MATRIX RANGES OF OPERATORS ON HILBERT SPACES

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TABLE OF CONTENTS

SUMMARY

Matrix ranges of operators on Hilbert spaces were first studied by W.B.Arveson in the paper "Subalgebras of C*-algebras,II" (Acta Mathematica 128). Arveson discussed a particular range, called the algebraic matrix range in this thesis, and showed that the sequence of these ranges forms a complete unitary invariant for irreducible compact operators. At the same time S.K.Parrott considered another range , called the spatial matrix range here. Parrott proved that the sequence of these ranges form a complete unitary invariant for all compact operators with trivial reducing null-space. His work is unpublished at present. This thesis examines the relations between the two ranges, and it is found that, for operators which are either compact or normal, the ranges are very closely related.

Much use is made of the theory of representations of a C*-algebra, so in chapter I a detailed account of this is given. Most of the material is well established in the standard texts, and the account given here leans heavily on Dixmier's book "Les C*-algèbres et leurs Représentations". The decomposition theorems are proved by the methods of Porta & Schwartz (Comm. on Pure & Applied Math. 20), §3 being largely a restatement of their paper. The only new material in the first chapter is that concerning the great universal representation. This concept is a slight variant of the universal representation, and it forms a crucial link between the two ranges.

In the second chapter, a detailed account of Arveson's work is given. Completely positive maps are discussed and Stinespring's characterisation theorem is proved. An extension theorem of Arveson is given, the algebraic matrix range is discussed in detail and the

complete unitary invariance theorem is proved. Much of the detail given here does **not** appear in Arveson's paper, but it was all known to him. Some of his generality has been omitted for the sake of clarity.

Chapter III begins with a discussion of spatial matrix ranges, and the complete unitary invariance is proved. Parrott's theorem is strengthened slightly. It is then shown that the two ranges are related via the great universal representation, and the decomposition theorems of §3 are used to elaborate this relation. In §10, a particularly close relation is deduced for compact operators, and also a theorem of de Barra, Giles & Sims is generalised. Finally, the relation between the spectrum and the matrix ranges is inspected, a new proof of a theorem of Arveson is given, and the relation holding for compact operators is deduced for normal operators.

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CHAPTER I

REPRESENTATIONS OF C*-ALGEBRAS

§1. C*-ALGEBRAS

In this section we set down the basic definitions concerning C*-algebras, and recall some fundamental propositions. This outlines the setting in which we discuss matrix ranges.

Throughout this thesis we shall denote the complex field by $\underset{\sim}{C}$ and the set of positive integers by $\underset{\sim}{N}$. Given $\lambda \in \underset{\sim}{C}$, λ^* denotes the complex conjugate of λ. For $n \in \underset{\sim}{N}$, $\underset{\sim}{C}^n$ will denote the vector space of n-tuples of complex numbers and $\underset{\sim}{\lambda}$ will denote the n-tuple $(\lambda_1, \ldots, \lambda_n)$. Unless otherwise specified, H and K will denote complex Hilbert spaces of arbitrary dimension. B(H) will denote the algebra of bounded linear operators on H and I_H will denote the identity of this algebra, though the suffix will often be omitted. We shall reserve the term "subspace" for closed linear manifolds.

For $T \in B(H)$, T^* denotes the adjoint of T defined by the relation

$$(Th, h') = (h, T^*h') \qquad (h, h' \in H).$$

The map $T \to T^*$ is an involution on B(H) and T is said to be self-adjoint if $T = T^*$; normal if $TT^* = T^*T$; and unitary if $TT^* = T^*T = I_H$. The involution is related to the usual norm in B(H) by

$$\|T^*\| = \|T\|, \quad \|T^*T\| = \|T\|^2 \qquad (T \in B(H)).$$

We shall also be concerned with bounded linear transformations from one Hilbert space into another. B(H,K) will denote the space of bounded linear maps from H into K, and for $V \in B(H,K)$, V^* denotes the adjoint of V given by

$$(Vh, k) = (h, V^*k) \qquad (h \in H,\ k \in K).$$

We have $V^* \in B(K,H)$ and

$$\|V^*\| = \|V\|, \quad \|V^*V\| = \|V\|^2.$$

These matters are dealt with in [15] (p.249).

A subalgebra A of B(H) is said to be self-adjoint if $T \in A$ implies $T^* \in A$. We define a C*- algebra to be a norm-closed self-adjoint subalgebra A of B(H) such that $I_H \in A$. Some authors omit this last requirement. Notice that the intersection of a family of C*-algebras on the same Hilbert space H is again a C*-algebra on H. Hence given a subset $M \subseteq B(H)$, we can define the C*-algebra generated by M to be the intersection of all C*-algebras containing M. This will be denoted by C*(M), or C*(T) when M = {T} for some $T \in B(H)$. It is not hard to see that C*(T) is the norm closure in B(H) of polynomials in T and T*, and also if T is normal, C*(T) is commutative.

Throughout, A and B will denote C*-algebras, not necessarily on the same Hilbert space. We define a morphism of A into B to be a map $\phi: A \to B$ satisfying

i) $\phi(S+T) = \phi(S)+\phi(T)$,

ii) $\phi(\lambda S) = \lambda\phi(S)$,

iii) $\phi(ST) = \phi(S)\phi(T)$,

iv) $\phi(S^*) = \phi(S)^*$

for all $S, T \in A$ and $\lambda \in \underline{C}$. A morphism is said to be a mono-, epi- or isomorphism according as it is one-one, onto or both.

Two valuable properties of C*-algebras are the automatic continuity of morphisms and the invariance of the spectrum:-

1.1 PROPOSITION:

Let A and B be C*-algebras and let $\phi: A \to B$ be a morphism, then

$$\|\phi(T)\| \leq \|T\| \qquad (T \in A).$$

Proof: See [5] (p.8).

We denote by $\sigma_A(T)$ the spectrum of T evaluated in the algebra A.

1.2 PROPOSITION:

Let A and B be C*-algebras with $A \subseteq B$. Then for $T \in A$ we have

$$\sigma_A(T) = \sigma_B(T).$$

Proof: See [6] (p.8).

Hence we may write $\sigma(T)$ without ambiguity.

Let A be a C*-algebra, then an element $T \in A$ is defined to be positive if there exists a self-adjoint element $S \in A$ such that $S^2 = T$. Note that a positive element must be self-adjoint. We shall denote by P_A the set of positive elements in A.

1.3 PROPOSITION:

The following five statements are equivalent:-

i) $T \in P_A$.

ii) $T = T^*$ and $\| (\|T\| I - T)\| \leq \|T\|$.

iii) $T = T^*$ and $\sigma(T)$ is contained in the positive reals.

iv) There exists $S \in A$ such that $T = S^*S$.

v) $(Th,h) \geq 0$ for every $h \in H$.

Proof: See [6] (p.12) for the equivalence of i) to iv), then [8] (p.41) and [9] (problem 169) show the equivalence of v) and iii).

It follows from this proposition that P_A is a norm-closed convex cone such that $P_A \cap (-P_A) = \{0\}$. We define a relation on A by $S \leq T$ if and only if $T - S \in P_A$, and then $\leq$ is a partial order.

§2. REPRESENTATIONS

In this section we discuss elementary representation theory, including an account of Hilbert space direct sums and the direct sums of representations. We construct a particular representation, which we call the great universal representation, and establish a few characteristic properties which will be used in the sequel.

Let $A \subseteq B(H)$ be a C*-algebra, then we define a representation π of A on a Hilbert space K to be a morphism of A into B(K). It is said to be non-degenerate if $\{\pi(T)k: T \in A, k \in K\}$ is dense in K; topologically cyclic if there exists $k_0 \in K$ such that $\{\pi(T)k_0: T \in A\}$ is dense in K (and in this case k_0 is said to be a topologically cyclic vector); and faithful if $\pi(T) = 0$ implies $T = 0$. A subspace $L \subseteq K$ is invariant for π if $\{\pi(T)\ell : T \in A, \ell \in L\} \subseteq L$, and it reduces π if L and $L^{\perp}$, the orthogonal complement of L, are both invariant for π.

Clearly, by proposition 1.1, any representation π of A satisfies $\|\pi(T)\| \leq \|T\|$ for every $T \in A$. Moreover, if π is topologically cyclic then it is non-degenerate, and if it is non-degenerate then $\pi(I_H) = I_K$. Two representations π_1 and π_2 of A on K_1 and K_2 are unitarily equivalent if there exists a unitary map $U : K_1 \to K_2$ with

$$\pi_1(T) = U^*\pi_2(T)U \qquad (T \in A).$$

We shall, in later sections, be involved in the decomposition of Hilbert spaces and representations into direct sums, and so we set down the basic definitions.

Let J be an index set of any cardinality, and for each $j \in J$ let r_j be a non-negative real number, then we define the sum

$$\sum_{j \in J} r_j = \sup\{\sum_{j \in I} r_j : I \text{ is a finite subset of } J\}.$$

For each $j \in J$, let H_j, $(\cdot,\cdot)_j$ be a Hilbert space and let $\underline{H}$ be the set of functions ϕ defined on J and satisfying:-

i) $\phi(j) \in H_j \qquad (j \in J)$,

ii) $\sum_{j \in J} (\phi(j), \phi(j))_j < \infty$.

Notice that for any $\phi \in \underline{H}$ the support of ϕ, $\{j \in J : \phi(j) \neq 0\}$, is countable. Define operations in $\underline{H}$ in pointwise fashion and an inner product by taking, for particular ϕ and ψ in $\underline{H}$, $\{j_n\}_{n=1}^{\infty}$ to be the union of the supports of ϕ and ψ, and then

$$(\phi, \psi) = \sum_{n=1}^{\infty} (\phi(j_n), \psi(j_n))_{j_n} .$$

This makes $\underline{H}$ a Hilbert space, which is called the Hilbert space direct sum of the H_j, and is denoted $\oplus_{j \in J} H_j$.

If also for every $j \in J$, π_j is a representation of A on H_j, then we define a representation $\oplus_{j \in J} \pi_j$ of A on $\oplus_{j \in J} H_j$ by

$$\{[(\oplus_{j \in J} \pi_j)(T)](\phi)\}(k) = \pi_k(T)[\phi(k)] \qquad (T \in A, k \in J, \phi \in \oplus_{j \in J} H_j).$$

This representation is the direct sum of the π_j .

Notice that if for every $j \in J$, H_j is a subspace of some fixed Hilbert space H such that H_i is orthogonal to H_j whenever $i \neq j$, then besides the Hilbert space direct sum we can define the orthogonal sum as

$$\sum_{j \in J} H_j = \{h \in H : h = \Sigma_{k \in J} h_k \text{ with } h_k \in H_k \text{ and } [k : h_k \neq 0] \text{ countable}\}.$$

Since the H_j are mutually orthogonal, we have

$$(h, h') = \sum_{j \in J} (h_j, h_j').$$

Given $\phi \in \oplus_{j \in J} H_j$, we can define $h_\phi \in \sum_{j \in J} H_j$ by $h_\phi = \Sigma_{j \in J} \phi(j)$ and then the map $h \to h_\phi$ is linear, one-one, onto and isometric. We shall occasionally confuse the two direct sums without explicitly using the map.

For any C*-algebra A, we shall denote by $D(A, I)$ the set of normalized states on A, i.e. the set of continuous linear functionals $f : A \to \underline{C}$ such that $\| f \| = f(I) = 1$. The next proposition shows that there is a correspondence between normalized states and topologically cyclic representations.

2.1 PROPOSITION:

Given $f \in D(A,I)$, there is a Hilbert space K_f and a topologically cyclic representation π_f of A on K_f with topologically cyclic unit vector u_f such that

$$f(T) = (\pi_f(T)u_f,u_f) \qquad (T \in A) .$$

Conversely, if τ is a topologically cyclic representation of A on K, then there is a topologically cyclic unit vector $v \in K$ and $f_v \in D(A,I)$ such that

$$f_v(T) = (\tau(T)v,v) \qquad (T \in A) .$$

Moreover, τ is unitarily equivalent to π_{f_v} .

Proof: First we remark that $f \in D(A,I)$ implies that f is a positive linear functional (i.e. it maps P_A into the positive reals), and this implies that

$$f(S^*T) = f(T^*S)^* , \ |f(S^*T)|^2 \leq f(S^*S)f(T^*T) \qquad (S,T \in A) .$$

Details of this remark will be found in [12] (pp. 213 & 247).

Given $f \in D(A,I)$, define $J_f = \{T \in A : f(T^*T) = 0\}$, then J_f is a linear manifold in A regarded as a linear space. Let $X_f = A/J_f$ and define an inner product on X_f by

$$(S+J_f, T+J_f) = f(T^*S) \qquad (S,T \in A) .$$

It is easy, using the remark, to verify that this is a well-defined inner product. Let $K_f,(\cdot,\cdot)$ be the Hilbert space completion of X_f . Define $\pi_f : A \to B(K_f)$ by

$$\pi_f(T)(S+J_f) = TS+J_f \qquad (S,T \in A)$$

Which defines $\pi_f(T)$ on X_f, and the definition is extended to K_f by continuity in the natural way. π_f is a representation of A on K_f and $\{\pi_f(T)(I+J_f) : T \in A\} = X_f$, which is dense in K_f, so π_f is topologically cyclic with topologically cyclic unit vector $u_f = I+J_f$.

Also $$(\pi_f(T)u_f, u_f) = (T+J_f, I+J_f) = f(T) \qquad (T \in A).$$

Conversely, if τ is a topologically cyclic representation of A on K, then choose any topologically cyclic unit vector $v \in K$.

Define $f_v : A \to \underline{C}$ by

$$f_v(T) = (\tau(T)v, v) \qquad (T \in A)$$

then proposition 1.1 shows that $\|\tau(T)\| \leq \|T\|$ and since τ is topologically cyclic we have $\tau(I) = I$, so $f_v \in D(A,I)$. Using the same notation as in the first half of this proof, we have

$$J_{f_v} = \{T \in A : f_v(T^*T)=0\} = \{T \in A : \tau(T)v=0\}.$$

Define $U_0 : \{\tau(T)v : T \in A\} \to X_{f_v}$ by

$$U_0(\tau(T)v) = T+J_{f_v} \qquad (T \in A).$$

It is easy to see that U_0 is a well defined linear isometry of $\{\tau(T)v : T \in A\}$ onto X_{f_v}, and these are respectively dense in K and K_{f_v}, so that U_0 extends to a unitary map $U : K \to K_{f_v}$. Finally

$$U\tau(T)[\tau(S)v] = U_0[\tau(TS)v] = TS+J_{f_v} = \pi_{f_v}(T)[S+J_{f_v}] = \pi_{f_v}(T)U[\tau(S)v]$$

and so by continuity

$$U\tau(T)k = \pi_{f_v}(T)Uk \qquad (k \in K),$$

which shows that τ and π_{f_v} are unitarily equivalent.

The correspondence $f \to \pi_f$, demonstrated above, enables us to construct representations in a standard way, discussion of which willbe found in [6] for example. Let A be a C*-algebra and let $\Lambda = \underline{N} \times D(A,I)$. Using the notation of proposition 2.1, for $\lambda = (n,f) \in \Lambda$, let $G_\lambda = K_f$ and $\gamma_\lambda = \pi_f$. We define the universal and great universal spaces of A to be respectively

$$H_A = \oplus_{f\in D(A,I)} K_f \quad , \quad G_A = \oplus_{\lambda\in\Lambda} G_\lambda .$$

Also we define the universal and great universal representations of A on H_A and G_A , respectively, to be

$$\pi_A = \oplus_{f\in D(A,I)} \pi_f \quad , \quad \gamma_A = \oplus_{\lambda\in\Lambda} \gamma_\lambda .$$

Notice that γ_A is unitarily equivalent to a countably infinite direct sum of copies of π_A . Now for each $f \in D(A,I)$, π_f is topologically cyclic and hence non-degenerate. Thus π_A and γ_A are non-degenerate and $\pi_A(I) = I_{H_A}$, $\gamma_A = I_{G_A}$. The following proposition demonstrates a fairly trivial property of the great universal representation which will be of use later.

2.2 PROPOSITION:

A countable direct sum of copies of γ_A is unitarily equivalent to γ_A .

Proof: If the direct sum is of a finite number, say r, of copies of γ_A , then let $J = \{1,2,\ldots,r\}$ and define

$$s_{mj} = 2^{j-1} + (m-1)2^j \qquad (j=1,2,\ldots,r-1 \;;m=1,2\ldots)$$

and let $\{s_{nr}\}_{n=1}^{\infty}$ be an enumeration of $\underline{N}\setminus\{s_{mj} : j=1,\ldots,r-1; \; m=1,\ldots\}$.

If the direct sum is infinite, let $J = \underline{N}$, and define

$$s_{mj} = 2^{j-1} + (m-1)2^j \qquad (j,m=1,2,\ldots).$$

In both cases $\{s_{mj} : j\in J, \; m=1,2,\ldots\} = \underline{N}$ and $j \neq k$ implies $s_{mj} \neq s_{nk}$ for all $m,n \in \underline{N}$.

Define $U : \oplus_{j\in J} G_A \to G_A$ by

$$[Ux](n,f) = [x(l)](m,f) \qquad (x \in \oplus_{j\in J} G_A \; , \; (n,f)\in\Lambda)$$

where $l\in J$ and $m\in\underline{N}$ are the unique elements such that $s_{ml}=n$.

U is clearly linear, and also

$$\| Ux\|^2 = \sum_{(n,f)\in\Lambda}(Ux(n,f), Ux(n,f)) = \sum_{m\in\underline{N}, j\in J, f\in D(A,I)} \| x(j)(m,f)\|^2$$

$$= \sum_{j\in J}\| x(j)\|^2 = \| x\|^2$$

so that U is an isometry. Moreover, given $g \in G_A$ we can define $z \in \oplus_{j\in J} G_A$ by $$z(j)(n,f) = g(s_{nj}, f) \qquad ((n,f) \in \Lambda)$$
and then $Uz = g$. Thus U is onto and hence unitary. Note that

$$(U^*g)(j)(n,f) = z(j)(n,f) = g(s_{nj}, f) \ .$$

Finally

$$\begin{aligned}[U^*\gamma_A(T)Ux](j)(n,f) &= [\gamma_A(T)Ux](s_{nj}, f) \\ &= \pi_f(T)[Ux(s_{nj}, f)] \\ &= \pi_f(T)[x(j)(n,f)] \\ &= [\gamma_A(T)x(j)](n,f) \\ &= [(\oplus_{j\in J}\gamma_A)(T)x](j)(n,f) \ ,\end{aligned}$$

which shows that $\oplus_{j\in J}\gamma_A$ is unitarily equivalent to γ_A .

The next theorem demonstrates that any countable direct sum of topologically cyclic representations is unitarily equivalent to a compression of the great universal representation. It is this which makes the great universal representation useful in the study of matrix ranges.

2.3 THEOREM:

Let A be a C*-algebra, J a countable index set and for each $j \in J$ let τ_j be a topologically cyclic representation of A on a Hilbert space K_j, then there is an ~~bounded~~ isometric linear map $V: \oplus_{j\in J}K_j \rightarrow G_A$ such that

$$(\oplus_{j\in J}\tau_j)(T) = V^*\gamma_A(T)V \qquad (T \in A)$$

Proof: Without loss of generality, we may suppose that $J \subseteq N$. By proposition 2.1 there exist $f_j \in D(A,T)$ and unitary maps U_j from K_j into K_{f_j} such that

$$\tau_j(T) = U_j^* \, \pi_{f_j}(T) \, U_j \qquad (T \in A,\ j \in J) \cdots\cdots(1).$$

Let $\Gamma = \{(j,f_j) : j \in J\} \subseteq \Lambda$, and define $V : \oplus_{j\in J} K_j \to G_A$ by

$$[V\phi](\lambda) = \begin{cases} U_j[\phi(j)] & \text{if } \lambda=(j,f_j) \\ 0 & \text{if } \lambda\notin\Gamma \end{cases} .$$

V is linear, and since each U_j is unitary, we have

$$\| V\phi\|^2 = \sum_{\lambda\in\Lambda}\|(V\phi)(\lambda)\|^2=\sum_{j\in J}\| U_j[\phi(j)]\|^2 = \sum_{j\in J}\| \phi(j)\|^2 = \| \phi\|^2.$$

Thus V is an isometry. Also note that given $x \in G_A$, we have

$$[V^*x](j) = U_j^* \, x(j,f_j) \qquad (j\in J) .$$

If $\lambda \notin \Gamma$, then

$$\{V[(\oplus_{j\in J}\tau_j)(T)\phi]\}(\lambda) = 0 = [\chi_A(T)V\phi](\lambda) , \quad (T\in A, \phi\in \oplus_{j\in J}K_j).$$

If $\lambda \in \Gamma$, then $\lambda = (k,f_k)$ and

$$\begin{aligned} \{V[(\oplus_{j\in J}\tau_j)(T)\phi]\}(\lambda) &= U_k\{[(\oplus_{j\in J}\tau_j)(T)\phi](k)\} \\ &= U_k\{[\tau_k(T)]\phi(k)\} \\ &= \pi_{f_k}(T)U_k\phi(k) \qquad \text{by}(1) \\ &= \pi_{f_k}(T)[(V\phi)(k)] \\ &= [\chi_A(T)V\phi](\lambda) , \quad (T\in A, \phi\in \oplus_{j\in J}K_j). \end{aligned}$$

Hence $V^*\chi_A(T)V=\oplus_{j\in J}\tau_j$, and since V is an isometry $\oplus_{j\in J}\tau_j$ is unitarily equivalent to the compression of χ_A to the range of V.

In the sequel we shall be concerned with the action of $\pi(T)$ on n-dimensional subspaces (for $n = 1,2,\cdots$), where π is a non-degenerate representation of $B(H)$ and $T \in B(H)$. The following result reduces the study of this action to the case when π is the great universal representation:-

2.4 THEOREM:

Let π be a non-degenerate representation of a C*-algebra $\underline{A} \subseteq B(H)$ on a Hilbert space K, and let L be a separable subspace of K, then there is a bounded linear map $Q : L \to G_A$ such that

$$(\pi(T)\ell,\ell') = (Q^*\gamma_A(T)Q\ell,\ell') \qquad (T\in A, \quad \ell,\ell'\in L).$$

Proof: Since L is separable, we can choose a maximal orthonormal subset $\{x_j : j\in J\}$ in L with $J \subseteq \underline{N}$. Let j_1 be the least integer in J, set $v_1 = x_{j_1}$ and let $K_1 = \{\pi(T)v_1 : T\in A\}^-$. K_1 is a closed linear manifold in K which reduces π. Define $\tau_1 : A \to B(K_1)$ by

$$\tau_1(T)k_1 = \pi(T)k_1 \qquad (T\in A, \ k_1\in K_1).$$

Notice that $v_1 = \pi(I)v_1 \in K_1$ and so τ_1 is topologically cyclic. If $K_1 \supseteq L$, then an application of theorem 2.3 proves the result.

If $L \not\subseteq K_1$, let P_2 be the projection of K onto $K_1^\perp$ and let j_2 be the least integer in $\{j\in J : P_2x_j\neq 0\}$. Set $v_2 = x_{j_2}$ and define $K_2 = \{\pi(T)v_2 : T\in A\}^-$. Again K_2 is a subspace which reduces π, and τ_2, the restriction of π to K_2, is topologically cyclic. If $L \subseteq K_1 \oplus K_2$, then an application of theorem 2.3 proves the result.

If $L \not\subseteq K_1 \oplus K_2$, let P_3 be the projection of K onto $(K_1 \oplus K_2)^\perp$ and proceed as above.

Either the result is proved at some n^{th} stage, or by the principle of induction, for each $j\in J$, there exists an integer m such that $j<j_m$. Hence $P_mx_j = 0$ and so $x_j \in K_1 \oplus K_2 \oplus \cdots \oplus K_m$. Thus $L \subseteq \oplus_{r=1}^{\infty} K_r$ and again theorem 2.3 completes the proof.

§3. DECOMPOSITION OF REPRESENTATIONS

The aim of this section is to exhibit the particularly convenient decomposition obtained when a non-degenerate representation is restricted to the algebra of compact operators. All the results are well known and will be found in [11] and [12]. Because our later work hinges on this material, we give full details of the proofs - following for the most part the account given by Porta and Schwartz.

We shall denote by t_s and t_w the strong and weak topologies on a Hilbert space H, and by τ_n, τ_s and τ_w the uniform, strong and weak operator topologies on B(H) respectively. Recall that a compact operator maps a weakly convergent sequence into a strongly convergent one.

By a projection on H we mean an operator $E \in B(H)$ such that $E = E^2 = E^*$, which is called an orthogonal projection by some authors. An idempotent is an operator $F \in B(H)$ such that $F^2 = F$. Commuting ~~projections~~ idempotents are ordered by the relation $E' \geq E$ if and only if $EE' = E$. A family Γ of commuting ~~projections~~ idempotents such that all products of pairs of elements of Γ are in Γ, is a directed set in the given ordering. Note that limiting processes take place along decreasing nets. Also note that the range of an idempotent is closed.

3.1 LEMMA:

Let π be a representation of B(H) on K and let $\Gamma \subseteq B(H)$ be a family of ~~projections~~ idempotents such that

i) There is a constant c such that $E \in \Gamma$ implies $\|E\| \leq c$,

ii) $E \in \Gamma$ and $E' \in \Gamma$ implies $EE' \in \Gamma$ and $EE' = E'E$,

then there is ~~a projection~~ a projection $F(\pi, \Gamma) \in B(K)$ such that

$$\pi(E) \to F(\pi, \Gamma) \quad (\tau_w) \quad \text{as } E \text{ decreases in } \Gamma.$$

Proof: Let $K_1 = \{ \cup_{E\in\Gamma} \operatorname{Ker} \pi(E) \}^-$ and $K_2 = \cap_{E\in\Gamma} \pi(E)K$, and suppose that $k \in K_1 \cap K_2$. Given $\delta > 0$, there exist $E_0 \in \Gamma$ and $\ell_1, \ell_2 \in K$ such that

$$\| k-\ell_1 \| < \delta , \quad \pi(E_0)\ell_1 = 0 , \quad k = \pi(E_0)\ell_2 .$$

But $E_0^2 = E_0$, so $\pi(E_0)\pi(E_0) = \pi(E_0)$, and also by proposition 1.1 $\| \pi(E_0)\| \leq \| E_0 \| \leq c$. Thus we have

$$\| k\| = \| k-\pi(E_0)\ell_1 \| = \| \pi(E_0)\pi(E_0)\ell_2-\pi(E_0)\ell_1 \| = \|\pi(E_0)(k-\ell_1)\| < \delta . c ,$$

from which it follows that $K_1 \cap K_2 = \{0\}$.

Given $k \in K$, $\{\pi(E)k : E\in\Gamma\}$ is contained in the ball in K of radius $c\| k\|$, which is weakly compact. Hence there is a subnet $\Delta \subseteq \Gamma$ and an element $k_2 \in K$ such that

$$\pi(E)k \to k_2 \quad (t_w) \quad \text{as } E \text{ decreases in } \Delta .$$

But Δ is cofinal, and $F \geq E$ implies $\pi(F)\pi(E) = \pi(E)$, so for each $F \in \Gamma$ we have

$$\pi(F)k_2 = \lim_\Delta \pi(F)\pi(E)k = \lim_\Delta \pi(E)k = k_2 .$$

Thus $k_2 \in K_2$. Let $k_1 = k - k_2$ and suppose that $\ell \in K_1^\perp$, then

$$(k_1, \ell) = (k-k_2, \ell) = \lim_\Delta (k-\pi(E)k, \ell) = 0$$

since $\pi(D)(k-\pi(D)k) = 0$ for every $D \in \Gamma$ and so $k-\pi(D)k \in K_1$. Thus $k_1 \in K_1$ and hence we have $K = K_1 \oplus K_2$.

Let $F(\pi,\Gamma)$ be the projection of K onto K_2, and let Δ be any subnet of Γ. For each $k \in K$, $\{\pi(E)k : E\in\Delta\}$ is norm-bounded, so there is a subnet $\Delta' \subseteq \Delta$ and an element $x \in K$ such that

$$\pi(E)k \to x \quad (t_w) \quad \text{as } E \text{ decreases in } \Delta'.$$

As above we can show that $x\in K_2$ and $k-x \in K_1$, so by the uniqueness of decomposition in a direct sum we have $x = F(\pi,\Gamma)k$. Thus

$$\pi(E)k \to F(\pi,\Gamma)k \quad (t_w) \quad \text{along } \Delta'$$

and so $F(\pi,\Gamma)k$ is the unique t_w-accumulation point of the norm-bounded set $\{\pi(E)k : E\in\Gamma\}$. Hence $\pi(E)k \to F(\pi, \Gamma)k$ (t_w) as E decreases in Γ. This holds for each $k\in K$ and so the result follows.

3.2 LEMMA:

Let $\Gamma \subseteq B(H)$ be a family of ~~projections~~ idempotents suchthat
i) there is a constant c such that $E\in\Gamma$ implies $\|E\| \leq c$,
ii) $E\in\Gamma$ and $E'\in\Gamma$ implies $EE'\in\Gamma$ and $EE' = E'E$,
iii) $\bigcap_{E\in\Gamma} EH = \{0\}$,

then for any compact operator $C\in B(H)$ we have $CE \to 0$ and $EC \to 0$ (τ_n) as E decreases in Γ.

Proof: Let $B_r = \{x\in H : \|x\| \leq r\}$ and $A_r = \{x\in H : \|x\| \geq r\}$. Suppose $z \in \bigcap_{E\in\Gamma}(CEB_1)^-$, then given $\delta>0$ we can choose $z(E) \in B_1$ such that

$$\| z - CEz(E)\| < \delta \qquad (E\in\Gamma) \quad \ldots\ldots\ldots(1).$$

The set $\{Ez(E) : E\in\Gamma\}$ is norm-bounded in H, so we may choose a weakly convergent subnet $\{Ez(E) : E\in\Delta\}$, and then let $y = \lim_\Delta Ez(E)$. Apply

Since $\bigcap_{E\in\Gamma} EH = \{0\}$, we have that $L = \bigcup_{E\in\Gamma}((EH)^\perp)$ is dense in H. Suppose $x\in L$, then $x\in (FH)^\perp$ for some $F\in\Gamma$. But $E\leq F$ implies $EH \subseteq FH$, so $(FH)^\perp \subseteq (EH)^\perp$, and thus $x \in (EH)^\perp$ for all $E \leq F$. Now Δ is cofinal, so

$$(y,x) = \lim_\Delta (Ez(E), x) = 0 .$$

This implies that y is orthogonal to L, which is dense in H, and hence $y=0$.

Thus $y = 0$, i.e. $Ez(E) \to 0$ (t_w) along Δ, and hence $CEz(E)\to 0$ (t_s) along Δ. It now follows that

$$\| z\| = \lim_\Delta \| z - CEz(E)\| \leq \delta \qquad \text{by (1)}.$$

Thus $z = 0$, and so $\bigcap_{E\in\Gamma}(CEB_1)^- = \{0\}$. Hence for any $r>0$ we have

$$A_r \cap \{ \bigcap_{E\in\Gamma}(CEB_1)^-\} = \emptyset \quad \ldots\ldots\ldots(2).$$

Choose $F \in \Gamma$, and let $X = A_r \cap (CFB_1)^-$, and then let $B_E = X \cap (CEB_1)^-$ for each $E \in \Gamma$. Each B_E is compact and by (2) $\bigcap_{E\in\Gamma} B_E = \emptyset$. By the finite intersection property there exist $E_1, \cdots, E_n$ such that $\bigcap_{i=1}^n B_{E_i} = \emptyset$, i.e. $A_r \cap \{\bigcap_{i=1}^n (CE_iB_1)^-\} = \emptyset$.

Let $E_r = E_1E_2\cdots E_n$, then $E_r \in \Gamma$ and if $E \leq E_r$ we have

$$CEB_{1/c^2} = CE_rEB_{1/c^2} \subseteq CE_rB_{1/c} = CE_jE_rB_{1/c} \subseteq CE_jB_1 \qquad (j=1,2,\cdots,n)$$

since $\|E\| \leq c$ and $\|E_r\| \leq c$. Thus for each $r > 0$, we have for sufficiently small E in Γ that $A_r \cap CEB_{1/c^2} = \emptyset$, or $\|CE\| < rc^2$. Thus $CE \to 0$ (τ_n) as E decreases in Γ.

Finally $\|EC\| = \|(EC)^*\| = \|C^*E^*\| \to 0$ (τ_n) along Γ, since C^* is also compact. and $\Gamma^* = \{E^* : E\in\Gamma\}$ is a family of idempotents having properties i), ii) and iii).

3.3 LEMMA:

Let Γ_1 and Γ_2 be two families of ~~projections~~ idempotents in B(H) such that

$$\left.\begin{array}{l} \text{i) there is a constant } c \text{ such that } E\in\Gamma_i \text{ implies } \|E\| \leq c \\ \text{ii) } E\in\Gamma_i \text{ and } E'\in\Gamma_i \text{ implies } EE'\in\Gamma_i \text{ and } EE' = E'E \\ \text{iii) } \bigcap_{E\in\Gamma_i} EH = \{0\} \\ \text{iv) } E\in\Gamma_i \text{ implies } I_H - E \text{ is of finite rank} \end{array}\right\} (i=1,2)$$

and let π be a non-degenerate representation of B(H) on K, then, in the notation of lemma 3.1, $F(\pi,\Gamma_1) = F(\pi,\Gamma_2)$.

Proof: Suppose $E\in\Gamma_1$, then $I_H - E$ has finite rank and so is compact. By lemma 3.2, $(I_H - E)F \to 0$ (τ_n) as F decreases in Γ_2. But π is non-degenerate so $\pi(I_H) = I_K$, and thus $(I_K - \pi(E))\pi(F) = \pi[(I_H - E)F] \to 0$ (τ_n) as F decreases in Γ_2. By definition of $F(\pi,\Gamma_2)$ we now have

$$F(\pi,\Gamma_2) = \pi(E)F(\pi,\Gamma_2) \qquad (E\in\Gamma_1).$$

Allowing E to decrease in Γ_1, this gives $F(\pi,\Gamma_2) = F(\pi,\Gamma_1)F(\pi,\Gamma_2)$.

Similarly, beginning with $E \in \Gamma_2$ and using the other half of the conclusion of lemma 3.2, we obtain $F(\pi,\Gamma_1) = F(\pi,\Gamma_1)F(\pi,\Gamma_2)$.

For a non-degenerate representation π of $B(H)$ on K, we define the splitting projection of π to be $F(\pi) = F(\pi,\Gamma)$ for any family Γ of idempotents in $B(H)$ satisfying the conditions i), ii) iii) and iv) of lemma 3.3 . To see that such families do exist, let $\{x_\alpha : \alpha\in J\}$ be a maximal orthonormal subset of H, and for each finite subset Q of J let F_Q be the projection of H onto the linear span of $\{x_\alpha : \alpha\in Q\}$, then $\Gamma = \{I_H - F_Q : Q \text{ is finite}\}$ has the required properties. This also shows that we may choose a family Γ of projections such that $\lim_\Gamma \pi(E) = F(\pi)$.

3.4 LEMMA:

Let π be a non-degenerate representation of $B(H)$ on K with splitting projection $F(\pi)$, then $F(\pi)\pi(T) = \pi(T)F(\pi)$ for every $T\in B(H)$.

Proof: Let U be an invertible operator in $B(H)$ and let Γ be a family of idempotents in $B(H)$ satisfying the four conditions of lemma 3.3 . Notice that $\Gamma' = \{UEU^{-1} : E\in\Gamma\}$ also satisfies these conditions, and so by lemma 3.3 it follows that

$$F(\pi) = \lim_\Gamma \pi(E) = \lim_\Gamma \pi(UEU^{-1}) = \pi(U)F(\pi)\pi(U)^{-1},$$

i.e.
$$F(\pi)\pi(U) = \pi(U)F(\pi) .$$

But for any $T\in B(H)$, there exists $\lambda\in\underline{C}$ such that $\lambda I_H + T$ is invertible, and hence $F(\pi)\pi(\lambda I_H + T) = \pi(\lambda I_H + T)F(\pi)$, and $\pi(I_H) = I_K$ so the result follows immediately.

This brings us to the first decomposition theorem which splits a non-degenerate representation into two direct summands.

3.5 THEOREM:

Let π be a non-degenerate representation of $B(H)$ on K with splitting projection $F(\pi)$. If K_1 and K_2 are respectively the kernel and range of $F(\pi)$, then the decomposition $K = K_1 \oplus K_2$ reduces π, so that if π_i is the restriction of π to K_i $(i=1,2)$ we have $\pi = \pi_1 \oplus \pi_2$ and furthermore :-

i) when $K_1 \neq \{0\}$, π_1 is faithful and has no non-faithful subrepresentations (other than the zero representation).

ii) if $C \in B(H)$ is compact, then $\pi_2(C) = 0$.

Proof: By lemma 3.4, $F(\pi)$ commutes with $\pi(T)$ for every $T \in B(H)$, so clearly the decomposition reduces π, and we have $\pi = \pi_1 \oplus \pi_2$.

Suppose $K' \subseteq K_1$ is a π_1-invariant linear manifold and the restriction π' of π_1 to K' is not faithful, then $\text{Ker}\ \pi'$ is a non-trivial two-sided ideal in $B(H)$, which must contain all finite rank operators. Let Γ be a family of idempotents in $B(H)$ satisfying the four conditions in lemma 3.3, then for each $E \in \Gamma$, $I-E$ is of finite rank so $\pi'(I-E) = 0$. Thus for each $x \in K'$ and each $E \in \Gamma$ we have $\pi(E)x = x$, and so by lemma 3.1, $F(\pi)x = x$. But $K' \subseteq K_1$, which is the kernel of $F(\pi)$, and thus $K' = \{0\}$. Hense π_1 has no non-faithful subrepresentations.

Let $C \in B(H)$ be a compact operator and let Γ be as above, then by lemma 3.2, $CE \to 0$ (τ_n) as E decreases in Γ. Thus $\pi(C)\pi(E) \to 0$ (τ_n) along Γ, and then by lemma 3.1, $\pi(C)F(\pi) = 0$. But $F(\pi)K = K_2$ and so $\pi_2(C) = 0$.

Next we show that the direct summand π_1 can be decomposed into representations which are unitarily equivalent to the identity representation. Firstly we need a lemma :-

3.6 LEMMA:

Let π be a non-degenerate representation of $B(H)$ on K and let π_1 and K_1 be as in theorem 3.5, then every non-null linear manifold M in K_1 which is invariant under π_1 contains a non-null subspace L, also invariant under π_1, such that there is a linear isometry W from H onto L with

$$WSh = \pi_1(S)Wh \qquad (h \in H,\ S \in B(H)) \ .$$

Proof: Choose $y \in H$ with $\| y \| = 1$, and let P be the rank one projection of H onto the linear span of $\{y\}$. Let M be a non-null linear manifold in K_1 which is invariant under π_1. The proof of theorem 3.5(i) shows that the restriction of π_1 to M is faithful, and thus we may choose $b \in M$ such that $\pi_1(P)b \neq 0$. Let $a = \dfrac{\pi_1(P)b}{\|\pi_1(P)b\|}$, then since M is π_1-invariant, we have $a \in M$.

For each $z \in H$, define $T_z \in B(H)$ by $T_z x = (x,y)z \quad (x \in H)$, and then define $W : H \to K_1$ by

$$Wz = \pi_1(T_z)a \qquad (z \in H) \ .$$

Now $T_{\lambda h + h'} = \lambda T_h + T_{h'}$, so W is linear, and also $\| \pi_1(T_z) \| \leq \| T_z \|$ while $\| T_z \| \leq \| y \| \, \| z \|$, $\| y \| = \| a \| = 1$, and so $\| Wz \| \leq \| z \|$ ...(1).

For $S \in B(H)$, $T_{Sz}x = (x,y)Sz = ST_z x \quad (x \in H)$, and so $WSz = \pi_1(S)Wz$ for every $z \in H$. Hence WH is invariant under π_1.

Given non-zero $x \in H$, define $A \in B(H)$ by $Ah = (h,x)y/\| x^2 \|$, then $Ax = y$ and so as above $Wy = WAx = \pi_1(A)Wx$. But $Wy = \pi_1(T_y)a$ and $T_y = P$, so $Wy = \pi_1(P)a = a$. Thus we have

$\|x\| = \|a\| \|x\| = \|Wy\| \|x\| = \|\pi_1(A)Wx\| \|x\| \leq \|A\| \|x\| \|Wx\| \leq \|Wx\|$.

This combined with (1) shows that W is an isometry.

Suppose $\{Wx_n\}$ is a Cauchy sequence in WH, then since W is an isometry, $\{x_n\}$ is a Cauchy sequence in H. But H is complete so $x_n \to x \in H$, thus $Wx_n \to Wx$ and so WH is closed. Notice that since $a \in M$ and M is π_1-invariant, we have $WH \subseteq M$. L = WH gives the result.

The further decomposition of the representation now follows:-

3.7 THEOREM:

Let π be a non-degenerate representation of B(H) on K with π_1 and K_1 as in theorem 3.5, then there is a decomposition $K_1 = \oplus_{j\in J} K_j$ which reduces π_1, so $\pi_1 = \oplus_{j\in J}\pi_j$, and for each $j \in J$, π_j is unitarily equivalent to ω, the identity representation of B(H) on H.

Proof: Let Δ be the set of all families $F = \{K_i : i\in I_F\}$ of non-null subspaces of K_1 satisfying :-

i) K_i is invariant under π_1 $(i\in I_F)$.

ii) There is an isometry W_i of H onto K_i with

$WS = \pi(S)W \quad (S \in B(H)) \qquad (i\in I_F)$.

iii) $\{K_i : i\in I_F\}$ are mutually orthogonal.

Apply lemma 3.6 to the linear manifold K_1 to obtain a subspace L such that $\{L\} \in \Delta$, and so Δ is not empty.

Δ is partially ordered by inclusion. Suppose $\Gamma = \{F_\alpha : \alpha \in A\}$ is a chain in Δ , then $F = \cup_{\alpha\in A} F_\alpha$ is an upper bound for Γ, and $F\in\Delta$. By Zorn's lemma, Δ contains a maximal element, say $G = \{K_j : j\in J\}$.

Suppose $\oplus_{j\in J} K_j \neq K_1$, then since the direct sum is closed,

$M = [\oplus_{j \in J} K_j]^{\perp}$ (the orthogonal complement in K_1) is a non-null linear manifold in K_1 which is invariant under π_1. By lemma 3.6 there is a subspace L such that $G \cup \{L\} \in \Delta$, which contradicts the maximality of G. Hence $\oplus_{j \in J} K_j = K_1$, and the result follows.

We can now easily furnish the description of the image of a compact operator under a non-degenerate representation :-

3.8 THEOREM:

Let π be a non-degenerate representation of B(H) on K, then the restriction of π to the algebra of compact operators in B(H) is unitarily equivalent to a direct sum of the zero representation on some Hilbert space with a family of copies of ω, the identity representation. i.e. there is an index set J, a Hilbert space L and a unitary map $U : K \to L \oplus (\oplus_{j \in J} H)$ such that for every compact operator $C \in B(H)$ we have

$$\pi(C) = U^*[\ 0 \oplus (\oplus_{j \in J} C)]U .$$

Proof: This is an easy consequence of theorems 3.5 and 3.7 .

In the case when H is finite dimensional, every $T \in B(H)$ is compact, so that the formula in theorem 3.8 holds for all $T \in B(H)$; and if π is non-degenerate we must then have $L = \{0\}$, so that we may suppress the zero representation and obtain the following :-

3.9 COROLLARY:

If H is finite dimensional and π is a non-degenerate representation of B(H) on K, then π is unitarily equivalent to a direct sum of copies of the identity representation.

Next we inspect more closely the decomposition of the universal and great universal representations of B(H) when H is infinite dimensional.

3.10 THEOREM:

Let H be infinite dimensional and let $\pi_{B(H)}$ be the universal representation of B(H) on $H_{B(H)}$, then the splitting projection satisfies $0 \neq F(\pi_{B(H)}) \neq I$, and so in the decomposition of theorem 3.5 $[H_{B(H)} = H_1 \oplus H_2]$ we have $H_1 \neq \{0\}$ and $H_2 \neq \{0\}$.

Proof: By definition and the remarks following the definition there is a net Γ of projections in B(H) of finite co-rank such that $\pi_{B(H)}(E) \to F(\pi_{B(H)})$ (τ_W) as E decreases in Γ.

Let C(H) denote the set of all compact operators in B(H), then I is at unit distance from C(H) (see [12] p.43) and so by the Hahn-Banach theorem there is a norm-continuous linear functional f on B(H) with $f(I) = 1$ and $f(C(H)) = \{0\}$. Now [6] (p.236) shows that there is a weak-operator continuous linear functional $\tilde{f}$ on the von Neumann algebra generated by $\pi_{B(H)}(B(H))$ such that $\tilde{f}(\pi_{B(H)}(S)) = f(S)$ $(S \in B(H))$. Thus $f(E) = \tilde{f}(\pi_{B(H)}(E)) \to \tilde{f}(F(\pi_{B(H)}))$ as E decreases in Γ. But $I-E \in C(H)$ and so $f(E) = 1$ for every $E \in \Gamma$. Thus we cannot have $F(\pi_{B(H)}) = 0$.

If $F(\pi_{B(H)}) = I$, then $H_1 = \{0\}$ and so $\pi_{B(H)}(S) = 0$ for every $S \in C(H)$, which contradicts the faithfulness of $\pi_{B(H)}$. Hence $F(\pi_{B(H)}) \neq I$.

The great universal representation $\gamma_{B(H)}$ of B(H) on $G_{B(H)}$ is unitarily equivalent to a countably infinite direct sum of copies of

$\pi_{B(H)}$, and so in the decomposition of theorem 3.5, $G_{B(H)} = G_1 \oplus G_2$, we have $G_j = \oplus_{n=1}^{\infty} H_j$ for j=1,2. Hence the next result:-

3.11 COROLLARY:

Let H be infinite dimensional and let $\gamma_{B(H)}$ be the great universal representation of B(H) on $G_{B(H)}$, then in the decomposition of theorem 3.5 both G_1 and G_2 are infinite dimensional.

CHAPTER II

ALGEBRAIC MATRIX RANGES

§4. COMPLETELY POSITIVE MAPS

We begin this chapter with a discussion of completely positive maps, as characterized by Stinespring in [14]. Familiarity with these is essential for the remainder of the chapter, which is a detailed account of Arveson's approach to matrix ranges, as given in [1] and [2].

We shall denote by $e_1, \ldots, e_n$ the usual basis for $\underline{C}^n$, and E_{ij} will denote the element of $B(\underline{C}^n)$ given by $(E_{ij}e_k, e_m) = \delta_{jk}\delta_{im}$, where δ_{xy} is the Kronecker delta.

Let X and Y be complex vector spaces, then we define sp(X×Y) to be the free complex vector space on $\{(x,y) : x \in X, y \in Y\}$ as basis. Let J denote the linear manifold in sp(X×Y) generated by all elements of the form

$$(\lambda x+x',y)-\lambda(x,y)-(x',y) \quad \text{or} \quad (x,\mu y+y')-\mu(x,y)-(x,y')$$

with $x,x' \in X$; $y,y' \in Y$ and $\lambda,\mu \in \underline{C}$. The algebraic tensor product of X and Y is defined to be the vector space

$$X \otimes Y = sp(X\times Y)/J \ .$$

The image of (x,y) under the canonical quotient map of sp(X×Y) onto sp(X×Y)/J is denoted $x \otimes y$. It is easy to see that

$$\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y) \ ,$$

$$(\lambda x+x') \otimes y = \lambda(x \otimes y) + x' \otimes y \ ,$$

$$x \otimes (\mu y+y') = \mu(x \otimes y) + x \otimes y' \ ,$$

$$0 \otimes y = 0 \otimes 0 = x \otimes 0 \ ,$$

for all $x,x' \in X$; $y,y' \in Y$ and $\lambda,\mu \in \underline{C}$.

For a Hilbert space H and $n \in \underline{N}$, we define the Hilbert space tensor product $H \otimes \underline{C}^n$ to be the space $\{\Sigma_{i=}^{n} h_i \otimes e_i : h_i \in H\}$, which is effectively the vector space of n-tuples of elements of H, with addition and scalar multiplication defined pointwise, i.e.

$$(\Sigma_{i=1}^{n} h_i \otimes e_i) + (\Sigma_{i=1}^{n} k_i \otimes e_i) = \Sigma_{i=1}^{n} (h_i + k_i) \otimes e_i ,$$

$$\lambda(\Sigma_{i=1}^{n} h_i \otimes e_i) = \Sigma_{i=1}^{n} \lambda h_i \otimes e_i ,$$

and an inner product defined by

$$(\Sigma_{i=1}^{n} h_i \otimes e_i , \Sigma_{j=1}^{n} k_j \otimes e_j) = \Sigma_{i=1}^{n} (h_i, k_i) .$$

Suppose $u_r = \Sigma_{i=1}^{n} h_i^r \otimes e_i$ and $\{u_r\}_{r=1}^{\infty}$ is a Cauchy sequence, then clearly $\{h_i^r\}_{r=1}^{\infty}$ is Cauchy for each $i = 1, 2, \cdots, n$. Thus there exist $h_i \in H$ with $h_i^r \to h_i$, and since i runs over a finite index set we have $u_r \to \Sigma_{i=1}^{n} h_i \otimes e_i$. This shows that $H \otimes \underline{C}^n$ is complete in the inner-product topology, and so it is a Hilbert space. It is not difficult to show that the algebraic tensor product of H and $\underline{C}^n$ coincides with the Hilbert space tensor product, so that we are justified in using the same symbol for both.

Given $T_{ij} \in B(H)$ for $i, j = 1, 2, \cdots, n$, we define a transformation $\Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij}$ on $H \otimes \underline{C}^n$ by

$$(\Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij})(\Sigma_{k=1}^{n} h_k \otimes e_k) = \Sigma_{i=1}^{n} (\Sigma_{j=1}^{n} T_{ij} h_j) \otimes e_i$$

for all $h_k \in H$. This transfomation is clearly linear and it is not difficult to show that it is bounded, hence it is an element of $B(H \otimes \underline{C}^n)$.

For a subset $B \subseteq B(H)$, we define $B \otimes B(\underline{C}^n)$ to be the subset of $B(H \otimes \underline{C}^n)$ consisting of all operators $\Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij}$ such that $T_{ij} \in B$ for $i, j = 1, 2, \cdots, n$. It is not difficult to verify that if B is respectively i) a linear manifold, ii) a subalgebra, iii) self adjoint, or iv) norm-closed, then so is $B \otimes B(\underline{C}^n)$. In particular, when B is a C*-algebra, so is $B \otimes B(\underline{C}^n)$. The adjoint in $B \otimes B(\underline{C}^n)$ is given by

$$\left(\Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij}\right)^* = \Sigma_{i,j=1}^{n} T_{ji}^* \otimes E_{ij}$$

(note particularly the transposition of suffices). Also the identity of $B(H \otimes \underline{C}^n)$ is $\Sigma_{i=1}^{n} I \otimes E_{ii}$.

Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be C*-algebras, and let $L \subseteq A$ be a linear manifold. Recall that in §1 we defined the cones P_A and P_B of positive elements in A and B respectively. A linear map ϕ from L into B is said to be positive if and only if $\phi(P_A \cap L) \subseteq P_B$.

4.1 PROPOSITION:

If L is a self-adjoint subalgebra containing I_H, and $\phi : L \to B$ is a positive linear map, then $\phi(T^*) = \phi(T)^*$ ($T \in L$) and also ϕ is uniformly continuous.

Proof: Given $S, T \in L$ and $\zeta, \eta \in \underline{C}$, let $U = \zeta S + \eta T$, then

$$\phi(U^*U) = |\zeta|^2\phi(S^*S) + \zeta^*\eta\phi(S^*T) + \zeta\eta^*\phi(ST^*) + |\eta|^2\phi(T^*T).$$

But ϕ is positive, so $\phi(U^*U)$, $\phi(S^*S)$ and $\phi(T^*T)$ must be positive and hence self-adjoint. Thus $\zeta^*\eta\phi(S^*T) + \zeta\eta^*\phi(ST^*)$ is self-adjoint for every $\zeta, \eta \in \underline{C}$. In particular, taking $\zeta = \eta = 1$ and then $\zeta = 1, \eta = i$ we have

$$\phi(S^*T) + \phi(ST^*) = R_1 \quad , \quad i\phi(S^*T) - i\phi(ST^*) = R_2$$

with R_1 and R_2 self-adjoint. Thus $\phi(S^*T) = \frac{1}{2}(R_1 - iR_2) = \phi(ST^*)^*$, and taking $S = I_H$ gives the result.

If $R \in L$ and $R = R^*$, then by [8] (p.41) we have

$$\|R\| = \sup\{|(Rh,h)| : h \in H \text{ and } \|h\| = 1\}.$$

Thus $-\|R\| I \leq R \leq \|R\| I$, and so $-\|R\| \phi(I) \leq \phi(R) \leq \|R\| \phi(I)$.

By the first part of this proposition, $\phi(R)^* = \phi(R)$, and so $\| \phi(R) \| \leq \| R \| \, \| \phi(I) \|$. Finally for any [redacted] $T \in L$,

$$\| \phi(T) \| \leq \| \phi(\tfrac{T+T^*}{2}) \| + \| \phi(\tfrac{T-T^*}{2i}) \|$$
$$\leq \tfrac{1}{2} \| \phi(I) \| \{ \| T+T^* \| + \| T-T^* \| \}$$
$$\leq 2 \| \phi(I) \| \, \| T \| .$$

Given a linear map $\phi : L \to B$, we define subordinate maps $\phi_n : L \otimes B(\underline{C}^n) \to B \otimes B(\underline{C}^n)$ for $n = 1, 2, \cdots$ by

$$\phi_n(\Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij}) = \Sigma_{i,j=1}^{n} \phi(T_{ij}) \otimes E_{ij} .$$

Then ϕ is said to be completely positive if and only if ϕ_n is positive for every $n \in \underline{N}$. Complete contractiveness and complete isometricity are defined in a similar way. Clearly ϕ is positive if and only if ϕ_1 is positive. We shall denote by $CP(L,B)$ the set of completely positive maps from L into B, and if $I_H \in L$, then $CP(L,B,R)$ will denote those completely positive maps which map I_H into $R \in B$.

4.2 LEMMA:

An element $\Sigma_{i,j=1}^{n} R_{ij} \otimes E_{ij} \in B(K) \otimes B(\underline{C}^n)$ is positive if and only if for every n-tuple $k_1, \cdots, k_n$ of elements of K we have

$$\Sigma_{i,j=1}^{n} (R_{ij} k_j, k_i) \geq 0 .$$

Proof: By definition $B(K) \otimes B(\underline{C}^n) \subseteq B(K \otimes \underline{C}^n)$, and by proposition 1.3 an operator $T \in B(H)$ is positive if and only if $(Th, h) \geq 0$ $(h \in H)$. The result now follows from the definitions of $\Sigma_{i,j=1}^{n} R_{ij} \otimes E_{ij}$ and of the inner product in $K \otimes \underline{C}^n$.

In the case when A is a C*-algebra, the elements of CP(A,B(K)) are characterized by the following theorem, due to Stinespring [14].

4.3 THEOREM:

Let $A \subseteq B(H)$ be a C*-algebra and let ϕ be a linear map of A into B(K), then $\phi \in CP(A,B(K))$ if and only if there exists a Hilbert space K', a representation π of A on K' and a bounded linear map $V : K \to K'$ such that

i) $\phi(T)k = V^*\pi(T)Vk \qquad (k \in K,\ T \in A)$,

ii) K' is the closed linear span of $\{\pi(T)Vk\ ;\ k \in K,\ T \in A\}$,

iii) π is non-degenerate (hence $\pi(I_H) = I_{K'}$) .

Proof: Suppose conditions i), ii) and iii) are satisfied. Given $n \in \underline{N}$, let $\underset{\sim}{T} = \Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij}$ be a positive element of $A \otimes B(\underline{C}^n)$, then by proposition 1.3 there exists $\underset{\sim}{S} = (\Sigma_{i,j=1}^{n} S_{ij} \otimes E_{ij}) \in A \otimes B(\underline{C}^n)$ such that $\underset{\sim}{T} = \underset{\sim}{S}^*\underset{\sim}{S}$. Hence for $k_1, \ldots, k_n \in K$ we have

$$\begin{aligned}\Sigma_{i,j=1}^{n} (\phi(T_{ij})k_j, k_i) &= \Sigma_{i,j,m=1}^{n} (\phi(S_{mi}^*S_{mj})k_j, k_i) \\ &= \Sigma_{i,j,m=1}^{n} (\pi(S_{mi}^*S_{mj})Vk_j, Vk_i) \\ &= \Sigma_{i,j,m=1}^{n} (\pi(S_{mj})Vk_j, \pi(S_{mi})Vk_i) \\ &= \Sigma_{m=1}^{n} \| \Sigma_{i=1}^{n} \pi(S_{mi})Vk_i \|^2 \\ &\geq 0 .\end{aligned}$$

Hence, by lemma 4.2, $\phi_n(\underset{\sim}{T})$ is positive, and so ϕ is completely positive.

Conversely, if ϕ is completely positive, then we consider A and K as vector spaces and construct the algebraic tensor product $A \otimes K = sp(A \times K)/J$, as detailed at the beginning of this section.

Define a bilinear form $[\cdot,\cdot]_0$ on sp(A×K) by

$$[\Sigma_{i=1}^{r}\lambda_i(T_i,k_i)\ ,\ \Sigma_{j=1}^{m}\mu_j(S_j,\ell_j)]_0 = \Sigma_{i=1}^{r}\Sigma_{j=1}^{m}\lambda_i\mu_j^*(\phi(S_j^*T_i)k_i,\ell_j)\ .$$

Notice that if one of the arguments is a generator of J, the linear manifold defined at the beginning of the section, then the form vanishes, and hence for $z\epsilon J$ and $x\epsilon sp(A\times K)$ we have $[z,x]_0=[x,z]_0=0$. Thus we may define a bilinear form $[\cdot,\cdot]$ on $A\otimes K$ by

$$[\Sigma_{i=1}^{r}T_i\otimes k_i\ ,\ \Sigma_{j=1}^{m}S_j\otimes\ell_j] = \Sigma_{i=1}^{r}\Sigma_{j=1}^{m}(\phi(S_j^*T_i)k_i,\ell_j)\ .$$

Proposition 4.1 shows that $[\cdot,\cdot]$ is hermitian, and the complete positivity of ϕ together with lemma 4.2 shows that $[\cdot,\cdot]$ is positive. A positive hermitian bilinear form satisfies the Schwarz inequality, $|[x,y]|^2 \leq [x,x][y,y]$. Let $N = \{u\epsilon A\otimes K : [u,u]=0\}$, then define a bilinear form $(\cdot,\cdot)$ on $(A\otimes K)/N$ by $(u+N,v+N) = [u,v]$. This is well defined and inherits positivity and the hermitian property from $[\cdot,\cdot]$, and also, from the Schwarz inequality, it is positive definite and hence an inner product. Let L denote the Hilbert space completion of $(A\otimes K)/N$, $(\cdot,\cdot)$.

Define for each $S\epsilon A$ a linear transformation $\rho_0(S)$ on sp(A×K) by

$$\rho_0(S)\{\Sigma_{i=1}^{r}\lambda_i(T_i,k_i)\} = \Sigma_{i=1}^{r}\lambda_i(ST_i,k_i)\ .$$

Notice that J is invariant under $\rho_0(S)$, so we may define a linear transformation $\rho(S)$ on $A\otimes K$ by

$$\rho(S)\{\Sigma_{i=1}^{r}T_i\otimes k_i\} = \Sigma_{i=1}^{r}ST_i\otimes k_i\ .$$

For $x = \Sigma_{i=1}^{r}T_i\otimes k_i$ and $S\epsilon A$, we show that

$$[\rho(S)x,\rho(S)x] \leq \|S\|^2[x,x] \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)\ .$$

If (1) does not hold, then there exist x_0 and S_0 such that $[x_0,x_0]\leq 1$, $\|S_0\| < 1$ and $[\rho(S_0)x_0,\rho(S_0)x_0] > 1$. Now $\rho(S^*) = \rho(S)^*$ and $\rho(ST) = \rho(S)\rho(T)$, so $[\rho(S_0^*S_0)x_0,x_0] > 1$. By the Schwarz inequality, we have $[\rho(S_o^*S_o)x_o, \rho(S_o^*S_o)x_o] > 1$, and then by iteration we obtain

$$\Sigma_{i,j=1}^{r} (\phi(T_{oj}^*\{S_o^*S_o\}^{2^k} T_{oi})k_{oi}, k_{oj}) = [\rho(\{S_o^*S_o\}^{2^k})x_o, x_o] > 1 \quad (k=1,2,\cdot\cdot)$$

where $x_o = \Sigma_{i=1}^{r} T_{oi} \otimes k_{oi}$. But by proposition 4.1, ϕ is uniformly continuous, and $\| S_o \| < 1$ so that an application of the Schwarz inequality shows that the left-hand side of the above equation converges to zero as $k \to \infty$. This clearly cannot happen and so (1) must hold.

By (1), $\rho(S)$ leaves N invariant and so we may define θ from A into the set of linear transformations on $(A \otimes K)/N$ by

$$\theta(S)(u + N) = \rho(S)u + N .$$

This is a well defined morphism and

$$(\theta(S)(u+N), \theta(S)(u+N)) \leq \| S \|^2 (u+N, u+N) ,$$

so $\theta(S)$ can be extended by continuity to $\pi_o(S) \in B(L)$. Thus we have a representation π_o of A on L such that

$$\pi_o(S)\{(\Sigma_{i=1}^{r} T_i \otimes k_i) + N\} = (\Sigma_{i=1}^{r} ST_i \otimes k_i) + N .$$

Notice that $\pi_o(I_H) = I_L$.

Define $V : K \to L$ by $Vk = I_H \otimes k + N$, then

$$\| Vk \|^2 = (Vk, Vk) = [I_H \otimes k, I_H \otimes k] = (\phi(I_H)k, k) \leq \| \phi(I_H) \| \| k \|^2,$$

so that V is bounded. It is clearly linear.

It is easy to verify that $\phi(S)k = V^*\pi_o(S)Vk$ $(k \in K, S \in A)$, and then taking K' to be the closed linear span of $\{\pi_o(S)Vk : k \in K, S \in A\}$ and π to be the restriction of π_o to K', the result follows.

4.4 COROLLARY:

If A is a C*-algebra and $\phi \in CP(A, B(K))$, then for every $T \in A$ $\phi(T^*)\phi(T) \leq \| \phi(I_H) \| \phi(T^*T)$, and $\| \phi \| = \| \phi_n \| = \| \phi(I_H) \|$ $(n=1,2\cdot\cdot\cdot)$.

Proof: By the theorem, there is a Hilbert space K', a representation π of A on K and a bounded linear map $V : K \to K'$ with $\phi(T) = V^*\pi(T)V$ for all $T\in A$ and $\pi(I_H) = I_K$. Thus

$$\begin{aligned}(\phi(T^*)\phi(T)k,k) &= (V^*\pi(T^*)VV^*\pi(T)Vk,k)\\ &= (V^*\pi(T)Vk, V^*\pi(T)Vk)\\ &\leq \| V^*\|^2(\pi(T)Vk,\pi(T)Vk)\\ &= \| V^*V\| \ (V^*\pi(T^*T)Vk,k)\\ &= \| \phi(I_H)\| \ (\phi(T^*T)k,k) \qquad (k\in K),\end{aligned}$$

which proves the first claim. Also

$$\|\phi(T)\| = \|V^*\pi(T)V\| \leq \|V^*\| \ \|\pi(T)\| \ \|V\| \leq \|V^*V\| \ \|T\| = \|\phi(I_H)\| \ \|T\|,$$

which shows that $\|\phi\| = \|\phi(I_H)\|$.

Given $n\in\underline{N}$, $\phi_n : A \otimes B(\underline{C}^n) \to B(K) \otimes B(\underline{C}^n)$ has subordinate maps $(\phi_n)_k : \{A \otimes B(\underline{C}^n)\} \otimes B(\underline{C}^k) \to \{B(K) \otimes B(\underline{C}^n)\} \otimes B(\underline{C}^k)$. But for both $Z = A$ and $Z = B(K)$, we have $\{Z \otimes B(\underline{C}^n)\} \otimes B(\underline{C}^k)$ is isometrically isomorphic to $Z \otimes B(\underline{C}^{n.k})$, and the isomorphism composed with $(\phi_n)_k$ gives $\phi_{n.k}$, which is positive since ϕ is completely positive. Thus ϕ_n is completely positive, and so by the second part of this proof $\|\phi_n\| = \|\phi_n(I_{B(H)\otimes B(\underline{C}^n)})\|$. Finally

$$\phi_n(I_{B(H)\otimes B(\underline{C}^n)}) = \phi_n(\Sigma_{i=1}^n I_H \otimes E_{ii}) = \Sigma_{i=1}^n \phi(I_H) \otimes E_{ii},$$

and hence $\|\phi_n\| = \|\phi(I_H)\|$.

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§5. ARVESON'S EXTENSION THEOREM

The key factor in Arveson's approach to matrix ranges is his theorem concerning the extension of completely positive maps from norm-closed self-adjoint linear manifolds to C*-algebras. We prove this theorem and a corollary, essentially following [1]. Two other results concerning the set of completely positive maps are given. These do not appear explicitly in [1] or [2], but were communicated privately by Arveson.

As usual $A \subseteq B(H)$ denotes a C*-algebra, and $M \subseteq A$ is a norm-closed self-adjoint linear manifold with $I_H \in M$. The Banach space of bounded linear maps from M into B(K) in the usual norm is denoted B_M. B_M^* denotes the set of linear functionals on B_M which admit a representation of the form

$$f(\phi) = \Sigma_{n=1}^{\infty} \rho_n(\phi(T_n)) \qquad (\phi \in B_M) \, ,$$

where $\{T_n\}$ is a bounded sequence in M and $\{\rho_n\}$ is a sequence of ultra-weakly continuous linear functionals on B(K) such that $\Sigma_{n=1}^{\infty} \|\rho_n\| < \infty$. We shall denote the ultra-weak, weak, strong and norm operator topologies on B(K) by $\tau_{\sigma w}$, τ_w, τ_s, τ_n respectively.

Define a topology τ_M on B_M by taking as base for the open sets the family

$$\{ \{\phi \in B_M : |f_i(\phi-\psi)| < \delta \quad (i=1,2,\cdots,n)\} : f_1, \cdots f_n \in B_M^*, \psi \in B_M, \delta > 0 \},$$

i.e. the weakest topology to make every element of B_M^* continuous. It is easy to see that (B_M, τ_M) is a locally-convex Hausdorff topological linear space.

We define a norm on B_M^* by $\| f\| = \sup\{|f(\phi)| : \phi \in B_M, \|\phi\| \leq 1\}$ and this makes B_M^* a [redacted] normed linear space.(+) Its norm continuous dual is denoted $(B_M^*)^n$.

For $\alpha = n, w, \sigma w, s$, $B(K)^{\alpha}$ denotes the τ_{α}-continuous dual of B(K). It is shown in [5] (p.34 et seq.) that the map $R \to R^{\wedge}$ given by $R^{\wedge}(\rho) = \rho(R)$ for $R \in B(K)$ and $\rho \in B(K)^{\sigma w}$ implements an isometric linear bijection between B(K) and $(B(K)^{\sigma w})^n$. The ideas in the next

(+) By showing that the representation of any $f \in B_M^*$ can be chosen so that $\sum_{n=1}^{\infty} \|\rho_n\| \leq \|f\|$ and $\|T_n\| \leq 1$ $(n=1,2,\cdots)$, it can be seen that B_M^* is in fact a complete normed linear space.

proposition are based on this.

5.1 PROPOSITION:

The unit ball $\{\phi \in B_M : \|\phi\| \leq 1\}$ is τ_M-compact.

Proof: To prove this we shall show that B_M is in fact isometrically isomorphic to $(B_M^*)^n$, so that τ_M is the weak-* topology on B_M, and the compactness of the unit ball then follows from [15] (p.228).

Define a map $\phi \to \phi^o$ from B_M to $(B_M^*)^n$ by $\phi^o(f) = f(\phi)$ $(f \in B_M^*)$. Clearly $\phi \to \phi^o$ is linear and $\|\phi^o\| \leq \|\phi\|$. ~~By the Hahn-Banach theorem,~~ By the footnote (†) we have $\|\phi^o\| = \|\phi\|$. ~~for each ϕ there exists $f \in B_M^*$ such that $\|f\| = 1$ and $f(\phi) = \|\phi\|$, hence $\|\phi^o\| = \|\phi\|$~~. Thus $\phi \to \phi^o$ is isometric and also one-one. It remains to show that it is onto.

Suppose $F \in (B_M^*)^n$. For each $\rho \in B(K)^{\sigma w}$ and each $T \in M$, define $f_{\rho,T}$ on B_M by $f_{\rho,T}(\phi) = \rho(\phi(T))$. By definition $f_{\rho,T} \in B_M^*$. Next define $\Pi_{F,T} \in (B(K)^{\sigma w})^n$ by $\Pi_{F,T}(\rho) = F(f_{\rho,T})$. It is easy to see that $\Pi_{F,T}$ is indeed linear and norm-continuous. As discussed before this proposition, there exists $R_{F,T} \in B(K)$ such that $\hat{R}_{F,T} = \Pi_{F,T}$. Define $\psi_F : M \to B(K)$ by $\psi_F(T) = R_{F,T}$. ψ_F is linear and

$$\|\psi_F(T)\| = \|R_{F,T}\| = \|\hat{R}_{F,T}\| = \|\Pi_{F,T}\| \leq \|F\| \, \|T\| \qquad (T \in M).$$

Thus $\psi_F \in B_M$. Finally, for each $f \in B_M^*$, we have a bounded sequence $\{T_n\}$ in M and a norm-summable sequence $\{\rho_n\}$ in $B(K)^{\sigma w}$ such that

$f(\phi) = \Sigma_{n=1}^{\infty} \rho_n(\phi(T_n))$ and so

$$\psi_F^o(f) = f(\psi_F) = \Sigma_{n=1}^{\infty} \rho_n(\psi_F(T_n)) = \Sigma_{n=1}^{\infty} \rho_n(R_{F,T_n})$$
$$= \Sigma_{n=1}^{\infty} \hat{R}_{F,T_n}(\rho_n) = \Sigma_{n=1}^{\infty} \Pi_{F,T_n}(\rho_n) = \Sigma_{n=1}^{\infty} F(f_{\rho_n,T_n}) = F(f).$$

Thus $\psi_F^o = F$ and so $\phi \to \phi^o$ is onto. Hence $(B_M^*)^n$ can be identified with B_M, and the result follows.

(†) For each $\delta > 0$, we may choose $T \in M$ with $\|T\| = 1$, $x \in K$ with $\|x\| = 1$ and $y = \varphi(T)x / \|\varphi(T)x\|$ such that $g_\delta \in B_M^*$, defined by $g_\delta(\psi) = (\psi(T)x, y)$, satisfies $|\varphi^o(g_\delta)| > \|\varphi\| \|g_\delta\| - 2\delta \|\varphi\|$. Thus $\|\varphi^o\| \geq \|\varphi\|$.

5.2 LEMMA:

If $\{\phi_\alpha\}$ is a net in B_M and $\phi_\alpha \to \phi_0$ (τ_M) , then $\phi_\alpha(T) \to \phi_0(T)$ (τ_W) for every $T \in M$. Conversely, if $\{\phi_\alpha\}$ is a bounded net in B_M and $\phi_\alpha(T) \to \phi_0(T)$ (τ_W) for every $T \in M$, then $\phi_\alpha \to \phi_0$ (τ_M) .

Proof: Suppose $\phi_\alpha \to \phi_0$ (τ_M) , and let U be a τ_W-neighbourhood of $\phi_0(T)$ for some $T\in M$, then there exist $k_1, \cdots, k_n, k_1', \cdots, k_n' \in K$ and $\delta > 0$ such that $V = \{R\in B(K) : |([R-\phi_0(T)]k_i, k_i')| < \delta \ (i=1, \cdots, n)\} \subseteq U$. Define linear functionals ρ_i on B(K) by $\rho_i(R) = (Rk_i, k_i')$ for $i=1,2,\cdots,n$. These are τ_W-continuous, and hence $\tau_{\sigma W}$-continuous. Thus $f_i(\phi) = \rho_i(\phi(T))$ $(\phi\in B_M)$ defines functionals $f_i \in B_M^*$. But $W = \{\phi\in B_M : |f_i(\phi-\phi_0)| < \delta \ (i=1,2,\cdots,n)\}$ is a τ_M-neighbourhood of ϕ_0, and so ϕ_α is eventually in W. Thus $\phi_\alpha(T)$ is eventually in V, hence in U, and so $\phi_\alpha(T) \to \phi_0(T)$ (τ_W) .

Conversely, suppose $\{\phi_\alpha\}$ is a bounded net and $\phi_\alpha(T) \to \phi_0(T)$ (τ_W) for every $T\in M$. Let U be a τ_M-neighbourhood of ϕ_0, then there exist $f_1, \cdots, f_m \in B_M^*$ and $\delta > 0$ such that $V = \{\phi\in B_M: |f_i(\phi-\phi_0)| < \delta \ (i=1, \cdots, m)\}$ satisfies $V \subseteq U$. Now $f_i(\phi) = \Sigma_{n=1}^{\infty} \rho_{in}(\phi(T_{in}))$ $(i=1, \cdots, m)$ with $\| T_{in}\| < \kappa$ for $i=1,2,\cdots,m$ and $n=1,2,\cdots$ and with $\rho_{in} \in B(K)^{\sigma W}$. Thus there exist $x_{ink} \in K$ and $y_{ink} \in K$ such that (see [5] p.38) $\rho_{in}(R) = \Sigma_{k=1}^{\infty}(Rx_{ink}, y_{ink})$ $(R\in B(K))$ and $\Sigma_{k=1}^{\infty} \| x_{ink}\|^2 < \infty$, $\Sigma_{k=1}^{\infty} \| y_{ink}\|^2 < \infty$. Hence

$$f_i(\phi) = \Sigma_{n=1}^{\infty} \Sigma_{k=1}^{\infty} (\phi(T_{in})x_{ink}, y_{ink}) \qquad (i=1, \cdots, m) .$$

Since $\{\phi_\alpha\}$ is a bounded net, there exist integers N and $Q_1, \cdots, Q_N$ such that for every α, including formally $\alpha = 0$, we have

$$|f_i(\phi_\alpha) - \Sigma_{n=1}^{N} \Sigma_{k=1}^{Q_n} (\phi_\alpha(T_{in})x_{ink}, y_{ink})| < \delta/4 \qquad (i=1, \cdots, m). \tag{1}$$

But for every T, $\phi_\alpha(T) \to \phi_0(T)$ (τ_W), and so

$$|f_i(\phi_\alpha) - \Sigma_{n=1}^{N} \Sigma_{k=1}^{Q_n} (\phi_o(T_{in})x_{ink}, y_{ink})| < \delta/2$$

for all sufficiently large α. Finally applying (1) to ϕ_o, we have

$$|f_i(\phi_\alpha) - f_i(\phi_o)| < \delta \qquad (i=1, \cdots, m)$$

for all sufficiently large α, and thus ϕ_α is eventually in V, and so in U. Hence $\phi_\alpha \to \phi_o \ (\tau_M)$

5.3 LEMMA:

$CP(A,B(K))|_M$ is a τ_M-closed convex cone in B_M, where $|_M$ denotes the restriction to M.

Proof: It is trivial to verify that $CP(A,B(K))$, and hence $CP(A,B(K))|_M$, is a convex cone.

Suppose $\phi_\alpha \in CP(A,B(K))$ and $\phi_\alpha \to \phi_o \ (\tau_A)$, then by lemma 5.2 applied to A, we have for every $T \in A$ that $\phi_\alpha(T) \to \phi_o(T) \ (\tau_W)$. Thus

$$\Sigma_{i,j=1}^{n} (\phi_\alpha(T_{ij})k_j, k_i) \quad \to \Sigma_{i,j=1}^{n} (\phi_o(T_{ij})k_j, k_i)$$

for every $n \in \underline{N}$, $T_{ij} \in A$ and $k_i \in K$, and so by lemma 4.2, $\phi_o \in CP(A,B(K))$. Thus $CP(A,B(K))$ is τ_A-closed in B_A, and by lemma 5.1 it follows that $CP(A,B(K)) \cap \{\phi \in B_A : \|\phi\| \leq r\}$ is τ_A-compact for every $r > 0$.

We show that restriction is τ_A to τ_M continuous. Let U be a τ_M-neighbourhood of $\phi_o|_M$ in B_M, then there exist $f_1, \cdots, f_n \in B_M^*$ such that $V = \{\phi \in B_M : |f_i(\phi - \phi_o|_M)| < \delta \ (i=1, \cdots, n)\} \subseteq U$ for some $\delta > 0$.
We may write, for each $i=1, \cdots, n$,

$$f_i(\phi) = \Sigma_{k=1}^{\infty} \rho_{ki}(\phi(T_{ki})) \qquad (\phi \in B_M)$$ with $T_{ki} \in M \subseteq A$ bounded, and $\rho_{ki} \in B(K)^{\sigma w}$: $\Sigma_{k=1}^{\infty} \|\rho_{ki}\| < \infty$.

Define $g_i \in B_A^*$ by

$$g_i(\psi) = \Sigma_{k=1}^{\infty} \rho_{ki}(\psi(T_{ki})) \qquad (\psi \in B_A),$$

then $W = \{\psi \in B_A : |g_i(\psi - \phi_o)| < \delta \ (i=1, \cdots, n)\}$ is a τ_A-neighbourhood of ϕ_0. But $g_i(\psi) = f_i(\psi|_M)$, so $W|_M = V \subseteq U$. Thus $\phi \to \phi|_M$ is τ_A to τ_M continuous, hence $[CP(A,B(K)) \cap \{\phi \in B_A: \|\phi\| \leq r\}]|_M$ is τ_M-closed for every $r>0$.

By corollary 4.4, since $I \in M$, we have that restriction is norm-preserving, and thus

$$[CP(A,B(K)) \cap \{\phi \in B_A: \|\phi\| \leq r\}]|_M = CP(A,B(K))|_M \cap \{\phi \in B_M: \|\phi\| \leq r\} .$$

The Krein-Smulian theorem ([7] p.429) now shows that $CP(A,B(K))|_M$ is τ_M-closed (recall that in proposition 5.1 we showed that τ_M is the weak-* topology on B_M).

5.4 LEMMA:

If g is a τ_M-continuous linear functional on B_M such that $g[CP(A,B(K))|_M] \geq 0$, then $g[CP(M,B(K))] \geq 0$.

Proof: Let J be the net of all finite rank projections $P \in B(K)$ directed in the increasing sense by the usual partial order. For $P \in J$ and $\phi \in B_M$, we define $P\phi P \in B_M$ by $(P\phi P)(T) = P\phi(T)P \quad (T \in M)$. Notice that if $\phi \in CP(M,B(K))$, then $P\phi P \in CP(M,B(K))$. Define also a linear functional g_P on B_M by $g_P(\phi) = g(P\phi P)$.

Now $P \to I_K \ (\tau_s)$, and multiplication is strongly continuous when one factor is in the unit ball, so $PXP \to X \ (\tau_s)$ for every $X \in B(K)$. In particular, $P\phi(T)P \to \phi(T)$ (τ_s, and hence τ_w) for each $T \in M$. But $\{P\phi P\}_{P \in J}$ is a bounded net, so by lemma 5.2, $P\phi P \to \phi \ (\tau_M)$ as $P \to I_K$. But g is τ_M-continuous, so $g(P\phi P) \to g(\phi)$, i.e.

$$g_P(\phi) \to g(\phi) \quad \text{as } P \to I_K \quad (\phi \in B_M) \quad \cdots\cdots\cdots\cdots\cdots\cdots (1).$$

Fix $P \in J$ and let n be the rank of P with $k_1, \cdots, k_n$ an orthonormal basis for PK . Define $R_{ij} \in B(K)$ by

$$R_{ij}(\lambda_1 k_1 + \cdots + \lambda_n k_n) = \lambda_j k_i \quad (i,j=1,\cdots,n)$$

$$\text{and} \quad R_{ij}(k) = 0 \quad \text{for } k\in(I-P)K \ .$$

For each i, j and each bounded linear functional $f \in M'$, define $\psi_{ij}(f) \in B_M$ by

$$[\psi_{ij}(f)](T) = f(T)R_{ij} \qquad (T\in M) \ .$$

Also define $F_{ij} \in (M')'$ by

$$F_{ij}(f) = g(\psi_{ij}(f)) \qquad (f\in M') \ .$$

Let $f_\nu \in M'$ be a net such that $\|f_\nu\| \leq 1$ and $f_\nu \to f$ (weak-*), then

$$f_\nu(T)R_{ij} \to f(T)R_{ij} \quad (\tau_W) \qquad (T\in M) \ .$$

Thus by lemma 5.2

$$\psi_{ij}(f_\nu) \to \psi_{ij}(f) \quad (\tau_M) \ .$$

But g is τ_M-continuous, so $F_{ij}(f_\nu) \to F_{ij}(f)$, which shows that each F_{ij} is weak-* continuous on the unit ball of M', and hence by the Krein-Smulian theorem ([7] p.429) it is weak-* continuous on M'. Thus there exists $T_{ij} \in M$ such that $f(T_{ij}) = F_{ij}(f)$ $(f\in M')$ ([10] p.155).

For $\phi\in B_M$, define $f_{ij} \in M'$ by $f_{ij}(T) = (\phi(T)k_j, k_i)$ $(T\in M)$, and let P_i denote the projection of K onto the linear span of $\{k_i\}$, then

$$P_i\phi(T)P_j = (\phi(T)k_j,k_i)R_{ij} = f_{ij}(T)R_{ij} = [\psi_{ij}(f_{ij})](T) \ .$$

Thus

$$g_P(\phi) = g(P\phi P) = \Sigma_{i,j=1}^n g(P_i\phi P_j) = \Sigma_{i,j=1}^n g[\psi_{ij}(f_{ij})]$$
$$= \Sigma_{i,j=1}^n F_{ij}(f_{ij}) = \Sigma_{i,j=1}^n f_{ij}(T_{ij}) = \Sigma_{i,j=1}^n (\phi(T_{ij})k_j,k_i) \quad \cdots\cdots\cdots(2) \ .$$

Choose any $h_1,\cdots,h_n \in H$, the Hilbert space underlying M and A. Define $V : K \to H$ by $Vk_i = h_i$ $(i=1,\cdots,n)$ and $Vk = 0$ $(k\in(I-P)K)$.

Then

$$\Sigma_{i,j=1}^n (T_{ij}h_j,h_i) = \Sigma_{i,j=1}^n (V^*T_{ij}Vk_j,k_i) = g_P(V^*\omega|_M V) \quad \text{by (2)},$$

where ω is the identity representation of A on H. Thus

$$\Sigma_{i,j=1}^{n}(T_{ij}h_j,h_i) = g(PV^*\omega|_M VP) = g(V^*\omega|_M V) \geq 0 \quad \cdots\cdots(3)$$

since by theorem 4.3 we have $V^*\omega V \in CP(A,B(K))$.

Suppose $\phi \in CP(M,B(K))$. By (3), $\underset{\sim}{T} = \Sigma_{i,j=1}^{n} T_{ij} \otimes E_{ij} \geq 0$, and thus $\phi_n(\underset{\sim}{T}) \geq 0$. Hence

$$\Sigma_{i,j=1}^{n}(\phi(T_{ij})k_j,k_i) \geq 0 \text{ .}$$

Finally, by (2), $g_P(\phi) \geq 0$ ($P \in J$) and so by (1), $g(\phi) \geq 0$.

Arveson's extension theorem now follows:-

5.5 THEOREM:

Let M be a norm-closed self-adjoint linear manifold in a C*-algebra A such that $I_H \in M \subseteq A \subseteq B(H)$, and let ϕ be in $CP(M,B(K))$, then there exists $\psi \in CP(A,B(K))$ such that $\psi|_M = \phi$.

Proof: Suppose on the contrary that $\phi \in CP(M,B(K))$ but $\phi \notin CP(A,B(K))|_M$. By lemma 5.3, $CP(A,B(K))|_M$ is a τ_M-closed convex cone in B_M , which is a locally convex Hausdorff topological linear space under τ_M . The strong separation theorem ([10] p.118) guarantees that there is a τ_M-continuous linear functional f on B_M such that

$$\text{Re } f(CP(A,B(K))|_M) \geq 0 \quad \text{and} \quad \text{Re } f(\phi) < 0 \quad \cdots\cdots\cdots\cdots(1) \text{ .}$$

Define an involution $\tilde{}$ on B_M by $\psi^\sim(T) = \psi(T^*)^*$ $(T \in M,\ \psi \in B_M)$. Suppose $\xi \in CP(M,B(K))$. For $T=T^*$ $\|T\|\, I_H$ and $\|T\|\, I_H - T$ are both positive, and thus $\xi(T) = \xi(\|T\|\, I_H) - \xi(\|T\|\, I_H - T)$ is a difference of positive elements, and so is self-adjoint. Hence for any $S \in M$

$$\xi^\sim(S)=\xi^\sim(\text{Re}S + i.\text{Im}S)=\xi(\text{Re}S - i.\text{Im}S)^*=\xi(\text{Re}S)^* + i\xi(\text{Im}S)^*=\xi(S) \text{ .}$$

Thus $\xi \in CP(M,B(K))$ implies $\xi^\sim = \xi$.

Next we show that $\psi \to \tilde{\psi}$ is τ_M-continuous. A basic τ_M neighbourhood of $\tilde{\psi_0}$ is $V = \{\psi \in B_M : |f_i(\psi - \tilde{\psi_0})| < \delta \ (i=1,\cdots,m)\}$ for $f_1,\cdots,f_n \in B_M^*$ and $\delta > 0$. By definition of $B_M{}^*$, there exist bounded sequences $\{T_{in}\}_{n=1}^{\infty}$ in M and norm-summable sequences $\{\rho_{in}\}_{n=1}^{\infty}$ in $B(K)^{\sigma w}$ such that

$$f_i(\psi) = \Sigma_{n=1}^{\infty} \rho_{in}(\psi(T_{in})) \quad (i=1,\cdots,m) .$$

For each i,n define ρ_{in}^* on B(K) by $\rho_{in}^*(R) = \rho_{in}(R^*) \quad (R \in B(K))$. The map $R \to R^*$ is $\tau_{\sigma w}$-continuous, so $\rho_{in}^* \in B(K)^{\sigma w}$. Thus we may define $g_i \in B_M^*$ by

$$g_i(\psi) = \Sigma_{n=1}^{\infty} \rho_{in}^*(\psi(T_{in}^*)) \quad (i=1,\cdots,m ; \psi \in B_M)$$

Let $W = \{\psi \in B_M : |g_i(\psi - \psi_0)| < \delta \ (i=1,\cdots,m)\}$, which is a τ_M-neighbourhood of ψ_0. But $f_i(\tilde{\psi}) = g_i(\psi)$, so $\tilde{W} = V$, and thus $\psi \to \tilde{\psi}$ is τ_M-continuous.

Define g on B_M by $g(\psi) = \frac{1}{2}[f(\psi) + f(\tilde{\psi})^*]$. Since f and $\psi \to \tilde{\psi}$ are τ_M-continuous, we have that g is τ_M-continuous. If $\xi \in CP(M,B(K))$ then as above $\tilde{\xi} = \xi$, and so $g(\xi) = \operatorname{Re} f(\xi)$. But by (1), $\xi \in CP(M,B(K))$ implies $\operatorname{Re} f(\xi) \geq 0$, and also $CP(A,B(K))|_M \subseteq CP(M,B(K))$, hence $g\{CP(A,B(K))|_M\} \geq 0$. Finally, by lemma 5.4, we have that $g\{CP(M,B(K))\} \geq 0$, and thus $\operatorname{Re} f(\phi) = g(\phi) \geq 0$. This contradicts (1), and so the theorem is proved.

5.6 COROLLARY:

Let N be a linear manifold in A with $I_H \in N$, and let $\phi : N \to B(K)$ be a completely contractive linear map such that $\phi(I_H) = I_K$, then there exists $\psi \in CP(A,B(K),I_K)$ such that $\psi|_N = \phi$.

Proof: For any $k \in K$ with $\| k \| = 1$, define $f_k : N \to \underline{C}$ by $f_k(T) = (\phi(T)k, k)$, then f_k is a bounded linear functional and $\| f_k \| = f_k(I_H) = 1$. By the Hahn-Banach theorem, there is a bounded linear functional $g_k : A \to \underline{C}$ such that $g_k|_N = f_k$ and $\| g_k \| = g_k(I_H) = 1$. Hence g_k is positive (see [6] p.25) and so by proposition 4.1 we have $g_k(T^*) = g_k(T)^*$ $(T \in A)$.

Let M denote the norm closure in A of $N + N^*$. Define θ from $N + N^*$ into $B(K)$ by $\theta(X + Y^*) = \phi(X) + \phi(Y)^*$ $(X, Y \in N)$. For $R \in B(K)$, the numerical radius $w(R)$ satisfies $\frac{1}{2}\| R \| \leq w(R) \leq \| R \|$ ([9] p.114) and hence

$$\begin{aligned} \|\theta(X + Y^*)\| &\leq 2 \sup\{ |(\theta(X + Y^*)k, k)| : \| k \| = 1 \} \\ &= 2 \sup\{ |(\phi(X)k, k) + (\phi(Y)k, k)^*| : \| k \| = 1 \} \\ &= 2 \sup\{ |g_k(X + Y^*)| : \| k \| = 1 \} \\ &\leq 2 \| X + Y^* \| . \end{aligned}$$

Thus by continuity we can extend θ to $\chi : M \to B(K)$. We show that χ is completely positive.

Choose $n \in \underline{N}$, and for any $\underset{\sim}{k} = \Sigma_{i=1}^{n} k_i \otimes e_i \in K \otimes \underline{C}^n$ with $\| \underset{\sim}{k} \| = 1$, define $f_{\underset{\sim}{k}} : N \otimes B(\underline{C}^n) \to \underline{C}$ by

$$f_{\underset{\sim}{k}}(\underset{\sim}{T}) = (\phi_n(\underset{\sim}{T})\underset{\sim}{k}, \underset{\sim}{k}) \qquad (\underset{\sim}{T} \in N \otimes B(\underline{C}^n)) .$$

Since ϕ is completely contractive and $\phi(I_H) = I_K$, we have $\| f_{\underset{\sim}{k}} \| = 1$ and $f_{\underset{\sim}{k}}(\underset{\sim}{I}_H) = 1$, where $\underset{\sim}{I}_H$ and $\underset{\sim}{I}_K$ denote respectively the identities of $B(H) \otimes B(\underline{C}^n)$ and $B(K) \otimes B(\underline{C}^n)$. As above, there exists a positive $*$-preserving linear functional $g_{\underset{\sim}{k}}$ on $A \otimes B(\underline{C}^n)$ with $\|g_{\underset{\sim}{k}}\| = g_{\underset{\sim}{k}}(\underset{\sim}{I}_H) = 1$, $g_{\underset{\sim}{k}}|_{N \otimes B(\underline{C}^n)} = f_{\underset{\sim}{k}}$. For $\underset{\sim}{X}, \underset{\sim}{Y} \in N \otimes B(\underline{C}^n)$,

$$(\chi_n(\underset{\sim}{X} + \underset{\sim}{Y}^*)\underset{\sim}{k}, \underset{\sim}{k}) = g_{\underset{\sim}{k}}(\underset{\sim}{X} + \underset{\sim}{Y}^*)$$

and so χ_n is positive on $(N + N^*) \otimes B(\underline{C}^n)$, hence on $M \otimes B(\underline{C}^n)$. Thus χ is completely positive, so $\chi \in CP(M, B(K))$ and $\chi|_N = \phi$. The result now follows from theorem 5.5 .

A further corollary shows that there are always completely positive maps which preserve the identity.

5.7 COROLLARY:

For any Hilbert space H and any $n \in \underline{N}$, $CP(B(H),B(\underline{C}^n),I_n)$ is not empty.

Proof: Let $M = \{\lambda I_H : \lambda \in \underline{C}\}$, then M is a norm-closed self-adjoint linear manifold in B(H), and so by theorem 5.5 we need only find $\phi \in CP(M,B(\underline{C}^n),I_n)$ to prove this corollary. Let m be the dimension of H.

If $m \geqslant n$, define $\phi : M \to B(\underline{C}^n)$ by $\phi(\lambda I_H) = \lambda I_n$. We can write $H = \underline{C}^n \oplus L$, for some Hilbert space L, and then given $\zeta_1, \cdots, \zeta_m \in \underline{C}^n$ we have $h_i = \zeta_i \oplus 0_L \in H$ with

$$\Sigma_{i,j=1}^{m}(\phi(\lambda_{ij}I_H)\zeta_j,\zeta_i) = \Sigma_{i,j=1}^{m}(\lambda_{ij}I_H h_j,h_i) \quad (\lambda_{ij} \in \underline{C}) .$$

An application of lemma 4.2 shows that ϕ is completely positive, and so $\phi \in CP(M,B(\underline{C}^n),I_n)$.

If $m < n$, then there are non-negative integers p and q such that $n = pm + q$ and $q < m$. Let $H_1 = \cdots = H_p = H$, then we can write $\underline{C}^n = H_1 \oplus \cdots \oplus H_p \oplus \underline{C}^q$ and then $B(H_1)\oplus \cdots \oplus B(H_p) \oplus B(\underline{C}^q) \subseteq B(C^n)$. By the first part, there exists $\phi_0 \in CP(B(H),B(\underline{C}^q),I_q)$. Define $\phi : M \to B(\underline{C}^n)$ by $\phi(\lambda I_H) = \lambda I_{H_1} \oplus \cdots \oplus \lambda I_{H_p} \oplus \phi_0(\lambda I_H)$, then we have $\phi \in CP(M,B(\underline{C}^n),I_n)$, which completes the proof.

We now show that there is a very useful relation between the completely positive maps which preserve the identity and those which fail to do so.

5.8 THEOREM:

Suppose $\phi \in CP(B(H),B(\underline{C}^n))$, then there exist $U \in B(\underline{C}^n)$ and $\theta \in CP(B(H),B(\underline{C}^n),I_n)$ such that $\phi(T) = U^*\theta(T)U \quad (T\in B(H))$.

Proof: Since ϕ is positive and I_H is positive, $\phi(I_H)$ is positive and so there exists $U \in B(\underline{C}^n)$ such that $U = U^*$ and $\phi(I_H) = U^2$. Let $M = U\underline{C}^n$, then M is a closed linear manifold in $\underline{C}^n$.

By theorem 4.3, there is a representation π of B(H) on K' and a bounded linear map $V : \underline{C}^n \to K'$ such that $\phi(T) = V^*\pi(T)V \quad (T\in B(H))$ and $\pi(I_H) = I_{K'}$. For $\underset{\sim}{\lambda} \in \underline{C}^n$ and $T \in B(H)$,

$$\|\phi(T)\underset{\sim}{\lambda}\|^2 = \|V^*\pi(T)V\underset{\sim}{\lambda}\|^2 \leq \|V^*\|^2\|\pi(T)\|^2(V^*V\underset{\sim}{\lambda},\underset{\sim}{\lambda})$$
$$= \|V^*\|^2\|\pi(T)\|^2(\phi(I_H)\underset{\sim}{\lambda},\underset{\sim}{\lambda}) \quad .$$

Thus $\operatorname{Ker}\phi(I_H) \subseteq \operatorname{Ker}\phi(T)$ and so $[\operatorname{Ker}\phi(T)]^{\perp} \subseteq [\operatorname{Ker}\phi(I_H)]^{\perp}$ $\cdots\cdots(1)$.

Suppose $\underset{\sim}{\mu} \in \operatorname{Ker}\phi(T^*)$ and $\underset{\sim}{\lambda} \in \underline{C}^n$, then by proposition 4.1

$$0 = (\underset{\sim}{\lambda},\phi(T^*)\underset{\sim}{\mu}) = (\underset{\sim}{\lambda},\phi(T)^*\underset{\sim}{\mu}) = (\phi(T)\underset{\sim}{\lambda},\underset{\sim}{\mu}) \quad .$$

Thus by (1), $\phi(T)\underline{C}^n \subseteq [\operatorname{Ker}\phi(T^*)]^{\perp} \subseteq [\operatorname{Ker}\phi(I_H)]^{\perp}$ $\cdots\cdots\cdots\cdots\cdots(2)$.

Next suppose $\underset{\sim}{\lambda} \in [\phi(T^*)\underline{C}^n]^{\perp}$ and $\underset{\sim}{\mu} \in \underline{C}^n$, then again by proposition 4.1, $(\phi(T)\underset{\sim}{\lambda},\underset{\sim}{\mu}) = (\underset{\sim}{\lambda},\phi(T^*)\underset{\sim}{\mu}) = 0$, and so $\underset{\sim}{\lambda} \in \operatorname{Ker}\phi(T)$. Thus $[\phi(T^*)\underline{C}^n]^{\perp} \subseteq \operatorname{Ker}\phi(T)$, and hence $[\operatorname{Ker}\phi(T)]^{\perp} \subseteq \phi(T^*)\underline{C}^n$. Apply this to $T = I_H$, then (2) becomes

$$\phi(T)\underline{C}^n \subseteq \phi(I_H)\underline{C}^n = U^2\underline{C}^n = UM \qquad (T\in B(H)) \quad \cdots\cdots\cdots(3) .$$

Let $e_1,\cdots,e_n$ be the usual basis for $\underline{C}^n$. By (3), for each $T\in B(H)$, there exist $\mu_i(T) \in M$ such that $\phi(T)e_i = U\mu_i(T) \quad (i=1,\cdots,n)$, and we may choose $\mu_i(I_H) = Ue_i$ - any fixed choice is taken for the other $\mu_i(T)$.

Define $\psi : B(H) \to B(M)$ by $\psi(T)(Ue_i) = \mu_i(T)$ $(i=1,\cdots,n; T\in B(H))$. The elements $Ue_1,\cdots,Ue_n$ span M, so $\psi(T)$ is defined on M. Also if $\Sigma_{i=1}^n \lambda_i Ue_i = \Sigma_{j=1}^n \xi_j Ue_j$, then since $\phi(I_H) = U^2$ we have

$$\phi(I_H)[\Sigma_{i=1}^n(\lambda_i-\xi_i)e_i] = U[\Sigma_{i=1}^n(\lambda_i-\xi_i)Ue_i] = 0 .$$

By (1), $\quad 0 = \phi(T)[\Sigma_{i=1}^n(\lambda_i-\xi_i)e_i] = U[\Sigma_{i=1}^n(\lambda_i-\xi_i)\mu_i(T)]$.

Thus $[\Sigma_{i=1}^n \lambda_i\mu_i(T) - \Sigma_{j=1}^n \xi_j\mu_j(T)] \in \text{Ker } U \cap U\underline{C}^n = \{0\}$, since U is self-adjoint. Hence ψ is well defined.

Now $U\mu_i(T + S) = \phi(T+S)e_i = \phi(T)e_i+\phi(S)e_i = U\mu_i(T) + U\mu_i(S)$, and so $[\mu_i(S+T) - \mu_i(S) - \mu_i(T)] \in \text{Ker } U \cap U\underline{C}^n$. Thus ,as above, we have $\mu_i(T+S) = \mu_i(T) + \mu_i(S)$. Similarly $\mu_i(\lambda T) = \lambda\mu_i(T)$ and thus ψ is linear. Also

$$(\psi(T)Ue_i,Ue_j) = (\mu_i(T),Ue_j) = (U\mu_i(T),e_j) = (\phi(T)e_i,e_j) ,$$

and thus for $\underset{\sim}{\lambda},\underset{\sim}{\xi} \in \underline{C}^n$ and $T \in B(H)$ we have

$$(\psi(T)U\underset{\sim}{\lambda},U\underset{\sim}{\xi}) = (\phi(T)\underset{\sim}{\lambda},\underset{\sim}{\xi}) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(4) .$$

Proposition 4.2, the complete positivity of ϕ and (4) show that ψ is completely positive. By choice, $\mu_i(I_H) = Ue_i$, so $\psi(I_H) = I_M$, thus $\psi \in CP(B(H),B(M),I_M)$.

Now U is self-adjoint, so $\underline{C}^n = U\underline{C}^n \oplus \text{Ker } U = M \oplus \text{Ker } U$. If $\text{Ker } U =\{0\}$, then $M = \underline{C}^n$ and $\theta = \psi$ fulfills the requirements of the corollary. If $\text{Ker } U \neq \{0\}$, then it is isometrically (linear space) isomorphic to $\underline{C}^k$ for some $k\in\underline{N}$, and so by corollary 5.7 there exists $\chi \in CP(B(H),B(\text{KerU}),I_{\text{KerU}})$. Define $\theta : B(H) \to B(\underline{C}^n)$ by $\theta = \psi \oplus \chi$, then $\theta \in CP(B(H),B(\underline{C}^n),I_n)$. Finally, for $\underset{\sim}{\lambda},\underset{\sim}{\xi} \in \underline{C}^n$, $T\in B(H)$,

$$(U^*\theta(T)U\underset{\sim}{\lambda}, \underset{\sim}{\xi}) = (\theta(T)U\underset{\sim}{\lambda},U\underset{\sim}{\xi}) = (\psi(T)U\underset{\sim}{\lambda},U\underset{\sim}{\xi}) = (\phi(T)\underset{\sim}{\lambda},\underset{\sim}{\xi}) \text{ by } (4) .$$

§6. ALGEBRAIC MATRIX RANGES

In this section we define the algebraic matrix range and obtain most of its elementary properties, merely expanding the account given in [2] . We deliberately omit a proposition concerning the algebraic matrix range of a normal operator, which Arveson deduces from a characterization of the extreme points of the cone of completely positive maps. An easier proof will be given in §11 .

Let $A \subseteq B(H)$ denote as usual a C*-algebra. For $T \in A$ and $n \in \underline{N}$, we define the n^{th} algebraic matrix range of T in A to be

$$V_n(A,T) = \{\phi(T) : \phi \in CP(A,B(\underline{C}^n),I_n)\} .$$

Notice that this is a subset of the bounded linear operators on $\underline{C}^n$. Arveson considered matrix representations of these operators, hence the term "matrix range". He studied the specific range $V_n(C^*(T),T)$. However, in view of his extension theorem (5.5), if $T \in A \subseteq B$ with A and B both C*-algebras, then

$$V_n(A,T) = V_n(B,T) \qquad (n=1,2,\cdots) .$$

Hence we may refer to the n^{th} algebraic matrix of an operator T, and may denote it $V_n(T)$.

We show that algebraic matrix ranges do generalise the algebraic numerical range. Recall that $D(A,I)$ denotes the set of normalized states on A, and the algebraic numerical range of $T \in B(H)$ is

$$V(T) = \{f(T) : f \in D(B(H),I)\} ,$$

so that $V(T) \subseteq \underline{C}$. The map $\chi : \underline{C} \to B(\underline{C})$ given by $\chi(\lambda)\mu = \lambda\mu$ is an isometric isomorphism, and for $R \in B(\underline{C})$, $\chi^{-1}(R) = R(1)$.

6.1 PROPOSITION:

Let T be in $B(H)$, then $\chi(V(T)) = V_1(T)$.

Proof: Suppose $R \in V_1(T)$, then $R = \phi(T)$ with $\phi \in CP(B(H), B(\underline{C}), I_1)$. Define $f : B(H) \to \underline{C}$ by $f(S) = \phi(S)(1)$. Since ϕ is positive, f is a positive linear functional, and $f(I_H) = 1$. By [6] (p.25) $\|f\| = 1$ and so $f \in D(B(H), I)$. Thus $R = \chi(R(1)) = \chi(f(T)) \in \chi(V(T))$.

Conversely, suppose $\lambda \in V(T)$, then $\lambda = f(T)$ with $f \in D(B(H), I)$. Define $\phi : B(H) \to B(\underline{C})$ by $\phi(S) = \chi(f(S))$. Clearly ϕ is a positive linear map and $\phi(I_H) = I_1$, $\phi(T) = \chi(\lambda)$. It remains only to show that ϕ is completely positive. Suppose $n \in \underline{N}$ and $\underset{\sim}{T} \in B(H) \otimes B(\underline{C}^n)$ is positive, then $\underset{\sim}{T} = \underset{\sim}{S}^*\underset{\sim}{S} = \Sigma_{i,j,k=1}^{n} S_{ki}^* S_{kj} \otimes E_{ij}$. Thus for $\underset{\sim}{\mu} = \Sigma_{i=1}^{n} \mu_i \otimes e_i \in \underline{C} \otimes \underline{C}^n$, we have

$$\begin{aligned}(\phi_n(\underset{\sim}{T})\underset{\sim}{\mu}, \underset{\sim}{\mu}) &= \Sigma_{i,j,k=1}^{n} (\phi(S_{ki}^* S_{kj})\mu_j, \mu_i) \\ &= \Sigma_{i,j,k=1}^{n} f(S_{ki}^* S_{kj})\mu_j \mu_i^* \\ &= f[\Sigma_{k=1}^{n} (\Sigma_{i=1}^{n} \mu_i S_{ki})^* (\Sigma_{j=1}^{n} \mu_j S_{kj})] \\ &\geqslant 0\end{aligned}$$

since f is positive and the argument is a sum of positive elements, hence is positive itself. Thus ϕ is completely positive.

In the proof of this last proposition we showed that a positive linear map of $B(H)$ into $B(\underline{C})$ is completely positive. This is part of a more general result, that if either A or B is commutative and ϕ is a positive linear map of A into B, then ϕ is completely positive (see [14]).

We proceed to demonstrate the elementary properties of the algebraic matrix range, most of which are natural generalisations of the numerical range situation.

6.2 PROPOSITION:

For $T \in B(H)$ and $n \in \underline{N}$, $V_n(T)$ is a compact set in $B(\underline{C}^n)$ and is contained in the ball of radius $\|T\|$.

Proof: By corollary 4.4, $\phi \in CP(B(H),B(\underline{C}^n),I_n)$ implies $\|\phi\| = 1$, and hence $V_n(T)$ is contained in the ball of radius $\|T\|$.

By lemma 5.3 applied to the case $M = A = B(H)$, $CP(B(H),B(\underline{C}^n))$ is $\tau_{B(H)}$-closed. Suppose $\{\phi_\alpha\}$ is a net in $CP(B(H),B(\underline{C}^n),I_n)$ and $\phi_\alpha \to \psi$ $(\tau_{B(H)})$, then $\psi \in CP(B(H),B(\underline{C}^n))$ and also by lemma 5.2, $\phi_\alpha(I_H) \to \psi(I_H)$ (τ_w) so $\psi(I_H) = I_n$. Thus $CP(B(H),B(\underline{C}^n),I_n)$ is a $\tau_{B(H)}$-closed subset of the unit ball in $B_{B(H)}$, and so by lemma 5.1 it is $\tau_{B(H)}$-compact.

Finally, by lemma 5.2, the restriction of the map $\phi \to \phi(T)$ to the unit ball in $B_{B(H)}$ is $\tau_{B(H)}$ to τ_w continuous, and so $V_n(T)$, being the image of a compact set by a continuous map, is compact.

The numerical range $V(T)$ is convex, and this generalises to $V_n(T)$. However, a stronger convexity property holds. We define the stronger concept as follows :- a subset $M \subseteq B(\underline{C}^n)$ is n-convex if for every subset $\{R_j : j=1,\cdots,N\} \subseteq M$ and every subset $\{U_j : j=1,\cdots,N\} \subseteq B(\underline{C}^n)$ such that $\Sigma_{j=1}^N U_j^* U_j = I_n$ we have $\Sigma_{j=1}^N U_j^* R_j U_j \in M$. We shall denote by $co_n(M)$ the set of all elements in the form $\Sigma_{j=1}^N U_j^* R_j U_j$ for some positive integer N, $R_j \in M$ and $U_j \in B(\underline{C}^n)$ with $\Sigma_{j=1}^N U_j^* U_j = I_n$.

Notice that if M is bounded, say contained in the ball of radius k, and $\{R_j : j=1,2,\cdots\} \subseteq M$, $\{U_j : j=1,2,\cdots\} \subseteq B(\underline{C}^n)$ is such that $\Sigma_{j=1}^\infty U_j^* U_j = I_n$, then $\Sigma_{j=1}^\infty U_j^* R_j U_j \in [co_n(M)]^-$. To see this, note that given $\delta > 0$, there exists an integer N such that

$$0 \leq I_n - \Sigma_{j=1}^{N} U_j^* U_j \; < \delta I_n/2k \, ,$$

and [redacted] $$\| (\Sigma_{j=1}^{\infty} U_j^* R_j U_j) - (\Sigma_{i=1}^{N} U_i^* R_i U_i) \| < \delta/2 \quad .$$

Let $X = (I_n - \Sigma_{j=1}^{N} U_j^* U_j)^{\frac{1}{2}}$, then $(\Sigma_{j=1}^{N} U_j^* U_j) + X^*X = I_n$ and

$$\begin{aligned} \| (\Sigma_{j=1}^{\infty} U_j^* R_j U_j) - [(\Sigma_{i=1}^{N} U_i^* R_i U_i) + X^* R_{N+1} X] \| & \\ < \delta/2 \; + \| R_{N+1} \| \, \| X^* \| \, \| X \| & \\ \leq \delta/2 \; + k \| X^* X \| & \\ < \delta \quad . & \end{aligned}$$

6.3 PROPOSITION:

For $T \in B(H)$ and $n \in \underline{N}$, $V_n(T)$ is n-convex.

Proof: Suppose $\{R_j : j=1, \cdots, N\} \subseteq V_n(T)$ and $\{U_j : j=1, \cdots, N\} \subseteq B(\underline{C}^n)$ with $\Sigma_{j=1}^{N} U_j^* U_j = I_n$, then there exist $\phi_j \in CP(B(H), B(\underline{C}^n), I_n)$ such that $R_j = \phi_j(T)$. Define $\psi : B(H) \to B(\underline{C}^n)$ by

$$\psi(S) = \Sigma_{j=1}^{N} U_j^* \phi_j(S) U_j \qquad (S \in B(H)) \; .$$

Clearly ψ is linear and $\psi(I_H) = I_n$. Also using lemma 4.2 it is easy to show that ψ is completely positive. Hence $\psi(T) \in V_n(T)$.

By proposition 6.2, $V_n(T)$ is closed and bounded, so infinite n-convex combinations of its elements will still be in $V_n(T)$.

A curious property of the sequence of algebraic ranges, which has no counterpart in numerical range theory, is that of coherence, as described in the next result.

6.4 PROPOSITION:

If $R \in V_n(T)$, then $V_k(R) \subseteq V_k(T)$ $\quad (k \in \underline{N},\ n \in \underline{N},\ T \in B(H))$.

Proof: There exists $\phi \in CP(B(H), B(\underline{C}^n), I_n)$ such that $\phi(T) = R$. Suppose $S \in V_k(R)$, then there exists $\psi \in CP(B(\underline{C}^n), B(\underline{C}^k), I_k)$ such that $\psi(R) = S$. Define $\theta : B(H) \to B(\underline{C}^k)$ by $\theta(X) = \psi(\phi(X))$. Clearly θ is linear and $\theta(I_H) = I_k$. Also the subordinate maps satisfy $\theta_r = \psi_r \circ \phi_r$, so θ is completely positive and $S = \theta(T) \in V_k(T)$.

The next proposition shows that $V_n(T)$ is invariant under unitary transformation of T . We shall show in §7 that for certain classes of operators, the sequence $\{V_n(T)\}_{n=1}^{\infty}$ forms a complete unitary invariant.

6.5 PROPOSITION:

For $T \in B(H)$, $n \in \underline{N}$ and $U \in B(H)$ a unitary operator, we have

$$V_n(U^*TU) = V_n(T) .$$

Proof: Suppose $R \in V_n(T)$, then $R = \phi(T)$ for some $\phi \in CP(B(H), B(\underline{C}^n), I_n)$. Define $\psi : B(H) \to B(\underline{C}^n)$ by $\psi(S) = \phi(USU^*)$. Notice that

$$\Sigma_{i,j=1}^{k} US_{ij}U^* \otimes E_{ij} = (\Sigma_{r=1}^{k} U \otimes E_{rr})(\Sigma_{i,j=1}^{k} S_{ij} \otimes E_{ij})(\Sigma_{m=1}^{k} U \otimes E_{mm})^*$$

and hence complete positivity of ψ follows easily . Also ψ is linear and $\psi(I_H) = \phi(U^*U) = \phi(I_H) = I_n$, so $\psi \in CP(B(H), B(\underline{C}^n), I_n)$. Thus $R = \phi(T) = \phi(UU^*TUU^*) = \psi(U^*TU) \in V_n(U^*TU)$, and so $V_n(T) \subseteq V_n(U^*TU)$ for every unitary $U \in B(H)$ and every $T \in B(H)$. Applying this to U^* and U^*TU gives $V_n(U^*TU) \subseteq V_n(UU^*TUU^*) = V_n(T)$.

If an operator can be split into direct summands, then its algebraic matrix range depends in a simple way on the matrix ranges of the summands, as shown by the next result.

6.6 THEOREM:

Let J be a countable index set, and for each $j \in J$ let H_j be a Hilbert space with $T_j \in B(H_j)$. Let $\underset{\sim}{H} = \oplus_{j \in J} H_j$, and define $\underset{\sim}{T} \in B(\underset{\sim}{H})$ by $\underset{\sim}{T} = \oplus_{j \in J} T_j$, then for $n \in \underline{N}$

$$V_n(\underset{\sim}{T}) = \overline{co}_n[\cup_{j \in J} V_n(T_j)]$$

where $\overline{co}_n$ denotes the closure of the n-convex hull.

Proof: Suppose $R \in V_n(\underset{\sim}{T})$, then $R = \phi(\underset{\sim}{T})$ for some $\phi \in CP(B(\underset{\sim}{H}), B(\underline{C}^n), I_n)$. For each $j \in J$, define $\phi_j : B(H_j) \to B(\underline{C}^n)$ by defining for $S \in B(H_j)$ an element $\underset{\sim}{S}_j \in B(\underset{\sim}{H})$ by

$$(\underset{\sim}{S}_j \underset{\sim}{h})(i) = \begin{cases} S(\underset{\sim}{h}(i)) & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad (\underset{\sim}{h} \in \underset{\sim}{H},\ i \in J),$$

and then setting

$$\phi_j(S) = \phi(\underset{\sim}{S}_j).$$

Suppose $(\Sigma_{p,q=1}^{r} S_{pq} \otimes E_{pq}) \in B(H_j) \otimes B(\underline{C}^r)$ is positive. Given $\underset{\sim}{\ell}_1, \cdots, \underset{\sim}{\ell}_r \in \underset{\sim}{H}$, we have

$$\Sigma_{p,q=1}^{r} ((\underset{\sim}{S}_{pq})_j \underset{\sim}{\ell}_q, \underset{\sim}{\ell}_p) = \Sigma_{p,q=1}^{r} (S_{pq}(\underset{\sim}{\ell}_q(j)), \underset{\sim}{\ell}_p(j)) \geq 0,$$

where $(\underset{\sim}{S}_{pq})_j$ is defined in the same way as $\underset{\sim}{S}_j$. Thus $\Sigma_{p,q=1}^{r} (\underset{\sim}{S}_{pq})_j \otimes E_{pq}$ is a positive element of $B(\underset{\sim}{H}) \otimes B(\underline{C}^r)$. Hence given $\underset{\sim}{\lambda}_1, \cdots, \underset{\sim}{\lambda}_r \in \underline{C}^n$, we have

$$\Sigma_{p,q=1}^{r} (\phi_j(S_{pq}) \underset{\sim}{\lambda}_q, \underset{\sim}{\lambda}_p) = \Sigma_{p,q=1}^{r} (\phi[(\underset{\sim}{S}_{pq})_j] \underset{\sim}{\lambda}_q, \underset{\sim}{\lambda}_p) \geq 0$$

using lemma 4.2. Thus $\phi_j \in CP(B(H_j), B(\underline{C}^n))$ for each $j \in J$. By theorem 5.8 there exist $U_j \in B(\underline{C}^n)$ and $\psi_j \in CP(B(H_j), B(\underline{C}^n), I_n)$ such that $\phi_j = U_j^* \psi_j U_j$ $(j \in J)$. Now $\psi_i(T_i) \in \cup_{j \in J} V_n(T_j)$ and

$$\Sigma_{j \in J} U_j^* U_j = \Sigma_{j \in J} \phi_j(I_{H_j}) = \phi(I_{\underset{\sim}{H}}) = I_n,$$

so

$$R = \phi(\underset{\sim}{T}) = \Sigma_{j \in J} \phi_j(T_j) = \Sigma_{j \in J} U_j^* \psi_j(T_j) U_j \in \overline{co}_n(\cup_{j \in J} V_n(T_j)),$$

using the fact that J is countable and the remark preceding 6.3 .

Conversely, if for some $k \in J$ we have $R \in V_n(T_k)$, then $R = \phi_k(T_k)$ where $\phi_k \in CP(B(H_k), B(\underline{C}^n), I_n)$. Let $M_k = \{\underset{\sim}{S} \in B(\underset{\sim}{H}) : H_k \text{ reduces } \underset{\sim}{S}\}$, which is a norm-closed self-adjoint linear manifold in $B(\underset{\sim}{H})$. Define $\psi_k : M_k \to B(\underline{C}^n)$ by $\psi_k(\underset{\sim}{S}) = \phi_k(\underset{\sim}{S}|_{H_k})$. ψ_k is clearly a completely positive linear map and $\psi_k(I_{\underset{\sim}{H}}) = I_n$. By theorem 5.5, there exists $\theta_k \in CP(B(\underset{\sim}{H}), B(\underline{C}^n), I_n)$ such that $\theta_k|_{M_k} \equiv \psi_k$. Now

$$\theta_k(\underset{\sim}{T}) = \psi_k(\underset{\sim}{T}) = \phi_k(\underset{\sim}{T}|_{H_k}) = \phi_k(T_k) = R ,$$

thus $V_n(T_k) \subseteq V_n(\underset{\sim}{T})$, which by propositions 6.2 and 6.3 is n-convex and closed, so $\overline{co}_n(\cup_{j \in J} V_n(T_j)) \subseteq V_n(\underset{\sim}{T})$.

We can now show that the sequence of algebraic matrix ranges is characterized by its four principal properties :- boundedness, closedness, n-convexity and coherence.

6.7 COROLLARY:

If $Z_n \subseteq B(\underline{C}^n)$ $(n \in \underline{N})$ is a sequence of sets such that

i) there is an $r>0$ (independent of n) such that $Z_n \subseteq \{R \in B(\underline{C}^n) : \|R\| \le r\}$

ii) Z_n is closed

iii) Z_n is n-convex

iv) if $R \in Z_n$, then $V_k(R) \subseteq Z_k$ $(k \in \underline{N})$

$(n \in \underline{N})$ [for i)–iv)]

then there is a separable Hilbert space H and $T \in B(H)$ with $V_n(T) = Z_n$ for every $n \in \underline{N}$.

Proof: Closed bounded sets in $B(\underline{C}^n)$ contain countable dense subsets, and so there exist sequences $\{R^n_j : j=1, \cdots\}$ dense in Z_n $(n \in \underline{N})$. Let $\{S_k : k \in \underline{N}\}$ be an enumeration of $\{R^n_j : j \in \underline{N} , n \in \underline{N}\}$. For each $k \in \underline{N}$, define $H_k = \underline{C}^n$ where n is the unique integer such that $S_k = R_j^n$, and let $H = \oplus_{k=1}^{\infty} H_k$. H is separable. Since the Z_n are uniformly bounded, we may define $T \in B(H)$ by $T = \oplus_{k=1}^{\infty} S_k$. By theorem 6.6,

$$V_m(T) = \overline{co}_m(\cup_{k=1}^{\infty} V_m(S_k)) \qquad (m \in \underline{N}) \qquad \cdots\cdots\cdots(1) .$$

But $S_k = R_j^n$ for some n,j and so by iv), $V_m(S_k) \subseteq Z_m$. But by ii) and iii), Z_m is m-convex and closed, hence $V_m(T) \subseteq Z_m$.

The identity map $\omega_n : B(\underline{C}^n) \to B(\underline{C}^n)$ satisfies $\omega_n \in CP(B(\underline{C}^n), B(\underline{C}^n), I_n)$ and so $R_j^n \in V_n(R_j^n) = V_n(S_k)$ for some k. Hence, by (1) we have $R_j^n \in V_n(T)$ for every $j \in \underline{N}$ and every $n \in \underline{N}$. But $\{R_j^n : j \in \underline{N}\}$ is dense in Z_n, and $V_n(T)$ is closed so $Z_n \subseteq V_n(T)$.

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§7. COMPLETE UNITARY INVARIANTS

We have shown in proposition 6.5 that the algebraic matrix range is unitarily invariant, and in this section we shall show that the sequence $\{V_n(T) : n \in \underline{N}\}$ is a complete unitary invariant for compact irreducible operators $T \in B(H)$. In [2], Arveson proves this for a slightly larger class of operators, but the scheme is more lucid in the restricted case, so we omit his generality.

A von Neumann algebra B is a τ_W-closed *-subalgebra of B(H) with $I_H \in B$, for some Hilbert space H. A positive linear map $\phi : B \to B$ is defined to be normal if it maps the least upper bound of an increasing net of positive operators, $\{X_\alpha\}$, into the least upper bound of $\{\phi(X_\alpha)\}$. In [5] (p.61) it is proved that for a normal positive linear map $\phi : B \to \underline{C}$, there exists a projection $P \in B$ such that

i) I-P is a maximal projection in Ker ϕ ,

ii) $\phi(X) = \phi(PX) = \phi(XP) \quad (X \in B)$,

iii) $\phi(X^*X) = 0$ if and only if $PX^*XP = 0$.

It is not hard to generalise this to the case of a normal positive linear map $\phi : B \to B$, and the projection P so obtained is called the support projection of ϕ .

An operator $T \in B(H)$ is defined to be irreducible if the only subspaces which are invariant for T and T* are $\{0\}$ and H, i.e. T has no reducing subspaces other than the trivial ones. If Λ is a subset of B(H), then $[\Lambda]$ will denote the linear span of Λ .

7.1 LEMMA:

Let B be a von Neumann algebra and let $\psi : B \to B$ be a normal completely positive linear map such that $\psi \circ \psi = \psi$ and $\|\psi\| \leq 1$, then the support projection P of ψ commutes with the fixed points of ψ .

Proof: Suppose $X \in B$ and $\psi(X) = X$. By proposition 4.1, we have $X^* = \psi(X)^* = \psi(X^*)$. The properties of P, corollary 4.4 and the fact that $\|\psi\| \leq 1$ show that $X^*X = \psi(X^*)\psi(X) = \psi(X^*P)\psi(PX) \leq \psi(X^*PX)$. Thus $X^*PX \leq X^*X \leq \psi(X^*PX)$ and so $PX^*PXP \leq PX^*XP \leq P\psi(X^*PX)P$. But $\psi \circ \psi = \psi$, so $\psi(\psi(X^*PX) - X^*PX) = 0$ while $\psi(X^*PX) - X^*PX$ is positive. By property iii) of P, we have $P\psi(X^*PX)P = PX^*PXP$, and substituting this in the last set of inequalities gives $PX^*PXP = PX^*XP$.

Let H be the Hilbert space underlying B, and suppose $h \in H$, then $\|(I-P)XPh\|^2 = \|XPh\|^2 - \|PXPh\|^2 = (PX^*XPh,h) - (PX^*PXPh,h) = 0$, and so $XP = PXP$. This holds for every X such that $\psi(X) = X$, in particular for X*, and so $PX = (X^*P)^* = (PX^*P)^* = PXP = XP$.

7.2 LEMMA:

Let B be a von Neumann algebra and let A be a norm-closed *-subalgebra which is weakly dense in B, contains the identity, and is such that every bounded linear functional f on A has an ultraweakly continuous linear extension to B. If $\rho : A \to A$ is a completely positive map with $\|\rho\| \leq 1$, then there exists a normal completely positive linear map $\psi : B \to B$ such that $\psi \circ \psi = \psi$, $\|\psi\| \leq 1$, $\psi(\rho(X)) = \psi(X)$ $(X \in A)$ and the fixed points of ψ include those of ρ .

Proof: For each $n \in \underline{N}$, let ρ^n denote the n-fold composition of ρ with itself. Let LIM denote a Banach limit on the space of bounded sequences of complex numbers, and let H denote the Hilbert space underlying B.

Define a bilinear form $[\cdot,\cdot]_X$ on H for each $X \in A$ by

$$[h,k]_X = \mathrm{LIM}\{(\rho^n(X)h,k)\}_{n=1}^{\infty} \quad (h,k \in H) .$$

Since $\|\rho\| \leq 1$, $|[h,k]_X| \leq \|X\| \, \|h\| \, \|k\|$. By Riesz's lemma (see [8] p.38), there is a bounded operator $\psi_0(X)$ on H such that $(\psi_0(X)h,k) = [h,k]_X$ $(h,k \in H)$. Clearly ψ_0 is a linear map of A into B(H) and $\|\psi_0\| \leq 1$. A standard separation argument shows that $\psi_0(X)$ is in the τ_W-closure of the convex hull of $\{\rho^n(X) : n \in \underline{N}\}$, and hence $\psi_0(X) \in B$. Also ρ is completely positive, so ρ^n is completely positive for each $n \in \underline{N}$. Thus if $\Sigma_{i,j=1}^{r} X_{ij} \otimes E_{ij}$ is a positive element in $A \otimes B(\underline{C}^r)$ and $h_1, \cdots, h_r \in H$, then we have

$$\Sigma_{i,j=1}^{r} (\psi_0(X_{ij})h_j,h_i) = \mathrm{LIM}\{\Sigma_{i,j=1}^{r} (\rho^n(X_{ij})h_j,h_i)\}_{n=1}^{\infty}$$
$$\geq 0 ,$$

and so ψ_0 is completely positive.

For each bounded linear functional f on A, let $\tilde{f}$ denote its

$\tau_{\sigma w}$-continuous extension to B. Since A is convex, its τ_s-closure coincides with the τ_w-closure (see [7] p.477), and so A is τ_s-dense in B. By the Kaplansky density theorem ([5] p.46), the unit ball of A is τ_s-dense in that of B. However, τ_s and $\tau_{\sigma s}$ induce the same topologies on the unit ball of B(H), and $\tau_{\sigma w} \subseteq \tau_{\sigma s}$, so the unit ball of A is $\tau_{\sigma w}$ dense in that of B. Hence $\|\tilde{f}\| = \|f\|$.

~~Now A^n and $B^{\sigma w}$ can be regarded as *-algebras by defining $(fg)(X) = f(X)g(X)$ and $f^*(X) = f(X)^*$. Note that if $g \in B^{\sigma w}$, then [illegible]. Thus the map $f \to \tilde{f}$ is an isometric isomorphism of A^n onto $B^{\sigma w}$.~~ If f is positive, then $\tilde{f}(I) = f(I) = \|f\| = \|\tilde{f}\|$ and so $\tilde{f}$ is positive.

The map $X \to X^{\wedge}$ of B into $(B^{\sigma w})^n$ given by $X^{\wedge}(g) = g(X)$ $(g \in B^{\sigma w})$ is an isometric isomorphism of B onto $(B^{\sigma w})^n$. Given $X \in B$, define $F_X \in (B^{\sigma w})^n$ by $F_X(g) = (g \circ \psi_0)^{\sim}(X)$ $(g \in B^{\sigma w})$, and then define $\psi : B \to B$ by $\psi(X)^{\wedge} = F_X$. Thus

$$g(\psi(X)) = (g \circ \psi_0)^{\sim}(X) \qquad (g \in B^{\sigma w},\ X \in B) .$$

Clearly ψ is a linear extension of ψ_0 to B. Suppose X is positive and g is positive, then $g \circ \psi_0$ is positive, and ~~since ~ is an isomorphism~~ as remarked above it follows that $(g \circ \psi_0)^{\sim}$ is positive. Hence $F_X(g) \geq 0$, and thus F_X is positive. But $Y \to Y^{\wedge}$ is an isomorphism, so $\psi(X)$ is positive, and hence ψ is a positive linear map.

To check complete positivity of ψ, we need only notice that $A \otimes B(\underline{C}^k)$ is a norm-closed *-subalgebra which is τ_w-dense in $B \otimes B(\underline{C}^k)$, and that $(A \otimes B(\underline{C}^k))^n$ can be identified with $A^n \otimes B(\underline{C}^k)$ and similarly $(B \otimes B(\underline{C}^k))^{\sigma w}$ with $B^{\sigma w} \otimes B(\underline{C}^k)$. The above process then shows that the subordinate map ψ_k is positive since $(\psi_0)_k$ is positive. Hence ψ is completely positive.

Suppose $\{X_\alpha\} \subseteq B$ and $X_\alpha \to X$ $(\tau_{\sigma w})$, then

$(g\circ\psi_0)^{\sim}(X_\alpha) \to (g\circ\psi_0)^{\sim}(X)$, and so $g(\psi(X_\alpha)) \to g(\psi(X))$ for every $g \in B^{\sigma w}$. Hence $\psi(X_\alpha) \to \psi(X)$ $(\tau_{\sigma w})$. Thus ψ is ultraweakly continuous, and so as in ([5] p.54) it is normal.

For $X \in A$, $\mathrm{LIM}\{(\rho^{n+1}(X)h,k)\}_{n=1}^{\infty} = \mathrm{LIM}\{(\rho^{n}(X)h,k)\}_{n=1}^{\infty}$, and so $\psi_0\circ\rho = \psi_0$, hence $\psi\circ\rho = \psi$ on A . By induction, $\psi\circ\rho^n(X) = \psi(X)$ for $X \in A$ and $n\in\underline{N}$, and thus $\psi(Y) = \psi(X)$ for all Y in the τ_W-closure of the convex hull of $\{\rho^n(X) : n\in\underline{N}\}$. But as remarked above, $\psi(X) = \psi_0(X)$ is in this τ_W-closure, so $\psi\circ\psi = \psi$ on A . By continuity we have $\psi\circ\psi = \psi$ on B.

Finally, if $X\in A$ and $\rho(X) = X$, then $\rho^n(X) = X$ $(n\in\underline{N})$, and so $(\psi_0(X)h,k) = \mathrm{LIM}\{(Xh,k)\}_{n=1}^{\infty} = (Xh,k)$. Thus $\psi(X) = \psi_0(X) = X$.

We now prove a special case of Arveson's boundary theorem (see [2]).

7.3 THEOREM:

If $T \in B(H)$ is a compact irreducible operator and ω is the identity map of B(H) into B(H), then the only completely positive linear extension of $\omega|_{[I,T]}$ to $C^*(T)$ is $\omega|_{C^*(T)}$.

Proof: Let π be the universal representation of B(H) on $\underset{\sim}{H} = \oplus_{f\in D(B(H),I)} K_f$ (see §2) and let B be the von Neumann algebra generated by $\pi(B(H))$. Let $\pi = \pi_1 \oplus \pi_2$ be the canonical decomposition of π (see §3) and let E be the projection onto the range of π_1 . E is a non-zero central projection in B. It was shown in §3 that π_1 decomposes further and is unitarily equivalent to a direct sum of copies of the identity representation of B(H) on H . Hence $BE = \pi_1(B(H))$ is isomorphic to B(H), and so its centre contains only the image of the scalar operators under the isomorphism. Thus the only central projections in BE are 0 and E (being the images of 0 and I). Also π_1 is faithful and so for $X \in B(H)$, $\pi(X)E = 0$ implies $X = 0$.

Let $\phi = \omega|_{[I,T]}$ and suppose θ is a completely positive linear extension of ϕ to [redacted] B(H) (Note that any extension to $C^*(T)$ can be extended to $B(H)$). Let $A = \pi(B(H))$, then A is a C^*-algebra which is τ_w-dense in B. Suppose $f \in A^n$, then f can be written as $f = \text{Re } f + i.\text{Im } f$, where Re f and Im f are self-adjoint elements of A^n, and by ([6] p.40) each of these is a difference of positive elements of A^n. Thus f is a linear combination of positive linear functionals. Let $g \in A^n$ be positive, then $\|g\| = g(\underset{\sim}{I_H})$ ([6] p.25). Define $q \in B(H)^n$ by $q(S) = g(\pi(S))/\|g\|$, then $q \in D(B(H), I_H)$. Recall from §2 that K_q is the completion of $B(H)/J_q$, where $J_q = \{S \in B(H) : q(S^*S) = 0\}$. Define $\underset{\sim}{h} \in \underset{\sim}{H}$ by

$$\underset{\sim}{h}(p) = \begin{cases} I_H + J_q & \text{if } p=q \\ 0 & \text{if } p \neq q \end{cases} \quad (p \in D(B(H), I_H)) ,$$

then $g(\pi(S)) = \|g\| q(S) = \|g\| (SI_H + J_q, I_H + J_q) = \|g\| (\pi(S)\underset{\sim}{h}, \underset{\sim}{h})$. Hence $g \in A^w$ and so also $f \in A^w$. By continuity, we can extend f to $\tilde{f} \in B^w \subseteq B^{\sigma w}$. Define $\rho : A \to A$ by $\rho = \pi \circ \theta \circ \pi^{-1}$. Since θ is completely positive and π and π^{-1} are morphisms, we have ρ completely positive. Also by corollary 4.4, $\|\theta\| = \|\theta(I_H)\| = 1$, since $\theta(I_H) = \phi(I_H) = I_H$. Thus $\|\rho\| \leq 1$.

By lemma 7.2, there exists a normal completely positive linear map $\psi : B \to B$ such that $\|\psi\| \leq 1$, $\psi \circ \pi \circ \theta = \psi \circ \pi$, $\psi \circ \psi = \psi$ and for $X \in B(H)$, if $\theta(X) = X$ then $\rho(\pi(X)) = \pi(X)$ and so $\psi(\pi(X)) = \pi(X)$. Let P be the support projection of ψ, then $P \in B$ and by lemma 7.1 it commutes with the fixed points of ψ. But for $X \in [I_H, T]$,

$\theta(X) = \phi(X) = X$, so $\psi(\pi(X)) = \pi(X)$, and hence P commutes with $\{\pi(X) : X \in [I_H, T]\}$. Since T is irreducible, $[I_H, T]$ generates B(H) as a von Neumann algebra. Also π_1 is unitarily equivalent to a direct sum of copies of the identity representation, so it is normal and hence $\pi_1([I_H, T])$ generates $BE = \pi_1(B(H))$ as a von Neumann algebra. Thus PE is a central projection in BE and so, as above, either PE = 0 or PE = E.

Suppose PE = 0, then using the defining properties of P we have

$$(\psi \circ \pi)(X) = \psi(\pi(X)) = \psi(P\pi(X)P) = \psi(P\pi_1(X)P \oplus P\pi_2(X)P)$$
$$= \psi(PE\pi_1(X)P \oplus P\pi_2(X)P) = \psi(0 \oplus P\pi_2(X)P) \qquad (X \in B(H)). \qquad \ldots\ldots (1)$$

But T is compact, so $\pi_2(T) = 0$. Also $\theta(T) = T$, so $(\psi \circ \pi)(T) = \pi(T)$.

Hence
$$\pi_1(T) \oplus 0 = \pi(T) = (\psi \circ \pi)(T)$$
$$= \psi(0 \oplus P\pi_2(T)P) \qquad \text{by (1)}$$
$$= \psi(0 \oplus 0) = 0 \oplus 0 .$$

Thus $\pi_1(T) = 0$, but π_1 is faithful so T = 0. This contradicts the hypothesis that T is irreducible, so we must have PE = E.

Suppose $X \in B(H)$ is such that $\theta(X) = X$, so by proposition 4.1 and corollary 4.4 we have $X^*X = \theta(X^*)\theta(X) \leq \theta(X^*X)$. Thus $\pi(\theta(X^*X) - X^*X)$ is a positive element in B, and $\psi \circ \pi \circ \theta = \psi \circ \pi$, so $\psi(\pi(\theta(X^*X) - X^*X)) = 0$. By the third defining property of P (see p.51), $P(\pi(\theta(X^*X) - X^*X))P = 0$, and so $EP(\pi(\theta(X^*X) - X^*X))PE = 0$. But PE = E, so also EP = E and hence $E(\pi(\theta(X^*X) - X^*X))E = 0$, i.e. $\pi_1(\theta(X^*X) - X^*X) = 0$. But π_1 is faithful, so $\theta(X^*X) = X^*X$ for every $X \in [I_H, T]$. Finally, for X, Y ∈ B(H) we have

$$Y^*X = \tfrac{1}{4}\{(X+Y)^*(X+Y) - (X-Y)^*(X-Y) + i(X+iY)^*(X+iY) - i(X-iY)^*(X-iY)\}$$

when $\theta(X) = X$ and $\theta(Y) = Y$.

so $\theta(Y^*X) = Y^*X$. Hence $\theta(Z) = Z$ for every $Z \in C^*(T)$, since $\theta(T) = T$ and $\theta(T^*) = T^*$.

A further two lemmas are required before we can prove the main theorem of this section. The first discusses the relation between matrix ranges and the norms of first order "matrix valued" polynomials. In [2], Arveson shows that the norms of n^{th} order polynomials of this type determine a class of operators (depending on n) up to unitary equivalence. Since we are restricting attention to compact irreducible operators, only first order polynomials are needed.

7.4 LEMMA:

Suppose $S \in B(K)$ and $T \in B(H)$ satisfy $V_n(S) \subseteq V_n(T)$ $(n \in \underline{N})$, then $\| I_K \otimes X + S \otimes Y \| \leq \| I_H \otimes X + T \otimes Y \|$ for every $X, Y \in B(\underline{C}^r)$ and every $r \in \underline{N}$.

Proof: Let $\{M_j : j \in J\}$ be an increasing directed set of finite dimensional linear manifolds in K such that $\cup_{j \in J} M_j$ is dense in K. Let P_j be the projection of K onto M_j. Define $\pi_j : B(K) \to B(M_j)$ by

$$\pi_j(R) = P_j R|_{M_j} \qquad (R \in B(K)) .$$

Now π_j is a linear map, and if $\Sigma_{r,s=1}^{m} R_{rs} \otimes E_{rs}$ is a positive element in $B(K) \otimes B(\underline{C}^m)$ with $k_1, \cdots, k_m$ an arbitrary m-tuple in M_j, then

$$\Sigma_{r,s=1}^{m} (\pi_j(R_{rs})k_s, k_r) = \Sigma_{r,s=1}^{m} (P_j R_{rs} k_s, k_r) = \Sigma_{r,s=1}^{m} (R_{rs} k_s, k_r) \geq 0 .$$

Thus π_j is completely positive, and for $k \in M_j$, $\pi_j(I_K)k = P_j k = k$, so $\pi_j \in CP(B(K), B(M_j), I_{M_j})$. But M_j is finite dimensional, so there is an integer $n(j)$ with M_j isometrically isomorphic to $\underline{C}^{n(j)}$, and then

$$P_j S|_{M_j} = \pi_j(S) \in V_{n(j)}(S) \subseteq V_{n(j)}(T) .$$

Hence there exists $\psi_j \in CP(B(H), B(M_j), I_{M_j})$ such that $\psi_j(T) = P_j S|_{M_j}$. By corollary 4.4, the subordinate maps satisfy $\|(\psi_j)_r\| = 1$ $(r \in \underline{N})$.

Suppose $X, Y \in B(\underline{C}^r)$, $\underset{\sim}{k} \in K \otimes \underline{C}^r$ and $\delta > 0$. Now $P_j \to I_K\ (\tau_s)$ and so there exists $i \in J$ such that

$$\|([I_K - P_i] \otimes X + [S - P_i S P_i] \otimes Y)\underset{\sim}{k}\| < \delta \ .$$

Thus we have

$$\begin{aligned}\|(I_K \otimes X + S \otimes Y)\underset{\sim}{k}\| &\leqslant \|(P_i \otimes X + P_i S P_i \otimes Y)\underset{\sim}{k}\| + \delta \\ &= \|(\psi_i)_r(I_H \otimes X + T \otimes Y)\underset{\sim}{k}\| + \delta \\ &\leqslant \|(I_H \otimes X + T \otimes Y)\underset{\sim}{k}\| + \delta \ .\end{aligned}$$

Hence the result.

We are grateful to T. A. Gillespie for providing a proof of the following lemma.

7.5 LEMMA:

Let $T \in B(H)$ be a compact irreducible operator, then $C^*(T)$ contains the algebra of all compact operators on H.

Proof: Let $F \in B(H)$ be any rank one operator, then there exist non-zero $y, z \in H$ such that $Fx = (x,z)y \quad (x \in H)$. Now $R = T^*T$ is a non-zero self-adjoint compact operator in $C^*(T)$. Let λ be a non-zero eigenvalue of R with corresponding spectral projection P, then P has finite rank and $P \in C^*(T)$. Let $x_1, \cdots, x_n$ be a basis for PH . Since T is irreducible, we have $C^*(T)$ irreducible, and then by ([12] p.253) it is strictly irreducible and further by ([12] p.62) it is strictly dense. Thus there exists $V \in C^*(T)$ such that $Vx_j = y \quad (j=1, \cdots, n)$, from which $VPH = \{\mu y : \mu \in \underline{C}\}$. Hence VP has rank one and so there exists non-zero $z_0 \in H$ such that $VPx = (x, z_0)y \quad (x \in H)$. Again by the strict density of $C^*(T)$, there exists $W \in C^*(T)$ such that $Wz_0 = z$ and hence

$$VPW^*X = (W^*x, z_0)y = (x,z)y = Fx \qquad (x \in H) .$$

Thus $F = VPW^* \in C^*(T)$. Hence $C^*(T)$ contains every rank one operator, therefore it also contains every finite rank operator, and consequently every compact operator.

We can now show that the sequence of algebraic matrix ranges forms a complete unitary invariant for compact irreducible operators.

7.6 THEOREM:

Suppose $S \in B(K)$ and $T \in B(H)$ are compact irreducible operators with $V_n(S) = V_n(T)$ $(n \in \underline{N})$, then S and T are unitarily equivalent.

Proof: Define $\phi_0 : [I_H, T] \to B(K)$ by $\phi_0(\alpha I_H + \beta T) = \alpha I_K + \beta S$. By lemma 7.4, ϕ_0 is completely contractive, and clearly $\phi_0(I_H) = I_K$. Thus by corollary 5.6 there exists $\phi_1 \in CP(C^*(T), B(K), I_K)$ such that $\phi_1(X) = \phi_0(X)$ for all $X \in [I_H, T]$.

Similarly define $\theta_0 : [I_K, S] \to B(H)$ by $\theta_0(\alpha I_K + \beta S) = \alpha I_H + \beta T$, then there exists $\theta_1 \in CP(B(K), B(H), I_H)$ such that $\theta_1(Y) = \theta_0(Y)$ for all $Y \in [I_K, S]$. By Stinespring's theorem (4.3) there is a Hilbert space M, a representation π of $C^*(\phi_1(C^*(T)))$ on M and a bounded linear map $V : H \to M$ such that

$$\theta_1(X)h = V^*\pi(X)Vh \qquad (X \in C^*(\phi_1(C^*(T))),\ h \in H) ,$$

$$\pi(I_K) = I_M ,$$

and

$$[\pi(X)Vh : X \in C^*(\phi_1(C^*(T))),\ h \in H]^- = M . \qquad \cdots\cdots(1)$$

For $Y \in [I_H, T]$ we have

$$Y = (\theta_1 \circ \phi_1)(Y) = V^*(\pi \circ \phi_1)(Y)V$$

and so $V^*(\pi o \phi_1)V$ is a completely positive linear extension to $C^*(T)$ of the identity map restricted to $[I_H, T]$. By theorem 7.3, it must be the identity map restricted to $C^*(T)$, i.e.

$$V^*(\pi o \phi_1)(X)V = X \qquad (X \in C^*(T)) .$$

Notice that $(\pi o \phi_1)(I_H) = I_M$, and so $V^*V = I_M$, thus V is an isometry.

Let $U \in C^*(T)$ be any unitary operator, then for any $h \in H$,

$$\begin{aligned} \|(\pi o \phi_1)(U)Vh - VUh\|^2 &= \|(\pi o \phi_1)(U)Vh\|^2 - 2\mathrm{Re}(V^*(\pi o \phi_1)(U)Vh, Uh) + \|VUh\|^2 \\ &= \|(\pi o \phi_1)(U)Vh\|^2 - 2\mathrm{Re}(Uh, Uh) + \|Uh\|^2 \\ &= \|(\pi o \phi_1)(U)Vh\|^2 - \|h\|^2 \\ &\leq \|h\|^2 - \|h\|^2 = 0 . \end{aligned}$$

Thus $(\pi o \phi_1)(U)V = VU$. But by a theorem of Russo and Dye [13] the closed linear span of the unitary operators in $C^*(T)$ is the whole of $C^*(T)$, so $(\pi o \phi_1)(Z)V = VZ \quad (Z \in C^*(T))$. Thus $\pi(X)VH \subseteq VH$ for every $X \in C^*(\phi_1(C^*(T)))$. By (1) we now have $M = VH$, so V is onto, hence unitary.

Since V is unitary, $\theta_1 = V^*\pi V$ is a representation of $C^*(\phi_1(C^*(T)))$ on H, and so $\theta_1|_{C^*(S)}$ is a representation of $C^*(S)$ on H. But $\theta_1(I_K) = I_H$ and $\theta_1(S) = T$ so $\theta_1(C^*(S)) = C^*(T)$, which is irreducible since T is irreducible. Thus $\theta_1|_{C^*(S)}$ is an irreducible representation of $C^*(S)$ on H. By lemma 7.5, $C^*(S)$ contains the algebra of compact operators in $B(K)$, and by theorem 3.8 an irreducible representation of the algebra of compact operators must be unitarily equivalent to the identity representation. Thus there is a unitary map $U : H \to K$ such that $U\theta_1(X)U^* = X$ for all compact X in $B(K)$. In particular, S is compact and $\theta_1(S) = T$, so S and T are unitarily equivalent.

CHAPTER III

SPATIAL MATRIX RANGES AND THEIR RELATION TO ALGEBRAIC MATRIX RANGES

§8. SPATIAL MATRIX RANGES

We begin this chapter by defining the spatial matrix ranges and describing which properties of the algebraic matrix ranges carry over and which fail to do so. The spatial ranges were first introduced by S.K.Parrott, whose work remains unpublished, using an equivalent formulation to the one given here. Parrott showed that the sequence of spatial matrix ranges forms a complete unitary invariant for compact operators with trivial reducing null-space. His proof is easily modified to show that a little more is true - in fact, that the sequence of closures of the spatial matrix ranges forms a complete unitary invariant. We give full details of the complete unitary invariance in this section.

For any $T \in B(H)$ and $n \in \underline{N}$, we define the n^{th} spatial matrix range of T to be the set $W_n(T) \subseteq B(\underline{C}^n)$ given by

$$W_n(T) = \{V^*TV : V \text{ is a linear isometry of } \underline{C}^n \text{ into } H\}.$$

Two alternative definitions of $W_n(T)$ are embodied in the following easy proposition:-

8.1 PROPOSITION:

i) Let $P \in B(H)$ be a projection of rank n, then there is a linear isometry $V : \underline{C}^n \to H$ such that $P = VV^*$.

ii) The operators belonging to the set $\{PT|_{PH} : P \text{ rank n projection}\}$ are unitarily equivalent to those in $W_n(T)$, and every operator in

$W_n(T)$ is unitarily equivalent to some such $PT|_{PH}$.

iii) If $\underset{\sim}{u} = (u_1, \cdots, u_n)$ is an orthonormal n-tuple in H, let $M(\underset{\sim}{u},T)$ denote the n×n matrix with i,j^{th} entry (Tu_j, u_i), then the set of all such $M(\underset{\sim}{u},T)$ coincides with the matrix representations of $W_n(T)$ with respect to the natural basis.

Proof: i) Let $x_1, \cdots, x_n$ be an orthonormal basis for PH and let $e_1, \cdots, e_n$ be the usual orthonormal basis for $\underline{C}^n$. Define $V : \underline{C}^n \to H$ by

$$V(\lambda_1 e_1 + \cdots + \lambda_n e_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n \qquad (\lambda_1, \cdots, \lambda_n \in \underline{C}),$$

then V is a linear isometry and for $h \in H$, $V^*h = (h,x_1)e_1 + \cdots + (h,x_n)e_n$. Hence $VV^*h = (h,x_1)x_1 + \cdots + (h,x_n)x_n = Ph$.

ii) Suppose $R \in B(PH)$ and $R = PT|_{PH}$ where P is a rank n projection. By i) there is a linear isometry $V : \underline{C}^n \to H$ such that $VV^* = P$ and V maps $\underline{C}^n$ onto PH. Define $U : \underline{C}^n \to PH$ by $U\underset{\sim}{\lambda} = V\underset{\sim}{\lambda}$, then U is onto and isometric, hence unitary. For $x \in PH$,

$$Rx = RPx = PTPx = V(V^*TV)V^*x = U(V^*TV)U^*x \ ,$$

and so R is unitarily equivalent to $V^*TV \in W_n(T)$.

Conversely, if $S \in W_n(T)$ then there is a linear isometry V of $\underline{C}^n$ into H such that $S = V^*TV$. Now $P = VV^*$ is a rank n projection and S is unitarily equivalent to $PT|_{PH}$.

iii) Given any orthonormal n-tuple $\underset{\sim}{u}$, define a linear isometry $V : \underline{C}^n \to H$ by $Ve_i = u_i$ $(i=1, \cdots, n)$, and then V^*TV has matrix representation $M(\underset{\sim}{u},T)$ with respect to the basis $e_1, \cdots, e_n$.

Conversely, if $S \in W_n(T)$, then $S = V^*TV$ for some linear isometry $V : \underline{C}^n \to H$. Setting $u_i = Ve_i$ $(i=1, \cdots, n)$ defines an orthonormal n-tuple in H, and $M(\underset{\sim}{u},T)$ is a matrix representation of S with respect to the basis $e_1, \cdots, e_n$.

We show that the spatial matrix range generalises the classical spatial numerical range, $W(T)$.

8.2 PROPOSITION:

For any $T \in B(H)$, $W_1(T)$ is the image of $W(T)$ in the natural identification of $\underline{C}$ with $B(\underline{C})$.

Proof: The identification of $\underline{C}$ with $B(\underline{C})$ is the isomorphism $\chi : \underline{C} \to B(\underline{C})$ given by $\chi(\lambda)\mu = \lambda\mu \quad (\lambda, \mu \in \underline{C})$. Notice that for $R \in B(\underline{C})$, we have $R = \chi(R(1))$.

Suppose $R \in W_1(T)$, then there is a linear isometry $V : \underline{C} \to H$ such that $R = V^*TV$. Let $u = V(1)$, then $V^*h = (h,u)$ $(h \in H)$ and so $R\lambda = \lambda(Tu,u)$. Thus $R = \chi((Tu,u))$ and $(Tu,u) \in W(T)$.

Conversely, if $\lambda \in W(T)$, then there is a unit vector $u \in H$ with $(Tu,u) = \lambda$. Define $V : \underline{C} \to H$ by $V\mu = \mu u$ $(\mu \in \underline{C})$, then V is a linear isometry and $\chi(\lambda) = V^*TV \in W_1(T)$.

If $V : \underline{C}^n \to H$ is a linear isometry, then $\|V^*TV\| \leq \|V^*\|\|T\|\|V\| \leq \|T\|$, so that for every $n \in \underline{N}$, $W_n(T)$ is contained in the ball of radius $\|T\|$. However, $W_n(T)$ may fail to be closed. The following operator demonstrates this :-

Let H be the Hilbert space of sequences $\{\alpha_r\}_{r=1}^{\infty}$ of complex numbers satisfying $\Sigma_{r=1}^{\infty} |\alpha_r|^2 < \infty$, with inner product given by $(\{\alpha_r\},\{\beta_r\}) = \Sigma_{r=1}^{\infty} \alpha_r\beta_r^*$. Define $T \in B(H)$ by $T\{\alpha_r\} = \{\alpha_r/r\}$. For fixed $n \in \underline{N}$, we shall show that $0 \in W_n(T)^-$ but $0 \notin W_n(T)$, so that all the spatial matrix ranges of this particular operator fail to be closed. Let $x_j \in H$ be the sequence whose j^{th} entry is 1 while all the other entries are zero. Given $\delta > 0$, choose $m \in \underline{N}$ such that $1/m < \delta$, and let $\underset{\sim}{u} = \{x_m, \cdots, x_{m+n}\}$. In the notation of proposition 8.1, we have $M(\underset{\sim}{u}, T) = \mathrm{diag}\{1/m, \cdots, 1/m+n\}$, and so the operator R with matrix representation $M(\underset{\sim}{u}, T)$ satisfies $\|R\| < \delta$. But by proposition 8.1 iii), $R \in W_n(T)$, and hence $0 \in W_n(T)^-$. However, for $\{\alpha_r\} \in H$, $(T\{\alpha_r\}, \{\alpha_r\}) = 0$ implies that $\Sigma_{r=1}^{\infty} |\alpha_r|^2/r = 0$, which implies $\alpha_r = 0$ $(r=1,2,\cdots)$. Thus for any orthonormal n-tuple $\underset{\sim}{u}$ in H, we have $M(\underset{\sim}{u}, T) \neq 0$, and so by proposition 8.1, $0 \notin W_n(T)$.

Notice that, unlike the algebraic matrix range, the spatial one can be void. For example, choose $k \in \underline{N}$ and let $H = \underline{C}^k$ with the Euclidean norm. For $n=1, \cdots, k$, there exist linear isometries $V : \underline{C}^n \to \underline{C}^k$, and then $V^* I_k V = V^* V = I_n$, so $W_n(I_k) = \{I_n\}$. However, for $n > k$, there are no linear isometries of $\underline{C}^n$ into $\underline{C}^k$, and so $W_n(I_k) = \emptyset$.

The Hausdorff-Toeplitz theorem ([9] problem 166) shows that $W(T)$, and hence $W_1(T)$, is convex for every $T \in B(H)$ and every Hilbert space H. The higher order spatial matrix ranges can fail to be convex, and in particular need not be n-convex, unlike the algebraic matrix ranges. For example, take $H = \underline{C}^2$ with the Euclidean norm, and let T be the operator with matrix representation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the usual basis. A linear isometry from $\underline{C}^2 \to \underline{C}^2$ must be onto, and hence unitary, so that every operator in $W_2(T)$ is unitarily equivalent to T, and in particular is unitary. Define linear isometries V_1 and V_2 from $\underline{C}^2$ into $\underline{C}^2$ by the matrix representations $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ respectively. Now $V_1^* T V_1$ and $V_2^* T V_2$ have matrix representations $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively, so $\frac{1}{2}V_1^* T V_1 + \frac{1}{2}V_2^* T V_2 = \frac{1}{\sqrt{2}}V_2$, which is not unitary, and so not in $W_2(T)$.

Thus $W_2(T)$ fails to be convex.

Despite its failure to be either compact or convex, the spatial matrix range is coherent. To see this, suppose that $T \in B(H)$, $R \in W_n(T)$ and $S \in W_k(R)$, then there are linear isometries $V : \underline{C}^n \to H$ and $U : \underline{C}^k \to \underline{C}^n$ such that $R = V^*TV$ and $S = U^*RU$. Hence $S = (VU)^*T(VU)$ and $VU : \underline{C}^k \to H$ is a linear isometry, so $S \in W_k(T)$. Thus $W_k(R) \subseteq W_k(T)$. The inclusion is vaccuously true if $k > n$.

Clearly if $S \in B(K)$ and $T \in B(H)$ are unitarily equivalent, then for every $n \in \underline{N}$, $W_n(S) = W_n(T)$. We now give the details of the converse of this statement. The reducing nullspace of $T \in B(H)$ is the intersection of the kernels of T and T^*, or equivalently $(T^*TH+TT^*H)^{\perp}$. For compact operators with trivial reducing null-space, we show that $W_n(S)^- = W_n(T)^-$ $(n \in \underline{N})$ implies that S and T are unitarily equivalent, which is a slight improvement on Parrot's theorem. Notice that since the spatial matrix ranges are unitary invariants, we may suppose that S and T act on the same Hilbert space.

Recall that τ_w, τ_s and τ_n denote respectively the weak operator, strong operator and uniform topologies on $B(H)$, and also that multiplication of operators is jointly continuous as a mapping

i) from $(B_1(H), \tau_w) \times (B(H), \tau_s)$ into $(B(H), \tau_w)$,

ii) from $(B_1(H), \tau_s) \times (B(H), \tau_s)$ into$((B(H), \tau_s)$,

where $B_1(H)$ denotes $\{T \in B(H) : \|T\| \leq 1\}$.

8.3 LEMMA:

If $S, T \in B(H)$ are such that $W_n(T) \subseteq W_n(S)^-$ for some $n \in \underline{N}$, and P is a rank n projection, then for every $\delta > 0$ there exists a partial isometry $G \in B(H)$ with initial projection P and $\|G^*SG-PTP\| < \delta$.

Proof: By proposition 8.1, there is a linear isometry $U : \underline{C}^n \to H$ such that $P = UU^*$. But $U^*TU \in W_n(T) \subseteq W_n(S)^-$, so given $\delta > 0$, there is a linear isometry $V : \underline{C}^n \to H$ such that $\| U^*TU - V^*SV\| < \delta$.
Define $G = VU^*$, then $G \in B(H)$ and $G^*G = UV^*VU^* = UU^* = P$, so G is a partial isometry with initial projection P. Also

$$\|G^*SG - PTP\| = \|UV^*SVU^* - UU^*TUU^*\| = \|U(V^*SV - U^*TU)U^*\| < \delta .$$

8.4 LEMMA:

If $S,T \in B(H)$ with S compact are such that $W_n(T) \subseteq W_n(S)^-$ $(n\in\underline{N})$, then T is compact and there exists $G \in B_1(H)$ such that $T = G^*SG$.

Proof: Let P be the projection of H onto an arbitrary closed separable linear manifold in H, then there exist projections $P_n \in B(H)$ of rank n $(n\in\underline{N})$ such that $P_n \to P$ (τ_s) as $n \to \infty$. By lemma 8.3, given $\delta > 0$, there exist partial isometries $G_n \in B(H)$ with initial projections P_n such that $\|P_nTP_n - G_n^*SG_n\| < 1/4n$ $(n\in\underline{N})$ ....(1).

Let H_0 denote the closed linear span of $\{PH, P_nH, G_nH : n\in\underline{N}\}$. Note that H_0 is separable, and also $G_n^*H = P_nH \subseteq H_0$ $(n\in\underline{N})$. Let $F_n = G_n|_{H_0}$, then F_n is a partial isometry in $B(H_0)$ with initial space P_nH. Now $\{F_n\}_{n=1}^{\infty}$ is a sequence in $B_1(H_0)$, which is τ_W-compact and (since H_0 is separable) metrisable (see [5] p.34), hence $B_1(H_0)$ is τ_W-sequentially-compact and so $\{F_n\}$ has a convergent subsequence. Without loss of generality, we may suppose that $F_n \to F \in B_1(H_0)$ (τ_W) as $n \to \infty$.

Since $H = H_0 \oplus H_0^{\perp}$, we may write a general element of H as x+y with $x\in H_0$ and $y\in H_0$, and then define $G\in B(H)$ by $G(x+y) = Fx$. Clearly

$G \in B_1(H)$, and $G_n(x+y) = G_n x$ since the initial space of G_n is $P_n H$ which is contained in H_0. Similarly, $G_n^*(x+y) = G_n^* x$. Thus

$$(G_n(x+y), x'+y') = (G_n x, x') = (F_n x, x') \rightarrow (Fx, x') \quad \text{as } n \rightarrow \infty,$$

and $(Fx, x') = (G(x+y), x'+y')$, so $G_n \rightarrow G\ (\tau_W)$ as $n \rightarrow \infty$. But the map $X \rightarrow X^*$ is τ_W to τ_W continuous, and so $G_n^* \rightarrow G^*\ (\tau_W)$ as $n \rightarrow \infty$. Now S is compact, so $SG_n \rightarrow SG\ (\tau_s)$, and hence $G_n^* SG_n \rightarrow G^*SG\ (\tau_W)$ as $n \rightarrow \infty$.

Given $z, z' \in H$, and $\delta > 0$, there exists $m \in \underline{N}$ such that $n \geqslant m$ implies that $|((G_n^* SG_n - G^*SG)z, z')| < \delta/4$, and combining this with (1) we have for $n \geqslant m'$, $|((P_n TP_n - G^*SG)z, z')| < \delta/2$. But $P_n TP_n \rightarrow PTP\ (\tau_s)$ as $n \rightarrow \infty$, so $|((PTP - G^*SG)z, z')| < \delta$. This holds for every $\delta > 0$ and for each $z, z' \in H$, so $PTP = G^*SG$. Since S is compact, we have G^*SG compact, and thus PTP is compact for every projection P onto a separable subspace.

(we m′ is … maximum … m and the …t integer …ger than $1/\delta$)

Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in H, and let P_0 be the projection of H onto the closed linear span of $\{x_n, Tx_n : n \in \underline{N}\}$. As above, P_0TP_0 is compact, so there is a subsequence $\{x_{nk}\}$ such that $\{P_0TP_0x_{nk}\}$ converges. But $P_0TP_0x_{nk} = Tx_{nk}$, and thus T is compact.

Now T compact implies that $\{TH + T^*H\}^-$ is separable, so let Q be the projection of H onto it, then as above there exists $G \in B_1(H)$ such that $QTQ = G^*SG$. But $QT = T$, so $TQ = G^*SG$ and hence $QT^* = G^*S^*G$. Finally $QT^* = T^*$, so $T^* = G^*S^*G$ and thus $T = G^*SG$.

8.5 THEOREM:

Let S and T be compact operators with trivial reducing null-spaces, and suppose that $W_n(S)^- = W_n(T)^-$ for every $n \in \underline{N}$, then S and T are unitarily equivalent.

Proof: Since the matrix ranges are unitary invariants, we may suppose that S and T act on the same Hilbert space H. By lemma 8.4 there exist $F,G \in B_1(H)$ such that $S = F^*TF$ and $T = G^*SG$. Let $Q = FG$, $J = T^*T$ and $K = (Q^*T)^*(Q^*T)$, then we have $\|Q\| \leq 1$, $T = Q^*TQ$, $J \geq K \geq 0$ and $J = Q^*KQ$.

If $J = 0$, then $T = 0$, and so $W_n(S)^- = W_n(0)^- = \{0\}$ $(n \in \underline{N})$. Given $x \in H$, $x \neq 0$, if $Sx \neq 0$ define a linear map $V : \underline{C}^2 \to H$ by $Ve_1 = x/\|x\|$ and $Ve_2 = Sx/\|Sx\|$. Notice that $(Ve_2, Ve_1) = \frac{\|x\|}{\|Sx\|}(S\frac{x}{\|x\|}, \frac{x}{\|x\|}) = 0$ since $W_1(S) = \{0\}$ and hence $W(S) = \{0\}$. Thus V is a linear isometry. But $W_2(S) = \{0\}$, so $V^*SV = 0$, and hence $\|Sx\| = \|x\|(V^*SVe_1, e_2) = 0$. Thus $S = 0$ and S and T are unitarily equivalent.

If $J \neq 0$, let E denote the eigenspace corresponding to the eigenvalue $\lambda = \|J\|$ of the positive compact operator J. If $x \in E$ is a unit eigenvector, then we have

$$\|J\| = (Jx,x) = (Q^*KQx,x) \leq (JQx,Qx) \leq \|J\| \, \|Qx\|^2 \leq \|J\| \quad \cdots\cdots(1).$$

Thus $\|Qx\| = 1 = \|x\|$, and so $\|Qy\| = \|y\|$ for all $y \in E$. Also from (1) it follows that $((\|J\| I - J)Qx, Qx) = 0$, but $\|J\| I - J \geq 0$ and so $(\|J\|I - J)Qx = 0$. Thus $Qx \in E$ for every unit vector $x \in E$, so $QE \subseteq E$. Now Q maps E isometrically into E, and E is finite dimensional, hence $QE = E$.

For any $x,y \in E$, we have $(x, Q|_E^* y) = (Qx,y) = (x,Q^*y)$, and thus $(Q|_E^* - Q^*)E \subseteq E^{\perp}$, from which $(I - Q^*Q)E = (Q|_E^* - Q^*)QE \subseteq E^{\perp}$. Hence $((I - Q^*Q)x,x) = 0$ $(x \in E)$, but $I - Q^*Q \geq 0$, so $(I - Q^*Q)x = 0$. This shows that $Q^*QE \subseteq E$, but $QE = E$, and so $Q^*E \subseteq E$. Thus E is a reducing subspace for Q and $Q|_E$ is unitary.

For a unit vector $x \in E$, we have from (1) that $((\|K\|I - K)Qx, Qx) = 0$ and $\|K\|I - K \geq 0$, so $(\|K\|I - K)Qx = 0$. Thus $KQE \subseteq QE$, but $QE = E$

and so $KE \subseteq E$. K is self-adjoint, hence E is a reducing subspace for K. Also, being an eigenspace of J, E is a reducing subspace for J. Thus we may restrict Q, K and J to $E^{\perp}$ and iterate the procedure with the next largest eigenvalue.

Now $(JH)^{-}$ is the orthogonal sum of the eigenspaces of J, and so $(JH)^{-}$ is a reducing subspace for Q, i.e. $(T^*TH)^{-}$ reduces Q and Q is unitary there. Similarly $(TT^*H)^{-}$ reduces Q and Q is unitary there. Thus Q^*Q and QQ^* both coincide with the identity on $(T^*TH + TT^*H)^{-}$. But $(T^*TH + TT^*H)^{\perp}$ is the reducing null-space of T, which is $\{0\}$ by hypothesis. Hence $(T^*TH + TT^*H)^{-} = H$, and so Q is unitary.

Recall that $Q = FG$ with $F,G \in B_1(H)$, so $I = Q^*Q = G^*F^*FG \leq G^*G \leq I$, thus G is an isometry. Also $I = QQ^* = FGG^*F^* \leq FF^* \leq I$, so that F^* is an isometry.

Repeating the whole procedure with $Q' = GF$, $J' = S^*S$ etc., we can show that G^* and F are isometries. Hence both F and G are unitary operators, and so S and T are unitarily equivalent.

Notice that if $T \in B(H)$ is irreducible, then $(T^*TH + TT^*H)^{-}$ is either H or $\{0\}$ since it is clearly a reducing subspace for T. If it is $\{0\}$ then $T = 0$, which is not irreducible. Hence we have $(T^*TH + TT^*H)^{\perp} = \{0\}$, i.e. T has trivial reducing null-space. This shows that the class of operators for which the spatial matrix ranges form a complete unitary invariant is larger than that for the algebraic matrix ranges.

§9. RELATIONS BETWEEN THE MATRIX RANGES

The first obvious relation is that of inclusion :-

9.1 PROPOSITION:

For any $T \in B(H)$ and every $n \in \underline{N}$, $W_n(T) \subseteq V_n(T)$.

Proof: Let $V : \underline{C}^n \to H$ be a linear isometry and let ω be the identity representation of $B(H)$ on H, then define $\phi : B(H) \to B(\underline{C}^n)$ by $\phi(T) = V^*\omega(T)V = V^*TV$, Note $\phi(I_H) = V^*V = I_n$, and then by theorem 4.3, $\phi \in CP(B(H), B(\underline{C}^n), I_n)$. Thus $V^*TV \in V_n(T)$, hence the result.

We have already noted that in general the spatial matrix range fails to be compact or n-convex. However, we can find C*-algebras in which these properties hold - for example, let γ be the great universal representation of B(H) (see §2), then $\gamma(B(H))$ is just such a C*-algebra. This is demonstrated by the next two propositions. We shall denote the great universal space of B(H) by G.

9.2. PROPOSITION:

For any $T \in B(H)$ and every $n \in \underline{N}$, $W_n(\gamma(T))$ is compact.

Proof: Let B denote the Banach space of bounded linear maps from $B(H)$ into $B(\underline{C}^n)$, where for $\phi \in B$, $\|\phi\| = \sup\{\|\phi(T)\| : \|T\| \leq 1\}$. Let B^* denote the set of linear functionals f on B which admit a representation $f(\phi) = \Sigma_{n=1}^{\infty} \rho_n(\phi(T_n))$ $(\phi \in B)$ with $\{T_n\}$ a norm bounded sequence in B(H) and $\{\rho_n\}$ a sequence of ultraweakly continuous linear functionals on $B(\underline{C}^n)$ such that $\Sigma_{n=1}^{\infty} \|\rho_n\| < \infty$. Let τ denote the weak topology on B given by B^*. It was shown in propositions 5.1 and 5.2 that the unit ball of B is τ-compact, and for a norm-bounded net $\{\phi_\alpha\}$, $\phi_\alpha \to \phi_0 \ (\tau)$ if and only if $\phi_\alpha(S) \to \phi_0(S) \ (\tau_W)$ for every $S \in B(H)$.

For each linear isometry $V : \underline{C}^n \to G$, define $\phi_V \in B$ by $\phi_V(S) = V^*\gamma(S)V$. Note that $\|\phi_V\| \leq 1$, so that the set D of all such ϕ_V is contained in the unit ball of B. Hence if D is τ-closed, then it is τ-compact. Suppose $\{\phi_\alpha\} \subseteq D$ and $\phi_\alpha \to \phi_0 \ (\tau)$, then for each $S \in B(H)$, $\phi_\alpha(S) \to \phi_0(S) \ (\tau_W)$. Hence, given $\Sigma_{i,j=1}^{k} S_{ij} \otimes E_{ij} \geq 0$ and $\underset{\sim}{\lambda}_1, \cdots, \underset{\sim}{\lambda}_k \in \underline{C}^n$, we have

$$\begin{aligned}\Sigma_{i,j=1}^{k} (\phi_0(S_{ij})\underset{\sim}{\lambda}_j \, , \, \underset{\sim}{\lambda}_i) &= \lim_\alpha \{\Sigma_{i,j=1}^{k} (\phi_\alpha(S_{ij})\underset{\sim}{\lambda}_j \, , \, \underset{\sim}{\lambda}_i)\} \\ &= \lim_\alpha \{\Sigma_{i,j=1}^{k} (V_\alpha^*\gamma(S_{ij})V_\alpha\underset{\sim}{\lambda}_j \, , \, \underset{\sim}{\lambda}_i)\} \\ &\geq 0 \quad \text{, by theorem 4.3 and lemma 4.2} \ .\end{aligned}$$

Thus, by lemma 4.2, ϕ_0 is completely positive, and so by theorem 4.3 there is a HIlbert space K, a representation π of B(H) on K and a bounded linear map $U : \underline{C}^n \to K$ such that $\phi_0(S) = U^*\pi(S)U \quad (S \in B(H))$. By theorem 2.4, there is a bounded linear map $W : U\underline{C}^n \to G$, the great universal space of B(H), such that $U^*\pi(S)U = U^*W^*\gamma(S)WU$, and so $\phi_0(S) = (WU)^*\gamma(S)(WU) \quad (S \in B(H))$. But $\phi_\alpha(I_H) = I_n$ and $\phi_\alpha(I_H) \to \phi_0(I_H)$ so $\phi_0(I_H) = I_n$. Thus WU is a linear isometry and hence D is τ-closed, therefore τ-compact.

Finally, $W_n(\gamma(T))$ is the image of D under the map $\phi \to \phi(T)$, which is τ to τ_W continuous, hence $W_n(\gamma(T))$ is ~~closed~~ compact. ~~But it is a bounded subset of $B(\underline{C}^n)$, so closed implies compact~~.

9.3 PROPOSITION:

For any $T \in B(H)$ and every $n \in \underline{N}$, $W_n(\gamma(T))$ is n-convex.

Proof: Suppose $\{U_j\}_{j=1}^{m} \subseteq B(\underline{C}^n)$ with $\Sigma_{j=1}^{m} U_j^* U_j = I_n$, and suppose $\{R_j\}_{j=1}^{m} \subseteq W_n(\gamma(T))$. There exist linear isometries $V_j : \underline{C}^n \to G$ such that $R_j = V_j^* \gamma(T) V_j$ $(j=1, \cdots, m)$. Define $Q_j : \underline{C}^n \to \oplus_{k=1}^{m} G$ by

$$(Q_j \underset{\sim}{\lambda})(k) = \left\{ \begin{matrix} V_j \underset{\sim}{\lambda} & \text{if } k=j \\ 0 & \text{if } k \neq j \end{matrix} \right\},$$

then Q_j is a linear isometry and $R_j = Q_j^*(\oplus_{k=1}^{m} \gamma(T)) Q_j$. Define $Q : \underline{C}^n \to \oplus_{k=1}^{m} G$ by $Q = \Sigma_{j=1}^{m} Q_j U_j$, then

$$Q^* Q = \Sigma_{i,j=1}^{m} U_i^* Q_i^* Q_j U_j = \Sigma_{j=1}^{m} U_j^* Q_j^* Q_j U_j = \Sigma_{j=1}^{m} U_j^* U_j = I_n ,$$

so that Q is a linear isometry. But

$$\begin{aligned} Q^*(\oplus_{k=1}^{m} \gamma(T)) Q &= \Sigma_{i,j=1}^{m} U_i^* Q_i^* (\oplus_{k=1}^{m} \gamma(T)) Q_j U_j \\ &= \Sigma_{j=1}^{m} U_j^* Q_j^* (\oplus_{k=1}^{m} \gamma(T)) Q_j U_j \\ &= \Sigma_{j=1}^{m} U_j^* R_j U_j . \end{aligned}$$

Thus $\Sigma_{j=1}^{m} U_j^* R_j U_j \in W_n(\oplus_{k=1}^{m} \gamma(T))$. By proposition 2.2, $\oplus_{k=1}^{m} \gamma(T)$ is unitarily equivalent to $\gamma(T)$, and the spatial matrix range is a unitary invariant, so $\Sigma_{j=1}^{m} U_j^* R_j U_j \in W_n(\gamma(T))$. Hence the result.

We now have a sequence of sets $W_n(\gamma(T)) \subseteq B(\underline{C}^n)$ such that

i) $W_n(\gamma(T)) \subseteq \{R \in B(\underline{C}^n) : \|R\| \leq \|\gamma(T)\| \leq \|T\|\}$.

ii) $W_n(\gamma(T))$ is closed.

iii) $W_n(\gamma(T))$ is n-convex.

iv) If $R \in W_n(\gamma(T))$, then $W_k(R) \subseteq W_k(\gamma(T))$ $(k \in \underline{N})$.

$(n \in \underline{N})$

This situation is very similar to that in corollary 6.7, which suggests that there may be operators S_T such that $W_n(\gamma(T)) = V_n(S_T)$ for every $n \in \underline{N}$. The next result shows that we may take $S_T = T$.

9.4 THEOREM:

For any $T \in B(H)$ and every $n \in \underline{N}$, $W_n(\gamma(T)) = V_n(T)$.

Proof: Recall that $V_n(T) = \{\phi(T) : \phi\in CP(B(H),B(\underline{C}^n),I_n)\}$. By Stinespring's theorem (4.3), $\phi\in CP(B(H),B(\underline{C}^n))$ if and only if there is a Hilbert space K, a representation π of B(H) on K and a bounded linear map $V : \underline{C}^n \to K$ such that $\phi(S) = V^*\pi(S)V \quad (S\in B(H))$. By theorem 2.4, there is a bounded linear map $U : V\underline{C}^n \to G$ such that

$$(\pi(S)V\underset{\sim}{\lambda} , V\underset{\sim}{\mu}) = (U^*\gamma(S)UV\underset{\sim}{\lambda} , V\underset{\sim}{\mu}) \qquad (\underset{\sim}{\lambda} , \underset{\sim}{\mu} \in \underline{C}^n),$$

and hence $\phi(S) = (UV)^*\gamma(S)(UV) \quad (S\in B(H))$. Now $\phi(I) = I_n$ if and only if UV is an isometry, thus $\phi\in CP(B(H),B(\underline{C}^n),I_n)$ if and only if there is a linear isometry $Q : \underline{C}^n \to G$ such that $\phi(S) = Q^*\gamma(S)Q \quad (S\in B(H))$. The result now follows from the definition of $W_n(\gamma(T))$.

This last result gives an elementary relation between the matrix ranges, but $\gamma(T)$ is in general just as difficult to compute as $CP(B(H),B(\underline{C}^n),I_n)$, so that we have not yet reduced the study of $V_n(T)$ to the more easily handled study of linear isometries. The results of §3 will now be of use in developing further relations, particularly when T is compact.

Recall that, by theorem 3.5, we can write $G = G_1 \oplus G_2$ and $\gamma = \gamma_1 \oplus \gamma_2$, where γ_2 annihilates the compact operators. Also, by theorem 3.8, there is an index set J such that γ_1 is unitarily equivalent to $\oplus_{j\in J}\omega$, where ω is the identity representation of B(H) on H. Thus for any $T \in B(H)$, we have $\gamma(T)$ unitarily equivalent to $\gamma_2(T) \oplus (\oplus_{j\in J}T)$, hence

$$V_n(T) = W_n(\gamma(T)) = W_n(\gamma_2(T) \oplus (\oplus_{j\in J} T)) \qquad (n\in\underline{N}) .$$

We shall show that J may be assumed countable. Notice that if H is finite dimensional, then by corollary 3.9 we have

$$V_n(T) = W_n(\gamma(T)) = W_n(\oplus_{j\in J} T) .$$

9.5 LEMMA:

Let I be any index set, and for each $i \in I$ let H_i be a Hilbert space with $T_i \in B(H_i)$. Let $\underset{\sim}{H} = \oplus_{i\in I} H_i$ and define $\underset{\sim}{T} \in B(\underset{\sim}{H})$ by $\underset{\sim}{T} = \oplus_{i\in I} T_i$, then for each $n \in \underline{N}$,

$$\cup_{i\in I} W_n(T_i) \subseteq W_n(\underset{\sim}{T}) .$$

Proof: Suppose $j\in I$ and $R \in W_n(T_j)$, then there is a linear isometry $V : \underline{C}^n \to H_j$ such that $R = V^*T_jV$. Define $U : \underline{C}^n \to \underset{\sim}{H}$ by

$$(U\underset{\sim}{\lambda})(i) = \begin{cases} V\underset{\sim}{\lambda} & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \quad (\underset{\sim}{\lambda} \in \underline{C}^n),$$

then U is a linear isometry and $R = U^*\underset{\sim}{T}U \in W_n(\underset{\sim}{T})$.

9.6 PROPOSITION:

If J is uncountable, then $W_n(\gamma_2(T) \oplus (\oplus_{j\in J} T)) = W_n(\gamma_2(T) \oplus (\oplus_{k=1}^{\infty} T))$.

Proof: Let $V : \underline{C}^n \to G_2 \oplus (\oplus_{j\in J} H)$ be a linear isometry, then since $\underline{C}^n$ is finite dimensional, $I = \{j\in J : (V\underset{\sim}{\lambda})(j) \neq 0 \text{ for some } \underset{\sim}{\lambda}\in\underline{C}^n\}$ is countable. Let $\{j_k : k\in I' \subseteq \underline{N}\}$ be an enumeration of I, and then define $U : \underline{C}^n \to G_2 \oplus (\oplus_{k=1}^{\infty} H)$ by $U\underset{\sim}{\lambda} = x(\underset{\sim}{\lambda}) \oplus y(\underset{\sim}{\lambda})$, where $x(\underset{\sim}{\lambda})$ is the component of $V\underset{\sim}{\lambda}$ in G_2 and $y(\underset{\sim}{\lambda}) \in \oplus_{k=1}^{\infty} H$ is given by

$$y(\underset{\sim}{\lambda})(k) = \begin{cases} (V\underset{\sim}{\lambda})(j_k) & \text{if } k \in I' \\ 0 & \text{if } k \notin I' \end{cases}.$$

Now U is a linear isometry, and

$$U^*(\gamma_2(T) \oplus (\oplus_{k=1}^{\infty} T))U = V^*(\gamma_2(T) \oplus (\oplus_{j\in J} T))V,$$

hence $W_n(\gamma_2(T) \oplus (\oplus_{j\in J} T)) \subseteq W_n(\gamma_2(T) \oplus (\oplus_{k=1}^{\infty} T))$.

The reverse inclusion follows from lemma 9.5 since we have

$$\gamma_2(T) \oplus (\oplus_{j\in J} T) = [\gamma_2(T) \oplus (\oplus_{k=1}^{\infty} T)] \oplus [\oplus_{j\in J\setminus \underline{N}} T] .$$

This shows that we may assume J is countable, and we then have for infinite dimensional spaces

$$V_n(T) = W_n(\gamma_2(T) \oplus (\oplus_{j\in J} T)) \quad (n\in \underline{N}) \quad \text{with J countable,}$$

and for finite dimensional spaces

$$V_n(T) = W_n(\oplus_{j\in J} T) \quad (n\in \underline{N}) \quad \text{with J countable.}$$

When H is finite dimensional, if J is also finite, then $\oplus_{j\in J} H$ is finite dimensional and hence $W_n(\oplus_{j\in J} T)$ is void for sufficiently large n. But by corollary 5.7 $V_n(T)$ is never void, so J must be countably infinite. Hence

$$V_n(T) = W_n(\oplus_{k=1}^{\infty} T) \qquad (T\in B(H),\ H \text{ finite dimensional}, n\in \underline{N}).$$

To obtain further relations between the matrix ranges, we clearly have to explore in more detail the spatial matrix range of a direct sum. We extend lemma 9.5 to an analogue of theorem 6.6, first establishing another lemma. Note that α denotes a cardinal which may take transfinite values, and so statements such as $n \leqslant \alpha$ should be interpreted in cardinal arithmetic.

9.7 LEMMA:

Let H be a Hilbert space with dimension α and let $n \in \underline{N}$ be such that $n \leqslant \alpha$. If $U : \underline{C}^n \to H$ is a bounded linear map, then there exist $A \in B(\underline{C}^n)$ and a linear isometry $V : \underline{C}^n \to H$ such that

$$U = VA \quad \text{and} \quad A^*A = U^*U .$$

Proof: Since $U^*U \in B(\underline{C}^n)$ is a positive operator, there exists $A \in B(\underline{C}^n)$ such that $A^*A = U^*U$ (proposition 1.3). Let $e_1, \cdots, e_n$ be the usual basis for $\underline{C}^n$. Notice that

$$(Ae_i, Ae_j) = (A^*Ae_i, e_j) = (U^*Ue_i, e_j) = (Ue_i, Ue_j) \qquad (i,j=1,\cdots,n)$$

so that $A\underline{C}^n$ is isometrically isomorphic (as a linear space) to $U\underline{C}^n$. Define $V_0 : A\underline{C}^n \to H$ by $V_0Ae_i = Ue_i$ $(i=1,\cdots,n)$, then V_0 is a well defined linear isometry. Now $\alpha \geq n$, so $\dim[(U\underline{C}^n)^\perp] \geq \dim[(A\underline{C}^n)^\perp]$, hence V_0 can be extended to a linear isometry $V : \underline{C}^n \to H$. Finally $VAe_i = V_0Ae_i = Ue_i$ $(i=1,\cdots,n)$, so $VA = U$.

9.8 THEOREM:

Let I be a countable index set, and for each $j \in I$ let H_j be an α_j-dimensional Hilbert space and then let $\underset{\sim}{H} = \oplus_{j\in I} H_j$. Given $T_j \in B(H_j)$ $(j \in I)$, define $\underset{\sim}{T} \in B(\underset{\sim}{H})$ by $\underset{\sim}{T} = \oplus_{j\in I} T_j$, then for $n \leq \min\{\alpha_j : j \in I\}$ we have

$$\cup_{j\in I} W_n(T_j) \subseteq W_n(\underset{\sim}{T}) \subseteq \overline{co}_n[\cup_{j\in I} W_n(T_j)] .$$

Proof: The first inclusion follows immediately from lemma 9.5. To demonstrate that the second one holds, suppose $n \leq \min\{\alpha_j : j \in I\}$ and $S \in W_n(\underset{\sim}{T})$, then $S = V^*\underset{\sim}{T}V$ for some linear isometry $V : \underline{C}^n \to \underset{\sim}{H}$. Let P_j denote the projection of $\underset{\sim}{H}$ onto H_j, then

$$S = V^*\underset{\sim}{T}V = \Sigma_{j\in I} V^*P_jT_jP_jV .$$

For each $j \in I$, $P_jV : \underline{C}^n \to H_j$ is a bounded linear map and $n \leq \alpha_j$, so by lemma 9.7 there exist $A_j \in B(\underline{C}^n)$ and linear isometries $V_j : \underline{C}^n \to H_j$ such that $P_jV = V_jA_j$ and $A_j^*A_j = V^*P_jV$. Thus

$$S = \Sigma_{j\in I} A_j^*(V_j^*T_jV_j)A_j$$

and $\Sigma_{j\in I} A_j^*A_j = \Sigma_{j\in I} V^*P_jV = V^*V = I_n$, so by the remarks before proposition 6.3 we have $S \in \overline{co}_n[\cup_{j\in I} W_n(T_j)]$.

9.9 THEOREM:

i) If H is finite dimensional and $T \in B(H)$, then for $n=1, \cdots, \dim H$ we have $V_n(T) = \overline{co}_n[W_n(T)]$.

ii) If H is infinite dimensional and $T \in B(H)$, then for $n \in \underline{N}$ we have $V_n(T) = \overline{co}_n[W_n(\gamma_2(T)) \cup W_n(T)]$.

Proof: i) We remarked on page 75 that, when H is finite dimensional, $V_n(T) = W_n(\oplus_{k=1}^{\infty} T)$. By theorem 9.8, for $n \in \underline{N}$ with $n \leq \dim H$,

$$W_n(T) \subseteq V_n(T) \subseteq \overline{co}_n[W_n(T)] .$$

But by propositions 6.2 and 6.3, $V_n(T)$ is closed and n-convex, so $V_n(T) = \overline{co}_n[W_n(T)]$.

ii) The great universal space decomposes as $G = G_1 \oplus G_2$ with $\gamma = \gamma_1 \oplus \gamma_2$, and then as on page 75 there is a countable index set J such that $V_n(T) = W_n(\gamma_2(T) \oplus (\oplus_{j \in J} T))$. By corollary 3.11, G_2 is infinite dimensional, and H is infinite dimensional by hypothesis, hence by theorem 9.8 for every $n \in \underline{N}$

$$W_n(\gamma_2(T)) \cup W_n(T) \subseteq V_n(T) \subseteq \overline{co}_n[W_n(\gamma_2(T)) \cup W_n(T)] .$$

Finally, by propositions 6.2 and 6.3, $V_n(T)$ is closed and n-convex, so the result follows.

This shows that the numerical range result - $V(T) = W(T)^-$ - has a natural generalisation to matrix ranges at least for finite dimensional spaces. We shall show in §10 that this generalisation holds for compact operators on infinite dimensional spaces, and in §11 that it holds for normal operators. The convex hull is suppressed in the numerical range case, since $W(T)$ is always convex.

§10. COMPACT OPERATORS ON INFINITE DIMENSIONAL HILBERT SPACES

For the results of this section, we need to know more about the concept of n-convexity introduced on page 45. It is well known that, for ordinary convexity, if A is a compact set in a finite dimensional Hilbert space, then the convex hull of A is also compact. We show that the same holds for n-convexity.

10.1 PROPOSITION:

If $A \subseteq B(\underline{C}^n)$ is compact, then $co_n(A)$ is compact.

Proof: First we show that every $R \in co_n(A)$ can be expressed as an n-convex combination of at most $(4n^2 + 1)$ elements in A. Suppose $R = \Sigma_{k=1}^{m} X_k^* S_k X_k$ with $m > 4n^2 + 1$, where $S_k \in A$, $X_k \in B(\underline{C}^n)$ and $\Sigma_{k=1}^{m} X_k^* X_k = I_n$. Since $B(\underline{C}^n)$ is $4n^2$-dimensional over $\underline{R}$ (the real number field), and $\{X_k^* S_k X_k - X_1^* S_1 X_1\}_{k=2}^{m}$ contains at least $4n^2 + 1$ elements, there exist $\lambda_2, \cdots, \lambda_m \in \underline{R}$ (not all zero) such that $\Sigma_{k=2}^{m} \lambda_k (X_k^* S_k X_k - X_1^* S_1 X_1) = 0$. Let $\lambda_1 = \Sigma_{k=2}^{m}(-\lambda_k)$, then

$$\Sigma_{k=1}^{m} \lambda_k X_k^* S_k X_k = 0 \quad \text{and} \quad \Sigma_{k=1}^{m} \lambda_k = 0 .$$

Choose r such that $\lambda_r = \max\{\lambda_1, \cdots, \lambda_m\}$, then $(1 - \lambda_k/\lambda_r) \geq 0$, and $X_r^* S_r X_r = \Sigma_{k \neq r}(-\lambda_k/\lambda_r) X_k^* S_k X_k$. Thus

$$R = \Sigma_{k \neq r}([1 - \lambda_k/\lambda_r]^{\frac{1}{2}} X_k)^* \, S_k([1 - \lambda_k/\lambda_r]^{\frac{1}{2}} X_k) ,$$

where
$$\Sigma_{k \neq r}(1 - \lambda_k/\lambda_r) X_k^* X_k = [\Sigma_{k=1}^{m} X_k^* X_k] - X_r^* X_r - [\Sigma_{k \neq r}(\lambda_k/\lambda_r) X_k^* X_k]$$
$$= I_n - [\Sigma_{k=1}^{m} (\lambda_k/\lambda_r) X_k^* X_k]$$
$$= I_n \qquad \text{since } \Sigma_{k=1}^{m} \lambda_k = 0 .$$

This expresses R as an n-convex combination of m-1 elements in A, and so by induction the expression can be reduced to $4n^2+1$ elements.

Let A_j denote the set of all $R \in B(\underline{C}^n)$ such that R can be expressed as an n-convex combination of precisely j elements in A . We have just shown that $co_n(A) = \cup_{j=1}^{n^2+1} A_j$.

Consider the Banach space Y_j of all j-tuples of elements of $B(\underline{C}^n)$ equipped with the norm $\|(X_1, \cdots, X_j)\| = (\Sigma_{k=1}^{j} \|X_k\|^2)^{\frac{1}{2}}$, and let $P_j = \{(X_1, \cdots, X_j) \in Y_j : \Sigma_{k=1}^{j} X_k^* X_k = I_n\}$. The map $(X_1, \cdots, X_j) \to \Sigma_{k=1}^{j} X_k^* X_k$ is clearly continuous, and P_j is the inverse image of the closed set $\{I_n\}$, so P_j is closed. But P_j is bounded, and Y_j is finite dimensional, so P_j is compact. By Tychanoff's theorem ([15] p.80), $P_j \times A \times \cdots \times A$ (j copies of A) is a compact subset of $Y_j \times B(\underline{C}^n) \times \cdots \times B(\underline{C}^n)$ in the product topology. The map $((X_1, \cdots, X_j), S_1, \cdots, S_j) \to \Sigma_{k=1}^{j} X_k^* S_k X_k$ is continuous, and A_j is the image of $P_j \times A \times \cdots \times A$ under this map. Hence A_j is compact for each j, and so $co_n(A) = \cup_{j=1}^{n^2+1} A_j$ is also compact.

10.2 COROLLARY:

If $A \subseteq B(\underline{C}^n)$ is bounded, then $co_n(A^-) = \overline{co}_n(A)$.

Proof: Since $B(\underline{C}^n)$ is finite dimensional, A bounded implies that A^- is compact. Clearly $co_n(A) \subseteq co_n(A^-)$, and so by proposition 10.1, $\overline{co}_n(A) \subseteq co_n(A^-)$.

Conversely, if $R \in co_n(A^-)$, then $R = \Sigma_{k=1}^{m} X_k^* S_k X_k$ with $S_k \in A^-$, $X_k \in B(\underline{C}^n)$ and $\Sigma_{k=1}^{m} X_k^* X_k = I_n$. Given $\delta > 0$, there exist $T_k \in A$ such that $\|T_k - S_k\| < \delta$ $(k=1, \cdots, m)$, and then

$$\|R - \Sigma_{k=1}^{m} X_k^* T_k X_k\| \leq \Sigma_{k=1}^{m} \|X_k^*\| \, \|S_k - T_k\| \, \|X_k\| < \delta[\Sigma_{k=1}^{m} \|X_k\|^2] .$$

Thus $R \in \overline{co}_n(A)$.

A property of compact operators on infinite dimensional spaces is that 0 is in the closure of their numerical ranges. This carries over to matrix ranges.

10.3 LEMMA:

Let H be infinite dimensional, $T \in B(H)$ a compact operator and $n \in \underline{N}$, then $0 \in W_n(T)^-$.

Proof: Since H has infinite dimension, there is an orthonormal sequence $\{x_k\}_{k=1}^{\infty}$ in H such that $x_k \to 0$ (weakly) as $k \to \infty$. But T is compact, so $\| Tx_k \| \to 0$ as $k \to \infty$. For each $k \in \underline{N}$, define $V_k : \underline{C}^n \to H$ by $V_k e_i = x_{k+i}$ $(i=1, \cdots, n)$, where $e_1, \cdots, e_n$ is the usual basis for $\underline{C}^n$. Each V_k is a linear isometry, and

$$|(V_k^* T V_k e_i, e_j)| = |(Tx_{k+i}, x_{k+j})| \leq \| Tx_{k+i} \| \to 0 \quad \text{as } k \to \infty .$$

Hence $\|V_k^* T V_k\| \to 0$ as $k \to \infty$, so $0 \in W_n(T)^-$.

We can now show that the result obtained in theorem 9.9 for operators on finite dimensional spaces also holds for compact operators in general.

10.4 THEOREM:

Let H be infinite dimensional, $T \in B(H)$ a compact operator and $n \in \underline{N}$, then $V_n(T) = \overline{co}_n W_n(T)$.

Proof: By theorem 9.9, $V_n(T) = \overline{co}_n[W_n(\gamma_2(T)) \cup W_n(T)]$, where γ_2 is part of the canonical decomposition of the great universal representation. But T is compact, so $\gamma_2(T) = 0$, and $W_n(0) = \{0\}$.

By lemma 10.3, $0 \in W_n(T)^-$, so $V_n(T) \subseteq \overline{co}_n(W_n(T)^-)$. But $W_n(T)$ is bounded, so by corollary 10.2, $V_n(T) \subseteq \overline{co}_n(W_n(T))$. Finally $W_n(T) \subseteq V_n(T)$ (9.1) and $V_n(T)$ is closed and n-convex (6.2 & 6.3), so $V_n(T) = \overline{co}_n(W_n(T))$.

In [3] de Barra, Giles and Sims show that, for a compact operator on an infinite dimensional Hilbert space, if $0 \in W(T)$ then $W(T)$ is closed, and also that $W(T)^- = co(\{0\} \cup W(T))$. We show that a similar result is true for matrix ranges.

10.5 THEOREM:

Let H be infinite dimensional, $T \in B(H)$ a compact operator and $n \in \underline{N}$, then i) $W_n(T)^- \subseteq co_n(W_n(T) \cup \{0\})$,

and ii) if $0 \in W_n(T)$, then $co_n(W_n(T))$ is closed.

Proof: i) Suppose $R \in W_n(T)^-$, then there exist linear isometries $V_k : \underline{C}^n \to H$ $(k \in \underline{N})$ such that $V_k^* T V_k \to R$ as $k \to \infty$. Let $e_1, \cdots, e_n$ be the usual basis for $\underline{C}^n$. Now $\{V_k e_1\}_{k=1}^{\infty}$ is contained in the unit ball of H, so by ([15] p.209) there exists $x_1 \in H$ with $\|x_1\| \leq 1$ and a subsequence $V_{s_1(k)} e_1 \to x_1$ (weakly) as $k \to \infty$. Similarly, $\{V_{s_1(k)} e_2\}_{k=1}^{\infty}$ is contained in the unit ball of H, so there exists $x_2 \in H$ with $\|x_2\| \leq 1$ and a subsequence such that

$$V_{s_2(k)} e_1 \to x_1 \quad \text{and} \quad V_{s_2(k)} e_2 \to x_2 \quad \text{(weakly) as } k \to \infty .$$

In this fashion, we obtain $x_1, \cdots, x_n$ in the unit ball of H and a subsequence such that

$$V_{s(k)} e_j \to x_j \quad \text{(weakly) as } k \to \infty \quad (j=1, \cdots, n). \quad \cdots\cdots\cdots (1)$$

Notice that for $\lambda_1, \cdots, \lambda_n \in \underline{C}$, $\{V_{s(k)}(\lambda_1 e_1 + \cdots + \lambda_n e_n)\}$ converges weakly to $\lambda_1 x_1 + \cdots + \lambda_n x_n$ as $k \to \infty$, and each $V_{s(k)}$ maps the unit ball of $\underline{C}^n$ into that of H, which is weakly closed, so

$$\|\lambda_1 e_1 + \cdots + \lambda_n e_n\| \leq 1 \quad \text{implies} \quad \|\lambda_1 x_1 + \cdots + \lambda_n x_n\| \leq 1 \ . \qquad \cdots\cdots(2)$$

Since T is compact, we have from (1) that

$$(V^*_{s(k)} T V_{s(k)} e_i, e_j) \to (Tx_i, x_j) \quad \text{as } k \to \infty \quad (i,j=1,\cdots,n) ,$$

and so $\quad (Tx_i, x_j) = (Re_i, e_j)$.

Define $U : \underline{C}^n \to H$ by $Ue_i = x_i$ $(i=1,\cdots,n)$. If $\underset{\sim}{\lambda} = \lambda_1 e_1 + \cdots + \lambda_n e_n$ and $\|\underset{\sim}{\lambda}\| \leq 1$, then by (2) , $\|U\underset{\sim}{\lambda}\| \leq 1$, and so $\|U\| \leq 1$. By lemma 9.7, there exist $A \in B(\underline{C}^n)$ and a linear isometry $V : \underline{C}^n \to H$ such that $U = VA$ and $A^*A = U^*U$. Now

$$(A^*A\underset{\sim}{\lambda}, \underset{\sim}{\lambda}) = (U^*U\underset{\sim}{\lambda}, \underset{\sim}{\lambda}) = \|U\underset{\sim}{\lambda}\|^2 \leq \|\underset{\sim}{\lambda}\|^2 = (I_n\underset{\sim}{\lambda}, \underset{\sim}{\lambda}) \qquad (\underset{\sim}{\lambda} \in \underline{C}^n),$$

and so $A^*A \leq I_n$, hence we can write

$$R = U^*TU = A^*(V^*TV)A = A^*(V^*TV)A + (I_n - A^*A)^{\frac{1}{2}}\, 0\, (I_n - A^*A)^{\frac{1}{2}} .$$

Thus $R \in co_n(W_n(T) \cup \{0\})$.

ii) If $0 \in W_n(T)$, then I) becomes $W_n(T)^- \subseteq co_n(W_n(T))$, and thus $co_n(W_n(T)^-) \subseteq co_n(W_n(T))$. But by corollary 10.2, $co_n(W_n(T)^-) = \overline{co}_n(W_n(T))$, so $\overline{co}_n(W_n(T)) \subseteq co_n(W_n(T))$ and hence $co_n(W_n(T))$ is closed.

The case $n = 1$ reduces to the theorem of de Barra, Giles and Sims since $W(T)$ is convex.

§11. THE SPECTRUM AND MATRIX RANGES

Some relations between the spectrum $\sigma(T)$ of $T \in B(H)$ and the two numerical ranges of T are given by :-

i) the point spectrum, $p\sigma(T)$, is contained in $W(T)$.

ii) $\sigma(T) \subseteq W(T)^- = V(T)$.

iii) If T is normal, then $V(T) = co(\sigma(T))$.

(see [4]). We establish similar results for matrix ranges.

11.1 PROPOSITION:

If $\lambda \in p\sigma(T)$ and $E_\lambda \subseteq H$ is the corresponding eigenspace, then $\lambda I_n \in W_n(T)$ for $n=1,2,\cdots,\dim E_\lambda$.

Proof: If $n \leq \dim E_\lambda$, then there exists an orthonormal n-tuple $\{x_1,\cdots,x_n\}$ of eigenvectors for λ. Define a linear isometry $V : \underline{C}^n \to H$ by $Ve_j = x_j$ $(j=1,\cdots,n)$ where $e_1,\cdots,e_n$ is the usual basis for $\underline{C}^n$, then $V^*TV = \lambda I_n$, hence the result.

An example shows that $\{\lambda I_n : \lambda \in \sigma(T)\}$ is not in general contained in $W_n(T)^-$:-

Take $H = \underline{C}^2$ with the Euclidean norm, and let $T \in B(H)$ have matrix representation $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ with respect to the standard basis. A linear isometry of $\underline{C}^2$ into $\underline{C}^2$ must be unitary, so $W_2(T)$ contains only unitary equivalents of T. But $\sigma(T) = \{0,1\}$, and $\sigma(1.I_2) = \{1\}$, so $1.I_2$ is not unitarily equivalent to T, hence is not in $W_2(T)$. Since H is finite dimensional, $W_2(T)$ is closed, and thus we have $1 \in \sigma(T)$ but $1.I_2 \notin W_2(T)^-$. However, notice that $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ are both in $W_2(T)$, and so denoting the projections onto the linear spans of e_1 and e_2 by P_1 and P_2 respectively, we have

$$P_1^*\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}P_1 + P_2^*\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}P_2 = 1.I_2 \quad \text{and } P_1^*P_1 + P_2^*P_2 = I_2 .$$

Thus $1.I_2 \in co_2W_2(T)$. This suggests the next result, which is a natural generalisation of ii) above.

11.2 PROPOSITION:

Let H be a m-dimensional Hilbert space, $T \in B(H)$ and $1 \leq n \leq m$, then $\{\lambda I_n : \lambda \in \sigma(T)\} \subseteq \overline{co}_n(W_n(T)) \subseteq V_n(T)$.

Proof: Suppose $\lambda \in \sigma(T)$, then $\lambda \in W(T)^-$ and so given $\delta > 0$, there exists $x \in H$ such that $|\lambda - (Tx,x)| < \delta$ and $\|x\| = 1$. Since $n \leq m$, there are linear isometries $V_i : \underline{C}^n \to H$ such that $V_i e_i = x$ $(i=1,\cdots,n)$. Let P_j denote the projection of $\underline{C}^n$ onto the linear span of $\{e_j\}$, then

$$(P_j^* V_j^* T V_j P_j e_i, e_k) = \begin{cases} (Tx,x) & \text{if } i = k = j \\ 0 & \text{otherwise} \end{cases} \quad (i,j,k=1,\cdots,n).$$

Thus $\|\lambda I_n - \Sigma_{j=1}^{n} P_j^* V_j^* T V_j P_j\| = \|(\lambda - (Tx,x))I_n\| < \delta$, and hence $\lambda I_n \in \overline{co}_n(W_n(T))$, and it was shown in §9 that $\overline{co}_n(W_n(T)) \subseteq V_n(T)$.

11.3 COROLLARY:

Let H be m-dimensional, $T \in B(H)$ and $1 \leq n \leq m$, then for $\lambda_1,\cdots,\lambda_n \in \sigma(T)$, we have $\mathrm{diag}\{\lambda_1,\cdots,\lambda_n\} \in \overline{co}_n(W_n(T))$.

We omit the easy proof of this corollary. Note that $\mathrm{diag}\{\lambda_1,\cdots,\lambda_n\}$ denotes the operator in $B(\underline{C}^n)$ having as matrix representation the diagonal matrix with entries $\lambda_1,\cdots,\lambda_n$.

The natural generalisation of the result for normal operators ((iii) above) was obtained by Arveson in [2] (proposition 2.4.1) . His proof uses the Krein-Milman theorem and a characterization of the extreme points of $CP(B(H),B(\underline{C}^n),I_n)$, which he proved in [1]. Using the relation $W_n(\gamma(T)) = V_n(T)$ (theorem 9.4) we provide an elementary proof of Arveson's result which only depends on the spectral theorem.

11.4 THEOREM:

Let $T \in B(H)$ be normal, then $V_n(T) = \overline{co}_n\{\lambda I_n : \lambda \in \sigma(T)\}$ $(n \in \underline{N})$.

Proof: By theorem 9.4, $V_n(T) = W_n(\gamma(T))$, where γ is the great universal representation of $B(H)$ on G. Clearly $\gamma(T)$ is normal and $\sigma(\gamma(T)) \subseteq \sigma(T)$. By the spectral theorem for normal operators ([8] p.71) there is a complex spectral measure $E(\cdot)$ such that $\gamma(T) = \int \lambda dE(\lambda)$. Now $E(\underline{C} \setminus \sigma(\gamma(T))) = 0$, and we can cover $\sigma(\gamma(T))$ by a finite number N of compact sets M_r with $\lambda_r \in (\text{int } M_r) \cap \sigma(\gamma(T))$ such that $M_r \cap M_s$ has spectral measure zero when $r \neq s$, (+) and for given $\delta > 0$,

$$|(\gamma(T)x,y) - \Sigma_{r=1}^{N} \lambda_r (E(M_r)x,y)| < \delta \quad (x,y \in G) .$$

Let $V : \underline{C}^n \to G$ be a linear isometry, then

$$|(V^*\gamma(T)Ve_i,e_j) - \Sigma_{r=1}^{N} \lambda_r (V^*E(M_r)Ve_i,e_j)| < \delta \quad (i,j=1,\cdots,n) ,$$

and thus $\|V^*\gamma(T)V - \Sigma_{r=1}^{N} \lambda_r V^*E(M_r)V\| < n^2\delta$. But $\cup_{r=1}^{N} M_r \supseteq \sigma(\gamma(T))$, so $\Sigma_{r=1}^{N} E(M_r) = I_G$, thus $\Sigma_{r=1}^{N} V^*E(M_r)V = I_n$. Also $\lambda_r \in \sigma(\gamma(T)) \subseteq \sigma(T)$ so $\Sigma_{r=1}^{N} \lambda_r V^*E(M_r)V \in co_n\{\lambda I_n : \lambda \in \sigma(T)\}$, and $V^*\gamma(T)V \in \overline{co}_n\{\lambda I_n : \lambda \in \sigma(T)\}$.

Conversely, if $\lambda \in \sigma(T)$, then since $\sigma(T) \subseteq W(T)^- \subseteq V(T)$ there exists $f \in D(B(H),I_H)$ such that $f(T) = \lambda$. As remarked on page 44, f is completely positive. Define $\phi : B(H) \to B(\underline{C}^n)$ by $\phi(S) = f(S)I_n$, then $\phi \in CP(B(H),B(\underline{C}^n),I_n)$ and $\phi(T) = \lambda I_n$, so $\lambda I_n \in V_n(T)$ and hence $\overline{co}_n\{\lambda I_n : \lambda \in \sigma(T)\} \subseteq V_n(T)$.

11.5 COROLLARY:

If $T \in B(H)$ is normal and $n \leqslant \dim H$, then $V_n(T) = \overline{co}_n(W_n(T)$.

Proof: Immediate from 11.2 and 11.4.

This corollary extends further the class of operators for which the matrix ranges are related in the natural fashion.

(+) We can arrange the covering with this condition since the spectral measure is finite and hence there are at most a countable number of point masses with respect to spectral measure.

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