ON THE STRENGTH DISTRIBUTION OF BUNDLES OF THREADS, AND ITS ASYMPTOTIC APPROXIMATION.

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INTRODUCTION.

The following paper is concerned with the distribution which arises in the theory of breaking strength of what we shall term bundles of threads. A bundle is defined to be a group of threads of equal length placed side by side with their corresponding ends clamped together so that when extension is applied to the bundle, all its constituent threads extend at an equal rate. It is required to relate the statistical distribution of breaking strength of bundles having a given number of threads of a given length to the known distribution of breaking strength of single threads of a standard length.

The problem presents itself naturally in the theory of strength testing of textile materials. The procedure of determining the breaking strength of bundles has similarities with a test widely used in practice for estimating the strength of textile yarns, known as the "hank" test for woollen and worsted yarns, and the "lea" test for cotton yarns. The method adopted is to reel a hank of yarn containing a stated number of turns of specified circumference, and to apply the breaking load to the hank by stretching it between two hooks. We have used the term "bundle", however, in preference to "hank" or "lea" for the reason that in the practical test just described the amount of slip at the hooks, although usually small, is indeterminate, so that the conditions of such a test are ill-defined and ambiguous.

The first detailed treatment of the subject appears to have been given in 1926 by F.T. Peirce (1). He deals exhaustively with its physical aspects and derives many useful formulae. Relevant experimental work on cotton yarns is published in previous papers of the same series.

The wider significance of the problem is also emphasised by Peirce. He points out that any study of the strength properties of materials must involve considerations fundamentally similar to those arising in the theory of bundles (called by him "composite specimens") since each element of the material may be thought of as made up of sub-elements arranged in both series and parallel along a given direction of stress. A recent notable attempt by W. Weibull, 1939 (2) to develop a statistical theory of Strength of Materials appears to us to fail in its assumption that when one sub-element breaks the remainder inevitably follow suit; it seems more likely that the remaining sub-elements would have a definite probability of bearing the redistributed stresses. The behaviour is, of course, complicated by the mutual support afforded by the contact of the sub-elements, and a realistic theory must necessarily involve a more sophisticated representation of the material.

The present work is a study of the mathematical properties of the bundle strength distribution. Apart from its practical application, the distribution itself is of considerable inherent interest and appears hitherto to have received little attention from mathematicians. In particular, the main object of the work has been to study the asymptotic form assumed by the distribution when the number of threads is large, as is most commonly the case in the practical applications.

Scope of the paper.

A preliminary discussion is given of the physical principles involved, and results are stated which apply in general to very large bundles whose constitution is indistinguishable from the population of single threads. The work in this section overlaps that of Peirce to a large extent, and many of the results quoted were also given by him.

The distribution of bundle strength is then obtained, and subsequently attention is concentrated on the case where the load distributes itself equally between the threads. Two peculiar forms of the single thread strength distribution are considered in detail.

The second main section of the paper is devoted to a study of the asymptotic behaviour of the bundle strength distribution for large numbers of threads, and a method is developed which reduces the problem to its simplest terms and provides a preliminary solution in the practically important case.

In the final section, the method of steepest descents independently confirms some of the results already obtained, and provides a means of studying the asymptotic behaviour in more detail.

PHYSICAL BACKGROUND.

In most threads the breaking strengths of small lengths taken contiguously along the thread are to some extent correlated, and it is convenient to take as fundamental a standard length ℓ of thread such that contiguous lengths are negligibly correlated in breaking strength; further, we shall assume that the length of the bundle is an exact multiple of the standard length ℓ. The practical problem then divides into two parts, namely (a) to relate the strength distribution of long single threads with the fundamental single thread distribution and (b) to discover how the strength distribution of a bundle of threads is related to the strength distribution of single threads of the same length.

(a) The first question is a relatively simple one.

Let $b(S)$ be the chance that a thread of length ℓ does not exceed S in strength. Then if a thread of length $n\ell$ is not to exceed S in strength, neither must any of the n sections of length ℓ, and the chance of such a thread occurring is therefore $b^n(S)$ provided its strength properties are distributed in a random way along the length of the thread. The special case where the fundamental distribution is nearly enough Normal, that is $b(S) = \int_{-\infty}^{S} e^{-\frac{(x-m)^2}{2}} \frac{dx}{\sqrt{2\pi}}$, was considered by Peirce (1) and later in much greater detail by Tippett, 1926 (3) who was interested in the analogous statistical problem, and tables of the distribution and its constants are available for n up to 1,000. Daniels, 1941 (4) has also shown that the mean M and standard deviation σ of this distribution satisfy the approximate relation $M = 2\cot\frac{\pi}{2}\sigma$ with good accuracy for all values of n. Furthermore, Fisher and Tippett, 1927 (5) considered the asymptotic form of $b^n(S)$ for arbitrary $b(S)$ when n is large, and conclude that with certain restrictions on $b(S)$, the only possible limiting forms which $b^n(S)$ can assume are $e^{-\frac{nK}{S^\alpha}}$ and $e^{-nKe^{-\alpha S}}$, the constants being determined by the way in which $b(S)$ approaches zero for small values of S. The second expression is actually a limiting form of the first which holds when $b(S)$ does not become zero until $S = -\infty$, and as values of $S < 0$ have no meaning in the present physical application, the second form is not relevant, except as an approximation.

It may be concluded that the effect of length of specimen on single thread strength distribution is well understood, and we need not further concern ourselves with it.

(b) We now turn to the distribution of bundle strength in relation to the corresponding distribution for single threads of the same length, and for this case it is necessary to consider in more detail the load-extension properties of the thread.

Whatever the assumptions made about these properties, the sequence of events when the load is applied is as follows. Suppose the load is increased gradually from zero to its final value. At first it is distributed in some way between all n threads and equilibrium may be reached without any thread giving way. If the load is great enough, however, one of the threads breaks at some stage, and the load is redistributed among the remaining $n-1$ threads, each bearing a rather larger share than before and making it more likely that others will give way. In similar fashion, a number of threads probably break successively, the situation finally resolving itself in one of two ways; either a point is reached where the remaining threads have sufficient strength to maintain between them the final load, or no such point is reached and all the threads finally give way, in which case, of course, the bundle is broken.

A test of breaking strength could no doubt be performed in this way, weights being added to the bundle in discrete units until all the threads go, and in fact the hank test is often thus carried out in mills. The more usual method, however, is to apply a continuously varying load in such a way that the rate of extension of the bundle is approximately constant. The recorded load then increases to a maximum and falls again, and at the maximum point, the values are taken as breaking load and breaking extension.

Mathematical formulation.

To formulate the problem more precisely it is necessary to know how the load distributes itself between the constituent threads of the bundle, and this depends on what assumptions are made regarding the elastic properties of the thread.

1. Inextensible threads. We observe first that if the threads are assumed to be inextensible, the problem has no definite solution. A more familiar example of the same state of affairs is that of a rigid beam supported horizontally at more than two points, in which case the pressures on the supports cannot be deduced. The threads must therefore be capable of extension.

2. Large bundles. General case. Denoting load by S and extension by e, the load-extension relation for a given thread can be written

$$S = f(e, \alpha)$$

where α represents, without real loss of generality, one parameter defining the curve and taking a value specific to each thread. The value of α for a particular thread will be in the general case correlated with the breaking load σ, or the breaking extension ϵ for that thread, where $\sigma = f(\epsilon, \alpha)$ so we write $\phi(\alpha, \epsilon)\, d\alpha\, d\epsilon$ for the joint distribution of α and ϵ over the population of single threads, and $\psi(\epsilon)\, d\epsilon = d\epsilon \int_0^\infty \phi(\alpha, \epsilon)\, d\alpha$ for the distribution of breaking extension for all threads.

Consider, therefore, a sample containing a large enough number n of threads not to be distinguishable in constitution from the population. Let a load S be applied to the bundle constructed from this sample and let the corresponding extension be e. If the bundle does not break, a state of equilibrium is set up at which there are r surviving threads, and r must satisfy the relation

$$\frac{r}{n} = \int_e^\infty \psi(\epsilon)\, d\alpha = \int_e^\infty d\epsilon \int_0^\infty \phi(\alpha, \epsilon)\, d\alpha \tag{1}$$

At extension e the load on a given thread is $f(e, \alpha)$ and when equilibrium is reached the total load is distributed over the surviving threads so as to make

$$S = r\, \frac{\int_e^\infty d\epsilon \int_0^\infty f(e, \alpha)\, \phi(\alpha, \epsilon)\, d\alpha}{\int_e^\infty d\epsilon \int_0^\infty \phi(\alpha, \epsilon)\, d\alpha} \tag{2}$$

Hence from (1) and (2) the load-extension relation for the bundle is

$$S = n \int_{e}^{\infty} d\epsilon \int_{0}^{\infty} f(e,\alpha)\, \phi(\alpha,\epsilon)\, d\alpha \tag{3}$$

The breaking extension for the bundle is attained when S has its greatest possible value S_τ in which case

$$\frac{d}{de} \int_{e}^{\infty} d\epsilon \int_{0}^{\infty} f(e,\alpha)\, \phi(\alpha,\epsilon)\, d\alpha = 0 \tag{4}$$

It is worth noting from (3) that for large bundles the load for a given extension is directly proportional to the number of constituent threads, the load-extension curve being similar for all large bundles.

3. Special forms of $f(e,\alpha)$

(1) In the present study we will confine ourselves to threads whose extension, up to the moment of breakage, is proportional to the load; setting $f(e,\alpha) = \alpha e$ in equation (3), we then have

$$S = ne \int_{e}^{\infty} d\epsilon \int_{0}^{\infty} \alpha\, \phi(\alpha,\epsilon)\, d\alpha$$

and

$$\int_{0}^{\infty} \alpha\, \phi(\alpha,\epsilon)\, d\alpha = a(\epsilon)\ \psi(\epsilon)$$

where $a(\epsilon)$ is the mean value of α over all threads having breaking extension equal to ϵ, and $\psi(\epsilon)$ as defined above is the distribution of breaking extension. Thus

$$S = ne \int_{e}^{\infty} a(\epsilon)\, \psi(\epsilon)\, d\epsilon \tag{5}$$

(2) It is found experimentally (cf. Peirce loc. cit.) that in a large number of cases the load-extension ratio is uncorrelated with the breaking extension for any strand (and hence, it is easy to show, with the breaking load also). Equation (5) then takes the simpler form

$$S = nae \int_{e}^{\infty} \psi(\epsilon)\, d\epsilon \tag{6}$$

and for breaking extension of the bundle,

$$\frac{d}{de} \left\{ e \int_{e}^{\infty} \psi(\epsilon)\, d\epsilon \right\} = 0 \tag{7}$$

(3) Finally, it may be that the dispersion of α is negligibly small compared with that of breaking strength σ*. That this is a reasonable assumption is evident if we consider a thread composed of a large number N of independent consecutive elements, such as, for example, a long chain, the r^{th} element having $\alpha = \alpha_r$, $\sigma = \sigma_r$. Under a load S, each element extends $e_r = \frac{S}{\alpha_r}$, and if the total extension is $e = \frac{1}{N}\sum_{r=1}^{N} e_r$, then the load-extension relation for the whole thread is $S = Ae$ where

$$\frac{1}{A} = \frac{1}{N} \sum_{r=1}^{N} \frac{1}{\alpha_r}$$

From this it is clear that as the number N of elements is increased, the dispersion of A decreases as $\sqrt{N}$. On the other hand, the dispersion of breaking strength in general decreases at a slower rate than $\sqrt{N}$ with increasing N.

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* This is found to be the case with hard twisted woollen yarns.

It is in fact possible to give a physical interpretation to the region between w_1 and w_2. When the load S is applied to the bundle, threads break until a fraction $a(w_2)$ have given way, at which point equilibrium is established. If now additional threads are cut, equilibrium is still maintained until the total fraction removed is $a(w_1)$; beyond this point, the cutting of an extra thread causes the whole bundle to collapse. The region between w_1 and w_2 thus indicates all possible states of equilibrium when a load S is applied to the bundle. Perhaps the argument is more clearly seen if it is noted that the ordinate of the line PQ gives the fraction of threads actually broken when the load on each thread is $1/w$, while the ordinate of the curve represents the fraction which must break under this load. Equilibrium is only possible when the former exceeds the latter.

The condition (9') to be satisfied at breaking load is made to hold when PQ is tangent to the curve at w_c. When PQ is steeper than this, so that it does not intersect the curve at any point other than at $w = 0$, there are no values of w for which equilibrium can be established.

The form of the curve for $a(w)$ can vary within wide limits, subject only to the conditions $a(0) = 1$, $a(\infty) = 0$, $a(w) \leq a(w')$ when $w > w'$

Small bundles.

So far we have only considered the behaviour of bundles which contain such a large number n of threads that the constitution of each bundle is taken to be the same as that of the parent population. When n is small and the sampling variation of the constituent threads is taken into account, the distribution of bundle strength becomes of interest.

A sample of n threads may itself be considered as a finite population in its own right, and the results just proved for large samples hold no less for small values of n. Suppose the strengths of the n threads are arranged in descending order of magnitude $\sigma_1, \sigma_2 \cdots \sigma_n$. Then the condition for equilibrium with r survivors under a load S is

$$S \leq r\sigma_r$$

and the strength of the bundle is given by

$$S_c = \max(r\sigma_r)$$

This rather curious rule may be illustrated by an example. Suppose that 6 threads when tested singly snapped at loads given in the first line of table 1.

Load	3.6	5.2	4.4	8.2	6.1	5.7
Order	6	4	5	1	2	3
Product	21.6	20.8	22.0*	8.2	12.2	17.1

Their descending order is written out in the second line, and finally the product of order x load is given. Then it follows that if all six threads has been tested as a bundle, its strength would have been given by the greatest of these products, that is, by 22.0.

The graphical representation is also of interest for small samples, the curve for $a(w)$ being replaced by the survivor curve of w_n for the sample (fig. 2) which has a step-like form, the height of each step being ( ). Note that the upper value is to be taken on each discontinuity.

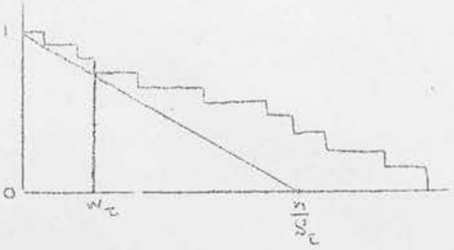


fig 2.

THE DISTRIBUTION OF BUNDLE STRENGTH.

An expression for the chance that a bundle has strength less than S will now be derived.* It is evident on considering the breakage of successive threads that if the bundle breaks under load S the conditions to be satisfied are

$$\left.\begin{array}{l} 0 \le s_n < \frac{S}{n} \\ s_n \le s_{n-1} < \frac{S}{n-1} \\ - - - - - \\ s_3 \le s_2 < \frac{S}{2} \\ s_2 \le s_1 < S \end{array}\right\} \qquad (10)$$

where $s_n, s_{n-1}, s_{n-2} \cdots$ are the strengths of threads breaking in order 1, 2, ... and as the distribution of single thread strength is $\theta(\sigma)d\sigma$ the chance of the event (10) occurring is seen to be

$$B_n = n!\int_0^{\frac{S}{n}} \theta(s_n)\,ds_n \int_{s_n}^{\frac{S}{n-1}} \theta(s_{n-1})\,ds_{n-1} \int_{s_{n-1}}^{\frac{S}{n-2}} \cdots \int_{s_3}^{\frac{S}{2}} \theta(s_2)\,ds_2 \int_{s_2}^{S} \theta(s_1)\,ds_1$$

the factor $n!$ allowing for all possible ways of arranging the threads. Setting $x_r = \int_0^{s_{r+1}} \theta(s_r)\,ds_r$ and $b_r = b(S/r)$ where $b(S) = \int_0^S \theta(x)\,dx$ as defined above, the formula is more simply written

$$B_n = n!\int_0^{b_n} dx_{n-1}\int_{x_{n-1}}^{b_{n-1}} dx_{n-2} \cdots \int_{x_2}^{b_2} dx_1 \int_{x_1}^{b_1} dx_0 \qquad (11)$$

The distribution may be expressed in a variety of other ways, some of which are now given.

Expression as determinant. Consider the slightly more general function

$$B_n(x) = n!\int_x^{b_n} dx_{n-1}\int_{x_{n-1}}^{b_{n-1}} dx_{n-2} \cdots \int_{x_1}^{b_1} dx_0 \qquad (12)$$

It satisfies the equation

$$\frac{\partial B_n(x)}{\partial x} = -\,n\,B_{n-1}(x) \qquad \text{with} \qquad B_0(x) = 1$$

and has the property that $B_n(b_n) = 0$, consequently a Taylor expansion leads to

$$0 = B_n(x) - n(b_n - x)B_{n-1}(x) + \frac{n(n-1)}{2!}(b_n - x)^2 B_{n-2}(x) - \cdots + (-)^{n-1} n (b_n - x)^{n-1} B_1 + (-)^n (b_n - x)^n$$

* The distribution for the more general case where the dispersion of the load-extension ratio α is not negligible is given in Appendix B.

and solving the set of equations obtained on replacing n by $n, n-1, n-2, \cdots$ we derive the result that

$$B_n(x) = n! \begin{vmatrix} (b_n - x) & \frac{(b_n-x)^2}{2!} & \frac{(b_n-x)^3}{3!} & \cdots & \frac{(b_n-x)^{n-1}}{(n-1)!} & \frac{(b_n-x)^n}{n!} \\ 1 & (b_{n-1}-x) & \frac{(b_{n-1}-x)^2}{2!} & \cdots & \frac{(b_{n-1}-x)^{n-2}}{(n-2)!} & \frac{(b_{n-1}-x)^{n-1}}{(n-1)!} \\ 0 & 1 & (b_{n-2}-x) & \cdots & \frac{(b_{n-2}-x)^{n-3}}{(n-3)!} & \frac{(b_{n-2}-x)^{n-2}}{(n-2)!} \\ - & - & - & - & - & - \\ 0 & 0 & 0 & \cdots & 1 & b_1 - x \end{vmatrix} \tag{13}$$

Alternatively the Taylor expansions

$$\left.\begin{aligned} B_n(x) &= B_n - nxB_{n-1} + \frac{n(n-1)}{2!}x^2B_{n-2} - \cdots \\ 0 &= B_n - nb_nB_{n-1} + \frac{n(n-1)}{2!}b_n^2B_{n-2} - \cdots \\ 0 &= \quad - B_{n-1} + (n-1)b_{n-1}B_{n-2} - \cdots \\ 0 &= \quad B_{n-2} - \cdots \\ &- - - - - \end{aligned}\right\} \tag{14}$$

lead to the equivalent form

$$B_n(x) = n! \begin{vmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ 1 & b_n & \frac{b_n^2}{2!} & \cdots & \frac{b_n^{n-1}}{(n-1)!} & \frac{b_n^n}{n!} \\ 0 & 1 & b_{n-1} & \cdots & \frac{b_{n-1}^{n-2}}{(n-2)!} & \frac{b_{n-1}^{n-1}}{(n-1)!} \\ - & - & - & - & - & - \\ 0 & 0 & 0 & \cdots & b_2 & \frac{b_2^2}{2!} \\ 0 & 0 & 0 & \cdots & 1 & b_1 \end{vmatrix} \tag{15}$$

In either form, the special case which concerns us is when $x = 0$, and the formula for B_n is then

$$B_n = n! \begin{vmatrix} b_n & \frac{b_n^2}{2!} & \cdots & \frac{b_n^{n-1}}{(n-1)!} & \frac{b_n^n}{n!} \\ 1 & b_{n-1} & \cdots & \frac{b_{n-1}^{n-2}}{(n-2)!} & \frac{b_{n-1}^{n-1}}{(n-1)!} \\ 0 & 1 & \cdots & \frac{b_{n-2}^{n-3}}{(n-3)!} & \frac{b_{n-2}^{n-2}}{(n-2)!} \\ - & - & - & - & - \\ 0 & 0 & \cdots & b_2 & \frac{b_2^2}{2!} \\ 0 & 0 & \cdots & 1 & b_1 \end{vmatrix} . \tag{16}$$

<u>Series expansions.</u> There are two interesting ways in which B_n can be expanded in series.

(i) The first is most simply arrived at from first principles. Consider the whole range of strength from 0 to ∞ to be divided up into intervals bounded by $\frac{S}{r}$, $r = 1$ to n, as in fig. 4.

The chance of a thread having strength between $\frac{S}{r-1}$ and $\frac{S}{r}$ is $b_{r-1} - b_r$.
If a bundle of n threads breaks under a load S, it must at least satisfy the condition that none of its threads exceeds S in strength, otherwise the last survivor would not break. The chance that the bundle contains p_1 threads between $\frac{S}{2}$ and S, p_2 between $\frac{S}{3}$ and $\frac{S}{2}$, ..., p_{n-1} between $\frac{S}{n}$ and $\frac{S}{n-1}$, and p_n less than $\frac{S}{n}$, where $p_1 + p_2 + \cdots + p_n = n$, is

$$\frac{n!\,(b_1-b_2)^{p_1}(b_2-b_3)^{p_2}\cdots(b_{n-1}-b_n)^{p_{n-1}}\,b_n^{p_n}}{p_1!\,p_2!\cdots p_{n-1}!\,p_n!}$$

But the bundle must have further restrictions if it is to break under load S. For equilibrium never to be possible, the number of threads in the bundle which are less than $\frac{S}{r}$ in strength must be at least equal to r, in which case the p's have to satisfy conditions (17a) or their equivalent form (17b), viz,

$$\left.\begin{aligned} &p_n \geq 1\\ &p_n + p_{n-1} \geq 2\\ &p_n + p_{n-1} + p_{n-2} \geq 3\\ &\cdots\cdots\\ &p_n + p_{n-1} + \cdots + p_2 \geq n-1\\ &p_n + p_{n-1} + \cdots + p_2 + p_1 = n \end{aligned}\right\} (17a) \qquad \left.\begin{aligned} &p_1 \leq 1\\ &p_1 + p_2 \leq 2\\ &p_1 + p_2 + p_3 \leq 3\\ &\cdots\cdots\\ &p_1 + p_2 + \cdots + p_{n-1} \leq n-1\\ &p_1 + p_2 + \cdots + p_{n-1} + p_n = n \end{aligned}\right\} (17b)$$

The chance of the bundle breaking under load S is therefore given by

$$B_n = \sum_p \frac{n!\,(b_1-b_2)^{p_1}(b_2-b_3)^{p_2}\cdots(b_{n-1}-b_n)^{p_{n-1}}\,b_n^{p_n}}{p_1!\,p_2!\cdots p_n!} \tag{18}$$

where the p's are summed over all values consistent with (17a) or (17b). The corresponding formula for $B_n(x)$ is obtained on writing $b_r - x$ for b_r.

(ii) The second series expansion is obtained from the multiple Taylor expansion of the determinant in powers of the b_r's. We have

$$B_n = (B_n)_0 + \sum_r b_r \left(\frac{\partial B_n}{\partial b_r}\right)_0 + \frac{1}{2!} \sum_{r>s} b_r b_s \left(\frac{\partial^2 B_n}{\partial b_r \partial b_s}\right)_0 + \cdots$$

where $(\ )_0$ means that b_r is replaced by 0 in the final expression. Many of the terms of this expansion vanish on account of two rows of the determinant becoming identical after differentiation when the b_r's are made zero, and in fact the only non-vanishing terms are obtained as follows. Numbering the rows from the bottom upwards, first the r_1^{th} row is differentiated r_1 times, then the $(r_1+r_2)^{th}$ row r_2 times, the $(r_1+r_2+r_3)^{th}$ row r_3 times and so on, till finally the n^{th} (top) row is differentiated r_m times where $r_1 + r_2 + \cdots + r_m = n$, it being essential to differentiate the n^{th} row at least once. If the resulting determinant is then rearranged with the r_1^{th} row at the bottom, the $(r_1+r_2)^{th}$ row in place of the r_1^{th}, the $(r_1+r_2+r_3)^{th}$ row in place of the $(r_1+r_2)^{th}$, and so on, the sign of the determinant is changed by $(-)^{r_1-1+r_2-1+\cdots+r_m-1} = (-)^{n-m}$ and its value becomes 1. The expansion is therefore

$$B_n = \sum_{m=1}^{n} \sum_{r} (-)^{n-m} \frac{n!\, b_{r_1}^{r_1}\, b_{r_1+r_2}^{r_2} \cdots b_{r_1+r_2+\cdots+r_{m-1}}^{r_{m-1}}\, b_n^{r_m}}{r_1!\, r_2! \cdots r_{m-1}!\, r_m!} \tag{19}$$

summed over all values of the r's such that $r_i \geqslant 1$ and $r_1+r_2+\cdots+r_m = n$

<u>Explicit formulae.</u> The recurrence formulae (14) provide the simplest way of obtaining explicit formulae for B_n, and for the first few values of n, we find

$$B_0 = 1, \quad B_1 = b_1, \quad B_2 = 2b_2b_1 - b_1^2,$$

$$B_3 = 6\, b_1b_2b_3 - 3b_1^2 b_3 - 3\, b_1 b_3^2 - b_3^3,$$

$$B_4 = 24\, b_1b_2b_3b_4 - 12\, b_1^2 b_3 b_4 - 12\, b_1 b_3^2 b_4 - 12\, b_1 b_2 b_4^2 - 4\, b_3^3 b_4$$
$$+\, 6\, b_1^2 b_4^2 - 4\, b_1 b_4^3 - b_4^4$$

With higher values of n these formulae rapidly become unmanageable. But it is precisely for large n that formulae are most required for the distribution of bundle strength, so that importance attaches to the possibility of obtaining some form of asymptotic expression for B_n. Before considering that question, however, we first of all turn our attention to two special forms for $b(s)$ which present certai simplifying features.

(i) <u>Dichotomous distribution.</u> A form of B_n which is specially simple for all values of n, although not a common one in practice, is that in which the single threads have only two possible strengths α and $\nu\alpha$, that is

$$b(s) = 0 \qquad 0 \leq S < \alpha$$
$$b(s) = b \qquad \alpha \leq S < \nu\alpha$$
$$b(s) = 1 \qquad \nu\alpha \leq S$$

We observe first that $b_n = 0$, and hence $B_n = 0$, whenever $S < n\alpha$. Divide up the range of S into the intervals $r\nu\alpha \leq S < (r+1)\nu\alpha$. Then for values of $S \geqslant n\alpha$ we have, when S is in the r^{th} interval,

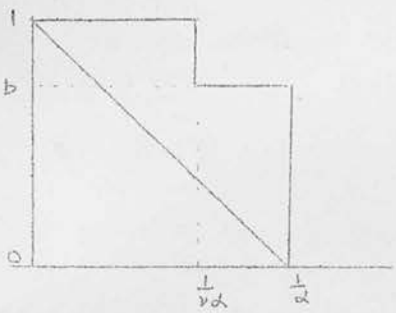
$$b_m = b\left(\tfrac{S}{m}\right) = 1, \quad m \leq r$$
$$= b, \quad m > r$$

This holds for all intervals having $r > [\frac{n}{\nu}]$, and also for that part of the interval $r = [\frac{n}{\nu}]$ in which $S \geqslant n\alpha$.

Consider now the series expansion (18). When $S \geqslant n\alpha$ and S is in the r^{th} interval, the only non-vanishing terms are those in which $P_m = 0$ except for $m = r$ and $m = n$. Writing $p_r = p$, $p_n = n - p$, conditions (17b) imply that $p \leq r$ and the expression for B_n becomes, with this special form for $b(s)$, $B_n = 0$ when $S < n\alpha$;

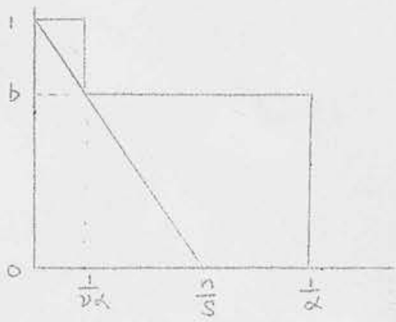
$$B_n = \sum_{p=0}^{n} \frac{n!\,(1-b)^p\, b^{n-p}}{p!\,(n-p)!} \quad \text{when} \quad r\nu\alpha \leq S < (r+1)\nu\alpha \quad \text{and} \quad S \geqslant n\alpha.$$

The frequency distribution is thus a partially collapsed binomial distribution; the frequencies at points $s = r\nu\alpha$ are $\frac{n!\,(1-b)^r\, b^{n-r}}{r!\,(n-r)!}$ provided $r > \left[\frac{n}{\nu}\right]$, but the remainder of the binomial terms at points where $r \leq \left[\frac{n}{\nu}\right]$ are grouped together at $s = n\alpha$. The approximate form of B_n when n is large is in this case easily deduced from the normal approximation to the binomial. For very large n there are two distinct limiting states according to whether the collapsed part contains the greater or lesser half of the binomial distribution, the appropriate conditions being $\nu(1-b) \leq 1$ or > 1 respectively. In the first case all values concentrate at $s = n\alpha$; in the second they concentrate at $s = n\nu\alpha(1-b)$. These ultimate limiting states are immediately obvious from the diagram (fig 4, 5) on applying the formula already given for very large bundles.



$\nu(1-b) \leq 1$

fig 4



$\nu(1-b) > 1$

fig 5

(ii) <u>Exponential form.</u> A distribution which appears to hold a central place in the present theory is

$$b(s) = e^{-\frac{k}{s}}$$

It will be noticed that it is a special case of Fisher and Tippett's limiting distribution for long threads. Its importance lies in the fact that the corresponding B_n can be linked up with the coefficients of a relatively simple generating function. On substituting $b_r = e^{-\frac{kr}{S}}$ in (19), the second of the two series expansions for B_n becomes

$$B_n = \sum_{m=1}^{n} \sum_{r} (-)^{n-m} \frac{n!\, e^{-\frac{K}{S}\left[r_1^2 + r_2^2 + \dots + r_m^2 + r_1 r_2 + r_1 r_3 + \dots + r_{m-1} r_m\right]}}{r_1!\, r_2! \cdots r_{m-1}!\, r_m!}$$

where

$$r_1 + r_2 + \dots + r_m = n$$

$$\text{ie}\quad B_n = e^{-\frac{kn^2}{2S}} \sum_{m=1}^{n} \sum_{r} (-)^{n-m} \frac{n!\, e^{-\frac{K}{2S}\left[r_1^2 + r_2^2 + \dots + r_m^2\right]}}{r_1!\, r_2! \cdots r_{m-1}!\, r_m!}$$

But the expression

$$\sum_{m=1}^{n} \sum_{r} (-)^{n-m} \frac{\xi_{r_1} \xi_{r_2} \cdots \xi_{r_m}}{r_1!\, r_2! \cdots r_m!}, \quad r_1 + r_2 + \dots + r_m = n$$

is the coefficient of t^n in the expansion of $\left(1 - \xi_1 t + \xi_2 \frac{t^2}{2!} - \xi_3 \frac{t^3}{3!} + \cdots\right)^{-1}$

from which it follows that $\qquad B_n = n!\, e^{-\frac{Kn^2}{2S}} C_n \qquad$ where C_n

is the coefficient of t^n in $1/M(t)$, $M(t)$ being defined by

$$M(t) = 1 - t e^{-\frac{K}{2S}} + \frac{t^2}{2!} e^{-\frac{2^2 K}{2S}} - \frac{t^3}{3!} e^{-\frac{3^2 K}{2S}} + \cdots \qquad (20)$$

The same result may also be obtained by manipulating the determinant in (16) when it is found that $e^{-\frac{Kn^2}{2S}}$ can be removed as a factor and the resulting determinant is in the well-known form which occurs in the expansion of the reciprocal of a series.

An extension of this result holds for $B_n(x)$ when $b(S) = e^{-\frac{K}{S}}$. The Tayler expansion of $B_n(x)$ was

$$B_n(x) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} (-)^r x^r B_{n-r}$$

and substituting the expression just obtained,

$$B_n(x) = \sum_{r=0}^{n} \frac{n!}{r!} (-)^r x^r e^{-\frac{(n-r)^2 K}{2S}} C_{n-r}$$

$$= n!\, e^{-\frac{n^2 K}{2S}} \sum_{r=0}^{n} \frac{(-)^r}{r!} \left(x e^{\frac{nK}{S}}\right)^r e^{-\frac{r^2 K}{2S}} C_{n-r}$$

showing that

$$B_n(x) = n!\, e^{-\frac{n^2 K}{2S}} C_n(x)$$

where $C_n(x)$ is the coefficient of t^n in $M(txe^{\frac{nx}{S}})/M(t)$ and $M(t)$ is defined in (20). This is not a true generating function, however, since it contains n explicitly.

It will be shown in a later section how these formulae may be used to gain information on the asymptotic behaviour of B_n for general forms of $b(s)$ by the method of steepest descents.

ASYMPTOTIC BEHAVIOUR OF B_n FOR LARGE n.

It has been proved that as n becomes very large the bundle strength concentrates round a value S_τ given by $S_\tau = n \max S[1 - b(s)]$ so that B_n tends to the form

$$B_n = 0 \quad , \quad S < S_\tau$$

$$B_n = 1 \quad , \quad S \geqslant S_\tau$$

No indication has so far been given, however, except in one special case, of the form assumed by B_n prior to this ultimate limiting form, and the remainder of the work is now devoted to a study of the asymptotic behaviour of B_n for general forms of the parent distribution $b(s)$. Compared with the corresponding work of Fisher and Tippett as applied to the limiting strength distribution of long threads, the problem is not an easy one. The following mode of attack, however, reduces the problem to its simplest terms and gives the dominant terms of the asymptotic expansion.

<u>An important identity.</u> It is first of all necessary to derive an identity of great value for the present purpose. We had

$$B_n = n! \begin{vmatrix} b_n & \frac{b_n^2}{2!} & \frac{b_n^3}{3!} & \cdots & \frac{b_n^{n-1}}{(n-1)!} & \frac{b_n^n}{n!} \\ 1 & b_{n-1} & \frac{b_{n-1}^2}{2!} & \cdots & \frac{b_{n-1}^{n-2}}{(n-2)!} & \frac{b_{n-1}^{n-1}}{(n-1)!} \\ 0 & 1 & b_{n-2} & \cdots & \frac{b_{n-2}^{n-3}}{(n-3)!} & \frac{b_{n-2}^{n-2}}{(n-2)!} \\ - & - & - & - & - & - \\ 0 & 0 & 0 & & b_2 & \frac{b_2^2}{2!} \\ 0 & 0 & 0 & & 1 & b_1 \end{vmatrix} \tag{16}$$

Let us introduce the notation

$$(n, m) = \int_0^{b_n} dx_{n-1} \int_{x_{n-1}}^{b_{n-1}} dx_{n-2} \cdots \int_{x_{m+1}}^{b_{m+1}} dx_m$$

$$= \begin{vmatrix} b_n & \frac{b_n^2}{2!} & \cdots & \frac{b_n^{n-m-1}}{(n-m-1)!} & \frac{b_n^{n-m}}{(n-m)!} \\ 1 & b_{n-1} & \cdots & \frac{b_{n-1}^{n-m-2}}{(n-m-2)!} & \frac{b_{n-1}^{n-m-1}}{(n-m-1)!} \\ - & - & - & - & - \\ 0 & 0 & \cdots & 1 & b_{m+1} \end{vmatrix} \tag{21}$$

so that, for example, $B_n = n!\,(n,0)$. Now in the determinant $(n,0)$, multiply the $(n-1)^{th}$ column by $-x$, the $(n-2)^{th}$ column by $\frac{x^2}{2!}$, ..., the 1st column by $(-)^{n-1}\frac{x^{n-1}}{(n-1)!}$ and add them to the n^{th} column, thus obtaining

$$(n,0) = \begin{vmatrix} b_n & \frac{b_n^2}{2!} & \frac{b_n^3}{3!} & \cdots & \frac{b_n^{n-1}}{(n-1)!} & \frac{(b_n - x)^n}{n!} - (-)^n \frac{x^n}{n!} \\ 1 & b_{n-1} & \frac{b_{n-1}^2}{2!} & \cdots & \frac{b_{n-1}^{n-2}}{(n-2)!} & \frac{(b_{n-1} - x)^{n-1}}{(n-1)!} \\ 0 & 1 & b_{n-2} & \cdots & \frac{b_{n-2}^{n-3}}{(n-3)!} & \frac{(b_{n-2} - x)^{n-2}}{(n-2)!} \\ - & - & - & - & - & - \\ 0 & 0 & 0 & & 1 & b_1 - x \end{vmatrix}$$

The expansion of the determinant in terms of the last column may then be rearranged to give

$$\frac{x^n}{n!} \equiv \sum_{m=0}^{n} (n, m) \frac{(x - b_m)^m}{m!} \qquad \text{for all } x \tag{22}$$

Differentiating r times and replacing $n-r$, $m-r$ by n, m gives the more general identity

$$\frac{x^n}{n!} \equiv \sum_{m=0}^{n} (n+r, m+r) \frac{(x - b_{m+r})^m}{m!} \tag{22a}$$

for all x, and $r \geq 0$.

A physical interpretation of (n,m). The chance of complete rupture of a bundle under a load S was $B_n = n!(n,0)$. Consider now the chance that under load S, m threads out of the n remain unbroken. The conditions to be satisfied for the first m threads to give way are

$$0 < s_n \le \frac{S}{n} \; ; \; s_n < s_{n-1} \le \frac{S}{n-1} \; ; \; s_{n-1} < s_{n-2} \le \frac{S}{n-2} \; ; \; \cdots \; s_{m+2} < s_{m+1} \le \frac{S}{m+1}$$

where s_n, s_{n-1}, s_{n-2} ...... are the strengths of threads breaking in order 1, 2, 3 ...... For the remaining threads not to break we must have $\frac{S}{m} < s_r$, $1 \le r \le m$. If we write as before $x_r = \int_0^{s_{r+1}} \theta(s_r)\, ds_r$ the chance of m threads surviving is therefore given by

$$B_{n,m} = n(n-1)\cdots(m+2)(m+1)\int_0^{b_n} dx_{n-1}\int_{x_{n-1}}^{b_{n-1}} dx_{n-2} \cdots \int_{x_{m+1}}^{b_{m+1}} dx_m \cdot (1-b_m)^m$$

i.e.

$$B_{n,m} = \frac{n!}{m!}(n,m)(1-b_m)^m \tag{23}$$

Substituting for (n,m) in (22) leads to

$$x^n \equiv \sum_{m=0}^{n} B_{n,m}\left(\frac{x-b_m}{1-b_m}\right)^m \qquad \text{for all } x$$

For example, when $x = 1$, the result $1 = \sum_{m=0}^{n} B_{n,m}$ simply expresses the fact that values of m from 0 to n cover all possible contingencies.

Quasi-binomial form. The identities (22) and (22a) are not in the forms most convenient for the present purpose. Writing $x = \frac{1}{\lambda}$, equation (22) becomes

$$1 \equiv \sum_{m=0}^{n} \frac{n!(n,m)}{m!}\lambda^{n-m}(1-\lambda b_m)^m \quad , \text{ all } \lambda$$

and by analogy with the binomial distribution, we are led to consider the function

$$Q_{n,m} = \frac{(n-m)!\,(n,m)}{b_m^{n-m}}$$

which satisfies, and is in fact uniquely determined by

$$1 \equiv \sum_{m=0}^{n} Q_{n,m}\,\frac{n!\,(\lambda b_m)^{n-m}(1-\lambda b_m)^m}{m!\,(n-m)!} \quad , \text{ all } \lambda , \tag{24}$$

or the more general identity

$$1 \equiv \sum_{m=0}^{n} Q_{n+r,\,m+r}\,\frac{n!\,(\lambda b_{m+r})^{n-m}(1-\lambda b_{m+r})^m}{m!\,(n-m)!}$$

$$\text{all } \lambda \;,\; r \ge 0 \tag{24a}$$

Note that $Q_{n,0} = B_n$. A physical interpretation of $Q_{n,m}$ for values of m other than zero is discussed in Appendix A.

<u>Behaviour for large n.</u> By means of (24) we are able to link up the behaviour of $Q_{n,m}$ for large n with that of

$$T_{n,m} = \frac{n!\,(\lambda b_m)^{n-m}\,(1-\lambda b_m)^m}{m!\,(n-m)!}$$

which we now consider. The important range of λ is $0 \leq \lambda \leq 1$, in which $T_{n,m}$ is always ≥ 0.

In the case of the ordinary binomial distribution $\dfrac{n!\,p^{n-m}(1-p)^m}{m!\,(n-m)!}$ there are two distinct limiting forms for large n. When neither p nor $1-p$ is small, the binomial is approximated to by a continuous normal distribution having the same first and second moments np and npq respectively. On the other hand when p is small the appropriate limiting form is the discrete Poisson distribution with parameter np, and similarly for small $1-p$.

In the present more general case it is useful to preserve the distinction, and we first discuss the limiting form corresponding to the normal approximation.

By Stirling's formula, assuming $m = O(n)$ but neither $\frac{m}{n}$ nor $1-\frac{m}{n}$ small, we have

$$T_{n,m} = \frac{n^{n+\frac{1}{2}}\,(\lambda b_m)^{n-m}\,(1-\lambda b_m)^m}{\sqrt{2\pi}\;m^{m+\frac{1}{2}}\;(n-m)^{n-m+\frac{1}{2}}}\left[1+O\left(\tfrac{1}{n}\right)\right]$$

The function $a(w)$ defined by $a(w) = a(\frac{1}{s}) = b(s)$ is most convenient to work with, and in the following work we confine our attention to the case where $a(w)$ is a continuous function of w. Writing $\frac{m}{n} = z$, $\frac{1}{n} = dz$, $s = n\zeta$, $T_{n,m} = T_n(z)$, the approximation is

$$T_n(z) = \sqrt{\frac{n}{2\pi}}\cdot\frac{dz}{\sqrt{z(1-z)}}\left[\frac{\lambda a(z/\zeta)}{1-z}\right]^{n(1-z)}\cdot\left[\frac{1-\lambda a(z/\zeta)}{z}\right]^{nz}\left[1+O(\tfrac{1}{n})\right] \quad (25)$$

Clearly the important regions of $T_n(z)$ are near the maxima of

$$F(z) = \left[\frac{\lambda a(z/\zeta)}{1-z}\right]^{(1-z)}\left[\frac{1-\lambda a(z/\zeta)}{z}\right]^z$$

at which the values of z must satisfy

$$0 = \frac{F'}{F} = -\log\frac{\lambda a}{(1-z)} + \log\frac{(1-\lambda a)}{z} + \frac{(1-z)a'}{\zeta a} - \frac{z\lambda a'}{\zeta(1-\lambda a)}$$

$$= -\log\left[1-\frac{(1-z-\lambda a)}{(1-z)}\right] + \log\left[1+\frac{(1-z-\lambda a)}{z}\right] + \frac{a'(1-z-\lambda a)}{\zeta a(1-\lambda a)}$$

The equation is evidently satisfied at least when

$$1 - z - \lambda a(z/\zeta) = 0 \quad (26)$$

and at the corresponding points $z = z_r$ we find that $F(z_r) = 1$. This suggests writing $F(z)$ in the form

$$F(z) = \left[1-\frac{(1-z-\lambda a)}{(1-z)}\right]^{(1-z)}\left[1+\frac{(1-z-\lambda a)}{z}\right]^z$$

$$= \exp\left\{-\tfrac{1}{2}(1-z-\lambda a)^2\left(\frac{1}{(1-z)}+\frac{1}{z}\right) - \tfrac{1}{3}(1-z-\lambda a)^3\left(\frac{1}{(1-z)^2}-\frac{1}{z^2}\right) - \cdots\right\} \quad (27)$$

from which we obtain

$$\frac{F'}{F} = \frac{(1-z-\lambda a)(1+\frac{\lambda}{\zeta}a')}{z(1-z)} \quad + \text{ higher powers of } (1-z-\lambda a)$$

$$\frac{F''}{F} - \left(\frac{F'}{F}\right)^2 = -\frac{(1+\frac{\lambda a'}{\zeta})^2}{z(1-z)} \quad + \text{ terms containing } (1-z-\lambda a)$$

and so at $z = z_r$,

$$F_r'' = -\frac{1}{z_r(1-z_r)}\left(1+\frac{\lambda}{\zeta}a'(z_r/\zeta)\right)^2 \tag{28}$$

It follows that <u>the points z_r satisfying</u> (26) <u>are all maxima of $F(z)$.</u>

Moreover at all other points in $0 < z < 1$ we must necessarily have $F(z) < 1$. For consider

$$F = \left(\frac{\lambda a}{1-z}\right)^{1-z}\left(\frac{1-\lambda a}{z}\right)^{z} = e^{G} \qquad \text{as a function of } \lambda .$$

Turning points occur when $G_\lambda = 0$, where

$$G_\lambda = a\left\{\frac{(1-z)}{\lambda a} - \frac{z}{(1-\lambda a)}\right\} = \frac{a(1-z-\lambda a)}{(1-z)^{1-z}(1-\lambda a)^{1-z}(z\lambda a)^{z}}$$

and for given z there is only one value of λ satisfying $G_\lambda = 0$, namely that making $1-z-\lambda a = 0$. Furthermore,

$$G_{\lambda\lambda} = -a^2\left[\frac{(1-z)}{(\lambda a)^2} + \frac{z}{(1-\lambda a)^2}\right]$$

showing that $G_{\lambda\lambda} < 0$ for all z and λ lying between 0 and 1. Hence G has a unique maximum at the point $\lambda = \frac{1-z}{a}$, and consequently so has F for the same λ. It takes the value $F = 1$ there, and decreases steadily on either side of it. No values of z and λ betwen 0 and 1 can therefore be found at which $F \geq 1$ and yet $1-z-\lambda a \neq 0$, for if such values existed, then at $\lambda = \frac{1-z}{a}$ we should also have $F = 1$, $1-z-\lambda a = 0$ which would imply $G_{\lambda\lambda} \geq 0$ at some point in the range.

Considering F once more as a function of z for fixed λ, we conclude that when n is large, F^n is negligible except near the roots $z = z_r$ of $1-z-\lambda a(z/\zeta) = 0$ at which it approximates to $F = 1$, and consequently $T_n(z)$ consists of a series of isolated peaks situated at these roots.

<u>Graphical representation of the roots.</u> The positions of the roots are conveniently examined by means of a diagram (fig. 6) similar to that already used in the previous discussion. A typical curve for $a(w)$ is shown. A line joining the point $\frac{1}{\lambda}$ on the vertical axis to the point $\frac{1}{\zeta}$ on the horizontal axis intercepts the curve, in the case shown, at points z_1/ζ, z_2/ζ, z_3/ζ and from the geometry of the figure it is evident that z_1, z_2, z_3 so found are the only possible roots of

$$1 - z - \lambda a(z/\zeta) = 0$$

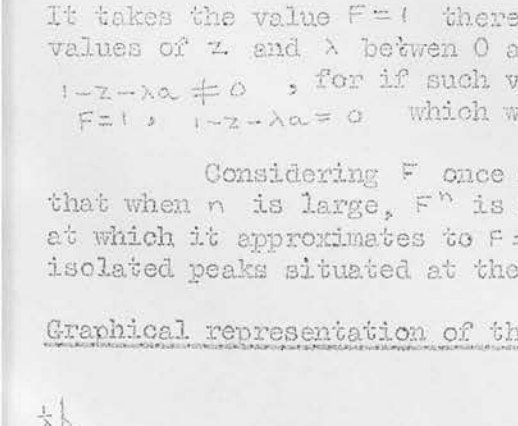


fig 6

With $a(w)$ curves of the form shown, a maximum of 3 distinct roots is possible with suitable values of ζ and λ, and in general the maximum number of possible roots is odd.

<u>Behaviour of $T_n(z)$ near a simple real root.</u> Let us first consider $F(z)$ near the point z_r. Since

$$1 - z - \lambda a(z/\zeta) = -\left[1 + \frac{\lambda}{\zeta} a'(z_r/\zeta)\right](z - z_r) + O(z - z_r)^2$$

the appropriate expansion of (27) is

$$F(z) = \exp\left\{-\frac{1}{2}\frac{\left[1 + \frac{\lambda}{\zeta} a'(z_r/\zeta)\right]^2}{z_r(1 - z_r)}(z - z_r)^2 + O(z - z_r)^3\right\}$$

and $T_n(z)$ may be written

$$T_n(z) = \sqrt{\frac{n}{2\pi}}\frac{dz}{\sqrt{z_r(1-z_r)}}\exp\left\{-\frac{n\left[1 + \frac{\lambda}{\zeta} a'(z_r/\zeta)\right]^2(z - z_r)^2}{2z_r(1 - z_r)}\right\}\left[1 + O(n(z-z_r)^3) + O(z-z_r\right.$$

Appreciable values of the exponential factor occur only when $O[n(z-z_r)^2] = O(1)$ so we need only consider the range in which $z - z_r = O(\frac{1}{\sqrt{n}})$. The formula is thus

$$T_n(z) = \sqrt{\frac{n}{2\pi}}\frac{dz}{\sqrt{z_r(1-z_r)}}\exp\left\{-\frac{n\left[1 + \frac{\lambda}{\zeta} a'(\frac{z_r}{\zeta})\right]^2(z - z_r)^2}{2z_r(1 - z_r)}\right\}\left[1 + O(\tfrac{1}{\sqrt{n}})\right] \quad (28)$$

Denoting $Q_{n,m}$ by $Q_n(z)$ when n is large, then when z_r are distinct, and neither z_r nor $1 - z_r$ is small, the identity (24) takes the approximate form

$$\sum_r \int_{-\infty}^{\infty} Q_n(z)\sqrt{\frac{n}{2\pi}}\frac{dz}{\sqrt{z_r(1-z_r)}}\exp - \frac{n\left[1 + \frac{\lambda}{\zeta} a'(\frac{z_r}{\zeta})\right]^2(z - z_r)^2}{2z_r(1 - z_r)} = 1 + O(\tfrac{1}{\sqrt{n}}) \quad (29)$$

summed over all real roots z_r.

It is necessary to point out that the function $Q_n(z)$ is not now uniquely defined by (29). For example, the function $Q_n(z) = 1 + \frac{(1-z)a'(z/\zeta)}{\zeta a(z/\zeta)}$ satisfies (29) over the whole range of z, and yet is obviously not relevant to our problem as it stands, since it takes a negative sign at every alternate z_r.

<u>Case of z_1 small.</u> We turn now to a case analogous to the Poisson limiting form. When z_1 is small, the approximation (28) is not valid, and we require to know the behaviour of

$$T_{n,m} = \frac{n!}{m!(n-m)!}(\lambda b_m)^{n-m}(1 - \lambda b_m)^m$$

for small $1 - \lambda$, when n is large but $m = O(1)$. Near $z = 0$ we have[*]

$$a(z/\zeta) = 1 + \frac{a'(0)}{\zeta}z + \frac{a''(0)}{2\zeta^2}z^2 + O(z^3)$$

i.e.

$$b_m = 1 - \frac{\zeta_0}{\zeta}\frac{m}{n} + \frac{a''(0)}{2\zeta^2}\frac{m^2}{n^2} + O(\tfrac{1}{n^3})$$

[*] The possibility of an expansion in non-integral powers of z near $z = 0$ is realised; such an expansion is in fact likely in many cases in view of the form of Fisher and Tippett's approximation for long threads. The more general analysis is not essentially different, however, and for simplicity integral powers only are considered.

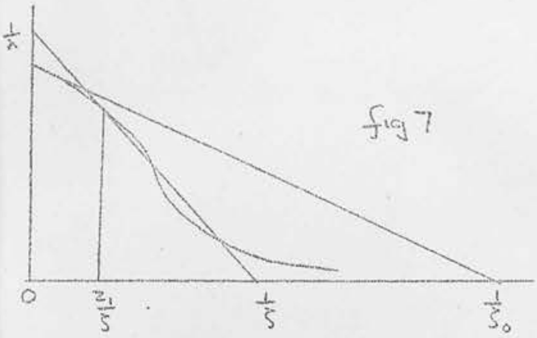


fig 7

where $1/\zeta_0$ is the intercept on the horizontal axis of the tangent to $a(w)$ at the origin (fig. 7). It is convenient to take $1-\lambda = O(\frac{1}{n})$, so put $\lambda = 1-\frac{\mu}{n}$ where $\mu = O(1)$. Then

$$(n-m)\log(\lambda b_m) = (n-m)\left[-\frac{\mu}{n} - \frac{\zeta_0}{\zeta}\frac{m}{n} + O(\tfrac{1}{n^2})\right]$$

$$= -(\mu + m\tfrac{\zeta_0}{\zeta}) + O(\tfrac{1}{n})$$

i.e. $$(\lambda b_m)^{n-m} = e^{-(\mu + m\frac{\zeta_0}{\zeta})}\left[1+O(\tfrac{1}{n})\right]$$

and $$1-\lambda b_m = \tfrac{1}{n}(\mu + \tfrac{m\zeta_0}{\zeta}) + O(\tfrac{1}{n^2})$$

so that $$(1-\lambda b_m)^m = \frac{(\mu + \frac{m\zeta_0}{\zeta})^m}{n^m}\left[1+O(\tfrac{1}{n})\right]$$

Hence

$$T_{n,m} = \frac{n^{n+\frac{1}{2}}e^{-m}(\mu+\frac{m\zeta_0}{\zeta})^m}{m!(n-m)^{n-m+\frac{1}{2}}\,n^m}\,e^{-(\mu+\frac{m\zeta_0}{\zeta})}\left[1+O(\tfrac{1}{n})\right]$$

and

$$(1-\tfrac{m}{n})^{n-m+\frac{1}{2}} = e^{-m}\left[1+O(\tfrac{1}{n})\right]$$

so that the appropriate limiting form when $1-\lambda = O(\frac{1}{n})$ and $m = O(1)$ is

$$T_{n,m} = \frac{(\mu+\frac{m\zeta_0}{\zeta})^m}{m!}\,e^{-(\mu+\frac{m\zeta_0}{\zeta})}\left[1+O(\tfrac{1}{n})\right] \tag{30}$$

When $\zeta > \zeta_0$, $T_{n,m}$ after a certain value of m decreases steadily to zero like $e^{\mu(\zeta/\zeta_0-1)+m(1-\zeta_0/\zeta)}\,\frac{1}{\sqrt{2\pi m}}\left(\frac{\zeta_0}{\zeta}\right)^m$ before m is as large as $O(n)$.

A simplification is introduced if $\zeta_0 = 0$. This occurs in the majority of distributions encountered in practice, for it implies that the chance of breaking strength exceeding S tends to zero more rapidly than $1/S$ as S increases. The form of $T_{n,m}$ degenerates in that case to the Poisson approximation

$$T_{n,m} = \frac{\mu^m}{m!}\,e^{-m}\left[1+O(\tfrac{1}{n})\right]$$

and when $\mu = 0$ we have simply $T_{n,0} = 1 + O(\frac{1}{n})$, $T_{n,m} = O(\frac{1}{n})$, $m = 1, 2, \ldots$. When $a'(0) = 0$ and $\lambda = 1$ the identity (24) therefore takes the approximate form

$$Q_{n,0} + \int_{\frac{1}{n}}^{1} Q_n(z)\,T_n(z)\,dz = 1 + O(\tfrac{1}{n}) \tag{31}$$

and if the real roots z_r of $1-z-a(z/\zeta)=0$, other than $z_1=0$, are distinct and such that neither z_r nor $1-z_r$ are small, (31) may be written

$$Q_{n,0} + \sum_r \int_{-\infty}^{\infty} Q_n(z)\,dz \sqrt{\frac{n}{2\pi z_r(1-z_r)}}\, \exp -\frac{n\left(1+a'(z_r/\zeta)\right)^2(z-z_r)^2}{2z_r(1-z_r)} = 1+O\left(\tfrac{1}{\sqrt{n}}\right) \quad (31a)$$

The asymptotic form of $B_n = Q_{n,0}$ is thus related to the behaviour of $Q_n(z)$ near the roots z_r of $1-z-a(z/\zeta)=0$. To study $Q_n(z)$ further for all values of z a more general form of (29) is now developed from (24a) on similar lines.

<u>Approximate form of generalised identity.</u> Equation (24a) is equivalent to

$$\sum_{m=r}^{n} Q_{n,m} \frac{(n-r)!\,(\lambda b_m)^{n-m}\,(1-\lambda b_m)^{m-r}}{(m-r)!\,(n-m)!} \equiv 1$$

Let

$$T_{n,m,r} = \frac{(n-r)!\,(\lambda b_m)^{n-m}\,(1-\lambda b_m)^{m-r}}{(m-r)!\,(n-m)!}$$

Provided neither $\frac{m-r}{n}$ nor $1-\frac{m}{n}$ is small, Stirling's approximation may be used as before to give

$$T_{n,m,r} = \frac{(n-r)^{n-r+\frac{1}{2}}\,(\lambda b_m)^{n-m}\,(1-\lambda b_m)^{m-r}}{\sqrt{2\pi}\,(m-r)^{m-r+\frac{1}{2}}\,(n-m)^{n-m+\frac{1}{2}}} \quad \left[1+O\left(\tfrac{1}{n}\right)\right]$$

With the previous notation, and putting in addition $r=nx$ this becomes

$$T_n(z,x) = \sqrt{\frac{n(1-x)}{2\pi(z-x)(1-z)}}\,dz\,(1-x)^{n(1-x)}\left[\frac{\lambda a(z/\zeta)}{1-z}\right]^{n(1-z)}\left[\frac{1-\lambda a(z/\zeta)}{z-x}\right]^{n(z-x)}\cdot\left[1+O\left(\tfrac{1}{n}\right)\right]$$

and we may write

$$F(z,x) = (1-x)^{(1-x)}\left[\frac{\lambda a(z/\zeta)}{1-z}\right]^{(1-z)}\left[\frac{1-\lambda a(z/\zeta)}{z-x}\right]^{(z-x)}$$

which it is convenient to arrange in the form

$$F(z,x) = \left\{1-\frac{[1-z-(1-x)\lambda a]}{(1-z)}\right\}^{(1-z)}\left\{1+\frac{[1-z-(1-x)\lambda a]}{(z-x)}\right\}^{(z-x)}$$

Putting $\kappa=(1-x)\lambda$, this becomes

$$F(z,x) = \exp\left\{-\tfrac{1}{2}(1-z-\kappa a)^2\left(\frac{1}{1-z}+\frac{1}{z-x}\right)-\tfrac{1}{3}(1-z-\kappa a)^3\left[\frac{1}{(1-z)^2}-\frac{1}{(z-x)^2}\right]-\cdots\right\}$$

The argument is followed through precisely as before and we obtain finally, writing λ in place of κ,

$$\sum_{x<z_r<1}\int_{-\infty}^{\infty} Q_n(z)\,dz\sqrt{\frac{n(1-x)}{2\pi(z_r-x)(1-z_r)}}\,\exp-\frac{n(1-x)\left(1+\lambda a'(z_r/\zeta)\right)^2(z-z_r)^2}{2(z_r-x)(1-z_r)} = 1+O\left(\tfrac{1}{\sqrt{n}}\right) \quad (32)$$

the roots z_r being supposed distinct, and the summation being taken only over those roots lying between x and 1. The approximation holds only when $1-z$, $z-x$, are not small.

The range within which $T_{n,m,r} \geq 0$ for $r \leq m \leq n$ was, with the previous λ, $0 \leq \lambda \leq 1/\zeta$, and substituting $(1-x)\lambda$ for λ we now have $0 \leq \lambda \leq \lambda_x = \frac{1-x}{a(x/\zeta)}$ the approximation failing near the ends of the range.

Behaviour of $Q_n(z)$ at non-critical points.

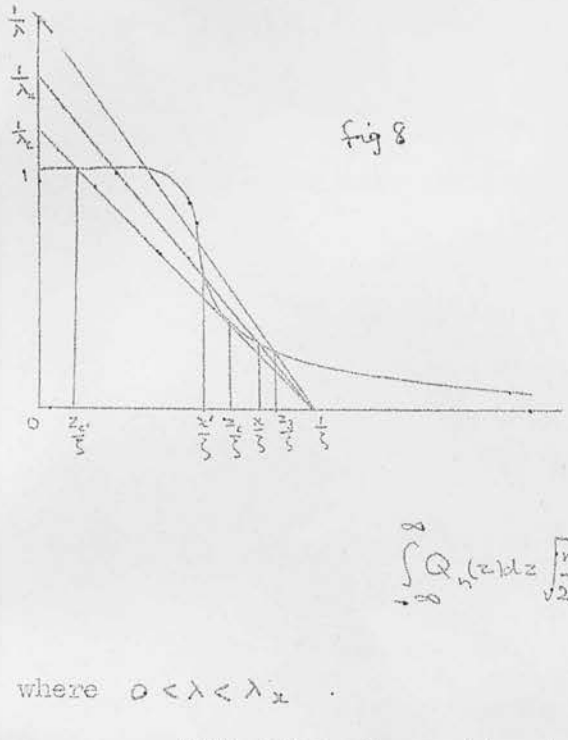


fig 8

Suppose for simplicity that $a(w)$ is the type of curve shown in fig. 8, such that the equation $1-z-\lambda a(z/\zeta) = 0$ has a maximum of 3 roots. Let us consider what happens as x changes from 1 to 0.

(i) $z_c < x$: Denoting by λ_c the value of λ for which roots z_2 and z_3 coincide at z_c, then when $x > z_c$ there is only one root, z_3, falling in the range $x \leq z \leq 1$ and (32) reduces to

$$\int_{-\infty}^{\infty} Q_n(z)\,dz \sqrt{\frac{n(1-x)}{2\pi(z_3-x)(1-z_3)}} \exp - \frac{n(1-x)\{1+\frac{1}{\zeta}a'(z_3/\zeta)\}^2 (z-z_3)^2}{2(z_3-x)(1-z_3)} = 1+O\left(\frac{1}{\sqrt{n}}\right) \quad (33)$$

where $0 < \lambda < \lambda_x$.

This integral equation is satisfied by

$$Q_n(z) = 1 + \frac{(1-z)a'(z/\zeta)}{\zeta a(z/\zeta)} + O\left(\frac{1}{\sqrt{n}}\right) \qquad (34)$$

which we may therefore take to hold for $z_c < z < 1$ except near $z = z_c$.*

(ii) $z_{c'} < x < z_c$: Avoiding for the moment the critical point z_c, consider now the range $z_{c'} < x < z_c$, where $z_{c'}$ is the value of z_1 when $z_2 = z_3 = z_c$. A typical value of x in the range is x' (fig. 8). When $\frac{1}{\lambda} > \frac{1}{\lambda_x}$ there is, as before, one root z_3, and the identity (33) holds. By making $\lambda = \lambda_x$, however, we gain information about $Q_n(x)$ but the approximation (32) no longer holds near $z = x$. The behaviour of $T_{n,m,r}$ near $m = r$ when $\lambda \doteq \lambda_x$ may, nevertheless, be investigated exactly as in the analogous less general case already discussed (to which this degenerates when $r = 0$). Putting $\lambda = \lambda_x(1-\frac{\mu}{n})$ where $\mu = O(1)$ and $\gamma = -\frac{a'(x/\zeta)}{a(x/\zeta)}$ it is found that

$$T_{n,m,r} = \frac{(1-x)^{(m-r)}\left[\mu+\frac{\gamma}{\zeta}(m-r)\right]^{(m-r)} e^{-(1-x)\left[\mu+\frac{\gamma}{\zeta}(m-r)\right]}}{(m-r)!} \cdot \left[1+O\left(\frac{1}{n}\right)\right] \quad (35)$$

when $m-r = O(1)$, and it is sufficient for our purpose to note that when $\lambda = \lambda_x$ and $m = r$, $T_{n,r,r} = 1 + O(\frac{1}{n})$. In place of (33) we now have

---

* The uniqueness of this solution has not so far been proved, but in a later section the same result is derived uniquely by an independent method.

$$Q_{n,r}T_{n,r,r} + Q_{n,r+1}T_{n,r+1,r} + \cdots$$

$$+ \int_{-\infty}^{\infty} Q_n(z)\,dz \sqrt{\frac{n(1-x')}{2\pi(z_3-x')(1-z_3)}} \exp - \frac{n(1-x')\left[1+\frac{\lambda x'}{\zeta}a'(z_3/\zeta)\right]^2 (z-z_3)^2}{2(z_3-x')(1-z_3)} = 1 + O\left(\frac{1}{\sqrt{n}}\right) \quad (36)$$

and hence, substituting (34),

$$Q_{n,r}T_{n,r,r} + Q_{n,r+1}T_{n,r+1,r} + \cdots = O\left(\frac{1}{\sqrt{n}}\right)$$

Each of these terms must therefore be $O\left(\frac{1}{\sqrt{n}}\right)$, and in particular since $T_{n,r,r} = 1 + O\left(\frac{1}{n}\right)$ we must have

$$Q_{n,r} = Q_n(x') = O\left(\frac{1}{\sqrt{n}}\right) \qquad \text{when} \quad z_{c'} < x' < z_c$$

(iii) $\underline{0 < x < z_{c'}}$: Finally when x lies in the range $0 < x < z_{c'}$ there is again only one root, this time z_1, and by the same argument as before we deduce that

$$Q_n(z) = 1 + \frac{(1-z)\,a'(z/\zeta)}{\zeta a(z/\zeta)} + O\left(\frac{1}{\sqrt{n}}\right) \qquad \text{when} \quad 0 < z < z_{c'}$$

It is not permissible, however, to assume that this necessarily holds near $z=0$, for when $x=0$, (32) reduces to (29) and the approximations fail near $z=0$. But when $m = O(1)$ we are enabled by (30) to write

$$\sum_{m=0}^{\infty} Q_{n,m} \frac{\left(\mu + \frac{m\zeta_0}{\zeta}\right)^m e^{-\left(\mu + m\frac{\zeta_0}{\zeta}\right)}}{m!} = 1 + O\left(\frac{1}{n}\right) \qquad (37)$$

in place of an equation of type (33). If $Q_n(z) \sim 1 + \frac{(1-z)\,a'(z/\zeta)}{\zeta a(z/\zeta)}$ were to hold right up to $z=0$ we should expect that when $m = O(1)$ i.e. $z = O\left(\frac{1}{n}\right)$, $Q_{n,m} \sim 1 - \frac{\zeta_0}{\zeta}$ for all m to the order considered, and we can show that this is in fact consistent with (37). For

$$\sum_{m=0}^{\infty} \frac{\left(\mu + \frac{m\zeta_0}{\zeta}\right)^m}{m!} e^{-\left(\mu + m\frac{\zeta_0}{\zeta}\right)} = -\sum_{m=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{e^{-\left(\mu + m\frac{\zeta_0}{\zeta}\right)u}}{(1-u)^{m+1}}\,du$$

where the contour C in the u plane encloses $u=1$,

$$= \frac{1}{2\pi i} \int_C \frac{e^{-\mu u}\,du}{u - 1 + e^{-u\frac{\zeta_0}{\zeta}}} \quad , \quad \zeta > \zeta_0$$

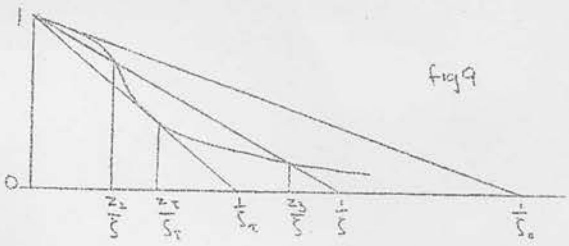
The integrand has a simple pole at $u=0$, which C must necessarily enclose for the expansion to be valid, and at which the residue is $1/\left(1-\frac{\zeta_0}{\zeta}\right)$. The sum of the series is thus $1/\left(1-\frac{\zeta_0}{\zeta}\right)$, which is consistent with the assertion that $Q_{n,m} = 1 - \frac{\zeta_0}{\zeta} + O\left(\frac{1}{n}\right)$.

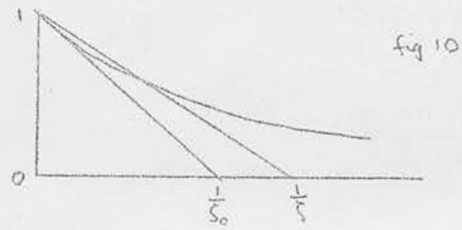
Our conclusions can therefore be summarised as follows. Let the form of $\alpha(w)$ be such that 3 distinct roots of $1 - z - \lambda\alpha(z/\zeta) = 0$ are the maximum possible, and let ζ be chosen so that when z_2 and z_3 are made to coalesce at z_c for suitable λ, the smallest root z_1 has a value $z_{c'} > 0$. Then the approximate form of $Q_n(z)$ is

$$\left.\begin{aligned} Q_n(z) &= 1 + \frac{(1-z)\alpha'(z/\zeta)}{\zeta\alpha(z/\zeta)} + O\left(\frac{1}{\sqrt{n}}\right) && \text{when } z_c < z < 1 \quad \text{(i)} \\ &= O\left(\frac{1}{\sqrt{n}}\right) && \text{when } z_{c'} < z < z_c \quad \text{(ii)} \\ &= 1 + \frac{(1-z)\alpha'(z/\zeta)}{\zeta\alpha(z/\zeta)} + O\left(\frac{1}{\sqrt{n}}\right) && \text{when } 0 \leq z < z_{c'} \quad \text{(iii)} \end{aligned}\right\} \qquad (38)$$

By varying ζ the range $z_{c'} < z < z_c$ may be made to vanish, or to include $z = 0$, but in all cases $Q_n(z) = O(\frac{1}{\sqrt{n}})$ throughout any part of this range, and $Q_n(z) = 1 + \frac{(1-z)\alpha'(z/\zeta)}{\zeta\alpha(z/\zeta)} + O(\frac{1}{\sqrt{n}})$ otherwise*. The argument may evidently be extended to cases where the number of roots is greater than 3.

<u>Application to $Q_{n,0}$ in non-critical regions.</u>





Consider a curve of the form shown in fig. 9, with $\zeta_0 \neq 0$; keep $\lambda = 1$ so that the lowest root is always $z_1 = 0$, and allow ζ to vary. When $\zeta < \zeta_\tau$, the remaining roots z_2 and z_3 are real and distinct; when $\zeta > \zeta_\tau$ they are imaginary. Since $\alpha(0) = 1$, $\alpha'(0) = -\zeta_0$, it may be stated immediately from the preceding discussion, on putting $z = 0$, that

$$\begin{aligned} B_n = Q_{n,0} &= O\left(\frac{1}{\sqrt{n}}\right) && \text{when} \quad \zeta < \zeta_\tau \\ &= 1 - \frac{\zeta_0}{\zeta} + O\left(\frac{1}{\sqrt{n}}\right) && \text{when} \quad \zeta > \zeta_\tau \end{aligned}$$

provided $|\zeta - \zeta_\tau| > O(\frac{1}{\sqrt{n}})$, and the curve has simple contact with its tangent at $\frac{z_\tau}{\zeta_\tau}$. This is an advance on the information previously gained from physical considerations that $B_n \sim 0$ when $\zeta \ll \zeta_\tau$ and $B_n \sim 1$ when $\zeta \gg \zeta_\tau$. It is now seen that when $\zeta_0 \neq 0$ B_n tends to 1 slowly when $\zeta > \zeta_\tau$ but falls off sharply to $O(\frac{1}{\sqrt{n}})$ when $\zeta < \zeta_\tau$. In the commonest case where $\zeta_0 = 0$, the transition from 0 to 1 occurs abruptly within a range of ζ which is $O(\frac{1}{\sqrt{n}})$. When $z_\tau = 0$, ζ_τ and ζ_0 coincide, as in fig. 10, and the result is $B_n = O(\frac{1}{\sqrt{n}})$ for $\zeta < \zeta_0$ and $B_n = 1 - \frac{\zeta_0}{\zeta} + O(\frac{1}{\sqrt{n}})$ for $\zeta > \zeta_0$ provided $|\zeta - \zeta_0| > O(\frac{1}{\sqrt{n}})$.

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* A similar, but cruder, conclusion is reached by a physical discussion in Appendix A.

Further information on the form of $B_n = Q_n$ must be obtained from a study of $Q_n(z)$ in the critical regions where it changes rapidly, and we proceed to examine these regions in detail.

Behaviour of $T_n(z)$ near a double root which is not small.

It was shown that $F(z)$ could be expressed in the form

$$F(z) = \exp\left\{-\tfrac{1}{2}(1-z-\lambda a)^2\left(\frac{1}{1-z}+\frac{1}{z}\right) - \tfrac{1}{3}(1-z-\lambda a)^3\left(\frac{1}{(1-z)^2}+\frac{1}{z^2}\right) - \cdots\right\}$$

so that $F(z) = 1$ when $1 - z - \lambda a(z/\zeta) = 0$. Suppose now that the two largest roots z_2 and z_3 are such that $\epsilon = |z_2 - z_3|$ is small. The roots may be real or complex provided their difference is small. Writing $f(z) = 1 - z - \lambda a(z/\zeta)$, let $\bar{z}$ be that value of z near z_2, z_3 which makes $f'(\bar{z}) = 0$. Then since

$$f(z) = f(\bar{z}) + \tfrac{1}{2}(z-\bar{z})^2 f''(\bar{z}) + O(z-\bar{z})^3$$

and $f(z_2) = f(z_3) = 0$, it follows, if $f''(\bar{z})$ is not small, that

$$\bar{z} = \tfrac{1}{2}(z_2+z_3) + O(\epsilon^3)$$

and

$$(z_2-z_3)^2 = -\frac{8f(\bar{z})}{f''(\bar{z})} + O(\epsilon^3)$$

Put $\xi = \frac{1}{4}(z_2-z_3)^2$, which will be positive or negative according as the roots are real or complex. Then when $z - \bar{z} = O(\epsilon)$

$$f(z) = \tfrac{1}{2}f''(\bar{z})\left[(z-\bar{z})^2 - \xi\right] + O(\epsilon^3)$$

$$= -\frac{\lambda a''(\bar{z}/\zeta)}{2\zeta^2}\left[(z-\bar{z})^2 - \xi\right] + O(\epsilon^3)$$

It may also be written $\quad f(z) = -\dfrac{\lambda a''(\bar{z}/\zeta)}{2\zeta^2}(z-z_2)(z-z_3) + O(\epsilon^3)$

Critical points. Two critical values of z may be defined as follows:-

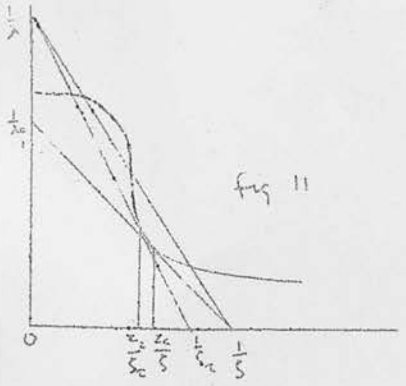

fig 11

1. Keep ζ fixed and change λ until the roots coincide at z_c, λ then being λ_c.

2. Keep λ fixed and change ζ until the roots coincide at z_τ, ζ then being ζ_τ.

It is almost obvious that both z_c and z_τ differ from $\bar{z}$ only by $O(\epsilon^2)$

1. In the first case,

$$\lambda(z) = \frac{1-z}{a(z/\zeta)} = \lambda_c + (z-z_c)\lambda'_c + \tfrac{1}{2}(z-z_c)^2\lambda''_c + O(z-z_c)^3,$$

$$\lambda(z_2) = \lambda(z_3) = \lambda, \quad \lambda'_c = 0, \quad \lambda''_c = -\frac{\lambda_c a''_c}{\zeta a_c}$$

which is in general not small,

and so as before, $\quad z_c = \tfrac{1}{2}(z_2+z_3) + O(\epsilon^2)$

and

$$\xi = \tfrac{1}{4}(z_2 - z_3)^2 = \frac{2\zeta a_c}{\lambda_c a''_c}(\lambda - \lambda_c) + O(\epsilon^3)$$

2. In the case of z_τ, put $w = \frac{z}{\zeta}$, $w_\tau = \frac{z_\tau}{\zeta_\tau}$,

$$\zeta(w) = \frac{1 - \lambda a(w)}{w} = \zeta_\tau + (w - w_\tau)\zeta'_\tau + \tfrac{1}{2}(w - w_\tau)^2 \zeta''_\tau + O(w - w_\tau)^3$$

where $\zeta(w_2) = \zeta(w_3) = \zeta$, $\zeta'_\tau = 0$, $\zeta''_\tau = \frac{-\lambda a''_\tau}{w_\tau}$, in general not small, so that $w_\tau = \frac{1}{2}(w_2 + w_3) + O(\epsilon^2)$, or

$$z_\tau = \tfrac{1}{2}(z_2 + z_3) + O(\epsilon^2)$$

to the same order, and

$$\xi = \tfrac{1}{4}\zeta_\tau^2 (w_2 - w_3)^2 + O(\epsilon^3) = -\frac{2 z_\tau \zeta_\tau}{\lambda a''_\tau}(\zeta - \zeta_\tau) + O(\epsilon^3)$$

Hence z_c and z_τ are interchangeable, as are also the two forms of ξ given in terms of $\lambda - \lambda_c$ and $\zeta - \zeta_\tau$. The latter is the more useful for our purpose. So in the vicinity of a double root z_τ, $F(z)$ takes the form,

$$F(z) = \exp\left\{ -\frac{1}{2z_\tau(1-z_\tau)}\left[\frac{\lambda a''_\tau}{2\zeta_\tau^2}(z - z_\tau)^2 + \frac{z_\tau}{\zeta_\tau}(\zeta - \zeta_\tau)\right]^2 + O(\epsilon^5)\right\}$$

where $z - z_\tau = o(\epsilon)$, but z_τ is not small, and when $z - z_\tau = O\left(\frac{1}{n^{1/4}}\right)$ the approximate value of $T_n(z)$ near z_τ is thus

$$T_n(z) = \sqrt{\frac{n}{2\pi z_\tau(1-z_\tau)}}\, e^{-\frac{n}{2z_\tau(1-z_\tau)}\left[\frac{\lambda a''_\tau}{2\zeta_\tau^2}(z - z_\tau)^2 + \frac{z_\tau}{\zeta_\tau}(\zeta - \zeta_\tau)\right]^2} \cdot \left[1 + O\left(\frac{1}{n^{1/4}}\right)\right]$$

provided z_τ is itself not small. An analogous result can be similarly derived for the more general function $T_n(z, x)$.

<u>Application to $Q_{n,0}$ when $\zeta - \zeta_\tau$ is small.</u>

The above expression with $\lambda = 1$, $y = z - z_\tau$, is now substituted into equation (31) to give

$$Q_{n,0} + \int_{-\infty}^{\infty} Q_n(y + z_\tau)\, dy \sqrt{\frac{n}{2\pi z_\tau(1-z_\tau)}} \cdot e^{-\frac{n}{2z_\tau(1-z_\tau)}\left[\frac{a''_\tau y^2}{2\zeta_\tau^2} + \frac{z_\tau}{\zeta_\tau}(\zeta - \zeta_\tau)\right]^2} = 1 + O\left(\frac{1}{n^{1/4}}\right) \quad (37)$$

As we already know that in the non-critical regions,

$$Q_n(z) = 1 + \frac{(1-z)\, a'(z/\zeta)}{\zeta a(z/\zeta)} + O\left(\frac{1}{\sqrt{n}}\right), \quad z \gg z_\tau$$

$$= O\left(\frac{1}{\sqrt{n}}\right), \quad z \ll z_\tau$$

it is natural to introduce the function $q(y)$ defined by

$$Q_n(z) = \left[1 + \frac{(1-z)\, a'(z/\zeta)}{\zeta a(z/\zeta)}\right] q(y)$$

so that $q(y) \sim 1$, $y \gg 0$; $q(y) \sim 0$, $y \ll 0$. Taking $\lambda = 1$ we find that

$1 + \frac{(1-z)a'}{\zeta a} = \frac{a''_\tau}{\zeta_\tau^2}\, y\left[1 + O\left(\frac{1}{n^{1/4}}\right)\right]$ in the important range of the integrand, and $Q_{n,0}$ satisfies

$$Q_{n,0} = 1 - \int_{-\infty}^{\infty} q(y)\frac{a''_c}{\zeta_c^2}\, y\, dy \sqrt{\frac{n}{2\pi z_c(1-z_c)}}\, e^{-n\left[\frac{a''_c y^2}{2\zeta_c^2} + \frac{z_c}{\zeta_c}(\zeta-\zeta_c)\right]^2} + O\left(\frac{1}{n^{1/4}}\right) \qquad (40)$$

or setting $v = \frac{a''_c y^2}{2\zeta_c^2}$,

$$Q_{n,0} = 1 - \int_{0}^{\infty} \left[q\left(\zeta_c\sqrt{\frac{2v}{a''_c}}\right) + q\left(-\zeta_c\sqrt{\frac{2v}{a''_c}}\right)\right] \sqrt{\frac{n}{2\pi z_c(1-z_c)}}\, dv\; e^{-\frac{n\left[v + \frac{z_c}{\zeta_c}(\zeta-\zeta_c)\right]^2}{2 z_c(1-z_c)}} + O\left(\frac{1}{n^{1/4}}\right) \qquad (41)$$

More generally, when $\zeta_0 \neq 0$ we have evidently to replace $Q_{n,0}$ by $\sum_{m=0}^{\infty} Q_{n,m}\left(\frac{m\zeta_0}{\zeta}\right)^m \frac{e^{-m\zeta_0/\zeta}}{m!}$ and it follows by reasoning similar to that used to derive (38 iii) near $z = 0$ that the above expression for $Q_{n,0}$ has to be multiplied throughout by $1 - \frac{\zeta_0}{\zeta}$. The situation is more complicated in the exceptional case when $\zeta_c - \zeta_0 = O(\frac{1}{\sqrt{n}})$ as there are then two critical regions, but the extension of the argument presents no new feature. Further information on $q(y)$ is obtained as follows.

Behaviour of $T_n(z,x)$ near a double root z_c when $x - z_c$ is small.

It has been shown that when $z_c < x$ and $x - z_c = O(1)$, the root z_3 is isolated, and it is possible to write

$$e^{-n(1-x)\left[\frac{1}{2}\frac{(1-z-\lambda a)^2}{(z-x)(1-z)} + \cdots\right]} = e^{-\frac{n(1-x)\left[1+\frac{\lambda}{\zeta}a'(z_3/\zeta)\right]^2 (z-z_3)^2}{2(z_3-x)(1-z_3)}} \cdot \left[1 + O(z-z_3)\right]$$

and to assume $z - z_3 = O\left(\frac{1}{\sqrt{n}}\right)$ over the important range of the integrand. Keeping ζ fixed, as λ is made to approach λ_c, z_3 comes into the vicinity of the double root z_c and it was than shown that

$$1 - z - \lambda a(z/\zeta) = -\frac{\lambda_c a''_c}{2\zeta^2}(z-z_2)(z-z_3) + O(\epsilon^3)$$

where $a''_c = a(z_c/\zeta)$, $\epsilon = z_3 - z_2 = 2(z_3 - z_c)$ to the same order. In the previous discussion, the roots ceased to be isolated when $\epsilon = O(\frac{1}{n^{1/4}})$. But as we are now considering the case where $z_c < x$ we must also have $z_3 - x \leq O(\epsilon)$, which has the effect of preventing the roots from "overlapping" until their difference ϵ is of smaller order than $O(\frac{1}{n^{1/4}})$. The roots may be said to "overlap" if $T_n(z,x)$ is not negligibly small at $z = z_c$ and the appropriate condition is that the exponent shall be $O(1)$ at $z = z_c$ i.e. $n\epsilon^3 = O(1)$ or $\epsilon = O(\frac{1}{n^{1/3}})$. We are, however, interested in the behaviour of $Q_n(z)$ over a range of $z - z_c = O(\frac{1}{n^{1/4}})$, which will depend on the behaviour of $T_n(z,x)$ when $z_3 - z_c = o(\frac{1}{n^{1/4}})$. Consequently we expect to find that the root z_3 may again be considered isolated and that the previous approximate form of $Q_n(z)$ holds over the range $z - z_c = O(\frac{1}{n^{1/4}})$ but to a lower order of accuracy: substitution of $q(y) = 1,\ y > 0$; $q(y) = 0,\ y < 0$ in (40) should thus give a first approximation to the form of $Q_{n,0} = B_n$ for large n.

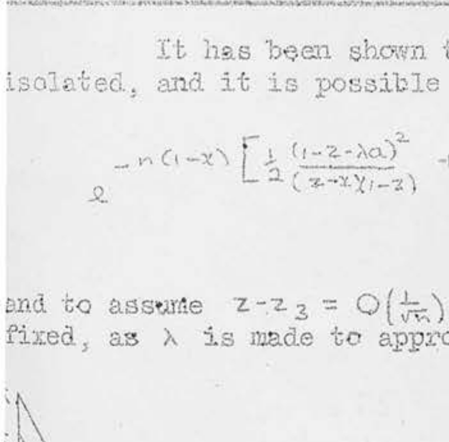

fig 12

Suppose then that $\epsilon = z_3 - z_2 = O(\frac{1}{n^{1/4}})$ and let us choose x so that $z_3 - x = O(\frac{1}{n^{1/4}})$ also, but not smaller. Since $z - x \leq O(\frac{1}{n^{1/4}})$ for the values of z that matter, we have at most $1 - z = 1 - x + O(\frac{1}{n^{1/4}})$, and so

$$T_n(z,x) = e^{-\frac{n\lambda_c^2 a_c''^2}{8\zeta^4}\frac{(z-z_2)^2(z-z_3)^2}{(z-x)}}\sqrt{\frac{n}{2\pi(z-x)}}\cdot\left[1+O\left(\frac{1}{n^{1/4}}\right)\right] \tag{42}$$

The range in which T_n is appreciable occurs when $\frac{n(z-z_2)^2(z-z_3)^2}{(z-x)} = O(1)$ and since $z - z_2 \geq x - z_2 = O(\frac{1}{n^{1/4}})$ it follows that $z - z_3 = O(\frac{1}{n^{3/8}})$. Hence

$$\frac{n(z-z_2)^2(z-z_3)^2}{(z-x)} = \frac{n(z_3-z_2)^2(z-z_3)^2}{(z_3-x)}\cdot\left[1+O\left(\frac{1}{n^{1/8}}\right)\right]$$

$$= \frac{4n(z_3-z_c)^2}{(z_3-x)}\cdot(z-z_3)^2\cdot\left[1+O\left(\frac{1}{n^{1/8}}\right)\right]$$

and

$$T_n(z,x) = e^{-\frac{n\lambda_c^2 a_c''^2}{2\zeta^4}\frac{(z_3-z_c)^2}{(z_3-x)}(z-z_3)^2}\sqrt{\frac{n}{2\pi(z_3-x)}}\cdot\left[1+O\left(\frac{1}{n^{1/8}}\right)\right]$$

The equation (33) is now replaced by

$$\int_{-\infty}^{\infty} Q_n(z)\,dz\sqrt{\frac{n}{2\pi(z_3-x)}}\;e^{-\frac{n\lambda_c^2 a_c''^2}{2\zeta^4}\frac{(z_3-z_c)^2}{(z_3-x)}(z-z_3)^2} = 1 + O\left(\frac{1}{n^{1/8}}\right)$$

which is satisfied if

$$Q_n(z) = \frac{\lambda_c a_c''}{\zeta^2}(z-z_c)\cdot\left[1+O\left(\frac{1}{n^{1/8}}\right)\right]$$

Application to $Q_{n,o}$.

Reverting to the notation of (40) and (41), we put to the same order, $\zeta = \zeta_\tau$, $z_c = z_\tau$, $\lambda_c = 1$ in the above expression for $Q_n(z)$ and deduce that $q(y) = 1 + O(\frac{1}{n^{1/8}})$ for values of $y > 0$ which are $O(\frac{1}{n^{1/4}})$ but not less. A similar extension of the argument in the range $z_{c'} < z < z_c$ can be made to show that $q(y) = O(\frac{1}{n^{1/8}})$ for $y < 0$, $|y| = O(\frac{1}{n^{1/4}})$.

It is seen that $Q_n(z)$ is $O(\frac{1}{n^{1/4}})$ for $z - z_c = O(\frac{1}{n^{1/4}})$; we anticipate for the moment the result that when $z - z_c < O(\frac{1}{n^{1/4}})$, $Q_n(z)$ is not greater than $O(\frac{1}{n^{1/4}})$ so that we may write $Q_n(z) \leq \frac{A}{n^{1/4}}$, A being $O(1)$. From (40) and (41) we then infer that

$$Q_{n,o} = 1 - \int_0^{\infty} e^{-\frac{n\left[u+\frac{z_\tau}{\zeta_\tau}(\zeta-\zeta_\tau)\right]^2}{2z_\tau(1-z_\tau)}}\sqrt{\frac{n}{2\pi z_\tau(1-z_\tau)}}\,du + R + O\left(\frac{1}{n^{1/8}}\right)$$

where the remainder term R has the form

$$R = \int_{-\frac{\alpha}{n^{1/4}}}^{\frac{\alpha}{n^{1/4}}} Q(z_c+y)\,dy\sqrt{\frac{n}{2\pi z_\tau(1-z_\tau)}}\cdot e^{-\frac{n}{2z_\tau(1-z_\tau)}\left[\frac{a_\tau'' y^2}{2\zeta_\tau^2}+\frac{z_\tau}{\zeta_\tau}(\zeta-\zeta_\tau)\right]^2}$$

α being $O(1)$. The most serious errors occur near $\zeta = \zeta_\tau$, for which value

$$R \leq \frac{A}{n^{1/4}}\int_{-\frac{\alpha}{n^{1/4}}}^{\frac{\alpha}{n^{1/4}}} dy\sqrt{\frac{n}{2\pi z_\tau(1-z_\tau)}}\left[1+O(ny^4)\right] = \frac{2\alpha A}{\sqrt{2\pi z_\tau(1-z_\tau)}}\left[1+O(\alpha^4)\right]$$

and α may be chosen to be small provided n is made large enough. If this is done it may be said that

$$Q_{n,0} \sim 1 - \int_0^\infty e^{-n\frac{\left[v + \frac{z_c}{\zeta_c}(\zeta - \zeta_c)\right]^2}{2z_c(1-z_c)}} \sqrt{\frac{n}{2\pi z_c(1-z_c)}}\, dv$$

$$\text{ie} \quad B_n = Q_{n,0} \sim \int_{-\infty}^{\zeta - \zeta_c} e^{-\frac{n z_c \chi^2}{2\zeta_c^2(1-z_c)}} \sqrt{\frac{n z_c}{2\pi \zeta_c^2(1-z_c)}}\, . \, d\chi \qquad (43)$$

for sufficiently large values of n. That is, for an $\alpha(w)$ curve satisfying the conditions stated, ζ tends for large values of n to be distributed according to a Normal error law about the mean value ζ_c with standard deviation $\sigma = \zeta_c\sqrt{\frac{1-z_c}{nz_c}} = \zeta_c\sqrt{\frac{a_c}{n(1-}}$

It is possible that in practice the formula may be useful for lower values of n than the above argument suggests for the errors are most serious for values of ζ near ζ_c which have little influence on the higher moments of the distribution. The fact remains, however, that the term $O(\frac{1}{n^{1/3}})$ arose from assuming $T_n(z,x)$ to be of Normal form when $z - z_c = O(\frac{1}{n^{1/3}})$, thus ignoring the appreciable skewness introduce by the factor $z - x$ in the denominator of the exponent. To take account of this it is necessary to keep $T_n(z,x)$ in the form (42), and an integral equation is then obtained for $Q_n(z)$ which is not of a familiar type (to the author). The weakness of the present approach is that it depends on such integral equations, which we have so far managed to solve somewhat intuitively, and for that reason the method of the succeeding section has been developed to provide a more direct attack on the behaviour of $Q_n(z)$ in the critical region.

The integral equation, however, is worth considering in a little more detail. It may be written in the form

$$\int_x^\infty Q_n(z)\, dz \sqrt{\frac{n}{2\pi(z-x)}}\; e^{-\frac{n\lambda_c^2 a_c''^2}{8\zeta^4}\frac{\left[(z-z_c)^2 - \xi\right]^2}{(z-x)}} = 1 + O\left(\frac{1}{n^{1/3}}\right)$$

the approximation holding to $O(\frac{1}{n^{1/3}})$ in the region where the roots overlap and $z - z_c = O(\frac{1}{n^{1/3}})$ but not smaller. A Fourier transform with respect to the parameter ξ casts it into the form

$$\frac{2\zeta^2}{\lambda_c a_c''} \int_x^\infty Q_n(z)\, dz\; e^{it(z-z_c)^2 - \frac{2\zeta^4(z-x)t^2}{n\lambda_c^2 a_c''^2}} = \delta(t,0) + O\left(\frac{1}{n^{1/3}}\right)$$

with the usual meaning for $\delta(t,0)$. Dividing out the factor $e^{\frac{2\zeta^4(x-z_c)t^2}{n\lambda_c^2 a_c''^2}}$ it becomes

$$\frac{2\zeta^2}{\lambda_c a_c''} \int_x^\infty Q_n(z)\, dz\; e^{it(z-z_c)^2 - \frac{2\zeta^4(z-z_c)^2 t^2}{n\lambda_c^2 a_c''^2}} = \delta(t,0) + O\left(\frac{1}{n^{1/3}}\right)$$

and so, differentiating x, $Q_n(x) = O(\frac{1}{n^{1/3}})$ when $x - z_c = O(\frac{1}{n^{1/3}})$.

THE GENERATING FUNCTION APPROACH.

The line of attack hitherto adopted has enabled us to obtain first approximations to the asymptotic form of B_n in the two characteristic cases where the value of w maximising $\frac{1-a(w)}{w}$ is, and is not, zero. More generally, we have reduced the asymptotic problem for any $a(w)$ to a consideration of the behaviour of $Q_n(z)$ when z is the greatest root of $1-z-\lambda a(z/\zeta)=0$. This restricted proble[m] may now be attacked by a more powerful method which independently reaffirms the result obtained in non-critical regions and yields the required information on $Q_n(z)$ for critical values of z.

The new method is based on the property of $\mathcal{B}_n = B_n$ when $b(s)$ has the special form $e^{-\frac{K}{s}}$, that $\mathcal{B}_n = n!\, e^{-\frac{Kn^2}{2s}} C_n$ where C_n is the coefficient of t^n in $1/M(t)$, an[d]

$$M(t) = 1 - t e^{-\frac{K}{2s}} + \frac{t^2}{2!} e^{-\frac{2^2 K}{2s}} - \frac{t^3}{3!} e^{-\frac{3^2 K}{2s}} + \cdots \qquad (20)$$

and, more generally, that $\mathcal{B}_n(x) = n!\, e^{-\frac{Kn^2}{2s}} C_n(x)$ where $C_n(x)$ is the coefficient of t^n in $M(tx e^{\frac{Kn}{s}})/M(t)$.

Let $z_j = w_j \zeta$ be the greatest root of $1-z-\lambda a(z/\zeta)=0$ and let us express $a(w)$ near w_j in the form

$$a(w) = \alpha + \beta e^{-K(w-w_j)} + O(w-w_j)^3$$

where $\quad K = -\frac{a_j''}{a_j'}, \quad \alpha = a_j - \frac{(a_j')^2}{a_j''}, \quad \beta = \frac{(a_j')^2}{a_j''}$

From the preceding work it is clear that $Q_n(z)$ remains unaltered to the order of approximation considered, when the form of $a(w)$ is varied at will in those regions where ζw is not, to the same order, near a root of $1-z-\lambda a=0$ provided that in so doing no new roots are introduced, and as the value of $Q_n(z)$ depends by definition only on $a(w)$ in which $w \geq z\zeta$, this need only apply to roots which are not less than z. We are therefore justified in replacing $a(w)$ by $\alpha + \beta e^{-K(w-w_j)}$ in examining $Q_n(z)$ near z_j. We may write $Q_{n,r}$ in the form

$$Q_{n,r} = (n-r)! \int_0^{\frac{b_n}{b_r}} dy_{n-1} \int_{y_{n-1}}^{\frac{b_{n-1}}{b_r}} dy_{n-2} \cdots \int_{y_{r+1}}^{\frac{b_{r+1}}{b_r}} dy_r$$

and if $m = nz$, $r = nz$, then

$$\frac{b_{m+r}}{b_r} = \frac{a\left(\frac{z+z_j}{\zeta}\right)}{a(z_j/\zeta)} = \frac{a(w+w_j)}{a(w_j)}$$

Consequently $Q_{n,r}$ is the same as that $Q_{n(1-z_j),0}$ which is constructed from the parent distribution

$$\frac{a(w+w_j)}{a(w_j)} = 1 - \frac{(a_j')^2}{a_j a_j''} + \frac{(a_j')^2}{a_j a_j''} e^{\frac{a_j'' w}{a_j'}}$$

$$= \frac{a_j'^2}{a_j a_j''} \left\{ e^{\frac{a_j'' w}{a_j'}} - \left(1 - \frac{a_j a_j''}{a_j'^2}\right) \right\} = \frac{1}{(1-x)} \left[e^{-Kw} - x \right], \text{ say}. \quad (44)$$

where $\quad -a_j' > 0 \quad$ that is, in the previous notation,

$$Q_{n,r} = \left[\frac{a_j'^2}{a_j a_j''} \right]^{n(1-z_j)} \mathcal{B}_{n(1-z_j)} \left(1 - \frac{a_j a_j''}{a_j'^2}\right) \qquad (44a)$$

A study of the asymptotic form of $\mathcal{B}_n(x)$ (when $b(s) = e^{-\frac{K}{S}}$ will thus provide the required information about $Q_n(z)$ when z is the greatest root of $1-z-\lambda a(z/\zeta)=0$

By virtue of the "generating function" property, we have

$$\mathcal{B}_n(x) = n!\, e^{-\frac{Kn^2}{2S}} \cdot \frac{1}{2\pi i}\int_C \frac{dt}{t^{n+1}} \cdot \frac{M(txe^{\frac{Kn}{S}})}{M(t)}$$

the contour C enclosing $t=0$ but no other singularity of the integrand, and the behaviour of $\mathcal{B}_n(x)$ for large n can be investigated by the method of steepest descents (see, for example, Jeffreys (6)) ; the function $M(t)$ is, however, an uncommon one, and we proceed first of all to discuss its relevant properties.

Integral representation of $M(t)$.

The identity;

$$\sqrt{\frac{S}{2\pi K}}\int_{-\infty}^{\infty} e^{-\frac{y^2 S}{2K} + riy}\, dy = e^{-\frac{Kr^2}{S}}$$

makes it possible to condense $M(t)$ into the form

$$M(t) = \sqrt{\frac{S}{2\pi K}}\int_{-\infty}^{\infty} e^{-te^{iy} - \frac{y^2 S}{2K}}\, dy$$

or, more generally[x],

$$M(t) = \frac{1}{i}\sqrt{\frac{S}{2\pi K}}\int_L e^{-te^z + \frac{z^2 S}{2K}}\, dz \quad , \text{ all } t \ ,$$

where L is Bromwich's contour extending from $c - i\infty$ to $c' + i\infty$, c and c' having any finite real values.

The Asymptotic form of $M(t)$ for large t and S .

A resemblance is at once noted between the integral form of $M(t)$ and the contour integrals arising in the theory of Bessel functions (see Watson (7)), the term z^2 appearing in place of z in the exponent of the integrand in our case. It is to be expected therefore that the behaviour of $M(t)$ when both t and S are large will be revealed by a study of contours analogous to the type used by Debye for Bessel functions of large order and argument. Writing $\frac{K}{S} = \frac{\nu}{n}$, $t = n\tau$ where ν , τ are $O(1)$, we have

$$M = \frac{1}{i}\sqrt{\frac{n}{2\pi\nu}} \cdot \int_L e^{n\left(\frac{z^2}{2\nu} - \tau e^z\right)} dz = \frac{1}{i}\sqrt{\frac{n}{2\pi\nu}}\int_L e^{\frac{n}{\nu}\phi(z)} dz$$

Following Debye, we choose L to be a curve of steepest descent passing through one of the saddle points of the integrand, which are given by the turning values of

$$\phi(z) = \frac{z^2}{2} - \nu\tau e^z$$

These occur at the roots $z = z_s$ of

$$\phi'(z) = z - \nu\tau e^z = 0$$

or

$$z e^{-z} = \nu\tau \qquad (46)$$

---

[x] It seems natural to use the symbol z to denote the complex variable although the notation conflicts with a previous use of z . Confusion is not likely to arise in practice between the two notations.

Real τ. We first discuss the form of the steepest descent curves when τ is real. Inspection of fig. 13 shows that equation (46) has no real roots when $\nu\tau > e^{-1}$, and two real roots $z_s = z_1, z_2$ when $\nu\tau < e^{-1}$ such that $z_1 < 1 < z_2$. It will transpire that only the latter case need concern us. The steepest descent curves are

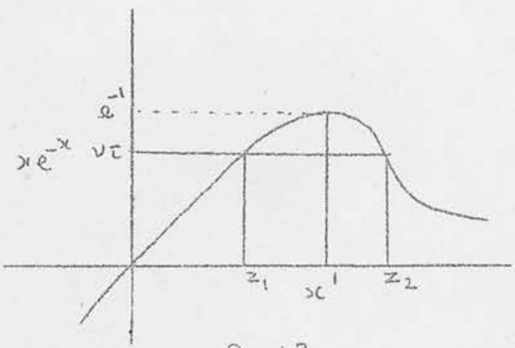

fig 13

$$\mathscr{J}\,\phi(z) = \mathscr{J}\,\phi(z_s) = 0$$

i.e. $\quad xy - \nu\tau e^{x}\sin y = 0$

or $\quad xe^{-x} = \nu\tau\dfrac{\sin y}{y}$ and $y = 0$

There is a branch through $(z_1, 0)$ crossing the real axis normally and oscillating with diminishing amplitude about $x = 0$, passing through the points $(0 \pm r\pi)$ (see fig. 14). Since $\phi''(z_s) = 1 - z_s$, the root z_1 is a minimum of $\phi(z)$ for real values of z and so is a maximum of $\phi(z)$ on the branch just described where $\phi(z)$ also takes real values. We choose this branch for our contour L.

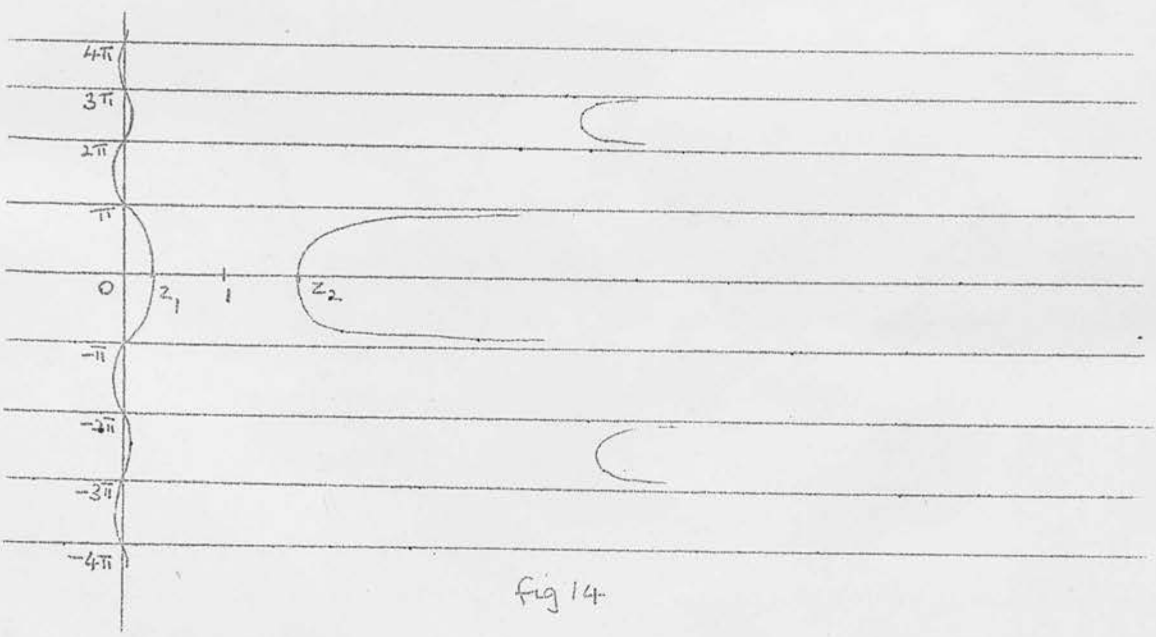

fig 14

The remainder of the curve consists of a semi-infinite loop between $y = -\pi$ and $y = \pi$ passing through $z_2 = 0$, and a series of semi-infinite loops lying in $y = \pm 2r\pi$ to $\pm(2r+1)\pi$. It is easy to show that there is no other saddle point besides z_1 lying on L, consequently $\phi(z)$ decreases steadily on either side of $(z_1, 0)$

Non-critical case when $1 - z_1 = O(1)$.

Near z_1, $\phi(z)$ takes the form

$$\phi(z) = \frac{z_1^2}{2} - z_1 + \frac{(z-z_1)^2}{2}(1-z_1) + O(z-z_1)^3$$

Provided $1 - z_1 = O(1)$, the curvature of L is negligible over the range in which $e^{\frac{n}{\nu}\phi(z)}$ is appreciable, and equation (45) becomes

$$M = e^{\frac{n}{\nu}\left(\frac{z_1^2}{2} - z_1\right)}\sqrt{\frac{n}{2\pi\nu}}\int_{-\infty}^{\infty} e^{-\frac{ny^2(1-z_1)}{2\nu}}\,dy\left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

i.e. $M = \dfrac{e^{\frac{n}{\nu}\left[\frac{z_1^2}{2}-z_1\right]}}{\sqrt{1-z_1}}\cdot\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$

Complex τ. Next we consider complex saddle points z_s. As before, $z_s-\nu\tau e^{z_s}=0$ where τ is now complex, and near z_s,

$$\phi(z) = \phi(z_s) + \frac{(1-z_s)}{2}(z-z_s)^2 - \frac{z_s(z-z_s)^3}{6} - \dots$$

In non-critical cases where $1-z_s$ is not small, $O(z-z_s)^3$ may be ignored. Let $z-z_s=\xi+i\eta=\rho e^{i\psi}$, $z_s=x_s+iy_s$. The steepest descent curve near z_s takes the form

$$\Im\left[\phi(z)-\phi(z_s)\right] = \xi\eta(1-x_s) - \tfrac{1}{2}(\xi^2-\eta^2)y_s = 0$$

to this order, which represents two perpendicular straight lines through z_s, and if $1-x_s=a\cos\alpha$, $y_s=a\sin\alpha$ the inclinations of the two branches satisfy $\sin(2\psi+\alpha)=0$ i.e. $\psi=\frac{\alpha}{2}$ or $\frac{\pi}{2}+\frac{\alpha}{2}$. On the branch $\psi=\frac{\pi}{2}+\frac{\alpha}{2}$ it is found that $\Re\left[\phi(z)-\phi(z_s)\right]=-\frac{1}{2}\rho^2 a$, so that the integrand passes through a maximum at z_s. This is, therefore, the correct branch to take, and $dz=e^{i\psi}d\rho=ie^{\frac{i\alpha}{2}}d\rho$ along it. The contribution to M from the saddle point z_s is consequently

$$\frac{1}{i}\sqrt{\frac{n}{2\pi\nu}}\, e^{\frac{n}{\nu}\phi(z_s)}\int_{-\infty}^{\infty} e^{-\frac{n\rho^2 a}{2}}\, i e^{\frac{i\alpha}{2}}d\rho = \frac{e^{\frac{n}{\nu}\phi(z_s)+\frac{i\alpha}{2}}}{\sqrt{a}} = \frac{e^{\frac{n}{\nu}\left[\frac{z_s^2}{2}-z_s\right]}}{\sqrt{1-z_s}}$$

as before, if we select that branch of $\sqrt{1-z_s}$ which is positive when z_s is real, the approximation holding to $O(\frac{1}{\sqrt{n}})$. This result is true for any saddle point, but we shall be interested in cases where $\arg\tau$ is small and $\nu|\tau|<e^{-1}$, when it may be shown that we have to integrate along a contour of Bromwich's type passing through z_s and containing no other saddle point. Then M is given entirely by

$$M = \frac{e^{\frac{n}{\nu}\left[\frac{z_s^2}{2}-z_s\right]}}{\sqrt{1-z_s}}\cdot\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$$

Behaviour of $M(t)$ in critical region.

When $1-z_s=O\left(\frac{1}{n^{\frac{1}{3}}}\right)$ it is no longer permissible to ignore $O(z-z_s)^3$ in the expansion of $\phi(z)$, thus,

$$\phi(z)-\phi(z_s) = \tfrac{1}{2}(1-z_s)(z-z_s)^2 - \tfrac{1}{6}z_s(z-z_s)^3 + O(z-z_s)^4$$

where $\phi'(z_s)=z_s-\nu\tau e^{z_s}=0$ and $\phi(z_s)=\frac{z_s^2}{2}-z_s$

It is convenient to introduce the new variables

$$Z = \frac{z_s}{(1-z_s)}(z-z_s) \quad , \quad \gamma e^{i\sigma} = \frac{(1-z_s)^3}{6z_s^2}$$

so that

$$\phi(z)-\phi(z_s) = \gamma e^{i\sigma}Z^2(3-Z) + O(z-z_s)^4$$

To the required order, therefore,

$$M = \frac{1}{i}\sqrt{\frac{n}{2\pi\nu}}\,\frac{(1-z_s)}{z_s}\,e^{\frac{n}{\nu}\left(\frac{z_s^2}{2}-z_s\right)}\int_{L'} e^{\frac{n}{\nu}\gamma e^{i\sigma}Z^2(3-Z)}\,dZ$$

This involves an integral of Airey's type, which we proceed to evaluate in the manner suggested by Watson (7) in his study of Nicholson's approximation to the Bessel function.

The steepest descent curve is such that $\gamma e^{i\sigma} z^2(3-z)$ is real and negative on it, taking the value 0 where the curve crosses the real axis at $z=0$, and decreasing indefinitely on either side. The curve is therefore given by

$$\sigma + 2\arg z + \arg(3-z) = (2r+1)\pi$$

$$\text{or} \qquad 2\arg z + \arg(z-3) = 2r\pi - \sigma$$

Since points on the exact steepest descent curve which are not in the vicinity of the saddle point make exponentially small contributions to the integral, it is considered legitimate to replace the exact curve by its approximate form for indefinitely large values of $|z|$. When $|z|$ is large, $\arg z = \frac{2r\pi-\sigma}{3}$ and the relevant branches are evidently those on which $\arg z \sim \frac{\pm 2\pi-\sigma}{3}$. The contour is next deformed into two straight lines through $z=1$ inclined at $\pm\frac{2\pi-\sigma}{3}$ to the real axis, that is, into $z = 1 + u\, e^{\frac{\pm 2\pi-\sigma}{3}}$ and the evaluation of the integral is carried out as indicated by Watson. Thus,

$$\int_{L'} e^{\frac{n}{\nu}\gamma e^{i\sigma} z^2(3-z)}\, dz = \int_1 - \int_2$$

where

$$\int_1 = -e^{\frac{2n}{\nu}\gamma e^{i\sigma} - \frac{(\pi+\sigma)i}{3}} \int_0^\infty e^{-\frac{n\gamma}{\nu}u^3 - 3\frac{n}{\nu}\gamma u e^{\frac{(2\sigma-\pi)i}{3}}}\, du$$

$$= -e^{\frac{2n}{\nu}\gamma e^{i\sigma} - \frac{(\pi+\sigma)i}{3}} \sum_0^\infty \frac{(-)^r}{r!}\left(\frac{3n\gamma}{\nu}\right)^r e^{\frac{(2\sigma-\pi)ri}{3}} \int_0^\infty e^{-\frac{n\gamma u^3}{\nu}} u^r\, du$$

$$= -\tfrac{1}{3} e^{\frac{2n}{\nu}\gamma e^{i\sigma}} \sum_0^\infty \frac{(-)^r}{r!} 3^r \left(\frac{n\gamma}{\nu}e^{i\sigma}\right)^{\frac{2r-1}{3}} . \, e^{-(r+1)\frac{i\pi}{3}}\, \Gamma\left(\tfrac{r+1}{3}\right)$$

and $\int_2$ is the same with $-\pi$ written for π.

Using $\Gamma\left(\frac{r+1}{3}\right)\Gamma\left(\frac{2-r}{3}\right) = \frac{\pi}{\sin\frac{(r+1)\pi}{3}}$, $\int_1 - \int_2$ reduces to

$$\int_1 - \int_2 = \frac{2i\pi}{3} e^{\frac{2n}{\nu}\gamma e^{i\sigma}} \sum_{r=0}^\infty \frac{(-)^r 3^r}{r!\,\Gamma\left(\frac{2-r}{3}\right)}\left(\frac{n\gamma e^{i\sigma}}{\nu}\right)^{\frac{2r-1}{3}}$$

$$= \frac{2i\pi}{3} e^{\frac{2n}{\nu}\gamma e^{i\sigma}} \left\{ I_{-\frac{1}{3}}\left(\frac{2n\gamma}{\nu}e^{i\sigma}\right) - I_{\frac{1}{3}}\left(\frac{2n\gamma e^{i\sigma}}{\nu}\right)\right\}$$

as may be seen on writing out the series and collecting terms. Hence in the critical region where $1-z_s = O(\frac{1}{n^{1/3}})$, M takes the form,

$$M = \frac{e^{\frac{n}{\nu}\left(\frac{z_s^3}{2}-z_s\right)}}{\sqrt{1-z_s}} \sqrt{\frac{2w}{\pi}}\, e^{w}\, K_{\frac{1}{3}}(w) \cdot \left[1 + O\left(\frac{1}{n^{1/3}}\right)\right]$$

where $w = \frac{n(1-z_s)^3}{3\nu z_s^2} = O(1)$, and $K_\nu(z)$ is the second solution of Bessel's equation with imaginary argument, defined by $K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\left\{I_{-\nu}(z) - I_\nu(z)\right\}$. Note that

$$\sqrt{\frac{2w}{\pi}}\, e^{w} K_{\frac{1}{3}}(w) = 1 - \frac{5}{72w} + O\left(\frac{1}{w^2}\right) ; \quad \sqrt{\frac{2w}{\pi}}\, e^{w} K_{\frac{1}{3}}(w) = \frac{\Gamma(-\frac{1}{3})}{\sqrt{\pi}}\left(\frac{w}{2}\right)^{\frac{1}{6}} + O\left(w^{\frac{5}{6}}\right)$$

for large and small ϖ respectively.

The function M does not become infinite at $z_s = 1$, instead, when $z_s = 1$,

$$M = e^{-\frac{n}{\nu}} \left(\frac{n}{6\nu}\right)^{\frac{1}{6}} \cdot \frac{\Gamma(-\frac{1}{3})}{\sqrt{\pi}} \left[1 + O\left(\frac{1}{n^{\frac{1}{3}}}\right)\right]$$

<u>Case of $\nu|\tau| > e^{-1}$.</u>

This case will not be considered in detail, but it may be mentioned that the steepest descents curves are of quite a different character. When τ is real, the relevant saddle points are complex conjugates z_1, z_1^* and in the non-critical region it is found that

$$M = \left\{ \frac{e^{\frac{n}{\nu}\left(\frac{z_1^2}{2} - z_1\right)}}{\sqrt{1 - z_1}} + \frac{e^{\frac{n}{\nu}\left(\frac{z_1^{*2}}{2} - z_1^*\right)}}{\sqrt{1 - z_1^*}} \right\} \left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

a similar result being obtained for complex τ .

<u>Application to $\mathcal{B}_n$ and $\mathcal{B}_n(x)$.</u> (when $b(s) = e^{-\frac{k}{s}}$)

The approximate expressions for M just obtained enable us to study the behaviour of $\mathcal{B}_n$ and $\mathcal{B}_n(x)$ for all $\nu \leq 1$. Although strictly speaking $\mathcal{B}_n$ is the special case of $\mathcal{B}_n(x)$ when $x = 0$, and so is included in the latter, it was thought best to present the discussion first in terms of $\mathcal{B}_n$ so as not to obscure the main argument with irrelevant detail.

1. <u>Asymptotic form of $\mathcal{B}_n$.</u>

$$\mathcal{B}_n = n!\, e^{-\frac{kn^2}{2S}} \cdot \frac{1}{2\pi i} \int_C \frac{dt}{t^{n+1}} \cdot \frac{1}{M(t)} = \frac{n!}{n^n} e^{-\frac{n\nu}{2}}\, I$$

where

$$I = \frac{1}{2\pi i} \int_{C'} \frac{d\tau}{\tau^{n+1} M}$$

the contour C' enclosing the origin, but no other singularity of $\frac{1}{M}$.

(i) Non-critical values of $\nu < 1$ such that $1 - \nu = O(1)$ are first considered. The asymptotic form of M suggests the transformation $\nu\tau = z e^{-z}$. When $\nu|\tau| < e^{-1}$ the contour C' transforms into a curve (fig. 15) consisting of a closed loop Γ through z_1 surrounding $z = 0$, and an infinite branch through z_2 , z_1 and z_2 being the real roots of $\nu\tau_0 = z e^{-z}$ where τ_0 is the value assumed by τ on the positive real axis.

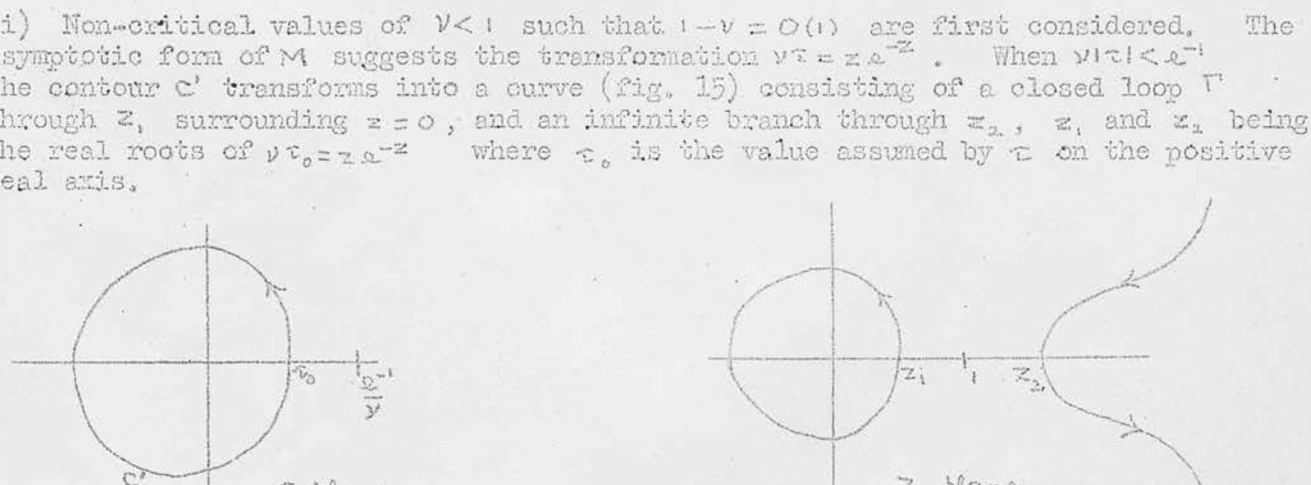

fig 15

From the form of M for large n we are led to take as the contour for z the closed loop Γ which is described once as τ describes C' once in the same sense. Then, since $\frac{d\tau}{\tau} = dz\left(\frac{1}{z} - 1\right)$, the integral becomes

$$I = \frac{\nu^n}{2\pi i} \int_\Gamma dz \left(\frac{1}{z} - 1\right) \sqrt{1 - z} \cdot z^{-n} e^{n\left[z\left(1 + \frac{1}{\nu}\right) - \frac{z^2}{2\nu}\right]} \cdot \left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

The steepest descents technique is once more applied to this integral. The saddle points are at the turning values of $e^{\psi(z)} = z^{-1} e^{z(1+\frac{1}{\nu}) - \frac{z^2}{2\nu}}$, that is, when

$$\psi'(z) = -\frac{1}{z} + \left(1+\frac{1}{\nu}\right) - \frac{z}{\nu} = 0 \qquad or \qquad (z-1)(z-\nu) = 0$$

The root $z=\nu$ makes the integrand a minimum along the real axis, and the contour is therefore chosen to pass through it, and to approximate in its vicinity to a curve of steepest descent. Near $z=\nu$, we have

$$\psi(z) = \psi(\nu) - \frac{(1-\nu)}{2\nu^2}(z-\nu)^2 + O(z-\nu)^3$$

so when $1-\nu = O(1)$ we ignore $O(z-\nu)^3$ and take Γ to be a straight line crossing the real axis normally at $z=\nu$; then

$$I = \frac{1}{2\pi}\frac{(1-\nu)^{3/2}}{\nu} e^{n(1+\frac{\nu}{2})}\int_{-\infty}^{\infty} e^{-\frac{n(1-\nu)y^2}{2\nu^2}}\,dy \cdot \left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$= \frac{(1-\nu)}{\sqrt{2\pi n}}\, e^{n(1+\frac{\nu}{2})} \cdot \left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$$

and using Stirling's approximation to $n!$ we arrive at the result

$$\mathcal{B}_n = 1-\nu + O\left(\frac{1}{\sqrt{n}}\right) = 1 - \frac{\zeta_0}{\zeta} + O\left(\frac{1}{\sqrt{n}}\right)$$

in our previous notation.

(ii) The critical region where $1-\nu$ is small is now discussed. Substituting in I the appropriate formula for M, we obtain

$$I = \frac{\nu^n}{2\pi i}\int_\Gamma dz\left(\frac{1}{z}-1\right)\cdot\sqrt{1-z}\cdot z^{-n} e^{n\left[z(1+\frac{1}{\nu}) - \frac{z^2}{2\nu}\right]}\sqrt{\frac{\pi}{2\varpi}}\cdot\frac{e^{-\varpi}}{K_{\frac{1}{3}}(\varpi)}$$

$$= \frac{\nu^n}{2\pi i}\sqrt{\frac{3\nu\pi}{2n}}\int_\Gamma e^{n\psi(z)}\,dz\cdot\frac{e^{-\varpi}}{K_{\frac{1}{3}}(\varpi)}$$

where $\varpi = \frac{n(1-z)^3}{3\nu z^2} = O(1)$, and $e^{\psi(z)} = z^{-1} e^{z(1+\frac{1}{\nu}) - \frac{z^2}{2\nu}}$

The saddle points of the integrand are not now exactly at the roots of $\psi'(z) = 0$ when $1-z = O(\frac{1}{n^{1/3}})$, but are displaced by an amount $O(\frac{1}{n^{1/3}})$ under the influence of the factor $e^{-\varpi}/K_{\frac{1}{3}}(\varpi)$. For relatively large $|1-z|$, however, the steepest descent curves are approximately the same as would be obtained if the factor in ϖ were absent. Let us therefore consider the latter curves in detail. The saddle points of $e^{\psi(z)}$ are still given by $\psi'(z) = \frac{(1-z)(z-\nu)}{\nu z^2} = 0$ and the branch through $z=\nu$ is chosen as before. Expanding $\psi(z)$ as far as $(z-\nu)^3$ we find

$$\psi(z) - \psi(\nu) = \frac{(1-\nu)(z-\nu)^2}{2\nu^2} - \frac{(z-\nu)^3}{3\nu^3} = \frac{(z-\nu)^2}{6\nu^3}(3-\nu-2z)$$

and the steepest descent curve through $z=\nu$ is

$$2\arg(z-\nu) + \arg(3-\nu-2z) = (2r+1)\pi$$

which for large z takes the form $3\arg z \sim 2r\pi$, the appropriate branches being $\arg z = \pm\frac{2\pi}{3}$. Employing Watson's device once more, we adopt for the contour Γ the two straight lines $z = 1 + u e^{\pm\frac{2i\pi}{3}}$, $u \geqslant 0$, upon which

$$\psi(z) - \psi(\nu) = \frac{(1-\nu)^3 + 3(1-\nu)u^2 e^{\pm\frac{i\pi}{3}} - 2u^3}{6\nu^3} + O\left(\frac{1}{n^{\frac{1}{3}}}\right)$$

Also $\nabla\nabla = \frac{n(1-z)^3}{3\nu z^2} = \frac{nu^3}{3\nu} + O\left(\frac{1}{n^{\frac{1}{3}}}\right)$, and since $e^{n\psi(\nu)} = \nu^{-n} e^{n(1+\frac{\nu}{2})}$ we finally obtain, using Stirling's approximation to $n!$,

$$\mathcal{B}_n = \frac{n!}{n^n} e^{-\frac{n\nu}{2}} I = \sqrt{3\nu}\, e^{\frac{n(1-\nu)^3}{6\nu^3}} \mathfrak{I} \int_0^\infty e^{\frac{n(1-\nu)u^2 e^{\frac{i\pi}{3}}}{2\nu^3} - \frac{nu^3}{3\nu}(1+\frac{1}{\nu^2}) + \frac{2i\pi}{3}} \frac{du}{K_{\frac{1}{3}}\left(\frac{nu^3}{3\nu}\right)} \cdot \left[1 + O\left(\frac{1}{n^{\frac{1}{3}}}\right)\right]$$

By retaining ν in the denominators in spite of the fact that $\nu = 1 + O(\frac{1}{n^{\frac{1}{3}}})$, a transitional approximate form for $\mathcal{B}_n$ has been arrived at which holds at worst to $O(\frac{1}{n^{\frac{1}{3}}})$ for all values of ν from $1-\nu = 0$ to $O(1)$, and tends to the simpler form in the non-critical region (although the transition is not obvious from the form of the integral as it stands). If we confine ourselves to $1-\nu = O(\frac{1}{n^{\frac{1}{3}}})$, however, the formula simplifies to

$$\mathcal{B}_n = \sqrt{3}\, e^{\frac{n(1-\nu)^3}{6}} \mathfrak{I} \int_0^\infty e^{\frac{n(1-\nu)u^2 e^{\frac{i\pi}{3}}}{2} - \frac{2nu^3}{3} + \frac{2i\pi}{3}} \frac{du}{K_{\frac{1}{3}}\left(\frac{nu^3}{3}\right)} \quad \left[1 + O\left(\frac{1}{n^{\frac{1}{3}}}\right)\right]$$

$$= \sqrt{3}\, e^{\frac{n(1-\nu)^3}{6}} \int_0^\infty e^{\frac{n(1-\nu)u^2}{4} - \frac{2nu^3}{3}} \frac{\sin\left[\frac{\sqrt{3}}{4}n(1-\nu)u^2 + \frac{2\pi}{3}\right] du}{K_{\frac{1}{3}}\left(\frac{nu^3}{3}\right)} \left[1 + O\left(\frac{1}{n^{\frac{1}{3}}}\right)\right]$$

The same formula holds over the range $1-\nu = O(\frac{1}{n^{\frac{1}{4}}})$, but to the lower order $O(\frac{1}{n^{\frac{1}{4}}})$

2. Asymptotic form of $\mathcal{B}_n(x)$.

The more general expression for $\mathcal{B}_n(x)$ was

$$\mathcal{B}_n(x) = n!\, e^{-\frac{\kappa n^2}{2s}} \cdot \frac{1}{2\pi i} \int_C \frac{dt}{t^{n+1}} \cdot \frac{M(t x e^{\frac{\kappa n}{s}})}{M(t)} = \frac{n!}{n^n} e^{-\frac{n\nu}{2}} I(x)$$

where
$$I(x) = \frac{1}{2\pi i} \int_{C'} \frac{d\tau}{\tau^{n+1}} \frac{M(n\tau x e^{\nu})}{M(n\tau)}$$

(i) As before, the non-critical region is first discussed. For it to be possible to substitute the approximate form of M in both numerator and denominator, we require $\nu|\tau| < e^{-1}$ and $\nu|\tau| x e^{\nu} < e^{-1}$. The implications of these restrictions are discussed below; accepting them, the formulae are

$$M(t) = \frac{e^{\frac{n}{\nu}(\frac{z^2}{2} - z)}}{\sqrt{1-z}} \left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right] \quad \text{and} \quad M(t x e^{\nu}) = \frac{e^{\frac{n}{\nu}\left[\frac{Z^2}{2} - Z\right]}}{\sqrt{1-Z}} \left[1 + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

where $\nu\tau = z e^{-z}$, $\nu\tau x e^{\nu} = Z e^{-Z}$

i.e. $Z e^{-Z} = z e^{-z} \cdot x e^{\nu}$

Transforming to z as before, we obtain

$$I(x) = \frac{\nu^n}{2\pi i}\int_\Gamma dz\left(\frac{1}{z}-1\right)\sqrt{\frac{1-z}{1-Z_1}}\; e^{n\Psi}\quad\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$$

where $e^{\Psi} = z^{-1}e^{z(1+\frac{1}{\nu})-\frac{z^2}{2\nu}-\frac{Z_1}{\nu}+\frac{Z_1^2}{2\nu}}$. Since $dZ_1\left(\frac{1}{Z_1}-1\right) = dz\left(\frac{1}{z}-1\right)$

the saddle points satisfy

$$\Psi'(z) = -\frac{1}{z}+\left(1+\frac{1}{\nu}\right)-\frac{z}{\nu}-\frac{(1-Z_1)}{\nu}\frac{dZ_1}{dz} = \frac{(1-z)(z-Z_1-\nu)}{z\nu}$$

and the appropriate saddle point is that given by $z_1-Z_1=\nu$ which makes $Z_1 = xz_1$ and so $z_1 = \frac{\nu}{1-x}$, $Z = \frac{x\nu}{1-x}$.

Ultimately we are going to substitute for x , κ , n the expressions implied in equation (44), and in particular x is to be replaced by $1-\frac{a_j a_j''}{a_j'^2}$ which is always less than unity for the kind of $a(w)$ curves we are considering. So wherever $z_1<1$ there must also be $Z_1<1$, and in the vicinity of z_1 the approximate expressions for M in numerator and denominator of the integrand are simultaneously valid. The restrictions on $|\tau|$ imposed above are thus interpreted and justified.

Proceeding, we find that

$$\Psi''(z) = \frac{1}{z^2}-\frac{(z-Z_1)^2}{\nu z^2(1-Z_1)}\quad ,\quad \Psi''(z_1) = \frac{(1-z_1)}{z_1^2(1-xz_1)}$$

and

$$e^{\Psi(z_1)} = z_1^{-1}e^{(1+\frac{\nu}{2})}$$

In the non-critical case, therefore,

$$I(x) = \frac{\nu^n}{2\pi}z_1^{-n}e^{n(1+\frac{\nu}{2})}\frac{(1-z_1)^{3/2}}{z_1(1-xz_1)^{\frac{1}{2}}}\int_{-\infty}^{\infty}e^{-\frac{n(1-z_1)y^2}{2z_1^2(1-xz_1)}}dy\quad\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$= \frac{(1-x)^n}{\sqrt{2\pi n}}(1-z_1)\,e^{n(1+\frac{\nu}{2})} + O\left(\frac{1}{\sqrt{n}}\right)$$

and so
$$\mathcal{B}_n(x) = (1-x)^n(1-z_1) + O\left(\frac{1}{\sqrt{n}}\right)$$

(ii) Turning next to the region where $1-z_1$ is small, we substitute the appropriate approximation for $M(t)$ in the denominator of the integrand while that for $M(txe^{\frac{\pi i}{3}})$ remains unchanged, and so obtain

$$I(x) = \frac{\nu^n}{2\pi i}\sqrt{\frac{3\nu\pi}{2n}}\int_\Gamma\frac{e^{n\Psi}dz}{\sqrt{1-Z_1}}\cdot\frac{e^{-W}}{K_{\frac{1}{3}}(W)}$$

where $W = \frac{n(1-z)^3}{3\nu z^2}$ and the other symbols are as defined above. As with $\mathcal{B}_n$, a transitional form can be arrived at by expanding $\Psi(z)$ near $z_1 = \frac{\nu}{1-x}$ as far as the cubic term, and integrating along the contour $z = 1+u\,e^{\pm\frac{2\pi i}{3}}$, $u\geq 0$. The third derivative of $\Psi(z)$ is found to be

$$\Psi'''(z) = -\frac{2}{z^3}-\frac{3(z-Z_1)Z_1}{\nu z^3(1-Z_1)^2}+\frac{2(z-Z_1)^3Z_1}{\nu z^3(1-Z_1)^3}$$

However, as we shall be concerned only with values of z_1 which are $1+O(\frac{1}{n^{1/3}})$, it is sufficient to take the simpler approximation, accurate to $O(\frac{1}{n^{1/3}})$ obtained on replacing z_1 by 1 except where it occurs as nz_1. Then, ignoring $(1-z_1)^4$,

$$\Psi(z)-\Psi(z_1) = \frac{(1-z_1)^3 + 3(1-z_1)u^2 e^{\pm\frac{i\pi}{3}} - 2u^3}{6(1-x)}, \quad w = \frac{nu^3}{3(1-x)}$$

and the final formula reduces to

$$\mathcal{B}_n(x) = (1-x)^n \sqrt{3}\, e^{\frac{n(1-z_1)^3}{6(1-x)}} \int\int_0^\infty e^{\frac{n(1-z_1)u^2 e^{\frac{i\pi}{3}}}{2(1-x)} - \frac{2nu^3}{3(1-x)} + \frac{2i\pi}{3}} \frac{du}{K_{1/3}\left(\frac{nu^3}{3(1-x)}\right)} \qquad (48)$$

to the desired order.

<u>The asymptotic form of B_n for general $b(s)$.</u>

We are now in a position to obtain a complete picture of the asymptotic behaviour of B_n for general forms of $b(s)$. Considering first the non-critical region, we revert to equation (44) and set $x = 1-\frac{a_j a_j''}{a_j'^2}$, $k = -\frac{a_j''}{a_j} > 0$ and write $n(1-z_j)$ for n, so that $\nu = -\frac{(1-z_j)a_j''}{5a_j'}$ and* $z_1 = \frac{\nu}{1-x} = -\frac{(1-z_j)a_j'}{5a_j}$. It then follows from (44a) that

$$Q_{n,r} = 1-z_1 + O(\tfrac{1}{\sqrt{n}}) = 1+\frac{(z-z_j)a_j'}{5a_j} + O(\tfrac{1}{\sqrt{n}})$$

which independently reaffirms the result previously obtained for non-critical values of ζ. Taking $z_j=0$, we deduce as before that $B_n = Q_{n,0} = 1-\zeta_0/5 + O\frac{1}{\sqrt{n}}$ where $a_0' = -\zeta_0$ is the gradient of $a(w)$ at $w=0$, and ζ is greater than, but not in the vicinity of, the critical value ζ_c.

When $\zeta-\zeta_c = O(\frac{1}{\sqrt{n}})$, a distinction is drawn between the cases where $\zeta_c=\zeta_0$ and $\zeta_c > \zeta_0$. (i) In the first case, the expressions for x and z_1 are $x = 1-\frac{a_0''}{\zeta_0^2}$ and $z_1 = \frac{\zeta_0}{5}$, while n remains n. From (44a) and (48) we then obtain

$$B_n = \sqrt{3}\, e^{\frac{n}{6}\frac{\zeta_0^2}{a_0''}\left(1-\frac{\zeta_0}{5}\right)^3} \int_0^\infty \frac{e^{\frac{n\zeta_0^2}{4a_0''}\left(1-\frac{\zeta_0}{5}\right)u^2 - \frac{2n\zeta_0^2}{3a_0''}u^3}\, du\, \sin\left[\frac{\sqrt{3}}{4}\frac{\zeta_0^2}{a_0''}\left(1-\frac{\zeta_0}{5}\right)u^2 + \frac{2\pi}{3}\right]}{K_{1/3}\left(\frac{n\zeta_0^2 u^3}{3a_0''}\right)} \left[1+O\left(\tfrac{1}{n^{1/3}}\right)\right]$$

At $\zeta=\zeta_0$, B_n is seen to be $O(\frac{1}{n^{1/3}})$ and so does not exceed that order when $\zeta<\zeta_0$

(ii) When $\zeta_c > \zeta_0$ and $\zeta_c-\zeta_0 = O(1)$, consider for simplicity the case when $\zeta_0 = 0$ We refer to equation (39) of the previous section and insert the appropriate formula for $Q_n(z_c+y)$. It was shown that $1-z_1 = 1+\frac{(1-z_j)a_j'}{5a_j}$ and for values of ζ near ζ_c such that $\zeta-\zeta_c = O(\frac{1}{\sqrt{n}})$ this becomes $1-z_1 = \frac{a_c''}{\zeta_c}y\left[1+O(\frac{1}{n^{1/3}})\right]$ where $y = z-z_c = O(\frac{1}{n^{1/3}})$ also $1-x = \frac{(1-z_c)}{\zeta_c^2}a_c''\left[1+O(\frac{1}{n^{1/3}})\right]$ and n has to be replaced by $n(1-z_j) = n(1-z_c)\left[1+O(\frac{1}{n^{1/3}})\right]$ (These relations follow on making use of the facts that $1-z_c-a_c=0$ and $1+\frac{a_c'}{\zeta_c}=0$).

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* I apologise for the clashing z notations.

With the above substitutions, equations (49) and (48) lead to

$$Q_n(z_\tau + y) = \sqrt{3}\, e^{\frac{n a_\tau''^2}{6\zeta_\tau^4} y^3}\, \mathcal{I} \int_0^\infty \frac{e^{\frac{n y u^2 e^{i\pi/3}}{2} - \frac{2n\zeta_\tau^2}{3a_\tau''} u^3 + \frac{2i\pi}{3}}\, du}{K_{\frac{1}{3}}\left(\frac{n\zeta_\tau^2 u^3}{3a_\tau''}\right)} \left[1 + O\left(\frac{1}{n^{\frac{1}{4}}}\right)\right]$$

Since $Q_n(z)$ is at most $O\left(\frac{1}{n^{\frac{1}{3}}}\right)$ for $z < z_\tau$ we insert this formula in (39) and arrive at the final result that

$$B_n = 1 - \int_0^\infty \frac{e^{-\frac{2n\zeta_\tau^2 u^3}{3a_\tau''}}\sqrt{3}\, du}{K_{\frac{1}{3}}\left(\frac{n\zeta_\tau^2 u^3}{3a_\tau''}\right)} \int_0^\infty e^{\frac{n a_\tau''^2}{6\zeta_\tau^4} y^3 + \frac{n y u^2}{4} - \frac{n}{2z_\tau(1-z_\tau)}\left[\frac{a_\tau'' y^2}{2\zeta_\tau^2} + \frac{z_\tau}{\zeta_\tau}(\zeta - \zeta_\tau)\right]^2} \sin\left[\frac{\sqrt{3}}{2} n y u^2 + \frac{2\pi}{3}\right] dy \cdot \sqrt{\frac{n}{2\pi z_\tau(1-z_\tau)}}$$

$$\cdot \left[1 + O\left(\frac{1}{n^{\frac{1}{4}}}\right)\right]$$

This rather complicated expression is the distribution of ζ required. Its moment-generating function is probably simpler, however, as a Fourier transform with respect to ζ converts the y integration into something which could be put into Airey's form and so expressed in terms of $K_{\frac{1}{3}}$, in which case the expression would reduce to a single integral.

When $\zeta_0 \neq 0$, the whole expression has to be multiplied by $1 - \zeta_0/\zeta$ as mentioned on p. 25. The case where $\zeta_\tau - \zeta_0$ is small presumably leads to a still more involved formula for B_n, but will occur relatively infrequently in practice, and is not worth considering.

APPENDIX A.

Physical interpretation of $Q_{n,m}$.

By definition

$$Q_{n,m} = \frac{(n-m)!\,(n,m)}{b_m^{n-m}} = \frac{(n-m)!}{b_m^{n-m}} \int_0^{b_n} dx_{n-1} \int_{y_{n-1}}^{b_{n-1}} dx_{n-2} \cdots \int_{y_{m+1}}^{b_{m+1}} dx_m$$

which by a simple transformation can be written

$$Q_{n,m} = (n-m)! \int_0^{\frac{b_n}{b_m}} dy_{n-1} \int_{y_{n-1}}^{\frac{b_n}{b_m}} dy_{n-2} \cdots \int_{y_{m+1}}^{\frac{b_{m+1}}{b_m}} dy_m$$

Now $b_m = b(\frac{S}{m})$ is the fraction of the population of threads which will break under a load S/m, and b_r/b_m is the fraction of the above fraction which will break under load $\frac{S}{r}$, where $r \leq m$. It is therefore possible to give the following somehwat artificial interpretation of $Q_{n,m}$.

Given S,
1. Select from the population all threads not exceeding $\frac{S}{m}$ in strength.
2. Take random sample of n of these.
3. Replace m threads of the sample at random by m unbreakable threads.

Then $Q_{n,m}$ is the chance that when a load S is applied to the resulting bundle all the $n-m$ breakable threads give way.

The population obtained by rejecting all threads stronger than $\frac{S}{m}$ has the distribution

$$\begin{aligned} \beta(S') &= \frac{b(S')}{b(\frac{S}{m})} && \text{when } S' \leq \frac{S}{m} \\ &= 1 && \text{when } S' > \frac{S}{m} \end{aligned}$$

Suppose that under a load S' the bundle comes to equilibrium when t threads survive, m of these being of course the unbreakable threads. When n is very large, t is determined as the largest root of

$$\frac{n-t}{n-m} = \beta\left(\frac{S'}{t}\right)$$

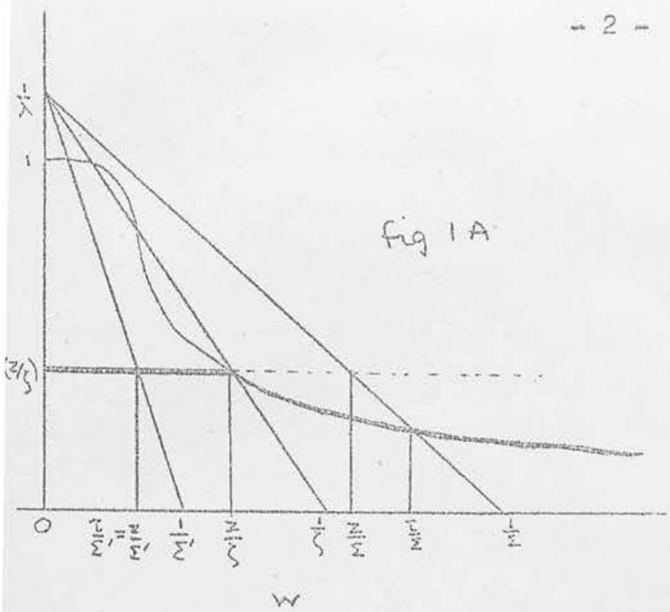
Put $t = n\tau$, $S' = n\Sigma$, $\alpha(w) = \beta(\frac{1}{w})$

and, as before,

$m = nz$, $s = n\zeta$, $a(w) = b(\frac{1}{w})$

The equilibrium condition becomes $\frac{1-\tau}{1-z} = \alpha\left(\frac{\tau}{\Sigma}\right)$, that is

$$\left.\begin{aligned} \frac{1-\tau}{1-z} &= \frac{a(\tau/\Sigma)}{a(z/\zeta)} && \text{when } \frac{\tau}{\Sigma} \geq \frac{z}{\zeta} \\ &= 1 && \text{when } \frac{\tau}{\Sigma} < \frac{z}{\zeta} \end{aligned}\right\} \quad (A1)$$

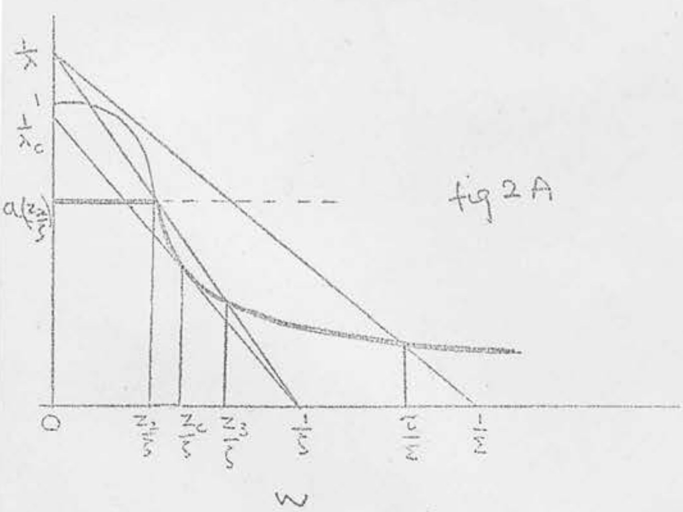


Consider now fig. 1A. The line joining $[\frac{z}{\zeta}, a(z/\zeta)]$ and $[\frac{1}{\zeta}, 0]$ cuts the vertical axis in $\frac{1}{\lambda}$ where $1 - z - \lambda a(z/\zeta) = 0$. Define the curve $c(w)$ to be

$$\begin{aligned} c(w) &= a(z/\zeta) & w &< \tfrac{z}{\zeta} \\ &= a(w) & w &\geq \tfrac{z}{\zeta} \end{aligned}$$

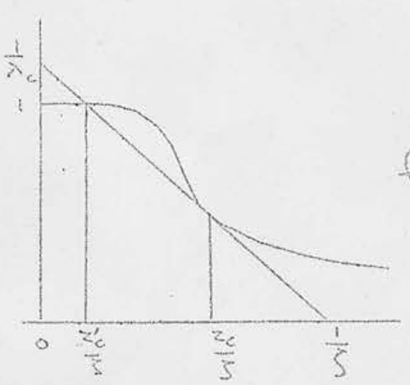
Then it is evident from the geometry of the figure that the line joining $[0, \frac{1}{\lambda}]$ and $[\frac{1}{\Sigma}, 0]$ intercepts $c(w)$ in points $\frac{\tau}{\Sigma}$ such that τ and Σ satisfy the condition (A1).

As the figure is drawn, z is the maximum root, z_3, of $1 - z - \lambda a(z/\zeta) = 0$ When $\frac{1}{\Sigma} > \frac{1}{\zeta}$, the number of survivors is $n\tau$ which is greater than nz, showing there still to be breakable threads which have not given way. The chance of all breakable threads giving way under load $n\Sigma > n\zeta$ is thus zero, to this order. When $\frac{1}{\Sigma} < \frac{1}{\zeta}$, we have $\tau = z$ for all $0 \leq \frac{1}{\Sigma} < \frac{1}{\zeta}$ (e.g. $\Sigma = \Sigma'$ as shown in fig. 1A) and thus the chance of all breakable threads giving way is now unity. When $\Sigma = \zeta$, however, the chance is indeterminate and so no information is given about $Q_n(z)$ when $z = z_3$.



But the case of z_2 is different (fig. 2A) for as Σ approaches ζ, τ becomes $z_3 > z_2$ and so the chance of all breakable threads giving way under load $n\zeta$ is zero to this order, i.e. $Q_n(z_2) \sim 0$. Similarly $Q_n(z_1) \sim 0$.

When all the roots are distinct, therefore, we conclude that $Q_n(z_1) \sim 0$ $Q_n(z_2) \sim 0$ and $Q_n(z_3)$ is indeterminate. When there is only one real root, z_3 , then $Q_n(z_3)$ is indeterminate.



Keeping ζ fixed and varying λ we are thus able to deduce that $Q_n(z) \sim 0$ when $z_c' < z < z_c$ where z_c' and z_c are the single and double roots at the tangent position corresponding to λ_c. (fig 3A) Outside this range of z we get no information about $Q_n(z)$ by the present argument.

APPENDIX B.

Distribution of Bundle Strength when the load-extension ratios are not equal.

Suppose the load-extension ratio is distributed, independently of breaking strength and extension, according to the law $\phi(\alpha)\,d\alpha$. Let the ratios for the n individual threads of a bundle be $\alpha_n, \alpha_{n-1}, \cdots, \alpha_2, \alpha_1$ where the threads break in the order $n, n-1, \ldots$. Under a load S, suppose that when r threads remain, the extension is ϵ, so that the load on the m^{th} thread is $s_m = \alpha_m \epsilon$. Then $S = \sum_{m=1}^{r} s_m = \epsilon \sum_{1}^{r} \alpha_m$. For the whole bundle to give way under load S, the breaking extensions must satisfy the relations

$$0 < \epsilon_n < \frac{S}{\alpha_1 + \alpha_2 + \cdots + \alpha_n}$$

$$\epsilon_n < \epsilon_{n-1} < \frac{S}{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}}$$

$$- - - - - -$$

$$\epsilon_2 < \epsilon_1 < \frac{S}{\alpha_1}$$

and thus for given $\alpha_1, \alpha_2, \cdots \alpha_n$ the chance of the bundle giving way is

$$B(\alpha_1, \alpha_2, \cdots, \alpha_n) = \int_0^{\frac{S}{\alpha_1 + \cdots + \alpha_n}} \psi(\epsilon_n)\, d\epsilon_n \int_{\epsilon_n}^{\frac{S}{\alpha_1 + \cdots + \alpha_{n-1}}} \psi(\epsilon_{n-1})\, d\epsilon_{n-1} \cdots \int_{\epsilon_2}^{\frac{S}{\alpha_1}} \psi(\epsilon_1)\, d\epsilon_1$$

$$= \int_0^{E\left(\frac{S}{\alpha_1 + \cdots + \alpha_n}\right)} dx_{n-1} \int_{x_{n-1}}^{E\left(\frac{S}{\alpha_1 + \cdots + \alpha_{n-1}}\right)} dx_{n-2} \cdots \int_{x_1}^{E\left(\frac{S}{\alpha_1}\right)} dx_0$$

where $E(e) = \int_0^e \psi(\epsilon)\, d\epsilon$ is the chance of single-thread breaking extension being less than e. Averaging over all α, the distribution of breaking strength for any bundle is found to be

$$B_n = \int_0^\infty \cdots \int_0^\infty \phi(\alpha_1)\phi(\alpha_2)\cdots\phi(\alpha_n)\, d\alpha_1 \cdots d\alpha_n \int_0^{E\left(\frac{S}{\alpha_1 + \cdots + \alpha_n}\right)} dx_{n-1} \int_{x_{n-1}}^{E\left(\frac{S}{\alpha_1 + \cdots + \alpha_{n-1}}\right)} dx_{n-2} \cdots \int_{x_1}^{E\left(\frac{S}{\alpha_1}\right)} dx_0 .$$

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