators:

4	ω_1^2	
3		
2	ω_1	a, b
1		
0	1	

and with the given relations, H_1 is clearly a Lefschetz algebra. Now define

$$H_2 = \left(H_1 \otimes \mathbb{Q}[\omega_2]/(\omega_2^2) \right) \# \mathbb{Q}[\alpha, \beta] \quad ,$$

where ω_2 is of degree 2, and α and β are both of degree 3. By (6.1.1.) and (6.1.2.), H_2 is a Lefschetz algebra, and as a vector space can be described by the picture:

6 $\omega_2\omega_1^{\,2}$

5

4 $\omega_1^{\,2}$ $\omega_2\omega_1$ $\omega_2 a$ $\omega_2 b$

3 α β

2 ω_2 ω_1 a b

1

0 1

and from the construction of a connected sum of algebras, $\alpha\beta = \omega_2\omega_1^{\,2}$.

Finally, define

$$H = H_2 \otimes Q[\omega_3]/(\omega_3^{\,2}) \quad ,$$

where again, ω_3 is of degree 2. Then by (6.1.1.), H is a Lefschetz algebra, and H is described as follows: As a vector space, the following can be taken as a basis for H;

8 $\omega_3\omega_2\omega_1^{\,2}$

7

6 $\omega_2\omega_1^{\,2}$ $\omega_3\omega_1^{\,2}$ $\omega_3\omega_2\omega_1$ $\omega_3\omega_2 a$ $\omega_3\omega_2 b$

5 $\omega_3\alpha$ $\omega_3\beta$

4 $\omega_1^{\,2}$ $\omega_2\omega_1$ $\omega_3\omega_1$ $\omega_3\omega_2$ $\omega_2 a$ $\omega_3 a$ $\omega_2 b$ $\omega_3 b$

3 α β

2 ω_3 ω_2 ω_1 a b

1

0 1

and the relations in H are $a^2 = 0 = b^2$, $ab = \omega_1^{\,2}$, $\omega_1 a = 0 = \omega_1 b$; $\omega_2^{\,2} = 0$; $\alpha^2 = 0 = \beta^2$, $\alpha a = 0 = \beta a$, $\alpha b = 0 = \beta b$, $\alpha\omega_1 = 0 = \beta\omega_1$, $\alpha\omega_2 = 0 = \beta\omega_2$,

$\alpha\beta = \omega_2\omega_1^2$; $\omega_3^2 = 0$.

In order to consider intrinsic formality, it is necessary to consider the Quillen model of this algebra, $L(s^{-1}H_*, \partial)$. In fact, for the purposes of showing that H is not intrinsically formal, it is sufficient to make a series of remarks about the differential in $L(s^{-1}H_*, \partial)$, and one remark about the Lie algebra $L(s^{-1}H_*)$. The remarks concerning the differential – i.e. remarks 1 to 5 below – will all be 'read-off' from the multiplicative structure of H and the basis written down above.

Remark 1. The formula for ∂ on the desuspension of the fundamental class of H, $s^{-1}\omega_3\omega_2\omega_1^2$, is as follows:

$$
\begin{aligned}
\partial(s^{-1}\omega_3\omega_2\omega_1^2) = \\
& - [s^{-1}\omega_3 , s^{-1}\omega_2\omega_1^2] - [s^{-1}\omega_2 , s^{-1}\omega_3\omega_1^2] - [s^{-1}\omega_1 , s^{-1}\omega_3\omega_2\omega_1] \\
& - [s^{-1}\omega_3\omega_2 , s^{-1}\omega_1^2] - [s^{-1}\omega_3\omega_1 , s^{-1}\omega_2\omega_1] \\
& - [s^{-1}a , s^{-1}\omega_3\omega_2 b] - [s^{-1}b , s^{-1}\omega_3\omega_2 a] \\
& - [s^{-1}\omega_3 a , s^{-1}\omega_2 b] - [s^{-1}\omega_3 b , s^{-1}\omega_2 a] \\
& + [s^{-1}\alpha , s^{-1}\omega_3\beta] - [s^{-1}\beta , s^{-1}\omega_3\alpha] .
\end{aligned}
$$

For a Lie algebra L(V), let (v) denote the (Lie) ideal in L(V) generated by v; then with this notation,

Remark 2. $\partial(s^{-1}H_5) \subset (s^{-1}\omega_3)$; and for all basis elements, $s^{-1}v$, of $s^{-1}H_4$, with the exception of $s^{-1}\omega_1^2$; $\partial(s^{-1}v) \subset (s^{-1}\omega_1, s^{-1}\omega_2, s^{-1}\omega_3)$. Also,

$$\partial(s^{-1}\omega_1^2) = - 1/2\, [s^{-1}\omega_1, s^{-1}\omega_1] - [s^{-1}a, s^{-1}b] .$$

Remark 3. $\partial(s^{-1}\omega_3\omega_2\omega_1) \subset (s^{-1}\omega_3 , s^{-1}\omega_2 , s^{-1}\omega_1)$.

Remark 4. $\partial(s^{-1}H_2) = 0$, and $\partial(s^{-1}H_3) = 0$.

Remark 5. $s^{-1}\omega_3\omega_2\omega_1{}^2$ is the only basis element of $s^{-1}H_*$, on which the formula for ∂ contains entries from $s^{-1}H_*$ of degree ≥ 4.

Remark 6. In $L(s^{-1}H_*)$, the following relation holds amongst brackets of length 4 and of degree 5 :-

$$0 = [s^{-1}\beta , [s^{-1}\omega_1 , [s^{-1}a , s^{-1}a]]] - [s^{-1}\omega_1 , [s^{-1}\beta , [s^{-1}a , s^{-1}a]]] + 2 [s^{-1}a , [s^{-1}a , [s^{-1}\beta , s^{-1}\omega_1]]] \quad .$$

Remark 6 follows since, firstly, the terms $s^{-1}\beta$, $s^{-1}\omega_1$ and $[s^{-1}a , s^{-1}a]$ are of even, odd and even degree respectively, and so by the Jacobi identity,

$$0 = [s^{-1}\beta , [s^{-1}\omega_1 , [s^{-1}a , s^{-1}a]]] + [s^{-1}\omega_1 , [[s^{-1}a , s^{-1}a] , s^{-1}\beta]] + [[s^{-1}a , s^{-1}a] , [s^{-1}\beta , s^{-1}\omega_1]] \quad ;$$

and secondly, the terms $s^{-1}a$ and $[s^{-1}\beta , s^{-1}\omega_1]$ are of odd degree, and so by the Jacobi identity,

$$0 = - 2 [s^{-1}a , [s^{-1}a , [s^{-1}\beta , s^{-1}\omega_1]]] + [[s^{-1}a , s^{-1}a] , [s^{-1}\beta , s^{-1}\omega_1]] \quad ,$$

and combining these two formulae gives the desired expression.

6.2.1. Lemma.

H as defined above is not intrinsically formal.

Proof. An explicit perturbation is defined, and then it is shown that this perturbation cannot be isomorphic to the formal Quillen model of H. Define P_3 as:

$$P_3(\ s^{-1}\omega_3\alpha\ ) = -\ [s^{-1}\omega_1\ ,\ [s^{-1}a\ ,\ s^{-1}a]]\quad ,$$

$$P_3(\ s^{-1}\omega_3\omega_2\omega_1\ ) = [s^{-1}\beta\ ,\ [s^{-1}a\ ,\ s^{-1}a]]\quad ,$$

$$P_3(\ s^{-1}\omega_3\omega_2 b\ ) = -\ 2\ [s^{-1}a\ ,\ [s^{-1}\beta\ ,\ s^{-1}\omega_1]]\quad ;$$

and $P_3 = 0$ on all other elements of $s^{-1}H_*$. First, it is necessary to check that this does indeed define a perturbation $L(s^{-1}H_*,\ \partial + P_3)$; i.e. that $(\partial+P)^2 = 0$. In order to do this, it is necessary to check that

$$\partial P_3 + P_3\partial = 0$$

on all elements of $s^{-1}H_*$. Remark 4 above implies that $\partial P_3 = 0$. Remark 5, in conjunction with the fact that $P_3 = 0$ on all elements of degree $\leqq 3$, implies that $P_3\partial = 0$ on all elements other than $s^{-1}\omega_3\omega_2\omega_1^2$. But it is clear from remarks 1 and 6 that $P_3\partial = 0$ on this element by construction. Therefore P_3 does define a perturbation.

Assume that there exists an isomorphism of DG Lie algebras

$$\Phi : L(s^{-1}H_*,\ \partial + P_3) \longrightarrow L(s^{-1}H_*,\ \partial)\ .$$

It is easy to show that, without loss of generality, Φ can be taken to be of the form $\Phi = 1 + \psi$, where ψ extends bracket length by at least one. Then since Φ must be a chain map, it is true that

$$\psi\partial + P_3 = \partial\psi \qquad\qquad (\dagger)$$

on all elements of $s^{-1}H_*$; modulo terms of length greater than 3. Further, ψ must equal zero on all elements of degree 1. Now consider the element $s^{-1}\omega_3\omega_2\omega_1$, of degree 5 in $s^{-1}H_*$. Remark 3 implies that $\partial(\ s^{-1}\omega_3\omega_2\omega_1\ )$ is contained in the ideal $(s^{-1}\omega_3,\ s^{-1}\omega_2,\ s^{-1}\omega_1)$, and therefore,

$$\psi\partial(s^{-1}\omega_3\omega_2\omega_1) \subset (\ s^{-1}\omega_1,\ s^{-1}\omega_2,\ s^{-1}\omega_3\ )\ .$$

Now $s^{-1}\omega_3\omega_2\omega_1$ is of degree 5,and so ψ of it must be a sum of brackets having entries of degree $\leqq$ 4. Thus remark 2 applies, and

$$\partial\psi(s^{-1}\omega_3\omega_2\omega_1) \equiv \partial(\ \lambda_1\ [s^{-1}\alpha,\ s^{-1}\omega_1{}^2] + \lambda_2\ [s^{-1}\beta,\ s^{-1}\omega_1{}^2]\ )$$

$$\Big(\ \mathrm{mod.}\ (s^{-1}\omega_1,\ s^{-1}\omega_2,\ s^{-1}\omega_3)\ \Big)\ .$$

But these two latter pieces of information, combined with (†) above, together imply that

$$P_3(s^{-1}\omega_3\omega_2\omega_1) \equiv \partial(\ \lambda_1\ [s^{-1}\alpha,\ s^{-1}\omega_1{}^2] + \lambda_2\ [s^{-1}\beta,\ s^{-1}\omega_1{}^2]\ )$$

$$\Big(\ \mathrm{mod.}\ (s^{-1}\omega_1,\ s^{-1}\omega_2,\ s^{-1}\omega_3)\ \Big)\ .$$

But it is clear from the definition of P_3 on this element, that this is false. Hence no such isomorphism Φ can exist.

So the above defines a perturbation $L(s^{-1}H_*,\ \partial + P)$ which is not isomorphic to $L(s^{-1}H_*,\ \partial)$, and so by the general theory of chapter 2, H is not intrinsically formal. **QED**

Finally, then, note that H as defined above is of type H(4,2) and also of type H(J;4,2). Thus, regarded as such, H lies immediately outside the ranges of dimensions dealt with by (5.1.9.) and (4.3.9.); hence those results, within the terms set up by them, are both best possible.

APPENDIX

This appendix is completely devoted to a proof of the 'homology' result quoted at (5.1.1.). The results stated there extend no further than the cases H is a Lefschetz algebra of degree 2n and type H(n,k) for $n \leqq 2k-1$. However, the method of computation is potentially of interest outside this range, for specific cases. Indeed, the first two results of the computation, although of a technical nature, are applicable to all Lefschetz algebras. To facilitate the calculation, a little notation is first introduced.

A.1. NOTATION.

Recall from (3.1.4.) that it is possible, given a Lefschetz algebra H, to perform a Lefschetz decomposition of H, and this will display H as:

$2n$	ω^n			
$2n-1$				
$2n-2$	ω^{n-1}	$\omega^{n-2}V_2$		
$2n-3$	$\cdot$		$\omega^{n-3}V_3$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	V_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	$\cdot$		V_3	
2	ω	V_2		
1				
0	1			

Furthermore, in a similar fashion to (3.2.2.), where a Poincaré-Lefschetz basis of H was constructed, it is possible to use such a decomposition to obtain a basis of H in an obvious way, by first choosing bases for the V_i's; then letting this determine a basis for H as:

$$H = \mathrm{sp}\{\ \omega^k,\ \omega^p v_{qr}\ \} \qquad (\dagger)\ ,$$

where $1 \leq k \leq n$; $2 \leq q \leq n$; $0 \leq p \leq n-q$ and $1 \leq r \leq \mathrm{rank}\ V_q$. Clearly, such a basis for H will be related to a Poincaré-Lefschetz basis for H in that, given a Lefschetz decomposition for H, a Poincaré-Lefschetz basis can be chosen to be the same as the above in degrees $\leq n$, but then in general a different choice must be made in higher degrees. In particular, the basis for V_n chosen as at (†) above, is identical to that chosen for a Poincaré-Lefschetz basis of H, as at (3.2.2.). In fact, for what follows, a Poincaré-Lefschetz basis would do as a choice of basis, but for notational purposes, the above is slightly simpler in that there are no elements of type $\omega^p v_i^*$.

With the above choice of basis for a Lefschetz algebra H, it is possible to define a grading on the induced basis of $s^{-1}H_*$, and hence a filtration on $L(s^{-1}H_*)$ – c.f. (4.2.1.). This is done by first giving all basis elements of $s^{-1}H_*$ of form $s^{-1}\omega^k$ grading 0, and all basis elements of $s^{-1}H_*$ of form $s^{-1}\omega^p v_{qr}$ grading −1. Then this induces a grading on a basis of $L(s^{-1}H_*)$, which in turn induces a filtration

$$L(s^{-1}H_*) = F_0 \supset F_{-1} \supset F_{-2} \supset \quad ..... \quad .$$

Further to this filtration, it is possible to use the Lefschetz decomposition to define a 'weighting' on elements of $s^{-1}H_*$. This is done by giving all elements of form $s^{-1}\omega^k$ weighting 0, and all elements in the vector space $s^{-1}\omega^p V_i$, for any p, weighting i, where $2 \leq i \leq n$.

Now from the way in which the differential in the Quillen model of H is constructed, it is possible to make some comments about this differential in the context of the above notation. Firstly, multiplication by ω in H only sends elements further up columns, and not 'from side to side'. Thus,for an element of weight i, say $s^{-1}\omega^p v_i$; the formula for ∂ of such an element in $L(s^{-1}H_*)$ may contain elements of weight greater or less than i, but not in that part of the formula for ∂ also involving entries in $s^{-1}\omega^k$ for the various k. That is;

A.1.1. Remark.

Let H be a Lefschetz algebra, and assume a Lefschetz decomposition has been performed, inducing a filtration of $L(s^{-1}H_*)$ and a weighting of $s^{-1}H_*$ as above. Then for any element $s^{-1}v$ of $s^{-1}H_*$, of filtration i for i = 0 or -1, and of weight m; that part of $\partial(v)$ in F_i/F_{i-1} has entries of weight $\leqq$ m only.

By considering the way in which the differential is 'read off' from the comultiplication in H_* and hence the multiplication in H - c.f. chapter 2, it is easy to see :-

A.1.2. Remark.

Let H be a Lefschetz algebra and assume a Lefschetz decomposition has been performed on H, inducing a basis for $s^{-1}H_*$ as above. Then if v is an element in the basis of H such that there is an element ωv also in the basis, then there is an element $s^{-1}\omega v$ in $s^{-1}H_*$ such that

$$\partial(s^{-1}\omega v) = - [s^{-1}\omega, s^{-1}v]$$

+ a sum of other brackets none of which have $s^{-1}\omega$ as an entry.

Of course, there are elements of H which multiply to zero with ω, and for such elements v in H, there will not be elements $s^{-1}\omega v$ in $s^{-1}H_*$ as above. However, multiplication by ω is 1-1 in H in degrees $\leqq$ n-1. Thus,

A.1.3. Remark.

Let H be a Lefschetz algebra of degree 2n, with basis chosen as above; let $s^{-1}v$ be an element of the induced basis of $s^{-1}H_*$ of degree $\leqq$ n-2. Then there exists an element $s^{-1}u$ in the basis of $s^{-1}H_*$ such that

$$\partial(s^{-1}u) = - [s^{-1}\omega, s^{-1}v]$$

+ a sum of other brackets none of which have $s^{-1}\omega$ as an entry.

If it were only necessary to calculate in degrees $\leqq$ n, these remarks made so far would be sufficient to perform the following calculations with success. However, for the purposes of (5.1.1.), it is necessary to go up to degrees $\leqq$ 2n−2, and so a further general comment is necessary to help in these higher degrees:

A.1.4. Lemma.

Let H be a Lefschetz algebra of degree 2n, and assume a Lefschetz decomposition has been performed so as to induce a weighting of elements of $s^{-1}H_*$ as above. Let X be a single bracket in $L(s^{-1}H_*)$, of degree $\leqq$ 2n−2 and of length $\geqq$ 3, such that there is an entry in X of maximal weight m, say.

If there is a unique entry in X of maximal weight, then for all other entries in the bracket, $s^{-1}v$ say, there exist elements $s^{-1}u$ such that

$$\partial(s^{-1}u) = - [s^{-1}\omega, s^{-1}v] \quad + \text{ other terms not involving } s^{-1}\omega;$$

and if there are any other entries of maximal weight m also in X, then all entries $s^{-1}v$ of X satisfy this latter property.

Proof. Consider the display of H given by the decomposition. This gives:

$2n$	ω^n			
⋮	⋮			
$2n-i$	⋮	$\omega^{n-i}V_i$		
⋮	⋮	⋮		
$2n-m$	⋮	⋮	$\omega^{n-m}V_m$	
⋮	⋮	⋮	⋮	
n	⋮	⋮	⋮	V_n
⋮	⋮	⋮		
m	⋮	⋮	V_m	
⋮	⋮	⋮		
i	⋮	V_i		
⋮				
0	1			

Now consider any entry $s^{-1}v$ in X of weight $\leqq$ m. For this not to have a corresponding element $s^{-1}u$, as in the statement, then the element v in H must multiply to zero with ω. But if v is of weight i, then it must be of degree 2n−i for this to be possible. Thus $s^{-1}v$ must be of degree 2n−i−1 in $s^{-1}H_*$; and since all elements of $s^{-1}H_*$ are of degree $\geqq 1$, then the bracket X is of degree $\geqq\ 2n - i - 1 + 1 + m\ \geqq 2n + m - i \geqq 2n$. But this contradicts the assumption that X is of degree $\leqq$ 2n−2. **QED**

With these above comments on the action of the differential in the Quillen model of Lefschetz algebras, it is now possible to begin the calculation. This is done in several stages, working progressively over the filtration defined above.

A.2. <u>THE F_0 AND F_{-1} COMPONENTS.</u>

Given a Lefschetz algebra, it is possible, as above, to define a basis of $s^{-1}H_*$ using a Lefschetz decomposition of H, and consequently a filtration of

$L(s^{-1}H_*)$. No claims are made about whether or not this induces a filtration on $H(L(s^{-1}H_*, \partial))$. However, to a certain extent, the result which this appendix is going to prove can be seen in terms of a statement about the non-exact cycles representing homology, and their filtration. To this extent, then, it is possible, as stated above, to work through the calculation progressively eliminating all representative cycles of higher and higher filtration. Some parts of the calculation hold as stated for general Lefschetz algebras, some parts could be adapted to hold for general Lefschetz algebras, whilst others hold only in the particular cases H is of type H(n,k) with $n \leqq 2k-1$. The two results of this section hold in general.

A.2.1. Lemma.

Let H be a Lefschetz algebra of degree 2n. Assume that a Lefschetz decomposition has been performed on H, inducing a basis of $s^{-1}H_*$ as in section A.1, and hence a filtration of $L(s^{-1}H_*)$ as in A.1 . Let η represent a class in $H(L(s^{-1}H_*, \partial))$ of bracket length $\geqq 3$. Then η can be chosen to be an element of F_{-1} .

Proof. Consider the inclusion of algebras

$$i: H^*(CP^n ; Q) \longrightarrow H .$$

This induces a projection of DG Lie algebras

$$p: L(s^{-1}H_*, \partial) \longrightarrow L(s^{-1}H_*(CP^n;Q), \partial)$$

which has kernel(p) = $L(\{s^{-1}\omega^p V_i\}, AD(s^{-1}\omega^r)\{s^{-1}\omega^p V_i\}, \partial)$. Here, the notation AD(W)(V) for vector spaces V and W means the vector space generated by all brackets $[w_1, [w_2, \ldots, [w_s, v]]..]]$ with the w_j in W, v in V and $s \geqq 1$. – c.f. [C–M–N (2.10) and (3.7)].

Since the differentials in the two right hand terms of the short exact sequence

$$0 \longrightarrow \ker(p) \xrightarrow{j} L(s^{-1}H_*,\partial) \xrightarrow{p} L(s^{-1}H_*(CP^n;Q),\partial) \longrightarrow 0$$

are quadratic, then the induced map p_* on homology in the long exact homology sequence preserves bracket length. But the homology of the right hand term is known; it is isomorphic to Q in degree 1, generated by $s^{-1}\omega$, and isomorphic to Q in degree 2n, generated by the length 2 term

$$\sum_i [s^{-1}\omega^i, s^{-1}\omega^{n+1-i}],$$

where the sum is over all i less than or equal to (n+1)/2, and if n is odd, then a 1/2 must be placed before the bracket $[s^{-1}\omega^{(n+1)/2}, s^{-1}\omega^{(n+1)/2}]$ in the sum. So the induced map j_* on homology is onto for bracket length ≥ 3. But $j_*([v]) = [\ j(v)\ ]$ by definition, and so any class in the image of j_* can be represented by an element of filtration F_{-1} . **QED**

A.2.2. Lemma.

Let H be a Lefschetz algebra of degree 2n, and let η be a class in $H(\ L(s^{-1}H_*, \partial)\ )$ of bracket length ≥ 3, and degree $\leq 2n-2$; where $L(s^{-1}H_*, \partial)$ is the Quillen model of H. Assuming a Lefschetz decomposition has been performed, then with the above notation, η can be represented by an element of F_{-2} .

Proof. By the above lemma, it is possible to write any representative of η as

$$\eta = \partial(\xi) + \zeta$$

where ζ is in F_{-1} ; and since η is a cycle, $\partial\zeta = 0$. Further, ζ can be written as

$$\zeta = A + B$$

with A in F_{-1}/F_{-2} and B in F_{-2} . Without loss of generality, it is possible to assume that A is of homogeneous bracket length m+1 say, for some m $\geqq$ 2. Then A can be written, using the Jacobi identity, in the form:

$$A = \sum \lambda_{(I,J)} \quad [s^{-1}\omega^{i_1} , [s^{-1}\omega^{i_2} , [\ldots\ldots , [s^{-1}\omega^{i_m} , s^{-1}\omega^{j_1} v_{j_2\ j_3}]\ldots]] ;$$

where $1 \leqq i_r < n$; $2 \leqq j_2 \leqq n$; and $0 \leqq j_1 \leqq n-j_2$.

<u>Claim.</u> Without loss of generality, it is possible to assume i_1 does not equal 1 for any i_1 in the above sum.

<u>Proof of claim.</u> If any bracket of A has i_1 = 1, perform the following manipulation: By the Jacobi identity, write that bracket of A as

$$[s^{-1}\omega, [s^{-1}\omega^{i_2} , [s^{-1}\omega^{i_3} [\ldots\ldots [s^{-1}\omega^{i_m} , s^{-1}\omega^{j_1} v_{j_2\ j_3}]..]]] =$$

$$\pm [\quad [s^{-1}\omega, s^{-1}\omega^{i_2}] \quad , [s^{-1}\omega^{i_3} , \ldots\ldots, \ [s^{-1}\omega^{i_m} , s^{-1}\omega^{j} {}_1 v_{j_2\ j_3}]\ldots\ldots]]$$

$$\pm [s^{-1}\omega^{i_2} , [s^{-1}\omega, [s^{-1}\omega^{i_3} , [\ldots\ldots\ldots, [s^{-1}\omega^{i_m} , s^{-1}\omega^{j_1} v_{j_2\ j_3}]..]]$$

and if i_2 is also = 1, then the last bracket may be omitted, and a 1/2 placed before the second bracket. But for degree reasons, $|s^{-1}\omega|$ + $|s^{-1}\omega^{i_2}|$ must be strictly less than 2n−2, and hence there exists a term $s^{-1}\omega^{i_2+1}$ in $s^{-1}H_*$. Thus, any bracket of A having i_1 = 1 can be further re-written

$$[s^{-1}\omega, [s^{-1}\omega^{i_2}, [\ \ldots\ldots , [s^{-1}\omega^{i_m}, s^{-1}\omega^{j_1} v_{j_2 j_3}]\ldots]] =$$

$$\pm\ \partial([s^{-1}\omega^{i_2+1}, [s^{-1}\omega^{i_3}, [\ldots\ldots, [s^{-1}\omega^{i_m}, s^{-1}\omega^{j_1} v_{j_2 j_3}]\ldots]]])$$

$$\pm \sum_{i_1 \neq 1} \mu_{(I,J)} [s^{-1}\omega^{i_1}, [s^{-1}\omega^{i_2}, \ldots\ldots, [s^{-1}\omega^{i_m}, s^{-1}\omega^{j_1} v_{j_2 j_3}]\ldots\ldots]]$$

$$+ \text{ sums of brackets in } F_{-2}\ .$$

So assume the above manipulation has been performed on all brackets of A for which $i_1 = 1$. Then η can be written

$$\eta = \partial\xi + \zeta = \partial\xi' + \zeta'$$

where $\zeta' = A' + B'$, and

$$A' = \sum \lambda'_{(I,J)} [s^{-1}\omega^{i_1}, [s^{-1}\omega^{i_2}, \ldots\ldots\ldots, [s^{-1}\omega^{i_m}, s^{-1}\omega^{j_1} v_{j_2 j_3}]\ldots]]]$$

with $i_1 \neq 1$ as required. end of proof of claim.

Now, A is the only part of η which can contribute terms to F_{-1}/F_{-2} under ∂, and so these contributions must cancel each other. But each bracket of A has an entry of unique maximal weight, viz. the element of weight greater than zero in each bracket. And furthermore, the contributions each bracket of A makes to F_{-1}/F_{-2} under ∂ will also have this property. In particular, from the first entry - $s^{-1}\omega^{i_1}$ - in each bracket of A, there will be contributions to ∂A, after rearranging by the Jacobi identity, of the form

$$\pm\ \lambda_{(I,J)} [s^{-1}\omega, [s^{-1}\omega^{i_1-1}, \ldots, [s^{-1}\omega^{i_m}, s^{-1}\omega^{j_1} v_{j_2 j_3}]\ldots]]$$

$$\pm \sum_{p > 1} \nu_{(p,I,J)} [s^{-1}\omega^{p}, [s^{-1}\omega^{i_1-p}, [\ldots\ldots\ldots, [s^{-1}\omega^{i_m}, s^{-1}\omega^{j_1} v_{j_2 j_3}]..]]]$$

and when all brackets of $\partial(A)$ in F_{-1}/F_{-2} are arranged so as to be 'nested' with their unique entry of maximum weight at the right hand end, the first bracket in this latter expression makes a unique form of contribution having first entry equal to $s^{-1}\omega$. But by (3.3.3.), with an obvious total order placed on the basis of $s^{-1}H_*$, all the brackets of $\partial(A)$ in F_{-1}/F_{-2}, when arranged as such, are linearly independent, and so for each bracket in A, either $\partial(s^{-1}\omega^{i_1}) = 0$, or $\lambda_{(I,J)} = 0$. If $\partial(s^{-1}\omega^{i_1}) = 0$, then $i_1 = 1$, which was ruled out by the claim; and so in fact each bracket of A remaining after the initial rearrangement must have zero constant term in front of it. So it is possible to write

$$\eta = \partial(\xi) + \zeta$$

where ζ is in F_{-2} , for any representative of the class η. **QED**

This last result sets the style for the forthcoming calculation. Whilst it will be necessary to specialise to get any further, and to introduce more tools into the argument, in particular the enveloping algebra; the calculation follows basically the same pattern at each stage.

A.3. <u>THE BRACKET LENGTH $\geq$ 4 CASE.</u>

By virtue of the fact that the differential in the Quillen model of H, for any algebra H, is quadratic, then it is possible to consider cycles of a homogeneous length only. For the remainder of this section, let H be a Lefschetz algebra of degree 2n and of type H(n,k) with $n \leqq 2k-1$. Also, assume for such an algebra, that a Lefschetz decomposition has been performed, inducing a basis of H and hence of $s^{-1}H_*$ as in A.1., and a weighting of elements of $s^{-1}H_*$ again as in A.1. .

A.3.1. Lemma.

Let H be a Lefschetz algebra as above, and let η be an element of $H(L(s^{-1}H_*, \partial))$ of degree $\leq$ 2n-2 and homogeneous bracket length m, for some m greater than 3. Then with the above notation, η can be represented by an element of F_{-m}.

Proof. By (A.2.2.), it is possible to write

$$\eta = \partial(\xi) + \zeta$$

for any representative of the class η, where ζ is in F_{-2}. So using the Jacobi identity, ζ can be written:

$$\zeta = \sum \lambda_{(I,J)} [u_{i1} , [u_{i2} , \ldots, [u_{im-1} , s^{-1}\omega^{j_1} v_{j2\ j3}]\ldots]]$$

where $s^{-1}\omega^{j_1} v_{j2\ j3}$ is of maximal weight in each bracket, and where at least one of the u_i's in each bracket comes from the set $\{ s^{-1}\omega^p v_{ij} \}$ for some p, i and j, for i $\leq$ j_2 in each bracket; and all of the u_i's come from the set $\{ s^{-1}\omega^r, s^{-1}\omega^p v_{ij} \}$ for all r,p and i $\leq$ j_2, in each bracket.

Now distinguish that part of this expression for ζ which has brackets with a unique maximal weight entry, from that part which has brackets with more than one maximal weight entry.

Write ζ = A + B with

$$A = \sum \lambda_{(I,J)} [a_{i1} , [a_{i2} ,\ldots, [a_{i\ m-1} , s^{-1}\omega^{j_1} v_{j2\ j3}]..]]]$$

where a_i is of strictly less weight than the right hand entry appearing in the same bracket as a_i; and

$$B = \sum \lambda'_{(I,J)} [b_{i1} , [b_{i2} , \ldots, [b_{i\ m-1} , s^{-1}\omega^{j_1} v_{j2\ j3}]\ldots]]$$

where each b_i is of weight less than or equal to that of the right hand entry appearing in the same bracket as b_i.

For the purposes of considering that part of $\partial\zeta$ lying in F_{-2}/F_{-3}, it is possible to consider the parts A and B separately. This is because, for considering contributions to F_{-2}/F_{-3} arising from ∂ of an entry, u, in some bracket of A or B, it is sufficient to consider that part of ∂u which does not lie in a lower filtration than that of u. And so for an entry u in each bracket of A or B, it is sufficient to consider that part of ∂u having entries of weight less than or equal to that of u only. - c.f. remark (A.1.1.). Thus, under ∂, those brackets of ζ having unique maximal weight entries - i.e. A - make contributions to the part of $\partial\zeta$ lying in F_{-2}/F_{-3} which has unique maximal weight entries in each bracket; and those brackets of ζ having more than one entry of maximal weight - i.e. B - make contributions to the part of $\partial\zeta$ in F_{-2}/F_{-3} which has more than one maximal weight entry in each bracket. And clearly such sets of contributions, under ∂, to F_{-2}/F_{-3} will be independent of each other.

So first consider the A part of ζ. ζ is a cycle, and so the part of ∂A which lies in F_{-2}/F_{-3} must cancel independently of everything else.

Claim. It is possible to assume $a_{i_1} \neq s^{-1}\omega$ in any bracket of A.

Proof of Claim. If $a_{i_1} = s^{-1}\omega$ in any bracket of A, perform a similar manipulation of the bracket as in the claim in the proof of (A.2.2.). First of all, use the Jacobi identity to rewrite that bracket as

$$[s^{-1}\omega, [a_{i_2}, \ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]]] =$$

$$\pm [\quad [s^{-1}\omega, a_{i_2}] \quad , [a_{i_3}, \ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]]$$

$$\pm [a_{i_2}, [s^{-1}\omega, [\ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]]] ;$$

where, if a_{i_2} also equals $s^{-1}\omega$, then the last bracket can be omitted, and a 1/2 placed before the second bracket. Now, because each bracket of A has a unique maximal weight entry, lemma (A.1.4.) applies, and there exists some element $s^{-1}\omega a_{i_2}$ in $s^{-1}H_*$. So it is possible to further rewrite the bracket as

$$[s^{-1}\omega, [a_{i_2}, [\ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]] =$$

$$\pm [a_{i_2}, [s^{-1}\omega, [\ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]]$$

$$\pm \partial([s^{-1}\omega a_{i_2}, [a_{i_3}, \ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]])$$

$$\pm \sum \mu_{(I,J)} \quad [a'_{i_1}, [a'_{i_2}, [a_{i_3}, \ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]]]$$

$$\pm [s^{-1}\omega a_{i_2}, \partial([a_{i_3}, [\ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}] \ldots]])]$$

$$+ \text{ elements of } F_{-3} ;$$

where, in the sum of terms appearing on the right hand side, $\mu_{(I,J)}$ is zero if $a'_{i_1} = s^{-1}\omega$. Furthermore, under such a manipulation, the only part of ∂ of each entry u, which does not result in contributions to F_{-3} is that part of ∂u having entries of weight less than or equal to that of u itself – c.f. remark (A.1.1.). So, to arrange all resulting brackets to be 'nested' with unique maximal weight entry at the right hand end will only require one application of the Jacobi identity for each bracket, and in particular, after doing so, the only first entries of the brackets on the right hand side will be entries of type $a_{i_2} \neq s^{-1}\omega$, $s^{-1}\omega a_{i_1}$, and a'_{i_1}, none of which equal $s^{-1}\omega$. End of proof of claim.

So ζ can be written, without loss of generality, as

$$\zeta = \sum \lambda_{(p,q,I,J)} [s^{-1}\omega^{p} v_{i_1\, q} , [a_{i_2} ,\ldots\ldots, [a_{i_{m-1}} , s^{-1}\omega^{j_1} v_{j_2\, j_3}] \;]]]$$

$$\sum \lambda_{(I,J)} [s^{-1}\omega^{i_1} , [a_{i_2} , \ldots\ldots, [a_{i_{m-1}} , s^{-1}\omega^{j_1} v_{j_2\, j_3}] \ldots]]]$$

$$+ B' ;$$

where $i_1 < j_2$ in each bracket in the first sum, $i_1 \neq 1$ in the second sum, and B' is now as B was before. Now consider the contributions which are made to $\partial\zeta$ and which have a unique maximal entry in each bracket and which lie in F_{-2}/F_{-3}. $\partial B'$ makes no such contribution, and after rearranging all such contributions made by A, under ∂, into the form whereby brackets are 'nested' and have the maximal weight entry as the right hand entry, it is easy to see that each element of A makes a unique contribution of the form

$$[s^{-1}\omega , [s^{-1}\omega^{p-1} v_{i_1\, q} , [a_{i_2} , [\ldots\ldots\ldots\ldots [a_{i_{m-1}} , s^{-1}\omega^{j_1} v_{j_2\, j_3}]..]]]$$

or

$$[s^{-1}\omega , [s^{-1}\omega^{i_1 -1} , [a_{i_2} [\ldots\ldots\ldots, [a_{i_{m-1}} , s^{-1}\omega^{j_1} v_{j_2\, j_3}]\ldots]]] ,$$

and contributions of this form will arise from ∂ of the first entry in each bracket of A, and no other entry. Since all terms in F_{-2}/F_{-3} must cancel, all terms in ∂A having $s^{-1}\omega$ as their first term must cancel independently, by the linear independence result (3.3.3.) - after placing an obvious total order on the basis of $s^{-1}H_*$. Thus, $\lambda_{(p,q,I,J)} = 0$, or $\partial(s^{-1}\omega^{p} v_{i_1\, q}) = 0$, for all p, q, I and J; and $\lambda_{(I,J)}$ 0, or $\partial(s^{-1}\omega^{i_1}) = 0$, for all I and J. The latter of these possibilities implies that $i_1 = 0$, and $i_1 = 1$ is not possible by the claim. Further, $\partial(s^{-1}\omega^p v_{i_1 q}) = 0$ implies that $p = 0$. Thus without loss of generality, it is in fact possible to write A as

$$A = \sum \lambda_{(q,I,J)} [s^{-1}v_{i_1 q}, [a_{i_2}, [\ldots\ldots, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}]\ldots]]] ;$$

where $i_1 < j_2$ in each bracket, and $\partial(s^{-1}v_{i_1 q}) = 0$.

But now, the above argument can be repeated over and over to show that first of all, it is possible to assume $a_{i_2} \neq s^{-1}\omega$, and then subsequently that in fact, each a_{i_2} must equal $s^{-1}v_{ij}$ for some $i < j_2$ in each bracket; and then subsequently that a_{i_3} is equal to some $s^{-1}v_{ij}$, and so on. Note that in order to do this, it is necessary that $\partial(a_{i_1}) = 0$, and subsequently that $\partial(a_{i_2}) = 0$, and so on, otherwise the above manipulations may result in taking backwards steps. And that furthermore, after showing a_{i_2} equals $s^{-1}v_{ij}$, then it will be necessary to consider the part of $\partial\zeta$ in F_{-3}/F_{-4} having a unique maximal entry in each bracket, and so on.

The above manipulations can be repeated as many times as necessary in order to be able to write A, without loss of generality, as:

$$A = \sum \lambda_{(I,J,K)} [s^{-1}v_{i_1 k_1}, \ldots\ldots [s^{-1}v_{i_{m-2} k_{m-2}}, [a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}]]\ldots]] .$$

where each $s^{-1}v_{ik}$ is of weight less than that of $s^{-1}v_{j_2 j_3}$ in any one bracket.

Then for dimensional reasons, as $|s^{-1}v_{ij}| \geqq k-1$, and $|A| \leqq 2n-2$, so $|[a_{i_{m-1}}, s^{-1}\omega^{j_1} v_{j_2 j_3}]| \leqq 2n-2-2k+2 \leqq n-1$. And hence $|s^{-1}\omega^{j_1} v_{j_2 j_3}| \leqq n-2$, so remark (A.1.3.) applies. Thus if any bracket remaining in A has $a_{i_{m-1}} = s^{-1}\omega$, this bracket can be rewritten as

$[s^{-1}v_{i_1\ k_1}, [\ .....\ [\ s^{-1}\omega\ ,\ s^{-1}\omega^{j_1}\ v_{j_2\ j_3}\ ].....]] =$

$$\pm\ \partial(\ [s^{-1}v_{i_1\ k_1}, [......, [s^{-1}v_{i_{m-2}\ k_{m-2}}, s^{-1}\omega^{j_1+1}\ v_{j_2\ j_3}\ ]...]]]\ )$$

$$\pm\ \sum\ \nu_{(I,J,K)}\ [s^{-1}v_{i_1\ k_1}, [......., [s^{-1}v_{i_{m-2}\ k_{m-2}}, [a'_{i_{m-1}}, s^{-1}\omega^{j'_1}\ v_{j_2\ j_3}\ ]...]]\ .$$

where the last sum has no non-zero terms with $a'_{i_{m-1}} = s^{-1}\omega$.

A can now be written, without loss of generality, as

$$A = \sum\ \lambda_{(I,J,K)}\ [s^{-1}v_{i_1\ k_1}, \ .....\ , [s^{-1}v_{i_{m-2}\ k_{m-2}}, [\ a_{i_{m-1}}, s^{-1}\omega^{j_1}\ v_{j_2\ j_3}\ ]...]]]\ ;$$

where no $a_{i_{m-1}}$ equals $s^{-1}\omega$ in any bracket. If A has any component left in $F_{-(m-1)}/F_{-m}$, then the entry $a_{i_{m-1}}$ for brackets in such a component equals $s^{-1}\omega^k$ for some $k > 1$. But for such brackets, by considering the contributions made to $\partial\zeta$, which lie in $F_{-(m-1)}/F_{-m}$ and which have a unique maximal entry in each bracket; it is easy to see that for each $a_{i_{m-1}}$ of the form $s^{-1}\omega^k$ in the above sum, a contribution to $\partial\zeta$ of the form

$$[s^{-1}v_1\ , [s^{-1}v_2\ , [.....\ , [s^{-1}\omega\ , [s^{-1}\omega^{k-1}\ , s^{-1}\omega^{j_1}\ v_{j_2\ j_3}\ ]]...]]]$$

will be made, and that furthermore this type of contribution will be made uniquely by $\partial(a_{m-1})$, and the linear independence result of (3.3.3.), after placing an obvious total order on the basis of $s^{-1}H_*$, then implies that for each $a_{i_{m-1}}$ of the form $s^{-1}\omega^k$, either $\partial a_{i_{m-1}} = 0$, or the corresponding constant term $\lambda_{(I,J,K)} = 0$. The former of these two implies that k=1, which is false by assumption on k, so A can in fact be written, without loss of generality as

$$A = \sum\ \lambda_{(I,J,J',K)}\ [s^{-1}v_{i_1\ k_1}, ...[s^{-1}v_{i_{m-2}\ k_{m-2}}, [\ s^{-1}\omega^{j'_1}\ v_{j'_2\ j'_3}, s^{-1}\omega^{j_1}\ v_{j_2\ j_3}\ ]..]]\ .$$

But once again, by considering the contributions made to $\partial\zeta$ in $F_{-m}/F_{-(m+1)}$, and having a unique maximal entry in each bracket; it is easy to see that for each $a_{i_{m-1}}$ in the above sum, of the form $s^{-1}\omega^{j'_1} v_{j'_2\, j'_3}$, a contribution of the form

$$[s^{-1}v_1 , [s^{-1}v_2 , \ldots\ldots\ldots , [s^{-1}\omega, [s^{-1}\omega^{j'_1-1} v_{j'2\, j'3} , s^{-1}\omega^{j_1} v_{j2\, j3}]\ldots]]$$

will be made, and that furthermore, this form of contribution will be made uniquely by $\partial(s^{-1}\omega^{j'_1} v_{j'_2\, j'_3})$, and the linear independence result of (3.3.3.) then implies that for each $a_{i_{m-1}}$ appearing in a bracket of A, either $\partial(a_{i_{m-1}}) = 0$, or the corresponding constant term $\lambda_{(I,J,J',K)} = 0$. The former of these two implies that $a_{i_{m-1}} = s^{-1}v_{qr}$ for some q and r.

So far, this implies that any terms of ζ appearing in A must be in F_{-m} as required. However, by considering one last time the contributions made to $\partial\zeta$ which have a unique maximal entry in each bracket and which lie in $F_{-m}/F_{-(m+1)}$, it is easy to see that each entry $s^{-1}\omega^{j_1} v_{j_2\, j_3}$ will result in a contribution, under ∂, of the form

$$[s^{-1}v_1 , [\ldots\ldots, [s^{-1}v_{m-1} , [s^{-1}\omega , s^{-1}\omega^{j_1-1} v_{j2\, j3}]]\ldots]] ;$$

and the linear independence result of (3.3.3.) again applies to imply that either $\partial(s^{-1}\omega^{j_1} v_{j2\, j3}) = 0$, or the corresponding constant term equals zero. The former of these implies that $j_1 = 0$ in each remaining bracket of A, and so not only is A an element of F_{-m} , but as a cycle of the form

$$A = \sum \lambda_{(I,K)} [s^{-1}v_{i_1\, k_1} , [\ldots\ldots, [s^{-1}v_{i_{m-1}\, k_{m-1}} , s^{-1}v_{i_m\, k_m}]\ldots]] .$$

It has so far been shown that it is possible to write

$$\eta = \partial(\xi') + A' + B ,$$

where A′ is a cycle in F_{-m} . It remains to show that B can be written as an element of F_{-m} . For terms in B, that is, all brackets having more than one entry of maximal weight, a similar method of proof to the above can be used. The only difference is that since there is no unique maximal entry in a bracket, it is not possible to use the linear independence result of (3.3.3.), which was key in the above. To surmount this problem, the universal enveloping algebra can be used. In addition, and this is crucial for success, the last part of (A.1.4.) applies.

Since B is a cycle in $L(s^{-1}H_* , \partial)$, then – c.f. the remarks at (3.3.2.) – the fact that

$$i: L(s^{-1}H_*) \longrightarrow T(s^{-1}H_*)$$

is a DG map implies that i(B) is a cycle in $T(s^{-1}H_*)$, the enveloping algebra of $L(s^{-1}H_*)$. And furthermore, the weighting of elements of $s^{-1}H_*$, and the filtering of elements of $L(s^{-1}H_*)$, carries over trivially into $T(s^{-1}H_*)$. Write i(B) in $T(s^{-1}H_*)$ as

$$i(B) = \sum \lambda_J \, b_{j1} \otimes b_{j2} \otimes \ldots\ldots \otimes b_{jm} \,.$$

Then because of (A.1.4.), for each entry b_j appearing in i(B), there exists some element of $s^{-1}H_*$, $s^{-1}\omega b_j$, with

$$\partial s^{-1}\omega b_j = - [s^{-1}\omega \, , b_j] \ + \text{terms not having } s^{-1}\omega \text{ as an entry.}$$

Hence, in $T(s^{-1}H_*)$, the following formula holds:

$$d(s^{-1}\omega b_j) = s^{-1}\omega \otimes b_j \pm b_j \otimes s^{-1}\omega$$

$$+ \text{ terms not having } s^{-1}\omega \text{ as an entry.}$$

Now, a similar argument to the above can be followed; first showing that without loss of generality, it is possible to assume that $b_{j1} \neq s^{-1}\omega$ in any tensor, by 'pulling out a boundary' according to the above formula. Then, by considering the contributions to $0 = d(i(B))$ of the form

$$s^{-1}\omega \otimes b'_{j1} \otimes b_{j2} \otimes \ldots\ldots \otimes b_{jm} ,$$

which must all cancel each other, it is easy to see that $d(b_{j1})$ must equal zero, and hence that b_{j1} must equal $s^{-1}v_{pq}$ for some p and q. Then these two steps can be repeated, to show b_{j2} must equal some $s^{-1}v_{pq}$, then b_{j3}, and so on. Ultimately, this will result in being able to write, without loss of generality,

$$i(B) = \sum \lambda_{(I,J)}\, s^{-1}v_{i_1\, j_1} \otimes \ldots\ldots \otimes s^{-1}v_{i_m\, j_m}$$

$$+ \sum \mu_{(I,J)}\, s^{-1}v_{i_1\, j_1} \otimes \ldots\ldots \otimes s^{-1}v_{i_{m-1}\, j_{m-1}} \otimes s^{-1}\omega$$

$$+ d\chi .$$

Now, a DG left inverse for i exists, r, say, such that

$$r\colon T(s^{-1}H_*) \longrightarrow L(s^{-1}H_*)$$

and r is a 'shuffle' type map – c.f. (3.3.2.). Thus $ri(B) = B$. But $r(s^{-1}v_{i_1} \otimes \ldots\ldots \otimes s^{-1}v_{i_{m-1}} \otimes s^{-1}\omega)$ can be written, using the Jacobi identity, as:

$$r(s^{-1}v_{i_1} \otimes \ldots \otimes s^{-1}v_{i_{m-1}} \otimes s^{-1}\omega) = \sum \nu_{(I,R)}\, [s^{-1}v_{r_1} , [\ldots\ldots, [s^{-1}v_{r_{m-1}} , s^{-1}\omega]\ldots\ldots]];$$

and in each bracket, there exists more than one entry of maximal weight. So lemma (A.1.4.) applies, and there exists some element $s^{-1}\omega v_{r_{m-1}}$ in $s^{-1}H_*$ for each bracket in this latter expression. So it is possible to write:

$$B = ri(B) = \sum \Lambda_{(I,J)} [s^{-1}v_{i_1\, j_1}, [\ldots\ldots, [s^{-1}v_{i_{m-1}\, j_{m-1}}, s^{-1}v_{i_m\, j_m}]\ldots]]$$

$$\pm\, \partial \left(\sum_{(I,J)} \sum_R \mu_{(I,J)} \nu_{(I,J,R)} [s^{-1}v_{r_1}, \ldots\ldots, [s^{-1}v_{r_{m-2}}, s^{-1}\omega v_{r_{m-1}}]\ldots]] \right)$$

$$+\, r(d(\chi)) \quad .$$

But r is a chain map, so $r(d(\chi)) = \partial(r(\chi))$, and thus

$$\eta = \partial(\xi'') + A' + \sum \Lambda_{(I,J)} [s^{-1}v_{i_1\, j_1}, \ldots., [s^{-1}v_{i_{m-1}\, j_{m-1}}, s^{-1}v_{i_m\, j_m}] \ldots]] ,$$

and since A′ is in F_{-m} , the result is proven. **QED**

A.3.2. Corollary.

Let H be a Lefschetz algebra of degree 2n and type H(n,k) where n is such that $n \leqq 2k - 1$. Then in degrees $\leqq 2n - 2$, the bracket length $\geqq 4$ terms of $H(L(s^{-1}H_*, \partial))$ are spanned, as a vector space, by the elements of form:

1. Bracket length $\geqq$ 5 – all terms zero.

2. Bracket length 4

    a. If $n = 2k-1$, then in degree $2n-2$, types $[s^{-1}v_{kp}, [s^{-1}v_{kq}, [s^{-1}v_{kr}, s^{-1}v_{ks}]]]$;

    b. zero otherwise.

Proof. By the above, any class in $H(L(s^{-1}H_*, \partial)$ of length $\geqq 4$ can be represented by elements of filtration F_{-4} . But for degree reasons, the only such elements in F_{-4} and of degree $\leq 2n-2$ are those as stated. **QED**

A.4 THE BRACKET LENGTH 3 CASE.

In this final section, the last piece of information is fitted into the picture to give the desired homology result. The argument followed is very similar to that of the previous section; again use is made of the weighting on $s^{-1}H_*$ and the filtration of $L(s^{-1}H_*)$ induced by a Lefschetz decomposition of H where H is any Lefschetz algebra. It is necessary to prove two preliminary results before going on to the main result. It will be understood that any Lefschetz algebra H has had a Lefschetz decomposition performed on it, inducing a weighting on $s^{-1}H_*$ and a filtration of $L(s^{-1}H_*)$ as in A.1 .

A.4.1. Lemma.

Let H be a Lefschetz algebra of type H(n,k) where $n \leq 2k-1$, and let $L(s^{-1}H_*, \partial)$ be the Quillen model of H. Let η be a bracket length 2 cycle in $L(s^{-1}H_*)$, of homogeneous degree $\leq n-2$, and of filtration F_{-1} . Then η is exact.

Proof. Let η be of degree r, say, where $r \leq n-2$. All elements of filtration F_{-2} are of degree $\geq 2k-2 \geq n-1$; and so η must be in F_{-1}/F_{-2} . Thus it is possible to write

$$\eta = \sum \lambda_J [s^{-1}\omega^{j_1} , s^{-1}\omega^{j_2} v_{j_3 j_4}] .$$

If any of the brackets in this sum have $j_1 = 1$, perform the following manipulation. By (A.1.3.), for all brackets $[s^{-1}\omega , s^{-1}\omega^{j_2} v_{j_3 j_4}]$ appearing in the sum, there exists a corresponding element $s^{-1}\omega^{j_2+1} v_{j_3 j_4}$ in $s^{-1}H_*$. Thus

– c.f. also remark (5.1.4.) – it is possible to write any bracket in η with $s^{-1}\omega$ as an entry as

$$[s^{-1}\omega\ ,\ s^{-1}\omega^{j_2} v_{j_3\ j_4}] =$$

$$\pm\ \partial\ \left(\ s^{-1}\omega^{j_2+1} v_{j_3\ j_4}\ \right)\quad \pm \text{ terms in } F_{-1}/F_{-2} \text{ not involving } s^{-1}\omega\ .$$

Thus without loss of generality, η can be written

$$\eta = \partial\xi + \sum\ \lambda'_J\ [s^{-1}\omega^{j_1}\ ,\ s^{-1}\omega^{j_2} v_{j_3\ j_4}\ ]\ \ ;$$

where $\lambda'_J = 0$ if $j_1 = 1$. Now $\partial\eta$ is contained in F_{-1}/F_{-2} – c.f. (5.1.4.), and so can be arranged as a sum of brackets, each having a unique entry of maximal weight at the right hand end. This maximal weight entry will of course be the entry in each bracket from F_{-1} . When $\partial\eta$ is arranged as such, using the Jacobi identity, it is clear that for each entry $s^{-1}\omega^{j_1}$ appearing in η, a contribution of the form

$$\pm\ [s^{-1}\omega\ ,\ [s^{-1}\omega^{j_1-1}\ ,\ s^{-1}\omega^{j_2} v_{j_3\ j_4}\ ]]$$

will be made to $\partial\eta$. And furthermore, that it will be possible to write $\partial\eta$ as

$$\partial\eta = \pm\ \sum\ \lambda'_J\quad [s^{-1}\omega\ ,\ [s^{-1}\omega^{j_1-1}\ ,\ s^{-1}\omega^{j_2} v_{j_3\ j_4}\ ]]$$

$$\pm\ \sum_{r_1\neq 1}\ \gamma_R\quad [s^{-1}\omega^{r_1}\ ,\ [s^{-1}\omega^{r_2}\ ,\ s^{-1}\omega^{r_3} v_{r_4\ r_5}\ ]]\ \ .$$

Now the linear independence result of (3.3.3.) applies to the terms in $\partial\eta$, when written as such, to imply that $\lambda'_J = 0$ for all J . That is, $\eta = \partial\xi$. QED

A.4.2. Lemma.

Let H be a Lefschetz algebra of type H(n,k) with $n \leqq 2k-1$, and let $L(s^{-1}H_*, \partial)$ be the Quillen model of H. Let η be a cycle in $L(s^{-1}H_*)$ of bracket length 3 and degree $\leqq n+k-3$. If η is an element of F_{-2}, then η is exact, and is the boundary of an element of F_{-2}.

Proof. First, since all elements of filtration F_{-3} must be of degree $\geqq 3k-3$, and $n+k-3 \leqq 3k-4$, then η must be in F_{-2}/F_{-3}. Furthermore, since all elements of $s^{-1}H_*$ are of degree $\geqq 1$, and all elements of filtration F_{-1} are of degree $\geqq k-1$, then any entry appearing in any bracket of η must be of degree $\leqq (n+k-3) - (k-1) - 1 \leqq n-3$. Thus remark (A.1.3.) applies, and for any entry u in a bracket of η, there exists some element of $s^{-1}H_*$, $s^{-1}\omega u$, such that

$$\partial s^{-1}\omega u = - [s^{-1}\omega , u] + \text{brackets in } F_0/F_{-2} \text{ not involving } s^{-1}\omega.$$

Furthermore, by (5.1.4.), if u in the above expression has filtration F_0, then so do all the brackets on the right hand side, and so does $s^{-1}\omega u$; and if u has filtration F_{-1}, then likewise, so do all the brackets on the right hand side and so does $s^{-1}\omega u$.

Now consider $i(\eta)$ under the map

$$i: L(s^{-1}H_* , \partial) \longrightarrow (T(s^{-1}H_*) , d)$$

where $T(s^{-1}H_*)$ is the universal enveloping algebra of $L(s^{-1}H_*)$. $i(\eta)$ is a d-cycle, since i is a chain map. The remarks of the paragraph above transfer in the obvious way to corresponding remarks about the elements of $i(\eta)$. In particular, if $i(\eta)$ is written:

$$i(\eta) = \sum \lambda_l\, a_{i_1} \otimes a_{i_2} \otimes a_{i_3} \;;$$

then by the usual sort of argument, c.f. the proofs of claims in (A.2.2.) and (A.3.1.), it is possible to assume that a_{i_1} does not equal $s^{-1}\omega$. More properly, if $a_{i_1} = s^{-1}\omega$ in any tensor of $i(\eta)$, then by transferring the above remarks into $T(s^{-1}H_*)$, there exists some element of $s^{-1}H_*$, $s^{-1}\omega a_{i_2}$, say, such that

$$d(s^{-1}\omega a_{i_2}) = \pm \; s^{-1}\omega \otimes a_{i_2} \;\pm\; a_{i_2} \otimes s^{-1}\omega$$

$$\pm \text{ terms not involving } s^{-1}\omega .$$

And, since a_{i_2} and a_{i_3} must both have filtration -1, otherwise η would not be in F_{-2}, then so does $s^{-1}\omega a_{i_2}$. Thus it is possible to rewrite any tensor having $a_{i_1} = s^{-1}\omega$ as:

$$a_{i_1} \otimes a_{i_2} \otimes a_{i_3} = d(s^{-1}\omega a_{i_2} \otimes a_{i_3}) \pm a_{i_2} \otimes s^{-1}\omega \otimes a_{i_3}$$

$$\pm \text{ a sum of tensors in } F_{-2} \text{ with } a_{i_1} \neq s^{-1}\omega .$$

And so without loss of generality, $i(\eta)$ can be written:

$$i(\eta) = d\xi + \sum \lambda_l\, a_{i_1} \otimes a_{i_2} \otimes a_{i_3}$$

where ξ is in F_{-2} and $a_{i_1} \neq s^{-1}\omega$. But $i(\eta)$ is a cycle, and so by considering the contribution to $d(i(\eta))$ arising from elements a_{i_1}, and of the form

$$s^{-1}\omega \otimes a'_{i_1} \otimes a_{i_2} \otimes a_{i_3} \;;$$

it is clear that when $i(\eta)$ is written in this way, $d(a_{i_1})$ must equal 0 or the corresponding constant term must equal 0. So without loss of generality, $i(\eta)$ can be written

$$i(\eta) = d(\xi) + \sum \lambda_l \;\; s^{-1}v_{i_1\, i_2} \otimes a_{i_3} \otimes a_{i_4} \,,$$

where ξ is in F_{-2} and $d(s^{-1}v_{i_1 i_2}) = 0$.

Now the above argument can be repeated, with suitable alterations, to obtain that, without loss of generality, $i(\eta)$ can in fact be written:

$$i(\eta) = d\xi + \sum \lambda_I \quad s^{-1}v_{i_1 i_2} \otimes s^{-1}v_{i_3 i_4} \otimes a_{i_5} ;$$

where ξ is in F_{-2} and $d(s^{-1}v_{pq}) = 0$. So since η is a cycle, it follows that $d(a_{i_5}) = 0$ for all i_5. As η is in F_{-2}, this implies that $a_{i_5} = s^{-1}\omega$ for all i_5, and thus that

$$i(\eta) = d(\xi) + \sum \lambda_I \quad s^{-1}v_{i_1 i_2} \otimes s^{-1}v_{i_3 i_4} \otimes s^{-1}\omega .$$

Now consider the map

$$i: L(s^{-1}H_*, \partial) \longrightarrow T(s^{-1}H_*),d$$

This has a left inverse r, such that r is a chain map - c.f. remark (3.3.2.). So writing $i(\eta) = d\xi + \zeta$,

$$\eta = ri(\eta) = r(d\xi) + r(\zeta) .$$

Further, since r is a 'shuffle' type map, $r(\zeta)$ can be written, using the Jacobi identity, as

$$r(\zeta) = \sum \lambda'_I \quad [s^{-1}v_{i_1 i_2} , [s^{-1}\omega , s^{-1}v_{i_3 i_4}]] ;$$

which, for the degree reasons explained above, and by (A.1.3.), can be written

$$r(\zeta) = \pm \sum \lambda'_I \quad \partial([s^{-1}v_{i_1 i_2} , s^{-1}\omega v_{i_3 i_4}]) .$$

Furthermore, in this last expression, all terms written on the right hand side are in F_{-2} . Also, since r is a chain map, $r(d(\xi)) = \partial(r(\xi))$, and $r(\xi)$ is certainly in F_{-2} since ξ is, and r simply 'shuffles' the entries. Thus

$$\eta = \partial\ \big(\ r(\xi) \pm \sum \lambda'_l\ \ [s^{-1}v_{i_1\, i_2}\, ,\, s^{-1}\omega\, v_{i_{m-1}\, i_4}\,]\ \big)\ ;$$

as required. **QED**

A.4.3. **Proposition.**

Let H be a Lefschetz algebra of type H(n,k) with $n \leqq 2k-1$, and let $L(s^{-1}H_*\ ,\ \partial)$ be the Quillen model of H. Then any bracket length 3 term in $H(\ L(s^{-1}H_*,\ \partial)\ )$ of degree $\leqq 2n-2$ can be represented as a sum of the following types of element:

1. $[s^{-1}v_{ij}\ ,\ [s^{-1}v_{pq}\ ,\ s^{-1}v_{rs}\ ]]$.

2. $[s^{-1}v_{ij}\ ,\ X_{ij}]$ where X_{ij} is ∂-closed and of degree $\geqq n-1$.

Proof. Let η be a cycle of bracket length 3 and of degree $\leqq 2n-2$ representing a class in $H(\ L(s^{-1}H_*,\ \partial)\ )$. Then by (A.2.2.), it is possible to assume that η is in F_{-2} . So write η as

$$\eta = A + B + C\ ,$$

where A is in F_{-2}/F_{-3} and all brackets in A have a unique maximal weight entry, B is in F_{-2}/F_{-3} and all brackets in B have more than one entry of maximal weight, and C is in F_{-3} . Then in a similar fashion to the above proof of (A.3.1.), it is possible to consider A, B and C separately by focussing attention on various components of $\partial\eta$, which must cancel seperately.

Consider first, the component of $\partial\eta$ which lies in F_{-2}/F_{-3}, and which has brackets with a unique entry of maximal weight in each bracket. A is the only part of η which can contribute such terms. Without loss of generality, use the Jacobi identity first to write A as

$$A = \sum \lambda_{(J,R)} \quad [a_{j1} , [a_{j2} , s^{-1}\omega^{r_1} v_{r_2\ r_3}]] ;$$

where the right hand entry is of strictly greatest weight in each bracket. If $a_{j1} = s^{-1}\omega$ in any bracket, it is possible to follow the, by now usual, kind of argument to rearrange such brackets as

$$[s^{-1}\omega , [a_{j2} , s^{-1}\omega^{r_1} v_{r_2\ r_3}]] =$$

$$\pm \quad [a_{j2} , [s^{-1}\omega , s^{-1}\omega^{r_1} v_{r_2\ r_3}]] \quad \pm \quad [[s^{-1}\omega , a_{j2}] , s^{-1}\omega^{r_1} v_{r_2\ r_3}]$$

$$= \pm [a_{j2} , [s^{-1}\omega , s^{-1}\omega^{r_1} v_{r_2\ r_3}]] \quad \pm \quad \partial \left([s^{-1}\omega a_{i2} , s^{-1}\omega^{r_1} v_{r_2\ r_3}] \right)$$

$$+ \sum \lambda'_{(J,R)} [a'_{j1} , [a'_{j2} , s^{-1}\omega^{r_1} v_{r_2\ r_3}]] ;$$

where, in the first rearrangement, if $a_{j2} = s^{-1}\omega$ also, then the first term on the right hand side may be omitted and a 1/2 placed before the second term; and further, in the second rearrangement, no $a'_{j1} = s^{-1}\omega$ in the last sum. Of course, in order to do such rearranging, it is necessary to know that an element $s^{-1}\omega a_{j2}$ exists, but this is guaranteed for all elements a_{j2} in A, by the remark (A.1.4.), since a_{j2} is not the entry of maximal weight. Now η can be written

$$\eta = \partial\xi + A' + B' + C' ,$$

where B′ and C′ have the same properties as B and C before, but A′ now has no first entries equal to $s^{-1}\omega$.

After rearranging $\partial\eta$ so that all brackets having a unique maximal weight entry have that entry at the right hand end; $\partial A'$ makes contributions to $\partial\eta$ of the form

$$\pm \quad [s^{-1}\omega \, , \, [a''_{j1} \, , \, [a'_{j2} \, , \, s^{-1}\omega^{r_1} \, v_{r_2 \, r_3} \,]]]$$

$$\pm \quad [a''_{j1} \, , \, [s^{-1}\omega \, , \, [a'_{j2} \, , \, s^{-1}\omega^{r_1} \, v_{r_2 \, r_3} \,]]]$$

$$\pm \quad [a'_{j1} \, , \, \partial \, \left(\, [a'_{j2} \, , \, s^{-1}\omega^{r_1} \, v_{r_2 \, r_3} \,] \, \right) \,] \, ;$$

where a'_{j1} and a''_{j1} do not equal $s^{-1}\omega$ in any bracket, and all brackets have their unique maximal entry at the right hand end. Now η is a cycle, and so all contributions to $\partial\eta$ from $\partial A'$ which lie in F_{-2}/F_{-3} must cancel each other separately from any contributions to $\partial\eta$ from $\partial B'$ and $\partial C''$. The linear independence result at (3.3.3.) now applies to these contributions to $\partial A'$, to imply that

$$\sum \lambda'_{(J,R)} \quad [s^{-1}\omega \, , \, [a''_{j1} \, , \, [a'_{j2} \, , \, s^{-1}\omega^{r_1} \, v_{r_2 \, r_3} \,]]] \quad = 0 \quad ;$$

and applying the result again for this sum, either $\partial a'_{j1} = 0$ in each bracket of A', or the corresponding constant term must be zero. The former of these two, together with the assumption that $a_{j1} \neq s^{-1}\omega$, implies that each $a'_{j1} = s^{-1}v_{pq}$, for some p and q, in each bracket of A'. So η can be written:

$$\eta = \partial\xi + \sum \lambda_{(J,R)} \quad [s^{-1}v_{j1 \, j2} \, , \, [s^{-1}\omega^{r_1} \, , \, s^{-1}\omega^{r_2} v_{r_3 \, r_4} \,]] \quad + B' + C' \, ,$$

where $\partial(s^{-1}v_{j1 \, j2}) = 0$.

In the above sum, write $\chi_J = \sum_R \lambda_{(J,R)} \, [s^{-1}\omega^{r_1} \, , \, s^{-1}\omega^{r_2} \, v_{r_3 \, r_4}]$; for each $J = (j_1 \, j_2)$. Then consider the component of $\partial\chi_J$ in F_{-1} for each J.

Claim. $\partial\chi_J$ has zero component in F_{-1}/F_{-2} , for each J.

Proof of Claim. The component of $\partial\chi_J$ in F_{-1}/F_{-2} has an entry in each bracket of maximal weight, and this is the unique such entry. Now,

$$\partial A' = \pm \sum [s^{-1}v_J , \partial \chi_J] \ ,$$

and the component of $\partial A'$ which lies in F_{-2}/F_{-3} must sum to zero independently of contributions to $\partial\eta$ from $\partial B'$ and $\partial C'$, and further, this component can be arranged so as to have the unique entry of maximal weight in each bracket at the right hand end of that bracket, and thus satisfy the hypotheses of (3.3.3.). Thus, by that result, the component of $\partial\chi_J$ in F_{-1}/F_{-2} for each J must be zero, or the corresponding constant term must vanish. End of proof of claim.

Now consider $\partial\chi_J$, for each J. Certainly $\partial(\partial\chi_J) = 0$. But $\partial\chi_J$ is in F_{-2}, by the above claim; therefore, (A.4.2.) above applies, and

$$\partial\chi_J = \partial x_J$$

for some x_J in F_{-2}. This is because $\partial\chi_J$ is of length 3, and since $|\eta| \leqq 2n-2$, then $|\partial\chi_J| \leqq 2n-2 - (k-1) - 1 \leqq n+k-3$. Therefore, for such an x_J, $\partial(\chi_J - x_J) = 0$. Now write

$$\eta = \partial\xi + \sum [s^{-1}v_J , \chi_J - x_J] + \sum [s^{-1}v_J , x_J] + B' + C' ,$$

$$= \partial\xi + \sum [s^{-1}v_J , X_J] + B' + C'' ;$$

where $X_J = \chi_J - x_J$, and C″ has incorporated the extra terms of F_{-3} from x_J, and $\partial X_J = 0$.

Claim. It is possible to assume $|X_J| \geqq n-1$.

Proof of Claim. For each J, it is possible to consider only homogeneous degree components of X_J, since these must all go to zero under ∂

independently of each other. For such a homogeneous degree term of any X_J of degree $\leqq n-2$, (A.4.1.) applies, to imply that this must also be exact. So each X_J can be written

$$X_J = \partial(\theta_J) + X_{(n-1)\ J}\ ,$$

where $X_{(n-1)\ J}$ is of degree $\geqq n-1$. Now the ∂ can be 'pulled past' the first term in each bracket of A′ containing the X_J 's, to give:

$$\eta = \partial\xi \pm \sum \partial\ \left([s^{-1}v_J\ ,\ \theta_J\ ]\ \right) \pm \sum [s^{-1}v_J\ ,\ X_{(n-1)\ J}\ ] + B' + C''\ .$$

End of proof of Claim.

So it has been proved that a length 3 element which is closed can be written

$$\eta = \partial\xi'' + \sum [s^{-1}v_J\ ,\ X_J] + B' + C''\ ,$$

where X_J is ∂-closed, so $B' + C''$ is a cycle, and B' is in F_{-2}/F_{-3} and consists of a sum of brackets having more than one entry of maximal weight in each bracket, and C'' is in F_{-3} . Since $B' + C''$ must be a ∂-cycle, it is possible to use an identical argument to the latter part of (A.3.1.) above, employing the universal enveloping algebra. In $L(s^{-1}H_*, \partial)$, $B' + C''$ is a ∂-cycle, and since $B' + C''$ is a sum of brackets which either have more than one entry of maximal weight in each bracket, or are in F_{-3}; then either (A.1.4.), or a simple degree argument and (A.1.3.) apply respectively, to give that there exists, for every entry u in $B' + C''$, some element $s^{-1}\omega u$ of $s^{-1}H_*$, such that

$$\partial s^{-1}\omega u\ \ = -\ [s^{-1}\omega\ ,\ u] + \text{terms not involving } s^{-1}\omega.$$

These facts transfer in an obvious way to corresponding facts about $T(s^{-1}H_*)$, the universal enveloping algebra of $L(s^{-1}H_*)$. With this information, writing

$$i(B' + C'') = \sum \lambda_J \; b_{j1} \otimes b_{j2} \otimes b_{j3} \, ,$$

it is possible, as in (A.3.1.), to write all tensors having first entry equal to $s^{-1}\omega$ as:

$$s^{-1}\omega \otimes b_{j2} \otimes b_{j3} = \pm\, d\,(s^{-1}\omega b_{j2} \otimes b_{j3}) \;\pm\; b_{j2} \otimes s^{-1}\omega \otimes b_{j3}$$

$$\pm\; s^{-1}\omega b_{j2} \otimes db_{j3} \;\pm \sum \; \mu_J \; b'_{j1} \otimes b'_{j2} \otimes b'_{j3} \;\; ;$$

where, if b_{j2} equals $s^{-1}\omega$, the second of the terms on the right hand side can be omitted, and a 1/2 placed before the first term; and in the last sum, b'_{j1} is not equal to $s^{-1}\omega$, for any b'_{j1}. So without loss of generality, it is possible to write

$$i(B' + C'') = d\gamma + \sum \; \lambda'_J \quad b_{j1} \otimes b_{j2} \otimes b_{j3} \, ;$$

where no b_{j1} equals $s^{-1}\omega$ in any tensor. Since $i(B' + C'')$ is a cycle, by considering the contributions to $d(i(B' + C''))$ of the form

$$s^{-1}\omega \otimes b'_{j1} \otimes b_{j2} \otimes b_{j3}$$

coming from $d(b_{j1})$ in each tensor, it is easy to see that $db_{j1} = 0$ for all b_{j1} . Because of the way that these entries were rearranged, this implies that $b_{j1} = s^{-1}v_{pq}$ for some p and q. So now it is possible to write

$$i(B' + C'') = d\gamma + \sum \; \lambda'_J \, s^{-1}v_{j1\; j2} \otimes b_{j3} \otimes b_{j4} \, ;$$

where $d(s^{-1}v_{j1\; j2}) = 0$. Repeating this last argument gives eventually that

$$i(B' + C'') \;= d(\gamma') + \sum \; \lambda''_J \, s^{-1}v_{j1} \otimes s^{-1}v_{j2} \otimes s^{-1}v_{j3}$$

$$+ \sum \; \mu'_J \, s^{-1}v_{j1} \otimes s^{-1}v_{j2} \otimes s^{-1}\omega \, .$$

The DG left inverse for i is simply a 'shuffle' type of map,

$$r: T(s^{-1}H_*) \longrightarrow L(s^{-1}H_*) ,$$

so $r(s^{-1}v_{j1} \otimes s^{-1}v_{j2} \otimes s^{-1}\omega)$ can be written , using the Jacobi identity,

$$r(s^{-1}v_{j1} \otimes s^{-1}v_{j2} \otimes s^{-1}\omega) = \sum \beta_{(J,R)} [s^{-1}v_{r1} , [s^{-1}v_{r2} , s^{-1}\omega]] .$$

All terms in B′ have more than one entry of maximal weight in each bracket, so the same is true of all brackets in this latter expression, and so by (A.1.4.), there exists some element of $s^{-1}H_*$, for each such bracket appearing in the expression, such that

$$\partial(s^{-1}\omega v_{r2}) = - [s^{-1}\omega , s^{-1}v_{r2}] \quad + \text{terms not involving } s^{-1}\omega .$$

Therefore, it is possible to write

$$B' + C'' = ri(B' + C'') = r(\sum \lambda''_J \, s^{-1}v_{j1} \otimes s^{-1}v_{j2} \otimes s^{-1}v_{j3})$$

$$\pm \sum \mu'_J \beta_{(J,R)} [s^{-1}v_{r1} , [s^{-1}v_{r2} , s^{-1}\omega]] \quad + \quad rd(\alpha) \quad .$$

But r is a DG map, and so $rd(\alpha) = \partial(r(\alpha))$; and remembering that r is a 'shuffle',

$$B' + C' = \partial \, \left(r(\alpha) \pm \sum \mu'_J \, \beta_{(J,R)} \, [s^{-1}v_{r1} , s^{-1}\omega v_{r2}] \right)$$

$$+ \sum \lambda'''_J [s^{-1}v_{j1} , [s^{-1}v_{j2} , s^{-1}v_{j3}]] .$$

for some λ'''_J. Finally, by putting the above pieces together, it is possible to write:

$$\eta = \partial \xi''' \; + \sum_J [s^{-1}v_J , X_J] + \sum \lambda'''_J [s^{-1}v_{j1} , [s^{-1}v_{j2} , s^{-1}v_{j3}]]$$

where $|X_J| \geqq n-1$, and $\partial X_J = 0$. **QED**

A.4.4. Corollary. (Proposition 5.1.1.)

Let H be a Lefschetz algebra of type H(n,k) with $n \leqq 2k-1$. Then in degrees $\leqq 2n-2$, the bracket length $\geqq 3$ terms of $H(L(s^{-1}H_* , \partial))$ are spanned, as a vector space, by the following elements:-

1. Bracket length $\geqq 5$ – all terms zero.

2. Bracket length 4

   a. if $n = 2k-1$, then in degree $2n-2$, types $[s^{-1}v_{kp} , [s^{-1}v_{kq} , [s^{-1}v_{kr} , s^{-1}v_{ks}]]]$.

   b. zero otherwise.

3. Bracket length 3

   a. types $[s^{-1}v_{ij} , [s^{-1}v_{pq} , s^{-1}v_{rs}]]$

   b. elements of type $[s^{-1}v_{ij} , X_{ij}]$; where X_{ij} is ∂-closed and of degree $\geqq n-1$.

Proof. Combine the results of (A.3.2.) and (A.4.3.) . QED

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