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Non-Concave and Behavioural Optimal Portfolio Choice Problems

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Declaration

This thesis, which was composed by myself, is submitted to The University of Edinburgh in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the School of Mathematics.

I hereby declare that the work presented in this thesis is, to the best of my knowledge and belief, original and my own, except where explicitly stated otherwise in the text. I further assert that none of the material contained in this thesis has been submitted, either in part or whole, for any other degree or professional qualification.

Andrea Sofia Meireles Rodrigues
Edinburgh, 30th April 2014

Não me esqueci de nada, mãe.
Guardo a tua voz dentro de mim.

Eugénio de Andrade, from 'Poema à Mãe' (ll. 35–36),
in *Os Amantes Sem Dinheiro* (1950).

I have not forgotten anything, mother.
I keep your voice inside of me.

Abstract

Our aim is to examine the problem of optimal asset allocation for investors exhibiting a behaviour in the face of uncertainty which is not consistent with the usual axioms of Expected Utility Theory. This thesis is divided into two main parts.

In the first one, comprising Chapter II, we consider an arbitrage-free discrete-time financial model and an investor whose risk preferences are represented by a possibly non-concave utility function (defined on the non-negative half-line only). Under straightforward conditions, we establish the existence of an optimal portfolio.

As for Chapter III, it consists of the study of the optimal investment problem within a continuous-time and (essentially) complete market framework, where asset prices are modelled by semi-martingales. We deal with an investor who behaves in accordance with Kahneman and Tversky's Cumulative Prospect Theory, and we begin by analysing the well-posedness of the optimisation problem. In the case where the investor's utility function is not bounded above, we derive necessary conditions for well-posedness, which are related only to the behaviour of the distortion functions near the origin and to that of the utility function as wealth becomes arbitrarily large (both positive and negative).

Next, we focus on an investor whose utility is bounded above. The problem's well-posedness is trivial, and a necessary condition for the existence of an optimal trading strategy is obtained. This condition requires that the investor's probability distortion function on losses does not tend to zero faster than a given rate, which is determined by the utility function. Provided that certain additional assumptions are satisfied, we show that this condition is indeed the borderline for attainability, in the sense that, for slower convergence of the distortion function, there does exist an optimal portfolio.

Finally, we turn to the case of an investor with a piecewise power-like utility function and with power-like distortion functions. Easily verifiable necessary conditions for well-posedness are found to be sufficient as well, and the existence of an optimal strategy is demonstrated.

Keywords: Attainability ; Behavioural finance ; Choquet integral ; Dynamic programming ; Finite horizon ; Non-concave utility ; Optimal portfolio ; Probability distortion ; Well-posedness.

American Mathematical Society–Mathematics Subject Classification (2010): 91G10 (Primary), 49J55 ; 49L20 ; 60H30 ; 91B16 ; 91G80 ; 93E20 (Secondary).

Lay summary

A standard problem in the literature of financial mathematics is that of choosing the best investment in the assets traded in the market. This optimal portfolio choice problem has been widely studied under the assumptions of Expected Utility Theory (henceforth abbreviated to EUT), which assumes in particular that all investors are averse to risk and that their risk preferences can be represented by a globally concave function, named utility.

Throughout the years, as many of EUT's key axioms have been challenged by paradoxes and experiments, various substitute theories for decision making in the face of uncertainty have been elaborated, such as Cumulative Prospect Theory (CPT). Within this framework, the existence of a reference point defining gains and losses is assumed, a feature that is absent in EUT. Moreover, the utility function is now assumed to have an S-shape, because even though investors generally exhibit risk aversion on gains, they tend to become risk-seeking when undergoing losses. Lastly, according to CPT, economic agents in the real world find it hard to evaluate the real probabilities of events, and instead perceive them in a biased way (which is modelled with so-called distortion functions).

This work presents a study of the portfolio optimisation problem for investors whose behaviour is not in agreement with the basic tenets of EUT. In the first part we treat the case of an investor with a non-concave utility (in a market where trading occurs only at a finite number of dates), whereas in the second part we incorporate the reference point and the distortion functions as well (in a market in which assets are traded continuously). The latter part in particular involves a considerable degree of complexity, especially since many of the most common mathematical tools used to solving the EUT portfolio problem are not suitable anymore.

Acknowledgements

To be great, be whole: exclude
Nothing, exaggerate nothing that is you.
Be whole in everything. Put all you are
Into the smallest thing you do.
The whole moon gleams in every pool,
It rides so high.

Ricardo Reis, 'To be great, be whole', in *Odes*,
tr. and ed. by E. Honig and S. M. Brown
(*Poems of Fernando Pessoa*, 1986).

More than ten years later, as I face these written pages, I remember that distant day when this thesis was born. I cannot think of a more perfect beginning to a piece of work, especially when it is that of years which were far from being of solitude. But let us not get ahead of the story.

The great Fernando Pessoa once wrote that 'God wills, man dreams, the work is born.'[1] So, with sixteen years of age and a very clear picture in my mind of who I wanted to be when I grew up, I decided to plan the whole rest of my life. As carefully and thoroughly as if it were one of my trip itineraries. Of course, I did not care much about randomness at that time; life can be funny that way. That is beside the point, though. Adapting Descartes, we can say that, on that day, this thesis was dreamed, therefore it was born.

That was the day that marked the beginning. Today, more than ten years later, as I face the beginning of the end written on these pages, I cannot help but remember the journey that brought me here. And I realise how long and demanding and sometimes intimidating it was. But solitude, that I cannot find, for I have been most fortunate in the people whose paths have intersected mine.

Starting with my first PhD supervisor, Dr Miklós Rásonyi, without whose guidance, wisdom, patience and encouragement this work would not have been done. I could write here how thankful I will always be to him for introducing me to a fascinating problem, for his help and concern and words of advice, for his understanding and generosity, for everything I was able to learn from him. But that would come short of showing the depth of my gratitude. So, instead, I shall thank him only (in the hope that it will sum it all up) for inspiring me, by example, to continue the pursuit of becoming not only a better mathematician, but also a better person. All of that while keeping in mind that 'le mieux est l'ennemi du bien'.[2] And that I should take as many walks as possible.

There have been many other great people who have influenced and motivated me during the course of my academic life, from my high-school Mathematics teacher (who, without knowing, awoke in me the wish to become a mathematician) to my university professors (whose engaging lectures only reinforced my belief that I could not have

[1] 'Deus quer, o homem sonha, a obra nasce.' from 'O Infante' (l. 1), in *Mensagem* (1934).
[2] Voltaire, from 'La Bégueule: conte morale' (1772, l. 2).

made a better choice). In particular, I am extremely obliged to Prof. João Pedro Nunes and to Prof. Isabel Simão, and it is not an overstatement to say that I would not be where I am today if it had not been for them. The Probability and Stochastic Analysis research group at the School of Mathematics at the University of Edinburgh also deserve my sincerest thank you, as it has been a privilege to be part of such a welcoming, gracious and supporting group. I am truly indebted as well to our Graduate School Administrator, Mrs Gill Law, for always being so kind, understanding and helpful.

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I would now like to address a heartfelt word of gratitude to all the people who constantly remind me that Mathematics is a wonderful part of life, but not the whole of life's wonders.

Such as my Portuguese friends back home, who despite the 2000 or so kilometres that separate Lisbon from Edinburgh, have stuck with me all these years. I could not have asked for better people with whom to grow up. You are my yesterday, my today, and I hope my tomorrow too. With new and obviously hilarious Math jokes.

Such as the Portuguese friends I have met in Edinburgh, who have brought a little bit of home to an initially foreign land. Thank you for everything we have shared, from the cod to the language mishaps, from the fado and football to the saudade. I have been so blessed that I have even won a new godmother and a new nephew.

Last, but most certainly not least, such as the non-Portuguese friends I was lucky to make here. In addition to beautiful places, Scotland is also full of beautiful people.

> *Edina! Scotia's darling seat!*
> . . . .
> *Thy sons, Edina, social, kind,*
> *With open arms the stranger hail;*
>
> Robert Burns, from "Address to Edinburgh" (1786, ll. 1, 17–18)

You were the ones who, in time, made leaving home feel like coming home. Thank you for enlarging my world, for giving me more reasons and occasions to celebrate, for bringing new words, music and flavours into my life. Knowing you has made me richer.

All in all, I wish to thank each and every one of my friends for everything we have lived together. And I do not just mean the big things, but the small ones as well, for it is of those that the 'spectacle of the world'[3] is made. So thank you! Asante! Dhanyavād! Dòjeh! Efharistó! Gracias! Grazie! Istuti! Köszönöm! Merci! Shukriya! Tak! Teşekkür ederim! Xièxie! Obrigada![4]

I must also mention my stronghold, my fixed point in life, my adorably crazy family, who love me with all my flaws and despite them.

A very special thank you is due to my solid, remarkable grandparents. I could not be more proud of whom I come from. In a time when life was already difficult enough, they made it a point that their children, and later their children's children, should be raised to become righteous and accomplished people. From them we all learned dignity

[3]From 'Sábio é o que se contenta com o espectáculo do mundo' (l. 1) by Ricardo Reis, in *Odes*.

[4]Special thanks to Alison for proofreading these Acknowledgements and for precious suggestions.

and humility[5], the value of everything and everyone, the importance of hard work and the power of perseverance. You did well. I may not listen to the rosary every day or be a 'home-fairy' yet, but trust me when I say that you did very well and that I cannot thank you enough for that. You can finally call me 'doctor', avó!

Secondly, I am very grateful for my cousins. I look at us all grown up and I cannot help but still see the children we used to be. You are the closest thing I have to siblings and you made my childhood a magical time. I will never be able to remember *Jeux sans Frontières*, Ken dolls, Tamagotchi toys, metal detectors, playing catch or stargazing without smiling the widest smile. Thank you for all the singing, fighting and laughing. The only thing I know in life which is as good as having old cousins is having new ones.

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Finally (I always like to save the best things for last, like a dessert), I wish to thank Isabel, my pillar of a mum. Thank you for being a model of strength, determination, commitment. And constancy, you know how much that means to me. Besides, when almost everyone was trying to convince me to go to medical school (fainting is clearly overrated), thank you for supporting my seemingly eccentric dream of becoming a mathematician (without ever giving me the slightest hint that you were less proud of me for that). Better yet, thank you for supporting all my dreams. Thank you for all the times you are strict and for all the times you spoil me. For being such a demanding and giving person. For every soup. Thank you, not only for your unshakable faith in me, but also for never letting me settle for less than what you believe I am capable of accomplishing.

Recomeça. . .	Start again. . .
Se puderes,	If you can,
Sem angústia e sem pressa.	Without anguish and without haste.
. . . .	. . . .
Enquanto não alcances	Until you reach [your goal]
Não descanses.	Do not rest.
De nenhum fruto queiras só metade.	Of no fruit should you want only half.

Miguel Torga, from "Recomeça" (ll. 1–3, 8–10)

Thank you for being the push that I need every time I resist flying solo. And for being my safety net (I never have to look down to be sure that you are there). Thank you so much for a love which helps me grow without smothering me (unlike mine for my cacti). I know it is a very small thing when compared to everything you have given me, but this thesis is for you. Of course, by now you must be thinking that these acknowledgements have already too many words, but the truth is that they will always have too few. So I shall finish by simply saying obrigada, mãe. For everything. I owe you all that I am.

To conclude, I would like to remark that 'the soul is divine and the work is imperfect.'[6] Thus, any errors or omissions which may remain in the content of this thesis are solely my responsibility. 'All else is up to God!'[7]

[5]'Perfection is impossible without humility. "Why should I strive for perfection, if I am already good enough?"', by Leo Tolstoy, in *A calendar of wisdom: daily thoughts to nourish the soul*, tr. and ed. by P. Sekirin (2010).

[6]'A alma é divina e a obra é imperfeita.', from 'Padrão' (l. 5) by Pessoa, in *Mensagem* (1934).

[7]'Tudo o mais é com Deus!', from 'D. Pedro' (l. 12) by Pessoa, in *Mensagem* (1934).

Contents

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List of figures

List of abbreviations

(NA)	Arbitrage-free condition
a.e.	Almost everywhere or almost every
a.s.	Almost surely
AE	Asymptotic elasticity
CDF	Cumulative distribution function
CPT	Cumulative Prospect Theory
càdlàg	Right-continuous with left-limits
	(from the French "continu(e) à droite, limité(e) à gauche")
DP	Dynamic programming
E(L)MM	Equivalent (local) martingale measure
EUT	Expected Utility Theory
HJB	Hamilton-Jacobi-Bellman
LLAD	Large-loss aversion degree
NFLVR	No free lunch with vanishing risk
PDE	Partial differential equation
r.v.	Random variable
RAE	Reasonable asymptotic elasticity
SDE	Stochastic differential equation

List of symbols

$\triangleq$ means that the expression on the left is *defined* to be equal to the expression on the right

General notation

$\mathbb{1}_X$ *indicator function* of the set X

$\bigsqcup$ *pairwise disjoint union*

$\circ$ function *composition*

$\mathrm{cl}\,(X)$ topological *closure* of the set X

$\diamond$ indicates the *end of a remark*

$\lceil\cdot\rceil$ (respectively, $\lfloor\cdot\rfloor$) *ceiling function* (respectively, *floor function*)

$\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ *Euclidean inner product*

$\|\cdot\|_{\mathbb{R}^n}$ *Euclidean norm*

$\log$ denotes the *natural logarithm*

$\mathbb{B}^n\,(x_0,r) \triangleq \{x\in\mathbb{R}^n\colon\ \|x-x_0\|_{\mathbb{R}^n}<r\}$ *open ball* of centre $x_0\in\mathbb{R}^n$ and radius $r>0$

sgn *signum function*

$\mathscr{P}\,(X)$ *power set* of X, i.e., the collection of all subsets of X

$\overline{\mathbb{R}} \triangleq \mathbb{R}\cup\{-\infty,+\infty\}=[-\infty,+\infty]$ *extended* real number line

$\square$ *end of proof* (or *Halmos*) mark

$\top$ (superscript) matrix *transposition*

$\varnothing \triangleq \{\}$ *empty set*

$\mathbf{0}_{\boldsymbol{n}}$ *null vector* of $\mathbb{R}^n$

$\vee$ (respectively, $\wedge$) *maximum operator* (respectively, *minimum operator*), i.e., $x\vee y \triangleq \max\{x,y\}$ (respectively, $x\wedge y\triangleq\min\{x,y\}$) for any $x,y\in\mathbb{R}$

$AE_+\,(f)$ *asymptotic elasticity* of the function f

$E_f\,(x)$ *elasticity* of the function f at the point x

$f\,(+\infty)$ *limit* (if it exists) of the real-valued function f as $x\to+\infty$

$x^+ \triangleq \max\{x,0\}$ (respectively, $x^-\triangleq-\min\{x,0\}$) for every $x\in\mathbb{R}$

X^c *complement set* of X

Measure theory

ℓ *Lebesgue measure* on $\mathbb{R}$

$\operatorname{ess\,sup}_{f\in\Theta} f$ (respectively, $\operatorname{ess\,inf}_{f\in\Theta} f$) *essential supremum* (respectively, *essential in-*

fimum) of the family Θ of measurable functions

$\operatorname{ess\,sup}_\mu f$ (respectively, $\operatorname{ess\,inf}_\mu f$) *essential supremum* (respectively, *essential infimum*) of the function f with respect to the measure μ

$\int_X f\,d\mu$ *Lebesgue integral* of f with respect to the measure μ

$\mathscr{B}(S)$ *Borel σ-algebra* of the topological space (S,τ)

$\mathbb{P}^{X|\mathscr{G}}$ *regular conditional distribution* of the function X given the σ-algebra $\mathscr{G}$, under the probability measure $\mathbb{P}$

$\operatorname{supp}(\mu)$ *topological support* of the measure μ

$\oint_X f\,d\mu$ *Choquet integral* of f with respect to the capacity μ

$\operatorname{Proj}_X(E) \triangleq \{x \in X\colon \exists\, y \in Y \text{ such that } (x,y) \in E\}$ *projection* of $E \subseteq X \times Y$ on X

$F_X^{\mathbb{P}}$ (respectively F_μ) *cumulative distribution function* of the r.v. X with respect to the probability measure $\mathbb{P}$ (respectively, of the probability measure μ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$)

$L^0(X,\Sigma,\mu)$ vector space of (equivalence classes of) measurable functions $f\colon X \to \mathbb{R}$

$L^p(X,\Sigma,\mu)$ vector space of (equivalence classes of) functions $f \in L^0(X,\Sigma,\mu)$ verifying $\int_X |f|^p\,d\mu < +\infty$ (with $p \in (0,+\infty)$)

$q_X^{\mathbb{P}}$ (respectively, q_μ) *quantile function* of the r.v. X with respect to the probability measure $\mathbb{P}$ (respectively, of the probability measure μ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$)

$U \overset{\mathbb{P}}{\sim} \mathscr{U}(I)$ means that the r.v. U has a *(continuous) uniform distribution* on the non-degenerate interval $I \subseteq \mathbb{R}$ with respect to the probability measure $\mathbb{P}$

Convex analysis

$\operatorname{aff}(X)$ *affine hull* of the set X

$\operatorname{conv}(X)$ *convex hull* of the set X

$\mathcal{E}(X)$ set of *extreme points* of the set X

$\operatorname{lin}(X)$ *linear span* of the set X

$\operatorname{ri}(X)$ *relative interior* of the set X

Multi-functions

$\operatorname{dom}\mathscr{S} \triangleq \{x \in X\colon \mathscr{S}(x) \neq \varnothing\}$ *domain* of the multi-function $\mathscr{S}$

$\operatorname{gph}\mathscr{S} \triangleq \{(x,y) \in X \times \mathbb{R}^n\colon x \in \operatorname{dom}\mathscr{S} \text{ and } y \in \mathscr{S}(x)\}$ *graph* of the multi-function $\mathscr{S}$

$\mathscr{S}\colon X \rightrightarrows \mathbb{R}^n$ *multi-function* from the set X to the Euclidean space $\mathbb{R}^n$

$\mathscr{S}^{-1}(U) \triangleq \{x \in X\colon \mathscr{S}(x) \cap U \neq \varnothing\}$ *inverse image* under the multi-function $\mathscr{S}$ of the set $U \subseteq \mathbb{R}^n$

Financial mathematics

β_t, κ_t see Proposition II.2.11

$\Delta S_t \triangleq S_t - S_{t-1}$ (or $\Delta S_t \triangleq S_t - S_{t-}$)

$\mathbb{F}$ *filtration* describing the information accruing to the agents in the economy

$\mathbb{P}$ *physical probability*

$\mathbb{P}^*$ ELMM

$\mathscr{A}(x_0)$ set of *feasible portfolios*

$\mathscr{M}_e(S)$ (respectively, $\mathscr{M}_e^{loc}(S)$) set of *all EMM* (respectively, *all ELMM*)

$\mathscr{W}$ class of real-valued random variables with finite moments of all (positive) orders, see eq. (II.4.5)

Ω *sample space*, that is, set of all possible market scenarios

$\overline{\Omega}_t$ see page 10

$\overline{\phi}$ *portfolio*

$\overline{\phi}^*$ *optimal portfolio*

$\overline{\varphi}_{x_0}$ *trivial portfolio*

Φ (respectively, $\Phi(x_0)$) class of *self-financing* portfolios (respectively, *starting from initial capital* x_0)

$\Pi^{\overline{\phi}}$ *wealth process* of the portfolio $\overline{\phi}$

Ψ (respectively, $\Psi(x_0)$) set of all *admissible* strategies (respectively, *for the initial capital* x_0)

$\rho \triangleq d\mathbb{P}^*/d\mathbb{P}$ *state price density*

$\widehat{\Omega}_t$ see Proposition II.2.10

$\widetilde{\Xi}_{t-1}^d$ see page 10

$\widetilde{S} \triangleq S/S^0$ *discounted value process* of S

$\Xi_{t-1}^d(H)$ (or $\Xi_{t-1}^d(x)$), see page 10

Ξ_t^n family of all $\mathscr{F}_t$-measurable, $\mathbb{R}^n$-valued random vectors

$D_t(\omega)$ see page 10

$M(x)$ see Lemma II.5.8

$M(x)^\circ, Z(x)$ see Lemma II.5.9

S *price process* of the risky assets

S_0 *price process* of the riskless asset

T *trading horizon*

u *utility function*

u_+ (respectively, u_-) *utility on gains* (respectively, *on losses*)

$V(X)$ *prospect functional* of the random variable X, see Definition III.2.10

$v^*(x_0)$ supremum of the optimal portfolio problem

$V_\pm(X)$ *Choquet integrals* of the random variable X, see eq. (III.2.7)

x_0 investor's *initial capital*

r *spot rate process*

Chapter I

Introduction and summary

1 Introduction

It seems reasonable to assume that any economic agent possessing a certain amount of wealth wishes to invest it in the most favourable way. Finding the most desirable investment strategies is precisely in what consists the optimal portfolio choice problem, a classic one in the field of financial mathematics.

A natural question that instantly arises concerns the criteria governing the investors' choices. After the ground-breaking work of Bernoulli [10], and ever since the seminal papers of Merton [39, 40] and Samuelson [55], Expected Utility Theory appears to have been one of the predominantly used theories in the literature for modelling decision making under uncertainty. This theory, also commonly known as EUT, was formulated by von Neumann and Morgenstern [62], and states that every rational investor's individual preferences (formally described by a binary relation satisfying certain properties) can be numerically measured and represented by a function, called utility (for a thorough treatment of this subject, the interested reader is referred e.g. to Föllmer and Schied [25, Sections 2.1–2.3]). Then, assuming that there is no intermediate consumption and that the investment horizon is finite, within this framework the investors will always choose a portfolio whose terminal value maximises their expected utility.

Different investors are allowed to possess different utilities, but it is assumed that these functions must all satisfy some common basic properties, with well-established economic meanings.

Firstly, the domain of the real-valued von Neumann-Morgenstern utility function has to include all possible outcomes of investment. For example, when wealth is restricted to be non-negative, the utility can be taken to be defined on the positive half-line only, whereas in the case where wealth is allowed to become negative, the domain must consist of the whole real line.

Moreover, every utility should be strictly increasing in wealth, to translate the fact that investors are greedy and non-satiable (or equivalently, their preference relations are monotone), that is, more is always better than less and no agent will ever have so much wealth that getting more would not be at least a little bit desirable.

In addition, the utilities are assumed to be continuous, because small modifications in wealth should have small impacts on preferences.

While these assumptions tend to be more or less undisputed, there is another one regarding the shape of the utility which has been met with a great deal of criticism, as we shall see below. Indeed, EUT postulates that all investors are risk-averse in the face of uncertainty at all times, which is reflected by a globally concave utility (so the marginal utility of wealth decreases as wealth increases).

Finally, according to EUT, all investors are completely rational and fully capable of objectively assessing probabilities.

Now, this theory is particularly appealing and relatively simple in that there exists a great variety of tools which can be employed for solving the portfolio optimisation problem.

One of the main approaches makes use of the dynamic programming (DP) principle, which roughly speaking allows for the problem to be broken into two related problems over two time intervals, the aim being to determine the optimal decision at each stage so that the resulting combined decision is globally optimal (we refer e.g. to Pham [44, Chapter 3]). This principle, whose proof relies not only on a Markov assumption on the asset prices and on measurable selection theorems, but also on the the dynamic consistency (or tower property) of the conditional expectation, is invoked to derive (through an application of Itô's formula) a partial differential equation (PDE), known in the literature as the Hamilton-Jacobi-Bellman (HJB) equation. Then, after obtaining a smooth solution (or showing its existence) by PDE methods, one concludes by checking that this smooth solution is indeed the value function of the optimisation problem (this is called the verification step, which again makes use of Itô's formula and gives, as a by-product, an optimal trading strategy).

Another common approach, which is more general (in the sense that it has weaker assumptions), involves martingale theory and convex duality techniques. The basic idea is the following: by appealing to the Itô martingale representation theorem, the original portfolio problem is transformed into a static optimisation problem subject to a linear budget constraint. Given that the utility function is globally concave, we are dealing with a concave maximisation problem, therefore we can apply methods of convex analysis to formulate and solve a dual problem, which in turn gives the solution to the primal problem (for a detailed account of this theory in continuous-time markets where prices are modelled by Itô processes, see for example Karatzas and Shreve [32, Chapter 3] and the references therein).

EUT has had several important applications, ranging from portfolio optimisation, to indifference pricing (thus providing an important method for valuing non-hedgeable claims) and to the theory of risk measures (a famous one being the entropic risk measure, which is associated with the exponential utility function).

However, over the years, some of EUT's fundamental principles have been systematically questioned by several paradoxes (e.g., Allais [2] paradox and Ellsberg [23] paradox) and empirical studies. Namely, the latter have demonstrated that decision makers do not evaluate wealth in terms of final asset states, but with respect to a reference point (which defines gains and losses). Also, whilst the investors are generally risk-averse on gains, they were found to exhibit a risk-seeking behaviour when making decisions involving losses. Furthermore, in reality, the economic agents are subjective and have a distorted perception of the actual probabilities (for example, small probabilities tend to be exaggerated, which may not only explain the Allais paradox, but also the attractiveness of insurance and gambling, as remarked for instance in Kahneman and Tversky [31]).

The attempts to incorporate the above psychological findings into economic theory have lead to the emergence of several alternative theories, amongst which is the Prospect Theory proposed by Kahneman and Tversky [31] (later amended in Tversky and Kahneman [61] to ensure that first order stochastic dominance is not violated[1], and re-named Cumulative Prospect Theory or CPT for short). The main tenets of this theory, for which Kahneman was awarded the Nobel Memorial Prize in Economics in the year 2002, are essentially the following: firstly, the existence of a reference point for each investor; secondly, the consideration of a utility function (now called prospect value function), defined on the real line, which is S-shaped (that is, concave and convex on the positive and negative half-lines, respectively); lastly, the use of probability weighting functions to model the way in which the investors miscalculate the physical probability measure (leading to the appearance of possibly non-linear Choquet [17] integrals). These make the study of the portfolio problem within the CPT framework substantially different from that of the EUT portfolio problem.

In this work, we shall therefore investigate the finite-horizon optimal portfolio problem for investors who violate the axioms of EUT, namely for a rational investor with a non-concave utility function in a discrete-time financial market, as well as for an agent behaving consistently with CPT in a continuous-time model.

2 Summary

A brief outline of this work is as follows.

Chapter II treats the discrete-time asset allocation problem when the investor has a non-concave utility function and when wealth is restricted to remain non-negative. In Section II.1 we present a short overview of some of the relevant literature on the subject. Section II.2 is dedicated to briefly recalling some of the basic concepts and results. In particular, the market model is specified, which is followed by the rigorous description of

[1] That is, an investor will choose a portfolio over another if the terminal value of the latter is second-order stochastically dominated by the terminal value of the former. We recall that, given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the random variable (r.v.) X has *first-order stochastic dominance* (respectively, *second-order stochastic dominance*) over the r.v. Y if $\mathbb{P}\{X \geq x\} \geq \mathbb{P}\{Y \geq x\}$ (respectively, $\int_{-\infty}^{x} (\mathbb{P}\{Y \leq y\} - \mathbb{P}\{X \leq y\})\, dy \geq 0$) for all x, with strict inequality for some x, and it is clear that first-order stochastic dominance implies second-order stochastic dominance.

the investor and by the mathematical formulation of the non-concave portfolio problem. Additionally, we construct a very simple toy example in which we emphasise some of the differences to EUT portfolio optimisation. Next, in Section II.3 we examine the problem in a one-step setting, while in Section II.4 we state our main result, which is proved using a dynamic programming approach. For the sake of exposition, all auxiliary results and proofs are compiled in Section II.5.

On the other hand, the topic of Chapter III is the continuous-time asset allocation problem for a behavioural investor in an essentially complete model. As before, Section III.1 provides a brief highlight of some of the existing literature, as well as of the main challenges of CPT. In the following Section III.2, we introduce a general model, make precise the tenets of CPT, rigorously state the behavioural portfolio problem, and return to the toy model of the preceding chapter to illustrate the importance of the changes introduced by CPT with respect to EUT in the optimisation problem. Then, in Section III.3, we restrict ourselves to the case of a continuous-time financial market, and we lay down the main assumptions which will be in force throughout the remainder on the chapter. The issues of well-posedness and attainability (see Definitions III.2.16 and III.2.17) are dealt with, and examples are included to demonstrate that the obtained results hold in important model classes. Moreover, in Section III.4 and Section III.5 we respectively consider the special cases where the utility on gains is bounded above and where both the utilities and the distortions are power functions. Once more, to improve readability, we collect the proofs and the auxiliary results in Section III.6.

Finally, Chapter IV, which concludes the present manuscript, summarises our most important findings and contributions, whilst suggesting ideas for further research.

Chapter II

Non-concave portfolio optimisation

1 Introduction

The work in this chapter will be the content of the future paper [14].

It focuses on the problem of choosing an optimal strategy for an investor who has a possibly non-concave utility function, defined on the non-negative half-line only. We shall be working in a finite-horizon and discrete-time frictionless market setting. Below is gathered a short description of some of the relevant research conducted in this subject.

Amongst several significant papers developed within the EUT framework, it is imperative to mention that by Kramkov and Schachermayer [35], as well its successor by Schachermayer [58]. In a continuous-time and not necessarily complete market model where it is assumed that asset prices are modelled with locally bounded semi-martingales and that the set of equivalent local martingale measures is non-empty, both papers show the existence of a solution to the optimal portfolio problem, provided that a certain condition involving the asymptotic elasticity of the utility is satisfied (famously named the reasonable asymptotic elasticity condition). This is accomplished via well-known tools of convex analysis. We further recall that, while in the first paper the utility is defined on the non-negative axis (for wealth is constrained to remain non-negative), the domain of the utility in the latter one is the whole real line. Despite their unquestionable importance, both papers, however, apply only to utility functions which are concave and smooth enough.

With regard to discrete-time and also possibly incomplete market models, we would like to point out the work of Rásonyi and Stettner [50] for a utility function on the whole real line, which is complemented by the paper [51] by the same authors for a utility on the non-negative half-line. In both cases, the existence of an optimal strategy is proved, this time by exploiting a dynamic programming technique. Moreover, weaker asymptotic elasticity conditions are imposed, and neither the smoothness of the utility nor the boundedness of the asset price processes are required. Yet, as before, the utilities are supposed to be concave.

A study of the optimal portfolio choice problem for a possibly non-concave and nonsmooth utility on the whole real line is presented in Carassus and Rásonyi [12]. There, again by a dynamic programming argument, it is established that a condition on the growth of the utility (associated with the notion of asymptotic elasticity) is sufficient to ensure that an optimal solution does in fact exist. Our goal here is to complement this paper, in the sense that we wish to solve the same problem, but for a non-concave utility whose domain is the non-negative axis (while borrowing ideas from Rásonyi and Stettner [51]).

In addition, we mention Reichlin [52, Chapter II], who studies the same optimisation problem for non-concave utility functions on the non-negative half-line which are of strictly sublinear growth[1]. Besides proving the existence of an optimal investment strategy under a condition involving the asymptotic elasticity and using methods of convex analysis, the author also examines some of its properties. Nonetheless, the existence result therein is not comparable to ours since, in Reichlin [52], the work is conducted with respect to a fixed martingale measure, whereas we impose some restrictions (see our Assumption II.2.16).

Finally, we remark that, in all of the aforementioned papers, rather than looking for conditions ensuring the well-posedness of the optimisation problem (i.e., that its supremum is finite), the authors simply assume that well-posedness holds true (which, as they remark, is not enough to imply the existence of an optimal strategy). This common practice will also be adopted in this chapter (we refer to our equation (II.4.3)).

2 Notation and set-up

2.1 The market

In what follows, we shall consider a *frictionless* and totally *liquid* financial market, that is, in which all costs and restraints associated with transactions are non-existent, where investors are allowed to short-sell stocks and to borrow money, and finally where it is always possible to buy or sell an unlimited number of shares of any asset.

Let the current time be denoted by 0, and let us fix a finite *trading horizon* $T \in \mathbb{N}$. We assume further that this is a discrete-time market, that is, that trading occurs only at finitely many points in time, or *trading dates*, given by $\{0, 1, \ldots, T\}$.

Also, the uncertainty in the economy is characterised by a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $\mathscr{F}$ a σ-algebra on the sample space Ω, and $\mathbb{P}$ the underlying probability measure (to be interpreted as the physical probability) on $(\Omega, \mathscr{F})$. Moreover, all the information accruing to the agents in the economy is described by a discrete filtration $\mathbb{F} = \{\mathscr{F}_t;\, t \in \{0, 1, \ldots, T\}\}$ such that $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets, where the fact that no information is lost is expressed by the non-decreasing nature of $\mathbb{F}$. Finally, we assume for convenience that the σ-algebra $\mathscr{F}_0$ is $\mathbb{P}$-trivial[2], and also that $\mathscr{F} = \mathscr{F}_T$.

[1] A function $f: [0, +\infty) \to \mathbb{R}$ is of *strictly sub-linear growth* if $\lim_{x \to +\infty} f(x)/x = 0$.

[2] Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a σ-algebra $\mathscr{G} \subseteq \mathscr{F}$ is $\mathbb{P}$-*trivial* if, for every $A \in \mathscr{G}$, either $\mathbb{P}(A) = 0$ or $\mathbb{P}(\Omega \setminus A) = 0$.

Next, we fix some strictly positive integer d, and we consider an $\mathbb{R}^d$-valued (not necessarily locally bounded) process $S = \{S_t;\, t \in \{0, 1, \ldots, T\}\}$, where S_t represents the prices at time t of d traded *risky assets*[3]. We shall consider as well the existence of a *risk-free asset*, also called money market, whose price process $S^0 = \left\{S_t^0;\, t \in \{0, 1, \ldots, T\}\right\}$ satisfies $S_0^0 = 1$ and evolves, as shown below, according to a non-negative *spot rate* process $r = \{r_t;\, t \in \{1, \ldots, T\}\}$,

$$S_t^0 \triangleq \prod_{s=1}^{t} (1 + r_s)\,, \qquad t \in \{1, \ldots, T\}\,,$$

where r_t is $\mathscr{F}_{t-1}$-measurable. Then, borrowing the notation[4]

$$\overline{S}_t \triangleq \left(S_t^0, S_t^\top\right)^\top = \left(S_t^0, S_t^1, \ldots, S_t^d\right)^\top, \qquad t \in \{0, 1, \ldots, T\}\,,$$

from Föllmer and Schied [25], and introducing, for each $n \in \mathbb{N}$ and $t \in \{0, 1, \ldots, T\}$, the family Ξ_t^n of all $\mathscr{F}_t$-measurable random vectors $\xi :\ \Omega \to \mathbb{R}^n$, we assume that the process $\overline{S}$ is adapted to the filtration $\mathbb{F}$, or in other words, $\overline{S}_t \in \Xi_t^{d+1}$ for every $t \in \{0, 1, \ldots, T\}$. Note that the adaptedness of the price process translates the fact that the price vector $\overline{S}_t$ is known at time t. In addition, let us designate the quantity $\left(S_t^0\right)^{-1}$ as the *discount factor* at time t, and define the $\mathbb{R}^d$-valued discounted value process of S as follows,

$$\widetilde{S} = \left\{\widetilde{S}_t = S_t / S_t^0;\, t \in \{0, 1, \ldots, T\}\right\}.$$

Finally, for each $t \in \{1, \ldots, T\}$, we define the $\mathscr{F}_t$-measurable, d-dimensional random vector $\Delta S_t \triangleq S_t - S_{t-1}$.

We now recall the following.

Definition II.2.1 (Trading strategy). We say that an $\mathbb{F}$-adapted, $\mathbb{R}^{d+1}$-valued stochastic process $\overline{\phi} = \left\{\overline{\phi}_t = \left(\phi_t^0, \phi_t^\top\right)^\top;\, t \in \{1, \ldots, T\}\right\}$, with $\phi_t = \left(\phi_t^1, \ldots, \phi_t^d\right)^\top$, is a *trading strategy* or *portfolio* if it is predictable with respect to $\mathbb{F}$, i.e., if it verifies

$$\overline{\phi}_t \in \Xi_{t-1}^{d+1}, \quad \forall\, t \in \{1, \ldots, T\}\,. \tag{II.2.1}$$

Then, for every $t \in \{1, \ldots, T\}$ and $i \in \{0, 1, \ldots, d\}$, $\overline{\phi}_t^i$ represents the number of shares of the i-th asset which are held by the investor at time t.

Remark II.2.2. *(i)* The predictability of the portfolio process means that the positions in the portfolio for each time $t \in \{1, \ldots, T\}$ are decided based on the information available at time $t-1$, and kept until time t, when the new prices S_t are revealed.

(ii) We recall that both short-selling and borrowing are allowed, which accounts for the fact that no constraints are imposed on the sign of $\overline{\phi}_t^i$. ◇

[3]These can be, for example, common stocks, commodities, foreign currencies, exchange rates or market indices, to name only a few.

[4]Here, the superscript $\top$ denotes matrix *transposition*.

To each portfolio is obviously associated a value, which is naturally defined as shown below.

Definition II.2.3 (Wealth process). The value of a trading strategy $\overline{\phi}$ at each time $t \in \{0, 1, \ldots, T\}$ is given by[5]

$$\Pi_t^{\overline{\phi}} \triangleq \begin{cases} \langle \overline{\phi}_1, \overline{S}_0 \rangle_{\mathbb{R}^{d+1}} = \phi_1^0 + \langle \phi_1, S_0 \rangle_{\mathbb{R}^d}, & \text{if } t = 0, \\ \langle \overline{\phi}_t, \overline{S}_t \rangle_{\mathbb{R}^{d+1}} = \phi_t^0 S_t^0 + \langle \phi_t, S_t \rangle_{\mathbb{R}^d}, & \text{otherwise,} \end{cases} \tag{II.2.2}$$

and we call $\Pi^{\overline{\phi}} = \left\{ \Pi_t^{\overline{\phi}};\, t \in \{0, 1, \ldots, T\} \right\}$ the *wealth process* of $\overline{\phi}$. Moreover, we say that $\overline{\phi}$ *starts from initial capital* $x_0 \in \mathbb{R}$ if $\Pi_0^{\overline{\phi}} = x_0$ almost surely (a.s.).

So we can introduce the following important concept.

Definition II.2.4 (Self-financing). A trading strategy $\overline{\phi}$ is called *self-financing* if, for every $t \in \{1, \ldots, T-1\}$, the equality

$$\Pi_t^{\overline{\phi}} = \langle \overline{\phi}_{t+1}, \overline{S}_t \rangle_{\mathbb{R}^{d+1}} \tag{II.2.3}$$

holds a.s.. We denote by Φ the class of all self-financing portfolios, and by $\Phi(x_0)$ the class of all self-financing portfolios starting from initial capital $x_0 \in \mathbb{R}$.

Remark II.2.5. The self-financing condition means that, before each time $t \in \{2, \ldots, T\}$ and after the prices $\overline{S}_{t-1}$ are known, the investor adjusts his position from $\overline{\phi}_{t-1}$ to $\overline{\phi}_t$ without injecting or withdrawing any wealth. It can be easily checked that a portfolio $\overline{\phi}$ is self-financing if and only if

$$\forall\, t \in \{1, \ldots, T\}, \quad \Pi_t^{\overline{\phi}} = \Pi_0^{\overline{\phi}} + \sum_{s=1}^{t} \left[\phi_s^0 \left(S_s^0 - S_{s-1}^0 \right) + \langle \phi_s, \Delta S_s \rangle_{\mathbb{R}^d} \right] \text{ a.s.}. \tag{II.2.4}$$

◇

We proceed with the mathematical statement of the notion of arbitrage, which can be described in lay terms as the opportunity to, out of nothing and without risk, make a sure profit.

Definition II.2.6 (Arbitrage opportunity). We say that a portfolio $\overline{\phi} \in \Phi(0)$ is an *arbitrage opportunity* if its value process satisfies both conditions below,

(i) for every $t \in \{1, \ldots, T\}$, $\Pi_t^{\overline{\phi}} \geq 0$ a.s.,

(ii) $\mathbb{P}\left\{ \Pi_T^{\overline{\phi}} > 0 \right\} > 0$.

In order to keep the notation simple, henceforth we shall suppose, without loss of generality, that the risk-free asset has constant price, that is, for every $t \in \{1, \ldots, T\}$ we have $S_t^0 = 1$ a.s. (i.e., the spot rate is null). Hence, we work directly with discounted prices and, like in Øksendal [43, Definition 12.1.1], we say that the market is *normalised*.

[5] We recall that, given $x, y \in \mathbb{R}^n$, the *Euclidean inner product* is defined by $\langle x, y \rangle_{\mathbb{R}^n} \triangleq \sum_{i=1}^n x_i y_i$. Moreover, we denote by $\|x\|_{\mathbb{R}^n} \triangleq \sqrt{\langle x, x \rangle_{\mathbb{R}^n}}$ the *Euclidean norm* of x.

Then, as in Rásonyi and Stettner [51], we shall impose the trading constraint that the value of a portfolio should not be allowed to become strictly negative.

Definition II.2.7 (Admissibility). Given $x_0 \geq 0$, we say that a portfolio $\overline{\phi} \in \Phi(x_0)$ is *admissible for* x_0 if

$$\forall t \in \{1, \ldots, T\}, \quad \Pi_t^{\overline{\phi}} \geq 0 \text{ a.s.}. \tag{II.2.5}$$

We denote by $\Psi(x_0)$ the set of all admissible strategies for x_0.

Remark II.2.8. Neither Rásonyi and Stettner [50], nor Carassus and Rásonyi [12] place any admissibility restriction on the portfolios, and therefore they allow portfolios whose value processes not only may take strictly negative values with positive probability, but also are not necessarily bounded from below. We observe, however, that our admissibility condition, despite considerably restricting the set of admissible portfolios (cf. Remark 2.1 in Carassus and Rásonyi [12]), is frequently adopted in the literature. Moreover, it is not a completely unreasonable or unrealistic one, as in pratice it means the existence of a credit limit (strictly negative positions, i.e. debts, are forbidden). ◇

Given that, in reality, arbitrage opportunities, when they occur, are immediately exploited by the investors, and thus tend to be eliminated quickly, it is only natural to impose the following throughout.

Assumption II.2.9 (Absence of arbitrage). *The market is arbitrage-free, that is,*

$$\textit{if } \overline{\phi} \in \Psi(0)\,, \textit{ then } \Pi_T^{\overline{\phi}} = 0 \textit{ a.s.}. \tag{NA}$$

Now fix $t \in \{1, \ldots, T\}$. We know that there exists a regular conditional distribution[6] of ΔS_t with respect to $\mathscr{F}_{t-1}$, under the physical measure $\mathbb{P}$ (see e.g. Theorem 10.2.2 in Dudley [22, pp. 345–346]), which we shall denote by[7]

$$\mathbb{P}^{\Delta S_t | \mathscr{F}_{t-1}} : \mathscr{B}\Big(\mathbb{R}^d\Big) \times \Omega \to [0, 1]\,.$$

[6]Here, $\mathbb{1}_A : X \to \{0, 1\}$ denotes the *indicator function* of a subset A of a set X, defined by

$$\mathbb{1}_A(x) \triangleq \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, letting $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, (S, Σ) be a measurable space, $X : \Omega \to S$ be a $(\mathscr{F}, \Sigma)$-measurable function, and $\mathscr{G} \subseteq \mathscr{F}$ be a σ-algebra over Ω containing all $\mathbb{P}$-null sets, we recall that a *regular conditional distribution* of X given $\mathscr{G}$, under $\mathbb{P}$, is a function $\mathbb{P}^{X|\mathscr{G}} : \Sigma \times \Omega \to [0, 1]$ for which the two conditions below hold true,

(i) for each $E \in \Sigma$, the function $\mathbb{P}^{X|\mathscr{G}}(E, \cdot) : \Omega \to [0, 1]$ is measurable with respect to $\mathscr{G}$, and there exists $\check{\Omega} \in \mathscr{G}$ with $\mathbb{P}(\check{\Omega}) = 1$ such that $\mathbb{P}^{X|\mathscr{G}}(E, \omega) = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_E(X) \,|\, \mathscr{G}](\omega)$ for every $\omega \in \check{\Omega}$,

(ii) there is some $\overline{\Omega} \in \mathscr{G}$ with $\mathbb{P}(\overline{\Omega}) = 1$ such that, for every $\omega \in \overline{\Omega}$, the function $\mathbb{P}^{X|\mathscr{G}}(\cdot, \omega) : \Sigma \to [0, 1]$ is a probability measure on (S, Σ).

[7]We recall that, given a non-empty set S, a collection τ of subsets of S is a *topology* on Y if both the empty set and S belong to τ, if any union of elements of τ belongs to τ, and if any finite intersection of elements of τ is also in τ. We call the elements of τ the *open sets* in S, and the complement of an open set is said to be *closed*. Then the *Borel σ-algebra* of S, denoted by $\mathscr{B}(S)$, is the σ-algebra generated by the open sets of τ.

By definition of regular conditional distribution, we can find a set $\overline{\Omega}_t \in \mathscr{F}_{t-1}$, with $\mathbb{P}(\overline{\Omega}_t^c) = 0$, such that for every $\omega \in \overline{\Omega}_t$, the map

$$\mathbb{P}^{\Delta S_t|\mathscr{F}_{t-1}}(\cdot,\omega) : \mathscr{B}\Big(\mathbb{R}^d\Big) \to [0,1]$$

is a probability measure on $\big(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)\big)$. It then follows that, for each $\omega \in \overline{\Omega}_t$, the support[8] of the probability measure $\mathbb{P}^{\Delta S_t|\mathscr{F}_{t-1}}(\cdot,\omega)$ exists and is non-empty. Thus, if $\omega \in \overline{\Omega}_t$, let $\operatorname{supp}\big(\mathbb{P}^{\Delta S_t|\mathscr{F}_{t-1}}(\cdot,\omega)\big)$ and $D_t(\omega)$ respectively denote the aforementioned support and its affine hull in $\mathbb{R}^d$, otherwise simply set $D_t(\omega) \triangleq \mathbb{R}^d$.

Clearly, each $D_t(\omega)$ is an affine subspace of $\mathbb{R}^d$. We further observe that, for every $\omega \notin \overline{\Omega}_t$, $D_t(\omega)$ is actually a linear space. The next result shows that, under the no-arbitrage Assumption II.2.9, so too is $D_t(\omega)$ for $\mathbb{P}$-almost every ($\mathbb{P}$-a.e.) $\omega \in \overline{\Omega}_t$.

Proposition II.2.10. *Suppose (NA) is verified. Then, for every $t \in \{1,\dots,T\}$, there exists a subset $\widehat{\Omega}_t$ of $\overline{\Omega}_t$ satisfying all of the following three conditions,*

(i) *$\widehat{\Omega}_t$ belongs to the σ-algebra $\mathscr{F}_{t-1}$,*

(ii) $\mathbb{P}\Big(\overline{\Omega}_t \setminus \widehat{\Omega}_t\Big) = 0$,

(iii) *for every $\omega \in \widehat{\Omega}_t$, the affine space $D_t(\omega)$ is actually a linear subspace of $\mathbb{R}^d$.*

Proof. See Section II.5, page 30. □

Furthermore, for each fixed $t \in \{1,\dots,T\}$, we define two important families of functions. Firstly, given any $\mathscr{F}_{t-1}$-measurable random variable $H \geq 0$ a.s., we set

$$\Xi^d_{t-1}(H) \triangleq \Big\{\xi \in \Xi^d_{t-1}\colon\ H + \langle \xi, \Delta S_t\rangle_{\mathbb{R}^d} \geq 0 \text{ a.s.}\Big\}.$$

In the particular case where $H = x$ a.s. for some $x \in \mathbb{R}^+_0$, we have

$$\Xi^d_{t-1}(x) \triangleq \Big\{\xi \in \Xi^d_{t-1}\colon\ x + \langle \xi, \Delta S_t\rangle_{\mathbb{R}^d} \geq 0 \text{ a.s.}\Big\}.$$

On the other hand, we take $\widetilde{\Xi}^d_{t-1}$ to be the class of all random vectors $\xi \in \Xi^d_{t-1}$ such that $\xi(\omega) \in D_t(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \overline{\Omega}_t$, and by abuse of language we shall write that 'ξ belongs to D_t a.s.'.

We end this subsection with an alternative and rather useful characterisation of the arbitrage-free condition (NA).

[8]Given a topological space (S,τ) and its Borel σ-algebra $\mathscr{B}(S)$, let μ be a measure on the measurable space $(S,\mathscr{B}(S))$. A τ-closed subset F of S is called the *(closed) topological support* of μ if it is the smallest closed set whose complement is a μ-null set. If it exists, we denote the support of μ by $\operatorname{supp}(\mu)$. The support of μ exists whenever (S,τ) is *second countable* (i.e., there exists a countable collection $\mathscr{O} = \{O_n\colon n \in \mathbb{N}\}$ of open sets in S such that every element of τ can be written as a union of elements of $\mathscr{O}$). Moreover, $\mu(S) = \mu(\operatorname{supp}(\mu))$, which implies in particular that $\operatorname{supp}(\mu) \neq \varnothing$ if $\mu(S) > 0$.

Proposition II.2.11. *The following two statements are equivalent,*

(i) (NA) *holds true,*

(ii) *for every $t \in \{1,\dots,T\}$, there exist two $\mathscr{F}_{t-1}$-measurable, real-valued random variables $\beta_t : \Omega \to \mathbb{R}$ and $\kappa_t : \Omega \to \mathbb{R}$ such that $\beta_t > 0$ a.s., $\kappa_t > 0$ a.s. and*

$$\mathbb{P}\big(\langle \xi, \Delta S_t\rangle_{\mathbb{R}^d} \le -\beta_t \,\|\xi\|_{\mathbb{R}^d} \,\big|\, \mathscr{F}_{t-1}\big) \ge \kappa_t \text{ a.s. on } \widetilde{\Omega}_t \triangleq \Big\{\omega \in \widehat{\Omega}_t \colon D_t(\omega) \neq \{\mathbf{0}_d\}\Big\} \tag{II.2.6}$$

for all $\xi \in \widetilde{\Xi}^d_{t-1}$.

Proof. This is essentially Proposition 3.3 in Rásonyi and Stettner [50]. □

Remark II.2.12. *(i)* It follows from the proof of Proposition II.2.11 that, for every $t \in \{1,\dots,T\}$, we have $\beta_t \le 1$ a.s. as well. It is also trivial that, for every $t \in \{1,\dots,T\}$, the inequality $\kappa_t \le 1$ holds a.s.. Furthermore, in the particular case where S has independent increments, the random variables β_t and κ_t can be taken to be deterministic.

(ii) Like in Remark 2.3 of Carassus and Rásonyi [12], we draw attention to the fact that the above 'quantitative' characterisation of (NA) holds true only for $\mathscr{F}_{t-1}$-measurable, $\mathbb{R}^d$-valued functions ξ which belong to D_t a.s.. This is what will motivate the use of orthogonal projections later on (cf. Proposition II.3.7). ◇

2.2 The investor

As explained in detail in the Introduction, every investor's risk preferences are described by a utility function, which must satisfy certain properties with well-established economic motivations.

Definition II.2.13 (Non-concave utility). A function $u : [0,+\infty) \to \mathbb{R} \cup \{-\infty\}$ is called a *utility* (on the non-negative half-line) if it is non-decreasing and continuous, and moreover its effective domain, given by

$$\operatorname{dom} u \triangleq \{x \in [0,+\infty) : u(x) > -\infty\}, \tag{II.2.7}$$

is a non-empty subset of $[0,+\infty)$.

Remark II.2.14. *(i)* Given that, in this chapter, we restrict wealth to be non-negative, we must consider utilities which are defined only over the non-negative real line.

(ii) We do not make any assumption concerning the differentiability of u. More importantly, we do not require that a utility function should be concave. ◇

We proceed by noticing that, since u is a monotone function, the limit

$$u(+\infty) \triangleq \lim_{x\to+\infty} u(x)$$

exists (though it may not be finite). Moreover, we present below some examples of utility functions, most of which being quite well-known in the literature.

Example II.2.15. (i) *The* (negative) exponential utility *with real parameter* $\alpha > 0$ *is the function* $u : [0, +\infty) \to [0, +\infty)$ *defined as*

$$u(x) \triangleq 1 - e^{-\alpha x}, \qquad x \geq 0. \tag{II.2.8}$$

We note that u *is strictly concave on its domain and also that* $u(+\infty) = 1 < +\infty$, *thus* $AE_+(u) = 0$ *(see Lemma A.12).*

(ii) *The* power utility *with parameter* $\alpha \in \mathbb{R} \setminus \{0\}$ *is the function given by*

$$u(x) \triangleq \begin{cases} x^\alpha, & \text{if } \alpha > 0, \\ 1 - (1+x)^\alpha, & \text{if } \alpha < 0, \end{cases} \tag{II.2.9}$$

for all $x \in [0, +\infty)$. *We note that* u *is strictly concave (respectively, strictly convex) on its domain if and only if* $\alpha < 1$ *(respectively,* $\alpha > 1$*). It is also trivial that* u *is bounded above if and only if* $\alpha < 0$. *In particular, this implies that* $AE_+(u) = 0$ *when* $\alpha < 0$ *(again by Lemma A.12). Finally, for* $\alpha > 0$ *we obtain* $AE_+(u) = \alpha$, *since in this case the elasticity of* u *at every point* $x > 0$ *equals* $E_u(x) = x\left(\alpha x^{\alpha-1}\right)/x^\alpha = \alpha$, *thus motivating the alternative designation of* isoelastic utility *(in fact, this is the only utility which has constant elasticity, see Proposition A.3).*

(iii) *The* logarithmic utility *is the function* $u : [0, +\infty) \to [0, +\infty)$ *defined by*

$$u(x) \triangleq \log(1 + x), \qquad x \geq 0. \tag{II.2.10}$$

Clearly, u *is concave on its domain and* $u(+\infty) = +\infty$. *Moreover,* $AE_+(u) = 0$.

(iv) *The* log-log utility *is the strictly concave function* $u : [0, +\infty) \to [0, +\infty)$ *given by*

$$u(x) \triangleq \log(1 + \log(1 + x)), \qquad x \geq 0. \tag{II.2.11}$$

This too is a function that goes to infinity as x *gets arbitrarily large, but it does so very slowly. In fact, the log-log utility grows more slowly (as* $x \to +\infty$*) than any strictly positive power of* $\log(x)$.

(v) *The* hyperbolic utility *with parameter* $\alpha \in (1, +\infty)$ *is the function defined as*

$$u(x) \triangleq \frac{\alpha x}{x + \alpha - 1}, \qquad x \geq 0. \tag{II.2.12}$$

It is straightforward to check that u *is strictly concave with* $u(+\infty) = \alpha < +\infty$, *thus another application of Lemma A.12 gives* $AE_+(u) = 0$.

(vi) *The function* $u : [0,+\infty) \to [0,+\infty)$ *defined by*

$$u(x) \triangleq \begin{cases} \sqrt{x}, & \text{if } x \in [0,1)\,, \\ 2x/\left(2+\log(x)\right), & \text{if } x \geq 1. \end{cases} \tag{II.2.13}$$

is a utility with $u(+\infty) = +\infty$. *Also,* u *is of strictly sub-linear growth, but it goes to infinity (as* $x \to +\infty$*) faster than* x^p *for any power* $p \in (0,1)$. *Finally,* $AE_+(u) = 1$.

(vii) *Let* $\alpha > 0$ *and* $\varpi \in (0,1)$, *and consider* $u : [0,+\infty) \to [0,+\infty)$ *given by*[9]

$$u(x) \triangleq \begin{cases} \exp\left\{\alpha \operatorname{sgn}(x-1)\,|\log(x)|^{\varpi}\right\}, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly, this utility function satisfies $u(+\infty) = +\infty$. *We also notice that it is concave on* $[x_0,+\infty)$, *for some* $x_0 = x_0(\alpha,\varpi) \geq 0$. *Furthermore, while* u *grows to infinity (as* $x \to +\infty$*) faster than any strictly positive power of* $\log(x)$, *it does so more slowly than* x^p *for any power* $p > 0$. *Lastly,* $AE_+(u) = 0$.

(viii) *The function* $u : [0,+\infty) \to [0,+\infty)$ *defined by*

$$u(x) \triangleq 1 + x - e^{-x}, \qquad x \geq 0,$$

is a strictly concave utility with $u(+\infty) = +\infty$ *and* $AE_+(u) = 1$. *Moreover,* $\lim_{x\to+\infty} u(x)/x = 1$.

Finally, we shall require the following.

Assumption II.2.16. *The value of* u *at* 0 *is a real number, i.e.,* $u(0) > -\infty$.

Remark II.2.17. Note that, under the assumption above, we have by monotonicity that $\operatorname{dom} u = [0,+\infty)$. We also recall that this assumption is not required in the treatment of the concave case given by Rásonyi and Stettner [51]. Even though we are aware that this is quite restrictive a condition (for example, it excludes the widely employed utility $u(x) \triangleq \log(x)$, $x \geq 0$), we do not see how to avoid it in the present setting. ◇

2.3 The optimal portfolio problem

As already described in the Introduction, the optimal portfolio problem consists in choosing the best investment in the assets, the criterion in the current chapter being the maximisation of expected utility. In other words, since we are assuming that there is no intermediate consumption, the investors with a given non-negative initial capital

[9] We recall that the *signum function*, $\operatorname{sgn} : \mathbb{R} \to \{-1,0,1\}$, is defined as follows,

$$\operatorname{sgn}(x) \triangleq \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

x_0 wish to select, from all allowable portfolios, the investment strategies whose terminal wealths give them the highest expected utility.

Thus, the first question which should be addressed is what feasible set $\mathscr{A}(x_0)$ to consider for the above problem. The first obvious observation is that the only self-financing portfolios which are available to an investor with initial capital x_0 are those which start from initial wealth x_0. Moreover, because we are imposing the condition that the wealth process should remain non-negative, we must further restrict ourselves to the set $\Psi(x_0)$. Thirdly, we can only allow those portfolios for which the expected utility is well-defined in the generalised sense[10]. This can be summarised below.

Definition II.2.18 (Feasible portfolios). A strategy $\overline{\phi} \in \Phi$ is said to be *allowable* (or *feasible*) for the non-concave portfolio problem (NCPP) with initial wealth $x_0 \in [0, +\infty)$ if it belongs to

$$\mathscr{A}(x_0) \triangleq \left\{ \overline{\phi} \in \Psi(x_0) \colon \mathbb{E}_{\mathbb{P}}\left[\left[u\Big(\Pi_T^{\overline{\phi}}\Big) \right]^+ \right] < +\infty \text{ or } \mathbb{E}_{\mathbb{P}}\left[\left[u\Big(\Pi_T^{\overline{\phi}}\Big) \right]^- \right] < +\infty \right\}. \tag{II.2.14}$$

We call $\mathscr{A}(x_0)$ the *feasible set* (or *set of feasible portfolios*).

Remark II.2.19. It is immediate to check that the feasible set is non-empty. Indeed, for every $x_0 \in [0, +\infty)$, the trivial portfolio $\overline{\varphi}_{x_0}$ given by

$$(\varphi_{x_0})_t^0(\omega) \triangleq x_0 \quad \text{and} \quad (\varphi_{x_0})_t^i(\omega) \triangleq 0, \quad \forall\, \omega \in \Omega,\ \forall\, t \in \{1, \ldots, T\},\ \forall\, i \in \{1, \ldots, d\}, \tag{II.2.15}$$

(that is, consisting in investing all of the wealth on the bond and none on the risky assets) belongs to $\mathscr{A}(x_0)$. ◇

Then the non-concave portfolio optimisation problem can be mathematically formalised as follows.

Definition II.2.20 (Non-concave portfolio choice problem). Given any $x_0 \geq 0$, the *non-concave portfolio problem* with initial wealth x_0 on a finite horizon T is written as

$$\sup\left\{ \mathbb{E}_{\mathbb{P}}\left[u\Big(\Pi_T^{\overline{\phi}}\Big) \right] \colon \overline{\phi} \in \mathscr{A}(x_0) \right\}. \tag{NCPP}$$

Setting $v^*(x_0) \triangleq \sup\left\{ \mathbb{E}_{\mathbb{P}}\left[u\Big(\Pi_T^{\overline{\phi}}\Big) \right] \colon \overline{\phi} \in \mathscr{A}(x_0) \right\}$, we say that $\overline{\phi}^* \in \mathscr{A}(x_0)$ is an *optimal strategy* if

$$v^*(x_0) = \mathbb{E}_{\mathbb{P}}\left[u\Big(\Pi_T^{\overline{\phi}^*}\Big) \right]. \tag{II.2.16}$$

[10] Here $x^+ \triangleq \max\{x, 0\}$ and $x^- \triangleq -\min\{x, 0\}$ for every $x \in \mathbb{R}$. Then, given a measure space (X, Σ, μ) and a measurable function $f \colon X \to \overline{\mathbb{R}}$, we recall that $\int_X f\, d\mu$ *exists* (or is *well-defined*) if $\int_X f^+\, d\mu < +\infty$ or $\int_X f^-\, d\mu < +\infty$, in which case we set

$$\int_X f\, d\mu < +\infty \triangleq \int_X f^+\, d\mu - \int_X f^-\, d\mu,$$

possibly taking the values $+\infty$ or $-\infty$.

Our aim for the remainder of this chapter is to establish, obviously under suitable conditions, the existence of an optimal portfolio. In order to do so, we shall employ a dynamic programming technique, which will allow us to split the original problem into several smaller sub-problems. Then, at each stage or time step, we shall try to find an optimal solution. At last, combining all of these optimal one-step solutions in an adequate way should give an optimal strategy for the original problem. Thus, the next chapter will be dedicated to the study of the one-step case.

Remark II.2.21. *(i)* One may wonder why the existence of an optimal $\overline{\phi}^*$ is relevant when the existence of ε-optimal strategies $\overline{\phi}^\varepsilon$ (i.e., ones that are ε-close to the supremum over all strategies) is automatic, for all $\varepsilon > 0$. There are at least two, closely related reasons for this.

Firstly, non-existence of an optimal strategy $\overline{\phi}^*$ usually means that an optimiser sequence $\left\{\overline{\phi}^{1/n}; n \in \mathbb{N}\right\}$ shows wild, extreme behaviour (e.g., they converge to infinity, see Example 7.3 of Rásonyi and Stettner [50]). Such strategies are both practically infeasible and economically counter-intuitive.

Secondly, existence of $\overline{\phi}^*$ normally goes together with some compactness property. Such a property seems necessary for the convergence of any potential numerical procedure to find an optimal (or at least an ε-optimal) strategy.

(ii) In particular, the preceding Remark II.2.19 implies that $v^*(x_0) \geq u(x_0)$. ◇

We conclude this section with the following result, which says that, if we wish to have any hope of finding an optimal portfolio, then (NA) cannot be dropped when the utility u is assumed to be strictly increasing.

Proposition II.2.22. *Let $x_0 \in [0, +\infty)$ be arbitrary. Suppose that u is strictly increasing on $[0, +\infty)$, and that $v^*(x_0) < +\infty$. Then, under Assumption II.2.16, there exists an optimal portfolio $\overline{\phi}^* \in \mathscr{A}(x_0)$ for (NCPP) only if (NA) holds true.*

Proof. This is a restatement of Proposition 3.1 in Rásonyi and Stettner [50]. □

2.4 Toy example: one-period binomial model

In this subsection, we present a very basic example, which is destined to show how having a non-concave utility (instead of a globally concave one) may affect the investors' decisions.

Thus, let us consider a one-period binomial model of a financial market with two assets, a risk-free asset and a risky asset, which are priced at the current time 0 and at the maturity $T = 1$. As in Section II.2, we may and shall assume that the bond has constant price $S_0^0 = S_1^0 = 1$. Moreover, we assume again for simplicity that S_1^1 can only take two possible values, a higher one h with probability $p \in (0, 1)$, and a lower one l with probability $1 - p$, where $0 < l < 1 < h$.

We are then considering the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $\Omega \triangleq \{0, 1\}$, $\mathscr{F} \triangleq \mathscr{P}(\Omega)$ the power set of Ω, $\mathbb{P}(\{1\}) = p$, and

$$S_1^1(\omega) = \begin{cases} l, & \text{if } \omega = 0, \\ h, & \text{if } \omega = 1. \end{cases}$$

Furthermore, we equip the probability space with a filtration $\mathbb{F} = \{\mathscr{F}_0, \mathscr{F}_1\}$, where $\mathscr{F}_0 = \{\varnothing, \Omega\}$ is the trivial σ-field and $\mathscr{F}_1 = \mathscr{F}$. It is a well-established fact in the literature that this model is free of arbitrage (we refer e.g. to Föllmer and Schied [25, Section 5.5]). In addition, assuming that the investor starts with non-negative initial wealth x_0, then set of all admissible strategies equals

$$\Psi(x_0) = \left\{ \left(\phi^0, \phi^1\right) \in \mathbb{R}^2 \colon \phi^0 = x_0 - \phi^1 \text{ and } \phi^1 \in \left[-\frac{x_0}{h-1}, \frac{x_0}{1-l} \right] \right\}.$$

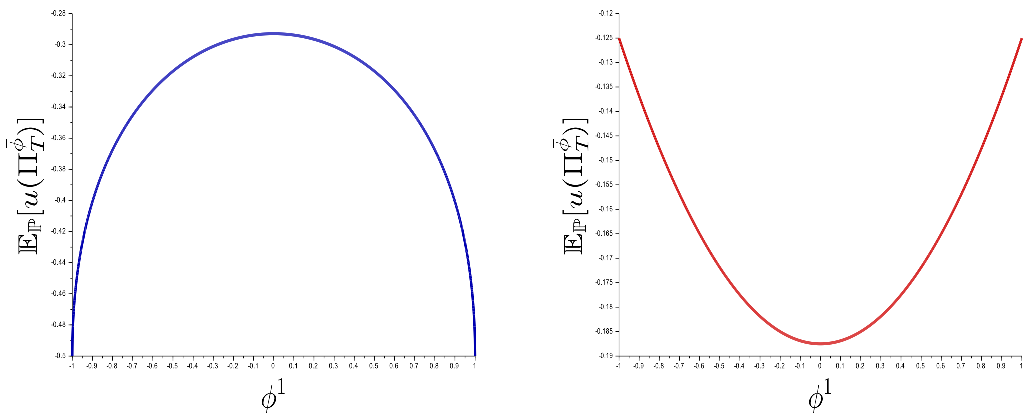
So suppose first that the investors' behaviour is consistent with EUT, and that they have a power utility (defined on the non-negative half-line only, for wealth is constrained to remain non-negative),

$$u(x) \triangleq x^\alpha - 1, \qquad x \geq 0,$$

with parameter $\alpha = 1/2$. Also, take $x_0 = 1/2$, $h = 3/2$, $l = 1/2$ and $p = 1/2$. Then, within this framework, the investors wish to find

$$v^*(1/2) = \sup \left\{ \frac{1}{2} \left(\sqrt{\frac{1+\phi^1}{2}} - 1 \right) + \frac{1}{2} \left(\sqrt{\frac{1-\phi^1}{2}} - 1 \right) : \phi^1 \in [-1, 1] \right\},$$

and we conclude that the best strategy for the investors is not to invest in the stock, but to allocate all of their wealth on the bond (i.e., $\overline{\phi}^* = (1/2, 0)$ and the optimal terminal wealth is $\Pi_1^{\overline{\phi}^*} = 1/2$, see Figure II.1a).



(a) EUT portfolio choice problem. **(b)** Non-concave portfolio choice problem.

Figure II.1: The functions to be maximised are plotted with respect to ϕ^1. The market parameters are $x_0 = 1/2$, $u = 3/2$, $l = 1/2$ and $p = 1/2$.

Next, let us suppose instead that the investors have the following non-concave utility function,

$$u(x) \triangleq \begin{cases} \left(x^2-1\right)/4, & \text{if } x \in [0,1], \\ x^{1/2}-1, & \text{if } x > 1. \end{cases}$$

so the function to be maximised over $\phi^1 \in [-1,1]$ is now

$$\mathbb{E}_{\mathbb{P}}\left[u\left(\frac{1}{2}+\phi^1\left(S_1^1-1\right)\right)\right] = \frac{1}{8}\left[\left(\frac{1+\phi^1}{2}\right)^2-1\right]+\frac{1}{8}\left[\left(\frac{1-\phi^1}{2}\right)^2-1\right].$$

It can be easily checked that, in this case, there exist two optimal portfolios: one that consists of borrowing money to buy one share of the stock (which has terminal wealth $(-1)+1/2=-1/2$ if the stock price goes down and $(-1)+3/2=1/2$ when the stock price goes up); the other one which involves short-selling one stock to invest in the bond (with terminal wealth $1-1/2=1/2$ if the stock price goes down and $1-3/2=-1/2$ if the stock performs well). Hence, not only did the optimal investment strategy change, but also it is no longer unique (it can be noted from Figure II.1b that the functional to be maximised is not globally concave any more). It is also worth pointing out that the optimal portfolio of EUT actually became the least attractive of all admissible portfolios.

Hence, this very simple example provides an important motivation for the study of the optimal portfolio problem for investors with a non-concave utility, since the conclusions may be completely different, perhaps even opposite, to those of EUT.

3 Optimal strategy in the one-step case

In this section, we consider an $\mathscr{F}$-measurable function $Y: \Omega \to \mathbb{R}^d$, and a σ-algebra $\mathscr{G} \subseteq \mathscr{F}$ containing all $\mathbb{P}$-null sets of $\mathscr{F}$. This setting will be applied in the next section with $\mathscr{G} = \mathscr{F}_{t-1}$ and $Y = \Delta S_t$, for every fixed $t \in \{1,\ldots,T\}$.

Keeping in line with the notation of the previous section, we denote by Ξ^d the family of all $\mathscr{G}$-measurable functions $\xi: \Omega \to \mathbb{R}^d$.

We shall also impose the following throughout, which can be regarded as absence of arbitrage at each single-time period (cf. Assumption II.2.9).

Assumption II.3.1. *For every $\xi \in \Xi^d$, if $\langle \xi, Y\rangle_{\mathbb{R}^d} \geq 0$ a.s., then $\langle \xi, Y\rangle_{\mathbb{R}^d} = 0$ a.s..*

Moreover, let $\mathbb{P}^{Y|\mathscr{G}}: \mathscr{B}(\mathbb{R}^d) \times \Omega \to [0,1]$ be the unique (up to a set of measure zero) regular conditional distribution for Y given $\mathscr{G}$ (its existence and uniqueness being ensured for example by Theorem 10.2.2 in Dudley [22]). We know, by definition of regular conditional distribution, that there exists some set $\overline{\Omega} \in \mathscr{G}$, with $\mathbb{P}\left(\overline{\Omega}^c\right) = 0$, such that, for any $\omega \in \overline{\Omega}$, the function

$$\begin{aligned} \mathbb{P}^{Y|\mathscr{G}}(\cdot,\omega) : \mathscr{B}\left(\mathbb{R}^d\right) &\to [0,1] \\ E &\mapsto \mathbb{P}^{Y|\mathscr{G}}(E,\omega) \end{aligned}$$

is a probability measure on $\left(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d)\right)$. Now, for each $\omega \in \overline{\Omega}$, let $\operatorname{supp}\left(\mathbb{P}^{Y|\mathscr{G}}(\cdot,\omega)\right)$ represent the support of $\mathbb{P}^{Y|\mathscr{G}}(\cdot,\omega)$ (which exists and is non-empty), and let $D(\omega)$ denote the affine hull of $\operatorname{supp}\left(\mathbb{P}^{Y|\mathscr{G}}(\cdot,\omega)\right)$, that is, $D(\omega) \triangleq \operatorname{aff}\left(\operatorname{supp}\left(\mathbb{P}^{Y|\mathscr{G}}(\cdot,\omega)\right)\right)$. On the other hand, when $\omega \notin \overline{\Omega}$, simply consider $D(\omega) \triangleq \mathbb{R}^d$. Then obviously each $D(\omega)$ is an affine subspace of $\mathbb{R}^d$. We note further that, for every ω outside $\overline{\Omega}$, $D(\omega)$ is actually a linear subspace. It is not difficult to check the following (the proof being identical to that of Proposition II.2.10).

Proposition II.3.2. *Under Assumption II.3.1, there exists a subset $\widehat{\Omega}$ of $\overline{\Omega}$, satisfying $\mathbb{P}\left(\overline{\Omega} \setminus \widehat{\Omega}\right) = 0$, and such that $D(\omega)$ is actually a vector (or linear) subspace of $\mathbb{R}^d$, for all $\omega \in \widehat{\Omega}$.* □

Remark II.3.3. Note that the hypothesis that $\mathscr{G}$ contains all $\mathbb{P}$-null subsets of Ω implies in particular that $\widehat{\Omega} = \overline{\Omega} \cap \left(\overline{\Omega} \setminus \widehat{\Omega}\right)^c \in \mathscr{G}$. We further remark that $\mathbb{P}\left(\widehat{\Omega}^c\right) = 0$. ◇

In addition, for every $\mathscr{G}$-measurable random variable $H : \Omega \to \mathbb{R}$ satisfying $H \geq 0$ a.s., define the set

$$\Xi^d(H) \triangleq \left\{\xi \in \Xi^d \colon \ \langle \xi, Y \rangle_{\mathbb{R}^d} \geq -H \text{ a.s.}\right\}.$$

Then in the particular case where $H = x$ a.s., for some $x \in [0, +\infty)$, we have

$$\Xi^d(x) \triangleq \left\{\xi \in \Xi^d \colon \ \langle \xi, Y \rangle_{\mathbb{R}^d} \geq -x \text{ a.s.}\right\}.$$

Finally, let $\widetilde{\Xi}^d$ denote the family of all functions $\xi \in \Xi^d$ such that $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \overline{\Omega}$.

Remark II.3.4. It is trivial to see that $\widetilde{\Xi}^d$ is non-empty. In fact, let $\xi_0 : \Omega \to \mathbb{R}^d$ be the *null function*, that is, $\xi_0(\omega) \triangleq \mathbf{0_d}$ for every $\omega \in \Omega$. Then we have by Proposition II.3.2 that, for every $\omega \in \widehat{\Omega}$, the affine space $D(\omega)$ is actually a vector space, and hence $\xi_0(\omega) = \mathbf{0_d} \in D(\omega)$. ◇

Like in the preceding section (cf. Proposition II.2.11), we can obtain the following.

Proposition II.3.5. *Under Assumption II.3.1, there exist two $\mathscr{G}$-measurable random variables $\beta : \Omega \to \mathbb{R}$ and $\kappa : \Omega \to \mathbb{R}$ such that $\beta > 0$ a.s., $\kappa > 0$ a.s. and*

$$\mathbb{P}(\langle \xi, Y \rangle_{\mathbb{R}^d} \leq -\beta \, \|\xi\|_{\mathbb{R}^d} | \mathscr{G}) \geq \kappa \text{ a.s. on } \widetilde{\Omega} \triangleq \left\{\omega \in \widehat{\Omega} \colon \ D(\omega) \neq \{\mathbf{0_d}\}\right\}, \qquad \text{(II.3.1)}$$

for all $\xi \in \widetilde{\Xi}^d$. □

Remark II.3.6. We know from Lemma II.5.6 that $\widetilde{\Omega} \in \mathscr{G}$. ◇

The next result essentially says that any portfolio can be replaced with its orthogonal projection[11] on D without altering its value (except possibly on a null set).

[11] Given a linear subspace D of the Euclidean space $\mathbb{R}^n$, the *orthogonal projection* on D is the linear map $\operatorname{pr}_D : \mathbb{R}^n \to D$ where, for every $x \in \mathbb{R}^n$, $\operatorname{pr}_D(x)$ is the (unique) vector in D such that $x - \operatorname{pr}_D(x) \in D^\perp \triangleq \left\{y \in \mathbb{R}^n \colon \ \langle x, y \rangle_{\mathbb{R}^n} = 0, \ \forall x \in D\right\}$.

Proposition II.3.7. *Let $x \geq 0$ and $\xi \in \Xi^d(x)$ be arbitrary, but fixed. Under Assumption II.3.1, if $\widehat{\xi} : \Omega \to \mathbb{R}^d$ is the function defined by*

$$\widehat{\xi}(\omega) \triangleq \begin{cases} pr_{D(\omega)}(\xi(\omega)) , & \text{if } \omega \in \widehat{\Omega}, \\ \xi(\omega) , & \text{otherwise,} \end{cases} \tag{II.3.2}$$

where $pr_M : \mathbb{R}^d \to M$ is the orthogonal projection onto the subspace M of $\mathbb{R}^d$, then $\widehat{\xi}$ belongs to $\Xi^d(x)$ and

$$x + \left\langle \widehat{\xi}, Y \right\rangle_{\mathbb{R}^d} = x + \langle \xi, Y \rangle_{\mathbb{R}^d} \ \ a.s.. \tag{II.3.3}$$

Proof. See Section II.5, page 31. □

Now let $V : [0, +\infty) \times \Omega \to \mathbb{R}$ be a function verifying the properties below.

Assumption II.3.8. *The function V satisfies the following,*

(i) *for any fixed $x \in [0, +\infty)$, the function $V(x, \cdot) : \Omega \to \mathbb{R}$ is measurable with respect to $\mathscr{F}$,*

(ii) *for $\mathbb{P}$-a.e. $\omega \in \Omega$, the function $V(\cdot, \omega) : [0, +\infty) \to \mathbb{R}$ is continuous on $(0, +\infty)$, right-continuous at 0, and increasing on $[0, +\infty)$, with $V(1, \omega) \geq 0$.*

Remark II.3.9. We observe that, for every $x \geq 0$ and for every $\xi \in \Xi^d(x)$, the function mapping each ω in Ω to $V(x + \langle \xi(\omega), Y(\omega) \rangle_{\mathbb{R}^d}, \omega)$ is well-defined, except possibly on a set of $\mathbb{P}$-measure zero. From this time forth, any function on Ω which is defined for $\mathbb{P}$-a.e. ω is considered to be well-defined. ◇

We shall also need the following integrability conditions.

Assumption II.3.10. *For every $x \in [0, +\infty)$,*[12]

$$\operatorname*{ess\,sup}_{\xi \in \Xi^d(x)} \mathbb{E}_{\mathbb{P}} \left[V^+(x + \langle \xi(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) \middle| \mathscr{G} \right] < +\infty \ \ a.s.. \tag{II.3.4}$$

Remark II.3.11. It is obvious that Assumption II.3.10 implies that, for all $x \geq 0$,

$$\operatorname*{ess\,sup}_{\xi \in \Xi^d(x)} \mathbb{E}_{\mathbb{P}} \left[V(x + \langle \xi(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) \middle| \mathscr{G} \right] < +\infty \text{ a.s..} \tag{II.3.5}$$

◇

Assumption II.3.12. *The conditional expectation, with respect to $\mathscr{G}$, of the function $V^-(0, \cdot) : \Omega \to [0, +\infty)$ is finite a.s., i.e.,*

$$\mathbb{E}_{\mathbb{P}} \left[V^-(0, \cdot) \middle| \mathscr{G} \right] < +\infty \ \ a.s.. \tag{II.3.6}$$

Finally, we impose the following growth condition on V.

[12] In order to make the notation less heavy, given any function $f : X \to \mathbb{R}$, we shall write henceforth $f^{\pm}(x) \triangleq [f(x)]^{\pm}$ for all $x \in X$.

Assumption II.3.13. *There exist constants $C > 0$ and $\gamma > 0$ such that*

$$\mathbb{P}\big\{\omega \in \Omega\colon\ V^+(\lambda x, \omega) \leq \lambda^\gamma V^+(x,\omega) + C\lambda^\gamma,\ \text{for all } \lambda \geq 1 \text{ and for all } x \geq 0\big\} = 1. \tag{II.3.7}$$

Next, we proceed with two technical lemmata. The first one essentially states that any admissible portfolio can be replaced with its projection on D without changing the desirability of the portfolio, whilst the second one shows that the set of all admissible strategies in D is bounded.

Lemma II.3.14. *Fix an arbitrary $x \geq 0$. Then, under Assumption II.3.1, for every $\xi \in \Xi^d(x)$, we have*

$$\mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)|\mathscr{G}] = \mathbb{E}_{\mathbb{P}}\Big[V\Big(x + \Big\langle \widehat{\xi}(\cdot)\,, Y(\cdot)\Big\rangle_{\mathbb{R}^d}\,, \cdot\Big)\Big|\mathscr{G}\Big] \ \text{a.s.}, \tag{II.3.8}$$

where $\widehat{\xi}$ is the projection given by (II.3.2).

Proof. See Section II.5, page 31. □

Lemma II.3.15. *Suppose that Assumption II.3.1 is in force. Given any $x_0 \geq 0$, there exists a $\mathscr{G}$-measurable, real-valued random variable K_{x_0} such that $K_{x_0} > x_0$ a.s., and for all $x \in [0, x_0]$ and all $\xi \in \Xi^d(x)$ with $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. ω in $\overline{\Omega}$,*

$$\|\xi\|_{\mathbb{R}^d} \leq K_{x_0} \ \text{a.s.}. \tag{II.3.9}$$

Proof. This is Lemma 2.1 in Rásonyi and Stettner [51]. For a proof, see Section II.5, page 32. □

As for the next lemma, it will allow us to apply certain convergence results for the conditional expectation later on, namely the reverse Fatou lemma.

Lemma II.3.16. *Suppose Assumptions II.3.1, II.3.8, II.3.10 and II.3.13 are in force. Given any $x \geq 0$, there is a non-negative random variable $L_x : \Omega \to \mathbb{R}$ such that $\mathbb{E}[L_x|\mathscr{G}] < +\infty$ a.s. and, for every $\xi \in \Xi^d(x)$ with $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, the inequality*

$$V^+(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \leq L_x \tag{II.3.10}$$

holds almost surely.

Proof. This is Lemma 2.3 in Rásonyi and Stettner [51], and its proof is reproduced in Section II.5, page 33. □

Furthermore, we establish the existence of a regular version of the conditional expectation.

Lemma II.3.17. *Let $x_0 \in [0, +\infty)$ be fixed, and let Assumptions II.3.8 and II.3.10 be in force. For every $\xi \in \Xi^d(x_0)$, there exists a function $G_\xi : [x_0, +\infty) \times \Omega \to \mathbb{R}$ such that*

(i) *for each fixed* $x \geq x_0$, $G_\xi(x,\cdot)$ *is a version of* $\mathbb{E}_{\mathbb{P}}\left[V(x+\langle\xi(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}\right]$,

(ii) *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$, $G_\xi(\cdot,\omega)\colon [x_0,+\infty) \to \mathbb{R}$ *is a non-decreasing and right-continuous function on* $[x_0,+\infty)$.

Proof. The proof, which is presented in Section II.5, page 36, is entirely analogous to that of Proposition 2.1 in Rásonyi and Stettner [51]. □

A regular version of the essential supremum[13] can be shown to exist as well.

Lemma II.3.18. *Under Assumptions II.3.1, II.3.10, II.3.12 and II.3.13, there exists a function* $G\colon [0,+\infty)\times\Omega \to \mathbb{R}$ *satisfying the two properties below,*

(i) $G(x,\cdot)$ *is a version of* $\operatorname{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x+\langle\xi(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}]$ *for each fixed* $x \in [0,+\infty)$,

(ii) *for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$, *the function* $G(\cdot,\omega)\colon [0,+\infty)\to\mathbb{R}$ *is non-decreasing and continuous on* $[0,+\infty)$.

Furthermore, given any $\mathscr{G}$*-measurable random variable* $H \geq 0$ *a.s.,*

$$G(H(\cdot),\cdot) = \operatorname*{ess\,sup}_{\xi\in\Xi^d(H)} \mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}] \quad a.s.. \tag{II.3.11}$$

Proof. See Section II.5, page 39. □

The following result demonstrates that a one-step optimal strategy can be constructed, while also stating some of its properties.

Proposition II.3.19. *Let* G *be the function given by Lemma II.3.18. Under Assumptions II.3.1, II.3.8, II.3.10, II.3.12 and II.3.13, there exists a* $\mathscr{B}([0,+\infty))\otimes\mathscr{G}$*-measurable function* $\widetilde{\xi}\colon [0,+\infty)\times\Omega\to\mathbb{R}^d$ *such that, for every* $x\in[0,+\infty)$, *the equality*

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x+\langle\xi(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}] = \mathbb{E}_{\mathbb{P}}\left[V\left(x+\left\langle\widetilde{\xi}(x,\cdot),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right] \tag{II.3.12}$$

holds a.s.. Moreover, given any $\mathscr{G}$*-measurable random variable* $H\geq 0$ *a.s., we have that* $\widetilde{\xi}(H(\cdot),\cdot)\in\Xi^d(H)$, *that* $\widetilde{\xi}(H(\omega),\omega)\in D(\omega)$ *for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$, *and finally that*

$$G(H(\cdot),\cdot) = \mathbb{E}_{\mathbb{P}}\left[V\left(H(\cdot)+\left\langle\widetilde{\xi}(H(\cdot),\cdot),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right] \quad a.s.. \tag{II.3.13}$$

[13] Given a measure space (X,Σ,μ) and an arbitrary (not necessarily countable) family Θ of measurable functions, we recall that a measurable function f^* is an *essential supremum* (respectively, *essential infimum*) of Θ with respect to μ if it satisfies both conditions below,

(i) for every $f\in\Theta$, $f\leq f^*$ a.e. (respectively, $f\geq f^*$ a.e.),

(ii) if g is a random variable such that, for every for every $f\in\Theta$, $f\leq g$ a.e. (respectively, $f\geq g$ a.e.), then $f^*\leq g$ a.e. (respectively, $f^*\geq g$ a.e.).

We denote by $\operatorname{ess\,sup}_{f\in\Theta} f$ (respectively, $\operatorname{ess\,inf}_{f\in\Theta} f$) the essential supremum (respectively, essential infimum) of Θ. Part (a) of Theorem A.32 in Föllmer and Schied [25] ensures that both the essential infimum and the essential supremum exist. Moreover, each one is unique (up to a μ-null set).

Proof. This is a compilation of Lemma 2.4, Lemma 2.5 and Proposition 2.4 of Rásonyi and Stettner [51]. The idea of the proof can be summarised in the following way. Firstly, we find a sequence of what can be regarded as stepwise-constant strategies, whose values tend to the optimal value for every initial endowment. Then, we apply a compactness argument to find a sub-limit of this sequence of strategies. Finally, we check that this sub-limit is actually an optimal strategy for the non-concave portfolio problem. The reader is referred to Section II.5, page 49 for the details. □

4 Dynamic programming

In this section, we shall follow the path of Rásonyi and Stettner [50], Rásonyi and Stettner [51], and Carassus and Rásonyi [12], and employ a dynamic programming approach to split the original optimisation problem into a number of sub-problems at different trading dates. Our goal is to invoke the results of the preceding section, thus allowing us to obtain an optimal solution at each stage. Combining them in an appropriate way should yield a globally optimal investment strategy.

We shall stipulate the following, for convenience only.

Assumption II.4.1. *The utility function satisfies* $u(1) = 0$*.*

Remark II.4.2. The reason why we may assume $u(1) = 0$ without loss of generality is that, given an arbitrary utility function u, we can translate it vertically and obtain a new utility $\overline{u} : [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ given by $\overline{u}(x) \triangleq u(x) - u(1)$. Then $\overline{u}$ satisfies Assumption II.4.1 and, since $\overline{u}$ differs from u only by a positive affine transformation[14], it is a well-established fact in the literature that these two utilities represent the exact same preferences (see e.g. Theorem 2.21 in Föllmer and Schied [25]). We note further that this does not affect our study of (NCPP) in any way (namely, it does not change the optimal strategies). ◇

We shall also make the following assumption on the growth of u.

Assumption II.4.3. *There exist some constants* $\overline{\gamma} > 0$*,* $\overline{x} > 0$ *and* $c \geq 0$ *such that*

$$u(\lambda x) \leq \lambda^{\overline{\gamma}} u(x) + c \tag{II.4.1}$$

for all $\lambda \geq 1$ *and* $x \geq \overline{x}$*.*

Remark II.4.4. By definition, any utility with finite asymptotic elasticity satisfies the assumption above (with $c = 0$). Thus, in particular, Assumption II.4.3 is true whenever the increasing function u, with $u(1) = 0$, is concave on $[x_0, +\infty)$ for some $x_0 \geq 0$ (by Lemma A.6).

It is also obvious that Assumption II.4.3 holds for all utilities which are bounded above. In fact, taking $c \triangleq u(+\infty) \in (0, +\infty)$, $\overline{x} \triangleq 1$ and any $\overline{\gamma} > 0$, we immediately obtain $u(\lambda x) \leq c \leq \lambda^{\overline{\gamma}} u(x) + c$ for all $x \geq \overline{x}$ and $\lambda \geq 1$.

[14]Given two real-valued functions f and g, we call g a *positive affine transformation* of f if there exist real numbers C_1 and $C_2 > 0$ such that $g(x) = C_1 + C_2 f(x)$ for all x in the domain.

Since there are bounded utilities with infinite AE (see Example A.13), having finite asymptotic elasticity is a sufficient, albeit not necessary, condition for a function to verify Assumption II.4.3.

Finally, we remark that Assumption II.4.3 does not imply that u has strictly linear growth (see e.g. Example II.2.15*(viii)*). ◇

We may now deduce the following auxiliary result, which provides an estimate for all $x \geq 0$, and not only for $x \geq \overline{x}$.

Lemma II.4.5. *Under Assumptions II.4.1 and II.4.3, there is some $C \in (0, +\infty)$ such that, for all $\lambda \geq 1$ and all $x \geq 0$,*

$$u^+(\lambda x) \leq \lambda^{\overline{\gamma}} u^+(x) + C\lambda^{\overline{\gamma}}. \tag{II.4.2}$$

Proof. See Section II.5, page 59. □

Hence, the next result, which is the main one of the present chapter, says that the problem (NCPP) admits an optimal strategy under certain conditions (namely, that the optimisation problem is well-posed, i.e., its supremum is finite).

Theorem II.4.6. *Let Assumptions II.2.9 and II.2.16 and Assumptions II.4.1 and II.4.3 hold true. Assume further that, for every $x_0 \in [0, +\infty)$,*

$$\sup_{\overline{\phi} \in \Psi(x_0)} \mathbb{E}_{\mathbb{P}}\left[u^+\left(\Pi_T^{\overline{\phi}}\right)\right] < +\infty. \tag{II.4.3}$$

Then, for each $x_0 \in [0, +\infty)$, there exists a strategy $\overline{\phi}^* = \overline{\phi}^*(x_0) \in \mathscr{A}(x_0)$ satisfying

$$\mathbb{E}_{\mathbb{P}}\left[u\left(\Pi_T^{\overline{\phi}^*}\right)\right] = v^*(x_0). \tag{II.4.4}$$

Proof. We give here a brief description of the proof, in which a dynamic programming technique is applied, as already mentioned above. In order to be able to use the results of the one-step case and thus get an optimal solution at each time step, we must recursively confirm that all of the required assumptions are fulfilled. Once this is accomplished, we paste together the aforementioned solutions to obtain an investment strategy, which we then show to be an optimal one for (NCPP). The details can be found in Section II.5, page 59. □

Remark II.4.7. *(i)* The case where $x_0 = 0$ is actually trivial. In fact, it is an obvious consequence of the no-arbitrage Assumption II.2.9 that, for every $\overline{\phi} \in \Psi(0)$, we have $\Pi_T^{\overline{\phi}} = 0$ a.s., and so $\mathbb{E}_{\mathbb{P}}\left[u^+\left(\Pi_T^{\overline{\phi}}\right)\right] = u^+(0) = 0$, because $u(0) \leq u(1) = 0$ (recall the monotonicity of u and Assumption II.4.1). Moreover, $\mathbb{E}_{\mathbb{P}}\left[u\left(\Pi_T^{\overline{\phi}}\right)\right] = u(0)$, hence $v^*(0) = u(0)$ and thus all strategies are optimal (or equivalently, the investor is indifferent amongst all admissible strategies).

(ii) Note that (II.4.3) implies in particular that the expectation $\mathbb{E}_{\mathbb{P}}\Big[u\Big(\Pi_T^{\overline{\phi}}\Big)\Big]$ is well-defined (possibly $-\infty$) for every $\overline{\phi} \in \Psi(x_0)$, hence all admissible strategies are feasible for (NCPP), i.e., $\mathscr{A}(x_0) = \Psi(x_0)$. ◇

As a very simple yet important example to which the preceding theorem clearly applies, we mention the case of u bounded above and satisfying Assumption II.2.16 and Assumption II.4.1. Another relevant example is given by the following.

Theorem II.4.8. *Let Assumptions II.2.9 and II.2.16 and Assumptions II.4.1 and II.4.3 hold true. Assume further that* $\|\Delta S_t\|_{\mathbb{R}^d}, 1/\beta_t \in \mathscr{W}$ *for every* $t \in \{1, \dots, T\}$, *where each* β_t *is the random variable given by Proposition II.2.11 and*

$$\mathscr{W} \triangleq \left\{Y \in \Xi_T^1 \colon \mathbb{E}_{\mathbb{P}}[|Y|^p] < +\infty \text{ for all } p > 0\right\}. \tag{II.4.5}$$

Then, for every $x_0 \in [0, +\infty)$, *condition* (II.4.3) *is satisfied and there exists an optimal strategy* $\overline{\phi}^* = \overline{\phi}^*(x_0) \in \mathscr{A}(x_0)$.

Proof. See Section II.5, page 65. □

5 Proofs and auxiliary results

5.1 Auxiliary results

Except where explicitly stated otherwise, assume everything is as in Section II.3.

The following elementary inequality will be used several times, not only in Chapter II, but also in Chapter III.

Lemma II.5.1. *Given any* $s > 0$, *the inequality*

$$|x + y|^s \le C\left(|x|^s + |y|^s\right) \tag{II.5.1}$$

holds for every $x, y \in \mathbb{R}$, *with* $C \triangleq 1 \vee 2^{s-1} > 0$. □

Proposition II.5.2. *Let the multi-function*[15] $\mathscr{S}\colon\ \Omega \rightrightarrows \mathbb{R}^d$ *be as given below,*

$$\mathscr{S}(\omega) \triangleq \begin{cases} \operatorname{supp}\big(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)\big), & \text{if } \omega \in \overline{\Omega}, \\ \mathbb{R}^d, & \text{otherwise.} \end{cases} \tag{II.5.2}$$

Then $\mathscr{S}$ *is measurable and closed-valued, with* $\operatorname{dom}\mathscr{S} = \Omega$.

Proof. That $\mathscr{S}$ is a closed-valued multi-function is a trivial consequence of the definition of support of a measure. Moreover, given that $\mathbb{P}^{Y|\mathscr{G}}\big(\mathbb{R}^d, \omega\big) = 1$ for every $\omega \in \overline{\Omega}$, it

[15] Given a measurable space (X, Σ), we recall that $\mathscr{S}\colon\ X \rightrightarrows \mathbb{R}^n$ is a *multi-function* (or a *set-valued mapping*) from X to the Euclidean space $\mathbb{R}^n$ if every $x \in X$ is associated with exactly one subset $\mathscr{S}(x)$ of $\mathbb{R}^n$. Moreover, we call a multi-function $\mathscr{S}$ *closed-valued* if, for all $x \in X$, the set $\mathscr{S}(x)$ is closed in $\mathbb{R}^n$. In addition, the *inverse image* under $\mathscr{S}$ of a set $U \subseteq \mathbb{R}^n$ is given by $\mathscr{S}^{-1}(U) \triangleq \{x \in X \colon \mathscr{S}(x) \cap U \neq \varnothing\}$, and so the *domain* of $\mathscr{S}$ is taken to be the inverse image of $\mathbb{R}^n$, that is, $\operatorname{dom}\mathscr{S} \triangleq \{x \in X \colon \mathscr{S}(x) \neq \varnothing\}$. Also, the *graph* of $\mathscr{S}$ is the set $\operatorname{gph}\mathscr{S} \triangleq \{(x, y) \in X \times \mathbb{R}^n \colon x \in \operatorname{dom}\mathscr{S} \text{ and } y \in \mathscr{S}(x)\}$. Finally, $\mathscr{S}$ is *measurable* if, for every open set $O \subseteq \mathbb{R}^n$, $\mathscr{S}^{-1}(O) \in \Sigma$.

follows immediately that $\operatorname{dom}\mathscr{S} = \Omega$. Besides, using the density[16] of the rationals, it is easy to check that[17]

$$\operatorname{gph}\mathscr{S} = \left(\bigcap_{\rho\in\mathbb{Q}^+} \bigcup_{q\in\mathbb{Q}^d} \left[\left(\left[\mathbb{P}^{Y|\mathscr{G}}\Big(\mathbb{B}^d(q,\rho)\,,\cdot\Big)^{-1}\big((0,1]\big) \right] \cap \overline{\Omega} \right) \times \mathbb{B}^d(q,\rho) \right] \right) \cup \left[\overline{\Omega}^c \times \mathbb{R}^d \right].$$

But, for every $q \in \mathbb{Q}^d$ and $\rho \in \mathbb{Q}^+$, $\mathbb{P}^{Y|\mathscr{G}}\big(\mathbb{B}^d(q,\rho)\,,\cdot\big)$ is a $\mathscr{G}$-measurable random variable, consequently $\operatorname{gph}\mathscr{S} \in \mathscr{G} \otimes \mathscr{B}\big(\mathbb{R}^d\big)$, which combined with the completeness of $(\Omega, \mathscr{G}, \mathbb{P})$ concludes the proof (we refer to Theorem 14.8 in Rockafellar and Wets [53]). □

Corollary II.5.3. *The multi-function* $\mathscr{D} : \Omega \rightrightarrows \mathbb{R}^d$ *defined by* $\mathscr{D}(\omega) \triangleq \operatorname{aff}(\mathscr{S}(\omega))$, *where* $\mathscr{S} : \Omega \rightrightarrows \mathbb{R}^d$ *is the mapping of Proposition II.5.2, is also closed-valued and measurable, with* $\operatorname{dom}\mathscr{D} = \Omega$. *In particular, this implies that*

$$D \triangleq \Big\{ (\omega, y) \in \Omega \times \mathbb{R}^d\colon\ y \in \mathscr{D}(\omega) \Big\} \in \mathscr{G} \otimes \mathscr{B}\Big(\mathbb{R}^d\Big)\,. \tag{II.5.3}$$

Proof. Measurability follows immediately from Proposition II.5.2 and Exercise 14.12 in Rockafellar and Wets [53], whereas closed-valuedness is a direct consequence of Exercise 2.11 in Rockafellar and Wets [53]. Finally, we notice that $D = \operatorname{gph}\mathscr{D}$, so the measurability of the set D follows from Theorem 14.8 in Rockafellar and Wets [53] and the fact that $\mathscr{D}$ is closed-valued. □

Proposition II.5.4. *Suppose* A *is a* $\mathscr{G}$*-random set*[18]. *Then* $\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A\}$ *belongs to the* σ*-algebra* $\mathscr{F}$ *and*

$$\mathbb{P}\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A\} = \int_\Omega \mathbb{P}^{Y|\mathscr{G}}(A_\omega, \omega)\, d\mathbb{P}(\omega)\,, \tag{II.5.4}$$

where $A_\omega \in \mathscr{B}\big(\mathbb{R}^d\big)$ *is the section*[19] *of* A *determined by* ω *in* Ω.

Proof. Let us consider any $A \in \mathscr{G} \otimes \mathscr{B}\big(\mathbb{R}^d\big)$. We shall begin by showing that the set $\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A\}$ is in $\mathscr{F}$. Indeed, setting $f(\omega) \triangleq \mathbb{1}_A(\omega, Y(\omega))$ for every $\omega \in \Omega$, it is clear that f is $\mathscr{F}$-measurable, therefore $\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A\} = f^{-1}(\{1\}) \in \mathscr{F}$.

Next, we turn to the proof of (II.5.4). In order to do so, let us start by supposing that $A = G \times E$, with $G \in \mathscr{G}$ and $E \in \mathscr{B}\big(\mathbb{R}^d\big)$, is a measurable rectangle. Then the

[16] A subset D of a topological space (S, τ) is *dense* in S if $\operatorname{cl}(D) = S$, where the topological *closure* $\operatorname{cl}(D)$ of D is the smallest closed set in S containing D.

[17] Here, $\mathbb{B}^n(x_0, r) \triangleq \big\{x \in \mathbb{R}^n\colon\ \|x - x_0\|_{\mathbb{R}^n} < r\big\}$ denotes the *open ball* of centre $x_0 \in \mathbb{R}^n$ and radius $r > 0$ in the Euclidean space $\mathbb{R}^n$.

[18] Given a measurable space (X, Σ) and $n \in \mathbb{N}$, we say that a subset E of $X \times \mathbb{R}^n$ is a Σ*-random set* if it belongs to the product σ-algebra $\Sigma \otimes \mathscr{B}(\mathbb{R}^n)$.

[19] Let X and Y be two non-empty sets, and consider a subset E of $X \times Y$. For every $x \in X$ (respectively, $y \in Y$), the x*-section* (respectively, y*-section*) of E is defined by $E_x \triangleq \{y \in Y\colon\ (x, y) \in E\}$ (respectively, $E_y \triangleq \{x \in X\colon\ (x, y) \in E\}$).

following holds a.s.,

$$\mathbb{P}\left\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A \middle| \mathscr{G}\right\} = \mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{G\cap[Y^{-1}(E)]}\middle|\mathscr{G}\right] = \mathbb{1}_G\,\mathbb{P}^{Y|\mathscr{G}}(E,\cdot)\,,$$

where the second inequality follows from the fact that $G \in \mathscr{G}$ and from the definition of regular conditional distribution.

Moreover, it is trivial to see that the section of A determined by every fixed $\omega \in \Omega$ equals

$$A_\omega = \begin{cases} E, & \text{if } \omega \in G, \\ \varnothing, & \text{otherwise,} \end{cases}$$

hence $\mathbb{P}^{Y|\mathscr{G}}(A_\omega, \omega) = \mathbb{1}_G(\omega)\,\mathbb{P}^{Y|\mathscr{G}}(E, \omega)$ for any $\omega \in \overline{\Omega}$.

Combining the above yields $\mathbb{P}\left\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A \middle| \mathscr{G}\right\}(\omega) = \mathbb{P}^{Y|\mathscr{G}}(A_\omega, \omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, so taking expectations on both sides of the equality allows us to conclude that (II.5.4) is true for all measurable rectangles of $\Omega \times \mathbb{R}^d$.

The validity of the result for an arbitrary $A \in \mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$ can now be derived in the classic way, using Dynkin's π-λ theorem (see, e.g., Lemma 4.11 in Aliprantis and Border [1]). $\square$

Corollary II.5.5. *The set $\{\omega \in \Omega\colon Y(\omega) \in D(\omega)\}$ belongs to the σ-algebra $\mathscr{F}$ and has $\mathbb{P}$-full measure.* $\square$

Lemma II.5.6. *Under Assumption II.3.1, the set $\widetilde{\Omega} \triangleq \left\{\omega \in \widehat{\Omega}\colon D(\omega) \neq \{\mathbf{0_d}\}\right\}$ belongs to the σ-algebra $\mathscr{G}$. Moreover, $\mathbb{P}\left(\widetilde{\Omega}\right) = 0$ if and only if $Y = \mathbf{0_d}$ a.s..*

Proof. Given that, for every $\omega \in \widehat{\Omega}$, the linear space $D(\omega)$ contains the vector $\mathbf{0_d}$ (see Proposition II.3.2), it is clear that $\widetilde{\Omega} = \left\{\omega \in \widehat{\Omega}\colon\ D(\omega) \cap \left(\mathbb{R}^d \setminus \{\mathbf{0_d}\}\right) \neq \varnothing\right\} = \widehat{\Omega} \cap \mathscr{D}^{-1}\left(\mathbb{R}^d \setminus \{\mathbf{0_d}\}\right) \in \mathscr{G}$, where $\mathscr{D}$ is the measurable multi-function of Corollary II.5.3.

Let us now prove the equivalence. We can use Proposition II.5.4 to first obtain the equality $\mathbb{P}\{\omega \in \Omega\colon\ Y(\omega) \neq \mathbf{0_d}\} = \int_\Omega \mathbb{P}^{Y|\mathscr{G}}\left(\mathbb{R}^d \setminus \{\mathbf{0_d}\}, \omega\right)\, d\mathbb{P}(\omega)$, therefore $Y = \mathbf{0_d}$ a.s. if and only if $\mathbb{P}^{Y|\mathscr{G}}\left(\mathbb{R}^d \setminus \{\mathbf{0_d}\}, \cdot\right) = 0$ a.s.. But if the latter holds, then it follows from the definition of support that $\operatorname{supp}\left(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)\right) \subseteq \{\mathbf{0_d}\}$ for $\mathbb{P}$-a.e. $\omega \in \overline{\Omega}$, which in turn implies that $\mathbb{P}\left(\widetilde{\Omega}\right) = 0$. Reciprocally, if $\mathbb{P}\left(\widetilde{\Omega}\right) = 0$ is true, then for $\mathbb{P}$-a.e. $\omega \in \widehat{\Omega}$, $\varnothing \neq \operatorname{supp}\left(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)\right) \subseteq D(\omega) = \{\mathbf{0_d}\}$, hence by the definition of support we have $\mathbb{P}^{Y|\mathscr{G}}\left(\mathbb{R}^d \setminus \{\mathbf{0_d}\}, \omega\right) = \mathbb{P}^{Y|\mathscr{G}}\left(\operatorname{supp}\left(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)\right)^c, \omega\right) = 0$. $\square$

Lemma II.5.7. *Let Θ be a collection (not necessarily countable) of $\mathscr{F}$-measurable random variables. If for every pair $f, g \in \Theta$ and for every $A \in \mathscr{F}$ we have $f\mathbb{1}_A + g\mathbb{1}_{A^c} \in \Theta$, then the family $\{\mathbb{E}_{\mathbb{P}}[f \mid \mathscr{G}]\,;\ f \in \Theta\}$ is directed both upwards and downwards[20].*

[20] For every $x, y \in \mathbb{R}$, let $x \vee y \triangleq \max\{x, y\}$ and $x \wedge y \triangleq \min\{x, y\}$. Then, given a measure space (X, Σ, μ), we say that an arbitrary family Θ (not necessarily countable) of measurable functions is *directed upwards* (respectively, *directed downwards*) if, for every $f_1, f_2 \in \Theta$, there exists some $g \in \Theta$ such that $g \geq f_1 \vee f_2$ (respectively, $g \leq f_1 \wedge f_2$) μ-a.e..

Proof. For every fixed pair of random variables f and g in Θ, take $h_u : \Omega \to \mathbb{R}$ and $h_d : \Omega \to \mathbb{R}$ to be the $\mathscr{F}$-measurable random variables respectively given by

$$h_u(\omega) \triangleq f(\omega)\, \mathbb{1}_{\{\mathbb{E}_{\mathbb{P}}[f\,|\mathscr{G}] \geq \mathbb{E}_{\mathbb{P}}[g\,|\mathscr{G}]\}}(\omega) + g(\omega)\, \mathbb{1}_{\{\mathbb{E}_{\mathbb{P}}[f\,|\mathscr{G}] < \mathbb{E}_{\mathbb{P}}[g\,|\mathscr{G}]\}}(\omega)\,, \text{ and}$$
$$h_d(\omega) \triangleq f(\omega)\, \mathbb{1}_{\{\mathbb{E}_{\mathbb{P}}[f\,|\mathscr{G}] < \mathbb{E}_{\mathbb{P}}[g\,|\mathscr{G}]\}}(\omega) + g(\omega)\, \mathbb{1}_{\{\mathbb{E}_{\mathbb{P}}[f\,|\mathscr{G}] \geq \mathbb{E}_{\mathbb{P}}[g\,|\mathscr{G}]\}}(\omega)\,.$$ □

Lemma II.5.8. *Let $x \geq 0$ be fixed. Under Assumption II.3.1, there exists a $\mathscr{G}$-random set $M(x)$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the section $M(x)_\omega$ of $M(x)$ determined by ω is non-empty, convex and compact*[21]*. Moreover, given any $\mathscr{G}$-measurable random variable $\xi : \Omega \to \mathbb{R}^d$, the following two statements are equivalent,*

(i) *$\xi \in \Xi^d(x)$ and $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

(ii) *$\xi(\omega) \in M(x)_\omega$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. This is Proposition 4.2 in Rásonyi and Stettner [51], we include its proof here for completeness.

Let us fix an arbitrary $x \in [0, +\infty)$. We know by the proof of Proposition II.5.2 that the graph of the multi-function $\mathscr{S} : \Omega \rightrightarrows \mathbb{R}^d$ given by (II.5.2) is in $\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$, hence $\mathscr{S}$ admits a Castaing representation[22] $\{\varsigma_n;\, n \in \mathbb{N}\}$ (this follows from Castaing and Valadier [16, Theorem III.22]). Now we define the set

$$M(x) \triangleq D \cap \left(\bigcap_{n\in\mathbb{N}} f_n^{-1}([-x, +\infty)) \right),$$

where D is given by (II.5.3) and $f_n : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is the $\big(\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)\,, \mathscr{B}(\mathbb{R})\big)$-measurable function defined by $f_n(\omega, y) \triangleq \langle y, \varsigma_n(\omega)\rangle_{\mathbb{R}^d}$, for every $n \in \mathbb{N}$.

Clearly, $M(x)$ belongs to $\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$. Furthermore, the section of $M(x)$ determined by each $\omega \in \overline{\Omega}$ equals

$$M(x)_\omega = D(\omega) \cap \left(\bigcap_{n\in\mathbb{N}} (\langle \cdot, \varsigma_n(\omega)\rangle_{\mathbb{R}^d})^{-1}([-x, +\infty)) \right),$$

which, being the countable intersection of convex and closed subsets of $\mathbb{R}^d$ (we recall that every affine space is closed and convex, and that convexity and closedness are preserved under inverse images of continuous functions), is itself convex and closed. Also, it is obvious that $M(x)_\omega \neq \varnothing$ for all $\omega \in \Omega$. In fact, if $\omega \in \widehat{\Omega}$, it is trivial that $\mathbf{0_d} \in M(x)_\omega$ (we invoke Proposition II.3.2), whereas for $\omega \notin \widehat{\Omega}$, either $\mathbf{0_d} \in D(\omega)$ or there must be at least one $n \in \mathbb{N}$ such that $\varsigma_n(\omega) \neq \mathbf{0_d}$, hence $\mathbf{0_d} \in M(x)_\omega$ or $\varsigma_n(\omega) \in M(x)_\omega$, respectively.

[21] Recall that a subset K of a topological space (S, τ) is *compact* if for every indexed (not necessarily countable) family $\{O_{i\in I};\, i \in I\}$ of open sets such that $K \subset \bigcup_{i\in I} O_i$, there is a finite subset J of I such that $K \subset \bigcup_{j\in J} O_j$ (that is, any open cover of Khas a finite subcover).

[22] We recall that a *Castaing representation* of a multi-function $\mathscr{S} : X \rightrightarrows \mathbb{R}^n$ is any countable family $\{\sigma_n;\, n \in \mathbb{N}\}$ of measurable functions $\sigma_n : \operatorname{dom}\mathscr{S} \to \mathbb{R}^n$ satisfying $\mathscr{S}(x) = \operatorname{cl}(\{\sigma_n(x)\,;\, n \in \mathbb{N}\})$ for every $x \in \operatorname{dom}\mathscr{S}$.

We proceed with the proof that *(i)* holds if and only if *(ii)* is true. To show necessity, consider any $\xi \in \Xi^d(x)$ verifying $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then, setting $A \triangleq \{(\omega, y) \in \Omega \times \mathbb{R}^d\colon\ \langle \xi(\omega)\,, y\rangle_{\mathbb{R}^d} < -x\} \in \mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$, we obtain

$$\int_\Omega \mathbb{P}^{Y|\mathscr{G}}(A_\omega, \omega)\ d\mathbb{P}(\omega) = \mathbb{P}\{\omega \in \Omega\colon\ (\omega, Y(\omega)) \in A\} = 0,$$

where we use Proposition II.5.4 to deduce the first equality. But this implies that $\mathbb{P}^{Y|\mathscr{G}}(A_\cdot, \cdot) = 0$ a.s., and so, given that $A_\omega = \{y \in \mathbb{R}^d\colon\ \langle \xi(\omega)\,, y\rangle_{\mathbb{R}^d} < -x\}$ is an open subset of $\mathbb{R}^d$, we must have by definition of support of a measure that

$$\operatorname{supp}\Big(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)\Big) \subseteq A^c_\omega = \Big\{y \in \mathbb{R}^d\colon\ \langle \xi(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} \geq -x\Big\}$$

for $\mathbb{P}$-a.e. $\omega \in \overline{\Omega}$. Thus, in particular, we get for $\mathbb{P}$-a.e. $\omega \in \Omega$ that

$$\langle \xi(\omega)\,, \varsigma_n(\omega)\rangle_{\mathbb{R}^d} \geq -x, \qquad \forall\, n \in \mathbb{N},$$

which establishes the desired implication. Next, we turn to the proof of sufficiency, so consider any $\xi \in \Xi^d$, and suppose there exists some measurable set $\Omega_1 \subseteq \Omega$, with $\mathbb{P}(\Omega_1^c) = 0$, such that $\xi(\omega) \in M(x)_\omega$ for every $\omega \in \Omega_1$. Then, in particular, we have $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. On the other hand, it is easy to check that, for all $\omega \in \Omega_1 \cap \overline{\Omega}$, the inclusion $\operatorname{supp}(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)) \subseteq \{y \in \mathbb{R}^d\colon\ x + \langle \xi(\omega)\,, y\rangle_{\mathbb{R}^d} \geq 0\}$ holds true. Indeed, the result is trivial when $\xi(\omega) = \mathbf{0}_{\boldsymbol{d}}$, thus let us assume instead that $\|\xi(\omega)\|_{\mathbb{R}^d} \neq 0$. Then, fixing any $y \in \operatorname{supp}(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega))$ and considering an arbitrary $\varepsilon > 0$, we can find some $n \in \mathbb{N}$ such that $\|y - \varsigma_n(\omega)\|_{\mathbb{R}^d} < \varepsilon / \|\xi(\omega)\|_{\mathbb{R}^d}$, hence we can use the Cauchy-Schwarz inequality to deduce

$$\begin{aligned} x + \langle \xi(\omega)\,, y\rangle_{\mathbb{R}^d} &= x + \langle \xi(\omega)\,, \varsigma_n(\omega)\rangle_{\mathbb{R}^d} + \langle \xi(\omega)\,, y - \varsigma_n(\omega)\rangle_{\mathbb{R}^d} \\ &\geq 0 - \|\xi(\omega)\|_{\mathbb{R}^d}\, \|y - \varsigma_n(\omega)\|_{\mathbb{R}^d} > -\varepsilon, \end{aligned}$$

and the arbitrarity of ε yields the claimed inclusion. Consequently, the monotonicity of any measure along with the definition of support yield

$$1 = \mathbb{P}^{Y|\mathscr{G}}\Big(\operatorname{supp}\Big(\mathbb{P}^{Y|\mathscr{G}}(\cdot, \omega)\Big), \omega\Big) \leq \mathbb{P}^{Y|\mathscr{G}}\Big(\Big\{y \in \mathbb{R}^d\colon\ x + \langle \xi(\omega)\,, y\rangle_{\mathbb{R}^d} \geq 0\Big\}, \omega\Big)$$

for $\mathbb{P}$-a.e. $\omega \in \overline{\Omega}$. As before, an obvious application of Proposition II.5.4 then gives the intended result $\mathbb{P}\{x + \langle \xi, Y\rangle_{\mathbb{R}^d} \geq 0\} = 1$.

Finally, we see that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the section $M(x)_\omega$ is a bounded set in $\mathbb{R}^d$. In order to do so, we begin by noticing that the multi-function

$$\begin{aligned} \mathscr{M}_x : \Omega &\rightrightarrows \mathbb{R}^d \\ \omega &\mapsto M(x)_\omega\,, \end{aligned}$$

is non-empty and closed-valued. We remark further that $\operatorname{gph} \mathscr{M}_x$ is measurable, hence we can invoke the von Neumann-Aumann selection theorem (see e.g. Theorem III.22 in Castaing and Valadier [16]) to obtain a Castaing representation $\{\mu_n;\, n \in \mathbb{N}\}$ for $\mathscr{M}_x$. Therefore, we have $\mu_n(\omega) \in M(x)_\omega$ for all $\omega \in \Omega$ and for every $n \in \mathbb{N}$, so by the equivalence proved above, we conclude that each $\mu_n \in \Xi^d(x)$ and $\mu_n(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Now fix any ω in the $\mathbb{P}$-full measure set $\bigcap_{n\in\mathbb{N}} \{\|\mu_n\|_{\mathbb{R}^d} \leq K_x\}$, with K_x the random variable found in Lemma II.3.15, and let $y \in M(x)_\omega$ be arbitrary. For each $\varepsilon > 0$, there exists some μ_n satisfying $\|y - \mu_n(\omega)\|_{\mathbb{R}^d} < \varepsilon$, hence the triangle inequality yields $\|y\|_{\mathbb{R}^d} < K_x(\omega) + \varepsilon$. It now follows from the arbitrarity of ε that $\|y\|_{\mathbb{R}^d} \leq K_x(\omega)$ for all $y \in M(x)_\omega$, and we conclude the proof with the well-known result that a subset of the Euclidean space $\mathbb{R}^d$ is compact if and only if it is closed and bounded. □

Lemma II.5.9. *Given any $x \geq 0$, let $M(x)$ be the set of Lemma II.5.8. Then, under Assumption II.3.1, the sets*

$$M(x)^\circ \triangleq \Big\{(\omega, y) \in \Omega \times \mathbb{R}^d\colon\ y \in \operatorname{ri}(M(x)_\omega)\Big\},\ \textit{and} \tag{II.5.5}$$

$$Z(x) \triangleq \Big\{(\omega, y) \in \Omega \times \mathbb{R}^d\colon\ y \in \operatorname{lin}(M(x)_\omega)\Big\}, \tag{II.5.6}$$

where $\operatorname{ri}(M(x)_\omega)$ and $\operatorname{lin}(M(x)_\omega)$ respectively denote the relative interior[23] *and the linear span of the section $M(x)_\omega$, belong to the product σ-algebra $\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$.*

Proof. This is the statement of Proposition 4.3 in Rásonyi and Stettner [51]. We provide an alternative proof below.

Fix an arbitrary $x \in [0, +\infty)$. In order to improve readability, we shall divide the proof into two parts.

(i) First, it is straightforward to show that $Z(x)$ is a $\mathscr{G}$-random set. Indeed, let us define the multi-function $\mathscr{Z}_x\colon\ \Omega \rightrightarrows \mathbb{R}^d$ by $\mathscr{Z}_x(\omega) \triangleq \operatorname{lin}(M(x)_\omega)$. It then follows from Lemma II.5.8 and Rockafellar and Wets [53, Theorem 14.8 and Exercise 14.12] that $\mathscr{Z}_x$ is measurable with $\operatorname{dom} \mathscr{Z}_x = \Omega$. In addition, since any linear space is an affine space, we have that $\mathscr{Z}_x$ is also closed-valued (we refer e.g. to Exercise 2.11 in Rockafellar and Wets [53]), thus Theorem 14.8 in Rockafellar and Wets [53] gives $Z(x) = \operatorname{gph} \mathscr{Z}_x \in \mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$.

(ii) Secondly, we verify that $M(x)^\circ$ belongs to $\mathscr{G}\otimes\mathscr{B}(\mathbb{R}^d)$ as well. Using the definition of relative interior and the density of $\mathbb{Q}^+$ in $\mathbb{R}^+$, it is immediate to check that

$$M(x)^\circ = \bigcup_{\rho\in\mathbb{Q}^+} \Big\{(\omega, y) \in \Omega \times \mathbb{R}^d\colon\ \mathbb{B}^d(y, \rho) \cap \operatorname{aff}(M(x)_\omega) \subseteq M(x)_\omega\Big\}.$$

Next, combining Lemma II.5.8 with Rockafellar and Wets [53, Theorem 14.8 and

[23] For any set $E \subseteq \mathbb{R}^n$, the *relative interior* of E, denoted by $\operatorname{ri}(E)$, is the interior of E relative to its affine hull, i.e., $\operatorname{ri}(E) \triangleq \{x \in \mathbb{R}^n\colon\ \mathbb{B}^n(x, r) \cap \operatorname{aff}(E) \subseteq E \text{ for some } r > 0\}$.

Exercise 14.12], as in the previous step, yields that the multi-function

$$\begin{aligned}\mathscr{N}_x : \Omega &\rightrightarrows \mathbb{R}^d \\ \omega &\mapsto \operatorname{aff}\big(M(x)_\omega\big)\,,\end{aligned}$$

is measurable and closed, so it admits a Castaing representation $\{\alpha_n;\, n \in \mathbb{N}\}$ (again, we use Theorem 14.8 in Rockafellar and Wets [53], and the von Neumann-Aumann selection theorem). Thus, for every $\rho \in \mathbb{Q}^+$ and $n \in \mathbb{N}$, let $f_n^\rho :\ \Omega \times \mathbb{R}^d \to \{0,1\}$ and $g_n :\ \Omega \times \mathbb{R}^d \to \{0,1\}$ be the $\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$-measurable functions respectively given by

$$(\omega, y) \mapsto (\alpha_n(\omega)\,, y) \mapsto \mathbb{1}_{E^\rho}(\alpha_n(\omega)\,, y)$$

with $E^\rho \triangleq \big\{(y,z) \in \mathbb{R}^d \times \mathbb{R}^d\!:\ \|y - z\|_{\mathbb{R}^d} \in [0,\rho)\big\} \in \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d)$, and

$$(\omega, y) \mapsto (\omega, \alpha_n(\omega)) \mapsto \mathbb{1}_{M(x)}(\omega, \alpha_n(\omega))\,.$$

It is now obvious that not only does $A \triangleq \bigcup_{q\in\mathbb{Q}^+} \bigcap_{n\in\mathbb{N}} (g_n - f_n^\rho)^{-1}([0,+\infty))$ belong to the product σ-algebra $\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)$, but also it contains $M(x)^\circ$. Conversely, let $(\omega, y) \in \Omega \times \mathbb{R}^d$ be such that, for some $\rho \in \mathbb{Q}^+$, we have $(g_n - f_n^\rho)(\omega, y) \geq 0$ for all $n \in \mathbb{N}$. Taking $\varrho \triangleq \rho/2 \in \mathbb{Q}^+$, let us fix an arbitrary $z \in \mathbb{B}^d(y, \varrho) \cap \operatorname{aff}\big(M(x)_\omega\big)$, as well as an arbitrary $\varepsilon > 0$. Therefore, choosing τ in $(0, 1 \wedge \varrho/\varepsilon)$, it is possible to find some $m \in \mathbb{N}$ for which $\|z - \alpha_m(\omega)\|_{\mathbb{R}^d} < \tau\varepsilon$. Consequently,

$$\|y - \alpha_m(\omega)\|_{\mathbb{R}^d} \leq \|y - z\|_{\mathbb{R}^d} + \|z - \alpha_m(\omega)\|_{\mathbb{R}^d} < \varrho + \tau\varepsilon < 2\varrho = \rho$$

by the triangle inequality, and so $1 = \mathbb{1}_{E^\rho}(\alpha_m(\omega)\,, y) \leq \mathbb{1}_{M(x)}(\omega, \alpha_m(\omega))$, which implies that $\alpha_m(\omega) \in M(x)_\omega$. But then, given that $(1-\tau)\,\varepsilon > 0$, there must exist some $n \in \mathbb{N}$ such that $\|\alpha_m(\omega) - \mu_n(\omega)\|_{\mathbb{R}^d} < (1-\tau)\,\varepsilon$, where the Castaing representation $\{\mu_n(\omega)\,;\, n \in \mathbb{N}\}$ for $\mathscr{M}_x$ is the one obtained in the proof of Lemma II.5.8. Combining all of the preceding results finally gives

$$\|z - \mu_n(\omega)\|_{\mathbb{R}^d} \leq \|z - \alpha_m(\omega)\|_{\mathbb{R}^d} + \|\alpha_m(\omega) - \mu_n(\omega)\|_{\mathbb{R}^d} < \tau\varepsilon + (1-\tau)\,\varepsilon = \varepsilon.$$

We then conclude that $z \in M(x)_\omega$, hence $A \subseteq M(x)^\circ$, and the measurability of the set $M(x)^\circ$ is established. □

5.2 Proofs of Section II.2

Proof of Proposition II.2.10. Fix $t \in \{1, \ldots, T\}$, and set $\widehat{\Omega}_t \triangleq \big\{\omega \in \overline{\Omega}_t\!:\ \mathbf{0}_d \in D_t(\omega)\big\}$. We remark that any affine space containing the origin is actually a linear space, so it suffices to see that *(i)* and *(ii)* hold true.

The first condition can be checked immediately. Indeed, since the multi-function

$\mathscr{D}_t : \Omega \rightrightarrows \mathbb{R}^d$ given by

$$\mathscr{D}_t(\omega) \triangleq \begin{cases} D_t(\omega), & \text{if } \omega \in \overline{\Omega}_t, \\ \mathbb{R}^d, & \text{otherwise,} \end{cases}$$

is closed-valued and measurable (see Corollary II.5.3), we can make use of Theorem 14.3 in Rockafellar and Wets [53] to conclude that $\widehat{\Omega}_t = \overline{\Omega}_t \cap \mathscr{D}_t^{-1}(\{\mathbf{0}_d\}) \in \mathscr{F}_{t-1}$.

As for the argument to show *(ii)*, it is essentially the same as that of the proof of Theorem 3 in Jacod and Shiryaev [28]. □

5.3 Proofs of Section II.3

Proof of Proposition II.3.7. Let us begin by showing that $\widehat{\xi}$ is in $\Xi^d(x)$. We know, by Corollary II.5.3, that the closed-valued mapping $\mathscr{D}$ given therein is measurable. Thus, Theorem 14.8 in Rockafellar and Wets [53] and the von Neumann-Aumann theorem ensure the existence of a countable sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of $\mathscr{G}$-measurable random variables $\sigma_n : \Omega \to \mathbb{R}^d$ such that, for every $\omega \in \Omega$, $\mathscr{D}(\omega) = \mathrm{cl}(\{\sigma_n(\omega)\,;\, n \in \mathbb{N}\})$.

Next, let us consider the set $E \triangleq \left[\left[\widehat{\Omega} \times \mathbb{R}^d\right] \cap D \cap \left(\bigcap_{n\in\mathbb{N}} h_n^{-1}(\{0\})\right)\right] \cup \left[\widehat{\Omega}^c \times \mathbb{R}^d\right]$, with $h_n : \Omega \times \mathbb{R}^d \to \mathbb{R}$ the measurable function given by $h_n(\omega, y) \triangleq \langle \xi(\omega) - y, \sigma_n(\omega)\rangle_{\mathbb{R}^d}$. Then it is clear that E is a $\mathscr{G}$-measurable random set. Moreover, for any fixed $\omega \in \Omega$, the section of E determined by ω is equal to

$$E_\omega = \begin{cases} \left\{\widehat{\xi}(\omega)\right\}, & \text{if } \omega \in \widehat{\Omega}, \\ \mathbb{R}^d, & \text{otherwise.} \end{cases}$$

Hence, the multi-function $\mathscr{E} : \Omega \rightrightarrows \mathbb{R}^d$ given by $\mathscr{E}(\omega) \triangleq E_\omega$ (and whose graph equals E) admits a measurable selector, that is, we can obtain a $\mathscr{G}$-measurable random variable $f : \Omega \to \mathbb{R}^d$ with $f(\omega) \in \mathscr{E}(\omega)$ for all $\omega \in \Omega$. But, since $\mathbb{P}\left(\widehat{\Omega}^c\right) = 0$, this implies that $\widehat{\xi} = f$ a.s. and so $\widehat{\xi}$ is a $\mathscr{G}$-measurable random variable.

We now proceed to prove that the inequality $x + \left\langle \widehat{\xi}, Y \right\rangle_{\mathbb{R}^d} \geq 0$ holds a.s. and, as a by-product, we shall deduce equation (II.3.3). For all $\omega \in \widehat{\Omega}$, we have that $\xi(\omega) - \widehat{\xi}(\omega) \in D(\omega)^\perp$. Furthermore, we know by Corollary II.5.5 that $B \triangleq \{\omega \in \Omega \colon Y(\omega) \in D(\omega)\}$ is in $\mathscr{F}$ and has full probability. Finally, given that $\xi \in \Xi^d(x)$ by hypothesis, we can find a measurable set $C \in \mathscr{F}$ with $\mathbb{P}(C^c) = 0$ and $x + \langle \xi(\omega), Y(\omega)\rangle_{\mathbb{R}^d} \geq 0$ for all $\omega \in C$. Consequently, if we consider any $\omega \in \Omega_1 \triangleq \widehat{\Omega} \cap B \cap C$, we obtain that

$$x + \left\langle \widehat{\xi}(\omega), Y(\omega) \right\rangle_{\mathbb{R}^d} = x + \langle \xi(\omega), Y(\omega)\rangle_{\mathbb{R}^d} \geq 0,$$

where the equality is due to $\left\langle \xi(\omega) - \widehat{\xi}(\omega), Y(\omega) \right\rangle_{\mathbb{R}^d} = 0$. The above equation combined with $\mathbb{P}(\Omega_1^c) = 0$ then yields the intended results. □

Proof of Lemma II.3.14. In order to check that equality (II.3.8) is true, let us start by showing that the $\mathbb{R}$-valued function given by

$$\omega \mapsto V(x + \langle \xi(\omega), Y(\omega) \rangle_{\mathbb{R}^d}, \omega)$$

is $\mathscr{F}$-measurable. In fact, let us define the functions $f : \Omega \to \mathbb{R} \times \Omega$, $g : \mathbb{R} \times \Omega \to \mathbb{R}^2$ and $\mathrm{pr}_2 : \mathbb{R}^2 \to \mathbb{R}$ as follows, $f(\omega) \triangleq (x + \langle \xi(\omega), Y(\omega) \rangle_{\mathbb{R}^d}, \omega)$, $g(y, \omega) \triangleq (y, V(y, \omega))$, and $\mathrm{pr}_2(y, z) \triangleq z$. Clearly, f is $(\mathscr{F}, \mathscr{B}(\mathbb{R}) \otimes \mathscr{F})$-measurable, because ξ and Y are $\mathscr{F}$-measurable, and the inner product and the sum in $\mathbb{R}^2$ are continuous functions. Moreover, we have that pr_2 is $\big(\mathscr{B}(\mathbb{R}^2), \mathscr{B}(\mathbb{R})\big)$-measurable, by continuity. On the other hand, is is trivial to check that $V(x + \langle \xi(\omega), Y(\omega) \rangle_{\mathbb{R}^d}, \omega) = (\mathrm{pr}_2 \circ g \circ f)(\omega)$ for every $\omega \in \Omega$, therefore we only need to check that g is $\big(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{B}(\mathbb{R}^2)\big)$-measurable. In order to do so, let O be an open subset of $\mathbb{R}^2$. Then $O = \bigcup_{n \in \mathbb{N}} O_n^1 \times O_n^2$, where O_n^i is an open subset of $\mathbb{R}$, for every $i \in \{1, 2\}$ and $n \in \mathbb{N}$. Furthermore, for every ω outside some $\mathbb{P}$-null set N, the function $V(\cdot, \omega)$ is continuous. Therefore,

$$g^{-1}(O) = \left[[\mathbb{R} \times (\Omega \setminus N)] \cap \left(\bigcup_{n \in \mathbb{N}} \left([O_n^1 \times \Omega] \cap \left[\bigcup_{\rho \in \mathbb{Q}^+} \bigcap_{q \in \mathbb{Q}} (h_{\rho,q}^n)^{-1}([0, +\infty)) \right] \right) \right) \right] \cup \left[g^{-1}(O) \cap [\mathbb{R} \times N] \right] \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{F},$$

where $h_{\rho,q}^n(y, \omega) \triangleq \mathbb{1}_{E_{\rho,q}^n}(\omega) - \mathbb{1}_{\mathbb{B}(q,\rho)}(y)$, $E_{\rho,q}^n \triangleq (V(q, \cdot))^{-1}\big(A_\rho^n\big) \in \mathscr{G}$ (because $V(q, \cdot)$ is $\mathscr{G}$-measurable), and $A_\rho^n \triangleq \bigcap_{z \notin O_n^2} (|z - \cdot|)^{-1}([\rho, +\infty)) \in \mathscr{B}(\mathbb{R})$ (because the intersection of closed sets is still a closed set).

It follows from a completely identical argument that the function defined by

$$\omega \mapsto V\Big(x + \Big\langle \widehat{\xi}(\omega), Y(\omega) \Big\rangle_{\mathbb{R}^d}, \omega\Big)$$

is $\mathscr{F}$-measurable as well.

It then suffices to note that $V(x + \langle \xi(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot)$ and $V\Big(x + \Big\langle \widehat{\xi}(\cdot), Y(\cdot) \Big\rangle_{\mathbb{R}^d}, \cdot\Big)$ are equal a.s., which is an immediate consequence of (II.3.3). □

Proof of Lemma II.3.15. Fix $x_0 \geq 0$ and let β be the real-valued random variable given by Proposition II.3.5. Then the function $K_{x_0} : \Omega \to \mathbb{R}$ defined by

$$K_{x_0}(\omega) \triangleq \frac{x_0 + 1}{\beta(\omega)} \mathbb{1}_{\{\beta > 0\}}(\omega), \qquad \omega \in \Omega,$$

is clearly non-negative and $\mathscr{G}$-measurable. Moreover, $K_{x_0} > 0$ a.s..

Next, let $x \in [0, x_0]$ be arbitrary, and consider any $\xi \in \Xi^d(x)$ satisfying $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. ω in $\overline{\Omega}$. Setting $B \triangleq \{\omega \in \Omega\colon \ \|\xi\|_{\mathbb{R}^d} > K_{x_0}(\omega)\}$, it can be easily checked that $B \in \mathscr{G}$ and

$$x + \langle \xi(\omega), Y(\omega) \rangle_{\mathbb{R}^d} \leq x - \beta(\omega) \|\xi(\omega)\|_{\mathbb{R}^d} < x - \beta(\omega) K_{x_0}(\omega) = x - (x_0 + 1) < 0$$

for every $\omega \in B \cap \{\omega \in \Omega\colon \beta(\omega) > 0\} \cap \{\omega \in \Omega\colon \langle \xi(\omega), Y(\omega) \rangle_{\mathbb{R}^d} \leq -\beta(\omega) \|\xi(\omega)\|_{\mathbb{R}^d}\}$.

Therefore,

$$\begin{aligned} 0 \le \mathbb{P}\Big(\widetilde{\Omega} \cap B \cap \{\langle \xi, Y\rangle_{\mathbb{R}^d} \le -\beta\, \|\xi\|_{\mathbb{R}^d}\}\Big) &\le \mathbb{P}(B \cap \{\langle \xi, Y\rangle_{\mathbb{R}^d} \le -\beta\, \|\xi\|_{\mathbb{R}^d}\}) \\ &= \mathbb{P}(B \cap \{\beta > 0\} \cap \{\langle \xi, Y\rangle_{\mathbb{R}^d} \le -\beta\, \|\xi\|_{\mathbb{R}^d}\}) \\ &\le \mathbb{P}\{x + \langle \xi, Y\rangle_{\mathbb{R}^d} < 0\} = 0, \end{aligned}$$

where the first and last equalities are due to $\beta > 0$ a.s. and $\xi \in \Xi^d(x)$, respectively.

On the other hand, we know again by Proposition II.3.5 that

$$\kappa\, \mathbb{1}_{\widetilde{\Omega} \cap B} \le \mathbb{P}\Big(\widetilde{\Omega} \cap B \cap \{\langle \xi, Y\rangle_{\mathbb{R}^d} \le -\beta\, \|\xi\|_{\mathbb{R}^d}\}\Big|\mathscr{G}\Big) \text{ a.s.,}$$

so taking expectations on both sides and invoking the preceding equality, we obtain $\mathbb{E}_{\mathbb{P}}\big[\kappa\, \mathbb{1}_{\widetilde{\Omega} \cap B}\big] = 0$.

But, given that $\kappa > 0$ a.s., this implies that the set $\widetilde{\Omega} \cap B$ has probability zero, hence

$$\begin{aligned} \mathbb{P}(B) = \mathbb{P}\Big(\widehat{\Omega} \cap B\Big) &= \mathbb{P}\Big(\Big\{\omega \in \widehat{\Omega}\colon\ D(\omega) = \{\mathbf{0_d}\}\Big\} \cap B\Big) \\ &= \mathbb{P}\Big(\Big\{\omega \in \widehat{\Omega}\colon\ D(\omega) = \{\mathbf{0_d}\}\Big\} \cap B \cap \big\{\omega \in \overline{\Omega}\colon\ \xi(\omega) \in D(\omega)\big\}\Big) = 0, \end{aligned}$$

as $\|\xi(\omega)\|_{\mathbb{R}^d} = 0 \le K_{x_0}(\omega)$ for all $\omega \in \Big\{\omega \in \widehat{\Omega}\colon\ D(\omega) = \{\mathbf{0_d}\}\Big\} \cap \big\{\omega \in \overline{\Omega}\colon\ \xi(\omega) \in D(\omega)\big\}$.

□

Proof of Lemma II.3.16. Fixing an arbitrary $x \ge 0$, let us consider the $\mathscr{G}$-random set $M(x)$ of admissible strategies in D, given by Lemma II.5.8. Then we know by Lemma II.5.9 that its linear span $Z(x)$ belongs to σ-algebra $\mathscr{G} \otimes \mathscr{B}(\mathbb{R}^n)$ as well.

Let us now organise the proof into four distinct parts.

(i) Firstly, it is not difficult to show that we can recursively choose $\mathscr{G}$-measurable, $\mathbb{R}^d$-valued functions $\zeta_1, \ldots, \zeta_d$ such that, for every $\omega \in \Omega$, $\{\zeta_1(\omega), \ldots, \zeta_d(\omega)\}$ is an orthogonal set[24] of vectors spanning $Z(x)_\omega$ (note that some, or even all, of them may eventually be the null vector), and $\|\zeta_i(\omega)\|_{\mathbb{R}^d} \in \{0, 1\}$ for every $i \in \{1, \ldots, d\}$.

(ii) Next, we show that $M(x)_\omega$ is contained in a simplex[25], for $\mathbb{P}$-a.e. $\omega \in \Omega$. Indeed, it follows from the proof of Lemma II.5.8 and from Lemma II.3.15 that, for all ω outside a $\mathbb{P}$-null set $N = N(x)$, $M(x)_\omega$ is contained in the set

$$P_x(\omega) \triangleq Z(x)_\omega \cap \Big\{y \in \mathbb{R}^d\colon\ A_x(\omega)\, y \le b_x(\omega)\Big\},$$

with A_x and b_x respectively the $(2d) \times d$ random matrix and the $d \times 1$ random vector with coefficients[26] $(A_x)_{ij} \triangleq (-1)^i \left(\zeta_{\lceil i/2 \rceil}\right)_j$ and $(b_x)_j \triangleq K_x$ for all $i \in \{1, \ldots, 2d\}$

[24]Given a vector space V with an inner product $\langle \cdot, \cdot \rangle$, we say that a set $E \subseteq V$ is an *orthogonal set* if, for every $x, y \in E$ with $x \ne y$, we have $\langle x, y\rangle = 0$.

[25]A set $S \subseteq \mathbb{R}^n$ is called a *simplex* if it is the convex hull of a finite number of points, that is, $S = \operatorname{conv}(\{x_1, \ldots, x_p\})$ for some $p \in \mathbb{N}_0$ and $x_1, \ldots, x_p \in \mathbb{R}^n$.

[26]The *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ and the *floor function* $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ are respectively given by

and $j \in \{1, \ldots, d\}$. It is also obvious that $\mathbf{0}_{\boldsymbol{d}} \in P_x(\omega)$, so $P_x(\omega)$ is non-empty. Moreover, it can be easily checked that $P_x(\omega)$ is closed and convex (recall that the pre-image of a closed set under a continuous function is closed as well, that every linear space is closed and convex, and finally that the intersection of closed and convex sets is still closed and convex). It is also immediate to see that $P_x(\omega)$ is contained in $\mathbb{B}^d\Big(\mathbf{0}_{\boldsymbol{d}}, \sqrt{d}\, K_x(\omega)\Big)$, that is, $P_x(\omega)$ is bounded.

As a consequence of the finite-dimensional version of the Krein-Milman theorem (we refer e.g. to Barnikov [7, Theorem II.3.3]), we conclude that every $P_x(\omega)$ is the convex hull of the set of its extreme points[27] (which is non-empty). Defining the 2^d random vectors

$$\theta_i(\omega) \triangleq K_x(\omega) \sum_{j=1}^{d} (\iota_i)_j\, \zeta_j(\omega)\,,$$

where $\{\iota_1, \ldots, \iota_{2^d}\} = \{-1, 1\}^d$ is the set of vectors in $\mathbb{R}^d$ whose components have absolute value equal to 1, it is easy to verify that $\mathcal{E}(P_x(\omega)) = \{\theta_1(\omega)\,, \ldots, \theta_{2^d}(\omega)\}$.

(iii) Thirdly, Lemma II.5.9 tells us that $M(x)^\circ$ is also a $\mathscr{G}$-random set, so we can apply the von Neumann-Aumann theorem to obtain a $\mathscr{G}$-measurable selector $\varrho:\ \Omega \to \mathbb{R}^d$ of $M(x)^\circ$. Next, for every $i \in \{1, \ldots, 2^d\}$, let us define the set

$$E_i \triangleq \{(\omega, \lambda) \in \Omega \times (0,1) \colon (1-\lambda)\, \varrho(\omega) + \lambda\, \theta_i(\omega) \in M(x)_\omega\} = \big(\mathbb{1}_{M(x)} \circ f_i\big)^{-1}(\{1\})\,,$$

where $f_i:\ \Omega \times (0,1) \to \Omega \times \mathbb{R}^d$ is the $\big(\mathscr{G} \otimes \mathscr{B}((0,1))\,, \mathscr{G} \otimes \mathscr{B}(\mathbb{R}^d)\big)$-measurable function given by $f_i(\omega, \lambda) \triangleq (\omega, (1-\lambda)\, \varrho(\omega) + \lambda\, \theta_i(\omega))$. Therefore, E_i belongs to $\mathscr{G} \otimes \mathscr{B}((0,1))$, and the multi-function $\mathscr{E}_i:\ \Omega \rightrightarrows \mathbb{R}$ given by

$$\mathscr{E}(\omega) \triangleq \begin{cases} (E_i)_\omega\,, & \text{if } \omega \in \widehat{\Omega}, \\ (0,1)\,, & \text{otherwise}, \end{cases}$$

and with graph E_i, is measurable and has $\operatorname{dom} \mathscr{E}_i = \Omega$. Hence, by the von Neumann-Aumann theorem, $\mathscr{E}_i$ admits a $\mathscr{G}$-measurable selector $\lambda_i:\ \Omega \to (0,1)$.

(iv) We are, at last, in condition to construct the desired random variable L_x. So let us define

$$L_x(\omega) \triangleq \sum_{i=1}^{2^d} \left(\frac{1}{\lambda_i(\omega)}\right)^{\gamma} \big[V^+(x + \langle (1-\lambda_i(\omega))\, \varrho(\omega) + \lambda_i(\omega)\, \theta_i(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d}\,, \omega) + C\big]$$

where the constants $C > 0$ and $\gamma > 0$ are those given by Assumption II.3.13. Then obviously $L_x(\omega) \geq 0$ a.s.. Moreover, recalling that each λ_i is measurable

$\lceil x \rceil \triangleq \min\{n \in \mathbb{Z} \colon x \leq n\}$ and $\lfloor x \rfloor \triangleq \max\{n \in \mathbb{Z} \colon x \geq n\}$ for all $x \in \mathbb{R}$.

[27] An *extreme point* of a set $E \subseteq \mathbb{R}^n$ is a point $x \in E$ with the property that, if $x = (1-\lambda)\, y + \lambda\, z$ with $y, z \in E$ and $\lambda \in (0,1)$, then $x = y = z$ (roughly speaking, it is a point that is not strictly between any two points in that set). We denote by $\mathcal{E}(E)$ the set of extreme points of E.

with respect to $\mathscr{G}$ yields

$$\mathbb{E}[L_x|\mathscr{G}]=\sum_{i=1}^{2^d}\left(\frac{1}{\lambda_i(\omega)}\right)^{\gamma}\mathbb{E}\big[V^+(x+\langle(1-\lambda_i(\omega))\varrho(\omega)+\lambda_i(\omega)\theta_i(\omega),Y(\omega)\rangle_{\mathbb{R}^d},\omega)\big|\mathscr{G}\big]$$
$$+C\sum_{i=1}^{2^d}\left(\frac{1}{\lambda_i(\omega)}\right)^{\gamma}<+\infty \text{ a.s.},$$

where the inequality is a straightforward consequence of Assumption II.3.10 (recall that $(1-\lambda_i(\omega))\,\varrho(\omega)+\lambda_i(\omega)\,\theta_i(\omega)\in M(x)_\omega$ for every $i\in\{1,\ldots,2^d\}$).

Finally, for every fixed $\omega\in\Omega\setminus N$, we know that the real-valued linear function $y\mapsto x+\langle y,Y(\omega)\rangle_{\mathbb{R}^d}$ attains its maximum on $P_x(\omega)$ at the extreme points $\{\theta_1(\omega),\ldots,\theta_{2^d}(\omega)\}$, consequently

$$x+\langle y,Y(\omega)\rangle_{\mathbb{R}^d}\leq x+\max\{\langle\theta_1(\omega)\,,Y(\omega)\rangle_{\mathbb{R}^d}\,,\ldots,\langle\theta_{2^d}(\omega)\,,Y(\omega)\rangle_{\mathbb{R}^d}\}$$

for all $y\in M(x)_\omega$. Therefore, the monotonicity of almost all sample paths of V implies that

$$V(x+\langle y,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\leq\bigvee_{i=1}^{2^d}V(x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\leq\sum_{i=1}^{2^d}V^+(x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)$$

holds a.s. (with the convention that $V^+(z,\omega)=0$ whenever $z<0$), thus

$$V^+(x+\langle y,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\leq\sum_{i=1}^{2^d}V^+(x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\ \text{ a.s..}$$

Now, fixing $i\in\{1,\ldots,2^d\}$, we get

$$V^+(x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)=V^+\left(\frac{1}{\lambda_i(\cdot)}\,\lambda_i(\cdot)\,[x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}]\,,\cdot\right)$$

(recall that $\lambda_i>0$). But, for every $\omega\in\{x+\langle\theta_i,Y\rangle_{\mathbb{R}^d}<0\}$, it holds that

$$V^+\left(\frac{1}{\lambda_i(\omega)}\,\lambda_i(\omega)\,[x+\langle\theta_i(\omega)\,,Y(\omega)\rangle_{\mathbb{R}^d}]\,,\omega\right)=0$$
$$<\left(\frac{1}{\lambda_i(\omega)}\right)^{\gamma}C=\left(\frac{1}{\lambda_i(\omega)}\right)^{\gamma}\big[V^+(\lambda_i(\omega)\,[x+\langle\theta_i(\omega)\,,Y(\omega)\rangle_{\mathbb{R}^d}]\,,\omega)+C\big]\,.$$

Combining this with $\lambda_i<1$ and Assumption II.3.13 then yields

$$V^+\left(\frac{1}{\lambda_i(\cdot)}\,\lambda_i(\cdot)\,[x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}]\,,\cdot\right)$$
$$\leq\left(\frac{1}{\lambda_i(\cdot)}\right)^{\gamma}\big[V^+(\lambda_i(\cdot)\,[x+\langle\theta_i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}]\,,\cdot)+C\big]$$

$$= \left(\frac{1}{\lambda_i(\cdot)}\right)^{\gamma} \left[V^+(\lambda_i(\cdot)\left[x + \langle \varrho(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d} + \langle \theta_i(\cdot) - \varrho(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\right], \cdot) + C\right] \text{ a.s..}$$

Additionally, since $\lambda_i < 1$, $x + \langle \varrho, Y\rangle_{\mathbb{R}^d} \geq 0$ a.s. (because $\varrho(\omega) \in M(x)_\omega$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, thus in particular the $\mathscr{G}$-measurable random variable ϱ is admissible, i.e., it belongs to $\Xi^d(x)$) and almost all sample paths of V are non-decreasing (Assumption II.3.8), we obtain

$$\begin{aligned} &V^+(\lambda_i(\cdot)\left[x + \langle \varrho(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\right] + \lambda_i(\cdot)\left[\langle \theta_i(\cdot) - \varrho(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\right], \cdot) \\ &\qquad\qquad \leq V^+(x + \langle \varrho(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d} + \langle \lambda_i(\cdot)\left[\theta_i(\cdot) - \varrho(\cdot)\right], Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \text{ a.s..} \end{aligned}$$

Hence, putting all the preceding equations together allows us to conclude that

$$\begin{aligned} &V^+(x + \langle \theta_i(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \\ &\quad \leq \left(\frac{1}{\lambda_i(\cdot)}\right)^{\gamma} \left[V^+(x + \langle (1-\lambda_i(\cdot))\,\varrho(\cdot) + \lambda_i(\cdot)\,\theta_i(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) + C\right] \text{ a.s.,} \end{aligned}$$

and so, for any arbitrary $\xi \in \Xi^d(x)$ satisfying $\xi(\omega) \in D(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $V^+(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \leq L_x$ a.s., as desired. □

Proof of Lemma II.3.17. Given an arbitrary $\xi \in \Xi^d(x_0)$, we start by fixing a version of the above conditional expectation for each $q \in \mathbb{Q}_0^+ \cap [x_0, +\infty)$, that is, let

$$F_\xi(q, \cdot) \triangleq \mathbb{E}_{\mathbb{P}}\left[V(q + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right] \text{ a.s..}$$

Then, for every $q_1, q_2 \in \mathbb{Q}_0^+$ with $x_0 \leq q_1 < q_2$, the inequality $F_\xi(q_1, \cdot) \leq F_\xi(q_2, \cdot)$ holds a.s.. To see that it is so, first observe that it follows from Assumption II.3.8 that

$$V(q_1 + \langle \xi(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d}\,, \omega) \leq V(q_2 + \langle \xi(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d}\,, \omega)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, hence the monotonicity of the conditional expectation gives us the claimed inequality. On the other hand, Assumption II.3.10 implies that, for every rational $q \geq x_0$,

$$F_\xi(q, \cdot) \leq \operatorname*{ess\,sup}_{\xi \in \Xi^d(q)} \mathbb{E}_{\mathbb{P}}\left[V(q + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right] < +\infty \text{ a.s.,}$$

where we invoke the obvious inclusion $\Xi^d(x_0) \subseteq \Xi^d(q)$ to deduce the first inequality.

Therefore, the set

$$N \triangleq \left(\bigcup_{\substack{q_1, q_2 \in \mathbb{Q}_0^+ \\ x_0 \leq q_1 < q_2}} E_{q_1, q_2}\right) \cup \left(\bigcup_{\substack{q \in \mathbb{Q}_0^+ \\ q \geq x_0}} E_q\right),$$

with $E_{q_1,q_2} \triangleq \{\omega \in \Omega\colon\ F_\xi(q_1,\omega) > F_\xi(q_2,\omega)\}$ and $E_q \triangleq \{\omega \in \Omega\colon\ F_\xi(q,\omega) = +\infty\}$ in $\mathscr{G}$, has probability zero, and we shall now define

$$G_\xi(x,\omega) \triangleq \begin{cases} \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>x}} F_\xi(q,\omega)\,, & \text{if } \omega \in \Omega \setminus N,\\ 0, & \text{otherwise,}\end{cases}$$

for every $x \geq x_0$.

It is obvious that, when $x \in \mathbb{Q}_0^+ \cap [x_0,+\infty)$, we have $G_\xi(x,\omega) \geq F_\xi(x,\omega)$ for all $\omega \in \Omega \setminus N$. Indeed, given any $q \in \mathbb{Q}_0^+ \cap (x,+\infty)$, it follows from the definition of the set N that $F_\xi(q,\omega) \geq F_\xi(x,\omega)$ for every $\omega \in \Omega \setminus N$, thus the claim follows from the definition of infimum.

Also, by construction, $G_\xi(x,\omega) < +\infty$ for all $x \geq x_0$ and $\omega \in \Omega$. Moreover, for every $\omega \in \Omega \setminus N$, $G(\cdot,\omega)$ is non-decreasing on $[x_0,+\infty)$, since

$$G_\xi(x,\omega) = \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>x}} F_\xi(q,\omega) \leq \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>y}} F_\xi(q,\omega) = G_\xi(y,\omega)$$

is true for all x and y such that $x_0 \leq x \leq y$.

The remainder of the proof will now be divided into two parts.

(i) We wish to show that, for every fixed $x \in [x_0,+\infty)$, $G_\xi(x,\cdot)$ is a version of $\mathbb{E}_{\mathbb{P}}\left[V(x + \langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\middle|\mathscr{G}\right]$.

In fact, on the one hand, we may deduce by an analogous argument to the one used above that, for every rational number $q > x$, the set

$$E_q \triangleq \{\omega \in \Omega\colon\ \mathbb{E}_{\mathbb{P}}\left[V(x + \langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\middle|\mathscr{G}\right](\omega) > F_\xi(q,\omega)\}$$

is a $\mathbb{P}$-null one, and so $\mathbb{P}\left((\Omega\setminus N)\cap\left(\bigcap_{\substack{q\in\mathbb{Q}_0^+\\ q>x}} E_q^c\right)\right) = 1$. But this implies that

$$\mathbb{E}_{\mathbb{P}}\left[V(x + \langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\middle|\mathscr{G}\right] \leq \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>x}} F_\xi(q,\cdot) = G_\xi(x,\cdot) \ \text{ a.s. on } \Omega\setminus N.$$

On the other hand, to check that the reverse inequality also holds a.s. on $\Omega\setminus N$, take a strictly decreasing sequence $\{q_n;\, n \in \mathbb{N}\}$ of positive rational numbers satisfying $x < q_n \leq x+1$ for every $n \in \mathbb{N}$, as well as $\lim_{n\to+\infty} q_n = x$ (where the existence of such a sequence is a trivial consequence of the density of the rationals in the reals). Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the continuity and monotonicity of the function $V(\cdot,\omega)\colon\ \mathbb{R} \to \mathbb{R}$ (see Assumption II.3.8) imply

$$\lim_{n\to+\infty} V(q_n + \langle\xi(\omega)\,,Y(\omega)\rangle_{\mathbb{R}^d}\,,\omega) = V(x + \langle\xi(\omega)\,,Y(\omega)\rangle_{\mathbb{R}^d}\,,\omega)\,,$$

with the limit being attained in a non-increasing way. Furthermore,

$$V(q_n + \langle \xi(\omega), Y(\omega)\rangle_{\mathbb{R}^d}, \omega) \leq V(x + 1 + \langle \xi(\omega), Y(\omega)\rangle_{\mathbb{R}^d}, \omega)$$

for all $n \in \mathbb{N}$.

But this implies that the sequence of non-negative random variables

$$\{V(x + 1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) - V(q_n + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\,;\, n \in \mathbb{N}\}$$

is non-decreasing, and that

$$\begin{aligned}\lim_{n\to+\infty} & [V(x + 1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) - V(q_n + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)] \\ &= V(x + 1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) - V(x + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) \text{ a.s..}\end{aligned}$$

So, we can apply the Monotone Convergence Theorem (for the conditional expectation) to conclude that

$$\begin{aligned}\lim_{n\to+\infty} & \mathbb{E}_{\mathbb{P}}[V(x + 1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) - V(q_n + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] \\ &= \mathbb{E}_{\mathbb{P}}[V(x + 1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) - V(x + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] \text{ a.s.,}\end{aligned}$$

and therefore, noting that $\mathbb{E}_{\mathbb{P}}[V(x + 1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] < +\infty$ a.s. by Assumption II.3.10 (and Remark II.3.11), we have

$$\lim_{n\to+\infty} \mathbb{E}_{\mathbb{P}}[V(q_n + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] = \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] \text{ a.s..}$$

Finally, we see that

$$\begin{aligned}G_\xi(x, \omega) &= \inf_{\substack{q\in\mathbb{Q}_0^+ \\ q<x}} F_\xi(q, \cdot) \leq \inf_{n\in\mathbb{N}} F_\xi(q_n, \cdot) \leq \liminf_{n\to+\infty} F_\xi(q_n, \cdot) \\ &= \liminf_{n\to+\infty} \mathbb{E}_{\mathbb{P}}[V(q_n + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] \text{ a.s.,}\end{aligned}$$

hence combining both results yields $G_\xi(x, \omega) \leq \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}]$ a.s., as intended.

(ii) It remains only to show that almost all sample paths of G_ξ are right-continuous by construction. In fact, fix any $\omega \in \Omega \setminus N$ and consider an arbitrary $x \geq x_0$. Then it follows from the definition of infimum that, for every $\varepsilon > 0$, there must exist some $q \in \mathbb{Q}_0^+ \cap (x, +\infty)$ satisfying $F_\xi(q, \omega) < G_\xi(x, \omega) + \varepsilon$. Hence, setting $\delta \triangleq q - x > 0$, we can use the previously established monotonicity of $G_\xi(\cdot, \omega)$ and the definition of infimum to obtain the inequality

$$0 \leq G_\xi(y, \omega) - G_\xi(x, \omega) \leq F_\xi(q, \omega) - G_\xi(x, \omega) < \varepsilon$$

for all $y \in (x, x+\delta) = (x, q)$, thus proving the right-continuity of $G_\xi(\cdot, \omega)$. □

Proof of Lemma II.3.18. Let us first choose, for each positive rational number q, a version $F(q, \omega)$ of $\operatorname{ess\,sup}_{\xi \in \Xi^d(q)} \mathbb{E}_{\mathbb{P}}[V(q + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G}]$.

Next, for any pair $q_1 < q_2$ of positive rational numbers, we define the set

$$E_{q_1,q_2} \triangleq \{\omega \in \Omega \colon F(q_1, \omega) > F(q_2, \omega)\} \in \mathscr{G},$$

which has probability zero. Indeed, if we consider any $\xi \in \Xi^d(q_1) \subseteq \Xi^d(q_2)$, obviously $q_1 + \langle \xi, Y\rangle_{\mathbb{R}^d} < q_2 + \langle \xi, Y\rangle_{\mathbb{R}^d}$, and so we have by Assumption II.3.8 that

$$V(q_1 + \langle \xi(\omega), Y(\omega)\rangle_{\mathbb{R}^d}, \omega) \leq V(q_2 + \langle \xi(\omega), Y(\omega)\rangle_{\mathbb{R}^d}, \omega)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. We then conclude from the monotonicity of the conditional expectation that

$$\mathbb{E}_{\mathbb{P}}[V(q_1 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G}] \leq \mathbb{E}_{\mathbb{P}}[V(q_2 + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G}] \leq F(q_2, \cdot) \text{ a.s.},$$

and it follows from the definition of essential supremum, combined with the arbitrarity of ξ, that $F(q_1, \cdot) \leq F(q_2, \cdot)$ a.s..

Similarly, for every $q \in \mathbb{Q}_0^+$, set $E_q \triangleq \{\omega \in \Omega \colon F(q, \omega) < +\infty\}$, which is also a $\mathbb{P}$-null set, by Assumption II.3.10 and Remark II.3.11.

But this implies that

$$N \triangleq \left(\bigcup_{\substack{q_1, q_2 \in \mathbb{Q}_0^+ \\ q_1 < q_2}} E_{q_1,q_2} \right) \cup \left(\bigcup_{q \in \mathbb{Q}_0^+} E_q \right),$$

being a countable union of null sets, has probability zero as well. So let us specify, for each $x \in [0, +\infty)$,

$$G(x, \omega) \triangleq \begin{cases} \inf_{\substack{q \in \mathbb{Q}_0^+ \\ q > x}} F(q, \omega), & \text{if } \omega \in \Omega \setminus N, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, when $x \in \mathbb{Q}_0^+$, $G(x, \cdot) \geq F(x, \cdot)$ on $\Omega \setminus N$. In addition, for each $x \geq 0$ we have that $G(x, \cdot)$ is $\mathscr{G}$-measurable (recall that the infimum of a countable family of measurable random variables is itself measurable). We shall split the remainder of the proof into five separate parts.

(i) With the above definition, it is straightforward to check that, for every $\omega \in \Omega$, the function $G(\cdot, \omega)$ is non-decreasing. In fact, let us consider an arbitrary $\omega \in \Omega \setminus N$ (the result being trivially true when $\omega \in N$). For every x and y such that

$0 \le x \le y$, the inequality below is obvious,

$$G(x,\omega) = \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>x}} F(q,\omega) \le \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>y}} F(q,\omega) = G(y,\omega)\,.$$

It is also clear that, for every ω outside the $\mathbb{P}$-null set N, it holds that $G(x,\omega) < +\infty$ for all $x \ge 0$.

(ii) We proceed to show that, for all $x \in [0,+\infty)$,

$$G(x,\cdot) = \operatorname*{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \text{ a.s..}$$

In order to do so, let us fix an arbitrary $x \in [0,+\infty)$. Then, using a similar reasoning to that above, we deduce that, for every $q \in \mathbb{Q}_0^+ \cap (x,+\infty)$, the inequality

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \le F(q,\cdot)$$

holds a.s., thus for $\mathbb{P}$-a.e. $\omega \in \Omega \setminus N$ we get

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}](\omega) \le \inf_{\substack{q\in\mathbb{Q}_0^+\\ q>x}} F(q,\omega) = G(x,\omega)\,.$$

It remains to verify that the reverse inequality is also true (except possibly on a set of measure zero). This will be achieved in three steps.

(a) So let us start by taking a strictly decreasing sequence $\{q_n;\, n \in \mathbb{N}\}$ of rational numbers satisfying $x < q_n < x+1$ and $\lim_{n\to+\infty} q_n = x$ (its existence being ensured by the density of $\mathbb{Q}$ in $\mathbb{R}$). Now, given any $n \in \mathbb{N}$, we know that the family $\left\{\mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}\right];\, \xi \in \Xi^d(q_n)\right\}$ is directed upwards (see Lemma II.5.7), therefore we can extract a sequence $\{\xi_k^n;\, k \in \mathbb{N}\} \subseteq \Xi^d(q_n)$ attaining the essential supremum in a non-decreasing way.

Consequently, we can recursively define the sets

$$A_k^n \triangleq \left\{\omega \in \Omega\colon\ \mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \xi_k^n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}\right](\omega) \ge F(q_n,\omega) - \frac{1}{n}\right\} \cap \left(\bigcup_{l=1}^{k-1} A_l^n\right)^c$$

(with the usual convention that an empty union is the empty set). It is clear that each one of the A_k^n belongs to the σ-algebra $\mathscr{G}$, and also that the family $\{A_k^n;\, k \in \mathbb{N}\}$ is pairwise disjoint[28]. These observations imply that the $\mathbb{R}^d$-valued random variable $\zeta_n \triangleq \sum_{k=1}^{+\infty} \xi_k^n\, \mathbb{1}_{A_k^n}$ is $\mathscr{G}$-measurable.

[28] A family of sets is said to be *pairwise disjoint* if, for every two sets E_1 and E_2 in the family with

Moreover, it satisfies $q_n + \langle \zeta_n, Y\rangle_{\mathbb{R}^d} \geq 0$ a.s., since for every ω outside the null set

$$\bigcup_{k=1}^{+\infty} \{\omega \in \Omega\colon\ q_n + \langle \xi_k^n(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} < 0\}\,,$$

either $\omega \in \bigsqcup_{k=1}^{+\infty} A_k^n$, in which case we get $q_n + \langle \zeta_n(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} = q_n + \langle \xi_k^n(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d}$ for some unique $k \in \mathbb{N}$, or $q_n + \langle \zeta_n(\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} = q_n$.

Thirdly, the following inequality is verified for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \zeta_n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right](\omega) \geq F(q_n, \omega) - \frac{1}{n}.$$

Indeed, it is immediate to obtain that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \zeta_n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right] \\
&\quad = \mathbb{E}_{\mathbb{P}}\left[\sum_{k=1}^{+\infty} V(q_n + \langle \xi_k^n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)\, \mathbb{1}_{A_k^n}(\cdot) + V(q_n, \cdot)\, \mathbb{1}_{\bigcap_{k=1}^{+\infty}(A_k^n)^c}(\cdot) \middle| \mathscr{G}\right] \\
&\quad = \mathbb{E}_{\mathbb{P}}\left[\sum_{k=1}^{+\infty} V(q_n + \langle \xi_k^n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)\, \mathbb{1}_{A_k^n}(\cdot) \middle| \mathscr{G}\right] \text{ a.s.,}
\end{aligned}$$

since

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{k=1}^{+\infty} (A_k^n)^c\right) \\
&= \mathbb{P}\left(\bigcap_{k=1}^{+\infty} \left\{\omega \in \Omega\colon\ \mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \xi_k^n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right](\omega) < F(q_n, \omega) - \frac{1}{n}\right\}\right) \\
&\leq \mathbb{P}\Bigg(\Bigg\{\omega \in \Omega\colon\ \lim_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \xi_k^n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right](\omega) \\
&\qquad\qquad = \left(\operatorname*{ess\,sup}_{\xi\in\Xi^d(q_n)} \mathbb{E}_{\mathbb{P}}[V(q_n + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}]\right)(\omega)\Bigg\} \\
&\quad \cap \left\{\omega \in \Omega\colon\ \left(\operatorname*{ess\,sup}_{\xi\in\Xi^d(q_n)} \mathbb{E}_{\mathbb{P}}[V(q_n + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}]\right)(\omega) = F(q_n, \omega)\right\} \\
&\quad \cap \left\{\omega \in \Omega\colon\ \lim_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[V(q_n + \langle \xi_k^n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G}\right](\omega) \leq F(q_n, \omega) - \frac{1}{n}\right\}\Bigg) \\
&= 0.
\end{aligned}$$

On the other hand, by Lemma II.3.16 and Assumption II.3.8, we have for

$E_1 \neq E_2$, we have $E_1 \cap E_2 = \varnothing$. Henceforth, we shall use the symbol $\bigsqcup$ to denote a pairwise disjoint union.

every $m \in \mathbb{N}$ that

$$\left| \sum_{k=1}^{m} V(q_n + \langle \xi_k^n(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) \, \mathbb{1}_{A_k^n} \right| \leq L_{q_n} + V^-(0, \cdot) \text{ a.s.},$$

so a straightforward application of Lebesgue's Dominated Convergence Theorem for the conditional expectation (recall Assumption II.3.12) yields

$$\mathbb{E}_{\mathbb{P}} \left[\sum_{k=1}^{+\infty} V(q_n + \langle \xi_k^n(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) \, \mathbb{1}_{A_k^n}(\cdot) \middle| \mathscr{G} \right] \\ = \sum_{k=1}^{+\infty} \mathbb{E}_{\mathbb{P}} \left[V(q_n + \langle \xi_k^n(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G} \right] \mathbb{1}_{A_k^n} \geq F(q_n, \cdot) - \frac{1}{n} \text{ a.s..}$$

(b) Next, fix an arbitrary natural n. It was observed above that $\zeta_n \in \Xi^d(q_n) \subseteq \Xi^d(x+1)$. Thus, taking $\widehat{\zeta}_n$ to be its projection on D given by Proposition II.3.7, we know from Lemma II.3.14 and the preceding equations that

$$\mathbb{E}_{\mathbb{P}} \left[V\Big(q_n + \Big\langle \widehat{\zeta}_n(\cdot), Y(\cdot) \Big\rangle_{\mathbb{R}^d}, \cdot\Big) \middle| \mathscr{G} \right] = \mathbb{E}_{\mathbb{P}} \left[V(q_n + \langle \zeta_n(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G} \right] \\ \geq F(q_n, \cdot) - \frac{1}{n} \text{ a.s..} \tag{II.5.7}$$

Moreover, Lemma II.3.15 allows us to conclude that $\|\widehat{\zeta}_n\|_{\mathbb{R}^n} \leq K_{x+1}$ a.s.. Therefore, we can invoke Proposition B.2 to extract a random subsequence $\left\{\widehat{\zeta}_{n_k};\, k \in \mathbb{N}\right\}$ such that $\lim_{k \to +\infty} \widehat{\zeta}_{n_k} = \zeta$ a.s., for some $\mathscr{G}$-measurable random variable ζ. But then

$$x + \langle \zeta(\omega), Y(\omega) \rangle_{\mathbb{R}^d} = \lim_{k \to +\infty} \Big(q_{n_k(\omega)} + \Big\langle \widehat{\zeta}_{n_k(\omega)}(\omega), Y(\omega) \Big\rangle_{\mathbb{R}^d} \Big) \geq 0$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, i.e. $\zeta \in \Xi^d(x)$, which in turn implies that

$$\operatorname*{ess\,sup}_{\xi \in \Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G}] \geq \mathbb{E}_{\mathbb{P}}[V(x + \langle \zeta(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{G}] \text{ a.s..} \tag{II.5.8}$$

(c) Finally, let us define the random variables $f_k : \Omega \to \mathbb{R}$ as follows,

$$f_k(\omega) \triangleq V\Big(q_{n_k(\omega)} + \Big\langle \widehat{\zeta}_{n_k(\omega)}(\omega), Y(\omega) \Big\rangle_{\mathbb{R}^d}, \omega\Big), \qquad \omega \in \Omega.$$

By virtue of the way the sequence $\{q_n;\, n \in \mathbb{N}\}$ and the random subsequence $\left\{\widehat{\zeta}_{n_k};\, k \in \mathbb{N}\right\}$ were produced, and of the continuity of the paths of V (see Assumption II.3.8), it is clear that $\lim_{k \to +\infty} f_k = V(x + \langle \zeta(\cdot), Y(\cdot) \rangle_{\mathbb{R}^d}, \cdot)$ a.s.. We further observe that, for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$f_k(\omega) \leq V\Big(x + 1 + \Big\langle \widehat{\zeta}_{n_k(\omega)}(\omega), Y(\omega) \Big\rangle_{\mathbb{R}^d}, \omega\Big) \leq L_{x+1}(\omega),$$

where the first inequality follows from the monotonicity of V (again we refer to Assumption II.3.8), and the second inequality is a simple consequence of Lemma II.3.16 combined with the fact that

$$x + 1 + \Big\langle \widehat{\zeta}_{n_k(\omega)}(\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} \geq q_{n_k(\omega)} + \Big\langle \widehat{\zeta}_{n_k(\omega)}(\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} \geq 0$$

for all ω outside the $\mathbb{P}$-null set $\bigcup_{n=1}^{+\infty} \Big\{\omega \in \Omega\colon\ q_n + \Big\langle \widehat{\zeta}_n(\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} < 0\Big\}$. Hence, we may apply the reverse Fatou lemma to conclude that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}[V(x + \langle \zeta(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)|\,\mathscr{G}] \geq \limsup_{k\to+\infty} \mathbb{E}_{\mathbb{P}}[\,f_k|\,\mathscr{G}] \\ &\quad \geq \liminf_{k\to+\infty} F\big(q_{n_k(\cdot)}, \cdot\big) + \liminf_{k\to+\infty} \left(-\frac{1}{n_k(\cdot)}\right) \geq \inf_{n\in\mathbb{N}} F(q_n, \cdot) \ \text{ a.s.,} \end{aligned} \tag{II.5.9}$$

where the second inequality is due to the fact that, for every $k \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[\,f_k|\,\mathscr{G}] &= \sum_{i=k}^{+\infty} \mathbb{E}_{\mathbb{P}}\Big[V\Big(q_i + \Big\langle \widehat{\zeta}_i(\cdot)\,, Y(\cdot)\Big\rangle_{\mathbb{R}^d}\,, \cdot\Big)\Big|\,\mathscr{G}\Big]\, \mathbb{1}_{\{n_k=i\}} \\ &\geq \sum_{i=k}^{+\infty} \left(F(q_i, \cdot) - \frac{1}{i}\right) \mathbb{1}_{\{n_k=i\}} = F\big(q_{n_k(\cdot)}, \cdot\big) - \frac{1}{n_k(\cdot)} \ \text{ a.s..} \end{aligned}$$

Combining equations (II.5.8) and (II.5.9) finally gives the intended inequality $\operatorname{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}[V(x + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)|\,\mathscr{G}] \geq \inf_{n\in\mathbb{N}} F(q_n, \cdot) \geq G(x, \cdot)$ a.s..

(iii) Thirdly, G is, by the way it was constructed, right-continuous a.s. (the reasoning being completely identical to that in part *(ii)* of the proof of Lemma II.3.17).

(iv) Now consider an arbitrary $\mathscr{G}$-measurable random variable $H \geq 0$ a.s.. We wish to see that

$$\mathbb{P}\left\{\omega \in \Omega\colon\ G(H(\omega)\,, \omega) = \operatorname*{ess\,sup}_{\xi\in\Xi^d(H)} \mathbb{E}_{\mathbb{P}}[V(H(\cdot) + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)|\,\mathscr{G}](\omega)\right\} = 1.$$

This follows immediately when H is a $\mathscr{G}$-measurable countable step-function, that is, of the form $H = \sum_{i=1}^{+\infty} x_i \mathbb{1}_{A_i}$, for some $x_i \geq 0$ with $x_i \neq x_j$ if $i \neq j$, as well as some $A_i \in \mathscr{G}$ with $\mathbb{P}\big(\bigcap_{i=1}^{+\infty} A_i^c\big) = 0$ and $A_i \cap A_j = \varnothing$ if $i \neq j$. In fact,

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}[V(H(\cdot) + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)|\,\mathscr{G}] \\ &\qquad = \sum_{i=1}^{+\infty} \mathbb{E}_{\mathbb{P}}\big[\,V\big(x_i + \langle \xi(\cdot)\, \mathbb{1}_{A_i}(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot\big)\, \mathbb{1}_{A_i}(\cdot)\big|\,\mathscr{G}\big] \\ &\qquad \leq \sum_{i=1}^{+\infty} G(x_i, \cdot)\, \mathbb{1}_{A_i} = G(H(\cdot)\,, \cdot) \ \text{ a.s.,} \end{aligned}$$

where the inequality is a straightforward consequence of the fact that each $G(x_i, \cdot)$

is a version of $\operatorname{ess\,sup}_{\xi\in\Xi^d(x_i)} \mathbb{E}_{\mathbb{P}}[V(x_i+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]$, combined with the fact that $x_i+\langle\xi\mathbb{1}_{A_i},Y\rangle_{\mathbb{R}^d}\geq(H+\langle\xi,Y\rangle_{\mathbb{R}^d})\,\mathbb{1}_{A_i}\geq 0$ a.s.. On the other hand, let X be such that, for every $\xi\in\Xi^d(H)$, we have $\mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]\leq X$ a.s.. Since, for every $i\in\mathbb{N}$, we have by one of the previous steps that

$$G(x_i,\cdot)=\operatorname*{ess\,sup}_{\xi\in\Xi^d(x_i)}\mathbb{E}_{\mathbb{P}}[V(x_i+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]\ \text{ a.s.},$$

we can find for every $n\in\mathbb{N}$, as before, some $\xi_n^i\in\Xi^d(x_i)$ such that

$$\mathbb{E}_{\mathbb{P}}\big[V\big(x_i+\big\langle\xi_n^i(\cdot)\,,Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\big)\big|\mathscr{G}\big]\geq G(x_i,\cdot)-\frac{1}{n}\ \text{ a.s..}$$

Thus, setting $\xi_n\triangleq\sum_{i=1}^{+\infty}\xi_n^i\mathbb{1}_{A_i}$, it is clear that

$$H+\langle\xi_n,Y\rangle_{\mathbb{R}^d}=\sum_{i=1}^{+\infty}\big(x_i+\big\langle\xi_n^i,Y\big\rangle_{\mathbb{R}^d}\big)\,\mathbb{1}_{A_i}\geq 0\text{ a.s.},$$

so $\xi_n\in\Xi^d(H)$. Moreover, for every $n\in\mathbb{N}$, we have by hypothesis that

$$\begin{aligned}X&\geq\sum_{i=1}^{+\infty}\mathbb{E}_{\mathbb{P}}\big[V\big(x_i+\big\langle\xi_n^i(\cdot)\,,Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\big)\big|\mathscr{G}\big]\,\mathbb{1}_{A_i}\\&\geq\sum_{i=1}^{+\infty}\left(G(x_i,\cdot)-\frac{1}{n}\right)\mathbb{1}_{A_i}=G(H(\cdot)\,,\cdot)-\frac{1}{n}\ \text{ a.s..}\end{aligned}$$

Consequently, $X\geq\lim_{n\to+\infty}(G(H(\cdot)\,,\cdot)-1/n)=G(H(\cdot)\,,\cdot)$ a.s., hence (II.3.11) follows from the definition of essential supremum.

Next, suppose H is any bounded, $\mathscr{G}$-measurable, non-negative (a.s.) random variable, so there exists some $M>0$ such that $H\leq M$ a.s.. It is a well-known fact that we can take a non-increasing sequence $\{H_n;\,n\in\mathbb{N}\}$ of $\mathscr{G}$-measurable step-functions converging to H a.s., and such that, for every $n\in\mathbb{N}$, $H_n\leq M$ a.s.. Then, fixing an arbitrary $\xi\in\Xi^d(H)$, we have for every $n\in\mathbb{N}$ that $H_n+\langle\xi,Y\rangle_{\mathbb{R}^d}\geq H+\langle\xi,Y\rangle_{\mathbb{R}^d}\geq 0$ a.s., therefore

$$\begin{aligned}G(H_n(\cdot)\,,\cdot)&=\operatorname*{ess\,sup}_{\zeta\in\Xi^d(H_n)}\mathbb{E}_{\mathbb{P}}[V(H_n(\cdot)+\langle\zeta(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]\\&\geq\mathbb{E}_{\mathbb{P}}[V(H_n(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]\ \text{ a.s.}\end{aligned}$$

(recall that the equality is true for step-functions), which in turn yields

$$\liminf_{n\to+\infty}G(H_n(\cdot)\,,\cdot)\geq\liminf_{n\to+\infty}\mathbb{E}_{\mathbb{P}}[V(H_n(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]\ \text{ a.s.}$$

But, on the one hand we get by the almost sure path right-continuity of G that $\lim_{n\to+\infty}G(H_n(\cdot)\,,\cdot)=G(H(\cdot)\,,\cdot)$ a.s.. On the other hand, we can apply both the

Fatou lemma and the reverse Fatou lemma (see Assumption II.3.12) to conclude

$$\liminf_{n\to+\infty} \mathbb{E}_{\mathbb{P}}[V(H_n(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \geq \mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \text{ a.s.},$$

hence $\operatorname{ess\,sup}_{\xi\in\Xi^d(H)} \mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \leq G(H(\cdot)\,,\cdot)$ a.s. (by the arbitrarity of $\xi \in \Xi^d(H)$). Now, to prove the reverse inequality, we can construct (as in part *(ii)* of this proof) a sequence $\{\zeta_n;\, n\in\mathbb{N}\}$ such that, for every $n\in\mathbb{N}$, we have $\zeta_n\in\Xi^d(H_n)$, $\zeta_n(\omega)\in D(\omega)$ for $\mathbb{P}$-a.e. $\omega\in\Omega$, and

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(H_n)} \mathbb{E}_{\mathbb{P}}[V(H_n(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] - \frac{1}{n} \leq \mathbb{E}_{\mathbb{P}}[V(H_n(\cdot)+\langle\zeta_n(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}]$$

We remark further that each ζ_n belongs to $\Xi^d(M)$ (because $M+\langle\zeta_n,Y\rangle_{\mathbb{R}^d} \geq H_n+\langle\zeta_n,Y\rangle_{\mathbb{R}^d} \geq 0$ a.s.), so by Lemma II.3.15 there exists a random variable K_M such that $\|\zeta_n\|_{\mathbb{R}^d} \leq K_M$ a.s.. Therefore we can use Proposition B.2 to get a random subsequence $\{\zeta_{n_k};\, k\in\mathbb{N}\}$ with $\lim_{k\to+\infty}\zeta_{n_k} = \zeta$ a.s., for some $\mathscr{G}$-measurable ζ. Clearly, $H+\langle\zeta,Y\rangle_{\mathbb{R}^d} = \lim_{k\to+\infty}\big(H_{n_k}+\langle\zeta_{n_k},Y\rangle_{\mathbb{R}^d}\big)$ a.s., and for every $k\in\mathbb{N}$,

$$H_{n_k}+\langle\zeta_{n_k},Y\rangle_{\mathbb{R}^d} = \sum_{i=k}^{+\infty}(H_i+\langle\zeta_i,Y\rangle_{\mathbb{R}^d})\,\mathbb{1}_{\{\omega\in\Omega:\ n_k(\omega)=i\}} \geq 0 \text{ a.s.},$$

hence $\zeta\in\Xi^d(H)$. Consequently,

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(H)} \mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \geq \mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\zeta(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \text{ a.s.},$$

by definition of essential supremum. Besides, we have by Lemma II.3.16 that, for every $k\in\mathbb{N}$,

$$V^+\Big(H_{n_k(\cdot)}(\cdot)+\big\langle\zeta_{n_k(\cdot)}(\cdot)\,,Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\Big) \leq V^+\Big(M+\big\langle\zeta_{n_k(\cdot)}(\cdot)\,,Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\Big) \leq L_M \text{ a.s.},$$

(note that $\zeta_{n_k}\in\Xi^d(M)$), so Fatou's lemma and the reverse Fatou lemma yield

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\zeta(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] &\geq \limsup_{k\to+\infty}\mathbb{E}_{\mathbb{P}}\Big[V\Big(H_{n_k(\cdot)}(\cdot)+\big\langle\zeta_{n_k(\cdot)}(\cdot)\,,Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\Big]\\ &\geq \limsup_{k\to+\infty} G\big(H_{n_k(\cdot)}(\cdot)\,,\cdot\big) = G(H(\cdot)\,,\cdot) \text{ a.s..}\end{aligned}$$

Combining the inequalities above, we establish (II.3.11) for any bounded H as well.

Finally, we extend the above result to an arbitrary $\mathscr{G}$-measurable $H \geq 0$ a.s.. Since $H = \sum_{n\in\mathbb{N}} H_n$, with each $H_n \triangleq H\mathbb{1}_{\{n-1\leq H<n\}}$ $\mathscr{G}$-measurable and bounded, we can obtain the desired equality using an argument entirely analogous to that of the countable step-function case.

(v) Lastly, we claim as well that almost all sample paths of G are left-continuous. To see this, let us begin with the remark that, as shown above, for every $x \geq 0$, the function $G(x,\cdot): \Omega \to \mathbb{R}$, being a version of the essential supremum of $\mathscr{G}$-measurable random variables, is itself measurable with respect to $\mathscr{G}$. In addition, almost every sample path of G is right-continuous. Therefore, it is a well known result that $G: [0,+\infty) \times \Omega \to \mathbb{R}$ is measurable with respect to the product σ-algebra $\mathscr{B}([0,+\infty)) \otimes \mathscr{G}$.

Next, defining for every $\omega \in \Omega$,

$$\overline{G}(x,\omega) \triangleq \begin{cases} \sup_{\substack{q\in\mathbb{Q}_0^+ \\ q<x}} G(q,\omega)\,, & \text{if } x>0, \\ G(0,\omega)\,, & \text{otherwise,} \end{cases}$$

it is obvious that $\overline{G}$ is $\mathscr{B}([0,+\infty)) \otimes \mathscr{G}$-measurable too. Besides, it is trivial to check that, for every $\omega \in \Omega$, the function $\overline{G}(\cdot,\omega)$ is non-decreasing on $(0,+\infty)$. We remark further that, by construction, all paths of G are left-continuous on $(0,+\infty)$. To see this, let us fix any real number $x > 0$ and consider an arbitrary $\varepsilon > 0$. By definition of supremum, we can find some positive rational number $q < x$ verifying $\overline{G}(x,\omega) - \varepsilon < G(q,\omega)$. Thus, setting $\delta \triangleq x - q > 0$, we obtain

$$0 \leq \overline{G}(x,\omega) - \overline{G}(y,\omega) \leq \overline{G}(x,\omega) - G(q,\omega) < \varepsilon$$

for all $y \in (q,x) = (x-\delta, x)$.

It then follows immediately from the monotonicity of all the sample paths of G that the inequality $G(x,\omega) \geq \overline{G}(x,\omega)$ holds true for every $x \geq 0$ and $\omega \in \Omega$. In particular, this gives that, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for all $x \geq 0$, it holds that $\overline{G}(x,\omega) < +\infty$. At last, by following the steps below, we shall show that $\mathbb{P}\big\{\omega \in \Omega\colon \forall\, x \geq 0,\ G(x,\omega) = \overline{G}(x,\omega)\big\} = 1$.

(a) The proof is by contradiction, so let us suppose that the set

$$\Omega_1 \triangleq \big\{\omega \in \Omega\colon \exists\, x > 0 \text{ s.t. } G(x,\omega) > \overline{G}(x,\omega)\big\}$$

has strictly positive measure, i.e., $\mathbb{P}(\Omega_1) > 0$. Note that, because $(\Omega, \mathscr{G}, \mathbb{P})$ is a complete measure space, we can apply the measurable projection theorem (see Theorem B.1) to deduce that $\Omega_1 = \mathrm{Proj}_\Omega\Big(\big(G - \overline{G}\big)^{-1}((0,+\infty))\Big)$ belongs to $\mathscr{G}$.

Thus, the multi-function $\mathscr{E}\colon\ \Omega \rightrightarrows [0,+\infty)$ given by

$$\mathscr{E}(\omega) \triangleq \begin{cases} \{x>0\colon\ G(x,\omega) > \overline{G}(x,\omega)\} & \text{if } \omega\in\Omega_1,\\ 1, & \text{otherwise,}\end{cases}$$

not only has $\operatorname{dom}\mathscr{E} = \Omega$, but also its graph

$$\operatorname{gph}\mathscr{E} = (\Omega_1^c \times \{1\}) \cup \Big([\Omega_1\times(0,+\infty)] \cap \Big[\big(G-\overline{G}\big)^{-1}((0,+\infty))\Big]\Big)$$

is a $\mathscr{G}$-random set. Consequently, we can apply the von Neumann-Aumann theorem to produce a $\mathscr{G}$-measurable selector $H:\ \Omega\to[0,+\infty)$ of $\mathscr{E}$. In particular, this implies that

$$\mathbb{P}\left\{\omega\in\Omega\colon\ G(H(\omega),\omega) > \overline{G}(H(\omega),\omega)\right\} \ge \mathbb{P}(\Omega_1) > 0. \tag{II.5.10}$$

Also, note that $H>0$. Furthermore, we may and shall assume, without loss of generality, that there exists some $\varepsilon\in(0,1]$ such that $H>\varepsilon$. Indeed, we have by the continuity from below[29] of the probability measure $\mathbb{P}$ that

$$\begin{aligned}\lim_{n\to+\infty}\mathbb{P}\left(\left\{H>\frac{1}{n}\right\}\cap\Omega_1\right) &= \mathbb{P}\left(\left[\bigcup_{n=1}^{+\infty}\left\{H>\frac{1}{n}\right\}\right]\cap\Omega_1\right)\\ &= \mathbb{P}(\{H>0\}\cap\Omega_1) = \mathbb{P}(\Omega_1),\end{aligned}$$

so there must exist some $p\in\mathbb{N}$ such that $\mathbb{P}\big(\{H>\frac{1}{n}\}\cap\Omega_1\big) > \mathbb{P}(\Omega_1)/2$ for all $n\ge p$. Thus, choosing $\varepsilon\in(0,1/p)$ and letting $\bar{H}:\ \Omega\to\mathbb{R}$ be the function defined by $\bar{H}(\omega)\triangleq H(\omega)\,\mathbb{1}_{\Omega_1^c\cup(\Omega_1\cap\{H>1/p\})}(\omega)+p^{-1}\mathbb{1}_{\Omega_1\cap\{H\le 1/p\}}(\omega)$, it is clear that $\overline{H}$ is $\mathscr{G}$-measurable and satisfies $\overline{H}>\varepsilon$. In addition,

$$\mathbb{P}\left\{\omega\in\Omega\colon\ G\big(\bar{H}(\omega),\omega\big) > \overline{G}\big(\bar{H}(\omega),\omega\big)\right\} \ge \mathbb{P}(\Omega_1\cap\{H>1/p\}) > 0.$$

(b) On the other hand, we shall see that $G(H(\omega),\omega)\le\overline{G}(H(\omega),\omega)$ holds for $\mathbb{P}$-a.e. $\omega\in\Omega$, thus contradicting (II.5.10).
Firstly, fix an arbitrary $n\in\mathbb{N}$. As in part *(ii)* of this proof, it is possible to construct some $\zeta_n\in\Xi^d(H)$ such that, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\zeta_n(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\,\mathscr{G}](\omega) \ge G(H(\omega),\omega) - \frac{1}{n}.$$

Next, setting for every $m\in\mathbb{N}$ (recall that $H>\varepsilon$),

$$f_n^m(\omega)\triangleq V\left(H(\omega)-\frac{\varepsilon}{m}+\frac{H(\omega)-\varepsilon/m}{H(\omega)}\langle\zeta_n(\omega),Y(\omega)\rangle_{\mathbb{R}^d},\omega\right),\qquad \omega\in\Omega,$$

[29] We recall that any measure μ on a measurable space (X,Σ) is *continuous from below* (respectively, *continuous from above*), that is, for every sequence $\{E_n;\ n\in\mathbb{N}\}\subseteq\Sigma$ which is non-decreasing (respectively, which is non-increasing and verifies $\mu(E_1)<+\infty$), we have $\lim_{n\to+\infty}\mu(E_n) = \mu\big(\bigcup_{n=1}^{+\infty}E_n\big)$ (respectively, $\lim_{n\to+\infty}\mu(E_n)=\mu\big(\bigcap_{n=1}^{+\infty}E_n\big)$).

it is trivial by continuity (see Assumption II.3.8) that $\{f_n^m;\, m \in \mathbb{N}\}$ converges a.s. to $V(H(\cdot) + \langle \zeta_n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)$, as $m \to +\infty$. Thus, Fatou lemma immediately gives

$$\liminf_{m\to+\infty} \mathbb{E}_{\mathbb{P}}\big[[f_n^m]^+ \big| \mathscr{G}\big] \geq \mathbb{E}_{\mathbb{P}}\big[V^+(H(\cdot) + \langle \zeta_n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)\big| \mathscr{G}\big] \text{ a.s..}$$

Secondly, we note that, for each $m \in \mathbb{N}$, the random vector $\zeta_n\,(H - \varepsilon/m)\,/H$ belongs to $\Xi^d(H - \varepsilon/m)$, because

$$H - \frac{\varepsilon}{m} + \left\langle \frac{H - \varepsilon/m}{H}\zeta_n, Y \right\rangle_{\mathbb{R}^d} = \frac{H - \varepsilon/m}{H}\,(H + \langle \zeta_n, Y\rangle_{\mathbb{R}^d}) \geq 0 \text{ a.s.}$$

(recall that $H > \varepsilon$ and $\zeta_n \in \Xi^d(H)$).

Therefore, given Assumption II.3.12 and the fact that, for every $m \in \mathbb{N}$, the inequality $[f_n^m]^- \leq V^-(0,\cdot)$ is true a.s., we can apply the reverse Fatou lemma to obtain

$$\limsup_{m\to+\infty} \mathbb{E}_{\mathbb{P}}\big[[f_n^m]^- \big| \mathscr{G}\big] \leq \mathbb{E}_{\mathbb{P}}\big[V^-(H(\cdot) + \langle \zeta_n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)\big| \mathscr{G}\big] \text{ a.s..}$$

Combining both inequalities and recalling the super-additivity[30] of the limit inferior yields $\liminf_{m\to+\infty} \mathbb{E}_{\mathbb{P}}[f_n^m | \mathscr{G}] \geq \mathbb{E}_{\mathbb{P}}[V(H(\cdot) + \langle \zeta_n(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)| \mathscr{G}]$ a.s..

Besides,

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(H-\varepsilon/m)} \mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot) - \frac{\varepsilon}{m} + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot\Big)\Big| \mathscr{G}\Big] \geq \mathbb{E}_{\mathbb{P}}[f_n^m | \mathscr{G}] \text{ a.s.}$$

for every $m \in \mathbb{N}$, and so

$$\liminf_{m\to+\infty} \operatorname*{ess\,sup}_{\xi\in\Xi^d(H-\varepsilon/m)} \mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot) - \frac{\varepsilon}{m} + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot\Big)\Big| \mathscr{G}\Big] \geq \liminf_{m\to+\infty} \mathbb{E}_{\mathbb{P}}[f_n^m | \mathscr{G}] \text{ a.s..}$$

On the other hand, let $m \in \mathbb{N}$ be arbitrary, but fixed. Then we know by the preceding step that

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d(H-\varepsilon/m)} \mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot) - \frac{\varepsilon}{m} + \langle \xi(\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot\Big)\Big| \mathscr{G}\Big](\omega) = G\Big(H(\omega) - \frac{\varepsilon}{m}, \omega\Big)$$

[30] A function $f:\ D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *super-additive* (respectively, *sub-additive*) if

$$f(x+y) \geq f(x) + f(y) \qquad \text{(respectively, } f(x+y) \leq f(x) + f(y)\text{)} \tag{II.5.11}$$

for every x, y in the domain D.

for every ω outside a $\mathbb{P}$-null set N'. Next, choosing $q_m \in \mathbb{Q}_0^+$ such that $H(\omega) - \varepsilon/m \leq q_m < H(\omega)$, it follows immediately from the definition of $\overline{G}$ (recall that $H > \varepsilon > 0$) and from the monotonicity of G (see the first part of this proof) that

$$\overline{G}(H(\omega)\,,\omega) = \sup_{\substack{q\in\mathbb{Q}_0^+ \\ q<H(\omega)}} G(q,\omega) \geq G(q_m,\omega)$$

$$\geq G(H(\omega)-\varepsilon/m,\omega) \geq \inf_{k\geq m} G(H(\omega)-\varepsilon/k,\omega)\,,$$

consequently,

$$\overline{G}(H(\cdot)\,,\cdot) \geq \sup_{m\in\mathbb{N}} \inf_{k\geq m} G(H(\cdot)-\varepsilon/k,\cdot) = \liminf_{m\to+\infty} G(H(\omega)-\varepsilon/m,\omega) \text{ a.s..}$$

So, putting together all the inequalities above finally yields that, for every $n \in \mathbb{N}$, $\overline{G}(H(\cdot)\,,\cdot) \geq G(H(\cdot)\,,\cdot) - 1/n$ a.s., hence

$$\overline{G}(H(\cdot)\,,\cdot) \geq \limsup_{n\to+\infty} \left(G(H(\cdot)\,,\cdot) - \frac{1}{n} \right) = G(H(\cdot)\,,\cdot) \text{ a.s.,}$$

as claimed. □

Proof of Proposition II.3.19. The proof will consist of five distinct parts.

(i) We shall begin by constructing a countable sequence of $\mathscr{B}([0,+\infty)) \otimes \mathscr{G}$-measurable functions $\xi_n : [0,+\infty) \times \Omega \to \mathbb{R}^d$ in such a way that they satisfy certain desired properties.

We know by Lemma II.5.7 that, for every $x \geq 0$, the family of $\mathscr{G}$-measurable random variables $\left\{ \mathbb{E}_{\mathbb{P}} \left[V(x + \langle \xi(\cdot)\,, Y(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G} \right] ; \xi \in \Xi^d(x) \right\}$ is directed upwards, thus it is possible to find a sequence $\{\eta_n(x,\cdot)\,; n \in \mathbb{N}\} \subseteq \Xi^d(x)$ such that

$$\lim_{n\to+\infty} \mathbb{E}_{\mathbb{P}} \left[V(x + \langle \eta_n(x,\cdot)\,, Y(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G} \right]$$

$$= \operatorname*{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}} \left[V(x + \langle \xi(\cdot)\,, Y(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G} \right] \text{ a.s.}$$

in an increasing way, i.e., such that for every $n \in \mathbb{N}$ we have

$$\mathbb{E}_{\mathbb{P}} \left[V(x + \langle \eta_n(x,\cdot)\,, Y(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G} \right] \leq \mathbb{E}_{\mathbb{P}} \left[V(x + \langle \eta_{n+1}(x,\cdot)\,, Y(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{G} \right] \text{ a.s..}$$

Let us fix such a countable sequence $\{\eta_n(q,\cdot)\,; n \in \mathbb{N}\}$ for every dyadic rational $q > 0$ (that is, of the form $i/2^j$, for some $i \in \mathbb{N}$, $j \in \mathbb{N}_0$).

Next, set $\xi_0(x,\omega) \triangleq 0$ for all $x \geq 0$ and all $\omega \in \Omega$, and for each $i \in \{1\} \cup (2\mathbb{N})$

arbitrary but fixed, choose some $\zeta_1^i \in \Xi^d(i/2)$ satisfying

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i}{2}+\langle\zeta_1^i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\right)\middle|\mathscr{G}\right]&\\ \geq \mathbb{E}_{\mathbb{P}}&\left[V\left(\frac{i}{2}+\left\langle\xi_0\left(\frac{i}{2},\cdot\right),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right]\\ \vee\,\mathbb{E}_{\mathbb{P}}&\left[V\left(\frac{i}{2}+\left\langle\eta_1\left(\frac{i}{2},\cdot\right),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right]\text{ a.s..}\end{aligned}$$

That such a ζ_1^i exists is a straightforward consequence of $\xi_0(i/2,\cdot)\,,\eta_1(i/2,\cdot)\in\Xi^d(i/2)$ and $\left\{\mathbb{E}_{\mathbb{P}}\left[V(i/2+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}\right];\,\xi\in\Xi^d(i/2)\right\}$ being directed upwards (again by Lemma II.5.7). In addition, if we take

$$\begin{aligned}f_1^{i+1}&\triangleq\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i+1}{2}+\langle\zeta_1^i(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\right)\middle|\mathscr{G}\right],\\ g_1^{i+1}&\triangleq\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i+1}{2}+\left\langle\eta_1\left(\frac{i+1}{2},\cdot\right),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right],\text{ and}\\ h_1^{i+1}&\triangleq\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i+1}{2}+\left\langle\xi_0\left(\frac{i+1}{2},\cdot\right),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right],\end{aligned}$$

for every even $i\in\mathbb{N}$, then we can use the fact that

$$\left\{\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i+1}{2}+\langle\xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\right)\middle|\mathscr{G}\right];\,\xi\in\Xi^d\left(\frac{i+1}{2}\right)\right\}$$

is directed upwards, along with the obvious inclusion $\Xi^d(i/2)\subseteq\Xi^d((i+1)/2)$, to find some $\zeta_1^{i+1}\in\Xi^d((i+1)/2)$ such that

$$\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i+1}{2}+\langle\zeta_1^{i+1}(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\right)\middle|\mathscr{G}\right]\geq f_1^{i+1}\vee g_1^{i+1}\vee h_1^{i+1}$$

holds a.s..

By construction, each ζ_1^i belongs to $\Xi^d(i/2)$, therefore so does its projection $\widehat{\zeta}_1^i$ given by Proposition II.3.7. Furthermore, it follows from Lemma II.3.17 that there exists a function $Z_1^i:[i/2,+\infty)\times\Omega\to\mathbb{R}$ such that, for every fixed $x\geq i/2$,

$$\mathbb{P}\left\{\omega\in\Omega\colon\,Z_1^i(x,\omega)=\mathbb{E}_{\mathbb{P}}\left[V\left(x+\left\langle\widehat{\zeta}_1^i(\cdot)\,,Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right](\omega)\right\}=1,$$

and such that, for every ω outside a $\mathbb{P}$-null set N_1^i, $Z_1^i(\cdot,\omega):\,[i/2,+\infty)\to\mathbb{R}$ is a non-decreasing and right-continuous function on $[i/2,+\infty)$.

Hence, the function $\xi_1:\,[0,+\infty)\times\Omega\to\mathbb{R}^d$ given by

$$\xi_1(x,\omega)\triangleq\sum_{i=1}^{+\infty}\widehat{\zeta}_1^i(\omega)\,\mathbb{1}_{\left[\frac{i}{2},\frac{i+1}{2}\right)}(x)$$

can be regarded as a countable step-function on the positive half-line, with each step taking on a $\mathscr{G}$-measurable, $\mathbb{R}^d$-valued random variable.

Moreover, it is straightforward to check that ξ_1 is $\mathscr{B}([0,+\infty))\otimes\mathscr{G}$-measurable. Another important observation is that $\xi_1(x,\cdot)\in\Xi^d(x)$ for every $x\geq 0$. Indeed, if $x\in[0,1/2)$, then for all $\omega\in\Omega$ we have $x+\langle\xi_1(x,\omega),Y(\omega)\rangle_{\mathbb{R}^d}=x\geq 0$, whereas when $x\in[i/2,(i+1)/2)$ for some $i\in\mathbb{N}$, $\xi_1(x,\cdot)=\widehat{\zeta}_1^i(\cdot)\in\Xi^d(i/2)\subseteq\Xi^d(x)$.

Lastly, let us define the function $G_1:\ [0,+\infty)\times\Omega\to\mathbb{R}$ by

$$G_1(x,\omega)\triangleq\sum_{i=1}^{+\infty}Z_1^i(x,\omega)\,\mathbb{1}_{\left[\frac{i}{2},\frac{i+1}{2}\right)}(x)\,.$$

It is clear that, for every ω outside the $\mathbb{P}$-null set $N_1\triangleq\bigcup_{i=1}^{+\infty}N_1^i$, the map $G_1(\cdot,\omega):\ [0,+\infty)\to\mathbb{R}$ is non-decreasing and right-continuous on each subinterval of the form $[i/2,(i+1)/2)$, $i\in\mathbb{N}_0$. Additionally, for every fixed $x\geq 1/2$, we have that x belongs to exactly one interval of the form $[i/2,(i+1)/2)$, $i\in\mathbb{N}$, and thus

$$\begin{aligned}G_1(x,\cdot)=Z_1^i(x,\cdot)&=\mathbb{E}_{\mathbb{P}}\left[V\Big(x+\Big\langle\widehat{\zeta}_1^i(\cdot),Y(\cdot)\Big\rangle_{\mathbb{R}^d},\cdot\Big)\Big|\mathscr{G}\right]\\&=\mathbb{E}_{\mathbb{P}}\left[V(x+\langle\xi_1(x,\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}\right]\ \text{a.s.},\end{aligned}$$

that is, $G_1(x,\cdot)$ is a version of $\mathbb{E}_{\mathbb{P}}\left[V(x+\langle\xi_1(x,\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}\right]$.

Next, proceeding recursively, let us assume that all functions $\xi_0,\xi_1,\ldots,\xi_{n-1}$, for some $n\in\mathbb{N}$, have been defined. Then, as before, for every $i\in\left\{1,2^1,\ldots,2^{n-1}\right\}\cup(2^n\mathbb{N})$ we can find some $\zeta_n^i\in\Xi^d(i/2^n)$ such that the inequality

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i}{2^n}+\left\langle\zeta_n^i(\cdot),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right]\\&\qquad\geq\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i}{2^n}+\left\langle\xi_{n-1}\left(\frac{i}{2^n},\cdot\right),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right]\\&\qquad\vee\,\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{i}{2^n}+\left\langle\eta_n\left(\frac{i}{2^n},\cdot\right),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right]\end{aligned}$$

holds a.s.. Now fix any $i\in\left\{2^1,\ldots,2^{n-1}\right\}\cup(2^n\mathbb{N})$. Again using the same reasoning as above, we can recursively choose for each

$$k\in\begin{cases}\{i+1,\ldots,2i-1\}\,, & \text{if } i\in\left\{2^1,\ldots,2^{n-1}\right\},\\\{i+1,\ldots,i+2^n-1\}\,, & \text{otherwise,}\end{cases}$$

some $\zeta_n^k\in\Xi^d(k/2^n)$ for which $\mathbb{E}_{\mathbb{P}}\left[V\big(k/2^n+\left\langle\zeta_n^k(\cdot),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\big)\middle|\mathscr{G}\right]\geq f_n^k\vee g_n^k\vee h_n^k$ holds a.s., with

$$f_n^k\triangleq\mathbb{E}_{\mathbb{P}}\left[V\left(\frac{k}{2^n}+\left\langle\zeta_n^{k-1}(\cdot),Y(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\mathscr{G}\right],$$

$$g_n^k \triangleq \mathbb{E}_{\mathbb{P}}\left[V\left(\frac{k}{2^n} + \left\langle \eta_n\left(\frac{k}{2^n},\cdot\right), Y(\cdot)\right\rangle_{\mathbb{R}^d}, \cdot\right)\middle| \mathscr{G}\right], \text{ and}$$
$$h_n^k \triangleq \mathbb{E}_{\mathbb{P}}\left[V\left(\frac{k}{2^n} + \left\langle \xi_{n-1}\left(\frac{k}{2^n},\cdot\right), Y(\cdot)\right\rangle_{\mathbb{R}^d}, \cdot\right)\middle| \mathscr{G}\right].$$

Hence, the function $\xi_n : [0,+\infty) \times \Omega \to \mathbb{R}^d$ defined by

$$\xi_n(x,\omega) \triangleq \sum_{i=1}^{+\infty} \widehat{\zeta}_n^i(\omega)\, \mathbb{1}_{\left[\frac{i}{2^n},\frac{i+1}{2^n}\right)}(x),$$

where each $\widehat{\zeta}_n^i \in \Xi^d(i/2^n)$ is the projection of ζ_n^i given by Proposition II.3.7, not only is $\mathscr{B}([0,+\infty)) \otimes \mathscr{G}$-measurable, but also satisfies $\xi_n(x,\cdot) \in \Xi^d(x)$ for every $x \geq 0$.

On the other hand, let $G_n : [0,+\infty) \times \Omega \to \mathbb{R}$ be the function given by

$$G_n(x,\omega) \triangleq \sum_{i=1}^{+\infty} Z_n^i(x,\omega)\, \mathbb{1}_{\left[\frac{i}{2^n},\frac{i+1}{2^n}\right)}(x),$$

with $Z_n^i : [i/2^n,+\infty) \times \Omega \to \mathbb{R}$ the regular version of the conditional expectation given by Lemma II.3.17 for each $\widehat{\zeta}_n^i$. Then, by an argument entirely analogous to that used above, we can see that for every ω outside some $\mathbb{P}$-null set N_n, the map $G_n(\cdot,\omega) : [0,+\infty) \to \mathbb{R}$ is non-decreasing and right-continuous on each subinterval of the form $[i/2^n,(i+1)/2^n)$, $i \in \mathbb{N}_0$. Besides, $G_n(x,\cdot)$ is a version of $\mathbb{E}_{\mathbb{P}}[V(x + \langle \xi_n(x,\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}]$ for every $x \geq 0$.

(ii) Secondly, we claim that the sequence of strategies $\{\xi_n;\, n \in \mathbb{N}\}$, given by the preceding step, is a maximising one, that is, it converges to the optimal value for every $x \in [0,+\infty)$.

So fix an arbitrary dyadic rational q. We shall also assume for simplicity that $q \in [1,2)$ (the argument being entirely analogous in the other cases), so we can write $q = 1 + i/2^j$ either for some $j \in \mathbb{N}$ and some odd $i \in \{1,3,\dots,2^j-1\}$, or for $j = 1$ and $i = 0$. It is trivial that, for every $n \in \mathbb{N}$,

$$\begin{aligned} G_n(q,\cdot) &= \mathbb{E}_{\mathbb{P}}[V(q + \langle \xi_n(q,\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] \\ &\leq \operatorname*{ess\,sup}_{\xi \in \Xi^d(q)} \mathbb{E}_{\mathbb{P}}[V(q + \langle \xi(\cdot), Y(\cdot)\rangle_{\mathbb{R}^d}, \cdot)|\mathscr{G}] = G(q,\cdot) \text{ a.s..} \end{aligned}$$

Moreover, it is clear by the preceding step that

$$\begin{aligned} G_{j+1}(q,\cdot) &= \mathbb{E}_{\mathbb{P}}\left[V\left(1 + \frac{2i}{2^{j+1}} + \left\langle \xi_{j+1}\left(\frac{2i}{2^{j+1}},\cdot\right), Y(\cdot)\right\rangle_{\mathbb{R}^d}, \cdot\right)\middle|\mathscr{G}\right] \\ &\geq \mathbb{E}_{\mathbb{P}}\left[V\left(1 + \frac{2i}{2^{j+1}} + \left\langle \xi_j\left(\frac{2i}{2^{j+1}},\cdot\right), Y(\cdot)\right\rangle_{\mathbb{R}^d}, \cdot\right)\middle|\mathscr{G}\right] = G_{j+1}(q,\cdot) \text{ a.s.,} \end{aligned}$$

so by a recursive argument we obtain that, for every $n \geq j$, $G_{n+1}(q,\cdot) \geq G_n(q,\cdot)$. Therefore, the sequence $\{G_n(q,\cdot)\,;\, n\in\mathbb{N}\}$, being monotone and bounded a.s., has a limit a.s., which is in turn no greater than $G(q,\cdot)$ a.s.. But, on the other hand, for every $n\in\mathbb{N}$ as before, it is trivial by construction that

$$G_n(q,\cdot) \geq \mathbb{E}_{\mathbb{P}}[V(q+\langle \eta_n(q,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] \text{ a.s..}$$

Hence, combining all of the above results with the definition of the sequence $\{\eta_n(q,\cdot)\,;\, n\in\mathbb{N}\}$ yields

$$G(q,\cdot) \geq \lim_{n\to+\infty} G_n(q,\cdot) \geq \lim_{n\to+\infty} \mathbb{E}_{\mathbb{P}}[V(q+\langle \eta_n(q,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\mathscr{G}] = G(q,\cdot) \text{ a.s..}$$

Given that $\mathbb{Q}$ is countable, we then conclude that for every ω outside some $\mathbb{P}$-null set N', we have $\lim_{n\to+\infty} G_n(q,\omega) = G(q,\omega)$ in a non-decreasing way for every dyadic rational q.

Next, let $x\in[1,2)$ be arbitrary, but fixed. Then we can find a non-increasing sequence $\{q_n^x;\, n\in\mathbb{N}\}$ of dyadic rationals converging to x in a way that $x < q_n^x < (\lfloor 2^n x\rfloor + 1)/2^n$ for every $n\in\mathbb{N}$. In addition, for every ω outside the $\mathbb{P}$-null set $N \triangleq \bigcup_{n\in\mathbb{N}} N_n$ we have $G_n(x,\omega) \leq G_n(q_n,\omega)$ for every $n\in\mathbb{N}$. Thus, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\limsup_{n\to+\infty} G_n(x,\omega) \leq \limsup_{n\to+\infty} G(q_n,\omega) = G(x,\omega)\,,$$

where the last equality is due to the almost sure path right-continuity of G.

On the other hand, given any dyadic rational $q^x\in[1,x)$, it can be easily checked that there exists some $m\in\mathbb{N}$ such that, for every integer $n\geq m$, we have $q^x\in[i/2^n,(i+1)/2^n)$ for some $i < \lfloor x2^n\rfloor$. Hence, for $\mathbb{P}$-a.e. $\omega\in\Omega$, we get $G_n(q^x,\omega)\leq G_n(x,\omega)$ for every $n\geq m$, and so

$$\liminf_{n\to+\infty} G_n(x,\omega) \geq \liminf_{n\to+\infty} G_n(q^x,\omega) = G(q,\omega)\,.$$

But by the arbitrarity of q and given that the sequence $\{q_k';\, k\in\mathbb{N}\}$, where $q_k' \triangleq \lfloor 2^k x\rfloor/2^k$, also converges to x as $k\to+\infty$, this time in a non-decreasing way, we conclude that, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\liminf_{n\to+\infty} G_n(x,\omega) \geq G(q_k,\omega)$$

for all $k\in\mathbb{N}$, thus

$$\liminf_{n\to+\infty} G_n(x,\omega) \geq \lim_{k\to+\infty} G(q_k,\omega) = G(x,\omega)\,,$$

where we use the almost sure path left-continuity of G to deduce the last equality.

In conclusion, for all ω outside some $\mathbb{P}$-null set, $\lim_{n\to+\infty} G_n(x,\omega)$ exists and equals $G(x, \omega)$ for all x.

(iii) Thirdly, we prove a result which is purely technical, but that will be useful later on. We claim that, for every $\mathscr{G}$-measurable random variable $H \geq 0$ a.s. and for every fixed $n \in \mathbb{N}$, the equality

$$G_n(H(\cdot)\,,\cdot) = \mathbb{E}_{\mathbb{P}}[V(H(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}]$$

holds a.s..

Indeed, if we consider any step-function $H = \sum_{i=1}^{+\infty} x_i \mathbb{1}_{A_i}$ as in the proof of Lemma II.3.18 (part *(iv)*), then it is trivial that

$$\begin{aligned} G_n(H(\cdot)\,,\cdot) &= \sum_{i=1}^{+\infty} \mathbb{E}_{\mathbb{P}}[V(x_i + \langle \xi_n(x_i,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\,\mathbb{1}_{A_i}(\omega)|\,\mathscr{G}] \\ &= \mathbb{E}_{\mathbb{P}}[V(H(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}] \text{ a.s.,} \end{aligned}$$

and we also observe that $\xi_n(H(\cdot)\,,\cdot) \in \Xi^d(H)$.

Thus, let us take instead any general $\mathscr{G}$-measurable $H \geq 0$ a.s.. By a similar argument to the preceding one, we may suppose without loss of generality that $H \leq M$ a.s., for some constant $M > 0$. Then, once again, it is possible to take a non-increasing sequence of $\mathscr{G}$-measurable step-functions $\{H_k;\, k \in \mathbb{N}\}$ verifying $\lim_{k\to+\infty} H_k = H$ a.s., as well as $H_k \in [i/2^n, (i+1)/2^n)$ on every set $\{\omega \in \Omega\colon\ H(\omega) \in [i/2^n, (i+1)/2^n)\}$ $(i \in \mathbb{N}_0)$. Recalling that ξ_n is piecewise constant, we have $\xi_n(H_k(\cdot)\,,\cdot) = \xi_n(H(\cdot)\,,\cdot)$ for every $k \in \mathbb{N}$, therefore

$$\begin{aligned} G_n(H_k(\cdot)\,,\cdot) &= \mathbb{E}_{\mathbb{P}}[V(H_k(\cdot) + \langle \xi_n(H_k(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}] \\ &= \mathbb{E}_{\mathbb{P}}[V(H_k(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}] \text{ a.s..} \end{aligned}$$

But we get by the a.s. path continuity of V (see Assumption II.3.8) that

$$\begin{aligned} &\lim_{k\to+\infty} V(H_k(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot) \\ &\qquad\qquad = V(H(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot) \text{ a.s..} \end{aligned}$$

Furthermore, for each $k \in \mathbb{N}$,

$$\begin{aligned} &|V(H_k(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)| \\ &\qquad\qquad \leq V^+(H_1(\cdot) + \langle \xi_n(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot) + V^-(0,\cdot) \text{ a.s.,} \end{aligned}$$

because $\xi_n(H(\cdot)\,,\cdot) = \xi_n(H_k(\cdot)\,,\cdot) \in \Xi^d(H_k)$, $\{H_k;\, k \in \mathbb{N}\}$ is a non-increasing sequence a.s., and the paths of V are monotone. Additionally, we may invoke

Lemma II.3.16 to deduce that

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\big[V^{+}(H_1(\cdot)+\langle\xi_n(H(\cdot)\,,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\big|\,\mathscr{G}\big]\\&=\sum_{i=1}^{+\infty}\mathbb{E}_{\mathbb{P}}\big[V^{+}\big(x_i^1+\big\langle\xi_n\big(x_i^1,\cdot\big)\,,Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\big)\big|\,\mathscr{G}\big]\,\mathbb{1}_{A_i^1}\leq\sum_{i=1}^{+\infty}\mathbb{E}_{\mathbb{P}}\Big[L_{x_i^1}\Big|\,\mathscr{G}\Big]\,\mathbb{1}_{A_i^1}<+\infty\end{aligned}$$

a.s.. Therefore, Lebesgue's Dominated Convergence Theorem (for the conditional expectation) yields

$$\begin{aligned}&\lim_{k\to+\infty}\mathbb{E}_{\mathbb{P}}[V(H_k(\cdot)+\langle\xi_n(H(\cdot)\,,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}]\\&\qquad=\mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi_n(H(\cdot)\,,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}]\ \text{a.s..}\end{aligned}$$

On the other hand, due to the a.s. path right-continuity of G_n, it holds that $\lim_{k\to+\infty}G_n(H_k(\cdot)\,,\cdot)=G_n(H(\cdot)\,,\cdot)$ a.s., which combined with the above results gives the intended equality.

(iv) Next, using a compactness argument, we find an optimal strategy. In order to do so, we shall start by proving that

$$\mathbb{P}\Big\{\omega\in\Omega\colon\ \forall\,x\geq 0,\ \sup_{n\in\mathbb{N}}\|\xi_n(x,\omega)\|_{\mathbb{R}^d}<+\infty\Big\}=1.$$

It is obvious by countability that it suffices to show that, for every ω outside a $\mathbb{P}$-null set,

$$\forall\,x\in[1,2)\ ,\ \sup_{n\in\mathbb{N}}\|\xi_n(x,\omega)\|_{\mathbb{R}^d}<+\infty$$

(the proof of the other cases being exactly the same). So fix arbitrary $n\in\mathbb{N}$ and $i\in\{0,1,\dots,2^n-1\}$. Given that, by construction, $\xi_n((2^n+i)\,/2^n,\cdot)\in\Xi^d((2^n+i)\,/2^n)$ and $\xi_n((2^n+i)\,/2^n,\omega)\in D(\omega)$ for $\mathbb{P}$-a.e. $\omega\in\Omega$, we can apply Lemma II.3.15 with $x_0\triangleq 2$ to conclude that there exists some random variable $K_{x_0}>x_0$ a.s. such that $\|\xi_n((2^n+i)\,/2^n,\cdot)\|_{\mathbb{R}^n}\leq K_{x_0}$ a.s.. Thus, recalling that ξ_n is stepwise constant by construction gives

$$\begin{aligned}&\mathbb{P}\{\omega\in\Omega\colon\ \forall\,x\in[(2^n+i)\,/2^n,(2^n+i+1)\,/2^n)\ ,\ \|\xi_n(x,\omega)\|_{\mathbb{R}^n}\leq K_{x_0}(\omega)\}\\&\qquad=\mathbb{P}\{\omega\in\Omega\colon\ \|\xi_n((2^n+i)\,/2^n,\omega)\|_{\mathbb{R}^n}\leq K_{x_0}(\omega)\}=1.\end{aligned}$$

But this implies that the set

$$\begin{aligned}&\Big\{\omega\in\Omega\colon\ \forall\,x\in[1,2)\ ,\ \sup_{n\in\mathbb{N}}\|\xi_n(x,\omega)\|_{\mathbb{R}^n}\leq K_{x_0}(\omega)\Big\}\\&=\bigcap_{n\in\mathbb{N}}\bigcap_{i=0}^{2^n-1}\Big\{\omega\in\Omega\colon\ \forall\,x\in\Big[\frac{2^n+i}{2^n},\frac{2^n+i+1}{2^n}\Big)\ ,\ \|\xi_n(x,\omega)\|_{\mathbb{R}^n}\leq K_{x_0}(\omega)\Big\}\end{aligned}$$

has $\mathbb{P}$-full measure, which in turn gives the intended result.

We then deduce that, for $\mathbb{P}$-a.e. ω and for all $x \geq 0$, $\liminf_{n\to+\infty} \|\xi_n(x,\omega)\|_{\mathbb{R}^n} < +\infty$, so by Proposition B.2 we can extract a strictly increasing sequence of $\mathscr{B}([0,+\infty)) \otimes \mathscr{G}$-measurable functions $n_k : [0,+\infty) \times \Omega \to \mathbb{N}$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\forall\, x \geq 0,\ \lim_{k\to+\infty} \xi_{n_k(x,\omega)}(x,\omega) = \widetilde{\xi}(x,\omega)\,,$$

for some $\mathscr{B}([0,+\infty)) \otimes \mathscr{G}$-measurable function $\widetilde{\xi} : [0,+\infty) \times \Omega \to \mathbb{R}$.

It is now clear by the piecewise-constant structure of $\{\xi_n;\, n \in \mathbb{N}\}$ that

$$\begin{aligned}
&\{\omega \in \Omega\colon\ x + \langle \xi_n(x,\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} \geq 0,\ \forall\, x \geq 0,\ \forall\, n \in \mathbb{N}\} \\
&= \bigcap_{n\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \{\omega \in \Omega\colon\ x + \langle \xi_n(x,\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} \geq 0,\ \forall\, x \in [i/2^n, (i+1)/2^n)\} \\
&= \bigcap_{n\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \Big\{\omega \in \Omega\colon\ x + \Big\langle \widehat{\zeta}^i_n(\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} \geq 0,\ \forall\, x \in [i/2^n, (i+1)/2^n)\Big\}.
\end{aligned}$$

But, for every $n \in \mathbb{N}$ and $i \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}\Big\{\omega \in \Omega\colon\ x + \Big\langle \widehat{\zeta}^i_n(\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} &\geq 0,\ \forall\, x \in [i/2^n, (i+1)/2^n)\Big\} \\
&\geq \mathbb{P}\Big\{\omega \in \Omega\colon\ i/2^n + \Big\langle \widehat{\zeta}^i_n(\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} \geq 0\Big\} = 1,
\end{aligned}$$

because $\widehat{\zeta}^i_n \in \Xi^d(i/2^n)$, so

$$\mathbb{P}\{\omega \in \Omega\colon\ x + \langle \xi_n(x,\omega)\,, Y(\omega)\rangle_{\mathbb{R}^d} \geq 0,\ \forall\, x \geq 0,\ \forall\, n \in \mathbb{N}\} = 1.$$

Hence,

$$\mathbb{P}\Big\{\omega \in \Omega\colon\ x + \big\langle \xi_{n_k(x,\omega)}(x,\omega)\,, Y(\omega)\big\rangle_{\mathbb{R}^d} \geq 0,\ \forall\, x \geq 0,\ \forall\, k \in \mathbb{N}\Big\} = 1, \tag{II.5.12}$$

which in turn implies

$$\mathbb{P}\Big\{\omega \in \Omega\colon\ \forall\, x \geq 0,\ x + \Big\langle \widetilde{\xi}(x,\omega)\,, Y(\omega)\Big\rangle_{\mathbb{R}^d} \geq 0\Big\} = 1, \tag{II.5.13}$$

and in particular $\widetilde{\xi}(x,\cdot) \in \Xi^d(x)$ for every $x \geq 0$.

Next, we check that this candidate $\widetilde{\xi}$ is indeed optimal, in the sense that, for every $x \geq 0$,

$$\mathbb{P}\Big\{\omega \in \Omega\colon\ G(x,\omega) = \mathbb{E}_{\mathbb{P}}\Big[V\Big(x + \Big\langle \widetilde{\xi}(x,\cdot)\,, Y(\cdot)\Big\rangle_{\mathbb{R}^d}, \cdot\Big)\Big|\,\mathscr{G}\Big](\omega)\Big\} = 1.$$

So fix an arbitrary $x \in [0,+\infty)$ and note that, on the one hand, it follows easily from Fatou's lemma that

$$\liminf_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[V^{-}\Big(x+\langle \xi_{n_k(\cdot)}(x,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \geq \mathbb{E}_{\mathbb{P}}\left[V^{-}\Big(x+\Big\langle \widetilde{\xi}(x,\cdot)\,,Y(\cdot)\Big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \text{ a.s..}$$

On the other hand, given that for all $k \in \mathbb{N}$,

$$V^{+}\Big(x+\langle \xi_{n_k(\cdot)}(x,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\Big) \leq L_x \text{ a.s.,}$$

where L_x is the random variable given by Lemma II.3.16 (note that it follows from equation (II.5.12) that $\xi_{n_k(\cdot)}(x,\cdot) \in \Xi^d(x)$), we can apply the reverse Fatou lemma as well to obtain

$$\limsup_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[V^{+}\Big(x+\langle \xi_{n_k(\cdot)}(x,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \leq \mathbb{E}_{\mathbb{P}}\left[V^{+}\Big(x+\Big\langle \widetilde{\xi}(x,\cdot)\,,Y(\cdot)\Big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \text{ a.s..}$$

Therefore, the preceding inequalities and the sub-additivity of the limit superior give

$$\begin{aligned} G(x,\cdot) = \limsup_{k\to+\infty} G_{n_k(\cdot)}(x,\cdot) &\leq \limsup_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[V\Big(x+\langle \xi_{n_k(\cdot)}(x,\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \\ &\leq \mathbb{E}_{\mathbb{P}}\left[V\Big(x+\Big\langle \widetilde{\xi}(x,\cdot)\,,Y(\cdot)\Big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \text{ a.s.,} \end{aligned}$$

where the first equality was proved in the preceding step *(ii)*.

Conversely, we have by the construction of G and by the definition of essential supremum that

$$\begin{aligned} G(x,\cdot) &= \operatorname*{ess\,sup}_{\xi\in\Xi^d(x)} \mathbb{E}_{\mathbb{P}}\left[V(x+\langle \xi(\cdot)\,,Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\middle|\mathscr{G}\right] \\ &\geq \mathbb{E}_{\mathbb{P}}\left[V\Big(x+\Big\langle \widetilde{\xi}(x,\cdot)\,,Y(\cdot)\Big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\mathscr{G}\right] \text{ a.s..} \end{aligned}$$

(v) Lastly, we check that the optimal strategy obtained above satisfies all of the properties stated in Proposition II.3.19. Indeed, let H be an arbitrary, but fixed, $\mathscr{G}$-measurable random variable satisfying $H \geq 0$ a.s.. It follows immediately from the $\mathscr{B}([0,+\infty))\otimes\mathscr{G}$-measurability of $\widetilde{\xi}$ and from the $\mathscr{G}$-measurability of H that the $\mathbb{R}^d$-valued function

$$\omega \mapsto \widetilde{\xi}(H(\omega)\,,\omega)$$

is $\mathscr{G}$-measurable. Besides, $\Big\{\omega\in\Omega\colon\ H(\omega)+\Big\langle\widetilde{\xi}(H(\omega)\,,\omega)\,,Y(\omega)\Big\rangle_{\mathbb{R}^d}\geq 0\Big\}$ contains the set in (II.5.13). Therefore, $\widetilde{\xi}(H(\cdot)\,,\cdot)\in\Xi^d(H)$.

In a completely analogous manner, we prove $\mathbb{P}\Big\{\omega\in\Omega\colon\ \widetilde{\xi}(H(\omega)\,,\omega)\in D(\omega)\Big\}=1$.

Also, given that $\widetilde{\xi}(H(\cdot)\,,\cdot)\in\Xi^d(H)$ by the preceding step, it is now trivial, by the definition of essential supremum, to see that

$$\mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot)+\Big\langle\widetilde{\xi}(H(\cdot),\cdot),Y(\cdot)\Big\rangle_{\mathbb{R}^d},\cdot\Big)\Big|\mathscr{G}\Big] \leq \operatorname*{ess\,sup}_{\xi\in\Xi^d(H)}\mathbb{E}_{\mathbb{P}}[V(H(\cdot)+\langle\xi(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot)|\mathscr{G}] \text{ a.s..}$$

At last, it remains only to prove that

$$G(H(\cdot),\cdot)\leq\mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot)+\Big\langle\widetilde{\xi}(H(\cdot),\cdot),Y(\cdot)\Big\rangle_{\mathbb{R}^d},\cdot\Big)\Big|\mathscr{G}\Big] \text{ a.s.,}$$

which together with (II.3.11) gives the claimed almost sure equality in (II.3.13). On the one hand, if we consider an arbitrary ω in the set

$$\Big\{\omega\in\Omega\colon\ \forall x\geq 0,\ \lim_{n\to+\infty}G_n(x,\omega)=G(x,\omega)\Big\} \cap\Big\{\omega\in\Omega\colon\ \forall x\geq 0\ \lim_{k\to+\infty}n_k(x,\omega)=+\infty\Big\}$$

of $\mathbb{P}$-full measure, it is trivial that $\lim_{k\to+\infty}G_{n_k(H(\omega),\omega)}(H(\omega),\omega)=G(H(\omega),\omega)$. On the other hand,

$$\limsup_{k\to+\infty}\mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot)+\langle\xi_{n_k(H(\cdot),\cdot)}(H(\cdot),\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot\Big)\Big|\mathscr{G}\Big] \leq\mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot)+\Big\langle\widetilde{\xi}(H(\cdot),\cdot),Y(\cdot)\Big\rangle_{\mathbb{R}^d},\cdot\Big)\Big|\mathscr{G}\Big] \text{ a.s..}$$

Indeed, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$\lim_{k\to+\infty}V\Big(H(\omega)+\langle\xi_{n_k(H(\omega),\omega)}(H(\omega),\omega),Y(\omega)\rangle_{\mathbb{R}^d},\omega\Big) =V\Big(H(\omega)+\Big\langle\widetilde{\xi}(H(\omega),\omega),Y(\omega)\Big\rangle_{\mathbb{R}^d},\omega\Big)$$

(recall the path continuity of V). Moreover, for every $k\in\mathbb{N}$,

$$\begin{aligned}&V\Big(H(\cdot)+\langle\xi_{n_k(H(\cdot),\cdot)}(H(\cdot),\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot\Big)\\&\leq\sum_{m=1}^{+\infty}V^+\Big(H_m(\cdot)+\langle\xi_{n_k(H_m(\cdot),\cdot)}(H_m(\cdot),\cdot)\mathbb{1}_{\{m-1\leq H<m\}}(\cdot),Y(\cdot)\rangle_{\mathbb{R}^d},\cdot\Big)\mathbb{1}_{\{m-1\leq H<m\}}\\&\qquad+V^-(0,\cdot)\\&\leq\sum_{m=1}^{+\infty}L_m\mathbb{1}_{\{m-1\leq H<m\}}+V^-(0,\cdot)\ \text{ a.s.,}\end{aligned}$$

where $H_m\triangleq H\mathbb{1}_{\{m-1\leq H<m\}}$ and L_m is the random variable of Lemma II.3.16 (note that it follows from (II.5.12) that $\xi_{n_k(H(\cdot),\cdot)}(H(\cdot),\cdot)\in\Xi^d(H)$, which in turn gives $\xi_{n_k(H_m(\cdot),\cdot)}(H_m(\cdot),\cdot)\,\mathbb{1}_{\{m-1\leq H<m\}}\in\Xi^d(m)$). Therefore, we may apply the

reverse Fatou lemma to obtain the claimed inequality.

Finally, we know by *(iii)* that, for each $k \in \mathbb{N}$,

$$G_{n_k(H(\cdot),\cdot)}(H(\cdot)\,,\cdot) = \sum_{i=1}^{+\infty} \mathbb{E}_{\mathbb{P}}[V(H(\cdot) + \langle \xi_i(H(\cdot)\,,\cdot)\,, Y(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)|\,\mathscr{G}]\,\mathbb{1}_{\{n_k(H(\cdot),\cdot)=i\}}$$
$$= \mathbb{E}_{\mathbb{P}}\Big[V\Big(H(\cdot) + \big\langle \xi_{n_k(H(\cdot),\cdot)}(H(\cdot)\,,\cdot)\,, Y(\cdot)\big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\,\mathscr{G}\Big] \text{ a.s..}$$

Combining all of the preceding results concludes. □

5.4 Proofs of Section II.4

Proof of Lemma II.4.5. Set $\widehat{x} \triangleq \max\{1, \overline{x}\}$, where $\overline{x}$ is that of Assumption II.4.3, and let us begin by considering an arbitrary $x \in [0, \widehat{x}]$. Then we can use the fact that u is non-decreasing, along with $u(1) = 0$ and inequality (II.4.1), to obtain $u^+(\lambda x) \leq u^+(\lambda\widehat{x}) = u(\lambda\widehat{x}) \leq \lambda^{\overline{\gamma}} u(\widehat{x}) + c$ for any $\lambda \geq 1$.

On the other hand, for every $x > \widehat{x}$, we have by the same arguments above that $u^+(\lambda x) = u(\lambda x) \leq \lambda^{\overline{\gamma}} u(x) + c = \lambda^{\overline{\gamma}} u^+(x) + c$ for all $\lambda \geq 1$.

Hence, choosing $C > u(\widehat{x}) + 2c$ and combining the two previous inequalities yields

$$u^+(\lambda x) \leq \max\left\{\lambda^{\overline{\gamma}} u(\widehat{x}) + c, \lambda^{\overline{\gamma}} u^+(x) + c\right\} \leq \lambda^{\overline{\gamma}} u^+(x) + \lambda^{\overline{\gamma}} u(\widehat{x}) + 2c \leq \lambda^{\overline{\gamma}} u^+(x) + \lambda^{\overline{\gamma}} C$$

for all $\lambda \geq 1$ and for all $x > 0$, as claimed. □

Proof of Theorem II.4.6. The proof will consist of verifying that the conditions of the one-step case (see Section II.3) are satisfied, which will then allow us to construct an optimal portfolio. This will be accomplished in several steps, in which a dynamic programming argument will be used. We follow the main ideas of Rásonyi and Stettner [51, Proposition 3.1].

In order to perform a dynamic programming procedure, we must prove that some crucial assumptions of Section II.3 are preserved at each time step. So let us start by defining the function $U_T : [0, +\infty) \times \Omega \to \mathbb{R}$ as follows,

$$U_T(x, \omega) \triangleq u(x)\,, \qquad x \geq 0,\ \omega \in \Omega.$$

We wish to apply the results of Section II.3 with $Y \triangleq \Delta S_T$, $\mathscr{G} \triangleq \mathscr{F}_{T-1}$ and $V \triangleq U_T$.

(i) It is trivial to see that Assumption II.3.1 is verified. In fact, if ξ is an arbitrary random vector in Ξ^d_{T-1} for which the inequality $\langle \xi, \Delta S_T\rangle_{\mathbb{R}^d} \geq 0$ holds a.s., then it is immediate to check that the portfolio $\overline{\phi}$ given by

$$\phi_t \triangleq \begin{cases} \xi, & \text{if } t = T, \\ \mathbf{0_d}, & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi^0_t \triangleq \begin{cases} -\langle \xi, S_{T-1}\rangle_{\mathbb{R}^d}\,, & \text{if } t = T, \\ 0, & \text{otherwise,} \end{cases}$$

belongs to $\Psi(0)$, hence by Assumption II.2.9 we must have that $\langle \xi, \Delta S_T\rangle_{\mathbb{R}^d} = \Pi_T^{\overline{\phi}} = 0$ a.s..

(ii) We note further that Assumption II.3.8 is also true. Indeed, if we fix any $x \geq 0$, then the function $U_T(x,\cdot): \Omega \to \mathbb{R}$, being constant on Ω, is $\mathscr{F}_T$-measurable. Secondly, for every $\omega \in \Omega$, we have by definition of U_T and Assumption II.4.1 that

$$U_T(1,\omega) = u(1) = 0.$$

In addition, since u is continuous and increasing on $[0,+\infty)$, it is also easy to check that, for each $\omega \in \Omega$, the function $U_T(\cdot,\omega): [0,+\infty) \to \mathbb{R}$ is increasing on $[0,+\infty)$, continuous on $(0,+\infty)$ and right-continuous at 0.

(iii) We now claim that Assumption II.3.10 is satisfied. In order to do so, fix an arbitrary $x \geq 0$. We have by Lemma II.5.7 that the family

$$\left\{\mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\middle|\, \mathscr{F}_{T-1}\right];\, \xi \in \Xi^d_{T-1}(x)\right\}$$

is directed upwards, so we can find a countable sequence of random vectors $\{\xi_n;\, n \in \mathbb{N}\} \subseteq \Xi^d(x)$ attaining the essential supremum, i.e., such that

$$\begin{aligned}\lim_{n\to+\infty} &\mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi_n(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\middle|\, \mathscr{F}_{T-1}\right] \\ &= \operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\middle|\, \mathscr{F}_{T-1}\right] \text{ a.s.}\end{aligned}$$

in a non-decreasing way. Therefore, it follows from the Monotone Convergence Theorem and from the definition of the conditional expectation that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}&\left[\operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\middle|\, \mathscr{F}_{T-1}\right]\right] \\ &= \lim_{n\to+\infty} \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi_n(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\middle|\, \mathscr{F}_{T-1}\right]\right] \\ &= \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi_n(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}, \cdot)\right].\end{aligned}$$

Now, it is straightforward to check that, given any $\xi \in \Xi^d(x)$, the $\mathbb{R}^{d+1}$-valued process $\overline{\phi}_\xi = \left\{\left((\phi_\xi)^0_t, (\phi_\xi)_t^\top\right)^\top;\, t \in \{1,\ldots,T\}\right\}$ defined by

$$(\phi_\xi)_t \triangleq \begin{cases} \xi, & \text{if } t = T, \\ \mathbf{0}_d, & \text{otherwise,}\end{cases} \quad \text{and} \quad (\phi_\xi)^0_t \triangleq \begin{cases} x - \langle \xi, S_{T-1}\rangle_{\mathbb{R}^d}, & \text{if } t = T, \\ x, & \text{otherwise,}\end{cases}$$

is a portfolio in $\Psi(x)$, with

$$\mathbb{E}_{\mathbb{P}}\left[u^+\left(\Pi_T^{\overline{\phi}_\xi}(\cdot)\right)\middle|\, \mathscr{F}_{T-1}\right] = \mathbb{E}_{\mathbb{P}}\left[u^+(x + \langle \xi(\cdot), \Delta S_T(\cdot)\rangle_{\mathbb{R}^d})\middle|\, \mathscr{F}_{T-1}\right]$$

$$= \mathbb{E}_{\mathbb{P}}\big[U_T^{+}(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) \big| \mathscr{F}_{T-1} \big] \text{ a.s..}$$

(where, to obtain the last equality, we recall that $x + \langle \xi, \Delta S_T \rangle_{\mathbb{R}^d} \geq 0$ a.s.). In particular, the preceding equality and condition (II.4.3) imply that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbb{E}_{\mathbb{P}}\big[U_T^{+}(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) \big| \mathscr{F}_{T-1} \big]\big] = \mathbb{E}_{\mathbb{P}}\Big[u^{+}\Big(\Pi_T^{\overline{\phi}_\xi}(\cdot) \Big) \Big] < +\infty,$$

and so $\mathbb{E}_{\mathbb{P}}\big[U_T^{+}(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) \big| \mathscr{F}_{T-1} \big] < +\infty$ a.s. (thus, the conditional expectation is well-defined, possibly $-\infty$, and finite a.s.).

At last, setting $\overline{\phi}_n \triangleq \overline{\phi}_{\xi_n}$, combining the results obtained above and invoking hypothesis (II.4.3) once again, we conclude that

$$\mathbb{E}_{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^{+}(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) \big| \mathscr{F}_{T-1} \right] \right] = \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}\Big[u^{+}\Big(\Pi_T^{\overline{\phi}_n}(\cdot) \Big) \Big] < +\infty,$$

hence $\operatorname{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^{+}(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) \big| \mathscr{F}_{T-1} \right] < +\infty$ a.s..

(iv) The next step consists of showing that we have Assumption II.3.12 as well. In fact, due to Assumption II.2.16, it is immediate that

$$\mathbb{E}_{\mathbb{P}}\big[U_T^{-}(0, \cdot) \big| \mathscr{F}_{T-1} \big] = \mathbb{E}_{\mathbb{P}}\big[u^{-}(0) \big| \mathscr{F}_{T-1} \big] = u^{-}(0) < +\infty \text{ a.s..}$$

(v) Lastly, let the constants $\overline{\gamma} > 0$ and $C > 0$ be those given by Assumption II.4.3 and Lemma II.4.5, respectively. Then, for every $\omega \in \Omega$, we obtain

$$U_T^{+}(\lambda x, \omega) = u^{+}(\lambda x) \leq \lambda^{\overline{\gamma}} u^{+}(x) + C \lambda^{\overline{\gamma}} = \lambda^{\overline{\gamma}} U_T^{+}(x, \omega) + C \lambda^{\overline{\gamma}}, \tag{II.5.14}$$

for all $\lambda \geq 1$ and $x \geq 0$.

Hence, by Lemma II.3.18, there exists a function $G_{T-1} : [0, +\infty) \times \Omega \to \mathbb{R}$ such that, for every ω in a $\mathbb{P}$-full measure set $\breve{\Omega}_{T-1}$, the function $G_{T-1}(\cdot, \omega) : [0, +\infty) \to \mathbb{R}$ is non-decreasing and continuous on $[0, +\infty)$. Moreover, for every $x \in [0, +\infty)$,

$$G_{T-1}(x, \cdot) = \operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}[U_T(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{F}_{T-1}] \text{ a.s..}$$

In addition, Proposition II.3.19 gives us an optimal $\mathscr{B}([0, +\infty)) \otimes \mathscr{F}_{T-1}$-measurable function $\widetilde{\xi}_T : [0, +\infty) \times \Omega \to \mathbb{R}^d$, i.e., for every $x \geq 0$,

$$\operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}[U_T(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{F}_{T-1}] = \mathbb{E}_{\mathbb{P}}\Big[U_T\Big(x + \Big\langle \widetilde{\xi}_T(x, \cdot)\,, \Delta S_T(\cdot) \Big\rangle_{\mathbb{R}^d}\,, \cdot \Big) \Big| \mathscr{F}_{T-1} \Big] \text{ a.s..}$$

So now let $U_{T-1} : [0,+\infty) \times \Omega \to \mathbb{R}$ be the function given by $U_{T-1}(x,\omega) \triangleq G_{T-1}(x,\omega)$. As before, we would like to use the results of Section II.3, this time with $Y \triangleq \Delta S_{T-1}$, $\mathscr{G} \triangleq \mathscr{F}_{T-2}$ and $V \triangleq U_{T-1}$.

(i) That Assumption II.3.1 is true follows in a similar way to that above. Indeed, considering any $\xi \in \Xi^d_{T-2}$ such that $\langle \xi, \Delta S_{T-1} \rangle_{\mathbb{R}^d} \geq 0$ a.s., we can construct a portfolio $\overline{\phi}$ as indicated below,

$$\phi_t \triangleq \begin{cases} \xi, & \text{if } t = T-1, \\ \mathbf{0_d}, & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi^0_t \triangleq \begin{cases} 0, & \text{if } t < T-1, \\ -\langle \xi, S_{T-2} \rangle_{\mathbb{R}^d}, & \text{if } t = T-1, \\ \langle \xi, \Delta S_{T-1} \rangle_{\mathbb{R}^d}, & \text{if } t > T-1. \end{cases}$$

Given that $\overline{\phi} \in \Psi(0)$ and Assumption II.2.9 holds true by hypothesis, we conclude that $\langle \xi, \Delta S_{T-1} \rangle_{\mathbb{R}^d} = \Pi^{\overline{\phi}}_T = 0$ a.s..

(ii) Next, we prove that Assumption II.3.8 holds. In fact, given any $x \geq 0$, the function $U_{T-1}(x,\cdot) : \Omega \to \mathbb{R}$ is $\mathscr{F}_{T-1}$-measurable. On the other hand, for every $\omega \in \breve{\Omega}_{T-1}$, we have by definition of U_{T-1} that $U_{T-1}(\cdot,\omega)$ is a non-decreasing function on $[0,+\infty)$, continuous on $(0,+\infty)$, and right-continuous at 0.

Furthermore, we know that

$$G_{T-1}(1,\cdot) = \operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(1)} \mathbb{E}_{\mathbb{P}}[U_T(1 + \langle \xi(\cdot), \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{F}_{T-1}] \text{ a.s.,}$$

that by the definition of essential supremum

$$\operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(1)} \mathbb{E}_{\mathbb{P}}[U_T(1 + \langle \xi(\cdot), \Delta S_T(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{F}_{T-1}] \geq \mathbb{E}_{\mathbb{P}}[U_T(1,\cdot) | \mathscr{F}_{T-1}] \text{ a.s.,}$$

(recall that the null function ξ_0 given in Remark II.3.4 belongs to $\Xi^d_{T-1}(x)$ for every $x \geq 0$), and finally that $\mathbb{E}_{\mathbb{P}}[U_T(1,\cdot) | \mathscr{F}_{T-1}] = u(1) = 0$ a.s., hence we obtain that, for every ω outside a $\mathbb{P}$-null set,

$$U_{T-1}(1,\omega) = G_{T-1}(1,\omega) \geq \mathbb{E}_{\mathbb{P}}[U_T(1,\cdot) | \mathscr{F}_{T-1}](\omega) = 0.$$

(iii) In what follows, we show that we also have Assumption II.3.10. Indeed, letting $x \geq 0$ be arbitrary, but fixed, it can be easily checked, in the same way as before (the construction of the portfolio becoming progressively involved, but totally analogous), that for every $\xi \in \Xi^d_{T-2}(x)$, the conditional expectation

$$\mathbb{E}_{\mathbb{P}} \left[U_{T-1}(x + \langle \xi(\cdot), \Delta S_{T-1}(\cdot) \rangle_{\mathbb{R}^d}, \cdot) | \mathscr{F}_{T-2} \right]$$

is not only well-defined (possibly $-\infty$), but also finite a.s.. Furthermore,

$$\operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-2}(x)} \mathbb{E}_{\mathbb{P}} \left[U^+_{T-1}(x + \langle \xi(\cdot), \Delta S_{T-1}(\cdot) \rangle_{\mathbb{R}^d}, \cdot) \middle| \mathscr{F}_{T-2} \right] < +\infty \text{ a.s.,}$$

as desired.

(iv) We proceed with the proof that Assumption II.3.12 is also verified. Now, given any $x \geq 0$, it is clear that

$$U_{T-1}(x,\cdot) = G_{T-1}(x,\cdot) \geq \mathbb{E}_{\mathbb{P}}\left[U_T(x,\cdot)\middle|\, \mathscr{F}_{T-1}\right] = u(x) \text{ a.s.},$$

where the inequality is due to $\xi_0 \in \Xi^d_{T-1}(x)$ and to the definition of supremum, thus in particular we obtain $\mathbb{E}_{\mathbb{P}}\left[U^-_{T-1}(0,\cdot)\middle|\, \mathscr{F}_{T-2}\right] \leq u^-(0) < +\infty$ a.s. (recall Assumption II.2.16).

(v) We finish by noting that, taking again $\overline{\gamma} > 0$ and $C > 0$ to be, respectively, the real numbers of Assumption II.4.3 and Lemma II.4.5, then we have that, for every $\lambda \geq 1$ and $x \geq 0$,

$$\begin{aligned} U^+_{T-1}(\lambda x,\cdot) &\leq \mathbb{E}_{\mathbb{P}}\left[U^+_T\left(\lambda x + \left\langle \widetilde{\xi}_T(\lambda x,\cdot)\,, \Delta S_T(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\, \mathscr{F}_{T-1}\right] \\ &\leq \lambda^{\overline{\gamma}}\mathbb{E}_{\mathbb{P}}\left[U_T\left(x + \left\langle \widetilde{\xi}_T(\lambda x,\cdot)\,/\lambda, \Delta S_T(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\, \mathscr{F}_{T-1}\right] \\ &\quad + \lambda^{\overline{\gamma}}\mathbb{E}_{\mathbb{P}}\left[U^-_T\left(x + \left\langle \widetilde{\xi}_T(\lambda x,\cdot)\,/\lambda, \Delta S_T(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\, \mathscr{F}_{T-1}\right] + \lambda^{\overline{\gamma}}C \text{ a.s.}, \end{aligned}$$

where the first inequality follows from the conditional Jensen inequality (for convex functions), and the second one uses (II.5.14). But it is easy to see that $\widetilde{\xi}_T(\lambda x,\cdot)\,/\lambda \in \Xi^d_{T-1}(x)$, and also that

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\left[U^-_T\left(x + \left\langle \widetilde{\xi}_T(\lambda x,\cdot)\,/\lambda, \Delta S_T(\cdot)\right\rangle_{\mathbb{R}^d},\cdot\right)\middle|\, \mathscr{F}_{T-1}\right](\omega) \\ &\qquad\qquad \leq \mathbb{E}_{\mathbb{P}}\left[U^-_T(0,\cdot)\middle|\, \mathscr{F}_{T-1}\right](\omega) \leq u^-(0)\,. \end{aligned}$$

Hence, setting $C' \triangleq C + u^-(0)$ (which is finite by Assumption II.2.16), we conclude that, for every $\lambda \geq 1$ and $x \geq 0$, $U^+_{T-1}(\lambda x,\cdot) \leq \lambda^{\overline{\gamma}} U^+_{T-1}(x,\cdot) + \lambda^{\overline{\gamma}}C'$ a.s..

But, using the regularity of the paths of U_{T-1}, we get that, for every ω in the set

$$\breve{\Omega}_{T-1} \cap \left(\bigcap_{\lambda \in \mathbb{Q}\cap[1,+\infty)} \bigcap_{q \in \mathbb{Q}^+_0} \left\{\omega \in \Omega\colon\ U^+_{T-1}(\lambda q,\omega) = \lambda^{\overline{\gamma}} U^+_{T-1}(q,\omega) + \lambda^{\overline{\gamma}}C'\right\}\right),$$

which has $\mathbb{P}$-full measure, the inequality $U^+_{T-1}(\lambda x,\omega) \leq \lambda^{\overline{\gamma}} U^+_{T-1}(x,\omega) + \lambda^{\overline{\gamma}}C'$ holds for all $\lambda \geq 1$ and $x \geq 0$, so Assumption II.3.13 is verified.

Consequently, we can apply Lemma II.3.18 and Proposition II.3.19 to obtain functions G_{T-2} and $\widetilde{\xi}_{T-1}$ satisfying some desired properties. Proceeding in a similar way for the remaining values of $t \in \{T-2,\dots,1\}$, we construct the functions $U_{T-2},\dots,U_1,U_0$ and $\widetilde{\xi}_{T-2},\dots,\widetilde{\xi}_1$.

The remainder of the proof is now dedicated to finding an optimal investment strategy. In order to achieve this, let us begin by considering an arbitrary, but fixed, $x_0 \geq 0$.

Then set, as in Rásonyi and Stettner [51, Proposition 3.2], $\phi_1^* \triangleq \widetilde{\xi}_1(x_0, \cdot)$, and define recursively, for $t \in \{2, \dots, T\}$,

$$\phi_t^* \triangleq \widetilde{\xi}_t\left(x_0 + \sum_{s=1}^{t-1} \langle \phi_s^*(\cdot)\,, \Delta S_s(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot\right).$$

Define as well

$$\phi_t^0 \triangleq x_0 + \sum_{s=1}^{t-1} \langle \phi_s^*, \Delta S_s\rangle_{\mathbb{R}^d} - \langle \phi_t^*, S_{t-1}\rangle_{\mathbb{R}^d}\,,$$

for every $t \in \{1, \dots, T\}$. We claim that the $d+1$-dimensional stochastic process $\overline{\phi}^* = \left\{\left(\phi_t^0, (\phi_t^*)^\top\right)^\top;\, t \in \{1, \dots, T\}\right\}$ constructed in this way gives an optimal portfolio for (NCPP) with initial wealth x_0.

(i) Firstly, we check that $\overline{\phi}^*$ is predictable, that is, ϕ_t^0 and ϕ_t^* are both $\mathscr{F}_t$-measurable, for every $t \in \{1, \dots, T\}$.

This will be done recursively. Starting with $t = 1$, it is trivial that $\phi_1^* \in \Xi_0^d(x_0)$, which in particular gives that ϕ_1^* is $\mathscr{F}_0$-measurable.

Then, taking $t = 2$, we see that $x_0 + \langle \phi_1^*, \Delta S_1\rangle_{\mathbb{R}^d} \geq 0$ a.s., and also that it is $\mathscr{F}_1$-measurable, so by Proposition II.3.19 we know that

$$\phi_2^* = \widetilde{\xi}_2(x_0 + \langle \phi_1^*(\cdot)\,, \Delta S_1(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot)$$

belongs to $\Xi_1^d\big(x_0 + \langle \phi_1^*, \Delta S_1\rangle_{\mathbb{R}^d}\big)$.

Suppose now that we have already established that, for some $t \in \{1, \dots, T-1\}$, each $x_0 + \sum_{i=1}^{s-1} \langle \phi_i^*, \Delta S_i\rangle_{\mathbb{R}^d}$ is $\mathscr{F}_{s-1}$-measurable and non-negative a.s., and that each ϕ_s^* belongs to $\Xi_{s-1}^d\left(x_0 + \sum_{i=1}^{s-1} \langle \phi_i^*, \Delta S_i\rangle_{\mathbb{R}^d}\right)$ (with $s \in \{1, \dots, t\}$). Then obviously $x_0 + \sum_{s=1}^{t} \langle \phi_s^*, \Delta S_s\rangle_{\mathbb{R}^d} = x_0 + \sum_{s=1}^{t-1} \langle \phi_s^*, \Delta S_s\rangle_{\mathbb{R}^d} + \langle \phi_t^*, \Delta S_t\rangle_{\mathbb{R}^d}$ is not only $\mathscr{F}_t$-measurable, but also non-negative a.s.. Moreover, another application of Proposition II.3.19 yields

$$\phi_{t+1}^* = \widetilde{\xi}_t\left(x_0 + \sum_{s=1}^{t} \langle \phi_s^*(\cdot)\,, \Delta S_s(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot\right) \in \Xi_t^d\left(x_0 + \sum_{s=1}^{t} \langle \phi_s^*, \Delta S_s\rangle_{\mathbb{R}^d}\right).$$

Hence, the claim that ϕ_t^* is $\mathscr{F}_{t-1}$-measurable is valid for all $t \in \{1, \dots, T\}$. This, in turn, immediately implies the $\mathscr{F}_{t-1}$-measurability of ϕ_t^0.

That the portfolio $\overline{\phi}^*$ is self-financing is obvious by construction. In addition, for every $t \in \{1, \dots, T\}$, we have that $\Pi_t^{\overline{\phi}^*} = x_0 + \sum_{s=1}^{t} \langle \phi_s^*, \Delta S_s\rangle_{\mathbb{R}^d} \geq 0$ a.s. by the preceding step, so it is also admissible for x_0, i.e., $\overline{\phi}^* \in \Psi(x_0) = \mathscr{A}(x_0)$ (recall Remark II.4.7).

(ii) Next, for any fixed $t \in \{1, \dots, T\}$, we can use the tower property of the conditional

expectation, the self-financing property of $\overline{\phi}^*$, and the definition of $\overline{\phi}^*_t$ to obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[U_t\Big(\Pi^{\overline{\phi}^*}_t(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big]\\
&\qquad = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\Big[U_t\Big(\Pi^{\overline{\phi}^*}_{t-1}(\cdot) + \Big\langle\widetilde{\xi}_t\Big(\Pi^{\overline{\phi}^*}_{t-1}(\cdot)\,,\cdot\Big)\,,\Delta S_t(\cdot)\Big\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\,\mathscr{F}_{t-1}\Big]\Big|\,\mathscr{F}_0\Big]\\
&\qquad = \mathbb{E}_{\mathbb{P}}\Big[U_{t-1}\Big(\Pi^{\overline{\phi}^*}_{t-1}(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] \text{ a.s.,}
\end{aligned}$$

where the last equality is due to equation (II.3.13).

But this implies $\mathbb{E}_{\mathbb{P}}\Big[U_T\Big(\Pi^{\overline{\phi}^*}_T(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] = \mathbb{E}_{\mathbb{P}}\Big[U_0\Big(\Pi^{\overline{\phi}^*}_0(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] = U_0(x_0,\cdot)$ a.s., which in turn gives $\mathbb{E}_{\mathbb{P}}\Big[u\Big(\Pi^{\overline{\phi}^*}_T(\cdot)\Big)\Big] = \mathbb{E}_{\mathbb{P}}\Big[U_T\Big(\Pi^{\overline{\phi}^*}_T(\cdot)\,,\cdot\Big)\Big] = \mathbb{E}_{\mathbb{P}}[U_0(x_0,\cdot)]$.

(iii) Thirdly, let the portfolio $\overline{\varphi}\in\mathscr{A}(x_0)$ and $t\in\{1,\dots,T\}$ be arbitrary. By the same arguments of the previous step, we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\Big[U_t\Big(\Pi^{\overline{\varphi}}_t(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big]\\
&\quad = \mathbb{E}_{\mathbb{P}}\Big[\mathbb{E}_{\mathbb{P}}\Big[U_t\Big(\Pi^{\overline{\varphi}}_{t-1}(\cdot) + \langle\varphi_t(\cdot)\,,\Delta S_t(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\,\mathscr{F}_{t-1}\Big]\Big|\,\mathscr{F}_0\Big]\\
&\quad \le \mathbb{E}_{\mathbb{P}}\Bigg[\operatorname*{ess\,sup}_{\xi\in\Xi^d_{t-1}(\Pi^{\overline{\varphi}}_{t-1})}\mathbb{E}_{\mathbb{P}}\Big[U_t\Big(\Pi^{\overline{\varphi}}_{t-1}(\cdot) + \langle\xi(\cdot)\,,\Delta S_t(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot\Big)\Big|\,\mathscr{F}_{t-1}\Big]\Bigg|\,\mathscr{F}_0\Bigg] \text{ a.s.,}
\end{aligned}$$

the inequality being a straightforward consequence of $\varphi_t\in\Xi^d_{t-1}\Big(\Pi^{\overline{\varphi}}_{t-1}\Big)$ and of the definition of essential supremum. Thus, (II.3.13) with $H=\Pi^{\overline{\varphi}}_{t-1}$ yields

$$\mathbb{E}_{\mathbb{P}}\Big[U_t\Big(\Pi^{\overline{\varphi}}_t(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] \le \mathbb{E}_{\mathbb{P}}\Big[U_{t-1}\Big(\Pi^{\overline{\varphi}}_{t-1}(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] \text{ a.s..}$$

Because the previous inequality is true for all $t\in\{1,\dots,T\}$, we then obtain

$$\mathbb{E}_{\mathbb{P}}\Big[U_T\Big(\Pi^{\overline{\varphi}}_T(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] \le \mathbb{E}_{\mathbb{P}}\Big[U_0\Big(\Pi^{\overline{\varphi}}_0(\cdot)\,,\cdot\Big)\Big|\,\mathscr{F}_0\Big] = U_0(x_0,\cdot) \text{ a.s..}$$

Finally, taking expectations on both sides allows us to get

$$\mathbb{E}_{\mathbb{P}}\Big[u\Big(\Pi^{\overline{\varphi}}_T(\cdot)\Big)\Big] = \mathbb{E}_{\mathbb{P}}\Big[U_T\Big(\Pi^{\overline{\varphi}}_T(\cdot)\,,\cdot\Big)\Big] \le \mathbb{E}_{\mathbb{P}}[U_0(x_0,\cdot)]$$

and so by the arbitrarity of $\overline{\varphi}$ we conclude that $v^*(x_0)\le\mathbb{E}_{\mathbb{P}}[U_0(x_0,\cdot)]$.

Hence, given that $\overline{\phi}^*\in\mathscr{A}(x_0)$, on the one hand we have $\mathbb{E}_{\mathbb{P}}\Big[u\Big(\Pi^{\overline{\phi}^*}_T(\cdot)\Big)\Big]\le v^*(x_0)$, whereas on the other we have $v^*(x_0)\le\mathbb{E}_{\mathbb{P}}[U_0(x,\cdot)] = \mathbb{E}_{\mathbb{P}}\Big[u\Big(\Pi^{\overline{\phi}^*}_T(\cdot)\Big)\Big]$, and the proof that $\overline{\phi}^*$ is optimal is completed. □

Proof of Theorem II.4.8. The proof unfolds exactly as that of Theorem II.4.6, the only difference residing, for each time stage $t\in\{T,\dots,1\}$, in the verification that Assumption II.3.10 is valid for the function $U_t:\ [0,+\infty)\times\Omega\to\mathbb{R}$).

Starting with $t = T$, it is straightforward by Assumption II.4.3 that, setting $\widehat{x} \triangleq 1 \vee \overline{x}$, we have $U_t(x, \omega) = u(x) \leq (x/\widehat{x})^{\overline{\gamma}} u(\widehat{x}) + c$ for all $x > \widehat{x}$ and $\omega \in \Omega$. On the other hand, it follows from the monotonicity of u that $U_t(x, \omega) = u(x) \leq u(\widehat{x})$ for all $x \in [0, \widehat{x}]$ and $\omega \in \Omega$. Therefore, setting $J_T \triangleq u(\widehat{x}) + c + 1 > 0$ gives

$$U_T(x, \omega) \leq J_T \left(x^{\overline{\gamma}} + 1\right), \quad \forall\, x \geq 0,\ \forall\, \omega \in \Omega.$$

Then, fixing an arbitrary $x \geq 0$, we may apply Proposition II.3.7, the Cauchy-Schwarz inequality, Lemma II.3.15 and the trivial inequality of Lemma II.5.1 to conclude that, for any $\xi \in \Xi^d_{T-1}(x)$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \middle| \mathscr{F}_{T-1}\right] &= \mathbb{E}_{\mathbb{P}}\left[U_T^+\left(x + \left\langle \widehat{\xi}(\cdot)\,, \Delta S_T(\cdot)\right\rangle_{\mathbb{R}^d}\,, \cdot\right) \middle| \mathscr{F}_{T-1}\right] \\ &\leq J_{T-1}\left(x^{\overline{\gamma}} + 1\right) \text{ a.s.,}\end{aligned}$$

with $J_{T-1} \triangleq \mathbb{E}_{\mathbb{P}}\left[C_{T-1} J_T \left(1 + (1/\beta_T(\cdot))^{\overline{\gamma}} \left\|\Delta S_T(\cdot)\right\|^{\overline{\gamma}}_{\mathbb{R}^d}\right) \middle| \mathscr{F}_{T-1}\right]$ and C_{T-1} a strictly positive real number. Consequently, we obtain by the definition of essential supremum that $\operatorname{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \middle| \mathscr{F}_{T-1}\right] \leq J_{T-1}\left(x^{\overline{\gamma}} + 1\right)$ a.s., which in turn implies that

$$\mathbb{E}_{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \middle| \mathscr{F}_{T-1}\right]\right] \leq \left(x^{\overline{\gamma}} + 1\right) \mathbb{E}_{\mathbb{P}}[J_{T-1}]\,. \quad \text{(II.5.15)}$$

But it is easy to see that $\mathbb{E}_{\mathbb{P}}[(J_{T-1})^p] < +\infty$ for every $p > 0$. Indeed, if $p \geq 1$ we may invoke Jensen's inequality (for the conditional expectation and for convex functions) and again Lemma II.5.1 to deduce

$$\mathbb{E}_{\mathbb{P}}[(J_{T-1})^p] \leq C_1 \mathbb{E}_{\mathbb{P}}[(J_T)^p] + C_2 \mathbb{E}_{\mathbb{P}}\left[(J_T)^p\,(1/\beta_T(\cdot))^{p\overline{\gamma}} \left\|\Delta S_T(\cdot)\right\|^{p\overline{\gamma}}_{\mathbb{R}^d}\right],$$

for some real numbers $C_1, C_2 > 0$. But then an obvious application of Hölder's inequality yields $\mathbb{E}_{\mathbb{P}}[(J_{T-1})^p] < +\infty$ (recall that J_T is a constant, and that $1/\beta_T, \|\Delta S_T\|_{\mathbb{R}^d} \in \mathscr{W}$ by hypothesis). In the case where $p \in (0, 1)$, we have by Jensen's inequality (this time for concave functions) and the preceding case that $\mathbb{E}_{\mathbb{P}}[(J_{T-1})^p] \leq \mathbb{E}_{\mathbb{P}}[J_{T-1}]^p < +\infty$. Therefore, the expectation on the right-hand side of equation (II.5.15) is finite, so as intended $\operatorname{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}\left[U_T^+(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) \middle| \mathscr{F}_{T-1}\right] < +\infty$ a.s..

Next, let us consider $t = T - 1$. We know by construction of U_{T-1} and by the previous discussion that, for every $x \geq 0$,

$$U_{T-1}(x, \cdot) = \operatorname*{ess\,sup}_{\xi \in \Xi^d_{T-1}(x)} \mathbb{E}_{\mathbb{P}}[U_T(x + \langle \xi(\cdot)\,, \Delta S_T(\cdot)\rangle_{\mathbb{R}^d}\,, \cdot) | \mathscr{F}_{T-1}] \leq J_{T-1}\left(x^{\overline{\gamma}} + 1\right) \text{ a.s..}$$

Thus, setting $J_{T-2} \triangleq \mathbb{E}_{\mathbb{P}}\left[C_{T-2} J_{T-1}(\cdot) \left(1 + (1/\beta_{T-1}(\cdot))^{\overline{\gamma}} \left\|\Delta S_{T-1}(\cdot)\right\|^{\overline{\gamma}}_{\mathbb{R}^d}\right) \middle| \mathscr{F}_{T-2}\right]$, again with C_{T-2} a strictly positive real number, we obtain as before that

$$\mathbb{E}_{\mathbb{P}}\left[\operatorname*{ess\,sup}_{\xi\in\Xi^d_{T-2}(x)} \mathbb{E}_{\mathbb{P}}\big[U^+_{T-1}(x+\langle\xi(\cdot)\,,\Delta S_{T-1}(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\big|\,\mathscr{F}_{T-2}\big]\right] \le \big(x^{\overline{\gamma}}+1\big)\,\mathbb{E}_{\mathbb{P}}[J_{T-2}]\,.$$

An argument entirely analogous to the one used above (this time taking into account that J_{T-1} also belongs to $\mathscr{W}$) allows us to conclude that Assumption II.3.10 is satisfied for $V = U_{T-1}$ as well.

Proceeding recursively in this way, we obtain for all $t \in \{T-2,\dots,1\}$ that, for every $x \ge 0$, both inequalities $U_t(x,\cdot) \le J_t\left(x^{\overline{\gamma}}+1\right)$ and

$$\operatorname*{ess\,sup}_{\xi\in\Xi^d_{t-1}(x)} \mathbb{E}_{\mathbb{P}}\big[U^+_{t}(x+\langle\xi(\cdot)\,,\Delta S_{t}(\cdot)\rangle_{\mathbb{R}^d}\,,\cdot)\big|\,\mathscr{F}_{t-1}\big] \le J_{t-1}\left(x^{\overline{\gamma}}+1\right)$$

hold a.s., for some random variables $J_t \in \mathscr{W}$ and

$$J_{t-1} \triangleq \mathbb{E}_{\mathbb{P}}\Big[C_{t-1}J_t(\cdot)\left(1+(1/\beta_t(\cdot))^{\overline{\gamma}}\,\|\Delta S_t(\cdot)\|^{\overline{\gamma}}_{\mathbb{R}^d}\right)\Big|\,\mathscr{F}_{t-1}\Big] \in \mathscr{W},$$

and some constant $C_{t-1} > 0$. One last iteration then gives, for each $x \ge 0$, that $U_0(x,\cdot) \le J_0\left(x^{\overline{\gamma}}+1\right)$ a.s., hence $\mathbb{E}_{\mathbb{P}}[U_0(x,\cdot)] < +\infty$.

Finally, given that for every $x_0 \ge 0$ we have $v^*(x_0) \le \mathbb{E}_{\mathbb{P}}[U_0(x_0,\cdot)]$ (which follows exactly as in the the proof of Theorem II.4.6 above), we conclude by the sub-additivity of the supremum that

$$\sup_{\overline{\phi}\in\Psi(x_0)} \mathbb{E}_{\mathbb{P}}\left[u^+\left(\Pi^{\overline{\phi}}_T\right)\right] \le v^*(x_0) + \sup_{\overline{\phi}\in\Psi(x_0)} \mathbb{E}_{\mathbb{P}}\left[u^-\left(\Pi^{\overline{\phi}}_T\right)\right] \le \mathbb{E}_{\mathbb{P}}[U_0(x_0,\cdot)] + u^-(0) < +\infty$$

(recall that, by admissibility, $\Pi^{\overline{\phi}}_T \ge 0$ for every $\overline{\phi} \in \Psi(x_0)$, and that Assumption II.2.16 is in force). Hence (II.4.3) holds and so an optimal portfolio exists by Theorem II.4.6. □

Chapter III

Behavioural portfolio optimisation

1 Introduction

In this chapter, we shall examine the finite-horizon optimal investment problem for an investor who behaves in accordance with CPT. A brief literature review is given in this section.

As mentioned in the Introduction, portfolio optimisation has been extensively studied under the assumptions of EUT. However, to the best of our knowledge, the existing mathematical literature on this problem within the framework of CPT is, despite its more than two decades of existence, fairly meagre, especially in continuous-time models.

It has been stressed by several authors (such as Bernard and Ghossoub [9], Carassus and Rásonyi [13], and Jin and Zhou [29], amongst others) that this scarcity is most certainly not due to a lack of interest or relevance of the problem. One possible explanation that is commonly offered is concerned with the problem's difficulty. Indeed, it has already been noted that, as a consequence of the presence of probability distortion functions, the behavioural agent's objective functional to be maximised involves (possibly non-linear) Choquet integrals. This, together with the lack of global concavity, raises new mathematically complex challenges, and the most common approaches to solving the EUT portfolio problem, such as dynamic programming or the use of convex duality methods, are not suitable anymore (as remarked in Jin and Zhou [29], the dynamic consistency, crucial to DP, is lost in this setting; what is more, there is not even an undisputed definition of the conditional Choquet expectation, see e.g. Kast, Lapied, and Toquebeuf [34]). We note further that the optimisation problem may be ill-posed (see Definition III.2.16), even in seemingly innocuous cases, which brings to light an additional issue.

According to Jin and Zhou [29], the majority of the first papers on CPT portfolio choice (see the references provided therein) dedicated themselves to finding experimental or numerical solutions in one-period markets, and it was frequent that some of the main principles of CPT were absent.

Optimal investment strategies are explicitly obtained by Bernard and Ghossoub [9] in a one-period model with one risky asset and one riskless asset, for a special reference point (the terminal wealth of the portfolio consisting of investing the totality of the initial wealth in the riskless asset) and a special utility function (the piecewise-power function originally used in CPT, in which the power of the gain part is not greater than the power of the loss part). They begin by dealing with the case where borrowing is allowed, but not short-selling, and they proceed to tackle the case where both are forbidden. Moreover, since closed-form expressions are derived, it is possible to study some properties of the optimal portfolio (such as homogeneity and the preservation of first-order stochastic dominance). Finally, as noted by the authors, their results depend on their choice of the reference point.

Another analytical treatment of the CPT portfolio problem in a one-period financial market with two assets (a risky one and a riskless one) is given in the paper by He and Zhou [27]. Two separate special cases are analysed: one where the reference point is the risk-free return and the utility is piecewise power; the other where a general reference point is considered, but the utility function is piecewise linear. The optimal solutions are explicitly derived for both cases. In addition, the authors conduct a thorough study of the well-posedness of the CPT portfolio problem, and conclude that it is essentially determined by a measure of loss aversion for large payoffs (which the authors call Large-Loss Aversion Degree, or LLAD). Finally, as in the previous paper, the closed-form solutions which are derived allow for an investigation of some properties of the optimal solutions (namely, their sensitivity to the reference point, to the trading horizon and to the investor's level of loss aversion). We conclude by remarking that assumptions of differentiability are imposed not only on the utility, but also on the distortions.

To our best knowledge, the first paper about CPT portfolio optimisation in a (generically incomplete) multi-period discrete-time model is that of Carassus and Rásonyi [13]. Assuming that the investor's utilities and distortions behave in a power-like way respectively at infinity and near zero, conditions for well-posedness are obtained (which not only are easy to verify, but also can be interpreted economically). In addition, even if an explicit solution is not determined, an optimal solution is shown to exist under two separate assumptions (one postulating that an independent external source of randomness can be found in the market, thus leading to the consideration of relaxed strategies using randomness, an idea borrowed from game theory; the other one that the filtration is rich enough and thus no external random source needs to be considered). We remark further that the reference point is taken to be arbitrary (though it must be sub-hedgeable, so that it can be related to the market in some way), and that no assumptions are made with respect to the continuity, monotonicity, differentiability or concavity of the utilities.

In the same discrete-time, generically incomplete market framework, and for a behavioural investor with power-like utilities and distortions, Rásonyi and Rodríguez-Villarreal [49] are capable of complementing and generalising the results of Carassus and Rásonyi [13]. They prove well-posedness under the parameter restriction of Rásonyi and Rodrigues [47], which is neither stronger nor weaker than the sufficient condition for well-posedness derived by Carassus and Rásonyi [13]. More importantly, by constructing an equivalent martingale measure whose density has some favourable integrability properties, the authors are able to drop the assumption concerning the external random source, and still establish the existence of an optimal portfolio under both parameter conditions.

Turning now to the continuous-time case, we would like to draw attention to the early work of Berkelaar, Kouwenberg, and Post [8]. Within a complete market framework in which asset prices follow Itô processes and wealth is restricted to be non-negative, a closed-form expression for the optimal terminal wealth is derived for a loss-averse investor (whose utility function is either a concave kinked power function or, like in Bernard and Ghossoub [9], the original piecewise-power utility of Tversky and Kahneman [61]) with a stochastic reference point (whose updating rule is known, though). Under additional assumptions (specifically, a constant market price of risk, a constant interest rate, and asset prices given by geometric Brownian motions), the authors further calculate the explicit optimal portfolio weights. This is accomplished with the martingale method, that is, by reducing the dynamic portfolio problem to a constrained non-concave static problem, which is in turn solved using a convexification method. Nonetheless, no probability distortions are considered in this paper, which substantially simplifies the problem.

Therefore, and as far as we know, the first analytical treatment of the CPT portfolio problem in continuous time which incorporates distortions is that of Jin and Zhou [29], where it is assumed that the market is complete and that the asset prices are Itô processes. Also, the admissibility condition which is imposed on the portfolios is that they should be tame[1]. Moreover, the utility is assumed to be any continuous S-shaped function, and differentiability assumptions are imposed both on the utility and on the distortions. Then, even though their study of well-posedness is far from being exhaustive, the authors are able to identify two ill-posed cases. Next, they set out to determine the optimal strategy and, in order to do so, they employ what they nickname a 'divide and conquer' procedure, which consists in dividing the original problem into three sub-problems. The idea behind this method is that, with any portfolio's terminal wealth, there are two associated parameters: an event (the set where it is non-negative, i.e., where it makes a gain) and a non-negative constant (the expectation of its positive part with respect to the unique equivalent martingale measure, see Definition III.2.1 below). Therefore, it is natural to expect that the optimal strategy should lead to parameters which are also optimal, in some sense. The three steps of their method are as follows. Firstly, they consider a gain part problem with two parameters, which is a Choquet maximisation problem, and they apply what they call a quantile formulation to change

[1] Recalling the nomenclature of [32], a portfolio is said to be *tame* if its wealth process is uniformly bounded below by some constant (possibly depending on the strategy). In practice, tameness means that there exists a limit on the credit which is needed to maintain the strategy until maturity, and that this limit is known in advance.

the decision variable of the problem from the random variable to its quantile function (in particular, this deals with the distortions and reduces the problem to a concave one with the usual linear expectation). Secondly, they consider the corresponding loss part problem (with the same parameters), which is a Choquet minimisation problem, which again they solve by employing a quantile formulation (followed by combinatorial optimisation techniques). Thirdly, having dealt with the two problems above, the authors then solve one last optimisation problem, in which they look for the parameters that maximise the difference of the optimal solutions to the two preceding problems. This technique gives explicit solutions, but under restrictive hypotheses (see Assumption 4.1 of Jin and Zhou [29]), whose economic interpretation is not obvious (we refer to the discussion in Subsection 6.2 of Jin and Zhou [29]) and which are not easy to check. Moreover, even though they are true for concave distortions and they are shown to be satisfied by a very specific example of an inverse S-shaped[2] distortion function (see their Example 6.1), it is not clear whether they hold for other distortions (namely, the original distortions of CPT, see Example III.2.9*(iv)*)). Furthermore, Campi and Vigna [11, Example 2.2] seem to be particularly critical of the fact that the Assumption 4.1 in Jin and Zhou [29] (and thus, the distortion on gains) depends on the market parameters.

We mention as well the paper of Carlier and Dana [15], whose Section 5 is devoted to the CPT problem for an investor with an S-shaped utility. We remark that continuity and differentiability assumptions are imposed both on the utility and on the distortions. Moreover, in this work, wealth is not allowed to become negative. Closed-form solutions are obtained, but again under a very restrictive assumption (see their equation (5.8) and Proposition 5.5).

Some related investigations have also been carried out in Campi and Vigna [11] within a complete market framework. Borrowing the method developed by Jin and Zhou [29] (and consequently under the same restrictive hypothesis which were referred to above), the explicit optimal terminal wealth is found. However, the authors use the risk-neutral (instead of the physical) probability in the definition of the objective function (as they start from the beginning with martingale prices, and so no change of measure is required), which leads to a problem that is entirely different from ours.

In addition, there is the work of Zhang, Jin, and Zhou [65], developed in the same setting as that of Jin and Zhou [29], the only difference being that here the losses are constrained to be uniformly bounded below by some universal constant. The optimal solution is obtained explicitly, essentially by the same 'divide and conquer' method of Jin and Zhou [29] (the main contribution resides in the fact that an additional constraint must now be taken into account).

See also Reichlin [52], where in Chapter IV sufficient conditions for the existence of an optimal portfolio are derived for weakly complete market models (see Schachermayer, Sîrbu, and Taflin [59, p. 59] for a definition) in which wealth is required to remain non-

[2]Recall that a function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *inverse* S-*shaped* if there exists some x_0 in the domain D such that f is concave on $(-\infty, x_0) \cap D$ and convex on $(x_0, +\infty) \cap D$.

negative.

As for our work, it concerns the optimal portfolio problem for a behavioural investor in a continuous-time financial market. Following the example of Carassus and Rásonyi [13], we shall require neither the differentiability nor the concavity of the utilities or of the distortions. Moreover, we point out that in Berkelaar et al. [8], Jin and Zhou [29], Carlier and Dana [15], and Reichlin [52], the wealth process of any admissible portfolio must be bounded below by some constant (which may eventually depend on the portfolio), which looks rather unnatural when we recall that, in classic utility maximisation problems on the whole real line, optimal strategies typically lead to arbitrarily large losses (we refer, e.g., to Schachermayer [58]). Therefore, as Carassus and Rásonyi [13], we choose not to impose restrictions on the portfolio losses, and we present our existence results for the optimiser in what we regard as a more natural class of admissible trading strategies (see our Definition III.3.3). Now, since in Berkelaar et al. [8], Carlier and Dana [15], and Reichlin [52] wealth is restricted to be non-negative, they only consider utilities which are defined on the positive real axis. Hence, apart from our work, the only other study we know in continuous time about the whole real line case is that of Jin and Zhou [29]. On the other hand, we shall work under assumptions that are only slightly weaker than market completeness (Assumptions III.3.5 and III.3.7), which not only will give us a large family of feasible portfolios (see Definition III.2.12), but will allows us to deal with the investor's reference point in a trivial way (we refer to Remark III.3.6). Finally, before tackling the issue of the existence of optimal portfolios and unlike many papers in which the well-posedness of the optimisation problem is assumed a priori, this work carefully examines the issue of well-posedness.

2 Notation and set-up

2.1 The market

Like in the preceding chapter, let us suppose that the financial market is *frictionless* and totally *liquid*. We represent by 0 the current time, and by $T \in (0, +\infty)$ the non-random *trading horizon*. We then let $\mathbb{T}$ denote the *trading set*, that is, the family of deterministic times at which trading occurs. Clearly, we must have $\{0, T\} \subseteq \mathbb{T}$, but no additional assumptions regarding $\mathbb{T}$ are made for the time being (for example, we can take it to be a finite collection of points as in Chapter II, or simply the interval $[0, T]$ as in the subsequent sections).

As usual, to model uncertainty we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where the space Ω is the set of all possible scenarios of market evolution, and $\mathbb{P}$ is the real-world probability measure. We suppose further that this probability space is equipped with a filtration $\mathbb{F} = \{\mathscr{F}_t;\, t \in \mathbb{T}\}$ modelling the evolution of information through time. Lastly, we shall assume that $\mathscr{F}_0$ not only contains all the $\mathbb{P}$-null sets, but is actually $\mathbb{P}$-trivial, and also that $\mathscr{F} = \mathscr{F}_T$.

Next, we fix an arbitrary $d \in \mathbb{N}$, and introduce a d-dimensional and $\mathbb{F}$-adapted process $S = \left\{S_t = \left(S_t^1, \ldots, S_t^d\right); t \in \mathbb{T}\right\}$. For each $i \in \{1, \ldots, d\}$, S_t^i represents the price of a certain *risky asset* i at time $t \in \mathbb{T}$. In addition to these d risky securities, we shall assume that the market contains a *riskless asset*, with constant price $S_t^0(\omega) = 1$ for any $t \in \mathbb{T}$ and $\omega \in \Omega$. Therefore, we shall work directly with discounted prices. The financial market then consists of $d+1$ traded assets.

Now we denote by Φ the set of all *portfolios* $\overline{\phi}$ which are *self-financing*, that is, such that the changes in the corresponding value processes $\Pi^{\overline{\phi}} = \left\{\Pi_t^{\overline{\phi}}; t \in \mathbb{T}\right\}$ are due only to changes in the asset prices and not to any injection or withdrawal of capital. We further note that it may be necessary (or we may choose) to restrict ourselves to a subset $\Psi \subseteq \Phi$ of *admissible strategies*. For the reason provided in Section III.1, we shall not, however, impose that wealth should be bounded below (in particular, wealth is allowed to become negative).

In addition, we recall that a *contingent claim* settling at time T (given by any $\mathscr{F}_T$-measurable random variable) is *hedgeable* if it is equal to the terminal value of some admissible portfolio (called a *replicating strategy*), and that a market is said to be *complete* if every contingent claim X with $X^- \in L^\infty(\Omega, \mathscr{F}_T, \mathbb{P})$[3] can be replicated (see, e.g., Ansel and Stricker [3, Section 2], or Delbaen and Schachermayer [20, Section 5]).

We also have the following important concept in financial mathematics.

Definition III.2.1 (Equivalent local martingale measure). We say that a probability measure $\mathbb{P}^*$ on $(\Omega, \mathscr{F})$ is an *equivalent martingale measure* (respectively, *equivalent local martingale measure*) for S, and we write EMM (respectively, ELMM) for short, if both conditions below are verified,

(i) $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ (in which case we write $\mathbb{P}^* \sim \mathbb{P}$)[4],

(ii) the (discounted) price process S is a martingale[5] (respectively, local martingale) with respect to $\mathbb{P}^*$.

We denote by $\mathscr{M}_e(S)$ (respectively, $\mathscr{M}_e^{loc}(S)$) the set of all EMM (respectively, ELMM).

We shall make the following crucial standing assumption throughout, which is essentially equivalent to the absence of arbitrage opportunities in the market (see Delbaen and Schachermayer [20] for the definition of the condition of *no free lunch with vanishing risk*, or NFLVR, and for the precise statement of the theorem).

Assumption III.2.2. *There exists at least one ELMM, i.e., $\mathscr{M}_e^{loc}(S) \neq \varnothing$.*

[3] We recall that, given a measure space (X, Σ, μ) and $p \in (0, +\infty)$, $L^0(X, \Sigma, \mu)$ denotes the vector space of (equivalence classes of) measurable functions $f : X \to \mathbb{R}$, whereas $L^p(X, \Sigma, \mu)$ denotes the vector space of (equivalence classes of) functions $f \in L^0(X, \Sigma, \mu)$ verifying $\int_X |f|^p \, d\mu < +\infty$. Lastly, for $p = +\infty$, the space $L^\infty(X, \Sigma, \mu)$ is the set of all (equivalence classes of) functions $f \in L^0(X, \Sigma, \mu)$ which are essentially bounded (i.e., bounded up to a set of μ-measure zero).

[4] That is, for all $A \in \mathscr{F}$ we have $\mathbb{P}^*(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$. Roughly speaking, we may say that $\mathbb{P}^*$ and $\mathbb{P}$ share the same 'impossible' and 'sure' events.

[5] I.e., the (discounted) price process S is a 'fair game', meaning that knowledge of the past does not help improve future gains.

Then, in what follows, let us fix a measure $\mathbb{P}^* \in \mathscr{M}_e^{loc}(S)$, or equivalently, let us fix a *state price density* $\rho \triangleq d\mathbb{P}^*/d\mathbb{P}$ (where $d\mathbb{P}^*/d\mathbb{P}$ is the unique, up to a $\mathbb{P}$-null set, Radon-Nikodým derivative of $\mathbb{P}^*$ with respect to $\mathbb{P}$).

2.2 The investor

As described in the Introduction, we are analysing a *CPT investor*, that is, a representative economic agent behaving in accordance with Cumulative Prospect Theory, introduced and developed by Kahneman and Tversky [31; 61]. In addition, we assume to be dealing with a small investor, whose behaviour has no influence on the movement of asset prices. We assume further that the investor has *initial capital* $x_0 \in \mathbb{R}$. The three fundamental principles of CPT will then be described below in mathematical terms.

Reference point

Firstly, the investor is assumed to have a reference point (also referred to in the literature as *benchmark* or *status quo*, see e.g. Bernard and Ghossoub [9], He and Zhou [27], Carassus and Rásonyi [13]) in wealth, with respect to which payoffs at the terminal time T are evaluated. Therefore, the investors' decision is not based on the terminal level of wealth (as it is assumed in EUT), but rather on the deviation of that wealth level from the reference point.

Definition III.2.3 (Reference point, gains and losses). A *reference point* is a fixed scalar-valued and $\mathscr{F}_T$-measurable random variable B satisfying

$$\mathbb{E}_{\mathbb{P}^*}[|B|] < +\infty. \tag{III.2.1}$$

Thus, given a payoff X at the terminal time T and a scenario $\omega \in \Omega$, the investor is said to make a *gain* (respectively, a *loss*) if the deviation from the reference level is strictly positive (respectively, strictly negative), that is, $X(\omega) > B(\omega)$ (respectively, $X(\omega) < B(\omega)$).

Note that B may be taken to be, for example, the constant x_0, that is, the terminal wealth of the portfolio consisting of investing all of the initial wealth in the riskless asset (this is the case in Bernard and Ghossoub [9]). The reference point can also be stochastic (for example, to reflect the fact that investor expects some random payoff at the maturity, or even that the reference point is updated through time).

Utility function

Secondly, according to this framework, the agent's preferences towards risk can be numerically described by a utility function (or, to follow the nomenclature of Tversky and Kahneman [61], a *prospect value function*), which is defined below.

Definition III.2.4 (Utility function). A *utility* is any function $u : \mathbb{R} \to \mathbb{R}$ given by

$$u(x) \triangleq u_+\big(x^+\big)\,\mathbb{1}_{[0,+\infty)}(x) - u_-\big(x^-\big)\,\mathbb{1}_{(-\infty,0)}(x)\,, \qquad x \in \mathbb{R}, \tag{III.2.2}$$

where the strictly increasing and continuous functions $u_\pm : [0,+\infty) \to [0,+\infty)$ satisfy $u_\pm(0) = 0$. The functions u_+ and u_- respectively represent the investor's *utility on gains* and *utility on losses*.

Remark III.2.5. Given that, in this work, we allow wealth to become negative, our utility function is defined on the whole real line, not just on the non-negative half-line.

Moreover, we impose minimal conditions on the utility, with clear financial meanings (as explained in Chapter I). For example, no differentiability is required. In addition, despite the fact that CPT specifically postulates that the utility should be S-shaped, no concavity assumptions are made on u_+ and u_- (we shall see in the next sections that the shape of the utility plays no role whatsoever in our treatment of the CPT portfolio problem). ◇

Moreover, it is clear that the utility functions $u_\pm$ have (possibly infinite) limits as $x \to +\infty$. In what follows, we shall use the notation $u_\pm(+\infty) \triangleq \lim_{x\to+\infty} u_\pm(x)$.

Probability distortion

The third, and perhaps most prominent, feature of CPT is that the investor systematically distorts (in a possibly non-linear way) the real probabilities.

Definition III.2.6 (Distortion functions). We call w_+ and w_-, both mapping from $[0,1]$ to $[0,1]$, *probability distortions* or *probability weighting functions* (*on gains* and *on losses*, respectively) if they are continuous and strictly increasing on their domain, with $w_\pm(0) = 0$ and $w_\pm(1) = 1$.

Remark III.2.7. The conditions $w_\pm(1) = 1$ (respectively, $w_\pm(0) = 0$) translates the fact that the investors, while distorting probabilities, are still able to identify the 'sure' events (respectively, 'impossible' events). ◇

Thus, the investor's subjective measures of the likelihood of gains and losses are given, respectively, by the *capacities* $w_+ \circ \mathbb{P}$ and $w_- \circ \mathbb{P}$ (which are set functions that may not be additive). We also have the following.

Definition III.2.8 (Overweighting or underweighting). An investor is said to *overweight* (respectively, *underweight*) small-probability losses if there is some $\varepsilon \in (0,1]$ such that, for all $x \in (0,\varepsilon)$,

$$w_-(x) > x \qquad \text{(respectively, } w_-(x) < x\text{)}. \tag{III.2.3}$$

Similarly, an investor *overweights* (respectively, *underweights*) large-probability losses if there exists an $\varepsilon \in (0,1]$ such that $w_-(x) > x$ (respectively, $w_-(x) < x$) for all $x \in (1-\varepsilon, 1)$. An entirely analogous definition can be given for small-probability or large-probability gains.

We conclude this subsection with two important examples of distortions and some of their properties.

Example III.2.9. (i) *The* power distortion *with real parameter* $\beta > 0$ *is the function* $w: [0,1] \to [0,1]$ *given by*

$$w(x) \triangleq x^{\beta}, \qquad x \in [0,1]\,. \tag{III.2.4}$$

Probabilities are always overweighted (respectively, underweighted) when $\beta < 1$ *(respectively,* $\beta > 1$*). Note that, when* $\beta = 1$*,* w *is simply the identity function, thus corresponding to the case where there is no distortion.*

(ii) *The* Prelec distortion *with parameters* $\varpi \in (0,1)$ *and* $\beta > 0$*, proposed by Prelec [45], is defined as*

$$w(x) \triangleq \begin{cases} 0, & \text{if } x = 0, \\ \exp\{-\beta\,[-\log(x)]^{\varpi}\}\,, & \text{if } 0 < x \leq 1. \end{cases} \tag{III.2.5}$$

We remark that allowing for the limiting case where $\varpi = 1$ *would yield the power distortion with parameter* β*.*

As observed in Prelec [45], this function is regressive[6]*, thus small probabilities are overweighted (where the smaller the parameter* ϖ*, the greater the elevation of the function with respect to the first diagonal), whereas large probabilities are underweighted.*

It is also worth noticing that the Prelec distortion is inverse S*-shaped, therefore implying that 'changes in probability have less impact as one moves away from the boundary of the probability interval' Prelec [45, p. 499, ll. 6–7].*

Besides, the Prelec distortion decreases to zero (as $x \to 0^+$*) more slowly than* x^p *for any power* $p > 0$*.*

(iii) *The function* $w: [0,1] \to [0,1]$ *given by*

$$w(x) \triangleq x + x\,(x-1)\,(x-1/2)\,, \qquad x \in [0,1]\,, \tag{III.2.6}$$

is also a regressive and inverse S*-shaped distortion.*

(iv) *The original CPT distortion proposed by Tversky and Kahneman [61, p. 309] is shown below,*

$$w(x) \triangleq \frac{x^{\beta}}{\left(x^{\beta} + (1-x)^{\beta}\right)^{1/\beta}}, \qquad x \in [0,1]\,,$$

[6] We recall that a function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is *regressive* if there is some x_0 in the domain D such that $f(x) > x$ for all $x \in (-\infty, x_0) \cap D$ and $f(x) < x$ for all $x \in (x_0, +\infty) \cap D$.

where $\beta \in (0,1)$. Note that this function is inverse S*-shaped. Additionally, it follows from the trivial inequality* (II.5.1) *that $w(x) \leq x^\beta$ for every $x \in [0,1]$. On the other hand, it is straightforward to check that $w(x) \geq 2^{1-1/\beta} x^\beta$. The mathematical treatment of this distortion can then turn out to be equivalent to* (i) *above.*

2.3 The optimal portfolio problem

In order to characterise how a CPT investor makes decisions in the face of uncertainty, we must start with the following.

Definition III.2.10 (Prospect functional). Given any $\mathscr{F}_T$-measurable random variable $X : \Omega \to \mathbb{R}$, let us denote by $V_\pm(X^\pm)$ the Choquet integral of $u_\pm \circ X^\pm$ with respect to the capacity $w_\pm \circ \mathbb{P}$, that is,

$$V_\pm(X^\pm) \triangleq \int_0^{+\infty} w_\pm\left(\mathbb{P}\{u_\pm(X^\pm) > y\}\right) \, dy. \tag{III.2.7}$$

Then, whenever $V_+(X^+) < +\infty$ or $V_-(X^-) < +\infty$, we define the *prospect functional* of X as follows,

$$V(X) \triangleq V_+(X^+) - V_-(X^-), \tag{III.2.8}$$

possibly taking the values $+\infty$ or $-\infty$.

Remark III.2.11. *(i)* An important observation is that the presence of the probability distortions $w_\pm$ leads to the appearance of the capacities $w_\pm \circ \mathbb{P}$ (not necessarily additive), which in turn results in the involvement of the Choquet integrals $V_\pm$ (not necessarily linear).

(ii) Note that, as remarked in Remark C.5, in the case where there are no probability distortions, that is, when $w_\pm(x) = x$ for Lebesgue-a.e. $x \in [0,1]$, we have that the equalities

$$V_\pm(X^\pm) = \mathbb{E}_\mathbb{P}[u_\pm(X^\pm)]$$

hold for any random variable X. Therefore, the prospect functional V can be regarded as a generalisation of the expected utility, and so our work covers the classic EUT portfolio optimisation. ◇

Now, we recall that, when evaluating a certain portfolio $\overline{\phi}$ in the absence of consumption, a CPT investor is not concerned about the terminal wealth $\Pi_T^{\overline{\phi}}$ (as it was assumed in EUT), but with the gain variable $\Pi_T^{\overline{\phi}} - B$ (which can also take negative values, thus representing a loss).

Next, in order to be able to study the optimisation problem, we must specify the family of all suitable trading strategies, $\mathscr{A}(x_0)$. Because the investor is assumed to have an initial wealth of x_0, we restrict ourselves to the set of trading strategies $\overline{\phi}$ in Ψ for which $\Pi_0^{\overline{\phi}} = x_0$. Another minimal assumption on the set of strategies is concerned with the Choquet integrals $V_+\left(\left[\Pi_T^{\overline{\phi}} - B\right]^+\right)$ and $V_-\left(\left[\Pi_T^{\overline{\phi}} - B\right]^-\right)$. In order to ensure that the prospect functional V given by (III.2.8) *exists* or is *well-defined*, we must require that either one of the aforementioned integrals is finite. All the above can be summarised below.

Definition III.2.12 (Feasible portfolios). A strategy $\overline{\phi} \in \Psi$ is said to be *allowable* or *feasible* for the behavioural portfolio problem (BPP) if it belongs to

$$\mathscr{A}(x_0) \triangleq \left\{\overline{\phi} \in \Psi \colon \Pi_0^{\overline{\phi}} = x_0 \text{ a.s. and } V\left(\Pi_T^{\overline{\phi}} - B\right) \text{ is well-defined}\right\}. \tag{III.2.9}$$

We call $\mathscr{A}(x_0)$ the *feasible set* (or *set of feasible portfolios*).

Remark III.2.13. Whenever X is a claim admitting a replicating portfolio that belongs to the set $\mathscr{A}(x_0)$, by abuse of language we may write 'X is in $\mathscr{A}(x_0)$' or 'X is feasible for (BPP)'. ◇

Consequently, the finite-horizon portfolio choice problem for an investor with CPT preferences consists of selecting the optimal trading strategies, from the set $\mathscr{A}(x_0)$ of all suitable portfolios (to be formally characterised later on), in terms of maximising the expected distorted payoff functional $V\left(\Pi_T^{\overline{\phi}} - B\right)$, which is written below in mathematical terms.

Definition III.2.14 (Behavioural portfolio choice problem). The *behavioural portfolio problem* with initial wealth $x_0 \in \mathbb{R}$ on a finite horizon $T \in (0, +\infty)$ is written as

$$\sup\left\{V\left(\Pi_T^{\overline{\phi}} - B\right) : \overline{\phi} \in \mathscr{A}(x_0)\right\}. \tag{BPP}$$

Setting $v^*(x_0) \triangleq \sup\left\{V\left(\Pi_T^{\overline{\phi}} - B\right) : \overline{\phi} \in \mathscr{A}(x_0)\right\}$, we say that $\overline{\phi}^* \in \mathscr{A}(x_0)$ is an *optimal strategy* if

$$v^*(x_0) = V\left(\Pi_T^{\overline{\phi}^*} - B\right). \tag{III.2.10}$$

Remark III.2.15. For a short discussion on the significance of the existence of an optimal strategy, once again see Remark II.2.21.

With regard to the conditions $u_\pm(0) = 0$, these are imposed for convenience only (cf. Remark II.4.2). ◇

To conclude this subsection, we notice that, in general, when studying optimisation problems such as (BPP) above, some issues may arise. In fact, first of all, it may happen that the supremum in (BPP) is infinite, in which case maximisation does not make sense. Intuitively speaking, it means that the investor can obtain an arbitrarily high degree of satisfaction from the trading strategies that are available in the market. For this reason, we introduce the following notion.

Definition III.2.16 (Well-posedness). A maximisation problem is termed *well-posed* if its supremum is finite. Problems that are not well-posed are said to be *ill-posed*.

Secondly, even if the optimisation problem (BPP) is well-posed, it may still happen that an optimal solution does not exist. Considering this, we state the following.

Definition III.2.17 (Attainability). A well-posed optimisation problem is said to be *attainable* if it admits an optimal solution.

Addressing these issues will be the purpose of the subsequent sections.

2.4 Toy example: one-period binomial model (revisited)

In order to illustrate the importance of the changes introduced by CPT with respect to EUT in the portfolio optimisation problem, let us return to the binomial market model of Subsection II.2.4.

It is a well-known fact that this model is not only arbitrage-free, but also complete. Moreover, the unique EMM is given by $\mathbb{P}^*(\{1\}) = (1-l)\,/\,(h-l) \in (0,1)$.

Finally, assume that the investor has initial wealth $x_0 = 0$ and reference point $B = 0$. Since we allow wealth to become negative, it is then trivial that the set of admissible portfolios coincides with that of self-financing portfolios, which is in turn equal to

$$\Phi(0) = \left\{ \left(\phi^0, \phi^1\right) \in \mathbb{R}^2 \colon \ \phi^0 = -\phi^1 \right\}.$$

We now study three interesting cases. In each figure, we shall plot the function to be maximised as a function of ϕ^1.

Case 1 Firstly, let us suppose that the investor has (negative) exponential utility with parameter $\alpha = 1$, that is,

$$u_+(x) \triangleq 1 - e^{-x} \quad \text{and} \quad u_-(x) \triangleq e^x - 1,$$

for all $x \in [0, +\infty)$. Also, let us take the market parameters $h = 3/2$, $l = 1/2$ and $p = 1/2$, exactly as before. Given that the utility is globally concave, it is trivial that the EUT portfolio choice problem is the following concave maximisation problem,

$$v^*(0) = \sup\left\{ \frac{1}{2}\left(1 - e^{-\phi^1/2}\right) + \frac{1}{2}\left(1 - e^{\phi^1/2}\right) \colon \ \phi^1 \in \mathbb{R} \right\},$$

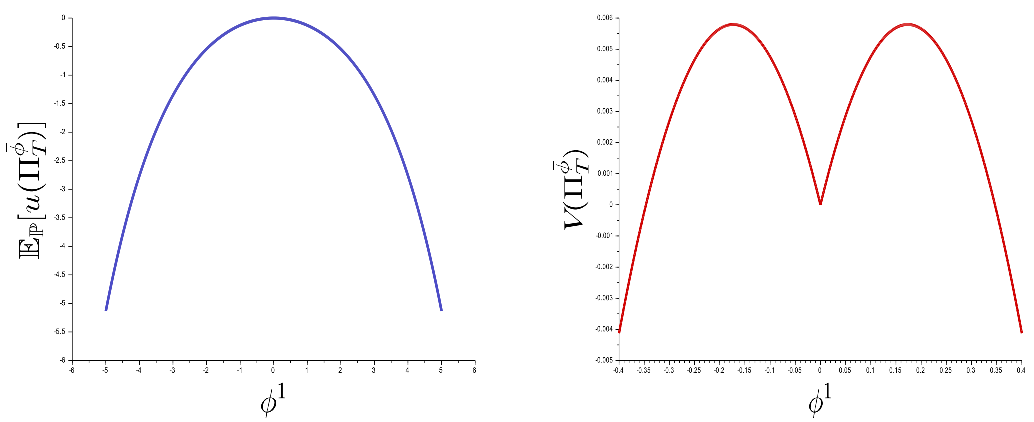
with the unique solution $\overline{\phi}^* = (0,0)$, as shown in Figure III.1a.

Let us now consider behavioural investors with the same exponential utility, and let us assume that they distort the probabilities of gains and losses according to the power functions below,

$$w_+(x) \triangleq x^{1/4} \quad \text{and} \quad w_-(x) \triangleq x^{1/2},$$

for all $x \in [0,1]$. Then the investors would like to find $\phi^1 \in \mathbb{R}$ maximising

$$V\left(\phi^1\left(S_1^1 - 1\right)\right) = \begin{cases} (1/2)^{1/4}\left(1 - e^{\phi^1/2}\right) - (1/2)^{1/2}\left(e^{-\phi^1/2} - 1\right), & \text{if } \phi^1 < 0, \\ (1/2)^{1/4}\left(1 - e^{-\phi^1/2}\right) - (1/2)^{1/2}\left(e^{\phi^1/2} - 1\right), & \text{if } \phi^1 \geq 0. \end{cases}$$



(a) EUT portfolio choice problem. **(b)** Behavioural portfolio choice problem.

Figure III.1: Market parameters $x_0 = 0$, $B = 0$, $u = 3/2$, $l = 1/2$ and $p = 1/2$. Power probability distortions, with parameters $1/4$ (for gains) and $1/2$ (for losses).

Note that, even though the utility is globally concave in this case, due to the presence of the distortion functions the behavioural portfolio problem is neither concave nor convex (see Figure III.1b). Besides, there are now two optimal trading strategies, $(-\log(2)/4, \log(2)/4)$ and $(\log(2)/4, -\log(2)/4)$.

Case 2 Secondly, let us suppose that the investors have a power utility on gains and a power utility on losses, with the same parameter $\alpha_\pm = 1/2$. Let us assume the same regarding the distortion functions, that is, $w_\pm \triangleq x^{1/2}$ for all $x \in [0,1]$. Moreover, let us take $h = 2$, $l = 1/2$ and $p = 1/(\sqrt{2}+1) \in (0, 1/2)$. It is not difficult to check that, for every $\phi^1 \in \mathbb{R}$,

$$V\left(\phi^1\left(S_1^1 - 1\right)\right) = \begin{cases} -\left[-\phi^1/\left(1+\sqrt{2}\right)\right]^{1/2}\left(1 - 2^{-1/4}\right), & \text{if } \phi^1 < 0, \\ \left[\phi^1/\left(1+\sqrt{2}\right)\right]^{1/2}\left(1 - 2^{-1/4}\right), & \text{if } \phi^1 \geq 0, \end{cases}$$

and so $v^*(0) = +\infty$ (cf. Figure III.2). Therefore, the issue of well-posedness is a relevant and recurring one, for even in such a basic example as this the richness of the feasible set can cause the portfolio optimisation problem to be ill-posed. We end with the remark that, had we not considered the distortions, then the problem would be well-posed (and the investor would be indifferent between all of the feasible portfolios).

Case 3 Thirdly, we assume that the investors have (negative) exponential utility, both on losses and on gains,

$$u_\pm(x) \triangleq 1 - e^{-\alpha_\pm x}, \qquad x \geq 0,$$

with $\alpha_+ = 1/2$ and $\alpha_- = 1$. Thus, their utility is bounded above and below. Besides, we suppose that their distortions on gains and on losses are power functions with parameters $\beta_+ = 1/2$ and $\beta_- = 1/4$, respectively. In addition, set $h = 2$, $l = 1/4$ and $p = 1/3$. It

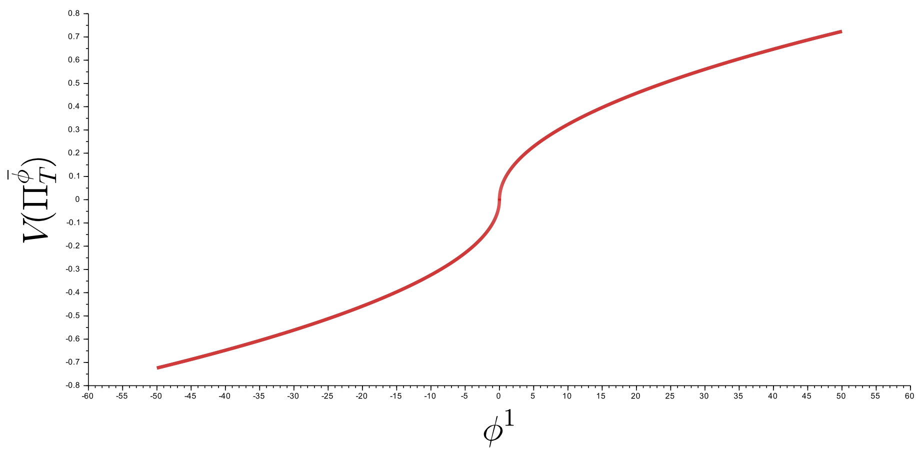


Figure III.2: Ill-posed behavioural portfolio choice problem. Market parameters $h = 2$, $l = 1/2$ and $p = 1/(\sqrt{2}+1)$. Power distortions and power utilities, with parameter $1/2$.

is straightforward to check that the investors aim to maximise

$$V\left(\phi^1\left(S_1^1 - 1\right)\right) = \begin{cases} (2/3)^{1/2}\left(1 - e^{3\phi^1/8}\right) - (1/3)^{1/4}\left(1 - e^{\phi^1}\right), & \text{if } \phi^1 < 0, \\ (1/3)^{1/2}\left(1 - e^{-\phi^1/2}\right) - (2/3)^{1/4}\left(1 - e^{-3\phi^1/4}\right), & \text{if } \phi^1 \geq 0, \end{cases}$$

over $\phi^1 \in \mathbb{R}$. It is trivial that, in this case, $v^*(0) = (2/3)^{1/2} - (1/3)^{1/4}$, so the problem is well-posed, and yet no optimal portfolio exists (as can be intuited from Figure III.3).

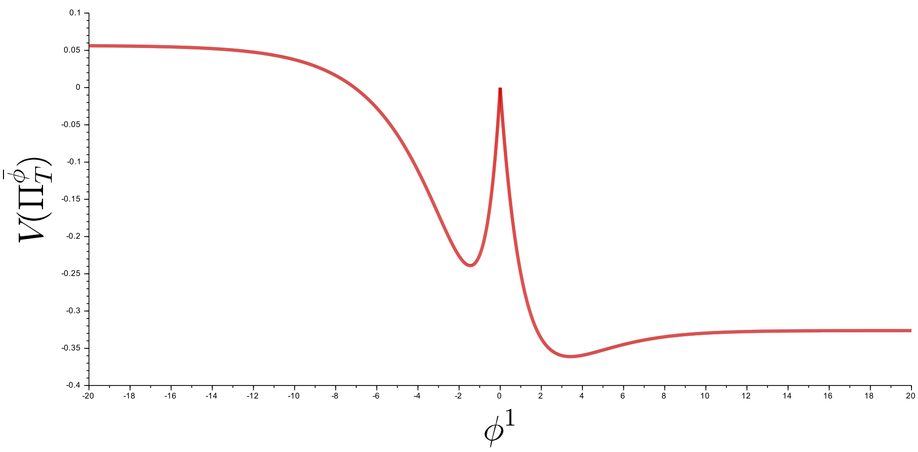


Figure III.3: Non-attainable behavioural portfolio choice problem. Market parameters $h = 2$, $l = 1/4$ and $p = 1/3$. Power distortions and (negative) exponential utilities.

3 General results in continuous time

In this section, as well as in the remaining ones of this chapter, we shall assume that trading occurs continuously in time, and thus $\mathbb{T} \triangleq [0, T]$. Furthermore, we assume that the process S is càdlàg, and that the filtration $\mathbb{F} = \{\mathscr{F}_t \colon t \in [0,T]\}$ satisfies the *usual conditions* of saturatedness and right-continuity. Also, we recall that the alternative characterisation (II.2.4) of the self-financing condition in discrete time can be extended to continuous-time markets as shown.

Definition III.3.1 (Self-financing portfolio). A *self-financing portfolio* is a $d+1$-dimensional, $\mathbb{F}$-predictable stochastic process $\overline{\phi} = \left\{\overline{\phi}_t = (\phi_t^0, \phi_t^\top)^\top ; t \in [0,T]\right\}$, with $\phi_t = (\phi_t^1, \ldots, \phi_t^d)^\top$, which is (S^0, S)-integrable and whose associated *wealth process* $\Pi^{\overline{\phi}} = \left\{\Pi_t^{\overline{\phi}}; t \in [0,T]\right\}$, given by $\Pi_t^{\overline{\phi}} \triangleq \sum_{i=0}^d \phi_t^i S_t^i$, fulfills the self-financing condition

$$\forall\, t \in [0,T]\,, \quad \Pi_t^{\overline{\phi}} = \Pi_0^{\overline{\phi}} + \int_0^t \phi_s \, dS_s \text{ a.s.}. \tag{III.3.1}$$

The family of all self-financing strategies is denoted by Φ.

Example III.3.2. *We recover the* trivial portfolio *of* (II.2.15)*, now in continuous time. As before, $\overline{\varphi}_{x_0}$ consists in investing all of the wealth on the riskless asset and none on the risky assets, i.e.,*

$$(\varphi_{x_0})_t^0 \triangleq x_0 \quad \text{and} \quad (\varphi_{x_0})_t^i \triangleq 0,$$

for all $t \in [0,T]$ and $i \in \{1, \ldots, d\}$. Then its wealth process equals $\Pi_t^{\overline{\varphi}_{x_0}} = x_0$ for all $t \in [0,T]$.

Next, it is a well-established fact in the literature that, without any further restrictions on the set of self-financing portfolios, such arbitrage opportunities as doubling schemes or suicide strategies (see, e.g., Harrison and Pliska [26]) can be found in the market. There are several approaches to ruling out these pathologies, one of the most frequent in the literature (and employed e.g. in Berkelaar et al. [8], Jin and Zhou [29], and Carlier and Dana [15]) being that the (discounted) wealth process of any portfolio should be bounded below by a constant (possibly depending on the portfolio). Although this restriction, which reflects the existence of a credit limit, is not an unrealistic one, it looks rather unnatural when we take into account what was explained in Section III.1. Therefore, in this work, we choose to adopt the following.

Definition III.3.3 (Admissible strategy). A self-financing trading strategy is said to be *admissible* if its (discounted) wealth process is a martingale under $\mathbb{P}^*$ (and not only a local martingale). We represent by Ψ the set of admissible strategies.

Remark III.3.4. The usual admissibility criterion has the advantage that it is invariant with respect to the EMM, whereas ours depends on the $\mathbb{P}^*$ we fixed above. ◇

Moreover, the following two assumptions will be in force throughout. The first one has to do with the reference point. As said before, B can be constant, deterministic or stochastic, but its evolution must be known in some way, which mathematically speaking means imposing the following.

Assumption III.3.5. *The reference point B is hedgeable.*

Remark III.3.6. As noted in Jin and Zhou [29, Remark 2.1], given that B admits a replicating portfolio $\overline{\phi}_B \in \Psi$ by virtue of Assumption III.3.5, the optimal portfolio problem can be reduced, for simplicity and without loss of generality, to one with reference level equal to zero (and possibly different initial capital). Indeed, defining the constant $y_0 \triangleq x_0 - \mathbb{E}_{\mathbb{P}^*}[B]$, it is trivial that a portfolio $\overline{\phi}^* \in \mathscr{A}(x_0)$ is optimal for (BPP) with initial wealth x_0 and reference point B if and only if $\overline{\phi}^* - \overline{\phi}_B \in \mathscr{A}(y_0)$ is optimal for (BPP) with starting capital y_0 and reference point equal to zero. Basically, as noted also in Reichlin [52, Chapter II, Remark 2.5], the investor buys today the hedging portfolio for B for the price $\mathbb{E}_{\mathbb{P}^*}[B]$, and then attempts to allocate the remaining wealth $x_0 - \mathbb{E}_{\mathbb{P}^*}[B]$ in an optimal way. ◇

The second assumption is a kind of completeness hypothesis on the market, although for a certain type of claims only.

Assumption III.3.7. *All random variables in $L^1(\Omega, \sigma(\rho), \mathbb{P}^*)$ are hedgeable.*

Remark III.3.8. It is worth noticing that Assumption III.3.7 is weaker (if only slightly) than the usual notion of market completeness. Indeed, let us suppose that our market model is complete, and consider an arbitrary claim $X \in L^1(\Omega, \sigma(\rho), \mathbb{P}^*)$. It is immediate that $\mathbb{P}^*$ is the only ELMM for S. Moreover, it is obvious that each

$$X_n \triangleq X\, \mathbb{1}_{\{|X| \leq n\}} + n\, \mathbb{1}_{\{X > n\}} - n\, \mathbb{1}_{\{X < -n\}},$$

with $n \in \mathbb{N}$, belongs to $L^\infty(\Omega, \mathscr{F}_T, \mathbb{P})$ (recall that $\sigma(\rho) \subseteq \mathscr{F}_T$), thus by completeness of the market we have that, for some $x_n \in \mathbb{R}$ and some admissible strategy ϕ^n,

$$X_n = x_n + \int_0^T \phi_t^n \, dS_t \text{ a.s..}$$

Now, setting $x \triangleq \mathbb{E}_{\mathbb{P}^*}[X]$ (which is a real number, since $\mathbb{E}_{\mathbb{P}^*}[|X|] < +\infty$ by hypothesis), it is not difficult to check (using Lebesgue's Dominated Convergence Theorem) that the sequence of stochastic integrals $\left\{ \int_0^T \phi_t^n \, dS_t;\, n \in \mathbb{N} \right\}$ converges strongly (hence weakly) in $L^1(\Omega, \mathscr{F}_T, \mathbb{P}^*)$ to $X - x$. In addition, it can be shown that, for every fixed $n \in \mathbb{N}$, the martingale $\left\{ \int_0^t \phi_s^n \, dS_s;\, t \in [0,T] \right\}$ is uniformly integrable. Therefore, we may invoke Corollary 2.5.2 in Yor [64] to deduce the existence of an admissible process ϕ such that $X - x = \int_0^T \phi_t \, dS_t$ a.s., that is, X is hedgeable.

Hence, it is clear that our Assumption III.3.7 holds within a complete market framework, but it can also be satisfied for incomplete financial models, as shown by the example provided in the upcoming Subsection III.3.4. ◇

We shall also require that both ρ and $1/\rho$ should have moments of all strictly positive orders, which will be needed frequently.

Assumption III.3.9. *Both ρ and $1/\rho$ belong to $\mathscr{W}$, where $\mathscr{W}$ is defined as in* (II.4.5) *(i.e., it is the family of all real-valued and $\mathscr{F}_T$-measurable random variables Y satisfying $\mathbb{E}_{\mathbb{P}}[|Y|^p] < +\infty$ for all $p > 0$).*

Finally, we shall impose the following technical assumption as well.

Assumption III.3.10. *The essential supremum of* ρ *with respect to* $\mathbb{P}$, $\operatorname{ess\,sup}_{\mathbb{P}} \rho$, *is infinite.*

3.1 Feasible portfolios

We note that, since the process $\Pi^{\overline{\phi}}$ is a $\mathbb{P}^*$-martingale for every admissible $\overline{\phi}$, having $\Pi_0^{\overline{\phi}} = x_0$ a.s. is equivalent to $\mathbb{E}_{\mathbb{P}^*}\left[\Pi_T^{\overline{\phi}}\right] = x_0$. Thus, we may rewrite

$$\mathscr{A}(x_0) \triangleq \left\{\overline{\phi} \in \Psi\colon\ \mathbb{E}_{\mathbb{P}^*}\left[\Pi_T^{\overline{\phi}}\right] = x_0 \text{ and } V\left(\left[\Pi_T^{\overline{\phi}} - B\right]\right) \text{ is well-defined}\right\}.$$

Now, before turning to the issues of well-posedness and attainability, we would like to ensure that the set of feasible portfolios is not empty, that is, there exists at least one portfolio satisfying all the conditions imposed above. This is guaranteed by the following, rather obvious result.

Lemma III.3.11. *Under Assumption III.3.5, the trivial portfolio of Example III.3.2 is feasible for problem* (BPP). □

Remark III.3.12. The preceding lemma implies $v^*(x_0) \geq -u_-(|x_0|) > -\infty$. ◇

3.2 Well-Posedness

It was mentioned above that it is common practice in the literature to assume a priori that a maximisation problem such as (BPP) should have a finite supremum. Here, on the contrary, before searching for the optimal portfolio for the behavioural investor, we address and provide a detailed study of the important issue of well-posedness, so as to identify and exclude the ill-posed cases. We shall see that ill-posedness may occur very frequently, even in cases that are apparently harmless, and we shall search for conditions which are necessary for well-posedness to hold.

So let us make the additional assumption below.

Assumption III.3.13. *The cumulative distribution function (CDF) of* ρ *under* $\mathbb{P}$*, which we denote by* $F_\rho^{\mathbb{P}}$*, is continuous.*

Remark III.3.14. Due to the fact that $\mathbb{P}^* \sim \mathbb{P}$, the above assumption is equivalent to $F_\rho^{\mathbb{P}^*}$ being continuous. ◇

We now proceed with the statement of a few results which will allow us to impose further conditions, not only on the utility functions u_+ and u_-, but also on the probability distortions w_+ and w_-. The first one is trivial.

Lemma III.3.15. *For every* $\overline{\phi} \in \Psi$, $V_+\left(\left[\Pi_T^{\overline{\phi}} - B\right]^+\right) \leq u_+(+\infty)$. □

Having established the well-posedness of the problem when the utility on gains is bounded above, let us now turn to the case where $u_+(+\infty) = +\infty$, that is, the pleasure that investors obtain from gains can become arbitrarily large. The following result says that, in this case, the utility on losses must also be unbounded, otherwise we will have an ill-posed problem.

Proposition III.3.16. *Suppose that $u_+(+\infty) = +\infty$ and $u_-(+\infty) < +\infty$. If there is an event $A \in \sigma(\rho)$ with $\mathbb{P}(A) \in (0,1)$, then under Assumptions III.3.5 and III.3.7 the problem is ill-posed.*

Proof. See Section III.6, page 109. □

The next proposition states that we must require not only that u_- grows to infinity as $x \to +\infty$, but also that it does so faster than u_+.

Proposition III.3.17. *Setting*

$$l \triangleq \liminf_{x \to +\infty} \frac{u_-(x)}{u_+(x)}, \tag{III.3.2}$$

let us assume that $l \in [0, +\infty)$, $u_+(+\infty) = +\infty$ and $AE_+(u_-) < +\infty$ (see Definition A.4). Suppose further that there exist some $\gamma \in (AE_+(u_-), +\infty)$ and some $A \in \sigma(\rho)$ such that both $\mathbb{P}^*(A) = 1/2$ and

$$w_+(\mathbb{P}(A)) > 2^\gamma l\, w_-(\mathbb{P}(A^c)) \tag{III.3.3}$$

hold true. Then, under Assumptions III.3.5 and III.3.7, the optimisation problem (BPP) is ill-posed, whatever distortions we consider.

Proof. The idea of the proof of this result is as follows. One constructs a sequence of payoffs which may lead to large losses, but also to large gains. Given that the pleasure of the gain overrides the pain of the potential loss, this results in an ill-posed problem. See Section III.6, page 110 for the details. □

Remark III.3.18. *(i)* Note that, had we imposed a loss limit on the wealth (that is, a universal constant such that all wealth should be bounded below by it), then the above argument would no longer hold, so one may say that the richness of our feasible set contributes to the prevalence of ill-posed cases.

(ii) It is worth pointing out that, whenever l in Proposition III.3.17 is equal to zero, it suffices to find an event $A \in \sigma(\rho)$ with $\mathbb{P}^*(A) = 1/2$, in which case the inequality (III.3.3) holds automatically for all γ. Indeed, since $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent, we must have $\mathbb{P}(A) > 0$, which combined with the fact that w_+ is strictly increasing, yields $w_+(\mathbb{P}(A)) > 0$ as well.

(iii) The last assumption in Proposition III.3.17, concerning the constant γ and the event A, is not too restrictive, as it can be shown to be easily satisfied in a large number of examples.

For instance, if $l = 0$, then under Assumption III.3.13 (recall Remark III.3.14) it is always possible to obtain some $a > 0$ such that $\mathbb{P}^*\{\rho \leq a\} = 1/2$. Setting $A \triangleq \{\rho \leq a\} \in \sigma(\rho)$, and making use of part *(ii)* of this remark concludes.

As for the case where l is a strictly positive real number, we provide below a very natural example of a financial market model in which a constant γ and an event A satisfying the conditions of Proposition III.3.17 can be found (see Example III.3.19). ◇

Example III.3.19. *If $AE_+(u_-) < +\infty$, then it is always possible to fix some strictly positive constant $\gamma > AE_+(u_-)$. So let us define the continuous function $f : [0,1] \to \mathbb{R}$ by $f(p) \triangleq w_+(p) - 2^\gamma l\, w_-(1-p)$. Clearly, there exists some $\varepsilon > 0$ such that $f(p) > 0$ for all $p \in (1-\varepsilon, 1]$. On the other hand, choosing $\mu > 0$ to be sufficiently large, we have*

$$\int_{-\infty}^{-\mu} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx < \varepsilon.$$

Next, let the continuous process $W = \{W_t;\, t \in [0,1]\}$ be a one-dimensional Wiener process *(with respect to the probability measure $\mathbb{P}$) starting from zero. We are assuming the flow of information in the market to be represented by $\mathbb{F}^W = \left\{\mathscr{F}_t^W;\, t \in [0,1]\right\}$, the* natural filtration *of W (which is the augmentation, by all $\mathbb{P}$-null sets, of the filtration generated by W). We further recall that the interest rate is here assumed to be null. Finally, the dynamics of the price process $S = \{S_t;\, t \in [0,1]\}$ of the risky asset are described, under the measure $\mathbb{P}$, by the Itô process with stochastic differential*

$$dS_t = \mu S_t dt + S_t dW_t, \qquad S_0 = s > 0,$$

for all $t \in [0,1]$. Thus, setting $\rho \triangleq \exp\left\{-\mu W_1 - \frac{\mu^2}{2}\right\}$, the probability measure $\mathbb{P}^*$ given by $d\mathbb{P}^*/d\mathbb{P} = \rho$ is the unique equivalent martingale measure, and the process $\tilde{W} = \left\{\tilde{W}_t;\, t \in [0,1]\right\}$ defined by $\tilde{W}_t \triangleq W_t + \mu t$ is a $\mathbb{P}^*$ Wiener process.

Now take $A \triangleq \left\{\tilde{W}_1 > 0\right\}$. Clearly $A \in \sigma(\rho)$, $\mathbb{P}^*(A) = 1/2$ and

$$\mathbb{P}(A) = \mathbb{P}\{W_1 + \mu > 0\} = 1 - \mathbb{P}\{W_1 \leq -\mu\} > 1 - \varepsilon,$$

which then guarantees that $w_+(\mathbb{P}(A)) - 2^\gamma l w_-(1 - \mathbb{P}(A)) > 0$, as intended.

Let us now investigate conditions on the distortion functions. We start by seeing that, in order to have well-posedness when $u_+(+\infty) = +\infty$, we must also require that $w_-(x)$ decreases to zero relatively 'slowly' as x goes to zero from above, in the sense that it must approach zero 'more slowly' than a given rate, determined by the utility on losses.

Proposition III.3.20. *Under Assumptions III.3.5, III.3.7, III.3.10 and III.3.13, as well as $u_+(+\infty) = +\infty$, the behavioural portfolio problem* (BPP) *is well-posed only if*

$$\lim_{x \to 0^+} w_-(x)\, u_-\left(\frac{1}{x}\right) = +\infty. \tag{III.3.4}$$

Proof. The basic idea of this proof can be summarised in the following way. We construct a sequence of payoffs, each of which leads, with large probability, to large gains. Nevertheless, there is a small chance that each one of them may lead to a large loss. But, since the pain of the potential loss is overridden by the distorted perception of the likelihood of the said loss, the loss part is controlled, thus yielding an ill-posed problem. See Section III.6, page 111 for the details. □

Example III.3.21. *An elementary calculation shows that the optimal portfolio problem of an investor with unbounded utility on gains, logarithmic utility on losses and Prelec distortion on losses (recall Example III.2.9(ii)) is ill-posed.*

The result below, which comes as an easy consequence of Proposition III.3.20, shows that, whenever the investors' pleasure of a gain can become arbitrarily large and their utility on losses does not grow 'too fast' (put precisely, it has at most linear growth), then overweighting the likelihood of small-probability losses is a necessary condition for the well-posedness of their optimal portfolio problem.

Corollary III.3.22. *Suppose $u_+(+\infty) = +\infty$ and assume further that the two conditions below are true,*

(i) *the utility u_- grows at most linearly, i.e., there exist real numbers $\gamma \in (0, 1]$, $\underline{x} > 0$, $C_1 > 0$ and $C_2 \geq 0$ such that $u_-(x) \leq C_1 x^\gamma + C_2$ for all $x \geq \underline{x}$,*

(ii) *for every $\varepsilon \in (0, 1]$, there exists some $x \in (0, \varepsilon)$ such that $w_-(x) \leq x$.*

Then, under Assumptions III.3.5, III.3.7, III.3.10 and III.3.13, the optimisation problem (BPP) is ill-posed. □

Remark III.3.23. *(i)* Consider the particular case where u_- is concave on $[x_0, +\infty)$ for some $x_0 \geq 0$ (that is, the investors exhibit risk seeking behaviours when facing sufficiently large losses). It is straightforward to check that such a function has at most linear growth, so it follows from the above corollary that a probability distortion on losses is a necessary condition for the well-posedness of (BPP). This is essentially the statement of Theorem 3.2 of Jin and Zhou [29]. So our result replaces the assumption of concavity with a slightly more relaxed one.

Another sufficient (albeit not necessary) condition for u_- to have at most linear growth is that u_- satisfies the celebrated RAE condition, that is, $AE_+(u_-) < 1$ (see Lemma A.8 and Remark A.9).

(ii) It should be noted that, under the assumptions of Theorem 4.4 in Carassus and Rásonyi [13], and regardless of the fact that there is a probability distortion on losses or not, the optimal portfolio problem in the multi-period incomplete financial market model under consideration can be well-posed. Also, Jin and Zhou [29, Remark 3.1] notice that the problem in Berkelaar et al. [8] is well-posed, even though no probability distortion is considered. This is because the wealth process is required to be non-negative in Berkelaar et al. [8]. Summing up these

arguments, we are led to believe that ill-posedness in our continuous-time model is likely to be due precisely to the richness of attainable payoffs (Carassus and Rásonyi [13, Remark 4.10]), as well as to the absence of any constraints on the wealth process. ◇

With respect for the next result, it states in particular that, when the distortion on gains w_+ is going to zero (as $x \to 0^+$) 'more slowly' than $[u_+(1/x)]^{-1}$, again we have ill-posedness.

Proposition III.3.24. *Under Assumptions III.3.5, III.3.7 and III.3.13, if*

$$\limsup_{x \to 0^+} w_+(x)\, u_+\left(\frac{1}{x}\right) = +\infty, \tag{III.3.5}$$

then the problem is ill-posed.

Proof. The idea behind this proof involves constructing a sequence of payoffs which may lead to small-probability large gains and whose large-probability losses are bounded below by a universal constant. Therefore, the pain of losses is controlled, whereas the pleasure of the potential gains can grow arbitrarily large (because the distortion on gains cannot outbalance it). See Section III.6, page 111 for the details. □

Remark III.3.25. *(i)* It is obvious that we have $\limsup_{x\to 0^+} w_+(x)\, u_+(1/x) > 0$ only if $u_+(+\infty) = +\infty$.

(ii) The final observation in the proof of this result shows that it would still be valid even if we imposed the constraint that the wealth should never fall below a certain wealth floor. ◇

Example III.3.26. *An easy calculation shows that, when the investors have a power utility and a Prelec distortion on gains, their optimal portfolio problem is ill-posed.*

An immediate consequence of the preceding result is the following, which says that, if the utility on gains grows 'quickly' (namely, faster than linearly), then underweighting the likelihood of small-probability gains is necessary for well-posedness.

Corollary III.3.27. *Let us assume the following,*

(i) *there exist real numbers $\gamma > 1$, $\underline{x} > 0$, $C_1 > 0$ and $C_2 \geq 0$ such that the inequality $u_+(x) \geq C_1\, x^\gamma + C_2$ is verified for all $x \geq \underline{x}$,*

(ii) *for every $\varepsilon \in (0, 1]$, there exists some $x \in (0, \varepsilon)$ such that $w_+(x) \geq x$.*

Then, under Assumptions III.3.5, III.3.7 and III.3.13, the optimisation problem (BPP) *is ill-posed.* □

Remark III.3.28. We note that this corollary complements Theorem 3.2 of Jin and Zhou [29] (and our Corollary III.3.22), but for the distortion on gains and when the utility on gains has super-linear growth. ◇

The main necessary conditions for well-posedness obtained so far can be briefly compiled as follows.

Theorem III.3.29 (Necessary conditions, unbounded utility on gains). *Suppose $u_+(+\infty) = +\infty$ and $AE_+(u_-) < +\infty$. Then, under Assumptions III.3.5, III.3.7, III.3.10 and III.3.13, the problem is well-posed only if the following three conditions are simultaneously satisfied,*

$$\liminf_{x\to+\infty} \frac{u_-(x)}{u_+(x)} > 0, \tag{III.3.6}$$

$$\limsup_{x\to 0^+} w_+(x)\, u_+\left(\frac{1}{x}\right) < +\infty, \tag{III.3.7}$$

$$\lim_{x\to 0^+} w_-(x)\, u_-\left(\frac{1}{x}\right) = +\infty. \tag{III.3.8}$$

□

Remark III.3.30. *(i)* Note that our conditions (III.3.6) to (III.3.8) are completely independent of the shapes of the functions. In particular, no concavity assumptions are needed, since what is really relevant for this discussion is the behaviour of the utilities as $x \to +\infty$, and that of the distortions for unlikely events (i.e., in a right-neighbourhood of zero). In addition, we observe that none of the conditions involve any market parameter, thus they are related (implicitly) to the market model under consideration only through the set of allowable strategies.

(ii) Furthermore, not only are these conditions very easy to be checked, but they also admit intuitive and financial interpretations.

As a matter of fact, the first condition means that the prospect value function is steeper in the negative domain than in the positive one, or economically speaking, that large losses are experienced more acutely than gains of the same order of greatness, indicating loss aversion. When the limit in (III.3.6) exists, it corresponds to the LLAD measure introduced in He and Zhou [27]. We point out, however, that unlike in the one-period model considered by He and Zhou [27, Theorem 1], in our continuous-time market the condition $\lim_{x\to+\infty} u_-(x)/u_+(x) = +\infty$ alone is insufficient to ensure well-posedness.

With regard to conditions (III.3.7) and (III.3.8), they reflect the fact that the investor's risk preferences and perceptions of reality, both on losses and on gains, have to be well-adjusted or well-calibrated, in the sense that, for instance, the distortion on losses cannot override the pain of a loss, nor can the distortion on gains be overwhelmed by the pleasure of a gain.

(iii) Lastly, a careful inspection of the proofs reveals that, if either one of the conditions (III.3.6) to (III.3.8) is not satisfied, then the maximisation problem is ill-posed even if our set of feasible strategies is restricted to the ones which are tame. ◇

3.3 Attainability

Let us now assume that the optimisation problem (BPP) is well-posed. As pointed out in Section III.2, it may still happen nonetheless that the problem is not attainable. Our aim is therefore to investigate the existence of an optimal trading strategy.

As before, Assumption III.3.5 will be in force. In addition, we introduce the following concept.

Definition III.3.31 (Maximising sequence). We say that a sequence of feasible portfolios $\{\overline{\phi}_n;\, n \in \mathbb{N}\} \subseteq \mathscr{A}(x_0)$ is a *maximising sequence* for (BPP) if

$$v^*(x_0) = \lim_{n\to+\infty} V\left(\Pi_T^{\overline{\phi}_n} - B\right). \tag{III.3.9}$$

Remark III.3.32. Obviously, by the definition of supremum, one can always find at least one such sequence. We note as well that the assumption of well-posedness and Remark III.3.12 imply that $v^*(x_0)$ is a real number. Given that any real sequence converging to a finite limit is bounded, in particular we get $\inf_{n\in\mathbb{N}} V\left(\Pi_T^{\overline{\phi}_n} - B\right) > -\infty$ for every maximising sequence $\{\overline{\phi}_n;\, n \in \mathbb{N}\}$. ◇

Henceforth, we shall impose the following important technical assumptions.

Assumption III.3.33. *For every maximising sequence* $\{\overline{\phi}_n;\, n \in \mathbb{N}\}$*,*

$$\sup_{n\in\mathbb{N}} V_+\left(\left[\Pi_T^{\overline{\phi}_n}\right]^+\right) < +\infty. \tag{III.3.10}$$

Assumption III.3.34. *For every maximising sequence* $\{\overline{\phi}_n;\, n \in \mathbb{N}\}$*, there exists an integrable function* $g : [0,+\infty) \to [0,+\infty)$ *such that the inequality*

$$w_+\left(\mathbb{P}\left\{u_+\left(\left[\Pi_T^{\overline{\phi}_n}\right]^+\right) > y\right\}\right) \leq g(y) \tag{III.3.11}$$

holds true for a.e. $y \geq 0$*.*

Remark III.3.35. *(i)* Under Assumption III.3.5 and the assumption of well-posedness, it is clear by the sub-additivity of the supremum that (III.3.10) implies

$$\sup_{n\in\mathbb{N}} V_-\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right) \leq \sup_{n\in\mathbb{N}} V_+\left(\left[\Pi_T^{\overline{\phi}_n}\right]^+\right) - \inf_{n\in\mathbb{N}} V\left(\Pi_T^{\overline{\phi}_n}\right) < +\infty \tag{III.3.12}$$

(recall Remark III.3.32). Conversely, if (III.3.12) is true, then under the same assumptions, $\sup_{n\in\mathbb{N}} V_+\left(\left[\Pi_T^{\overline{\phi}_n}\right]^+\right) \leq \sup_{n\in\mathbb{N}} V\left(\Pi_T^{\overline{\phi}_n}\right) + \sup_{n\in\mathbb{N}} V_-\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right) < +\infty$. Hence, (III.3.10) and (III.3.12) are equivalent.

(ii) It is clear that Assumption III.3.34 will allow us to use, later on, the reverse Fatou lemma. ◇

So we are finally in the position to state the main result of this subsection, which under certain conditions and whenever (BPP) is well-posed, establishes the existence of an optimal investment strategy.

Theorem III.3.36. *Suppose that* (BPP) *is well-posed, and that Assumptions III.3.5, III.3.7, III.3.9, III.3.13, III.3.33 and III.3.34 are in force. If for every maximising sequence* $\{\overline{\phi}_n;\, n \in \mathbb{N}\}$ *there exists some constant* $\eta > 1$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)^{\eta}\right] < +\infty, \tag{III.3.13}$$

then the problem (BPP) *is attainable.*

Proof. The intuition behind the proof can be summarised as follows. Firstly, we use a compactness property (more specifically, the tightness of the family of laws of the terminal wealths of a maximising sequence of portfolios) to extract a limit distribution. A natural candidate for an optimal portfolio is then one whose terminal wealth has the said distribution. See Section III.6, page 112 for the details. □

Remark III.3.37. Note that combining (III.3.13) with de la Vallée-Poussin's lemma yields in particular that the sequence of losses of a maximising sequence is uniformly integrable with respect to $\mathbb{P}$. If, in addition, Assumption III.3.9 also holds, then this is true under $\mathbb{P}^*$ as well. ◇

3.4 Examples

In this subsection we present two examples of important non-trivial market models to which our results can be applied, for they verify the assumptions imposed above.

Multi-dimensional diffusion model

Let $k \in \mathbb{N}$, and let the continuous process $W = \left\{W_t = \left(W_t^1, \ldots, W_t^k\right)^\top;\, t \in [0,T]\right\}$, taking values in $\mathbb{R}^k$, be a k-dimensional Wiener process (with respect to $\mathbb{P}$), which is initialised at zero a.s.. As in Example III.3.19, we are considering the natural filtration $\mathbb{F}^W = \left\{\mathscr{F}_t^W;\, t \in [0,T]\right\}$. We further recall that the interest rate is here assumed to be null.

In this particular example, the price process of the i-th stock $S^i = \left\{S_t^i;\, t \in [0,T]\right\}$ follows the stochastic differential equation (SDE)

$$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^{k} \sigma_t^{ij} S_t^i dW_t^j, \qquad S_0^i = s_i > 0, \tag{III.3.14}$$

for any $i \in \{1, \ldots, d\}$. Here we assume that the coefficients $\mu^i = \left\{\mu_t^i;\, t \in [0,T]\right\}$ and $\sigma^i = \left\{\sigma_t^i = \left(\sigma_t^{i1}, \ldots, \sigma_t^{id}\right)^\top;\, t \in [0,T]\right\}$, respectively the *appreciation rate* process and the $\mathbb{R}^d$-valued *volatility* process of the i-th risky asset, are $\mathbb{F}^W$-adapted and satisfy $\int_0^T \left|\mu_t^i\right| dt + \int_0^T \sum_{j=1}^k \left|\sigma_t^{ij}\right|^2 dt < +\infty$ a.s., thus ensuring the existence and uniqueness of strong solutions to the SDE (III.3.14). Finally, writing σ_t to denote the $d \times k$ volatility matrix with entries σ_t^{ij}, we assume that, for Lebesgue a.e. $t \in [0,T]$, $\sigma_t \sigma_t^\top$ is non-singular a.s..

We study two cases separately.

Complete Market Let us suppose that there are as many risky assets as sources of randomness, that is, $k = d$. Then it is trivial that there exists a uniquely determined d-dimensional progressively measurable process $\theta = \left\{\theta_t = \left(\theta_t^1, \dots, \theta_t^d\right)^\top ; \, t \in [0,T]\right\}$ such that, for a.e. $t \in [0,T]$,

$$-\mu_t^i = \sum_{j=1}^{d} \sigma_t^{ij} \theta_t^j \text{ a.s.}$$

holds simultaneously for all $i \in \{1, \dots, d\}$. In addition, let us further assume that the condition $0 < \int_0^T \sum_{i=1}^d \left|\theta_t^i\right|^2 dt < +\infty$ is satisfied a.s. and that the strictly positive local martingale $\{\rho_t ; \, t \in [0,T]\}$ given by

$$\rho_t = \exp\left\{\sum_{i=1}^{d} \int_0^t \theta_s^i dW_s^i - \frac{1}{2} \int_0^t \sum_{i=1}^{d} \left|\theta_s^i\right|^2 ds\right\} \text{ a.s.}$$

is indeed a martingale under $\mathbb{P}$. Hence, by the Cameron-Martin-Girsanov theorem and its converse, it is well-known that the probability measure $\mathbb{P}^*$, with Radon-Nikodým derivative $d\mathbb{P}^*/d\mathbb{P} = \rho_T$ a.s., is the unique ELMM for S, and the process defined by $\widetilde{W}_t^i = W_t^i - \int_0^t \theta_s^i \, ds$, for all $t \in [0,T]$ and $i \in \{1, \dots, d\}$, is also a d-dimensional Wiener process, but on the complete filtered probability space $\left(\Omega, \mathscr{F}_T^W, \mathbb{F}^W, \mathbb{P}^*\right)$. Such a financial market is arbitrage-free and complete, hence any integrable contingent claim is hedgeable (so Assumptions III.3.5 and III.3.7 are trivially true). Therefore, provided that Assumptions III.3.9 and III.3.13 are satisfied, our Theorem III.3.36 applies. That is, for example, the case when the market price of risk process θ is deterministic (in which case it is straightforward to check that ρ_T is log-normally distributed both under $\mathbb{P}$ and under $\mathbb{P}^*$). The reader is also referred to Nualart [42, Chapter 2] for conditions ensuring the existence of a continuous law for ρ.

Incomplete Market Assume now that $1 \le d < k$, so there exist more sources of risk than traded stocks. Moreover, in what follows, we shall consider only the case where all coefficients $\mu^i = \left\{\mu^i(t) \, ; \, t \in [0,T]\right\}$ and $\sigma^i = \left\{\sigma^i(t) = \left(\sigma^{i1}(t), \dots, \sigma^{id}(t)\right)^\top ; \, t \in [0,T]\right\}$ are, as made explicit by the notation, deterministic Borelian functions of t. It is then clear that a martingale measure for the d-dimensional process S is not unique. Indeed, the system of linear equations

$$-\mu^i(t) = \sum_{j=1}^{k} \sigma^{ij}(t) \, \theta^j(t) \, , \qquad i \in \{1, \dots, d\} \, , \text{ Lebesgue a.e. } t \in [0,T] \, , \qquad \text{(III.3.15)}$$

has infinitely many solutions, and any $\mathbb{R}^k$-valued process $\theta = \{\theta(t) \, ; \, t \in [0,T]\}$ satisfying (III.3.15) defines a martingale measure $\mathbb{P}_\theta^*$ for S, under suitable conditions. Therefore, albeit admitting no arbitrage opportunities, the market is incomplete. Nonetheless, it is possible to construct a standard k-dimensional Brownian motion (with respect to the probability measure $\mathbb{P}$) starting at zero a.s., which we denote by $\bar{W} = \left\{\bar{W}_t = \left(\bar{W}_t^1, \ldots, \bar{W}_t^k\right)^\top ; t \in [0,T]\right\}$, whose natural filtration coincides with that of W and such that the price process S^i follows (under $\mathbb{P}$) the dynamics

$$dS_t^i = \mu^i(t)\, S_t^i dt + \sum_{j=1}^{d} \bar{\sigma}^{ij}(t)\, S_t^i d\bar{W}_t^j,$$

for all $i \in \{1, \ldots, d\}$, where $\bar{\sigma}(t) = \left[\bar{\sigma}^{ij}(t)\right]_{i,j\in\{1,\ldots,d\}}$ is a deterministic and invertible square matrix of order d, for Lebesgue a.e. $t \in [0,T]$, with entries satisfying $\int_0^T \sum_{i,j=1}^d \left|\bar{\sigma}^{ij}(t)\right|^2 dt < +\infty$. As a consequence, there exists a (unique) deterministic process $\bar{\theta} = \left\{\bar{\theta}(t) = \left(\bar{\theta}^1(t), \ldots, \bar{\theta}^d(t)\right)^\top ; t \in [0,T]\right\}$, taking values in $\mathbb{R}^d$, that solves $-\mu^i(t) = \sum_{j=1}^d \bar{\sigma}^{ij}(t)\, \bar{\theta}^j(t)$, for all $i \in \{1, \ldots, d\}$ and for Lebesgue a.e. $t \in [0,T]$. If we impose the additional condition $0 < \int_0^T \sum_{i=1}^d \left|\bar{\theta}^i(t)\right|^2 dt < +\infty$, then as above the probability measure $\mathbb{P}^*$ given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{\sum_{i=1}^{d} \int_0^T \bar{\theta}^i(s)\, d\bar{W}_s^i - \frac{1}{2}\int_0^T \sum_{i=1}^{d} \left|\bar{\theta}^i(s)\right|^2 ds\right\} \text{ a.s.,}$$

is an ELMM for S, and the process $\widetilde{W} = \left\{\widetilde{W}_t = \left(\widetilde{W}_t^1, \ldots, \widetilde{W}_t^k\right)^\top ; t \in [0,T]\right\}$, with $\widetilde{W}_t^i = \bar{W}_t^i - \int_0^t \bar{\theta}^i(t)\, ds$ if $i \in \{1, \ldots, d\}$ and $\widetilde{W}_t^i = \bar{W}_t^i$ if $i \in \{d+1, \ldots, k\}$, is a k-dimensional Brownian motion on the probability space $\left(\Omega, \mathscr{F}_T^W, \mathbb{P}^*\right)$. Moreover, it is clear that $\mathbb{F}^W$ and the natural filtration of $\widetilde{W}$ agree, and that the Radon-Nikodým derivative $d\mathbb{P}^*/d\mathbb{P}$ is log-normally distributed (under the measures $\mathbb{P}$ and $\mathbb{P}^*$), so again Assumptions III.3.9, III.3.10 and III.3.13 are verified. Finally, let $\mathbb{G} = \{\mathscr{G}_t;\, t \in [0,T]\}$ be the natural filtration of the $\mathbb{R}^d$-valued Brownian motion $\left\{\left(\widetilde{W}_t^1, \ldots, \widetilde{W}_t^d\right)^\top ; t \in [0,T]\right\}$, which is a sub-filtration of $\mathbb{F}^W$. It is obvious that $d\mathbb{P}^*/d\mathbb{P}$ is measurable with respect to $\mathscr{G}_T$, and that any $\mathscr{G}_T$-measurable and $\mathbb{P}^*$-integrable random variable is hedgeable, hence Assumption III.3.7 is valid (and weaker, if only slightly, than the standard notion of market completeness).

A three-asset jump-diffusion model with constant coefficients

It is possible to construct examples of models with jumps to which Theorem III.3.36 applies, a very simple one being presented below.

Let us assume that the market contains a riskless asset with null interest rate, and two risky assets whose price processes have dynamics given by

$$dS_t^1 = S_{t^-}^1 \left(\sigma_1\, dW_t^1 + \nu\, dM_t\right), \qquad S_0^1 = s_1 > 0,$$
$$dS_t^2 = S_{t^-}^2 \left(\mu\, dt + \sigma_2\, dW_t^2\right), \qquad S_0^2 = s_2 > 0,$$

where W^1, W^2 are two independent Wiener processes, M is the compensated martingale associated with a Poisson process N with constant intensity λ (i.e., $M_t \triangleq N_t - \lambda t$, $t \in [0, T]$) which is independent of both W^1 and W^2, and where the coefficients satisfy $\mu \neq 0$, $\sigma_1, \sigma_2 > 0$, and $\nu > -1$ (note that this ensures that the price process S^1 remains strictly positive, since its jumps occur when N jumps and $S^1_t = S^1_{t-}(1 + \nu\,\Delta N_t)$). Note that we are considering the filtration $\mathbb{F}$ to be the natural filtration of the triple (W^1, W^2, M). We observe further that S^1 is already a martingale with respect to the historical probability measure.

It also is trivial to see that there exist infinitely many ELMM for (S^1, S^2), and thus the market is not complete. Nevertheless, if we fix the ELMM $\mathbb{P}^*$ with Radon-Nikodým density given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{-\frac{\mu}{\sigma_2} W^2_T - \frac{\mu^2}{2\sigma_2^2} T\right\} \text{ a.s.},$$

it is obvious that all of the Assumptions III.3.7, III.3.9, III.3.10 and III.3.13 are fulfilled. We do not enter into more details here as the examples we can construct in the discontinuous case look rather artificial.

4 Bounded utility on gains

The results presented in this section will appear in Rásonyi and Rodrigues [48].

As it is widely stated in the literature, the paper by Menger [37] (whose English translation can be found in [38]) appears to have been the first to assert the necessity of a boundedness assumption on the utility function in order to avoid a St. Petersburg-type paradox. Even though this has lead to a considerable amount of debate, several authors have since advocated and made further arguments for considering bounded utilities (see e.g. Arrow [4, 5, 6], Markowitz [36], and Savage [57], to cite only a few). The existence in the real world of utilities which are unbounded above is peremptorily dismissed as 'absurd' by Muraviev and Rogers [41, footnote 3, p. 272], who provide a strong argument against unbounded utilities (which they attribute to Kenneth Arrow). In the light of the above, we shall make the following.

Assumption III.4.1 (Bounded utility on gains). *The utility on gains is bounded, i.e., $u_+(+\infty) < +\infty$.*

Remark III.4.2. Some of the best-known and most commonly used utilities in the literature satisfy this assumption, see Example II.2.15*(i)*, *(ii)* and *(v)*. ◇

As Remark III.4.6 below shows, we cannot impose that the utility should be bounded below as well, because this would contradict the existence of an optimiser.

4.1 Feasible portfolios

If follows easily from Lemma III.3.15 that, when the investors' utility is bounded above, the feasible set consists of all admissible portfolios starting from initial wealth x_0.

Proposition III.4.3. *Under Assumption III.4.1,* $\mathscr{A}(x_0) = \left\{\overline{\phi} \in \Psi \colon \mathbb{E}_{\mathbb{P}^*}\left[\Pi_T^{\overline{\phi}}\right] = x_0\right\}$. □

4.2 Well-posedness

Another immediate consequence of Lemma III.3.15, as observed in the previous section, is that well-posedness is trivial in our current setting (the best that can happen to the investor is always bounded above).

Proposition III.4.4. *Under Assumption III.4.1,* $v^*(x_0) < +\infty$. □

4.3 Attainability

It may still be the case that an optimal solution does not exist, so we must now study whether or not the finite supremum $v^*(x_0)$ is indeed a maximum. A first and important answer is given by the following result.

Theorem III.4.5 (Necessary condition I). *Under Assumptions III.3.5, III.3.7, III.3.10 and III.3.13 and Assumption III.4.1, there exists an optimal portfolio for problem* (BPP) *only if*

$$\liminf_{x \to 0^+} w_-(x)\, u_-\left(\frac{1}{x}\right) > 0. \tag{III.4.1}$$

Proof. See Section III.6, page 116. □

Remark III.4.6. *(i)* This result can be seen as the analogue of Proposition III.3.20, but for bounded u_+ and on the issue of attainability. We notice once more that the necessary condition (III.4.1), involving only the losses, depends neither on the market parameters nor on the shapes of the functions. Moreover, besides being easy to verify, it has an evident financial interpretation (already given in Remark III.3.30).

(ii) In particular, Theorem III.4.5 implies that, if we have $u_-(+\infty) < +\infty$ as well, then the optimisation problem is not attainable. Although many authors argue in favour of such u_- (see e.g. Muraviev and Rogers [41]), for the remainder of this section we shall only consider the case where u_- is not bounded above.

(iii) Considering the specific case where both u_- and w_- are power functions, respectively with parameters $\alpha_- > 0$ and $\beta_- > 0$, there is an optimal strategy only if $\alpha_- \geq \beta_-$. Moreover, trivial modifications in the proof of Theorem III.4.5 show that, when $\alpha_- = \beta_-$ (and thus $\lim_{x\to 0^+} w_-(x)\, u_-(1/x) = 1$), existence of an optimal portfolio still does not hold.

(iv) Another interesting conclusion which can be drawn is that, under additional conditions on the growth of u_-, the investor must overweight the likelihood of small-probability losses, otherwise there is no optimal portfolio (the hypothesis on u_- and the proof being exactly the same as those of Corollary III.3.22, except that

now linear growth of u_- is not allowed). This complements Theorem 3.2 of Jin and Zhou [29], but for a bounded u_+ and on the issue of attainability. ◇

Example III.4.7. *An investor with a logarithmic utility and a Prelec distortion on losses does not admit an optimal trading strategy (cf. Example III.3.21).*

Motivated by Theorem III.4.5, we introduce the following concept.

Definition III.4.8 (Associated distortion). Given a real number $\delta > 0$ and a continuous, strictly increasing function $u : [0, +\infty) \to [0, +\infty)$ with $u(+\infty) = +\infty$, let us define the function $w_u^\delta : [0, 1] \to [0, 1]$ in the following way,

$$w_u^\delta(x) \triangleq \begin{cases} 0, & \text{if } x = 0, \\ \left[u(1) / u(1/x)\right]^\delta, & \text{if } x \in (0, 1]. \end{cases} \tag{III.4.2}$$

We call w_u^δ the *distortion associated* with u with parameter δ.

Remark III.4.9. It is a trivial exercise to check that, for every appropriate choice of u and δ, the function w_u^δ defined above is continuous and strictly increasing on $[0, 1]$, with $w_u^\delta(0) = 0$ and $w_u^\delta(0) = 0$, whence a valid probability distortion. We note further that investors having distortion w_u^δ overweight (respectively, underweight) small probabilities if their utility function verifies $u(x) < u(1)\, x^{1/\delta}$ (respectively, $u(x) > u(1)\, x^{1/\delta}$) for sufficiently large x. Similarly, they underweight (respectively, overweight) large probabilities whenever $u(x) < u(1)\, x^{1/\delta}$ (respectively, $u(x) > u(1)\, x^{1/\delta}$) for all x in a right-neighbourhood of 1. ◇

Example III.4.10. (i) *If u is a power function with parameter $\alpha > 0$, then its associated distortion with parameter $\delta > 0$ is also a power function (with exponent $\alpha\delta > 0$).*

(ii) *Let u be the utility of Example II.2.15(vii) with parameters $\alpha > 0$ and $\varpi \in (0, 1)$. Then, for every $\delta > 0$, its associated distortion is the Prelec distortion with parameters $\alpha\delta > 0$ and $\varpi \in (0, 1)$.*

The following corollary to Theorem III.4.5 is now immediate and tells us that, in the particular case where the distortion on losses is the distortion associated with u_- for some parameter $\delta > 0$, a necessary condition for attainability is that $\delta \leq 1$.

Corollary III.4.11 (Necessary condition II). *Let $u_-(+\infty) = +\infty$ and $\delta > 0$. Suppose that the investor's probability weighting on losses satisfies*

$$w_-(x) \leq w_{u_-}^\delta(x), \qquad x \in [0, 1]. \tag{III.4.3}$$

Under Assumptions III.3.5, III.3.7, III.3.10 and III.3.13 and Assumption III.4.1, the optimal portfolio problem (BPP) is attainable only if $\delta \leq 1$. □

Therefore, when the parameter δ is strictly greater than 1, by the preceding result we know that the supremum in (BPP) is never attained. The same conclusion also holds with $\delta = 1$ for some fairly typical utility functions (see Remark III.4.6*(iii)* above).

The remainder of this section will be devoted to arguing that the condition $\delta < 1$ is not only 'almost necessary', but also sufficient to ensure that an optimal trading strategy does in fact exist, under an additional hypothesis on u_- below.

Assumption III.4.12. *For every $\delta \in (0,1)$, there is some $\xi > 1$ such that*

$$\lim_{x \to +\infty} \frac{\left[u_-(x^\xi)\right]^\delta}{u_-(x)} = 0. \tag{III.4.4}$$

As an almost reciprocal of Corollary III.4.11, we have the following.

Theorem III.4.13 (Sufficient condition). *Suppose u_- and $w^\delta_{u_-}$ are respectively as in the statement of Corollary III.4.11 and Definition III.4.8, and that*

$$w_-(x) \geq w^\delta_{u_-}(x)\,, \quad \text{for all } x \in [0,1]. \tag{III.4.5}$$

Under Assumptions III.3.5, III.3.7, III.3.9 and III.3.13, as well as Assumptions III.4.1 and III.4.12, if $\delta \in (0,1)$, then there exists an optimal strategy.

Proof. See Section III.6, page 116. □

Hence, Corollary III.4.11 and Theorem III.4.13 show that $[u_-(1/x)]^{-1}$ can be regarded as the threshold for the distortion function as far as the existence of an optimal portfolio is concerned. Below this, in the sense of $\delta < 1$, attainability holds. Above this, when $\delta > 1$ (or, for some cases, also when $\delta = 1$), it does not.

Finally, the aim of the last result of this section is to show that, however artificial Assumption III.4.12 may seem at first glance, it is actually associated with the renowned concept of asymptotic elasticity.

Lemma III.4.14. *Suppose $u_-(+\infty) = +\infty$, and let $z_- : [0,+\infty) \to [0,+\infty)$ be the transform of u_- given by $z_-(x) \triangleq \log(u_-(e^x))$, for all $x \geq 0$. If there exist $\gamma > 0$ and $\underline{x} > 0$ such that*

$$z_-(\lambda x) \leq \lambda^\gamma z_-(x)\,, \quad \text{for all } \lambda \geq 1 \text{ and } x \geq \underline{x}, \tag{III.4.6}$$

then Assumption III.4.12 is satisfied.

Proof. See Section III.6, page 116. □

Remark III.4.15. When z_- is continuously differentiable on $(x_0, +\infty)$, for some $x_0 \geq 0$, we know by Proposition A.10 that condition (III.4.6) is equivalent to $AE_+(z_-) < +\infty$. ◇

We finish this section with an important example demonstrating that Assumption III.4.12 is satisfied by a large class of functions, namely it is true for some of the most frequently considered utilities in the literature.

Example III.4.16. (i) *Suppose u_- is continuously differentiable and $AE_+(u_-) < +\infty$. If, in addition, there exist constants $C > 0$, $\gamma > 0$ such that $u_-(x) \geq C\,x^\gamma$ holds true for all x sufficiently large, then u_- satisfies Assumption III.4.12. Indeed,*

$$\frac{x\ (z_-)'(x)}{z_-(x)} \leq \frac{x\ (z_-)'(x)}{\log(C) + \gamma x} = \frac{(z_-)'(x)}{(\log(C)\,/x) + \gamma}$$

for every sufficiently large x, therefore $AE_+(z_-) \leq \gamma^{-1} \limsup_{x\to+\infty} (z_-)'(x)$. But, as noted in Kramkov and Schachermayer [35, p. 946], it is trivial to check that $\limsup_{x\to+\infty} (z_-)'(x) = AE_+(u_-)$, which is finite by hypothesis, hence the claimed result is given by Lemma III.4.14.

In particular, this implies that the power utility function with parameter $\alpha > 0$ (not necessarily strictly less than one), having asymptotic elasticity equal to α, verifies Assumption III.4.12. Moreover, so do the utilities of items (vi) *and* (viii) *of Example II.2.15, which have asymptotic utility* 1.

(ii) *Let u_1 be the utility of Example II.2.15*(vii) *with parameters $\alpha > 0$ and $\varpi \in (0, 1)$, u_2 the logarithmic utility, and u_3 the log-log utility. Their transforms, z_1, z_2 and z_3, respectively, equal $z_1(x) = \alpha\, x^\varpi$, $z_2(x) = \log(\log(1 + e^x))$, and $z_3(x) = \log(\log(1 + \log(1 + e^x)))$, for all $x \geq 0$. It can be checked that these functions are strictly concave, hence $AE_+(z_i) \leq 1$ for all $i \in \{1, 2, 3\}$ (see, e.g., Kramkov and Schachermayer [35, Lemma 6.1]).*

(iii) *Assume $u_-(+\infty) = +\infty$, and also that $(u_-)'$ exists and tends to 0 fast enough as $x \to +\infty$, or more precisely, $(u_-)'(x) \leq C\,[x\ \log(x)]^{-1}$ for some $C > 0$ and for x large enough. Then Assumption III.4.12 is fulfilled. Indeed,*

$$\limsup_{x\to+\infty} \frac{x\ (z_-)'(x)}{z_-(x)} = \limsup_{x\to+\infty} \frac{x\, e^x\ (u_-)'(e^x)}{u_-(e^x)\log(u_-(e^x))} \leq \lim_{x\to+\infty} \frac{C}{u_-(e^x)\log(u_-(e^x))} = 0.$$

5 Power utilities and distortions

Most of the material in this section has been published in the paper [47].

It was briefly mentioned in Section III.4 that the requirement that utilities must be bounded has not been unanimously accepted in the literature. On the contrary, there are divided opinions, giving origin to some productive and resolute discussions (see, for example, the paper by Ryan [54] and the response of Arrow [6]). One of the recurrent objections to the assumption of boundedness is that it excludes power and logarithmic utilities, two of the most often used functions in economics. Thus, many papers have been dedicated to supporting and trying to accommodate the use of unbounded utilities (we refer to the works, e.g., of Fishburn [24], Samuelson [56] and Toulet [60]). Hence, having dealt with bounded utilities on gains in the preceding section, we now turn to investors whose utility function is unbounded above. We then know by Section III.3 that they must also have a utility function which is unbounded below (otherwise the problem is automatically ill-posed, see Proposition III.3.16). Inspired by Carassus and Rásonyi [13], we shall focus our study on power-like utilities (with strictly positive parameters, to ensure that the functions are strictly increasing).

Assumption III.5.1 (Power-like utilities). *The investor has a piecewise power-like utility function, that is, there exist strictly positive real numbers α_+, α_-, κ_+ and κ_- such that, for all $x \in [0, +\infty)$,*

$$u_+(x) \leq \kappa_+ x^{\alpha_+} \quad \text{and} \quad u_-(x) \geq \kappa_- x^{\alpha_-}. \tag{III.5.1}$$

Remark III.5.2. This choice is not as artificial or arbitrary as it may seem. Firstly, the power utility is one of the most applied in finance and economics. Secondly, if we assume that the utility on gains is a power function, then we know by Proposition III.3.17 that taking for the utility on losses a function which grows more slowly than any strictly positive power (such as the logarithmic or the log-log utilities) would inevitably lead, under additional assumptions, to an ill-posed problem. Conversely, if u_- is a power function, then taking for u_+ a function which grows more quickly than any strictly positive power would also yield an ill-posed problem (again by Proposition III.3.17). ◇

With regard to the distortions, and motivated by Definition III.4.8, we shall assume that the investor's distortion on gains and distortion on losses behave like the distortions associated with power functions. Recalling Example III.4.10*(i)*, that means imposing the following.

Assumption III.5.3 (Power-like distortions). *The investor has power-like weighting functions, i.e., there exist real numbers $\beta_+, \beta_- > 0$, $k_+ \geq 1$ and $k_- \in (0, 1]$ such that,*

$$w_+(x) \leq k_+ x^{\beta_+} \qquad \text{and} \qquad w_-(x) \geq k_- x^{\beta_-}. \tag{III.5.2}$$

for all $x \in [0, 1]$.

5.1 Well-posedness

Like in the preceding Section III.3, we are concerned with seeking conditions on the parameters under which the portfolio problem is a well-posed one. In order to do so, we shall make the additional assumption below.

Assumption III.5.4. *The essential infimum of ρ with respect to $\mathbb{P}$, $\operatorname{ess\,inf}_{\mathbb{P}} \rho$, is zero.*

Then our first result, which is a direct consequence of Theorem III.3.29 and of an easy adaptation of its proof, essentially recovers Theorem 3.9 in Rásonyi and Rodrigues [47], while also including the borderline cases $\alpha_\pm = \beta_\pm$.

Theorem III.5.5 (Necessary conditions). *Assume that both the utilities u_+, u_- and the distortions w_+, w_- are power functions with strictly positive parameters α_+, α_-, β_+ and β_-, respectively. Suppose further that there exist a real number $\gamma > \alpha_-$ and an event $A \in \sigma(\rho)$ verifying both $\mathbb{P}^*(A) = 1/2$ and $[\mathbb{P}(A)]^{\beta_+} > 2^\gamma [1 - \mathbb{P}(A)]^{\beta_-}$. Under As-*

sumptions III.3.5, III.3.7, III.3.10 and III.3.13 and Assumption III.5.4, the behavioural portfolio problem (BPP) *is well-posed only if*

$$\alpha_+ < \alpha_- \qquad \text{and} \qquad \frac{\alpha_+}{\beta_+} < 1 < \frac{\alpha_-}{\beta_-}. \tag{III.5.3}$$

Proof. See Section III.6, page 117. □

Remark III.5.6. *(i)* Again we stress that these conditions are extremely easy to verify and are not associated with any of the market parameters.

(ii) The condition $\alpha_+ < \alpha_-$ is also mentioned in Bernard and Ghossoub [9] and He and Zhou [27] for one-period models, and in Carassus and Rásonyi [13] and Rásonyi and Rodríguez-Villarreal [49] for multi-period markets. In particular, in He and Zhou [27], the authors conclude that the restriction $\alpha_+ < \alpha_-$ alone (without any further conditions on the distortions) is enough to ensure well-posedness. This statement is no longer true in our continuous-time market model.

(iii) The financial meaning of the parameter restrictions given by (III.5.3) has already been discussed in Remark III.3.30. Here we explore further some interesting special cases.

Suppose that the investor is risk-seeking on losses ($\alpha_- < 1$). Thus, it follows from loss aversion (the first inequality in (III.5.3)) that investor must be risk-averse when gaining ($\alpha_+ < 1$). Moreover, the investor needs to inflate the probabilities of losses occurring ($\beta_- < 1$, cf. Corollary III.3.22). With respect to the distortion on gains, we remark that all three cases $\beta_+ < 1$ or $\beta_+ = 1$ or $\beta_+ > 1$ are possible (so long as $\alpha_+ < \beta_+$ holds).

On the other hand, it may happen that the investors exaggerate the likelihood of gains ($\beta_+ < 1$), in which case they must be risk-averse when gaining ($\alpha_+ < 1$).

Note, however, that we do not impose that the exponents must be strictly less than one. In particular, we allow for the case where the investor is risk-neutral or risk-seeking on gains ($\alpha_+ \geq 1$), which in turn implies that the perceived probabilities of gains must be no greater than the actual probabilities ($\beta_+ > 1$, cf. Corollary III.3.27). In addition, loss aversion gives that the investor cannot be risk-seeking on losses ($\alpha_- > 1$).

Another possible case is that in which either there is no distortion on losses or the investor underweights the probabilities of losses ($\beta_- \geq 1$). Then, the problem is ill-posed unless the investor is risk-averse when losing ($\alpha_- > 1$).

In any case, the condition reflecting loss aversion always introduces a kink in the utility function at zero. What is more, the investor's utility must be steeper on losses than on gains. ◇

The rest of this subsection is devoted to the proof of the reciprocal of Theorem III.5.5, and thus to showing that the inequalities in (III.5.3) above are sharp.

Theorem III.5.7 (Sufficient conditions). *Under Assumption III.3.9 and Assumptions III.5.1 and III.5.3, if*

$$\alpha_+ < \alpha_- \qquad \textit{and} \qquad \frac{\alpha_+}{\beta_+} < 1 < \frac{\alpha_-}{\beta_-}, \tag{III.5.3}$$

then the behavioural portfolio problem (BPP) *is well-posed.*

Proof. See Section III.6, page 118. □

Remark III.5.8. It is worth comparing Theorem III.5.7 to a result of Carassus and Rásonyi [13]. In a discrete-time, multi-period market model, and adapting their notation to ours, well-posedness is obtained in Carassus and Rásonyi [13, Theorem 4.4] whenever

$$\frac{\alpha_+}{\beta_+} < \alpha_-, \tag{III.5.4}$$

modulo some integrability conditions related to the price process. One can check, using Proposition 7.1 of Rásonyi and Stettner [50], that the integrability conditions of Carassus and Rásonyi [13] imply the existence of a risk-neutral measure $\mathbb{P}^* \sim \mathbb{P}$ with $d\mathbb{P}^*/d\mathbb{P}$ and $d\mathbb{P}/d\mathbb{P}^*$ in $\mathscr{W}$, hence our Theorem III.5.7 ensures well-posedness also in the case

$$\alpha_+ < \alpha_- \qquad \text{and} \qquad \frac{\alpha_+}{\beta_+} < 1 < \frac{\alpha_-}{\beta_-}. \tag{III.5.3}$$

Note that (III.5.4) and (III.5.3) are incomparable conditions, as none of them implies the other one. It is also worth emphasizing that the domains of optimization are different in Carassus and Rásonyi [13] and in the present work. Hence, in the discrete-time multi-period case, our Theorem III.5.7 complements, but does not subsume, the corresponding results of Carassus and Rásonyi [13]. ◇

5.2 Attainability

The last theorem of this section states that, if the parameters satisfy the inequalities in (III.5.3), then under suitable assumptions not only is (BPP) well-posed, but also it is attainable.

Theorem III.5.9. *Suppose Assumptions III.3.5, III.3.7, III.3.9 and III.3.13 and Assumptions III.5.1 and III.5.3 are in force. If*

$$\alpha_+ < \alpha_- \qquad \textit{and} \qquad \frac{\alpha_+}{\beta_+} < 1 < \frac{\alpha_-}{\beta_-}, \tag{III.5.3}$$

then there is an optimal portfolio for (BPP)*.*

Proof. See Section III.6, page 118. □

Remark III.5.10. Our Theorems III.5.7 and III.5.9 apply, in particular, to the original CPT distortion functions proposed by Tversky and Kahneman [61] with $k_+ = 1$ and $k_- = 2^{1-1/\beta} \in (0, 1]$ (recall Example III.2.9*(iv)*). ◇

6 Proofs and auxiliary results

6.1 Auxiliary results

Unless otherwise stated, we stay in the setting of Section III.3.

Lemma III.6.1. *Suppose Assumptions III.3.5, III.3.7, III.3.10 and III.3.13 hold true, and set*

$$l \triangleq \liminf_{x \to 0^+} w_-(x)\, u_-\left(\frac{1}{x}\right). \tag{III.6.1}$$

If $l < +\infty$, then $v^*(x_0) \geq u_+(+\infty) - l$.

Proof. Given that both w_- and u_- are non-negative functions, it is trivial that the limit inferior in (III.6.1) must belong to $[0, +\infty]$. Using the hypothesis that $l < +\infty$, it is possible to recursively construct a strictly decreasing sequence $\{a_n;\, n \in \mathbb{N}\}$, whose terms satisfy $a_n \in (0, 1/n)$ and $w_-(a_n)\, u_-(1/a_n) < l + 1/n$ for every $n \in \mathbb{N}$. It is also clear that this sequence must converge to zero as n goes to infinity. Moreover, we can use Assumption III.3.13 to find, for every $n \in \mathbb{N}$, some real number b_n such that $\mathbb{P}\{\rho \leq b_n\} = 1 - a_n$. We remark that, since $\rho > 0$ a.s., each b_n must be strictly positive. We further notice that strict monotonicity of $\{a_n;\, n \in \mathbb{N}\}$ implies that the sequence $\{b_n;\, n \in \mathbb{N}\}$ is strictly increasing. Lastly, we claim that $\lim_{n \to +\infty} b_n = +\infty$. Indeed, denoting by b the supremum of $\{b_n;\, n \in \mathbb{N}\}$, the continuity from below of $\mathbb{P}$ yields

$$\mathbb{P}\{\rho \leq b\} \geq \mathbb{P}\left(\bigcup_{n=1}^{+\infty} \{\rho \leq b_n\}\right) = \lim_{n \to +\infty} \mathbb{P}\{\rho \leq b_n\} = 1 - \lim_{n \to +\infty} a_n = 1,$$

therefore we must have $b = +\infty$ (otherwise b would be an essential upper bound for ρ, thus contradicting Assumption III.3.10).

Now, for every $n \in \mathbb{N}$, we define the event $A_n \triangleq \{\rho \leq b_n\} \in \sigma(\rho)$ (we remark that $\mathbb{P}^*(A_n) \in (0,1)$, because $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent measures), as well as the non-negative and $\sigma(\rho)$-measurable random variable

$$X_n \triangleq \frac{b_n}{2\,\mathbb{P}^*(A_n)} \mathbb{1}_{A_n}.$$

In addition, combining the continuity from below with the equivalence of $\mathbb{P}$ and $\mathbb{P}^*$, it is straightforward to see that $\lim_{n \to +\infty} \mathbb{P}^*(A_n) = \mathbb{P}^*\big(\bigcup_{n=1}^{+\infty} \{\rho \leq b_n\}\big) = 1$, so

$$V_+(X_n) = u_+\left(\frac{b_n}{2\,\mathbb{P}^*(A_n)}\right) w_+(\mathbb{P}(A_n)) \xrightarrow[n \to +\infty]{} u_+(+\infty)\, w_+(1).$$

Next, let

$$Y_n \triangleq \frac{b_n - 2x_0}{2\,\mathbb{P}^*(A_n^c)} \mathbb{1}_{A_n^c},$$

which is also $\sigma(\rho)$-measurable. Given that $\lim_{n \to +\infty} b_n = +\infty$ as seen above, there exists an integer n_0 such that $b_n > 2x_0$ for any $n \geq n_0$, thus $Y_n \geq 0$ for every $n \geq n_0$.

Furthermore, it follows from $\lim_{n\to+\infty}(b_n - 2x_0)/(2b_n) = 1/2$ that there must be some $n_1 \in \mathbb{N}$ such that $(b_n - 2x_0)/(2b_n) < 1$ for all $n \geq n_1$. Combining these facts with the inequality $\mathbb{P}^*(A_n^c) = \mathbb{E}_{\mathbb{P}}\left[\rho \mathbb{1}_{A_n^c}\right] \geq b_n\, \mathbb{P}(A_n^c)$ and with the monotonicity of u_- yields

$$V_-(Y_n) = u_-\left(\frac{b_n - 2x_0}{2\,\mathbb{P}^*(A_n^c)}\right) w_-(\mathbb{P}(A_n^c)) \leq u_-\left(\frac{1}{\mathbb{P}(A_n^c)}\right) w_-(\mathbb{P}(A_n^c)) < l + \frac{1}{n}$$

for every $n \geq \max\{n_0, n_1\}$.

Hence, setting $Z_n \triangleq X_n - Y_n$, for $n \in \mathbb{N}$, it is obvious that Z_n is $\sigma(\rho)$-measurable, and also that $\mathbb{E}_{\mathbb{P}^*}[Z_n] = x_0$ by construction. Besides, for every $n \geq n_0$, we have $\mathbb{E}_{\mathbb{P}^*}[|Z_n|] = b_n - x_0 < +\infty$, therefore Z_n is replicable from initial capital x_0. Moreover, each Z_n (for n sufficiently large) is actually feasible for (BPP), because $V_-(Z_n^-) = V_-(Y_n) < +\infty$ (note that also $V_+(Z_n^+) = V_+(X_n) < +\infty$). Finally, recalling that $w_+(1) = 1$, we obtain $\sup_{n\in\mathbb{N}} V(Z_n) \geq \liminf_{n\to+\infty} V(Z_n) \geq u_+(+\infty) - l$, which implies the desired inequality. □

Lemma III.6.2. *Under Assumption III.4.1, there exists an optimal portfolio for problem (BPP) only if $v^*(x_0) < u_+(+\infty)$.*

Proof. Let us suppose that there exists a feasible portfolio $\overline{\phi}^* \in \mathscr{A}(x_0)$ such that $v^*(x_0) = V(X_*)$, where $X_* \triangleq \Pi_T^{\overline{\phi}^*} - B$. Since u_+ is assumed to be strictly increasing, we have that $\varnothing = \{u_+(X_*^+) \geq u_+(+\infty)\} = \bigcap_{n=1}^{+\infty} \{u_+(X_*^+) > u_+(+\infty) - 1/n\}$.

Thus, the continuity from above of the probability measure implies the existence of some $n_0 \in \mathbb{N}$ for which both $\mathbb{P}\{u_+(X_*^+) > u_+(+\infty) - 1/n_0\} < 1$ and $u_+(+\infty) - 1/n_0 > 0$ hold true.

As a consequence, $0 \leq \mathbb{P}\{u_+(X_*^+) > y\} \leq \mathbb{P}\{u_+(X_*^+) > u_+(+\infty) - 1/n_0\} < 1$ for all $y \in \left[u_+(+\infty) - \frac{1}{n_0}, u_+(+\infty)\right]$, whence

$$\begin{aligned} u_+(+\infty) - V(X_*) &\geq u_+(+\infty) - V_+\big(X_*^+\big) \\ &\geq \int_{u_+(+\infty)-1/n_0}^{u_+(+\infty)} \left[1 - w_+\big(\mathbb{P}\{u_+\big(X_*^+\big) > y\}\big)\right] dy > 0, \end{aligned}$$

because w_+ is assumed to be strictly increasing. □

Lemma III.6.3. *Let w be a distortion and $u : [0,+\infty) \to [0,+\infty)$ a utility such that $u(+\infty) = +\infty$. Suppose $f : [0,+\infty) \to [0,+\infty)$ is a continuous and strictly increasing function, satisfying both $f(0) = 0$ and $f(+\infty) = +\infty$. Then*

$$w(\mathbb{P}\{f(X) > t\}) \leq \frac{1}{u(f^{-1}(t))} \int_0^{+\infty} w(\mathbb{P}\{u(X) > y\})\, dy \tag{III.6.2}$$

for any $t > 0$ and for any random variable $X \geq 0$.

Proof. Let $t > 0$ be arbitrary, but fixed. Then, for any random variable $X \geq 0$, we have by the monotonicity of the integral, as well as that of u and f (and their inverses) that

$$\int_0^{+\infty} w(\mathbb{P}\{u(X) > y\})\, dy \geq \int_0^{u(f^{-1}(t))} w\big(\mathbb{P}\big\{f(X) > f\big(u^{-1}(y)\big)\big\}\big)\, dy$$
$$\geq \int_0^{u(f^{-1}(t))} w(\mathbb{P}\{f(X) > t\})\, dy,$$

where the last inequality follows from the inclusion $\{f(X) > t\} \subseteq \big\{f(X) > f\big(u^{-1}(y)\big)\big\}$ for all $y \in \big[0, u\big(f^{-1}(t)\big)\big]$. □

Corollary III.6.4. *Suppose u is as in Lemma III.6.3, and let $\delta > 0$ be arbitrary. Then for every $s > 0$ we have*

$$\mathbb{P}\{X^s > t\} \leq \left[u^{-1}\left(u(1)\left[\frac{u(t^{1/s})}{V_u^\delta(X)}\right]^{1/\delta}\right)\right]^{-1} \tag{III.6.3}$$

for all $t > 0$ and for all random variables $X \geq 0$, where

$$V_u^\delta(X) \triangleq \int_0^{+\infty} w_u^\delta(\mathbb{P}\{u(X) > y\})\, dy \tag{III.6.4}$$

and w_u^δ is the associated distortion of Definition III.4.8. □

Remark III.6.5. Note that, if $X \geq 0$ and $V_u^\delta(X) = 0$, then $X = 0$ a.s. (see Proposition C.6*(vi)* and Remark C.7), and so (III.6.3) is trivially valid. ◇

Lemma III.6.6. *Let u be a utility function with $u(+\infty) = +\infty$. The following three statements are equivalent,*

(i) *Assumption III.4.12 holds true for u,*

(ii) *for each $\delta \in (0,1)$, there exist a real number $\zeta > 1$ and a non-increasing function $G : (0,+\infty) \to [1,+\infty)$ such that, for every $\lambda > 0$,*

$$u\big(x^\zeta\big) \leq [\lambda\, u(x)]^{1/\delta}, \tag{III.6.5}$$

for all $x \geq G(\lambda)$,

(iii) *for every $\delta \in (0,1)$, there is $\varsigma > 1$ for which $\lim_{x\to+\infty} [z(x) - \delta\, z(\varsigma x)] = +\infty$, where z is the transform of u defined in Lemma III.4.14.*

Proof. We begin by showing *(i)*⇔*(ii)*. To prove that *(ii)* implies *(i)*, fix $\delta \in (0,1)$ and let $\varepsilon > 0$ be arbitrary. Then, for all $x \geq G(\varepsilon/2) \geq 1$, we have $\big[u\big(x^\zeta\big)\big]^\delta / u(x) \leq \varepsilon/2$, and therefore $\lim_{x\to+\infty} \big[u\big(x^\zeta\big)\big]^\delta / u(x) = 0$.

Now we prove the reverse implication. Again, let $\delta \in (0,1)$ be fixed, and consider an arbitrary $\lambda > 0$. Since, by hypothesis, $\lim_{x\to+\infty} \big[u\big(x^\xi\big)\big]^\delta / u(x) = 0$, there exists some $L \triangleq L(\lambda) \geq 1$ such that $u\big(x^\xi\big) < [\lambda\, u(x)]^{1/\delta}$ for all $x \geq L$. Next define, for each $\lambda > 0$, the non-empty set

$$\mathscr{S}_\lambda \triangleq \left\{L \geq 1\colon\ u\left(x^\xi\right) < [\lambda\, u(x)]^{1/\delta} \text{ for all } x \geq L\right\},$$

which is bounded below by 1, so it admits an infimum. Then let $G : (0, +\infty) \to \mathbb{R}$ be the function given by $G(\lambda) \triangleq \inf \mathscr{S}_\lambda$, for any $\lambda > 0$. Clearly, by construction, $G \geq 1$. Furthermore, it can be easily checked that, for every $\lambda > 0$ and for all $x \geq G(\lambda)$, the inequality $u(x^\xi) \leq [\lambda\, u(x)]^{1/\delta}$ holds true. In fact, if $x > G(\lambda)$, then by definition of infimum, we can find some $M \in \mathscr{S}_\lambda$ satisfying $x > M$, and thus $u(x^\xi) < [\lambda\, u(x)]^{1/\delta}$. So suppose, on the other hand, that $x = G(\lambda)$. Again it follows from the definition of infimum that there exists a sequence $\{M_n;\, n \in \mathbb{N}\} \subseteq \mathscr{S}_\lambda$ such that $\lim_{n\in\mathbb{N}} M_n = x$ and hence, by continuity, we obtain

$$u\left(x^\xi\right) = \lim_{n\to+\infty} u\left(M_n^\xi\right) \leq \lim_{n\to+\infty} [\lambda\, u(M_n)]^{1/\delta} = [\lambda\, u(x)]^{1/\delta}.$$

Finally, it remains to show that G is indeed a non-increasing function of λ. To see this, let $0 < \lambda_1 \leq \lambda_2$. Then, for all $x \geq G(\lambda_1) \geq 1$, we have $u(x^a) \leq [\lambda_1 u(x)]^{1/\delta} \leq [\lambda_2 u(x)]^{1/\delta}$, hence $G(\lambda_1)$ belongs to $\mathscr{S}_{\lambda_2}$. Consequently, we must have, by the definition of the infimum, that $G(\lambda_1) \geq G(\lambda_2)$.

We now prove *(i)* $\Leftrightarrow$ *(iii)*. To show that *(i)* implies *(iii)*, let us start by considering an arbitrary $M > 0$. Then we can use the hypothesis to find some $x_0 > 1$ such that $\left[u\left(e^{\xi x}\right)\right]^\delta / u(e^x) < e^{-M}$ holds true for all $x \geq x_0$, and applying logarithms to both sides of the above inequality yields $-M > \delta\, z(\xi x) - z(x)$. The reverse implication can also be obtained in a straightforward and analogous manner. □

Lemma III.6.7. *Suppose u is a utility with $u(+\infty) = +\infty$ and satisfying Assumption III.4.12. Let $\delta \in (0, 1)$, and let the non-increasing function $G : (0, +\infty) \to [1, +\infty)$ and the real number $\zeta > 1$ be those given by Lemma III.6.6. Then, for every $\eta \in (1, \zeta)$, there exist constants $C > 0$ and $D > 0$ such that, for all random variables $X \geq 0$,*

$$\mathbb{E}_\mathbb{P}[X^\eta] \leq C + \frac{\left[G\left(D\,[V_u^\delta(X)]^{-1}\right)\right]^\eta}{u^{-1}\left(D\,[V_u^\delta(X)]^{-1/\delta}\right)}, \tag{III.6.6}$$

with $V_u^\delta(X)$ given by (III.6.4)*.*

Proof. Fix $\delta \in (0, 1)$ and $\eta \in (1, \zeta)$, and let X be any non-negative random variable. If $X = 0$ a.s., then $\mathbb{E}_\mathbb{P}[X^\eta] = 0$ and $V_u^\delta(X) = 0$ (by Proposition C.6*(v)*), hence the inequality (III.6.6) is satisfied trivially for any $C > 0$ and $D > 0$. So suppose now that $\mathbb{P}\{X > 0\} > 0$, which implies $V_u^\delta(X) > 0$ (by Proposition C.6*(vi)*). Using Corollary III.6.4,

$$\mathbb{E}_\mathbb{P}[X^\eta] = \int_0^\infty \mathbb{P}\{X^\eta > t\}\, dt \leq 1 + \int_1^{+\infty} \left[u^{-1}\left(u(1)\left[\frac{u(t^{1/\eta})}{V_u^\delta(X)}\right]^{1/\delta}\right)\right]^{-1} dt. \tag{III.6.7}$$

We apply Lemma III.6.6 with $\lambda \triangleq [u(1)]^\delta / V_u^\delta(X) > 0$ to obtain, for every $x \geq G\Big([u(1)]^\delta / V_u^\delta(X)\Big)$,

$$u^{-1}\left(u(1)\left[\frac{u(x)}{V_u^\delta(X)}\right]^{1/\delta}\right) \geq x^\zeta,$$

where we have also made use of the fact that u^{-1} is strictly increasing. On the other hand, it follows again from the monotonicity of both u and u^{-1} that

$$u^{-1}\left(u(1)\left[\frac{u\big(t^{1/\eta}\big)}{V_u^\delta(X)}\right]^{1/\delta}\right) \geq u^{-1}\left(u(1)\left[\frac{u(1)}{V_u^\delta(X)}\right]^{1/\delta}\right)$$

for all $t \geq 1$. Thus, the preceding facts and the change of variables $x = t^{1/\eta}$ yield

$$\begin{aligned}
&\int_1^{+\infty}\left[u^{-1}\left(u(1)\left[\frac{u\big(t^{1/\eta}\big)}{V_u^\delta(X)}\right]^{1/\delta}\right)\right]^{-1} dt \\
&\qquad \leq \int_1^{\left[G\left([u(1)]^\delta / V_u^\delta(X)\right)\right]^\eta}\left[u^{-1}\left([u(1)]^{1+1/\delta}\left[\frac{1}{V_u^\delta(X)}\right]^{1/\delta}\right)\right]^{-1} dt \\
&\qquad\quad + \eta\int_{G\left([u(1)]^\delta / V_u^\delta(X)\right)}^{+\infty}\frac{\left[u^{-1}\left(\left[u(1)\, u(x) / V_u^\delta(X)\right]^{1/\delta}\right)\right]^{-1}}{x^{1-\eta}}\, dx \\
&\qquad \leq \frac{\left[G\left([u(1)]^\delta\left[V_u^\delta(X)\right]^{-1}\right)\right]^\eta - 1}{u^{-1}\left([u(1)]^{1+1/\delta}\left[V_u^\delta(X)\right]^{-1/\delta}\right)} + \eta\int_1^{+\infty}\frac{1}{x^{1+\zeta-\eta}}\, dx,
\end{aligned} \tag{III.6.8}$$

and we note that the integral appearing in the second term is finite (because $\zeta - \eta > 0$).

Hence, plugging (III.6.8) into (III.6.7), setting

$$C \triangleq 1 + \eta\int_1^{+\infty} x^{\eta-\zeta-1}\, dx \in (1, +\infty) \qquad \text{and} \qquad D \triangleq [u(1)]^\delta \wedge [u(1)]^{1+1/\delta} > 0,$$

and noting that $\left[G\left(D\left[V_u^\delta(X)\right]^{-1}\right)\right]^\eta - 1 \leq \left[G\left(D\left[V_u^\delta(X)\right]^{-1}\right)\right]^\eta$, allows us to finally deduce the claimed inequality. □

Lemma III.6.8. *Suppose that u is the power utility with parameter $\alpha > 0$. Then, for every $\gamma > 0$ and $\eta \in (0, \alpha/\gamma)$, there exists a real number $D > 0$ such that*

$$\mathbb{E}_{\mathbb{P}}[X^\eta] \leq 1 + D\left(\int_0^{+\infty}\mathbb{P}\{X^\alpha > y\}^\gamma\, dy\right)^{1/\gamma} \tag{III.6.9}$$

for all random variables $X \geq 0$.

Proof. Fix arbitrary $\gamma > 0$ and $\eta \in (0, \alpha/\gamma)$. Noticing that $u(+\infty) = +\infty$ and recalling

Example III.4.10*(i)*, we may apply Corollary III.6.4 with $\delta \triangleq \gamma/\alpha > 0$ to obtain

$$\mathbb{E}_{\mathbb{P}}[X^\eta] = \int_0^\infty \mathbb{P}\{X^\eta > t\}\, dt \le 1 + \left(\int_0^{+\infty} \mathbb{P}\{X^\alpha > y\}^\gamma\, dy\right)^{1/\gamma} \int_1^{+\infty} t^{-\alpha/(\gamma\eta)}\, dt.$$

Set $D \triangleq \int_1^{+\infty} t^{-\alpha/(\gamma\eta)}\, dt$, which is finite (note $\alpha/(\gamma\eta) > 1$) and strictly positive. □

Lemma III.6.9. *Suppose Assumption III.3.9 is in force. If* (III.5.3) *holds, then there exist real numbers* $\eta \in (\alpha_+ \vee \beta_-, \alpha_-)$ *and* $L_1, L_2 > 0$ *such that*

$$\int_0^{+\infty} \{(X^+)^{\alpha_+} > y\}^{\beta_+}\, dy \le L_1 + L_2 \int_0^{+\infty} \{(X^-)^\eta > y\}^{\beta_-}\, dy \tag{III.6.10}$$

for all random variables X *with* $\mathbb{E}_{\mathbb{P}^*}[X] = x_0$.

Proof. We start by noticing that the hypothesis $\alpha_+ < \beta_+$ implies that $1/\alpha_+ > 1/\beta_+$. Moreover, since $\alpha_+ < \alpha_-$ and $\beta_- < \alpha_-$, there exists η such that $\max\{\alpha_+, \beta_-\} < \eta < \alpha_-$. In particular, we deduce that $1 < \eta/\alpha_+$, and thus $1/\beta_+ < \eta/(\alpha_+\beta_+)$. Hence, we may choose λ such that $1/\beta_+ < \lambda < \min\{1/\alpha_+, \eta/(\alpha_+\beta_+)\}$. Then, given that $1 < 1/(\lambda\alpha_+)$, there exists some p satisfying $1 < p < 1/(\lambda\alpha_+)$. Finally, we note that $1 < \eta/\beta_-$ and $(\alpha_+\beta_+\lambda)/\beta_- < \eta/\beta_-$ (because $\lambda < \eta/(\alpha_+\beta_+)$, that is, $\alpha_+\beta_+\lambda < \eta$), so we can take q such that $\max\{1, \alpha_+\lambda\beta_+/\beta_-\} < q < \eta/\beta_-$.

Next, since for all $y \ge 1$ we have $\mathbb{P}\{(X^+)^{\alpha_+} > y\} \le \mathbb{E}_{\mathbb{P}}\left[(X^+)^{\alpha_+\lambda}\right]/y^\lambda$ by Chebyshev's inequality, it follows from the monotonicity of the integral that

$$\int_0^{+\infty} \mathbb{P}\{(X^+)^{\alpha_+} > y\}^{\beta_+}\, dy \le 1 + C_1\, \mathbb{E}_{\mathbb{P}}\left[(X^+)^{\alpha_+\lambda}\right]^{\beta_+},$$

with the strictly positive constant $C_1 \triangleq \int_1^{+\infty} y^{-\lambda\beta_+} dy$ (we recall that $\lambda\beta_+ > 1$, so the integral is finite). Also, applying Hölder's inequality (with $p > 1$) yields

$$\mathbb{E}_{\mathbb{P}}\left[(X^+)^{\alpha_+\lambda}\right] = \mathbb{E}_{\mathbb{P}}\left[\frac{1}{\rho^{1/p}}\rho^{1/p}\,(X^+)^{\alpha_+\lambda}\right] \le C_2\, \mathbb{E}_{\mathbb{P}^*}\left[(X^+)^{\alpha_+\lambda p}\right]^{1/p},$$

where $C_2 \triangleq \mathbb{E}_{\mathbb{P}}\left[\rho^{-1/(p-1)}\right]^{(p-1)/p}$ is finite (recall Assumption III.3.9) and strictly positive. Thus, combining the previous equation and Jensen's inequality for concave functions (we note that $\alpha_+\lambda p < 1$), we obtain

$$\mathbb{E}_{\mathbb{P}}\left[(X^+)^{\alpha_+\lambda}\right]^{\beta_+} \le C_3\, \mathbb{E}_{\mathbb{P}^*}\left[(X^+)^{\alpha_+\lambda p}\right]^{\beta_+/p} \le C_3\, \mathbb{E}_{\mathbb{P}^*}\left[X^+\right]^{\alpha_+\lambda\beta_+}.$$

But, using the hypothesis that $\mathbb{E}_{\mathbb{P}^*}[X] = x_0$, it is clear that

$$\mathbb{E}_{\mathbb{P}^*}\left[X^+\right]^{\alpha_+\lambda\beta_+} = \left(x_0 + \mathbb{E}_{\mathbb{P}^*}\left[X^-\right]\right)^{\alpha_+\lambda\beta_+} \le C_4 + C\, \mathbb{E}_{\mathbb{P}^*}\left[X^-\right]^{\alpha_+\lambda\beta_+},$$

where the inequality follows from the trivial inequality of Lemma II.5.1 with $\alpha_+\lambda\beta_+ > 0$, and $C_4, C \in (0, +\infty)$.

Now, we use Hölder's inequality once more (this time with $q > 1$) to see that

$$\mathbb{E}_{\mathbb{P}^*}\big[X^-\big]^{\alpha_+\lambda\beta_+} \leq C_5\,\mathbb{E}_{\mathbb{P}}\big[\big(X^-\big)^q\big]^{(\alpha_+\lambda\beta_+)/q}$$

(here $C_5 \triangleq \mathbb{E}_{\mathbb{P}}\big[\rho^{q/(q-1)}\big]^{\alpha_+\lambda\beta_+(q-1)/q}$ is also strictly positive and finite). Moreover, we have by the trivial inequality already mentioned above (first with $(\alpha_+\lambda\beta_+)/q > 0$ and then with $\beta_- > 0$) that, for some universal constants $C_6, C_7 > 0$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\big[\big(X^-\big)^q\big]^{(\alpha_+\lambda\beta_+)/q} &\leq C\left(\left|-1\right|^{(\alpha_+\lambda\beta_+)/q} + \left|1 + \mathbb{E}_{\mathbb{P}}\big[\big(X^-\big)^q\big]\right|^{(\alpha_+\lambda\beta_+)/q}\right)\\ &\leq C_6 + C_7\,\mathbb{E}_{\mathbb{P}}\big[\big(X^-\big)^q\big]^{\beta_-},\end{aligned}$$

where the second inequality is due to $(\alpha_+\lambda\beta_+)/q < \beta_-$.

Therefore, combining the above inequalities yields

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[\big(X^+\big)^{\alpha_+\lambda}\right]^{\beta_+} &\leq C_8 + C_9\,\mathbb{E}_{\mathbb{P}}\big[\big(X^-\big)^q\big]^{\beta_-}\\ &\leq C_8 + C_9\left[1 + D\left(\int_0^{+\infty}\mathbb{P}\big\{\big(X^-\big)^\eta > y\big\}^{\beta_-}\,dy\right)^{\frac{1}{\beta_-}}\right]^{\beta_-}\\ &\leq M_1 + M_2\int_0^{+\infty}\mathbb{P}\big\{\big(X^-\big)^\eta > y\big\}^{\beta_-}\,dy, \qquad \text{(III.6.11)}\end{aligned}$$

where to obtain the second inequality we use the auxiliary Lemma III.6.8 above (note that $q < \eta/\beta_-$), and where the last inequality follows from one last application of the trivial inequality to which we referred previously (once more with $\beta_- > 0$). Furthermore, all constants C_8, C_9, D, M_1, M_2 belong to $(0, +\infty)$.

Hence,

$$\int_0^{+\infty}\mathbb{P}\big\{\big(X^+\big)^{\alpha_+} > y\big\}^{\beta_+}\,dy \leq L_1 + L_2\int_0^{+\infty}\mathbb{P}\big\{\big(X^-\big)^\eta > y\big\}^{\beta_-}\,dy,$$

with L_1 and L_2 positive constants (that do not depend on the random variable X, but on the parameters only), as intended. ☐

Lemma III.6.10. *Let α, γ and η be strictly positive real numbers satisfying $\eta < \alpha < \gamma$. Then there exist real numbers $\zeta \in (0,1)$ and $R_1, R_2 > 0$ such that, for all random variables $X \geq 0$, the following inequality is true,*

$$\int_0^{+\infty}\mathbb{P}\{X^\alpha > y\}^\eta\,dy \leq R_1 + R_2\left(\int_0^{+\infty}\mathbb{P}\{X^\gamma > y\}^\eta\,dy\right)^\zeta \qquad \text{(III.6.12)}$$

In particular, this implies that we have $\int_0^{+\infty}\mathbb{P}\{X_n^\gamma > y\}^\eta\,dy \to +\infty$ (as n goes to $+\infty$) whenever $\int_0^{+\infty}\mathbb{P}\{X_n^\alpha > y\}^\eta\,dy \to +\infty$ (as $n \to +\infty$) for any sequence $\{X_n;\, n \in \mathbb{N}\}$ of non-negative random variables.

Proof. We start by fixing some χ satisfying $1/\eta < \chi < \gamma/(\eta\alpha)$. Such a χ exists, because

$1 < \gamma/\alpha$ and $\eta > 0$ imply that $1/\eta < \gamma/(\eta\alpha)$. We also note that, since $\chi\alpha < \gamma/\eta$, we can choose a real number ξ such that $\chi\alpha < \xi < \gamma/\eta$.

Next, let X be an arbitrary non-negative random variable. Given that $\xi < \gamma/\eta$, we know from Lemma III.6.8 that

$$\mathbb{E}_{\mathbb{P}}\left[X^{\xi}\right] \leq 1 + D\left(\int_0^{+\infty} \mathbb{P}\{X^{\gamma} > y\}^{\eta}\, dy\right)^{1/\eta}$$

for some strictly positive real number D (not depending on X, but only on the parameters γ, η and ξ). Therefore, recalling that $\eta > 0$, it follows from the trivial inequality (II.5.1) that

$$\mathbb{E}_{\mathbb{P}}\left[X^{\xi}\right]^{\eta} \leq C + C_1 \int_0^{+\infty} \mathbb{P}\{X^{\gamma} > y\}^{\eta}\, dy, \tag{III.6.13}$$

with $C, C_1 \in (0, +\infty)$. Now, by Jensen's inequality for concave functions (note that $(\alpha\chi)/\xi < 1$), we obtain

$$\mathbb{E}_{\mathbb{P}}[X^{\alpha\chi}] = \mathbb{E}_{\mathbb{P}}\left[\left(X^{\xi}\right)^{\frac{\alpha\chi}{\xi}}\right] \leq \mathbb{E}_{\mathbb{P}}\left[X^{\xi}\right]^{\frac{\alpha\chi}{\xi}}. \tag{III.6.14}$$

Moreover, using Chebyshev's inequality, we get

$$\int_0^{+\infty} \mathbb{P}\{X^{\alpha} > y\}^{\eta}\, dy \leq 1 + C_2\, \mathbb{E}_{\mathbb{P}}[X^{\alpha\chi}]^{\eta}, \tag{III.6.15}$$

where $C_2 \triangleq \int_1^{+\infty} y^{-\eta\chi}\, dy$ is a strictly positive real number (note that $\eta\chi > 1$).

Thus, combining inequalities (III.6.13) to (III.6.15) yields

$$\begin{aligned}
\int_0^{+\infty} \mathbb{P}\{X^{\alpha} > y\}^{\eta}\, dy &\leq 1 + C_2\left(\mathbb{E}_{\mathbb{P}}\left[X^{\xi}\right]^{\eta}\right)^{\frac{\alpha\chi}{\xi}} \\
&\leq 1 + C_2\left(C + C_1\int_0^{+\infty} \mathbb{P}\{X^{\gamma} > y\}^{\eta}\, dy\right)^{\frac{\alpha\chi}{\xi}} \\
&\leq R_1 + R_2\left(\int_0^{+\infty} \mathbb{P}\{X^{\gamma} > y\}^{\eta}\, dy\right)^{\frac{\alpha\chi}{\xi}}
\end{aligned}$$

where the last inequality is due again to the trivial inequality (II.5.1) mentioned above, and the positive constants R_1, R_2 depend only on the parameters. Setting $\zeta = (\alpha\chi)/\xi \in (0,1)$ completes the proof. □

6.2 Proofs of Section III.3

Proof of Proposition III.3.16. The combination of the hypothesis $\mathbb{P}(A) \in (0,1)$ with the fact that $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent measures ensures that $\mathbb{P}^*(A) \in (0,1)$. Thus, for every $n \in \mathbb{N}$, we can set

$$X_n \triangleq n\,\mathbb{1}_A \qquad \text{and} \qquad Y_n \triangleq \frac{n\,\mathbb{P}^*(A) - x_0}{\mathbb{P}^*(A^c)}\,\mathbb{1}_{A^c}.$$

Now, each X_n is a non-negative, $\sigma(\rho)$-measurable random variable with

$$V_+(X_n) = u_+(n)\, w_+(\mathbb{P}(A))\,.$$

On the other hand, it is straightforward to see that, for n sufficiently large, each one of the $\sigma(\rho)$-measurable random variables Y_n is also non-negative. Moreover,

$$V_-(Y_n) = u_-\left(\frac{n\,\mathbb{P}^*(A) - x_0}{\mathbb{P}^*(A^c)}\right) w_-(\mathbb{P}(A^c))\,.$$

Consequently, if for each sufficiently large integer n we define $Z_n \triangleq X_n - Y_n$, then not only is this claim hedgeable by Assumption III.3.7, but actually it is feasible for (BPP). Indeed, $\mathbb{E}_{\mathbb{P}^*}[Z_n] = x_0$ and $V_-(Z_n^-) = V_-(Y_n) < +\infty$ (also, $V_+(Z_n^+) = V_+(X_n) < +\infty$).

But this implies that $v^*(x_0) = +\infty$, since

$$\liminf_{n\to+\infty} V(Z_n) = u_+(+\infty)\, w_+(\mathbb{P}(A)) - u_-(+\infty)\, w_-(\mathbb{P}(A^c)) = +\infty,$$

where we use $\mathbb{P}(A) > 0$ and the fact that the distortion on gains is strictly increasing to deduce $w_+(\mathbb{P}(A)) > 0$. □

Proof of Proposition III.3.17. Given that the utilities u_+ and u_- are both non-negative functions, it is obvious that we must have $l \in [0, +\infty]$. So let us start by noticing that, if we assume l to be a non-negative real number, then we can construct a sequence of reals $\{a_n;\, n \in \mathbb{N}\}$ such that both $a_n \geq n$ and $u_-(a_n)\,/u_+(a_n) < l + 1/n$ are satisfied for every strictly positive integer n. It is obvious, by the way this sequence was constructed, that $\lim_{n\to+\infty} a_n = +\infty$. This implies, in particular, the existence of an $n_0 \in \mathbb{N}$ such that $a_n \geq 2\,|x_0|$ for all $n \geq n_0$.

On the other hand, since $AE_+(u_-)$ is finite and $\gamma > AE_+(u_-)$ by hypothesis, there exists some $\underline{x} = \underline{x}(\gamma) \geq 0$ such that

$$u_-(\lambda x) \leq \lambda^\gamma u(x) \tag{III.6.16}$$

for all $\lambda \geq 1$ and $x \geq \underline{x}$. In addition, we can find $n_1 = n_1(\gamma) \in \mathbb{N}$ such that $a_n \geq \underline{x}$ for every $n \geq n_1$.

Let us now define, for each $n \in \mathbb{N}$, the $\sigma(\rho)$-measurable random variables

$$X_n \triangleq a_n\, \mathbb{1}_A \qquad \text{and} \qquad Y_n \triangleq (a_n - 2x_0)\, \mathbb{1}_{A^c}.$$

Clearly each X_n is non-negative, and so is Y_n for every $n \geq n_0$. Moreover, we have

$$V_+\left(X_n\right) = u_+(a_n)\, w_+(\mathbb{P}(A))$$

for all $n \in \mathbb{N}$, and

$$V_-(Y_n) = u_-(a_n - 2x_0)\, w_-(\mathbb{P}(A^c)) \leq u_-(2a_n)\, w_-(\mathbb{P}(A^c)) \leq 2^\gamma u_-(a_n)\, w_-(\mathbb{P}(A^c))$$

for all $n \geq \max\{n_0, n_1\}$, where the first inequality is a simple consequence of the monotonicity of u_- and the second inequality follows from (III.6.16).

Hence, setting $Z_n \triangleq X_n - Y_n$ for every $n \in \mathbb{N}$, it is easy to check that $Z_n \in L^1(\Omega, \sigma(\rho), \mathbb{P}^*)$, thus each claim Z_n is hedgeable. Furthermore, we obtain

$$E_{\mathbb{P}^*}[Z_n] = a_n \mathbb{P}^*(A) - (a_n - 2x_0)\,\mathbb{P}^*(A^c) = x_0,$$

as well as $V_+(Z_n^+) = V_+(X_n) < +\infty$ and $V_-(Z_n^-) = V_-(Y_n) < +\infty$ for all $n \geq n_0$, therefore allowing us to deduce that each Z_n is feasible for (BPP), for sufficiently large n. Finally, we see that

$$\begin{aligned} V(Z_n) &\geq u_+(a_n)\left[w_+(\mathbb{P}(A)) - 2^\gamma \frac{u_-(a_n)}{u_+(a_n)} w_-(\mathbb{P}(A^c))\right] \\ &\geq u_+(a_n)\left[w_+(\mathbb{P}(A)) - 2^\gamma \left(l + \frac{1}{n}\right) w_-(\mathbb{P}(A^c))\right] \end{aligned}$$

for all $n \geq \max\{n_0, n_1\}$, so $\lim_{n\to+\infty} V(Z_n) = +\infty$, and consequently $v^*(x_0) = +\infty$.

We conclude the proof with the observation that, for sufficiently large values of n, each Z_n is bounded from below by $-(a_n - 2x_0)$. □

Proof of Proposition III.3.20. The proof is by contraposition. For this, let us suppose that (III.3.4) is not true, so we must have $\liminf_{x\to 0^+} w_-(x)\, u_-(1/x) \in [0, +\infty)$. Hence, it follows immediately from Lemma III.6.1 and from the hypothesis $u_+(+\infty) = +\infty$ that $v^*(x_0) = +\infty$, as claimed. □

Proof of Proposition III.3.24. Using the hypothesis given by (III.3.5), it is easy to obtain some $a_1 \in (0, 1)$ such that $w_+(a_1)\, u_+(1/a_1) > 1$. Next, it is possible to find $a_2 \in (0, a_1 \wedge 1/2)$ for which the inequality $w_+(a_2)\, u_+(1/a_2) > 2$ holds true. Repeating this process, we can recursively construct a strictly decreasing sequence $\{a_n;\, n \in \mathbb{N}\}$ whose terms satisfy $a_n \in (0, 1/n)$ as well as $w_+(a_n)\, u_+(1/a_n) > n$ for every n. It is also clear that $\lim_{n\to+\infty} a_n$ exists and is equal to zero. Moreover, it follows from Assumption III.3.13 and $\rho > 0$ a.s. that, for each $n \in \mathbb{N}$, we can find some $b_n > 0$ such that $\mathbb{P}\{\rho \leq b_n\} = a_n$. Finally, the strict monotonicity of $\{a_n\}_{n\in\mathbb{N}}$ implies that the sequence $\{b_n;\, n \in \mathbb{N}\}$ is strictly decreasing as well.

Now, for every strictly positive integer n, define

$$X_n \triangleq \frac{|x_0| + b_n}{\mathbb{P}^*(A_n)} \mathbb{1}_{A_n},$$

with $A_n \triangleq \{\rho \leq b_n\} \in \sigma(\rho)$ (note that, since the probability measures $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent, $\mathbb{P}^*(A_n) \in (0, 1)$). Then every X_n is a non-negative, $\sigma(\rho)$-measurable random variable and

$$V_+(X_n) = u_+\left(\frac{|x_0| + b_n}{\mathbb{P}^*(A_n)}\right) w_+(\mathbb{P}(A_n)) \geq u_+\left(\frac{1 + |x_0|/b_n}{a_n}\right) w_+(a_n) \geq u_+\left(\frac{1}{a_n}\right) w_+(a_n),$$

where the first inequality follows easily from $\mathbb{P}^*(A_n) = \mathbb{E}_{\mathbb{P}}[\rho \mathbb{1}_{A_n}] \leq b_n \mathbb{P}(A_n)$ combined with the monotonicity of u_+, and the second inequality is a simple consequence of u_+ being strictly increasing and of $\{b_n;\, n \in \mathbb{N}\}$ being a strictly positive sequence.

On the other hand, setting

$$Y_n \triangleq \frac{|x_0| - x_0 + b_n}{\mathbb{P}^*(A_n^c)} \mathbb{1}_{A_n^c},$$

it is obvious that every Y_n is also a non-negative random variable. Furthermore, we can use the monotonicity of $\{b_n;\, n \in \mathbb{N}\}$ and that of u_- to obtain

$$V_-(Y_n) = u_-\left(\frac{|x_0| - x_0 + b_n}{\mathbb{P}^*(A_n^c)}\right) w_-(\mathbb{P}(A_n^c)) \leq u_-\left(\frac{|x_0| - x_0 + b_1}{\mathbb{P}^*(A_n^c)}\right) w_-(\mathbb{P}(A_n^c))\,.$$

Finally, we observe that the sequence of events $\{A_n^c;\, n \in \mathbb{N}\}$ is non-decreasing, so the continuity from below of the probability measures $\mathbb{P}$ and $\mathbb{P}^*$ yields $\mathbb{P}\big(\bigcup_{n=1}^{+\infty} A_n^c\big) = \lim_{n\to+\infty} \mathbb{P}(A_n^c) = \lim_{n\to+\infty} (1 - a_n) = 1$, and $\mathbb{P}^*\big(\bigcup_{n=1}^{+\infty} A_n^c\big) = \lim_{n\to+\infty} \mathbb{P}^*(A_n^c)$. But $\mathbb{P}^*$ is equivalent to $\mathbb{P}$, therefore we must have $\mathbb{P}^*\big(\bigcup_{n=1}^{+\infty} A_n^c\big) = 1$.

Hence, defining $Z_n \triangleq X_n - Y_n$, for $n \in \mathbb{N}$, we get

$$\liminf_{n\to+\infty} V(Z_n) \geq \liminf_{n\to+\infty} \left[n - u_-\left(\frac{|x_0| - x_0 + b_1}{\mathbb{P}^*(A_n^c)}\right) w_-(\mathbb{P}(A_n^c))\right] = +\infty.$$

Since it can be easily checked that every Z_n is feasible for (BPP) (recall Assumption III.3.7), the claim follows.

We finish by drawing attention to the fact that $Z_n \geq -\left(2\,|x_0| + b_1\right)/\mathbb{P}^*(A_1^c)$ for all $n \in \mathbb{N}$. □

Proof of Theorem III.3.36. Essentially, we shall follow the proof Theorem 4.7 in Rásonyi and Rodrigues [47] (which in turn is inspired in the proof of Theorem 6.8 of Carassus and Rásonyi [13]), while borrowing some key ideas from Reichlin [52] (who in turn follows the paths of Jin and Zhou [29], He and Zhou [27] and Carlier and Dana [15]).

Let us begin by fixing a maximising sequence $\{\overline{\phi}_n;\, n \in \mathbb{N}\} \subseteq \mathscr{A}(x_0)$. For the sake of convenience of writing, we shall henceforth denote by X_n the terminal wealth of the n-th portfolio $\overline{\phi}_n$.

Using Assumption III.3.9 and equation (III.3.13), it is straightforward to obtain that, for every $\tau \in (0,1)$, $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[|X_n|^\tau] < +\infty$. Indeed, given any $\tau \in (0,1)$, we start by using Hölder's inequality (first with $1/\tau > 1$ and then with $\eta > 1$), $\mathbb{E}_{\mathbb{P}^*}[X_n] = x_0$, the trivial inequality (II.5.1), and Assumption III.3.9 to derive

$$\mathbb{E}_{\mathbb{P}}\left[\left(X_n^+\right)^\tau\right] \leq C_1 \mathbb{E}_{\mathbb{P}^*}\left[X_n^+\right]^\tau \leq C_2 + C_1 \mathbb{E}_{\mathbb{P}^*}\left[X_n^-\right]^\tau \leq C_2 + C_3 \mathbb{E}_{\mathbb{P}^*}\left[\left(X_n^-\right)^\eta\right]^{\tau/\eta},$$

and so we have

$$\mathbb{E}_{\mathbb{P}}[|X_n|^\tau] \leq \mathbb{E}_{\mathbb{P}}\left[\left(X_n^+\right)^\tau\right] + \mathbb{E}_{\mathbb{P}}\left[\left(X_n^-\right)^\tau\right] \leq C_2 + C_3 \mathbb{E}_{\mathbb{P}^*}\left[\left(X_n^-\right)^\eta\right]^{\tau/\eta} + C_4 \mathbb{E}_{\mathbb{P}^*}\left[\left(X_n^-\right)^\eta\right]^{\tau/\eta},$$

where the second inequality is again a trivial consequence of Hölder's inequality (with $\eta/\tau > 1$). Thus, the claimed result follows from the definition of supremum.

From this, it is now trivial to deduce the tightness of the family $\{\mathbb{P}_{X_n};\, n \in \mathbb{N}\}$, where $\mathbb{P}_{X_n}$ denotes the law of the random variable X_n with respect to $\mathbb{P}$. In fact, let $\varepsilon > 0$ be arbitrary. Given that $N \triangleq \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\Big[|X_n|^{1/2}\Big] \in [0, +\infty)$, choosing $M = M(\varepsilon)$ such that $M > (S/\varepsilon)^2$ and setting $K = [-M, M]$, we obtain by Chebyshev's inequality

$$\mathbb{P}_{X_n}(K^c) = \mathbb{P}\{|X_n| > M\} \leq \frac{\mathbb{E}_{\mathbb{P}}\Big[|X_n|^{1/2}\Big]}{M^{1/2}} < \varepsilon.$$

Thus, by Prokhorov [46] theorem, we can extract a weakly convergent subsequence $\Big\{\mathbb{P}_{X_{n_k}};\, k \in \mathbb{N}\Big\}$, and we write $\mathbb{P}_{X_{n_k}} \xrightarrow{w} \nu$ for some probability measure ν on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Next, let $q_\rho^{\mathbb{P}}$ denote the quantile function of ρ with respect to $\mathbb{P}$, which is unique up to a set of Lebesgue measure zero. Then, by our Assumption III.3.13, the $\sigma(\rho)$-measurable random variable $U \triangleq F_\rho^{\mathbb{P}}(\rho)$ follows under $\mathbb{P}$ a uniform distribution[7] on the interval $(0, 1)$, and moreover $\rho = q_\rho^{\mathbb{P}}(U)$ $\mathbb{P}$-a.s. (recall Lemma B.8).

So take $X_* \triangleq q_\nu(1 - U)$, which is clearly a $\sigma(\rho)$-measurable random variable. In addition, because $1 - U$ is also uniformly distributed on $(0, 1)$ under $\mathbb{P}$, we conclude by Lemma B.7 that X_* has probability law ν, hence the subsequence of random variables $\{X_{n_k};\, k \in \mathbb{N}\}$ converges in distribution to X_* as $k \to +\infty$, and we write $X_{n_k} \xrightarrow{\mathscr{D}} X_*$.

We shall now check, in three separate steps, that X_* satisfies all the right properties.

(i) Firstly, we see that the Choquet integrals $V_\pm(X_*^\pm)$ are both finite. In fact, given that the maximum function and u_+ are continuous, by the mapping theorem we have that the sequence of random variables $\big\{u_+\big(X_{n_k}^+\big);\, k \in \mathbb{N}\big\}$ converges in distribution to $u_+(X_*^+)$. Therefore, $\lim_{k\to+\infty} \mathbb{P}\big\{u_+\big(X_{n_k}^+\big) > y\big\} = \mathbb{P}\{u_+(X_*^+) > y\}$ for every $y \in \mathbb{R}$ at which the CDF of $u_+(X_*^+)$ is continuous (and we recall that any non-decreasing function has, at most, countably many discontinuities). Similarly, we deduce that $\lim_{k\to+\infty} \mathbb{P}\big\{u_-\big(X_{n_k}^-\big) > y\big\} = \mathbb{P}\{u_-(X_*^-) > y\}$ for all y outside a countable set. Since the distortion functions $w_\pm$ are continuous as well, it is again obvious by the mapping theorem that $\lim_{k\to+\infty} w_\pm\big(\mathbb{P}\big\{u_\pm\big(X_{n_k}^\pm\big) > y\big\}\big) = w_\pm(\mathbb{P}\{u_\pm(X_*^\pm) > y\})$ for Lebesgue a.e. y. Therefore, applying Fatou's lemma (we recall that the distortion functions are non-negative) we get

$$V_\pm\big(X_*^\pm\big) = \int_0^{+\infty} w_\pm\big(\mathbb{P}\big\{u_\pm\big(X_*^\pm\big) > y\big\}\big)\, dy$$

[7] Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a non-degenerate interval $I \subseteq \mathbb{R}$, we recall that a random variable $U : \Omega \to$ is said to have a *(continuous) uniform distribution* on I with respect to $\mathbb{P}$, and we write $U \overset{\mathbb{P}}{\sim} \mathscr{U}(I)$, if for every $B \in \mathscr{B}(\mathbb{R})$,

$$\mathbb{P}\{U \in B\} = \frac{\ell(B \cap I)}{\ell(I)},$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$. In the special case where $I = (0, 1)$, we write that U has a *standard uniform distribution.*

$$\leq \liminf_{k\in\mathbb{N}} \int_0^{+\infty} w_\pm\big(\mathbb{P}\big\{u_\pm\big(X^\pm_{n_k}\big) > y\big\}\big)\, dy = \liminf_{k\in\mathbb{N}} V_\pm\big(X^\pm_{n_k}\big)\,.$$

But $\liminf_{k\in\mathbb{N}} V_\pm\big(X^\pm_{n_k}\big) \leq \sup_{k\in\mathbb{N}} V_\pm\big(X^\pm_{n_k}\big) \leq \sup_{n\in\mathbb{N}} V_\pm(X^\pm_n)$, and we know by Assumption III.3.33 (as well as Remark III.3.37) that $\sup_{n\in\mathbb{N}} V_\pm(X^\pm_n) < +\infty$, so we have the intended result.

(ii) Secondly, we obtain that the inequality $V(X_*) \geq v^*(x_0)$ holds. We already know, by the previous step, that $V_-(X^-_*) \leq \liminf_{k\in\mathbb{N}} V_-\big(X^-_{n_k}\big)$. We note further that, due to Assumption III.3.34, we can apply the reverse Fatou lemma to obtain

$$V_+\big(X^+_*\big) \geq \limsup_{k\in\mathbb{N}} \int_0^{+\infty} w_+\big(\mathbb{P}\big\{u_+\big(X^+_{n_k}\big) > y\big\}\big)\, dy = \limsup_{k\in\mathbb{N}} V_+\big(X^+_{n_k}\big)\,.$$

Combining the previous inequalities then yields

$$\begin{aligned} V(X_*) &\geq \limsup_{k\in\mathbb{N}} V_+\big(X^+_{n_k}\big) - \liminf_{k\in\mathbb{N}} V_-\big(X^-_{n_k}\big) \\ &\geq \limsup_{k\in\mathbb{N}} \big\{V_+\big(X^+_{n_k}\big) - V_-\big(X^-_{n_k}\big)\big\} = v^*(x_0)\,, \end{aligned}$$

where the last inequality is a straightforward consequence of $\liminf_{k\in\mathbb{N}} V_-\big(X^-_{n_k}\big) = -\limsup_{k\in\mathbb{N}}\big\{-V_-\big(X^-_{n_k}\big)\big\}$ combined with the sub-additivity of the limit superior and with the fact that any subsequence of a convergent sequence is also convergent (to the same limit).

(iii) Lastly, it remains only to show that $\mathbb{E}_{\mathbb{P}^*}[X_*] \leq x_0$, i.e., that the $\mathbb{P}^*$-price of the claim X_* does not exceed the investor's wealth. This will be done using an argument of Reichlin [52, Subsection III.4.1, p. 63], who in turn borrows ideas from other authors, as cited above. We remark, however, that some modifications are required to account for the fact that here wealth is allowed to become negative.

First, it is trivial by the way X_* was chosen that $\mathbb{E}_{\mathbb{P}^*}[X^\pm_*]$ equals

$$\mathbb{E}_{\mathbb{P}}\big[\rho\, X^\pm_*\big] = \mathbb{E}_{\mathbb{P}}\Big[q^{\mathbb{P}}_\rho(U)\,[q_\nu(1-U)]^\pm\Big] = \int_0^1 q^{\mathbb{P}}_\rho(x)\,[q_\nu(1-x)]^\pm\, dx$$

(recall that $\rho > 0$ a.s., therefore $q^{\mathbb{P}}_\rho$ is non-negative a.e. on $(0,1)$). Furthermore, the fact that the family $\{X_{n_k};\, k\in\mathbb{N}\}$ converges in distribution to X_* implies that the sequence of quantile functions $\Big\{q^{\mathbb{P}}_{X_{n_k}};\, k\in\mathbb{N}\Big\}$ converges to q_ν a.e. on $(0,1)$ (see, e.g., Cinlar [18, Proposition III.5.7]).

Now it is quite immediate to deal with the positive part $\mathbb{E}_{\mathbb{P}^*}[X^+_*]$. Indeed, since the positive part function is non-decreasing and continuous, we can combine Fatou's lemma with one of the Hardy-Littlewood inequalities (see Theorem B.10) to obtain

$$\int_0^1 q^{\mathbb{P}}_\rho(x)\,[q_\nu(1-x)]^+\, dx \leq \liminf_{k\to+\infty} \int_0^1 q^{\mathbb{P}}_\rho(x)\,\Big[q^{\mathbb{P}}_{X_{n_k}}(1-x)\Big]^+\, dx$$

$$= \liminf_{k\to+\infty} \int_0^1 q_\rho^{\mathbb{P}}(x)\, q_{X_{n_k}^+}^{\mathbb{P}}(1-x)\ dx \le \liminf_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[\rho\, X_{n_k}^+\right],$$

where the equality is a trivial consequence of the fact that $\left[q_{X_{n_k}}^{\mathbb{P}}(x)\right]^+ = q_{X_{n_k}^+}^{\mathbb{P}}(x)$ for a.e. $x \in (0,1)$, as recalled in Lemma B.9.

Turning to the negative part $\mathbb{E}_{\mathbb{P}^*}[X_*^-]$, we first note the uniform integrability of the family of a.e. non-negative functions $\left\{q_\rho^{\mathbb{P}} q_{X_{n_k}^-}^{\mathbb{P}}\,;\, k \in \mathbb{N}\right\}$ on $(0,1)$. To see this, choose some $\eta' \in (1,\eta)$, and then Hölder's inequality with $\eta/\eta' > 1$ yields, for every $k \in \mathbb{N}$,

$$\int_0^1 \left[q_\rho^{\mathbb{P}}(x)\, q_{X_{n_k}^-}^{\mathbb{P}}(x)\right]^{\eta'} dx \le \mathbb{E}_{\mathbb{P}}\left[\left(q_\rho^{\mathbb{P}}(U)\right)^{\frac{\eta\,\eta'}{\eta-\eta'}}\right]^{\frac{1}{\eta'}-\frac{1}{\eta}} \mathbb{E}_{\mathbb{P}}\left[\left(q_{X_{n_k}^-}^{\mathbb{P}}(U)\right)^{\eta}\right]^{\frac{\eta'}{\eta}}$$

$$= C\, \mathbb{E}_{\mathbb{P}}\left[\left(X_{n_k}^-\right)^{\eta}\right]^{\frac{\eta'}{\eta}} \le C\left(\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\left[\left(X_n^-\right)^{\eta}\right]\right)^{\frac{\eta'}{\eta}} < +\infty,$$

where we use that each random variable $q_{X_{n_k}^-}^{\mathbb{P}}(U)$ has the same distribution as $X_{n_k}^-$, and we invoke Assumption III.3.9 to ensure that $C \triangleq \mathbb{E}_{\mathbb{P}}\left[\rho^{\eta\eta'/(\eta-\eta')}\right]^{1/\eta'-1/\eta}$ is a strictly positive real number. Hence, by de la Vallée-Poussin lemma, the claim follows.

But the negative part function is non-increasing, so $q_{X_{n_k}^-}^{\mathbb{P}}(x) = \left[q_{X_{n_k}}^{\mathbb{P}}(1-x)\right]^-$ for a.e. $x \in (0,1)$ and for any $k \in \mathbb{N}$ (again by Lemma B.9). Moreover, it is a continuous function as well, thus $\lim_{k\to+\infty}\left[q_{X_{n_k}}^{\mathbb{P}}(1-x)\right]^- = [q_\nu(1-x)]^-$ for a.e. $x \in (0,1)$. These facts combined with uniform integrability give L^1-convergence, therefore

$$\lim_{k\to+\infty} \int_0^1 q_\rho^{\mathbb{P}}(x)\, q_{X_{n_k}^-}^{\mathbb{P}}(x)\ dx = \int_0^1 q_\rho^{\mathbb{P}}(x)\, [q_\nu(1-x)]^-\ dx.$$

On the other hand, it follows from the second Hardy-Littlewood inequality that

$$\mathbb{E}_{\mathbb{P}}\left[\rho\, X_{n_k}^-\right] \le \int_0^1 q_\rho^{\mathbb{P}}(x)\, q_{X_{n_k}^-}^{\mathbb{P}}(x)\ dx,$$

for every $k \in \mathbb{N}$.

Combining all of the above inequalities, using the super-additivity of the limit inferior, and recalling the feasibility of each X_{n_k}, we conclude

$$\mathbb{E}_{\mathbb{P}^*}[X_*] \le \liminf_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[\rho\, X_{n_k}^+\right] - \limsup_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\left[\rho\, X_{n_k}^-\right] \le \liminf_{k\to+\infty} \mathbb{E}_{\mathbb{P}}[\rho\, X_{n_k}] = x_0.$$

Finally, it is also straightforward to check that X_* belongs to $L^1(\mathbb{P}^*)$, since

$$\mathbb{E}_{\mathbb{P}^*}[|X_*|] = \mathbb{E}_{\mathbb{P}^*}[X_*] + 2\,\mathbb{E}_{\mathbb{P}^*}\left[X_*^-\right] \leq x_0 + 2 \lim_{k\in\mathbb{N}} \int_0^1 q_\rho^{\mathbb{P}}(x)\, q^{\mathbb{P}}_{X^-_{n_k}}(x)\; dx < +\infty,$$

hence, by Assumption III.3.7, X_* admits a replicating portfolio $\overline{\phi}^*$ from initial capital $\mathbb{E}_{\mathbb{P}^*}[X_*] \leq x_0$. A fortiori, with initial capital x_0 one also has $V(\Pi_T^{\overline{\phi}^*}) \geq v^*(x_0)$, so $\overline{\phi}^*$ is an optimal strategy. □

6.3 Proofs of Section III.4

Proof of Theorem III.4.5. By contraposition. Suppose $\liminf_{x\to 0^+} w_-(x)\, u_-(1/x) = 0$. Combining the proof of Proposition III.4.4 with Lemma III.6.1 gives that $v^*(x_0) = u_+(+\infty)$, and so it follows from Lemma III.6.2 that (BPP) is not attainable. □

Proof of Theorem III.4.13. Fix an arbitrary $\delta \in (0, 1)$. The idea is to apply Theorem III.3.36, so we must ensure that all hypothesis are fulfilled. So consider any maximising sequence $\{\overline{\phi}_n;\, n \in \mathbb{N}\}$. That Assumption III.3.33 holds, is trivial by the elementary Lemma III.3.15. Thus, we know by Remark III.3.37 that we also have $\sup_{n\in\mathbb{N}} V_-\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right) < +\infty$.

Besides, it is obvious that $g : [0, +\infty) \to [0, +\infty)$ defined by $g \triangleq \mathbb{1}_{[0,u_+(+\infty)]}$ satisfies the conditions of Assumption III.3.34.

Finally, since Assumption III.4.12 is in force, we have by Lemma III.6.7 that there exist real numbers $\eta > 1$ and $C, D > 0$, as well as a non-increasing function G such that

$$\mathbb{E}_{\mathbb{P}}\left[\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)^\eta\right] \leq C + \frac{\left[G\left(D\left[V^\delta_{u_-}\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)\right]^{-1}\right)\right]^\eta}{(u_-)^{-1}\left(D\left[V^\delta_{u_-}\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)\right]^{-1/\delta}\right)}$$

for every $n \in \mathbb{N}$ (with $V^\delta_{u_-}\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)$ as in (III.6.4)). Noting that $w_- \geq w^\delta_{u_-}$ implies

$$\frac{\left[G\left(D\left[V^\delta_{u_-}\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)\right]^{-1}\right)\right]^\eta}{(u_-)^{-1}\left(D\left[V^\delta_{u_-}\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)\right]^{-1/\delta}\right)} \leq \frac{\left[G\left(D\left[V_-\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)\right]^{-1}\right)\right]^\eta}{(u_-)^{-1}\left(D\left[V_-\left(\left[\Pi_T^{\overline{\phi}_n}\right]^-\right)\right]^{-1/\delta}\right)}$$

for every $n \in \mathbb{N}$ yields equation (III.3.13), and so the proof is completed. □

Proof of Lemma III.4.14. Fix $\delta \in (0, 1)$ arbitrary and choose $\varsigma \in \left(1, \delta^{-1/\gamma}\right)$. Then, for every $x \geq \underline{x}$, we have $z(x) - \delta\, z(\varsigma x) \geq z(x)\,[1 - \delta\,\varsigma^\gamma]$. Given that $z(+\infty) = +\infty$ and $\delta\,\varsigma^\gamma < 1$, we obtain that $\liminf_{x\to+\infty}\,[z(x) - \delta\, z(\varsigma x)] = +\infty$, and finally we use Lemma III.6.6 to infer that Assumption III.4.12 holds true. □

6.4 Proofs of Section III.5

Proof of Theorem III.5.5. We shall divide the proof into three separate parts.

(i) It is obvious that $u_+(+\infty) = +\infty$ and $AE_+(u_-) = \alpha_- < +\infty$. Therefore, we are in condition to apply Theorem III.3.29, which gives that, if $\alpha_- < \alpha_+$ or $\beta_+ < \alpha_+$ or $\beta_- \geq \alpha_-$, then the behavioural problem is ill-posed.

(ii) Under the stated hypothesis, Proposition III.3.17 further tells us that (BPP) is still ill-posed when $\alpha_- = \alpha_+$.

(iii) It remains only to examine what happens for $\beta_+ = \alpha_+$. A slight modification of the proof of Proposition III.3.24, under the additional Assumption III.5.4, will allow us to obtain the ill-posedness of the optimisation problem in this case as well. Indeed, letting w_+ be an arbitrary distortion and setting

$$l \triangleq \limsup_{x \to 0^+} \frac{w_+(x)}{x^{\alpha_+}},$$

let us assume that $l \in (0, +\infty)$. Then, as before, we can use the definition of limit superior and Assumption III.3.13 to find two strictly decreasing sequences of strictly positive real numbers $\{a_n;\, n \in \mathbb{N}\}$ and $\{b_n;\, n \in \mathbb{N}\}$, whose terms satisfy all three conditions $a_n \in (0, 1/n)$, $w_+(a_n) \,/\, (a_n)^{\alpha_+} > l - 1/n$ and $\mathbb{P}\{\rho \leq b_n\} = a_n$. It is also clear by construction that $\lim_{n\to+\infty} a_n$ exists and is equal to zero. Moreover, we must have $\lim_{n\to+\infty} b_n = 0$, otherwise

$$\mathbb{P}\left\{\rho < \inf_{n\in\mathbb{N}} b_n\right\} \leq \mathbb{P}\left(\bigcap_{n=1}^{+\infty} \{\rho \leq b_n\}\right) = \lim_{n\to+\infty} \mathbb{P}\{\rho \leq b_n\} = \lim_{n\to+\infty} a_n = 0$$

and thus $\inf_{n\in\mathbb{N}} b_n \in (0, +\infty)$ would be an essential lower bound for ρ, which would in turn contradict Assumption III.5.4.

Now, for each $n \in \mathbb{N}$, let A_n be the event $A_n \triangleq \{\rho \leq b_n\} \in \sigma(\rho)$. Note that, by the equivalence of the measures $\mathbb{P}$ and $\mathbb{P}^*$, we must have $\mathbb{P}^*(A_n) \in (0, 1)$. Besides,

$$\lim_{n\to+\infty} \mathbb{P}^*(A_n) = \mathbb{P}^*\left(\bigcap_{n=1}^{+\infty} A_n\right) = 0.$$

Therefore, we can define the non-negative, $\sigma(\rho)$-measurable random variables

$$X_n \triangleq \frac{|x_0| + b_n}{\mathbb{P}^*(A_n)} \mathbb{1}_{A_n} \qquad \text{and} \qquad Y_n \triangleq \frac{|x_0| - x_0 + b_n}{\mathbb{P}^*(A_n^c)} \mathbb{1}_{A_n^c}.$$

It is easy to see that

$$V_+(X_n) = \left(\frac{|x_0| + b_n}{\mathbb{P}^*(A_n)}\right)^{\alpha_+} w_+(\mathbb{P}(A_n)) \geq \left(\frac{|x_0| + b_n}{b_n}\right)^{\alpha_+} \frac{w_+(a_n)}{(a_n)^{\alpha_+}},$$

where the inequality follows from $\mathbb{P}^*(A_n) = \mathbb{E}_{\mathbb{P}}[\rho\, \mathbb{1}_{A_n}] \leq b_n\, \mathbb{P}(A_n)$ and $\alpha_+ > 0$. On the other hand,

$$V_-(Y_n) = u_-\left(\frac{|x_0| - x_0 + b_n}{\mathbb{P}^*(A_n^c)}\right) w_-(\mathbb{P}(A_n^c))\,.$$

Hence, taking Z_n to be the feasible claim given by $Z_n \triangleq X_n - Y_n$, it is straightforward to check that, for every $n \in \mathbb{N}$,

$$V(Z_n) \geq \left(\frac{|x_0| + b_n}{b_n}\right)^{\alpha_+} \left(l - \frac{1}{n}\right) - u_-\left(\frac{|x_0| - x_0 + b_n}{\mathbb{P}^*(A_n^c)}\right) w_-(\mathbb{P}(A_n^c))\,,$$

whence $\liminf_{n\to+\infty} V(Z_n) \geq +\infty - u_-(|x_0| - x_0) = +\infty$. □

Proof of Theorem III.5.7. The proof is by contradiction. Let us suppose that the optimisation problem is ill-posed, that is, $v^*(x_0) = +\infty$. Then we can find a sequence of trading strategies $\{\overline{\phi}_n;\, n \in \mathbb{N}\}$ in the feasible set $\mathscr{A}(x_0)$ with $\lim_{n\to+\infty} V\left(\Pi_T^{\overline{\phi}_n} - B\right) = +\infty$.

Since we have $V_+\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^+\right) \geq V\left(\Pi_T^{\overline{\phi}_n} - B\right)$ for every $n \in \mathbb{N}$, we deduce that

$$\lim_{n\to+\infty} \int_0^{+\infty} \mathbb{P}\left\{\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^+\right)^{\alpha_+} > y\right\}^{\beta_+} dy \geq \frac{1}{\kappa_+ k_+} \lim_{n\to+\infty} V_+\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^+\right) = +\infty$$

as well. Thus, it follows from Lemma III.6.9 that

$$\int_0^{+\infty} \mathbb{P}\left\{\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right)^{\eta} > y\right\}^{\beta_-} dy \xrightarrow[n\to+\infty]{} +\infty,$$

for some $\eta \in (\alpha_+ \vee \beta_-, \alpha_-)$. Consequently, we apply the second part of Lemma III.6.10 to conclude that also

$$\lim_{n\to+\infty} V_-\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right) \geq \kappa_- k_- \lim_{n\to+\infty} \int_0^{+\infty} \mathbb{P}\left\{\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right)^{\alpha_-} > y\right\}^{\beta_-} dy = +\infty.$$

One final application of Lemmata III.6.9 and III.6.10 yields

$$\begin{aligned} V\left(\Pi_T^{\overline{\phi}_n} - B\right) &\leq L_1 + L_2 \int_0^{+\infty} \mathbb{P}\left\{\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right)^{\eta} > y\right\}^{\beta_-} dy - V_-\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right) \\ &\leq (R_1 + L_2 R_1) + L_2 R_2 \left[V_-\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right)\right]^{\zeta} - V_-\left(\left[\Pi_T^{\overline{\phi}_n} - B\right]^-\right), \end{aligned}$$

for some $0 < \zeta < 1$, and this inequality combined with the previous equation gives $\lim_{n\to+\infty} V\left(\Pi_T^{\overline{\phi}_n} - B\right) = -\infty$, which is absurd. Hence, as claimed, the problem is well-posed. □

Proof of Theorem III.5.9. In order to be able to use Theorem III.3.36, first we must show that its hypothesis hold true. So consider an arbitrary maximising sequence

$\{\overline{\phi}_n;\, n \in \mathbb{N}\}$, and let $\lambda > 0$ be exactly as in the proof of Lemma III.6.9.

Denoting the terminal wealth of each portfolio $\overline{\phi}_n$ by X_n for simplicity of notation, we shall begin by showing that $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\Big[(X_n^+)^{\alpha+\lambda}\Big] < +\infty$, the idea being quite similar to that of the proof of Theorem III.5.7. Assume by contradiction that $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\Big[(X_n^+)^{\alpha+\lambda}\Big] = +\infty$. Then it is possible to choose a subsequence such that $\lim_{k\to+\infty} \mathbb{E}_{\mathbb{P}}\Big[\big(X_{n_l}^+\big)^{\alpha+\lambda}\Big] = +\infty$. By equation (III.6.11) in the proof of Lemma III.6.9, we conclude that

$$\int_0^{+\infty} \mathbb{P}\big\{\big(X_{n_l}^-\big)^{\eta} > y\big\}^{\beta_-}\, dy \xrightarrow[l\to+\infty]{} +\infty,$$

where η is as defined in the proof. Therefore, using Lemma III.6.10 we also obtain that $\lim_{k\to+\infty} V_-\big(X_{n_k}^-\big) = +\infty$, and hence

$$\begin{aligned} V\left(X_{n_k}\right) &\leq L_1 + L_2 \int_0^{+\infty} \mathbb{P}\big\{\big(X_{n_k}^-\big)^{\eta} > y\big\}^{\beta_-}\, dy - V_-\big(X_{n_k}^-\big) \\ &\leq C_1 + C_2 \left(V_-\big(X_{n_k}^-\big)\right)^{\zeta} - V_-\big(X_{n_k}^-\big) \xrightarrow[k\to+\infty]{} -\infty, \end{aligned}$$

where the first and second inequalities follow, respectively, from Lemma III.6.9 and Lemma III.6.10 (note that C_1 and C_2 are strictly positive constants depending only on the parameters, and that $\zeta \in (0,1)$). But this is absurd, because $\lim_{n\in\mathbb{N}} V(X_n) = v^*(x_0) > -\infty$ and therefore any subsequence of $V(X_n)$ must also converge to $v^*(x_0)$.

Next, combining Chebyshev's inequality with the fact that $\beta_+ > 0$ gives

$$\mathbb{P}\big\{\big(X_n^+\big)^{\alpha_+} > y\big\}^{\beta_+} \leq \frac{\mathbb{E}_{\mathbb{P}}\Big[(X_n^+)^{\lambda\alpha_+}\Big]^{\beta_+}}{y^{\lambda\beta_+}}$$

for all $y > 0$ and $n \in \mathbb{N}$.

Therefore, on the one hand, taking $C_3 \triangleq k_+\kappa_+^{\lambda\beta_+} \int_0^{+\infty} y^{-\lambda\beta_+}\, dy$, which is finite (recall $\lambda\beta_+ > 1$) and strictly positive, it is trivial that $V_+(X_n^+) \leq 1 + C_3\, \mathbb{E}_{\mathbb{P}}\Big[(X_n^+)^{\alpha+\lambda}\Big]^{\beta_+}$ for every $n \in \mathbb{N}$, so using the definition of supremum we conclude that $\sup_{n\in\mathbb{N}} V_+(X_n^+) < +\infty$, thus establishing Assumption III.3.33.

On the other hand, choosing $g\colon\ [0,+\infty) \to [0,+\infty)$ to be the function defined by $g(y) \triangleq 1$ if $y \in [0,1]$, and $g(y) \triangleq k_+\kappa_+^{\lambda\beta_+} \Big(\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\Big[(X_n^+)^{\lambda\alpha_+}\Big]\Big)^{\beta_+} / y^{\lambda\beta_+}$ when $y > 1$, it is straightforward to check that g is integrable (again because $\lambda\beta_+ > 1$). Hence, Assumption III.3.34 is verified.

Finally, since the parameters fulfill (III.5.3) by hypothesis, we know by Theorem III.5.7 that (BPP) is well-posed, therefore Remark III.3.37 gives $\sup_{n\in\mathbb{N}} V_-(X_n^-) < +\infty$ as well. Additionally, we may choose $\xi \in (1, \alpha_-/\beta_-)$, and so it follows from Lemma III.6.8 that there exists some $D \in (0,+\infty)$ such that $\mathbb{E}_{\mathbb{P}}\Big[(X_n^-)^{\xi}\Big] \leq 1 + D\left(V_-(X_n^-)\right)^{1/\beta_-}$ for all $n \in \mathbb{N}$, which in turn implies that $\sup_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}\Big[(X_n^-)^{\xi}\Big] < +\infty$. Consequently, condition (III.3.13) is true as well, and invoking Theorem III.3.36 establishes the existence of an optimal strategy, as claimed. □

Chapter IV

Conclusions and future work

As already stated in previous chapters, in this thesis we analysed the optimal portfolio choice problem for non-EUT economic agents in finite-horizon financial markets.

We started by considering a stochastic model with only a finite number of trading dates, which was free of arbitrage opportunities, but possibly (and generically) incomplete. We also assumed that the investors' preferences were described by utility functions satisfying continuity and monotonicity, but not necessarily concavity. In addition, because the condition that wealth should remain non-negative was imposed, the domain of these utilities was restricted to the non-negative half-line (instead of the whole real line). Then, under an assumption ensuring the well-posedness of the optimisation problem and another one on the growth of the utility (related to the well-known concept of asymptotic elasticity), the existence of an optimal trading strategy was established using a dynamic programming method.

As for the second part of this work, which was developed independently of the first one, the optimal portfolio problem was studied for an investor behaving fundamentally in accordance with CPT, this time assuming an essentially complete continuous-time market and that asset prices were modelled with general semi-martingales. Unlike in the preceding chapter and in a vast body of the literature on portfolio optimisation, rather than taking the well-posedness of the behavioural portfolio problem as an assumption, we began with the derivation of necessary conditions for well-posedness, which besides being fairly straightforward to verify, have a clear financial meaning. Only after that did we go on to demonstrate the existence of an optimal strategy under suitable conditions.

Next, we restricted our attention to two important special cases, one where the investor's utility on gains was bounded above, and the other one in which the utilities and the probability distortions were power-like functions. While well-posedness was trivial in the first case, in the second one we identified several situations which were apparently harmless and yet ill-posed. We were then able to find conditions on the parameters for well-posedness, which turned out to be not only necessary, but also sufficient. We remark that these are very similar to others already obtained, or at least referred to, in earlier works in discrete-time markets (see Bernard and Ghossoub

[9], He and Zhou [27] and Carassus and Rásonyi [13]). With respect to the issue of attainability in the first case, our main contribution consisted of the introduction of a special type of distortion which, under an additional growth condition on the utility of losses (associated with the celebrated notion of asymptotic elasticity, and satisfied by a large class of functions, including some of the most popular utilities in the literature), allowed us to obtain the borderline for existence. We would like to extend this result to general unbounded u_+.

We wish to stress, however, that our results in both chapters are abstract existence results, and that no optimal portfolio or terminal wealth was explicitly obtained. It would be interesting if, as in Jin and Zhou [29] for instance, an optimal solution could be characterised explicitly, in order to compare it with that of EUT or to analyse its sensitivity to the market parameters (such as the finite trading horizon). Nonetheless, it should be noted that our existence results are shown to be valid in quite a large number of cases. In particular, just to mention an example, we are capable of accommodating the original CPT distortions in our study of the behavioural portfolio problem, whereas it is not clear whether the results of Jin and Zhou [29] apply in this case (again we refer to Assumption 4.1 therein).

In addition, we are aware that we did not address here the natural question whether these optimal portfolios are unique or not, but taking into account the absence of uniqueness in very simple examples such as the ones presented in our Subsections II.2.4 and III.2.4, it seems hopeless to get uniqueness.

Furthermore, we included in this work (see Subsection III.3.4) an application of our results of Chapter III to a multidimensional diffusion model, as well as to a very particular jump-diffusion model. Extending the existence result to more general models, namely to incomplete diffusion markets with stochastic coefficients, is work in progress.

It should be noted as well that, even though we allowed for our model in Chapter III to be incomplete, we did ultimately fix one ELMM and developed our results with respect to that measure (in particular, our feasible set for the optimisation problem was not measure invariant, in that it strongly depended on which specific ELMM we considered). It would be interesting if we could study the problem over another domain of optimisation.

Finally, there exist several other potential and logical directions for further research. The first one involves studying these problems in markets with frictions, for example illiquid markets. Another one would be to see whether one can use these results for pricing claims, as it is done in the context of EUT. Obviously, for the reasons already explained, this problem appears to be very difficult to solve. A third suggestion would consist of studying new risk measures, for instance, an entropic risk measure with distortions (and to examine if, or under which conditions, this would be a convex or coherent risk measure, see e.g. Föllmer and Schied [25, Chapter 4]).

Appendices

These appendices contain a collection of some widely established definitions and results in the literature, which are used in the main body of this work.

A Asymptotic elasticity

We follow Section 6 of Kramkov and Schachermayer [35].

In this appendix, let us consider a non-decreasing function $f: [0, +\infty) \to \mathbb{R} \cup \{-\infty\}$ with non-empty effective domain. We start with the following.

Definition A.1 (Elasticity). Suppose f is differentiable on $(0, +\infty)$. The *elasticity* of f at a point $x > 0$ with $f(x) > 0$ is defined as $E_f(x) \triangleq x f'(x) / f(x)$.

Remark A.2. The elasticity of f at a point $x > 0$ with $f(x) > 0$ gives us, approximately, the relative change of f with respect to small relative changes in x. Note that the monotonicity of f ensures that the elasticity of f at any point $x > 0$ with $f(x) > 0$ is a non-negative real number. ◇

The next result, whose proof is straightforward, identifies the functions with constant elasticity.

Proposition A.3. *Let $k \in [0, +\infty)$ and suppose f is differentiable on $(0, +\infty)$. Then f has constant elasticity k on $(0, +\infty)$ if and only if $f(x) = f(1)\, x^k$ for all $x \in (0, +\infty)$.* □

Next, we present the definition of asymptotic elasticity, a concept which was first introduced in the financial mathematics literature by Kramkov and Schachermayer [35, Definition 2.2] (even though close notions can already be found in the earlier papers of Karatzas, Lehoczky, Shreve, and Xu [33, Assumption 4.3] and Cvitanić and Karatzas [19, Equation (5.4)]).

Definition A.4 (Asymptotic elasticity). Suppose that $f(x_0) \geq 0$ for some $x_0 \geq 0$. Then we call

$$AE_+(f) \triangleq \inf\{\gamma > 0 \colon \exists\, \overline{x} \geq 0 \text{ such that } f(\lambda x) \leq \lambda^\gamma f(x)\,,\ \forall\, \lambda \geq 1,\ \forall\, x \geq \overline{x}\} \qquad \text{(A.1)}$$

(with the usual convention that the infimum of the empty set is $+\infty$) the *asymptotic elasticity* (AE) of f at $+\infty$.

Remark A.5. By definition, the asymptotic elasticity of any function (verifying the required condition) is non-negative (and possibly infinite). ◇

The following lemma, which has an immediate proof, identifies a special class of functions with asymptotic elasticity not exceeding one.

Lemma A.6. *If there exists some $x_0 \geq 0$ so that $f(x_0) \geq 0$ and f is concave on $[x_0, +\infty)$, then $AE_+(f) \in [0, 1]$.* □

Adapted from Schachermayer [58, Definition 1.5], we have the following.

Definition A.7 (Reasonable asymptotic elasticity). A function f with $f(x_0) \geq 0$ for some $x_0 \geq 0$ is said to satisfy the *reasonable asymptotic elasticity* (or *RAE*) condition if $AE_+(f) < 1$.

The next result, which has a trivial proof, is essentially the statement of Lemma 6.5 in Kramkov and Schachermayer [35], but without requiring the differentiability of f.

Lemma A.8. *Let us assume that $f(x_0) \geq 0$ for some $x_0 \geq 0$. If the RAE condition is true for f, then there exist $\gamma \in (0, 1)$, $\overline{x} > 0$ and $C > 0$ such that $f(x) \leq C\, x^\gamma$ for all $x \geq \overline{x}$.*

Remark A.9. Note that the reciprocal of Lemma A.8 is not true in general, as shown in Kramkov and Schachermayer [35, Lemma 6.5], or in our Example A.13 below. ◇

The subsequent result provides an alternative characterisation of asymptotic elasticity, which makes explicit its connection with the notion of elasticity.

Proposition A.10. *Let us assume that there is some $x_0 \geq 0$ for which $f(x_0) > 0$ and f is continuously differentiable on $(x_0, +\infty)$. Then,*

$$AE_+(f) = \limsup_{x \to +\infty} E_f(x)\,. \tag{A.2}$$

Proof. This is part of Lemma 6.3 in Kramkov and Schachermayer [35]. We draw attention to the fact that the proof only uses the monotonicity and the continuous differentiability of f, not its concavity or the Inada conditions on its first derivative[1]. □

Remark A.11. It should be pointed out that the definition of asymptotic elasticity which is commonly found in the literature is not the one given in Definition A.4, but that given by (A.2). The reason why we choose to use the former is that it does not require differentiability. ◇

Lemma A.12. *Let f and g be as in Proposition A.10. The following assertions are true,*

(i) *if f is concave on $[x_0, +\infty)$ with $f(+\infty) \in (0, +\infty)$, then $AE_+(f) = 0$,*

(ii) *if f and g differ only by a positive affine transformation, then $AE_+(f) = AE_+(g)$.*

[1] The first derivative of a strictly increasing and differentiable function f is said to satisfy the *Inada conditions* if $\lim_{x\to 0^+} f'(x) = +\infty$ and $f'(+\infty) = 0$.

Proof. This is part (ii) of Lemma 6.1, and Lemma 6.2 in Kramkov and Schachermayer [35]. □

We finish this section with the following example, showing that a non-decreasing and continuously differentiable function can be bounded above and yet have non-zero (actually, infinite) asymptotic elasticity, so the assumption of concavity in the previous lemma cannot be dropped. The construction of this example is inspired by the proof of Lemma 6.5 of Kramkov and Schachermayer [35].

Example A.13. *Let $f : [0, +\infty) \to \mathbb{R}$ be the continuous and strictly increasing function which takes the values*

$$f(n) \triangleq \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)},$$

$$f(n+a_n) \triangleq \frac{n-1}{2(n+1)} - \frac{a_n}{2(n+1)(n+2)},$$

$$f(n+1-a_n) \triangleq \frac{n}{2(n+2)} - \frac{a_n}{2(n+2)(n+3)},$$

with $a_n \triangleq \left(n^3+6n^2+10n+3\right) / \left(2n^3+12n^2+21n+10\right) \in (0, 1/2)$, and which is linear on the intervals $[n, n+a_n]$, $[n, n+a_n]$, $[n+a_n, n+1-a_n]$ and $[n+1-a_n, n+1]$, for every $n \in \mathbb{N}_0$.

Clearly $f(+\infty) = 1/2$ and $f(1) = 0$. We also note that $f(0) = -1/2 > -\infty$. Moreover, the piecewise linearity of f and trivial computations yield

$$f'(x) = \frac{f(n+1-a_n) - f(n+a_n)}{1-2a_n} = 1$$

for any $x \in (n+a_n, n+1-a_n)$, so in particular $f'(n+1/2)$ equals 1. Furthermore, we have the following inequality,

$$\frac{f(n+1/2)}{n+1/2} \le \frac{f(n+1)}{n+1/2} = \frac{n}{(n+2)(2n+1)},$$

thus combining all of the above gives $\lim_{n\to+\infty} (n+1/2) f'(n+1/2) / f(n+1/2) = +\infty$, and hence $AE_+(u) = +\infty$. We finish by noticing that, as in the proof of Lemma 6.5 in Kramkov and Schachermayer [35], f can be slightly modified in such a way that it becomes smooth enough and our conclusion is still valid.

B Measure theory

B.1 General results

Theorem B.1 (Measurable projection). *Let (X, Σ, μ) be a complete measure space and $n \in \mathbb{N}$. If E belongs to the product σ-algebra $\Sigma \otimes \mathscr{B}(\mathbb{R}^n)$, then its projection on X,*

$$\mathrm{Proj}_X(E) \triangleq \{x \in X \colon \exists y \in \mathbb{R}^n \text{ such that } (x,y) \in E\}, \tag{B.1}$$

belongs to Σ.

Proof. This is a simplified version of Theorem III.23 in Castaing and Valadier [16]. □

The following results states that any sequence of random variables which is bounded in a certain sense admits an almost surely convergent random subsequence, so it can be regarded as a '"randomized" version of the Bolzano-Weierstraß theorem', as noted by Föllmer and Schied [25, Lemma 1.63, pp. 38-39].

Proposition B.2 (Convergent random subsequence). *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and let* $d \in \mathbb{N}$. *If* $\{f_n;\ n \in \mathbb{N}\}$ *is a sequence of* $\mathscr{B}([0,+\infty)) \otimes \mathscr{F}$*-measurable functions* $f_n :\ [0,+\infty) \times \Omega \to \mathbb{R}^d$ *satisfying*

$$\mathbb{P}\left\{\omega \in \Omega\colon\ \forall\, x \geq 0,\ \liminf_{n\to+\infty} \|f_n(x,\omega)\|_{\mathbb{R}^d} < +\infty\right\} = 1, \tag{B.2}$$

then there exists a sequence $\{n_k;\ k \in \mathbb{N}\}$ *of* $\mathscr{B}([0,+\infty)) \otimes \mathscr{F}$*-measurable random variables* $n_k :\ [0,+\infty) \times \Omega \to \mathbb{N}$ *with* $\mathbb{P}\{\omega \in \Omega\colon\ \forall\, k \in \mathbb{N},\ \forall\, x \geq 0,\ n_{k+1}(x,\omega) > n_k(x,\omega)\} = 1$, *as well as some* $\mathscr{B}([0,+\infty)) \otimes \mathscr{F}$*-measurable function* $f :\ [0,+\infty) \times \Omega \to \mathbb{R}^d$, *such that for every* ω *outside some* $\mathbb{P}$*-null set,*

$$\forall\, x \geq 0, \qquad \lim_{k\to+\infty} f_{n_k(x,\omega)}(x,\omega) = f(x,\omega)\,. \tag{B.3}$$

Proof. This is essentially Lemma 2 in Kabanov and Stricker [30]. □

Remark B.3. Notice that, for every ω outside a $\mathbb{P}$-null set we have that $\lim_{k\in\mathbb{N}} n_k(x,\omega) = +\infty$ for all $x \geq 0$. Indeed, for $\mathbb{P}$-a.e. $\omega \in \Omega$, it can be shown by mathematical induction that $n_k(x,\omega) \geq k$ for all $k \in \mathbb{N}$ and $x \geq 0$, which implies the desired result. ◇

B.2 Cumulative distribution function and quantile function

A thorough survey of quantile functions, their properties and related results can be found in Föllmer and Schied [25, Appendix A.3].

Here, $(\Omega, \mathscr{F}, \mathbb{P})$ will be a probability space, $X :\ \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$ will be a random variable such that $|X| < +\infty$ a.s., and μ will be a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Definition B.4 (Cumulative distribution function). The *cumulative distribution function* of X with respect to $\mathbb{P}$ (respectively, of μ) is the non-decreasing and right-continuous function $F_X^{\mathbb{P}} :\ \mathbb{R} \to [0,1]$ (respectively, $F_\mu :\ \mathbb{R} \to [0,1]$) given by

$$F_X^{\mathbb{P}}(x) \triangleq \mathbb{P}(X \leq x) \qquad \text{(respectively, } F_\mu(x) \triangleq \mu(\,(-\infty, x]\,)\,) \tag{B.4}$$

for every $x \in \mathbb{R}$.

Definition B.5 (Quantile function). A function $q_X^{\mathbb{P}} : (0,1) \to \mathbb{R}$ is called a *quantile function* of the random variable X with respect to $\mathbb{P}$ if it is a generalised inverse of $F_X^{\mathbb{P}}$, i.e., if it is such that

$$F_X^{\mathbb{P}}\left(q_X^{\mathbb{P}}(p)-\right) \leq p \leq F_X^{\mathbb{P}}\left(q_X^{\mathbb{P}}(p)\right) \tag{B.5}$$

for any level $p \in (0,1)$, where $F_X^{\mathbb{P}}(x-) \triangleq \lim_{s\uparrow x} F_X^{\mathbb{P}}(s) = \mathbb{P}\{X < x\}$. Analogously, a quantile function q_μ of μ is a generalised inverse of F_μ.

Remark B.6. We know by Lemma A.15 in Föllmer and Schied [25] that all quantile functions are non-decreasing (thus measurable). The same lemma also tell us that any quantile function is unique, up to a subset of $(0,1)$ of Lebesgue measure zero. ◇

Lemma B.7. *Let U be a standard uniform random variable on $(\Omega, \Sigma, \mathbb{P})$. Then the distribution function with respect to $\mathbb{P}$ of the random variable given by*

$$Y(\omega) \triangleq q_X^{\mathbb{P}}(U(\omega)) \qquad \text{(respectively, } Y(\omega) \triangleq q_\mu(U(\omega))\text{)}, \tag{B.6}$$

for every $\omega \in \Omega$, is $F_X^{\mathbb{P}}$ (respectively, F_μ).

Proof. See Lemma A.19 in Föllmer and Schied [25]. □

The next result, which is a standard one in the literature, shows us how to find a uniformly distributed random variable on $(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma B.8. *If there exists a random variable X on $(\Omega, \mathscr{F}, \mathbb{P})$ with a continuous CDF, then the random variable*

$$U(\omega) \triangleq F_X^{\mathbb{P}}(X(\omega)), \qquad \omega \in \Omega, \tag{B.7}$$

has a standard uniform distribution with respect to $\mathbb{P}$. Moreover, $X = q_X^{\mathbb{P}}(U)$ $\mathbb{P}$-a.s..

Proof. This is Lemma A.21 in Föllmer and Schied [25]. □

Lemma B.9. *Let $f: \mathbb{R} \to \mathbb{R}$ be a function, and set $Y \triangleq f(X)$. If f is non-decreasing (respectively, non-increasing), then for a.e. $p \in (0,1)$,*

$$q_Y^{\mathbb{P}}(p) = f\left(q_X^{\mathbb{P}}(p)\right) \qquad \text{(respectively, } q_Y^{\mathbb{P}}(p) = f\left(q_X^{\mathbb{P}}(1-p)\right)\text{)}. \tag{B.8}$$

Proof. See Lemma A.23 in Föllmer and Schied [25]. □

Theorem B.10 (Hardy-Littlewood inequalities). *Given any two random variables X and Y on $(\Omega, \mathscr{F}, \mathbb{P})$, we have*

$$\int_0^1 q_X^{\mathbb{P}}(1-p)\, q_Y^{\mathbb{P}}(p)\; dp \leq \mathbb{E}[XY] \leq \int_0^1 q_X^{\mathbb{P}}(p)\, q_Y^{\mathbb{P}}(p)\; dp, \tag{B.9}$$

so long as all the integrals are well-defined.

Proof. Again, the reader is referred to Föllmer and Schied [25, Theorem A.24]. □

C Choquet integral

For a detailed treatment of this subject, the reader is referred e.g. to Denneberg [21, Chapter 5], or Wang and Klir [63, Chapter 11].

Let us consider a measurable space (X, Σ). The following concept was introduced by Choquet [17].

Definition C.1 (Capacity). A set function $\mu : \Sigma \to [0, +\infty]$ is called a *capacity* (or *fuzzy measure*) on (X, Σ) if it satisfies the properties below,

(i) $\mu(\varnothing) = 0$,

(ii) $\mu(E_1) \leq \mu(E_2)$ for every $E_1, E_2 \in \Sigma$ with $E_1 \subseteq E_2$ (*monotonicity*).

We say that a capacity is *normalised* if $\mu(X) = 1$. Moreover, μ is *continuous from below* (respectively, *continuous from above*) if, for any non-decreasing (respectively, non-increasing) sequence $\{E_n;\, n \in \mathbb{N}\} \subseteq \Sigma$

$$\lim_{n\to+\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{+\infty} E_n\right) \qquad \left(\text{respectively, } \lim_{n\to+\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{+\infty} E_n\right)\right). \tag{C.1}$$

Remark C.2. Note that a capacity may not be additive, that is, there may exist disjoint measurable sets E_1 and E_2 such that $\mu(E_1 \cup E_2) \neq \mu(E_1) + \mu(E_2)$. ◇

Example C.3. *Given any probability measure $\mathbb{P}$ on (X, Σ), and any non-decreasing function $w : [0,1] \to [0,1]$ verifying $w(0) = 0$, the set function $\mu : \Sigma \to [0,1]$ given by $\mu(E) \triangleq (w \circ \mathbb{P})(E)$ is a normalised capacity. If, in addition, $w(1) = 1$, then μ is normalised. Lastly, when w is continuous, it follows from the continuity from below and from above of the probability measure $\mathbb{P}$ that μ is also continuous, both from below and from above.*

Due to the possible non-additivity of a capacity μ, the Lebesgue integral with respect to μ may not be well-defined (see Wang and Klir [63, p. 224]). So, instead, we have the following.

Definition C.4 (Choquet integral). Let μ be a capacity with $\mu(X) < +\infty$, and let $f : X \to [0, +\infty)$ be a measurable function. The *Choquet integral* of f with respect to μ is defined as

$$\oint_X f \, d\mu \triangleq \int_0^{+\infty} \mu\{x \in X\colon\, f(x) > y\}\, dy, \tag{C.2}$$

where the integral on the right-hand side of the above equation is the Lebesgue integral with respect to the Lebesgue measure (possibly infinite).

Remark C.5. *(i)* Note that, since f is measurable, the set $\{x \in X\colon\, f(x) > y\}$ belongs to Σ for every $y \geq 0$, and so $\mu\{x \in X\colon\, f(x) > y\}$ makes sense. In addition, the monotonicity of μ implies that the function $g : [0, +\infty) \to [0, +\infty)$ given by $g(y) \triangleq \mu\{x \in X\colon\, f(x) > y\}$ is non-increasing, thus Borel measurable.

(ii) In the particular case where μ is actually a measure, it is straightforward by Tonelli's theorem that the Choquet integral coincides with the usual Lebesgue integral, that is, $\oint_X f\, d\mu = \int_X f\, d\mu$. For this reason, the Choquet integral can be regarded as a generalisation of the Lebesgue integral. ◇

Given that the capacity μ is not necessarily additive, it may happen that the Choquet integral is non-linear (as shown in Example 11.2 of Wang and Klir [63]). However, it still satisfies some important properties.

Proposition C.6. *Suppose μ is a capacity with $\mu(X) < +\infty$. Given any non-negative measurable functions f and g, the following hold,*

(i) $\oint_X f\, d\mu \geq 0$ *(*positivity*),*

(ii) *if $f \leq g$, then $\oint_X f\, d\mu \leq \oint_X g\, d\mu$ (*monotonicity*),*

(iii) *if c is a real number so that $f + c \geq 0$, then $\oint_X (f + c)\, d\mu = \oint_X f\, d\mu + c\,\mu(X)$ (*translatability*),*

(iv) *if $\mu\{x \in X\colon f(x) \neq c\} = 0$ for some real number $c \geq 0$, then $\oint_X f\, d\mu \leq c\,\mu(X)$ (with equality if and only if $\mu\{x \in X\colon f(x) > y\} = \mu(X)$ for a.e. $y \in [0, c)$),*

(v) *if $\mu\{x \in X\colon f(x) > 0\} = 0$, then $\oint_X f\, d\mu = 0$,*

(vi) *if $\oint_X f\, d\mu = 0$ and μ is continuous from below, then $\mu\{x \in X\colon f(x) > 0\} = 0$.*

Proof. Conditions *(i)* and *(ii)* are trivial to prove (note, however, that the monotonicity property of the Choquet integral does not follow directly from its positivity, because the Choquet integral may be non-linear). Condition *(iii)* is Theorem 11.4 in Wang and Klir [63]. As for conditions *(v)* and *(vi)*, these can also be found in Wang and Klir [63, Theorem 11.3].

So it remains to show *(iv)*. Suppose $\mu\{x \in X\colon f(x) \neq c\} = 0$ for some $c \geq 0$. Thus, for every $y \geq c$, we have by the monotonicity of μ that $\mu\{x \in X\colon f(x) > y\} \leq \mu\{x \in X\colon f(x) \neq c\} = 0$, and so

$$\oint_X f\, d\mu = \int_0^c \mu\{x \in X\colon f(x) > y\}\; dy \leq \int_0^c \mu(X)\; dy = c\,\mu(X)\,,$$

where the inequality is due to the fact that μ is monotone. Now consider $y \in [0, c)$. It follows again from monotonicity that $\mu\{x \in X\colon f(x) \leq y\} \leq \mu\{x \in X\colon f(x) \neq c\} = 0$, but because μ is possibly non-additive, we cannot conclude that $\mu\{x \in X\colon f(x) > y\} = \mu(X)$. If, however, we take it as an hypothesis, then the claimed equality follows. ☐

Remark C.7. When the capacity μ is as in Example C.3 with w strictly increasing, it is clear that $\mu(E) = 0$ if and only if E is a $\mathbb{P}$-null set. Therefore, for any non-negative measurable function f, we have in particular that $\mu\{x \in X\colon f(x) \neq c\} = 0$ for some real number $c \geq 0$ if and only if $f = c$ a.s. (in which case $\mu\{x \in X\colon f(x) > y\} = 1$ for all $y \in [0, c)$). ◇

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