The Deformation Theory of a Birationally Commutative Surface of Gelfand-Kirillov Dimension 4

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Doctoral Dissertation

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise. This work has not been submitted for any other degree or professional qualification.

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Lay Abstract

In the pure mathematical field of noncommutative algebra we are interested in understanding systems of numbers where the rules that are true for the counting numbers fail to hold. In particular, we examine abstract number systems where the order in which you multiply can effect the final result. For example, although in the counting numbers $3 \times 6 = 6 \times 3$, there are more exotic mathematical settings in which this equation fails to hold. We call these exotic number systems *rings*. One of the main aims of noncommutative algebra is to classify all of the possible rings.

Unfortunately, this goal is currently unattainable. Indeed, this question of classification is so hard that we do not even know what kinds of rings *could* possibly exist, let alone how to categorise them. For this reason generating new examples of rings allows us to test our preconceptions of what must or must not be true about them.

One method for generating new examples of rings is to 'deform' rings that are already well understood. For example, the ring of polynomials is made up of elements like $xy + 1$ and $x^2 + 1$. In this ring, $xy = yx$ is a rule that always holds. One can ask however what happens if instead the rule was $xy = 2yx$. Amazingly, this simple change to the rule has connections as far afield as quantum mechanics in physics.

In this thesis, we approach a well understood ring and deform it using a method similar to the above. In our case we adopt a recipe for using symmetries of objects like the sphere to determine deformations of this ring. In order to do so we develop new ideas in one large class of rings of interest across mathematics. We then describe how to apply these ideas in the specific case of this well understood ring.

Abstract

Let $\mathbb{K}$ be the field of complex numbers. In this thesis we construct new examples of noncommutative surfaces of GK-dimension 4 using the language of formal and infinitesimal deformations as introduced by Gerstenhaber. Our approach is to find families of deformations of a certain well known GK-dimension 4 birationally commutative surface defined by Zhang and Smith in unpublished work cited in [YZ06], which we call A.

Let B_* and K_* be respectively the bar and Koszul complexes of a PBW algebra $C = \frac{\mathbb{K}\langle V\rangle}{(R)}$. We construct a graph whose vertices are elements of the free algebra $\mathbb{K}\langle V\rangle$ and edges are relations in R. We define a map $m_2 : B_2 \to K_2$ that extends to a chain map $m_* : B_* \to K_*$. This map allows the Gerstenhaber bracket structure to be transferred from the bar complex to the Koszul complex. In particular, m_2 provides a mechanism for algorithmically determining the set of infinitesimal deformations with vanishing primary obstruction.

Using the computer algebra package 'Sage' [Dev15] and a Python package developed by the author [Cam], we calculate the degree 2 component of the second Hochschild cohomology of A. Furthermore, using the map m_2 we describe the variety $U \subseteq \mathrm{HH}^2_2(A)$ of infinitesimal deformations with vanishing primary obstruction. We further show that U decomposes as a union of 3 irreducible subvarieties V_g, V_q and V_u.

More generally, let C be a Koszul algebra with relations R, and let E be a localisation of C at some (left and right) Ore set. Since R is homogeneous in degree two, there is an embedding $R \hookrightarrow C \otimes C$ and in the following we identify R with its (nonzero) image under this map. We construct an injective linear map $\tilde{\Lambda} : \mathrm{HH}^2(C) \to \mathrm{HH}^2(E)$ and prove that if $f \in \mathrm{HH}^2(E)$ satisfies $f(R) \subseteq C$ then $f \in \mathrm{Im}(\tilde{\Lambda})$. In this way we describe a relationship between infinitesimal deformations of C with those of E.

Rogalski and Sierra [RS12] have previously examined a family of deformations of A arising from automorphism of the surface $\mathbb{P}^1 \times \mathbb{P}^1$. By applying our understanding of the map $\tilde{\Lambda}$ we show that these deformations correspond to the variety of infinitesimal deformations V_g. Furthermore, we show that deformations defined similarly by automorphisms of other minimal rational surfaces also correspond to infinitesimal deformations lying in V_g.

We introduce a new family of deformations of A, which we call A_q. We show that elements of this family have families of deformations arising from certain quantum analogues of geometric automorphisms of minimal rational surfaces, as defined by Alev and Dumas [AD95]. Furthermore, we show that after taking the semi-classical limit $q \to 1$ we obtain a family of deformations of A whose infinitesimal deformation lies in V_q.

Finally, we apply a heuristic search method in the space of Hochschild 2-cocycles of A. This search yields another new family of deformations of A. We show that elements of this family are non-noetherian PBW noncommutative surfaces with GK-dimension 4. We further show that elements of this family can have as function skew field the division ring of the quantum plane $\mathbb{K}_q(u, v)$, the division ring of the first Weyl algebra $D_1(\mathbb{K})$ or the commutative field $\mathbb{K}(u, v)$.

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Contents

Chapter 1

Introduction

In this thesis we study the infinitesimal deformation theory of certain noncommutative algebras, with a view to applying this theory to find novel families of GK-dimension 4 noncommutative surfaces. We develop algorithmic tools to approach the problem before applying these to one specific birationally commutative surface. As a result we describe a new family of noncommutative surfaces, elements of which have either the q-division ring or the division ring of the first Weyl algebra as their function skew fields.

1.1 Motivation

Throughout we will assume $\mathbb{K}$ is the field of complex numbers, although outside of Chapter 6 all of the results hold over any algebraically closed field of characteristic 0. One of the aims of noncommutative projective geometry it to classify so-called noncommutative surfaces. In this thesis, we mean by this finitely graded $\mathbb{K}$-algebras with function skew fields that are division rings of 'transcendence degree' 2 over $\mathbb{K}$ (see Section 2.2). This classification problem is open and very difficult. A possible first step is to find all the possible division rings of transcendence degree 2 that occur as the function skew field for noncommutative surfaces.

This approach is known as the birational classification of noncommutative projective surfaces, and is also very much an open problem. In [Art97], Artin made the bold conjecture that these division rings fall into the following four classes.

1. a field of transcendence degree 2
2. a division ring finite-dimensional over a central field of transcendence degree 2
3. the full quotient division ring of an Ore extension $K[x; \sigma, \delta]$, where K is a field of transcendence degree 1
4. the function skew field of a Sklyanin algebra. (We will not define Sklyanin algebras here as they are not relevant for this thesis.)

This conjecture remains unproven, but substantial progress has been made regarding algebras within each class.

Amongst the strongest of these results are those concerning birationally commutative surfaces. These are the noncommutative surfaces whose function skew field is commutative. In [Rog09], Rogalski showed that if C is a birationally commutative surface with finite GK-dimension, then its GK-dimension must be 3, 4 or 5. The birationally commutative surfaces in GK-dimension 3 are completely classified, whilst those in GK-dimension 5 are very well understood.

Relatively little is known about birationally commutative surfaces with GK-dimension 4. For example, it was incorrectly conjectured by Rogalski and Stafford [RS09] that these could never be noetherian. This was shown to be false by Rogalski and Sierra [RS12]. Part of the problem is that we simply do not know of many examples of GK-dimension 4 noncommutative surfaces. This paucity of examples is the main motivation behind the research in this thesis. The approach is inspired by work of Rogalski and Sierra [RS12] and can be be loosely summarised as 'deform what you know'. We recall an algebra central to [RS12] which is the main object of study in this thesis.

Definition 1.1.1. We define the algebra

$$A = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4 \rangle}{(R)}$$

where R is the set comprising six relations

$$R = \left\{ \begin{array}{lll} r_1 := x_3x_1 - x_1x_3, & r_2 := x_4x_2 - x_2x_4, & r_3 := x_4x_1 - x_2x_3 \\ r_4 := x_1x_2 - x_2x_3, & r_5 := x_3x_2 - x_1x_4, & r_6 := x_4x_3 - x_1x_4 \end{array} \right\}.$$

This algebra was shown by Zhang and Smith [YZ06] to be a birationally commutative surface of GK-dimension 4 that is neither left nor right noetherian.

Rogalski and Sierra found that A has a family of deformations whose generic element is noetherian, disproving a conjecture of Rogalski and Stafford. We take this result as a strong hint that the algebra A is interesting, and that it has families of deformations with surprising properties. With that in mind, our aim is to deform the algebra A to find new families of GK-dimension 4 noncommutative surfaces.

1.2 Summary of Approach

We provide here a brief narrative of the structure of the research in this thesis. We assume for this section only that the reader is comfortable with all definitions from Chapter 2. The main aim of the research was to find families of deformations of the algebra A (see Definition 1.1.1). This algebra is therefore of central importance to the entire document.

The approach we take is to apply the theory of formal deformations, first introduced by Murray Gerstenhaber in the middle of the last century. Informally, this theory works on the intuition of having a 'moduli space' of algebras and finding families of algebras that lie on curves passing through a chosen algebra of interest. More formally, this moduli space often does not exist, and one must work over the power series ring $\mathbb{K}[[s]]$. However, the intuition is still useful to keep in mind.

The relevant notion for us is that of infinitesimal deformations. These can be thought of as tangent vectors to the space of algebras at A, and provide a sense of what 'directions' one can deform an algebra in. In his foundational papers [Ger63, Ger64], Gerstenhaber showed that the space of isomorphism classes of infinitesimal deformations is parametrised by the second Hochschild cohomology group $\mathrm{HH}^2(A)$. For this reason, we start the thesis by calculating the second Hochschild cohomology of A. Since we are interested in graded deformations, we restrict our attention to the degree two component of the cohomology space.

There is a slight wrinkle in the theory of infinitesimal deformations in that there exist infinitesimal deformations that do not arise as tangent vectors to any formal deformations. This fact is measured by the obstruction theory of the algebra. Obstructions are also measured by Hochschild cohomology, in this case by the third Hochschild cohomology group. In particular, the primary obstruction to the so-called integration problem of finding formal deformations with a given infinitesimal deformation as tangent is a cohomology class in HH^3 (see Section 2.3 for a formal statement of this).

In general, determining the set U of infinitesimal deformations with vanishing primary obstruction is a difficult process. In the case of A there is a useful property that we can leverage; Sierra and Rogalski established in [RS12] that A is PBW. The relevance of this is that PBW algebras come equipped with a particularly nice locally finite resolution called the Koszul complex. Since cohomology theories are independent of the particular resolution used to define them, this allows us to reduce the general problem considerably.

There are already known families of deformations of A discovered by Rogalski and Sierra [RS12]. In order to avoid rediscovering these families we follow a four step process:

1. Calculate $\mathrm{HH}^2_2(A)$.

2. Find the set in $\mathrm{HH}^2_2(A)$ of cohomology classes with vanishing primary obstruction.

3. Establish which tangent directions in $\mathrm{HH}^2_2(A)$ correspond to families of deformations studied by Rogalski and Sierra.

4. 'Follow' the other tangent directions to discover new families of deformations of A.

In terms of the thesis: step 1 is the content of Chapter 3, step 2 is the content of Chapters 4 and 5, step 3 is the content of Chapter 6 and step 4 is the content of Chapters 7 and 8.

Steps 1 and 2 are mostly computational. However, Chapter 4 details the techniques used to reduce the generally difficult calculations of primary obstructions to a problem in finite dimensional linear algebra. This work relies heavily on the work of Bergman in his famous Diamond Lemma [Ber78].

We establish with these calculations that the variety of infinitesimal deformations of A with vanishing primary obstructions decomposes as a union of three irreducible subvarieties which we call V_g, V_q and V_u. We now explain the names of these varieties.

The following paragraph should be read as an informal discussion for intuition purposes. Consider $\mathrm{Q}_{\mathrm{gr}}(A) = \mathbb{K}(u,v)[t,t^{-1};\sigma]$. There are two ways one might naively attempt to deform $\mathrm{Q}_{\mathrm{gr}}(A)$:

(a) 'Deform' σ by composing it with some parametrised family of automorphisms of $\mathbb{K}(u,v)$. We refer to these as *geometric* deformations.

(b) 'Deform' $\mathbb{K}(u,v)$ to some division ring D so that σ still defines an automorphism of D. We call these *quantum* deformations.

Since there is a homomorphism of algebras $A \hookrightarrow \mathrm{Q}_{\mathrm{gr}}(A)$, one may hope that the Hochschild cohomology (and so the infinitesimal deformations) of A and $\mathrm{Q}_{\mathrm{gr}}(A)$ may be related to one another. Since Hochschild cohomology is not functorial, this is not as simple as one might at first expect. However, we establish in Section 6.2 that in this case a comparison can be made because the embedding is in particular a localisation.

Applying this work, step 3 of our overview corresponds to taking tangent vectors in Lie algebras of automorphism groups of minimal rational surfaces and deforming A using these. We find that these deformations, which are infinitesimals of those studied by Rogalski and Sierra, correspond precisely to the variety V_g. Therefore the g in V_g stands for geometric, as these are deformations of type (a).

In contrast, we define in Chapter 3 a family of deformations of A whose function skew field is the q-division ring. This family arises precisely as a quantum deformation as defined above. This family has infinitesimal lying in V_q and the q in V_q therefore stands for quantum as this family is of type (b). Unfortunately, we have been unable to find any families corresponding to vectors lying in V_u, and therefore the u in V_u stands for unknown.

Finally in step 4, we use a heuristic computer based search in order to discover a new family of deformations of A whose associated infinitesimals lie in V_q. Unlike those with infinitesimal lying in V_g, the members of this family are not birationally commutative. Furthermore, we find that algebras in this family are noncommutative surfaces of GK dimension 4 that are PBW but not noetherian.

1.3 Summary of General Results

Before discussing results specific to the algebra A, we explain the main theorems of the thesis that hold for more general algebras. Note that Chapter 2 contains the necessary background material that provides the foundation for the work in this thesis.

1.3.1 An Algorithmic Approach to Calculating Primary Obstructions for PBW Algebras

If C is a $\mathbb{K}$-algebra then there is a natural resolution of C as a C-bimodule called the *bar complex* which is defined as $B_* = C^{\otimes *+2}$ (see Definition 2.1.2 for the full definition). The bar complex is often used to define the Hochschild cohomology of C as the cohomology of $\mathrm{Hom}_{C^e}(B_*, C)$.

In [Ger63], Gerstenhaber showed that there exists a graded Lie algebra structure on B_* called the Gerstenhaber bracket. Moreover, this descends to a graded Lie algebra on Hochschild cohomology $\mathrm{HH}^*(C)$. Gerstenhaber established that if an infinitesimal deformation f integrates to a formal deformation then $[f, f] = 0 \in \mathrm{HH}^3$. For this reason we call the cohomology class of $[f, f]$ the primary obstruction of f.

The bar complex is unwieldy to say the least. Much effort has been spent trying to move the Gerstenhaber bracket to other complexes in which calculations might be more efficiently carried out. One particularly nice set of algebras is the *Koszul algebras*. For a full definition of Koszul algebras please see Definition 2.1.4, but it suffices to say here that these algebras come equipped with a resolution called the *Koszul complex* K_*.

It is known that for a Koszul algebra, there exists a chain map $m_* : B_* \to K_*$ that would allow the bracket structure to be moved across to the (often locally finite) Koszul complex K_*. An explicit map m_* has not been found, but the existence of the map m_* has been used to establish strong theorems about the deformation theory of Koszul algebras, most notably by Braverman and Gaitsgory in [BG96]. In Chapter 4, our approach is to attack the problem head on, but in the restricted setting of PBW algebras.

Let $C = \frac{\mathbb{K}\langle V\rangle}{R}$ be a PBW algebra with Koszul complex K_*. We define the Bergman graph to be a certain weighted directed acyclic graph whose vertices correspond to elements of the free algebra $\mathbb{K}\langle V\rangle$ and whose edges are relations $r \in R$. This graph builds upon the theory of reduction systems and the Diamond Lemma introduced by Bergman [Ber78].

We write $\langle V\rangle$ for the free monoid on the finite set of generators V. To every element of $\langle V\rangle$ we associate a certain set of paths in the Bergman graph called simplification paths. The main result of Chapter 4 is the following:

Theorem 1.3.1 (Theorem 4.7.1)**.** *Let $P = \{p_{x,y}\}$ be a choice of simplification path of xy for every $x, y \in \langle V\rangle$. Then P determines an explicitly constructable map $m_2 : B_2 \to$*

K_2 that extends to a chain map $m_* : B_* \to K_*$.

The details of the construction of m_2 are too involved to discuss in this summary, but are entirely algorithmic. In fact we implement this function in Python in Appendix B. The utility of m_2 is that it reduces the problem of calculating the cohomology class of $[f, f]$ to an application of m_2 and finite dimensional linear algebra. Since this calculation allows us to verify when primary obstructions of infinitesimal deformations vanish, this map makes the calculations in the rest of the thesis possible and amenable to computer calculation.

1.3.2 Relating Infinitesimal Deformations of an Algebra to those of a Localisation

Let C be a Koszul domain with relations R. If S is a (left and right) Ore set in C then we may consider the localisation $E := C_S$ and ask what relation the deformation theory of C has to that of E. We pay particular attention to the infinitesimal deformations of C and E.

Unlike Hochschild homology, Hochschild cohomology is not functorial. For this reason, it is not enough that we have an algebra map $C \hookrightarrow E$ to deduce that there exists a corresponding mapping of cohomology spaces $\mathrm{HH}^2(C) \to \mathrm{HH}^2(E)$. However, in Chapter 6 we establish the existence of a map $\tilde{\Lambda} : \mathrm{HH}^2(C) \to \mathrm{HH}^2(E)$ using the assumption that E is a localisation of C. Therefore a more formal statement of the question we address is how to determine which $f \in \mathrm{HH}^2(E)$ lie in $\mathrm{Im}\,(\tilde{\Lambda})$.

For a Hochschild cocycle f we write $[f]$ for the Hochschild cohomology class of f. An infinitesimal deformation of E is determined by a cocycle $f \in \mathrm{Hom}(E^{\otimes 4}, E)$. Under the canonical localisation map $C \hookrightarrow E$ we can consider the restriction $f|_{1 \otimes C^{\otimes 2} \otimes 1}$. It follows from the definition of the bar complex that if $f(1 \otimes C^{\otimes 2} \otimes 1) \subseteq C$ then $[f] \in \mathrm{Im}(\tilde{\Lambda})$. In this way, we have a sufficient condition for $f \in \mathrm{HH}^2(E)$ to lie in $\mathrm{Im}(\tilde{\Lambda})$.

However, since f is nothing more than a linear map on an infinite dimensional space, this condition does not provide a particularly useful test in practice. We address this by utilising the Koszul complex and establish the following result which provides an efficiently computable condition in the setting of Koszul algebras.

Theorem 1.3.2 (Theorem 6.2.9)**.** *If $f \in \mathrm{Hom}(E^{\otimes 4}, E)$ is a cocycle and $f(1 \otimes R \otimes 1) \subseteq C$ then $[f] \in \mathrm{Im}(\tilde{\Lambda})$.*

1.4 The Deformation Theory of A

We now discuss results that are specific to the algebra A and are built upon the general results of the preceding section.

1.4.1 The Second Hochschild Cohomology of A and Obstructions

In this section we elucidate the results that provide a road map for finding new deformations of A. The first of these concerns the second Hochschild cohomology. Since we are interested in families of deformations that are quadratic algebras, we only concern ourselves with the degree 2 piece of $\mathrm{HH}^2(A)$. The calculation of this space is the content of Chapter 3.

Theorem 1.4.1 (Theorem 3.3.1)**.** *The vector space $\mathrm{HH}^2_2(A)$ is 8-dimensional. Furthermore, all infinitesimal deformations of A are defined by generators and relations over the dual numbers $S = \frac{\mathbb{K}[\epsilon]}{(\epsilon^2)}$ as:*

$$\frac{S\langle x_1, x_2, x_3, x_4\rangle}{(R)}$$

where R is the set of relations

$$R = \left\{ \begin{array}{c} r_1 := x_3x_1 - (1+a\epsilon)x_1x_3 - b\epsilon x_3^2 - c\epsilon x_1^2, \\ r_2 := x_4x_2 - (1+d\epsilon)x_2x_4 - e\epsilon x_4^2 - f\epsilon x_2^2, \\ r_3 := x_4x_1 - x_2x_3 - b\epsilon x_1x_4 \\ r_4 := x_1x_2 - (1+a\epsilon)x_2x_3 - c\epsilon x_2x_1 - g\epsilon x_1^2 + h\epsilon x_2^2, \\ r_5 := x_3x_2 - (1+a\epsilon+d\epsilon)x_1x_4 - \epsilon(c+f)x_2x_3 - e\epsilon x_3x_4, \\ r_6 := x_4x_3 - x_1x_4 - b\epsilon x_3x_4 + g\epsilon x_1x_3 - h\epsilon x_2x_4 \end{array} \right\}$$

for constants $a, \ldots, h \in \mathbb{K}$.

Our proof of this theorem is carried out using ‘Sage’ [Dev15] and a noncommutative algebra software package written by the author called ‘Polygnome’ [Cam].

Following Theorem 1.4.1 and utilising the map m_2 described in Theorem 1.3.1, in Chapter 5 we establish which elements of $\mathrm{HH}^2_2(A)$ have vanishing primary obstruction. Let U be the set of $f \in \mathrm{HH}^2_2(A)$ such that $[f, f] = 0 \in \mathrm{HH}^3$.

Theorem 1.4.2 (Theorem 5.3.2)**.** *The variety $U \subseteq \mathrm{HH}^2_2(A)$ decomposes as a union of three irreducible subvarieties: V_g, V_q and V_u.*

The three varieties V_g, V_q and V_u form the foundation of our approach to finding deformations of A. We know that these varieties give us some information about what families of deformations can possibly exist. We turn our attention first to the families of deformations that we already know about, i.e. those discussed by Rogalski and Sierra [RS12].

1.4.2 Families of Deformations Arising from Automorphisms of Surfaces

The families of deformations discovered by Rogalski and Sierra are constructed by deforming $Q_{gr}(A)$, the graded quotient ring of A. By work of Yekutieli and Zhang [YZ06], $Q_{gr}(A)$ is isomorphic to $\mathbb{K}(u,v)[t,t^{-1};\sigma]$, where σ is a certain automorphism of $\mathbb{K}(u,v)$. If we are to find new families of deformations, we certainly want to avoid rediscovering those families that are already well understood. For this reason, we establish which elements of HH^2_2 appear as infinitesimals of these previously studied families.

Rogalski and Sierra restricted their attention to deformations of $Q_{gr}(A)$ of the form

$$\mathbb{K}(u,v)[t,t^{-1};\sigma\circ\tau],$$

where τ is the pull back of some automorphism of the surface $\mathbb{P}^1\times\mathbb{P}^1$. In Chapter 6 we examine the more general case of τ being an automorphism of any minimal rational surface, in the hopes of finding families with distinct infinitesimals.

Since $Q_{gr}(A)$ is a localisation of A, and A is a Koszul domain, we may apply Theorem 1.3.2 in order to determine which infinitesimals are tangent to these families. The combined result of this work is the following.

Theorem 1.4.3 (Theorems 6.4.1, 6.5.1 and 6.6.2)**.** *The deformations of A arising from automorphisms of minimal rational surfaces correspond to the space of infinitesimal deformations V_g. Furthermore, the set of deformations studied by Rogalski and Sierra comprises all of V_g.*

This result is a signal that in order to find new families of deformations of A, one must concentrate on those families whose infinitesimals lie in V_q and V_u. The rest of the results concern such deformations.

1.4.3 The Family A_q and its Infinitesimal Deformations

The archetypal example in deformation theory is the quantum plane $\mathbb{K}_q[u,v]$. This is the algebra

$$\mathbb{K}_q[u,v] := \frac{\mathbb{K}\langle u,v\rangle}{(vu-quv)} \text{ where } q\in\mathbb{K}^*.$$

The quantum plane specialises to the polynomial ring $\mathbb{K}[u,v]$ at the semi-classical limit $q\to 1$.

A is isomorphic to a subalgebra of $Q_{gr}(A)=\mathbb{K}(u,v)[t,t^{-1};\sigma]$ generated by a set $E=\{t,ut,vt,uvt\}$, where

$$\sigma(u)=uv \text{ and } \sigma(v)=v. \tag{1.1}$$

Let $\mathbb{K}_q(u,v)$ be the full division ring of $\mathbb{K}_q[u,v]$. The equations (1.1) also define an automorphism of $\mathbb{K}_q(u,v)$. Therefore we may define a family of algebras generated by $E_q = \{t, ut, vt, uvt\}$ in the graded division ring

$$\mathbb{K}_q(u,v)[t, t^{-1}; \sigma].$$

We call this family of algebras A_q, and prove that the family A_q is a family that deforms A in Corollary 3.4.2.

Using similar methods to those of Theorems 1.4.1 and 1.4.2, in Chapters 3 and 5 we find the following theorem.

Theorem 1.4.4 (Theorem 3.4.3 and Proposition 5.4.1)**.**

1. $\mathrm{HH}^2_2(A_q)$ *is a four dimensional space.*
2. *All infinitesimal deformations of* A_q *have vanishing primary obstruction.*

We use A_q to generate more families of A by mimicking the work of Rogalski and Sierra. Since $\mathbb{K}_q(u,v)$ is noncommutative, it is not the function field of any projective surface. However, there is work of Alev and Dumas [AD95] that describes subgroups of $\mathbb{K}_q(u,v)$ that are quantum analogues to certain automorphism groups of projective surfaces. We define families of deformations of A_q by taking appropriate subalgebras of

$$\mathbb{K}_{q'}(u,v)[t, t^{-1}; \sigma \circ \tau]$$

for τ an automorphism arising from these 'quantum' geometric automorphism groups. Unlike in the case of A, these families have infinitesimals that comprise all of $\mathrm{HH}^2_2(A_q)$.

Theorem 1.4.5 (Theorem 7.3.1)**.** *For every isomorphism class of infinitesimal deformations L of A_q there exists a family of deformations of $Q_{gr}(A_q)$ such that the associated infinitesimal F_1 satisfies:*

$$[F_1|_{R_q}] = L.$$

Once these families are described, we take the semi-classical limit $q \to 1$ and find that these families provide new families of deformations of the algebra A.

Proposition 1.4.6 (Proposition 7.3.2)**.** *The semi-classical limits of the families of deformations of A_q arising from quantum geometric automorphisms correspond to a 2 dimensional subspace of infinitesimal deformations of A lying in V_q.*

1.4.4 A Family of Deformations of A with the PBW Property

The final result in the thesis is that we define a new family of deformations of A all of whose elements satisfy the PBW property. In order to find this family, we carried out a heuristic search of the space V_q for cocycles that would define such a family. The details of this search are beyond the scope of this summary. The main result of Chapter 8 is the following.

Theorem 1.4.7 (Theorem 8.2.1, Corollary 8.2.8 and Corollary 8.2.3)**.** *Let*

$$A(a,c,d,f) := \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4\rangle}{(R_{a,c,d,f})}$$

where $a, c, d, f \in \mathbb{K}$ and $R_{a,c,d,f}$ is the set of relations

$$R_{a,c,d,f} = \left\{ \begin{array}{ll} r_1 := x_3x_1 - (1+a)x_1x_3 - cx_1^2, & r_2 := x_4x_2 - (1+d)x_2x_4 - fx_2^2, \\ r_3 := x_4x_1 - (1+d)x_2x_3 - fx_2x_1, & r_4 := x_1x_2 - x_2x_3, \\ r_5 := x_3x_2 - (1+a)x_1x_4 - cx_2x_3, & r_6 := x_4x_3 - (1+a)x_1x_4 - cx_2x_3 \end{array} \right\}.$$

If $af - cd = 0$ and $a \neq -1 \neq d$ then $A(a,c,d,f)$ is a non-noetherian GK-dimension 4 domain that is PBW with respect to the lexicographic ordering induced by $x_2 < x_1 < x_3 < x_4$.

In particular we establish that $A(a,c,d,f)$ is a flat family of algebras deforming A.

Theorem 1.4.8 (Corollary 8.2.2)**.** *$A(a,c,d,f)$ is a flat family of algebras deforming A over the ring*

$$\frac{\mathbb{K}[a,c,d,f,\frac{1}{1+a},\frac{1}{1+d}]}{(af-cd)}.$$

We analyse this family further by describing the function skew field, and thereby classifying these algebras up to birational equivalence. In particular, we demonstrate that $A(a,c,d,f)$ is a GK-dimension 4 noncommutative surface.

Theorem 1.4.9 (Corollary 8.2.7 and Proposition 8.2.9)**.** *The function skew field of $A(a,c,d,f)$ is isomorphic to*

1. *$\mathbb{K}_q(u,v)$ where $q = \frac{1+d}{1+a}$ if $a \neq d$.*

2. *$D_1(\mathbb{K})$, the division ring of the first Weyl algebra, if $a = d = 0$ and $f \neq c$.*

3. *$\mathbb{K}(u,v)$ if $a = d$ and $c = f$.*

The methods used to find this family are interesting in their own right, but fail to yield results for infinitesimal deformations lying in the variety V_u. However, since $V_q \cap V_u$ and $V_u \cap V_g$ are large subvarieties in V_u, we have already seen several families of algebras whose infinitesimal lies in V_u. In conclusion, we have seen families whose infinitesimals comprise all of V_g, a substantial portion of V_q and large subvarieties of V_u.

1.5 Further Work

1.5.1 The map m_*

In the setting of PBW algebras we may apply basic homological algebra and deduce that there exist maps m_n for all $n \in \mathbb{N}$ that extend m_2 to the entire bar resolution. Although we have not discussed this explicitly, our work in Chapter 4 implies an algorithm for calculating a continuation to B_3 of m_2 by taking 'paths' of 2-cells in the Bergman graph. It is a firm belief of the author that the maps m_n may be defined algorithmically by taking choices of sequences of n-cells in the Bergman graph that have as boundary $(n-1)$-cells that are specified by the choice of m_{n-1}.

The more general problem of finding a map $\phi_* : B_* \to K_*$ for Koszul algebras is interesting and known to be difficult. The map m_2 provides an obvious starting point for further study. The combinatorial theory underlying the Bergman Diamond Lemma is well known in computer science, where an almost identical theorem known as the Church-Rosser theorem is fundamental in the theory of λ-calculus. In the λ-calculus theory there are situations where the diamond lemma only holds for certain subsets of the 'terms' under study. This failure reflects somewhat the difference between a general Koszul algebra and a PBW algebra and there may be already developed theory that one may apply in the setting of noncommutative algebras.

1.5.2 Deformations of A

There are immediate questions for investigation in the family $A(a, c, d, f)$. None of the algebras in this family are noetherian, in contrast to those examined by Rogalski and Sierra. It would be extremely surprising if they were AS-Gorenstein as none of those in the Rogalski-Sierra family had this property. These algebras are an obvious launching point for further work on GK-dimension 4 noncommutative surfaces.

Turning to A, we have not discussed the formal deformations of A for the most part as our aims have been realising flat families over non-local rings. However, we have evidence that many of the infinitesimal deformations with vanishing primary obstruction integrate to formal deformations. An interesting property for algebras to have is that infinitesimal deformations integrate if and only if they have vanishing primary obstruction. This would be worth investigating in A, and the classification of the infinitesimal deformations in Chapter 5 would provide a useful starting point for such work.

As for other families of deformations over non-local rings, we conjecture that there exist families of algebras that deform A with associated infinitesimals that cover all of V_q. We have discussed in Chapter 8 that answering this question will require new techniques as the resulting algebra cannot be PBW with the specified basis that we have studied.

Finally, are there any flat families of algebras whose associated infinitesimal lies in V_u? We have been unable to answer this question, except in the case of the overlaps with V_q and V_g. The author believes the answer to be positive, but this is not based on anything but optimism. The main conjecture we make about any such algebras is that they will have function skew field that is isomorphic $\mathbb{K}_q(u, v)$ for some q. This reasoning behind this is that the generic element of all the families discussed in the thesis has such a skew field. These algebras are worth investigation not least because examples of transcendence degree 2 division rings 'in the wild' are of interest as testing grounds for the theory of noncommutative surfaces and Artin's conjecture.

Chapter 2

Background

In this chapter we present the background material, theory and theorems upon which the rest of the thesis is based. In particular we will provide an overview of the fundamentals of algebraic deformation theory, the theory of Koszul and PBW algebras, and introduce the algebra A that will play a key role in the thesis.

Notation 2.0.1. We adopt the global convention that $\mathbb{K}$ is a field and unadorned tensor products are to be considered as tensor products over $\mathbb{K}$. Furthermore, we will sometimes use the symbol $|$ for the tensor product $\otimes_{\mathbb{K}}$.

2.1 Koszul Algebras

First introduced by Priddy [Pri70], Koszul algebras have proved fertile ground for research. The motivation in studying them has been that whilst they are a large class of quadratic algebras, their behaviour is considerably less wild than the general case. The definitive text on Koszul algebras is by Polishchuk and Positelskei [PP05], although we also take definitions from [BG96] as they are more suited to the work in this thesis.

Definition 2.1.1. For a $\mathbb{K}$-algebra C we call the $\mathbb{K}$-algebra $C^e := C \otimes_{\mathbb{K}} C^{\mathrm{op}}$ the *enveloping algebra* of C.

The useful property of enveloping algebras is that the category of C-bimodules is equivalent to the category of right C^e-modules.

Before considering Koszul algebras themselves, we define the bar resolution which plays a central role in deformation theory and will appear throughout the thesis.

Definition 2.1.2. For a $\mathbb{K}$-algebra C, let $B_n(C)$ be the C-bimodule $C^{\otimes n+2}$, which we often just write as B_n if the context is clear. We define the *bar* (or *standard*) *resolution* of C to be the complex of right C^e-modules

$$B_* \to C,$$

where the boundary map $b_n : B_n \to B_{n-1}$ is given by

$$\begin{aligned} b_n(c_0 \otimes c_1 \otimes \ldots \otimes c_n \otimes c_{n+1}) = & \; c_0c_1 \otimes c_2 \otimes \ldots \otimes c_{n+1} \\ & + \sum_i (-1)^i c_0 \otimes \ldots \otimes c_ic_{i+1} \otimes \ldots \otimes c_{n+1} \\ & + (-1)^{n+1} c_0 \otimes c_1 \otimes \ldots \otimes c_nc_{n+1}, \end{aligned}$$

and the final map $C \otimes C \to C$ is the multiplication map.

Note that the above definition is completely general and requires no assumptions about the structure of C. From this point forwards however, we will be dealing only with quadratic algebras.

Notation 2.1.3. If V is a finite set then we denote by $T(V)$ the tensor algebra on $\mathrm{sp}_{\mathbb{K}}(V)$. Note that $T(V) = \bigoplus_i T(V)_i$ is a graded algebra with each generator lying in $T(V)_1$.

Definition 2.1.4. For a set of relations $R \subseteq \mathrm{sp}_{\mathbb{K}}(V) \otimes \mathrm{sp}_{\mathbb{K}}(V)$ we consider an algebra

$$C := \frac{\mathbb{K}\langle V \rangle}{(R)}.$$

Since $R \subseteq T(V)_2$, C is $\mathbb{N}$-graded and we write this grading as

$$C = \bigoplus_{i \in \mathbb{N}} C_i.$$

The *Koszul complex* is given by $K_n = C \otimes \overline{K}_n \otimes C$, where $\overline{K}_n$ is defined as:

$$\overline{K}_n = \begin{cases} \mathbb{K} & \text{if } n = 0 \\ C_1 & \text{if } n = 1 \\ \bigcap_{j=0}^{n-2} C_1^{\otimes j} \otimes R \otimes C_1^{\otimes i-j-2} & \text{otherwise.} \end{cases}$$

The inclusion of R into $C_1 \otimes C_1$ gives a C-bimodule embedding $i_n : K_n \hookrightarrow B_n$. The differential of the Koszul complex is induced by restriction from that of the bar complex; this is well defined since $\operatorname{Im} b_n \circ i_n \subseteq i_{n-1}(K_{n-1})$.

An algebra is *Koszul* if the Koszul complex is a resolution of C as a right C^e-module.

Remark 2.1.5. There are many equivalent definitions of a Koszul algebra. We choose a homological definition, since much of the work in the thesis is homological in nature. For a comparison of other definitions of Koszul algebras we direct the reader to [BG96, Appendix A].

2.1.1 PBW Algebras and the Diamond Lemma

A well studied sub-class of Koszul algebras, also introduced by Priddy [Pri70], are the PBW algebras. Named for Poincaré, Birkhoff and Witt, these are algebras which admit a specified basis of monomials, and first arose as the enveloping algebras of Lie algebras. Our definition is somewhat restricted as we are only interested in the quadratic case, and is that adopted in [PP05].

Definition 2.1.6. Let $V := \{v_1, \ldots, v_\gamma\}$ and $R \subseteq \mathrm{sp}_{\mathbb{K}}(V) \otimes \mathrm{sp}_{\mathbb{K}}(V)$ be a finite set of relations of the form

$$v_j v_i - \sum_{(r,s)\prec(j,i)} c_{rs} v_r v_s,$$

where $j > i$ and $\prec$ is the lexicographic ordering. The algebra $C := \frac{T(V)}{(R)}$ is a *PBW algebra* if it has a basis of all monomials of the form $v_1^{i_1} \cdots v_\gamma^{i_\gamma}$. Consider an element $p \in T(V)$. We say p is in *PBW order* (with respect to R) if it is a sum of monomials of the form $c_i v_1^{i_1} \cdots v_\gamma^{i_\gamma}$ for some constant $c_i \in \mathbb{K}^*$.

Example 2.1.7. The simplest example of a PBW algebra is the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ with the ordering on the generators given by $x_1 < x_2 < \ldots < x_n$.

Theorem 2.1.8 ([Pri70, Theorem 5.3])**.** *A PBW algebra is Koszul.*

One of the most useful properties that PBW algebras have is that they generalise the theory of Gröbner bases from commutative algebra. This fact is known as the Diamond Lemma and is often used to prove that a given algebra is PBW. First explicitly proved by Bergman [Ber78], the Diamond Lemma allows one to test the relations of an algebra for the PBW property by finding the PBW form of a finite set of monomials. Before stating the Diamond Lemma we require a few preliminaries.

In the following we adopt notation in keeping with that of Bergman's original paper. We have chosen a basis of R of the form $\{W_\sigma - f_\sigma\}$, where W_σ is a monomial and f_σ is a polynomial in PBW order.

Definition 2.1.9. We define the *reduction system* of C to be the set of pairs $\{\sigma = (W_\sigma, f_\sigma)\}$. Furthermore, we define a *reduction* $r_{B\sigma D}$ for monomials $B, D \in T(V)$ as the linear map on $T(V)$ such that for a monomial M we have:

$$r_{B\sigma D}(M) = \begin{cases} Bf_\sigma D & \text{if } M = BW_\sigma D \\ M & \text{otherwise.} \end{cases}$$

We call a composition of any number of reductions a *reduction sequence*.

If a relation is of the form

$$v_j v_i - \sum_{(r,s)\prec(j,i)} c_{rs} v_r v_s,$$

then we label the associated element of the reduction system $\sigma_{j,i}$.

The idea of the Diamond Lemma is that in order to verify that a set of relations defines a PBW algebra, one only needs to check that the PBW forms of a finite set of monomials are unique.

Definition 2.1.10. If V and R are as in Definition 2.1.6 then an *overlap ambiguity* is a degree 3 monomial of the form

$$v_j v_k v_l \text{ with } j > k > l.$$

By definition, for an overlap ambiguity w, there are two reductions one may apply to w: $r_{v_j \sigma_{k,l}}$ and $r_{\sigma_{j,k} v_l}$.

Definition 2.1.11. An overlap ambiguity $w = v_j v_k v_l$ is said to be *resolvable* if there exist two reduction sequences $r_n \cdots r_1$ and $s_m \cdots s_1$ such that $s_1 = r_{v_j \sigma_{k,l}}$, $r_1 = r_{\sigma_{j,k} v_l}$ and

$$s_m \cdots s_1(w) = r_n \cdots r_1(w).$$

We are now ready to state the Diamond Lemma. Note that the version we use is not as powerful as the original result as we are only interested in graded quadratic algebras in this thesis. For this reason we have modified the theorem slightly.

Theorem 2.1.12 ([Ber78, Theorem 1.2])**.** *If V and R are as in Definition 2.1.6, then the algebra $C = \frac{\mathbb{K}\langle V\rangle}{(R)}$ is PBW if and only if all overlap ambiguities are resolvable.*

Example 2.1.13. We show using the Diamond Lemma that the algebra A defined in Definition 1.1.1 is PBW with respect to the ordering given by $x_2 < x_1 < x_3 < x_4$. Note that this was previously established by Rogalski and Sierra [RS12, Proof of Lemma 5.7]. The overlaps of A are precisely the monomials

$$\{x_4 x_1 x_2, x_3 x_1 x_2, x_4 x_3 x_2, x_4 x_3 x_1\}.$$

We show the resolution in full for $x_4 x_1 x_2$ but only give the two sequences for the other three cases.

1.

$$\begin{aligned} r_{x_2 r_6} r_{r_2 x_3} r_{x_4 r_4}(x_4 x_1 x_2) &= r_{x_2 r_6} r_{r_2 x_3}(x_4 x_2 x_3) = r_{x_2 r_6}(x_2 x_4 x_3) = x_2 x_1 x_4 \\ &= r_{x_2 r_5}(x_2 x_3 x_2) = r_{x_2 r_5} r_{r_3 x_2}(x_4 x_1 x_2) \end{aligned}$$

2.

$$r_{x_1 r_6} r_{r_5 x_3} r_{x_3 r_4}(x_3 x_1 x_2) = x_1^2 x_4 = r_{x_1 r_5} r_{r_1 x_2}(x_3 x_1 x_2)$$

3.
$$r_{r_3x_4}r_{x_4r_5}(x_4x_3x_2) = x_2x_3x_4 = r_{r_4x_4}r_{x_1r_2}r_{r_6x_2}(x_4x_3x_2)$$

4.
$$r_{r_3x_3}r_{x_4r_1}(x_4x_3x_1) = x_2x_3^2 = r_{r_4x_3}r_{x_1r_3}r_{r_6x_1}(x_4x_3x_1)$$

Since all four overlaps are resolvable, we can apply the Diamond Lemma and conclude that A is PBW with the basis $\{x_2^i x_1^j x_3^k x_4^l\}$.

2.2 The Algebra A

The main focus of this thesis is applying deformation theory to study algebras 'close' to the algebra A as defined in Definition 1.1.1. Before discussing A itself we provide some context and definitions that will allow us to state fully the properties that A has that we are particularly interested in.

2.2.1 Preliminary Definitions from Noncommutative Projective Geometry

In the field of noncommutative projective geometry the main idea is to generalise the sheaf theoretic approach of modern algebraic geometry to noncommutative algebras. We will not discuss the field in detail, as the technical heart of the thesis is Gerstenhaber's deformation theory. However, many of the concepts are relevant. In particular, the growth and dimension of a ring will be used throughout the thesis.

Gelfand-Kirillov Dimension

There are many definitions of dimension for noncommutative rings which go some way to generalising commutative notions of dimension. The most relevant in this thesis is Gelfand-Kirillov dimension.

Definition 2.2.1. The *Gelfand-Kirillov dimension* (hereafter GK-dimension) of an algebra C is
$$\mathrm{GKdim}(C) = \sup_V \limsup_{n\to\infty}(\log_n(\dim_{\mathbb{F}}(V^n))),$$
where the supremum ranges over all finite dimensional subspaces V of C.

GK-dimension generalises Krull dimension in the sense that they agree for finitely generated commutative domains. Unfortunately, GK-dimension has some bad properties in general. For example, for any real number r larger than 2, there is an algebra with GK-dimension r [KL00, Theorem 2.9]. In the algebraic setting of this thesis the GK-dimension of an algebra is strongly related to its Hilbert series.

Definition 2.2.2. For a graded $\mathbb{K}$-algebra $C = \bigoplus_i C_i$ the *Hilbert series* of C is the power series

$$h_C(p) = \sum_i \dim(C_i)p^i.$$

In other words, $h_C(p)$ is the generating function for $\dim(C_i)$.

Lemma 2.2.3 ([Rog, Lemma 2.7])**.** *Suppose that $C = \bigoplus_i C_i$ is a graded algebra such that $C_0 = \mathbb{K}$. If the Hilbert series of C is of the form $h_C(t) = 1/p(t)$, where $p \in \mathbb{Z}[t]$, then C has finite GK-dimension if and only if all roots of p in the complex plane lie on the unit circle. Moreover, in this case, the GK-dimension is an integer, and it is equal to the multiplicity of vanishing of $p(t)$ at $t = 1$.*

Noncommutative Localisation

Localisation refers to the process of adding inverses to a ring. In the noncommutative setting there are problems with naively inverting given sets of elements. the condition required to avoid these issues is the Ore conditions.

Definition 2.2.4. A set $S \subseteq C$ is a right Ore set with respect to a subset $B \subseteq C$ if S is a multiplicatively closed set such that for any $a \in B$ and $s \in S$, there exist $b \in B$ and $r \in S$ such that the following holds:

$$ar = sb.$$

S is a right Ore set if B is the whole ring C. A left Ore set is defined in the same way mutatis mutandis.

If S is a right Ore set in a domain C then we can consider the localisation of C at S, which we define with a universal property.

Definition 2.2.5. For S a right Ore set in a domain C then the localisation of C at S is written C_S and is a ring with a homomorphism $\theta : C \hookrightarrow C_S$ satisfying the following universal property: if R is a ring and $\phi : C \to R$ is a ring homomorphism so that $\phi(s)$ is a unit for every element $s \in S$, then ϕ factors through C_S under θ.

The existence of a ring satisfying this property is the content of [MR01, Theorem 2.1.12]. Noncommutative localisation is the tool that allows us to consider birational geometry in a noncommutative setting.

Noncommutative Birational Equivalence

Noncommutative birational equivalence generalises the commutative notion of birational equivalence between two projective varieties to noncommutative graded domains.

Definition 2.2.6. For C a graded domain of finite GK-dimension we define the *graded quotient ring*

$$\mathrm{Q_{gr}}(C) := C\langle h^{-1} \mid 0 \neq h \in C \text{ is homogeneous}\rangle$$

It is not immediately obvious that such a ring is well defined, in that localisation in noncommutative rings depends on the Ore condition being satisfied. This can be verified by combining [NvO82, Theorem C.I.1.6] and [KL00, Theorem 4.12].

The utility of this construction arises from the following theorem.

Theorem 2.2.7 ([NvO82, Theorems A.I.5.8 and C.I.1.6])**.** *For C a graded domain of finite GK-dimension, there exists a division ring D and τ an automorphism of D such that*

$$Q_{gr}(C) \cong D[t, t^{-1}; \tau].$$

Definition 2.2.8. We call the division ring D in Theorem 2.2.7 the *function skew field* of C. We will often refer to this simply as the *function field* by abuse of language, although we remind the reader that this is not intended to suggest D is commutative. Two algebras are said to be *birationally equivalent* if they have isomorphic function skew fields.

Definition 2.2.9. We call a finitely generated graded $\mathbb{K}$-algebra $C = \bigoplus_i C_i$ with $C_0 = \mathbb{K}$ *finitely graded*. A finitely graded domain with a function skew field of transcendence degree 2 over the ground field $\mathbb{K}$ is called a *noncommutative projective surface* or simply a *noncommutative surface*.

Classifying noncommutative surfaces is an open and difficult problem, even up to birational equivalence (see [SVdB01]). One particularly well understood class of noncommutative surfaces are the birationally commutative surfaces.

Definition 2.2.10. If a finitely graded domain has a function field that is commutative then we call it *birationally commutative.* We will refer to noncommutative projective surfaces that are birationally commutative as birationally commutative surfaces.

2.2.2 Definition of A and Basic Properties

Although A was first defined by Yekutieli and Zhang in [YZ06, Section 7] as a subgroup of a group algebra, we prefer to follow Rogalski and Sierra [RS12] and we have defined A by a finite presentation in Definition 1.1.1.

The following result is due to Smith and Zhang, although the proof is not published, and establishes many of the basic properties of A. Note that although due to Smith and Zhang, the result was published in a paper of Yekutieli and Zhang.

Proposition 2.2.11 ([YZ06, Proposition 7.6])**.** *A is a finitely graded $\mathbb{K}$-algebra with the following properties.*

(a) A is a Koszul algebra.

(b) A is a domain with Hilbert series

$$h_A(p) = \frac{1}{(1-p)^4}.$$

(c) A is neither left nor right noetherian.

Note that we have already established Part (a) of Proposition 2.2.11 in Example 2.1.13 since PBW algebras are Koszul by Theorem 2.1.8. Of particular interest from a noncommutative geometry perspective is the following birational classification of A.

Notation 2.2.12. For a set X the symbol $\mathbb{K}\langle X\rangle$ is used to denote the subalgebra generated by X if X is a subset of a $\mathbb{K}$-algebra and the free algebra on X otherwise. It will be clear from the context which is meant.

Proposition 2.2.13 ([YZ06, Proposition 7.8 and its proof])**.** *A is a GK-dimension 4 birationally commutative surface. In particular, there exists a isomorphism:*

$$Q_{gr}(A) \cong \mathbb{K}(u,v)[t,t^{-1};\sigma]$$

where $\sigma \in \mathrm{Aut}(\mathbb{K}(u,v))$ *is defined by*

$$\sigma(u) = uv \text{ and } \sigma(v) = v.$$

Indeed, A is embedded in $\mathrm{Q_{gr}}(A)$ in the following manner:

Lemma 2.2.14 ([RS12, Lemma 5.7 (i)])**.** *Let* $E := \{t, ut, vt, uvt\}$*. Then the map* $\phi : A \to \mathbb{K}(u,v)[t,t^{-1};\sigma]$ *given by*

$$\phi(x_1) = t, \phi(x_2) = ut, \phi(x_3) = vt, \phi(x_4) = uvt$$

defines an isomorphism:

$$A \cong \mathbb{K}\langle E\rangle.$$

This particular embedding of A into $\mathrm{Q_{gr}}(A)$ is a useful tool that will appear throughout the thesis. There are several properties of A that are best understood when considering A as a subalgebra of $\mathrm{Q_{gr}}(A)$, whereas the presentation of A given in Definition 1.1.1 is often easier to make calculations with.

2.2.3 Deformations Of A

By basic theory from birational geometry [Har77, Theorem 4.4], the automorphism $\sigma \in \mathrm{Aut}(\mathbb{K}(u,v))$ defined in Proposition 2.2.13 defines a birational self-map for any rational surface. Rogalski and Sierra consider one such birational self-map in [RS12]. In the following we abuse notation and write σ for both the automorphism of $\mathbb{K}(u, v)$ and the birational map of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.2.15. Define the birational map $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$\sigma([x : y][z : w]) = [xz : yw][z : w].$$

Then choose the chart on $\mathbb{P}^1 \times \mathbb{P}^1$ given by $u = x/y$ and $v = z/w$.

It is immediate that this choice of chart on $\mathbb{P}^1 \times \mathbb{P}^1$ corresponds to the automorphism of $\mathbb{K}(u, v)$ defined in Proposition 2.2.13. The paper [RS12] concerns families of algebras that 'deform' A by certain a set of automorphisms $\{\tau\} \subseteq \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

Definition 2.2.16. If $\tau \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ then consider the set

$$E(\tau) := \{t, ut, vt, uvt\} \subseteq \mathbb{K}(u, v)[t; \sigma \circ \tau].$$

Then define $A(\tau)$ to be the algebra $\mathbb{K}\langle E(\tau)\rangle$.

Intuitively, one expects $A(\tau)$ to be an algebra with similar properties to A. In fact the main theorem of [RS12] is that whilst it is true that (for certain τ) $A(\tau)$ has GK-dimension 4, almost all of these algebras are noetherian.

Theorem 2.2.17 ([RS12, Theorem 1.6])**.** *There exists a subgroup of automorphisms in* $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ *comprised of elements* $\tau = \tau(\rho, \theta)$ *parametrised by* $\rho, \theta \in \mathbb{K}$*, such that if* ρ *and* θ *are algebraically independent over the prime subfield of* $\mathbb{K}$ *then* $A(\tau)$ *is a GK-dimension 4 noetherian finitely graded domain which is Koszul.*

This theorem was surprising since at the time no examples of noetherian birationally commutative surfaces were known to exist in GK-dimension 4. The algebras $A(\tau)$ can be considered as deformations of A in an obvious way. One goal of this thesis is to discover whether or not there are other families of algebras that deform A.

2.3 Deformation Theory

The theory of formal deformations of algebras was first introduced by Gerstenhaber [Ger64], and has had wide ranging impact in both mathematics and physics. Inspired by ideas from analytic deformation theory [KNS58, KS58] and examples from quantum mechanics [Moy49], Gerstenhaber defined families of algebras that can be thought of as 'close' to a given algebra.

There are several readable surveys of Gerstenhaber's deformation theory available. Szendroi has written a very readable introduction with applications to Calabi-Yau manifolds in [Sze99]. However, for more detailed and algebraic considerations of the theory the reader is directed to [Fox93] for an introduction and [Gia11] for a high level historical survey. Furthermore, the original papers by Gerstenhaber [Ger63, Ger64] are still extremely relevant and accessible.

2.3.1 Hochschild Cohomology

The main thrust of Gerstenhaber's foundational papers [Ger63, Ger64] is that the deformation theory of a $\mathbb{K}$-algebra is intimately related to its Hochschild cohomology. Furthermore, the Hochschild cohomology comes with a wealth of algebraic structures defined upon it and these also have relation to the deformation theory of the algebra in question. For that reason we first define Hochschild cohomology, which was introduced by Hochschild in [Hoc45].

Definition 2.3.1. The *Hochschild cohomology* of a $\mathbb{K}$-algebra C, written $\mathrm{HH}^*(C)$, is the homology of the dual complex to the bar complex (see Definition 2.1.2). That is to say that it is the cohomology of the bar complex under the contravariant functor

$$\mathrm{Hom}_{C^e}(-, C).$$

Example 2.3.2. 1. $\mathrm{HH}^0(C)$ is the centre of the algebra C. To see this, note that $f \in \mathrm{Hom}_{C^e}(C^{\otimes 2}, C)$ is determined by $f(1 \otimes 1) = c \in C$ so that $\mathrm{Hom}_{C^e}(C^{\otimes 2}, C) \cong C_{C^e}$. Then f is a Hochschild 0-cocycle if and only if for every $d \in C$

$$b^1(f)(1 \otimes d \otimes 1) = f(d \otimes 1) - f(1 \otimes d) = dc - cd = 0.$$

Therefore f is a cocycle if and only if c is central.

2. $\mathrm{HH}^1(C)$ is the quotient of the group of derivations of C by the group of inner derivations of C.

Of course, by the Comparison Theorem [Wei94, Theorem 2.2.6] the explicit dependence on the bar complex in the definition of Hochschild cohomology is illusory, in that $\mathrm{HH}^*(C) \cong \mathrm{Ext}^*_{C^e}(C, C)$. One could replace this resolution with any other projective resolution of C as a right C^e-module and the resulting cohomology would be the same. Indeed, much of the work in this thesis depends upon that fact. However, there are several algebraic structures defined on the bar complex that have significance in Gerstenhaber's theory of formal deformations, as we shall shortly see.

2.3.2 Deformations

Definition 2.3.3. Let R be a $\mathbb{K}$-algebra with an augmentation map $R \to \mathbb{K}$ so that $\mathbb{K}$ has a canonical R-module structure. For a $\mathbb{K}$-algebra C we define a *flat family* of algebras that deforms C over R to be a flat R-algebra A_R with an algebra isomorphism $A_R \otimes_R \mathbb{K} \cong A$.

We define a *formal deformation* of C to be a flat family of algebras that deforms C over $R = \mathbb{K}[[s]]$. Equivalently, a formal deformation of C is an associative $\mathbb{K}[[s]]$-algebra structure on the $\mathbb{K}[[s]]$ vector space $C_s := C\widehat{\otimes}\mathbb{K}[[s]]$ with an isomorphism of $\mathbb{K}$-algebras

$$\frac{C_s}{(s)} \cong C.$$

Intuitively, one can think of a formal deformation of an algebra as a one-parameter family of algebras such that for any choice of $q \in \mathbb{K}$ one obtains a new algebra $\frac{C_s}{(s-q)}$. Unfortunately, in general this raises issues of convergence that are best left aside. However, we can certainly write the algebra structure on $C_s = C\widehat{\otimes}\mathbb{K}[[s]]$ as a power series which proves useful.

Definition 2.3.4. A formal deformation of a $\mathbb{K}$-algebra C is defined by an associative $\mathbb{K}[[s]]$-bilinear map $F : C_s \otimes_{\mathbb{K}[[s]]} C_s \to C_s$ which is given on elements of $a, b \in C$ by

$$F(a, b) = \sum_i F_i(a, b)s^i,$$

where each F_i is a bilinear map on C. By the definition of a formal deformation we know that $F_0(a, b) = ab$, i.e. F_0 is the multiplication map of C. We call the first F_i that is nonzero for $i > 0$ the *infinitesimal* of F. By an abuse of language, we will often refer to the power series F as a formal deformation.

An infinitesimal is best thought of as a tangent vector lying in the tangent space to some moduli space of algebras. Using noncommutative differentials one can make this statement more formal and accurate [Art96, Section 9], but the intuition suffices for this thesis.

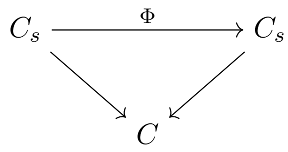
Now we repeat the definition of formal deformation, but this time over the ring $\frac{\mathbb{K}[[s]]}{(s^i)}$.

Definition 2.3.5. For any $i \in \mathbb{N}_{\geq 0}$, an ith level deformation of $\mathbb{K}$-algebra C is an associative $\frac{\mathbb{K}[[s]]}{(s^i)}$-algebra structure on $C_s^{(i)} := C \otimes \frac{\mathbb{K}[[s]]}{(s^i)}$, given by a $\frac{\mathbb{K}[[s]]}{(s^i)}$-bilinear map F', with an isomorphism

$$\frac{C_s^{(i)}}{(s)} \cong C.$$

One can think of ith level deformations of an algebra C as approximations of formal deformations for C. However, one should be aware that it is not always true that an ith level deformation arises as $\frac{C_s}{(s^i)}$ for some formal deformation structure on C_s.

Definition 2.3.6. A homomorphism of formal (respectively ith level) deformations from F to G is a $\mathbb{K}[[s]]$ (resp. $\frac{\mathbb{K}[[s]]}{(s^i)}$) bilinear map, Φ, on C_s (resp. $C_s^{(i)}$) so that the following diagram commutes:



With this definition of morphisms, formal deformations form a category $\mathcal{E}$ and ith level deformations form a category $\mathcal{E}_i$. We also naturally obtain functors $\mathcal{E} \to \mathcal{E}_i$ for all $i \in \mathbb{N}$ and $\mathcal{E}_j \to \mathcal{E}_i$ for any pair $i, j \in \mathbb{N}$ with $i < j$ arising from the canonical quotient maps

$$\mathbb{K}[[s]] \to \frac{\mathbb{K}[[s]]}{(s^i)} \text{ and } \frac{\mathbb{K}[[s]]}{(s^j)} \to \frac{\mathbb{K}[[s]]}{(s^i)}.$$

The category $\mathcal{E}_2$ is of particular interest.

Definition 2.3.7. A 2nd level deformation F' is called an *infinitesimal deformation.* Furthermore, let $F'_1 : C \otimes C \to C$ be the linear map in the expansion of F' as a series, so that for all $a, b \in C$

$$F'(a, b) = ab + F'_1(a, b)s \in C_s^{(2)}.$$

Then by a further abuse of language we refer to F'_1 as an infinitesimal deformation. Note that $F'_1 \in B^2$, the space of Hochschild 2-cochains.

The following proposition is a reformulation of the fact that a functor $\mathcal{E} \to \mathcal{E}_2$ exists.

Proposition 2.3.8 ([Ger64, Section 1])**.** *An infinitesimal of a formal deformation of C is an infinitesimal deformation of C.*

We now come to the first connection to Hoschshild cohomology. Recall that we would like to think of infinitesimals of a formal deformation as tangent vectors to some moduli space of algebras. The following fundamental theorem is a statement of the fact that this tangent space is precisely the second Hochschild cohomology space.

Proposition 2.3.9 ([Ger64, Section 1])**.** *An infinitesimal deformation is a Hochschild 2-cocycle. Furthermore, two infinitesimal deformations are isomorphic in the sense of Definition 2.3.5 if and only if they are cohomologous in Hochschild cohomology.*

2.3.3 Integrating Deformations

We turn to the question of whether something akin to a converse of Proposition 2.3.8 ever holds true. More generally, we ask when is an object in $\mathcal{E}_i$ the image of some object in $\mathcal{E}$ under the functor $\mathcal{E} \to \mathcal{E}_i$. Since

$$\mathbb{K}[[s]] \cong \varprojlim \frac{\mathbb{K}[[s]]}{(s^i)},$$

this question can be approached by answering the weaker question of when does an object in $\mathcal{E}_i$ lie in the image of some object in $\mathcal{E}_{i+1}$.

Definition 2.3.10. If $F(a,b) = \sum_j^i F_j(a,b)s^j$ is an ith level deformation of C then we say F *integrates* to an $(i+1)$st level deformation if there exists an $F_{i+1} \in \mathrm{Hom}_{\mathbb{K}}(C^{\otimes 2}, C)$ such that

$$F'(a,b) = \sum_j^{i+1} F_j(a,b)s^j$$

is an $(i+1)$st level deformation.

We will see that the answer to this question is once again related to the Hochschild cohomology of the algebra in question. However, we must first define some terms. The following proposition is simply a reformulation of the fact that a formal deformation defines an associative algebra structure. Note that in the case $i = 1$ this proposition is a restatement of Proposition 2.3.8.

Proposition 2.3.11 ([Ger64, Section 1])**.** *If $F = \sum_i F_i s^i$ is a formal deformation of C then for each $i \in \mathbb{N}$ and every $a, b, c \in C$ the following equation holds:*

$$\sum_{p+q=i} F_p(F_q(a,b),c) - F_p(a, F_q(b,c)) = 0. \qquad (\nu_i)$$

We can rewrite this formula as:

$$\sum_{\substack{p+q=i \\ p,q>0}} F_p(F_q(a,b),c) - F_p(a,F_q(b,c)) = aF_i(b,c) - F_i(ab,c) + F_i(a,bc) - F_i(a,b)c$$

$$= b^2(F_i)(a,b,c). \qquad (\nu_i')$$

For this reason, the expression on the left hand side of equation (ν_i') is of particular interest as its value controls whether or not an F_i can exist that integrates a given $(i-1)$st level deformation.

Definition 2.3.12. For any $i \in \mathbb{N}_{\geq 1}$, we call the expression

$$\sum_{\substack{p+q=i \\ p,q>0}} F_p(F_q(a,b),c) - F_p(a,F_q(b,c))$$

the *associator* of the $(i-1)$st level deformation $F = \sum_j^{i-1} F_j s^j$.

The following proposition takes the problem of integration and places it firmly into the realm of Hochschild cohomology.

Proposition 2.3.13 ([Ger64, Proposition 3])**.** *The associator is a Hochschild 3-cocycle.*

Definition 2.3.14. For an ith level deformation F we call the cohomology class of the associator the *obstruction* of F. In the case of an infinitesimal deformation, we call the obstruction the *primary* obstruction. If G is the associator of an infinitesimal deformation F_1 then the primary obstruction is by definition the cohomology class of

$$G(a,b,c) = F_1(F_1(a,b),c) - F_1(a,F_1(b,c)).$$

From this it immediately follows from (ν_i') that an ith level deformation integrates to an $(i+1)$st level deformation if and only if its associator is a coboundary (i.e. is cohomologous to 0).

2.3.4 The Gerstenhaber Bracket

The final piece of technology we require is the Gerstenhaber bracket. Gerstenhaber defined several interrelated functions on the bar complex, the full content of which is beyond the scope of this overview. However, what is relevant here is that there is a graded Lie algebra structure on the bar complex B^* which is usefully connected to the question of integration of ith level deformations.

Definition 2.3.15. For $0 \leq i \leq n$ let $\circ_i : B^n \otimes B^m \to B^{n+m-1}$ be the map defined for $f \in B^n$ and $g \in B^m$ by

$$f \circ_i g(c_0 \otimes c_1 \otimes \ldots \otimes c_{n+m}) = f(c_0 \otimes \ldots \otimes c_{i-1} \otimes g(1 \otimes c_i \otimes \ldots \otimes c_{i+m} \otimes 1) \otimes c_{i+m+1} \otimes \ldots \otimes c_{n+m}).$$

Then $f \circ g = \sum_i (-1)^{(m-1)i} f \circ_i g$ defines a bilinear map on B^*, which in turn defines the *Gerstenhaber bracket* given by:

$$[f,g] = f \circ g - (-1)^{(n-1)(m-1)} g \circ f.$$

Proposition 2.3.16 ([Ger63, Theorems 1 and 4])**.** *The Gerstenhaber bracket defines a graded Lie algebra structure on the bar complex that descends to a graded Lie algebra structure on Hochschild cohomology.*

The main theorem regarding this bracket is the following, which relates the question of integration of deformations to the cohomology of the Gerstenhaber bracket of certain elements.

Proposition 2.3.17. *An ith level deformation integrates to an $(i+1)$st level deformation if and only if*

$$\sum_{\substack{p+q=i \\ p,q>0}} [F_p, F_q]$$

is a coboundary in B_3.

In particular, an infinitesimal deformation f *has vanishing primary obstruction if and only if* $[f, f]$ *is a coboundary.*

Chapter 3

The Second Hochschild Cohomology Space for Two Algebras of Interest

3.1 Introduction

In this chapter we explain and carry out some calculations regarding the Hochschild cohomology structure for two specific PBW algebras, A and A_q. Recall from Definition 2.3.1 that the Hochschild cohomology of an algebra C is $\mathrm{Ext}_{C^e}(C, C)$, where C^e is the enveloping algebra of C as defined in Definition 2.1.1.

A classical result of Gerstenhaber (see Proposition 2.3.9) is that the second Hochschild cohomology space $\mathrm{HH}^2(C)$ parametrises the isomorphism classes of infinitesimal deformations of C. In this thesis we study only graded deformations where the parameter of deformation has degree 0. This means that we only need concern ourselves with calculating the degree 2 component of the second cohomology space.

Throughout the chapter we make use of the PBW property of the algebras. Specifically, since PBW algebras are Koszul we use the Koszul complex as the projective resolution of C in calculating $\mathrm{Ext}_{C^e}(C, C)$. The terms in the Koszul complex are finitely generated and so make the calculations tractable by computer. Much of the leg work is carried out using two symbolic algebra packages: for linear algebra calculations we use 'Sage' [Dev15] whereas for noncommutative algebra calculations we use 'Polygnome', a Python [Ros95] package written by the author (see Appendix B). The code for these calculations is included in Appendix A.1.

Throughout the discussion of computer calculations we will assume a basic understanding of object oriented programming concepts (objects, classes etc.) and will not explain any Python syntax. For references on these topics we refer the reader to [Ros95].

3.2 Calculations of Hochschild Cohomology Spaces

We consider a PBW algebra C. The Hochschild cohomology is calculated by a routine Ext calculation as described for example in [Wei94, Chapter 3]. We include the details of the calculation for completeness here as they are carried out by computer. The Koszul resolution (see Definition 2.1.4) is denoted by $K_n = C \otimes \overline{K_n} \otimes C$, with maps $k_n : K_n \to K_{n-1}$. The dual complex is denoted by

$$K^n = \mathrm{Hom}_{C^e}(K_n, C)$$

with the chain map written as k^n.

We will repeatedly make use of the isomorphism

$$\mathrm{Hom}_{C^e}(K_n, C) \cong \mathrm{Hom}_{\mathbb{K}}(\overline{K_n}, C) \tag{3.1}$$

which arises naturally from the adjunction between the free right C^e-module functor and the corresponding forgetful functor.

The calculation then will proceed as follows. Firstly, we choose bases for $\overline{K}_n$ for $n = 1, 2$ and 3. By the isomorphism (3.1) this is equivalent to choosing a free generating set for each of these K^n's.

Once these generating sets are chosen, we form the matrix of k^n for $n = 2$ and $n = 3$. We use this to calculate the kernel of k^3 and the image of k^2 in degree two. From bases of these two vector spaces we calculate a basis for the quotient space, which is the second Hochschild cohomology space in degree two.

Notation 3.2.1. Our convention is to write elements of K^n as column vectors. If $Z = \{z_1, \ldots, z_m\}$ is the chosen ordered free generating set of K_n then the vector

$$\begin{pmatrix} \Theta_1 \\ \vdots \\ \Theta_m \end{pmatrix}$$

represents the unique function in K^n that sends each $1 \otimes z_i \otimes 1$ to $\Theta_i \in A$. This determines the function completely because of the isomorphism (3.1).

3.2.1 Implementation Details

A full discussion of the source code of 'Polygnome' is beyond the scope of this thesis. However, we will discuss here the details of how the boundary maps in the Koszul resolutions are defined. We have made the full source code freely available in an online repository [Cam]. The defining code for the boundary maps `k_1` and `k_3` (and their corresponding dual maps) can be found in Appendix B.1. We discuss `k_2` and its dual map as examples here.

The following code uses a *decorator* `bimoduleMapDecorator`. A decorator is a Python class that modifies a function defined by the user in a predetermined way, perhaps depending on some variables. In this case `bimoduleMapDecorator` takes as variables the domain and codomain of a function between two free bimodules over an algebra. The decorator allows the user to define a function on a bimodule generating set of the domain and the program will automatically extend the function to the entire bimodule, thereby reducing the amount of repetition in the source code.

Let C be a PBW algebra with generating space V and relation space R. Recall that k_2 has domain $C \otimes R \otimes C$ and codomain $C \otimes V \otimes C$. In 'Polygnome' this information is stored as both the codomain and domain being the `tensorAlgebra` $C \otimes F \otimes C$, where F is merely a placeholder that is implemented as the `algebra` with no relations.

Relations in 'Polygnome' are implemented in a class `relation` that allows the storage of expressions of the form:

$$\texttt{leadingMonomial} = xy \text{ and } \texttt{lowerOrderTerms} = \sum c_i f_i^1 f_i^2.$$

See Section 4.1 for a discussion of the notation used here. Since K_2 has a free bimodule generating set of the form $\{1 \otimes (xy - \sum c_i f_i^1 f_i^2) \otimes 1\}$ we define `k_2` on this set only, recalling that

$$\begin{aligned} k_2(1 \otimes (xy - \sum c_i f_i^1 f_i^2) \otimes 1) = x \otimes y \otimes 1 - \sum c_i f_i^1 \otimes f_i^2 \otimes 1 \\ + 1 \otimes x \otimes y - \sum c_i \otimes f_i^1 \otimes f_i^2. \end{aligned}$$

```
def k_2(tens, alg):
    freeAlgebra = algebra()
    K1 = K2 = tensorAlgebra([alg, freeAlgebra, alg])

    @bimoduleMapDecorator(K2, K1)
    def k_2Inner(tens): #tens = 1 | relation | 1
        assert isinstance(tens, pureTensor)
        answer= tensor()
        rel =tens.monomials[1]
        for i in rel.leadingMonomial:
            answer = answer + i.coefficient \
                * pureTensor((i[0], i[1], 1))
            answer = answer + i.coefficient \
                * pureTensor((1, i[0], i[1]))
        for i in rel.lowerOrderTerms:
            answer = answer - i.coefficient \
                * pureTensor((i[0], i[1], 1))
            answer = answer - i.coefficient \
                * pureTensor((1, i[0], i[1]))
        return answer
    return k_2Inner(tens)
```

In order to dualise a map defined on the Koszul complex, we define a function that takes as parameter a function, and returns a function. Python allows functions to be passed and returned as variables in this manner. The following code makes use of `functionOnKn`, a class that encapsulates the vector notation for functions on K_n and which needs to be told the algebra and the chosen basis for K_n.

```
def koszulDualMap(inputMap):
    def functionFactory(func, knBasis):
        # func --> func o inputMap
        images = [func(inputMap(i, func.algebra)) \
                    for i in knBasis]
        return functionOnKn(func.algebra, knBasis, images)
    return functionFactory

k_2Dual = koszulDualMap(k_2)
```

3.3 The Second Hochschild Cohomology of A

Recall the algebra A from Definition 1.1.1. The algebra A is a PBW algebra with the PBW basis $\{x_2^i x_1^j x_3^k x_4^l\}$ [RS12, Proof of Lemma 5.7]. By work in [RS12, Section 5], we know that the Koszul complex of A is isomorphic to:

$$0 \to A[-4] \to A[-3]^{\oplus 4} \to A[-2]^{\oplus 6} \to A[-1]^{\oplus 4} \to A \to 0.$$

With this in mind, to find the doubly defined relations (i.e. $\overline{K_3}$) we only need to give four linearly independent elements of

$$V \otimes R \cap R \otimes V.$$

By observation one can see that the set

$$D := \left\{ \begin{array}{ll} d_1 := x_3 r_4 + x_1(r_6 - r_5) = r_1 x_2 - r_5 x_3, & d_2 := x_4 r_1 - x_1 r_3 = r_6 x_1 + (r_4 - r_3) x_3 \\ d_3 := x_4 r_5 - x_1 r_2 = r_6 x_2 + (r_4 - r_3) x_4 & d_4 := x_4 r_4 + x_2(r_6 - r_5) = r_3 x_2 - r_2 x_3 \end{array} \right\}$$

is linearly independent and therefore provides a basis of $\overline{K_3}$.

We walk through the Sage script that calculates first the basis of $\operatorname{Ker}(k^3)_2$ followed by the basis of $\operatorname{Im}(k^2)_2$. After some manipulations this allows us to read off a basis of $\operatorname{HH}^2_2(A)$.

3.3.1 Explanation of Computer Calculations

At the start of the script the variable `bases` is a list so that `bases[i]` is the PBW basis for A_i and `KnBases` is a list so that `KnBases[i]` is the chosen generating set for K_i. The first step is to build the generating set for K^2 and K^1. We include the K^2 case here as the code is almost identical in either case. Recall that `functionOnKn` is a class that encapsulates the vector notation for functions on K_n and which needs to be told the algebra and the chosen basis for K_n.

```
K2DualBasis = []
for i in range(6):
    for element in bases[2]:
        vectorRepresentation = [0] * i \
                              + [element] \
                              + [0] * (6-i-1)
        function = functionOnKn(A,
                              KnBases[2],
                              vectorRepresentation)
        K2DualBasis.append(function)
```

Now to find the kernel of k^3 we form the matrix of k^3 and use Sage to calculate a basis of the kernel. Note that we must convert the vector representations from those used internally by ‘Polygnome’ to those used by ‘Sage’ and vice versa. This is done using the helper functions `polygnomeVectorToSage` and `sageVectorToPolygnome`. `K` is defined as the field $\mathbb{Q}$ since we assume that $\mathbb{K}$ has characteristic 0.

```
matrixOfk_3Dual = [k_3Dual(vect, KnBasis[3])
                        for vect in K2DualBasis]
matrixOfk_3Dual = [polygnomeVectorToSage(vect, 3, 3)
                        for vect in matrixOfk_3Dual]
matrixOfk_3Dual= sage.Matrix(K, matrixOfk_3Dual)
```

Once we have k^3 stored as a matrix, it is a simple matter to ask 'Sage' to calculate the basis of $\mathrm{Ker}(k^3)$. Of course, we must convert this basis into a more readable form before printing it out.

```
kernelOfK_3Dual = matrixOfk_3Dual.left_kernel().basis()
kernelOfK_3Dual = [sageVectorToPolygnome(vect, 2)
                            for vect in kernelOfK_3Dual]
```

The output of this script can be found in Appendix A.1.1.

Now we repeat the above procedure, but instead we calculate the image of k^2 instead of the kernel of k^3. Most of the script is identical with a few integers changed. The salient portion of code is the following:

```
matrixOfk_2Dual = [k_2Dual(vect, KnBases[2])
                              for vect in K1DualBasis]
matrixOfk_2Dual = [polygnomeVectorToSage(vect,2,2)
                          for vect in matrixOfk_2Dual]
matrixOfk_2Dual= sage.Matrix(K, matrixOfk_2Dual)
imageOfk_2Dual = matrixOfk_2Dual.row_space().basis()
imageOfk_2Dual = [sageVectorToPolygnome(vect, 2)
                            for vect in imageOfk_2Dual]
```

The output of this script can be found in Appendix A.1.2. We gather the relevant information in the following theorem.

Theorem 3.3.1. *Let k^n be the chain map of the Koszul complex of A. Then*

1. $\dim(\mathrm{Im}(k^2)_2) = 14$
2. $\dim(\ker(k^3)_2) = 22$
3. $\dim(\mathrm{HH}^2_2(A)) = 8$
4. *The images of the following cocycles are a basis of* $\mathrm{HH}^2_2(A)$*:*

$$\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_3^2 \\ 0 \\ x_1x_4 \\ 0 \\ 0 \\ x_3x_4 \end{pmatrix} \begin{pmatrix} x_1^2 \\ 0 \\ 0 \\ x_2x_1 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \\ -x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x_2^2 \\ 0 \\ x_2x_4 \end{pmatrix}$$

The vector notation here is defined with reference to the chosen basis R above, as discussed in Notation 3.2.1.

Proof. This follows directly from the preceding computer calculations. We have ordered the output of the scripts discussed here in Appendices A.1.1 and A.1.2 so that the basis of the cohomology space is easy to read off by eye. □

3.4 A q-Deformation of A and its Second Hochschild Cohomology

The central problem of the thesis is to find families of deformations of the algebra A. We define here A_q, a one-parameter family of deformations of A that will be of particular interest in Chapter 7. We shall see that the family A_q itself has families of deformations which specialise to families of deformations of A in the semi-classical limit $q \to 1$. For that reason we record the details of the calculation of the degree two component of the second Hochschild cohomology of A_q. Since the calculations of the Hochschild cohomology of A_q are analogous to those for A we include them here.

3.4.1 Motivation

There are two methods of defining A that are both useful. The first, given in Section 3.3 is as a finitely presented algebra with four generators. However, another approach is taken in [RS12], in which the authors start with the graded division ring

$$Q_{gr}(A) = \mathbb{K}(u,v)[t, t^{-1}; \sigma]$$

where $\sigma \in \operatorname{Aut}(\mathbb{K}(u,v))$ is the map defined by

$$\sigma(u) = uv \text{ and } \sigma(v) = v.$$

Then one may define A to be the subalgebra generated by

$$E := \{x_1 = t,\ x_2 = ut,\ x_3 = vt,\ x_4 = uvt\}.$$

We define A_q in two ways. Consider the quantum plane

$$k_q[u,v] = \frac{\mathbb{K}\langle u,v\rangle}{vu - quv} \text{ for some } q \in \mathbb{K}^*$$

It is a basic fact in deformation theory that these algebras form a one-parameter family of deformations of $k[u,v]$. One may define an automorphism σ' of $\mathbb{K}_q(u,v)$ induced by setting

$$\sigma'(u) = uv \text{ and } \sigma'(v) = v. \tag{3.2}$$

Since this satisfies:

$$\sigma'(vu) = \sigma'(v)\sigma'(u) = vuv = quv^2 = q\sigma'(u)\sigma'(v) = \sigma'(quv)$$

in $\mathbb{K}_q(u, v)$, we can deduce that σ' does indeed define an automorphism of $\mathbb{K}_q(u, v)$. Since the defining equations (3.2) of σ' are the same for that of $\sigma \in \mathrm{Aut}(\mathbb{K}(u, v))$, we abuse notation and refer to both maps as σ with the algebra on which σ acts clear by the context. We define the algebra A_q to be the subalgebra of $\mathbb{K}_q(u, v)[t, t^{-1}; \sigma]$ generated by E_q, where

$$E_q := \{t, ut, vt, uvt\}.$$

We will show that A_q is a quantised version of A in the sense that $k_q[u, v]$ is a quantised affine plane, in particular A_q is a family of deformations of A.

3.4.2 Presentation of A_q

For a parameter $q \in \mathbb{K}^*$ consider the algebra

$$A'_q = \frac{\mathbb{K}\langle x_1, x_2, x_3, x_4\rangle}{(R_q)}$$

where R_q is the set consisting of the six relations

$$R_q = \left\{ \begin{array}{lll} r_1 := x_3x_1 - x_1x_3, & r_2 := x_4x_2 - qx_2x_4, & r_3 := x_4x_1 - x_2x_3 \\ r_4 := x_1x_2 - x_2x_3, & r_5 := x_3x_2 - qx_1x_4, & r_6 := x_4x_3 - x_1x_4 \end{array} \right\}.$$

By comparison with the definition of A it is clear that A'_q is a PBW algebra with the PBW basis $\{x_2^i x_1^j x_3^k x_4^l\}$ (see Section 3.3). Our next result is that A_q and A'_q are isomorphic.

Proposition 3.4.1. *The algebras A_q and A'_q are isomorphic. Furthermore, if $\overline{K_3} = \overline{K_3}(A_q)$ then $\dim(\overline{K_3}) = 4$.*

Proof. We first show that $A_q \cong A'_q$ by establishing that the map $\phi : A_q \to A'_q$ defined by

$$x_1 \mapsto t, x_2 \mapsto ut, x_3 \mapsto vt, x_4 \mapsto uvt$$

is in fact an isomorphism of algebras.

A simple calculation shows that the elements in E_q satisfy the relations in R_q, so that ϕ is a surjective algebra homomorphism. As for injectivity, we compare the Hilbert series of the two algebras.

Since A'_q is a PBW algebra with basis $\{x_2^i x_1^j x_3^k x_4^l\}$, we know that

$$h_{A'_q}(p) = \frac{1}{(1-p)^4}.$$

An argument similar to the proof of [RS12, Lemma 4.12 (1)] shows that the degree d component of $\mathbb{K}\langle E_q\rangle$ has the following set as a basis:

$$\left\{ u^i v^j t^d \;\middle|\; 0 \leq i \leq d,\, a(i) \leq j \leq b(i) \right\},$$

where

$$a(i) = \binom{i}{2} \text{ and } b(i) = \binom{d+1}{2} - \binom{d-i}{2}.$$

Thus

$$\dim_{\mathbb{K}}(\langle E_q\rangle_d) = \sum_{i=0}^{d} \left[\binom{d+1}{2} - \binom{d-i}{2} - \binom{i}{2} + 1 \right] = \binom{d+3}{3},$$

which implies

$$h_{\langle E_q\rangle}(p) = \frac{1}{(1-p)^4}.$$

Let $V = \mathrm{sp}_{\mathbb{K}}(x_1, x_2, x_3, x_4)$. Note that the canonical mapping $T(V) \to A_q$ is a graded homomorphism. We know that $R_q \otimes V$ and $V \otimes R_q$ are both $(6 \times 4 = 24)$-dimensional, and that $(A_q)_3$ is $(\binom{6}{3} = 20)$-dimensional. Recall that by definition

$$\overline{K_3} = R_q \otimes V \cap V \otimes R_q$$

Therefore we have the following equations:

$$\begin{aligned} \dim((A_q)_3) = 20 &= \dim(T(V)_3) - \dim(R_q \otimes V) - \dim(V \otimes R_q) + \dim(\overline{K}_3) \\ &= 64 - 24 * 2 + \dim(\overline{K_3}). \end{aligned}$$

This implies that $\dim(\overline{K_3}) = 4$ as required. $\square$

Corollary 3.4.2. *A_q is a flat family of algebras over $\mathbb{K}[q]$ that deforms A.*

Proof. This is a standard consequence of the fact that the Hilbert series of A_q is independent of q [Har77, Theorem III.9.9]. $\square$

As before, we can see that the set

$$D_q := \left\{ \begin{array}{ll} d_1 := x_3 r_4 + x_1(q r_6 - r_5) = r_1 x_2 - r_5 x_3, & d_2 := x_4 r_1 - x_1 r_3 = r_6 x_1 + (r_4 - r_3) x_3 \\ d_3 := x_4 r_5 - x_1 r_2 = r_6 x_2 + q(r_4 - r_3) x_4 & d_4 := x_4 r_4 + x_2(q r_6 - r_5) = r_3 x_2 - r_2 x_3 \end{array} \right\}$$

is linearly independent and so provides a basis for $\overline{K_3}$.

3.4.3 Explanation of Computer Calculations for A_q

As the form of the script is almost identical to that in Section 3.3.1 we will not go over it in as much detail as in that section. The main difference is that now we have a field `Kq` which is the field $\mathbb{Q}(q)$ since we are working in a field of characteristic zero with one indeterminate q.

At the start of the script the variable `bases` is a list so that `bases[i]` is the PBW basis for $(A_q)_i$ and `qKnBases` is a list so that `qKnBases[i]` is the chosen generating set for K_i. As before we construct a list which is a basis of K^2. The logic here is identical to that in Section 3.3.1.

```
qK2DualBasis = []
for i in range(6):
    for element in bases[2]:
        vectorRepresentation = [0] * i \
                               + [element] \
                               + [0] * (6-i-1)
        function = functionOnKn(A,
                                qKnBases[2],
                                vectorRepresentation)
        qK2DualBasis.append(function)
```

The script now neatly divides into two pieces. The first calculates the kernel of k^3, and the output of this can be found in Appendix A.2.1.

```
matrixOfk_3Dual = [k_3Dual(vect, qKnBasis[3])
                        for vect in qK2DualBasis]
matrixOfk_3Dual = [polygnomeVectorToSage(vect, 3, 3)
                        for vect in matrixOfk_3Dual]
matrixOfk_3Dual= sage.Matrix(Kq, matrixOfk_3Dual)

kernelOfK_3Dual = matrixOfk_3Dual.left_kernel().basis()
kernelOfK_3Dual = [sageVectorToPolygnome(vect, 2)
                        for vect in kernelOfK_3Dual]
```

The second piece calculates the image of k^2:

```
matrixOfk_2Dual = [k_2Dual(vect, qKnBases[2])
                        for vect in qK1DualBasis]
matrixOfk_2Dual = [polygnomeVectorToSage(vect,2,2)
                        for vect in matrixOfk_2Dual]
matrixOfk_2Dual= sage.Matrix(Kq, matrixOfk_2Dual)
imageOfk_2Dual = matrixOfk_2Dual.row_space().basis()
imageOfk_2Dual = [sageVectorToPolygnome(vect, 2)
                        for vect in imageOfk_2Dual]
```

The output of this script can be found in Appendix A.2.2. We gather the relevant information in the following theorem.

Theorem 3.4.3. *Let k^n be the chain map of the Koszul complex of A_q. Then*

1. $\dim(\mathrm{Im}(k^2)_2) = 18$
2. $\dim(\ker(k^3)_2) = 14$
3. $\dim(\mathrm{HH}^2_2(A_q)) = 4$
4. *The images of the following cocycles form a basis of* $\mathrm{HH}^2_2(A_q)$

$$\left\{ \begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ qx_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \right\}$$

The vector notation here is defined with reference to the chosen basis R_q above, as discussed in Notation 3.2.1.

Proof. As with Theorem 3.3.1, this is a direct consequence of the calculations discussed above. The particular bases are recorded in Appendices A.2.1 and A.2.2, ordered to make the basis of HH^2_2 easily read off by eye. □

As a point of interest, we note that under the semi-classical limit $q \to 1$ we can see that these vectors have as limits elements of the chosen basis of $\mathrm{HH}^2_2(A)$.

Chapter 4

A Partial Chain Map from the Bar Resolution to the Koszul Resolution for a PBW-Algebra

In this chapter we define a section to the canonical inclusion $i_* : K_* \to B_*$ of the Koszul complex into the bar complex for a PBW algebra. We do this to reduce the generally difficult problem of calculating Gerstenhaber brackets to a computer calculation. Our main application of this work will be to calculate which infinitesimal deformations of A and A_q have vanishing primary obstruction; for a detailed discussion of this please see Section 5.2.

The work in this chapter is based mostly upon Bergman's Diamond Lemma [Ber78, Theorem 2.1] and the theory of reduction systems. Please see Section 2.1.1 for a discussion of these.

4.1 Preliminaries

Take $\mathbb{K}$ to be a field of characteristic 0 and unadorned tensor products to be over $\mathbb{K}$. We will use both $a \otimes b$ and $a|b$ to represent elements of tensor structures and will freely interchange between the two.

Recall the definitions from Section 2.1.1 relating to PBW algebras. Throughout this chapter we assume that A is a PBW algebra generated by a finite set $V = \{v_1, \ldots, v_\gamma\}$. We choose a finite set $R \subseteq \mathrm{sp}_{\mathbb{K}}(V) \otimes \mathrm{sp}_{\mathbb{K}}(V)$ of relations, so that $A = T(V)/(R)$ with R chosen so that each relation is of the form

$$xy - \sum c_i f_i^1 f_i^2 \text{ where } c_i \in \mathbb{K},\ x, y, f_i^1, f_i^2 \in V \text{ and } f_i^1 f_i^2 \text{ is in PBW order.}$$

Note that since R is homogeneous, $A = \bigoplus_0^\infty A_i$ is a graded algebra.

Notation 4.1.1. We adopt the notation that for an element $v \in T(V)$ we write $\left[v\right]$ for v in PBW reduced form with respect to the relations R. Note that $\left[v\right]$ is an element of $T(V)$; we write $\pi(v)$ for the image of v under the canonical projection from $T(V)$ to A.

We introduce a dot product on $T(V)$ for convenience, by extending the following bilinearly:

$$(m_1 \bullet m_2) = \delta_{m_1,m_2} \text{ for monomials } m_1, m_2 \in T(V).$$

Example 4.1.2. We recall the definition of the boundary map of the Koszul complex in the specific degrees 1,2 and 3 since they will be used extensively throughout this chapter.

1. The map $k_1 : A \otimes V \otimes A \to A \otimes A$ is defined on the free generating set $\{1 \otimes v_i \otimes 1\}_{i=1}^{\gamma}$ for K_1 as :

$$k_1(1 \otimes v_i \otimes 1) = \pi(v_i) \otimes 1 - 1 \otimes \pi(v_i)$$

2. Let the chosen basis of R be written as $\{\rho_\sigma\}_\sigma$ where each

$$\rho_\sigma = W_\sigma - f_\sigma = xy - \sum_i c_i f_i^1 f_i^2 \text{ for } c_i \in \mathbb{K}^* \text{ and } x, y, f_i^j \in V.$$

Then $k_2 : A \otimes R \otimes A \to A \otimes V \otimes A$ is defined on an element of the generating set $\{1 \otimes \rho_\sigma \otimes 1\}_\sigma$ as:

$$k_2(1 \otimes \rho_\sigma \otimes 1) = \pi(x) \otimes y \otimes 1 - \sum_i c_i \pi\left(f_i^1\right) \otimes f_i^2 \otimes 1 + 1 \otimes x \otimes \pi(y) - \sum_i c_i \otimes f_i^1 \otimes \pi\left(f_i^2\right)$$

3. Let a chosen basis of $\overline{K}_3$ be $\left\{\sum_i c_i x_i \otimes r_i = \sum_j c'_j \otimes r'_j z_j\right\}$ where $c_i, c'_j \in \mathbb{K}^*$, $r_i, r'_j \in R$ and $x_i, z_j \in V$. Then k_3 is defined on the corresponding generating set for K_3 as:

$$k_3(1 \otimes \sum_i c_i x_i r_i \otimes 1) = \sum_i c_i \pi(x_i) \otimes r_i \otimes 1 - \sum_j c'_j \otimes r'_j \otimes \pi(z_j).$$

Definition 4.1.3. A partial ordering $\prec$ on a monoid N is a *partial monoid ordering* if for every $B, C, M_1, M_2 \in N$:

$$M_1 \prec M_2 \implies BM_1C \prec BM_2C.$$

We write the free monoid on V as $\langle V \rangle$. A partial monoid ordering on $\langle V \rangle$ is said to be *compatible* with R if for every σ in the reduction system, each monomial M with nonzero coefficient in f_σ satisfies $M \prec W_\sigma$.

Example 4.1.4. Lexicographic ordering given by $v_1 \prec v_2 \prec \ldots \prec v_\gamma$ is a compatible monoid ordering on $\langle V \rangle$ satisfying the descending chain condition. This follows immediately from the definition of a PBW-algebra.

Since B_* and K_* are both resolutions of A as an A^e-module we have by the comparison theorem [Wei94, Theorem 2.2.6] that they are quasi-isomorphic. Van den Bergh [VdB94, Proposition 3.3] showed that the map natural inclusion $i_* : K_* \to B_*$ provides such a quasi-isomorphism. Also, we know by work in [Ger64, Section 1.1] that the infinitesimal deformation structure of A can be studied by calculations on the Hochschild cohomology groups $\mathrm{Ext}^*_{A^e}(A, A)$. Since each K_* is finitely generated, we wish to study the Hochschild cohomology by using K_* instead of B_*. However, since the Gerstenhaber bracket is defined with explicit reference to B_*, we require a map to transfer this structure from B_* to K_*.

Therefore the aim of this chapter is to (partially) define a chain map $m_* : B_* \to K_*$ that is a section of i_* for use in calculations. We only require this map to be defined in positions 1 and 2, and so we concentrate on those positions in the following. A discussion of some of the basic properties that m_* must satisfy can be found in [HSSÁ14, Definition 1.4].

4.2 The Map in Position 1

We define $m_1 : B_1 \to K_1$ first on a bimodule generating set. Let $x_1 \cdots x_n \in \langle V \rangle$ with each $x_i \in V$ be in PBW order. Then

$$m_1(1|x_1 \cdots x_n|1) = \sum_{i=1}^{n} x_1 \cdots x_{i-1}|x_i|x_{i+1} \cdots x_n.$$

We now extend m_1 A-bilinearly to all of B_1. That the square

$$\begin{array}{ccc} B_1 & \xrightarrow{b_1} & B_0 \\ {\scriptstyle m_1}\downarrow & & \| \\ K_1 & \xrightarrow{k_1} & K_0 \end{array}$$

commutes is easy to show:

$$
\begin{aligned}
b_1(1|x_1\cdots x_n|1) &= x_1\cdots x_n|1 - 1|x_1\cdots x_n\\
&= x_1\cdots x_n|1 - x_1\cdots x_{n-1}|x_n + x_1\cdots x_{n-1}|x_n - x_1\cdots x_{n-2}|x_{n-1}x_n\\
&+ \ldots + x_1|x_2\cdots x_n - 1|x_1\cdots x_n\\
&= k_1(\sum_{i=1}^{n} x_1\cdots x_{i-1}|x_i|x_{i+1}\cdots x_n)\\
&= k_1\circ m_1(1|x_1\cdots x_n|1).
\end{aligned}
$$

We note that this map is also defined in [HSSÁ14, Definition 1.4].

4.3 The Bergman Graph

In order to define m_2 we use the theory developed by Bergman in [Ber78] concerning some of the combinatorial structures arising from a PBW algebra and its relations. Although the following few sections deal with building a map which is described mathematically, it may be helpful to motivate the description from an algorithmic point of view.

A common approach in algorithm design is to find or define a data structure that models the important information for the problem at hand (see e.g. [Ski08, Section 4.3]). This not only improves the efficiency of algorithms, it also provides a starting point for how to approach algorithmic problems. In this case a directed graph models the relevant data very well because there is an implied partial order structure defined by the relations as we shall see shortly. Therefore we introduce a directed acyclic graph called the Bergman graph. Choosing certain paths through this graph will allow us to define choices of the map m_2.

Definition 4.3.1. We construct a weighted directed $(n-1)$-coloured graph G that has as vertices all elements of $T(V)$ in degree n. The edges of this graph correspond to reductions $r_{B\sigma C}$ for the algebra A. There is an arrow of colour i and weight $w \in \mathbb{K}^*$ from a to b exactly when $a \neq b$ and there exists some reduction $r := r_{B\sigma C}$ such that $r(a) = b$ with $deg(B) = i-1$ and $(BW_\sigma C \bullet a) = w \neq 0$. We call this graph the *Bergman graph* (in degree n).

Note that the connected components of the Bergman graph correspond precisely to elements of the algebra A, since two elements are connected by a path precisely when one is some reduction of the other.

Notation 4.3.2. We define four functions, the head function h, tail function t, weight function wt and colour function c which all return values as the names suggest when applied to an arrow.

Definition 4.3.3. A path in G from vertex v_1 to vertex v_2 is an ordered tuple of arrows $(a_1, \ldots, a_n)$ with the following holding for each: $h(a_i) = t(a_{i+1})$ for each $1 \leq i < n$, $t(a_1) = v_1$ and $h(a_n) = v_2$. We also allow the empty path, which can start and finish at any vertex.

For a given element a in $T(V)$ we can consider the subgraph of the Bergman graph generated by taking all arrows out of each vertex recursively starting from a. We call this subgraph of the Bergman graph $\mathrm{BG}(a)$. The PBW condition then means that this graph has a unique terminal node, $\big[a\big]$, that all paths will reach if extended enough. We call any path from a to $\big[a\big]$ a *simplification path.* Note that an empty path starting at a monomial in PBW order is considered a simplification path. For any simplification path $(p_1, \ldots, p_m)$ we get a corresponding reduction sequence for $t(p_1)$.

Definition 4.3.4. Let $\mathrm{Path}(G)$ be the free A-bimodule generated by the set of paths in G. We define a function $\nu : \mathrm{Path}(G) \to K_2$ by extending the following map on a path $p = (p_1, \ldots, p_m)$ A-bilinearly:

$$\nu(p) = \begin{cases} 0 & \text{if } p \text{ is the empty path} \\ \mathrm{wt}(p_1)\pi(B) \otimes \rho_\sigma \otimes \pi(C) & \text{if } m = 1 \text{ and } p_1 = r_{B\sigma C} \\ \nu(p_1) + \nu(p_2, \ldots, p_m) & \text{for } m \geq 2 \end{cases}$$

Because the upcoming work can get quite complicated when the underlying procedure is actually very simple, we give a few examples of calculating these paths and their images in K_2. In the following we use red to represent the colour 1, blue for the colour 2 and green for the colour 3.

Example 4.3.5. Consider the quantum plane

$$A = \mathbb{K}_q[x, y] = \frac{\mathbb{K}\langle x, y\rangle}{yx - qxy} \text{ for } q \in \mathbb{K}^*$$

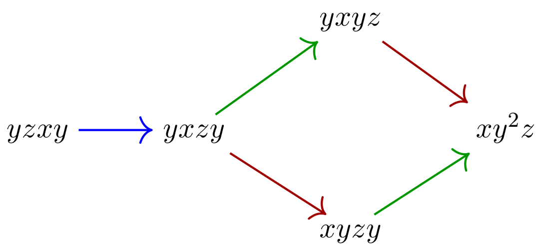
with the PBW basis $\left\{x^i y^j \;\middle|\; i, j \in \mathbb{N}\right\}$. If we take the element $a := yx^2$ then the Bergman graph of a is:

$$yx^2 \xrightarrow{1} qxyx \xrightarrow{q} q^2x^2y$$

where we have labelled the arrows with their weights. The unique path from $a = yx^2$ to $\big[a\big]$ has an image under ν of $1 \otimes (yx - qxy) \otimes x + qx \otimes (yx - qxy) \otimes 1$.

Example 4.3.6. Consider the commutative algebra $A = \mathbb{K}[x, y, z]$ with the PBW basis $\left\{x^i y^j z^k \;\middle|\; i, j, k \in \mathbb{N}\right\}$. If we take the element $a := yxyz$ then the Bergman graph of a

is:



where all of the weights are 1 in this case. Let p be the top path and q be the bottom path. Then:

$$\nu(p) = \nu(q) = y \otimes [z, x] \otimes y + yx \otimes [z, y] \otimes 1 + 1 \otimes [y, x] \otimes yz.$$

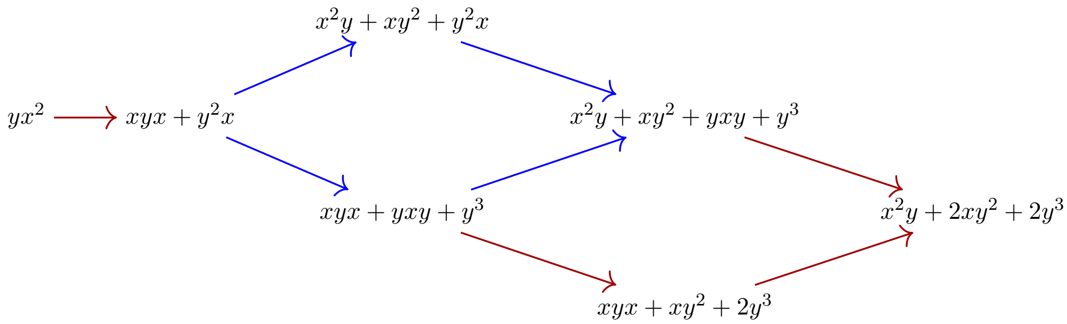
The fact that these two paths have the same image under ν is no accident and is an example of a somewhat subtle phenomenon which will be investigated below.

Example 4.3.7. Consider the Jordan plane:

$$A = \frac{\mathbb{K}\langle x, y \rangle}{(yx - xy - y^2)}$$

with the PBW basis $\{x^i y^j \mid i, j \in \mathbb{N}\}$. Set $r := xy - yx - y^2 \in V \otimes V$.

If we take the element $a := yx^2$ then the Bergman graph of a is:



where all of the weights are 1 in this case. The three different choices of path all have the same image under ν, which is

$$1 \otimes r \otimes x + x \otimes r \otimes 1 + y \otimes r \otimes 1 + 1 \otimes r \otimes y.$$

4.4 The Minimal Partial Monoid Order on $\langle V \rangle$

In this section we show that the definition of a compatible monoid order actually suffices to define a compatible monoid order that is the minimal partial order with that property. This minimal ordering provides a very useful tool for reasoning about paths in the Bergman graph because, as we shall prove, a monomial x is related to another monomial y in this ordering precisely when x there is some vertex v of the Bergman graph of y with $(x \bullet v)$ nonzero.

Proposition 4.4.1. *In a PBW algebra A, for any infinite sequence of reductions $(r_1, r_2, \ldots)$ and any element $x \in T(V)$ there exists some $i \in \mathbb{N}$ such that for all $j \in \mathbb{N}_{\geq i}$*

$$r_j r_{j-1} \cdots r_1(x) = r_i \cdots r_1(x).$$

Proof. Since A is PBW and Example 4.1.4 gives us a compatible monoid ordering with the descending chain condition, this proposition is a simple application of Bergman's Diamond Lemma [Ber78, Theorem 1.2]. □

We define a partial monoid ordering on $\langle V \rangle$. This order is defined with respect to R in the chosen form $W_\sigma - f_\sigma$. For monomials $x, y \in \langle V \rangle$, we write:

$$x \leq' y \iff \text{there exists a reduction } r \text{ such that } (r(y) \bullet x) \neq 0.$$

Then we define the relation $\leq$ on $\langle V \rangle$ to be the transitive closure of $\leq'$.

Definition 4.4.2. If $x, y \in \langle V \rangle$ with $y \leq x$ then we say a sequence of monomials $\{N_j\}_{j=0}^{i}$ *connects* x and y if

$$N_0 = x \leq' N_1 \leq' \ldots \leq' N_i = y.$$

Lemma 4.4.3. *If $y, x \in \langle V \rangle$ and $y \leq x$ then choose a sequence of monomials $\{N_j\}_j^i$ that connects x and y of minimal length. For each $j \in \{1, \ldots i\}$ choose a reduction r_j so that*

$$(N_j \bullet r_j(N_{j-1})) \neq 0.$$

Then the reduction sequence $r_i r_{i-1} \cdots r_1$ satisfies

$$(y \bullet r_i r_{i-1} \cdots r_1(x)) \neq 0.$$

Proof. We argue by induction on i. If $i = 1$ then by the definition of $\leq'$

$$(y \bullet r_1(x)) \neq 0.$$

Assume the result holds if $i = L - 1$ and take any y, x as in the lemma with a sequence of N_j of length L. Write $r_L(N_{L-1}) = ay + Z$ where $a \in \mathbb{K}^*$ and $Z \in T(V)$ such that $(y \bullet Z) = 0$.

Since $N_{L-1} \leq x$ and we have a sequence of length $L - 1$ connecting x to N_{L-1}, by induction we deduce that:

$$(r_{L-1} r_{L-2} \cdots r_1(x) \bullet N_{L-1}) \neq 0.$$

In other words

$$r_{L-1}r_{L-2}\cdots r_1(x) = bN_{L-1} + \Theta$$

where $b \in \mathbb{K}^*$ and $\Theta \in T(V)$ satisfies $(\Theta \bullet N_{L-1}) = 0$. In particular we know that $r_L(\Theta) = \Theta$. Of course, $(\Theta \bullet y)$ must be 0 since otherwise L would not be the length of a minimal path.

From this we obtain

$$r_L r_{L-1}\cdots r_1(x) = aby + bZ + \Theta,$$

which implies

$$(r_L r_{L-1}\cdots r_1(x) \bullet y) = ab + b(Z \bullet y) + (\Theta \bullet y) = ab \neq 0.$$

□

Lemma 4.4.4. *For $x, y \in \langle V\rangle$ we have that $y \leq x$ if and only if there exists a vertex v in* $\mathrm{BG}(x)$ *such that*

$$(v \bullet y) \neq 0.$$

Proof. If $y \leq x$ then we obtain a reduction sequence $r_i \cdots r_1$ by applying Lemma 4.4.3 such that $r_i \cdots r_1(x)$ is a vertex in $\mathrm{BG}(x)$ and

$$(r_i \cdots r_1(x) \bullet y) \neq 0.$$

As for the converse, this follows from the definition of the Bergman graph. If such a vertex exists then there is a path from x to v which corresponds to a reduction sequence $r_i \cdots r_1$. Write $r_j = r_{A_j \sigma_j B_j}$. Then

$$y \leq' A_{i-1}W_{\sigma_{i-1}}B_{i-1} \leq' \ldots \leq' A_1 W_{\sigma_1} B_1 \leq' x.$$

□

Proposition 4.4.5. *$\leq$ is the unique minimal partial monoid order on $\langle V\rangle$ that is compatible with R, in the sense that if $\prec$ is any other partial monoid order that is compatible with R and $a \leq b$ then $a \prec b$.*

Proof. We first show that $\leq$ is a partial order, i.e. that it is reflexive, transitive and antisymmetric. That $\leq$ is reflexive is trivial since we may take, for example, any relation $r_{B\sigma C}$ such that $\deg(B) > \deg(y)$ and be sure that $r(y) = y$, so that $y \leq' y$. Of course, the transitive closure of a relation is transitive so this is also trivial. As for antisymmetry, this does require some work.

If $x, y \in \langle V \rangle$ with $x \leq y$ and $y \leq x$ then we prove by contradiction that $x = y$. If $x \neq y$ then take a sequence $M_1, \ldots M_{j-1} \in \langle V \rangle$ of minimal length such that

$$x \leq' M_{j-1} \ldots \leq' M_1 \leq' y$$

and likewise a sequence $N_1 \ldots N_{i-1} \in \langle V \rangle$ of minimal length connecting y to x:

$$y \leq' N_{i-1} \leq' \ldots \leq' N_1 \leq' x.$$

We induct on $i + j$.

Base Case: If $x \leq' y$ and $y \leq' x$ then there are two reductions s, r such that $s(y) = bx + B$ and $r(x) = cy + C$ where

$$(B \bullet x) = (B \bullet y) = (C \bullet x) = (C \bullet y) = 0.$$

Now we claim that $(r, s, r, s, \ldots)$ is an infinite sequence of reductions that never stabilises when applied to x. Note that

$$sr(x) = bcx + cB + C.$$

Therefore if $d, e \in \mathbb{K}$ satisfy:

$$(sr)^n(x) = b^i c^j x + dB + eC,$$

then

$$r(sr)^n(x) = b^i c^{j+1} y + (b^i c^j + e)C + dB$$

and

$$(sr)^{n+1}(x) = b^{i+1} c^{j+1} x + (b^i c^j + e)C + (b^i c^{j+1} + d)B,$$

since B and C are orthogonal to both x and y. Therefore $(r, s, r, s, r \ldots)$ never stabilises when applied to x. This contradicts Proposition 4.4.1.

Inductive case: We reduce to the case that $j = 1$ by noting that $x \leq' M_{j-1}$ by definition and $M_{j-1} \leq x$ is given by considering the sequence $M_{j-2}, \ldots, M_1, y, N_{i-1}, \ldots, N_1$. If there is a shorter sequence connecting M_{j-1} and x then by induction $M_{j-1} = x$ and the original sequences were not minimal. Therefore we may pass to the case that $x \leq' y$ and $y \leq' N_{i-1} \leq' \ldots \leq' N_1 \leq' x$.

By Lemma 4.4.3 we obtain a reduction sequence $r_i r_{i-1} \cdots r_1$ such that

$$r_i r_{i-1} \cdots r_1(x) = by + B \text{ for some } b \in \mathbb{K}^* \text{ and } B \in T(V) \text{ satisfying } (B \bullet y) = 0.$$

We claim that furthermore $(B \bullet N_l) = 0$ for all l. Indeed if there is some N_l for which $(N_l \bullet B) \neq 0$ then consider that there must be a smallest $m > l$ such that

$$(r_{m+1} r_m \cdots r_l \cdots r_1(x) \bullet N_l) \neq 0$$

which by definition would imply that $(r_m(N_m) \bullet N_l) \neq 0$ and so $N_m \leq' N_l$.

But $N_m \leq N_l$ by definition, and so we have $N_l = N_m$ by induction, contradicting the minimality of i.

By the same argument, if s is a reduction such that $s(y) = cx + C$ with $c \in \mathbb{K}^*$ and C orthogonal to both of x and y, then

$$(N_j \bullet C) = 0 \text{ for all } j.$$

Therefore $(r_1, \ldots, r_n, s, \ldots)$ forms an infinite sequence of reductions that never stabilises when applied to x, since it must always pass through elements of the form:

$$dx + eB + fC \text{ and } gy + hB + iC \text{ for some } e, f, h, i \in \mathbb{K} \text{ and } d, g \in \mathbb{K}^*.$$

This contradicts Proposition 4.4.1 and provides the inductive step.

Therefore $\leq$ is a partial order on $\langle V \rangle$. That it is also a monoid order is trivial since for any $M, N \in \langle V \rangle$

$$r_{C\sigma D}(y) = ax + Z \text{ for } a \in \mathbb{K}^* \text{ and } Z \in T(V) \text{ orthogonal to } x \text{ and } y$$

implies

$$(r_{MC\sigma DN}(MyN) \bullet MxN) = (M(ax + Z)N \bullet MxN) = a \neq 0.$$

That minimal orders are unique is obvious, and that $\leq$ is minimal is clear since the definition of compatible is that if $r(y) = bx + B$ for some r then $x \prec y$. Since any partial order must be transitive, then $x \leq y$ implies that $x \prec y$.

□

As a simple corollary to Proposition 4.4.5 and Lemma 4.4.4, the Bergman graph is a directed acyclic graph. Since there are a finite number of monomials of any given degree, and the relations of A are homogeneous, we also know that $\leq$ satisfies the descending chain condition. We also record the following useful fact.

Lemma 4.4.6. *There is an upper limit for the length of a path in* $\mathrm{BG}(a)$ *that depends only upon* $a \in T(V)$.

Proof. Call this limit $U(a)$. We make an induction argument. If a is in PBW order then this limit is zero. Otherwise, assume the result holds for all b such that there is a reduction r with $r(a) = b$. Then the upper limit for a is

$$\max \{U(b) \mid a \to_r b\} + 1.$$

□

4.5 A Few Operations on Paths

We define some useful operations on edges in the Bergman graph. Since the Bergman graph has vertices that are elements of $T(V)$, there is an obvious $T(V)$-bimodule structure on the free group of vertices. We shall see below that this defines a $T(V)$-bimodule structure on the free group of edges as well and that ν 'respects' this structure in a useful way.

Furthermore, one would expect there to be some relationship between $\mathrm{BG}(a)$ and $\mathrm{BG}(a + X)$ for some $a, X \in T(V)$. Although a full description of this relationship would be cumbersome (and unhelpful), we note a particular case in which $\mathrm{BG}(a)$ is a subgraph of $\mathrm{BG}(a + X)$ and this fact is very useful in reasoning about paths in the Bergman graph.

Definition 4.5.1. Let $X \in T(V)$ and $a \in \langle V\rangle$. If for every $v \in \langle V\rangle \setminus \{a\}$ such that $(X \bullet v) \neq 0$ we have that $v \leq a$ implies that $v = \lfloor v \rceil$, then we say X *doesn't interfere* with a. In other words, if $X = \sum_i c_i m_i$ for constants $c_i \in \mathbb{K}^*$ and monomials $m_i \in \langle V\rangle$, then either $m_i = \lfloor m_i \rceil$ or $m_i \not\leq a$.

Otherwise, we say X *interferes* with a.

Note that non-interference is an inheritable property in the sense that if $m \leq a$ and $X \in T(V)$ doesn't interfere with a then X doesn't interfere with m.

Definition 4.5.2. If p is a path in the Bergman graph starting from $a \in \langle V\rangle$, and $X \in T(V)$ doesn't interfere with a, then we define the *translation* of p by X, written p^X, to be the path corresponding to the same reductions as p starting from $a + X$.

Lemma 4.5.3. *1. If $x \leq y$ then there exists a vertex v in* $\mathrm{BG}(y)$ *such that*

$$v = ax + X$$

where $a \in \mathbb{K}^*$ and $X \in T(V)$ doesn't interfere with x.

2. If $x \leq y$ then $\mathrm{BG}(x)$ *is isomorphic to a subgraph of* $\mathrm{BG}(y)$ *rooted at any vertex $ax + X$ in* $\mathrm{BG}(y)$ *where $a \in \mathbb{K}^*$ and $X \in T(V)$ doesn't interfere with x.*

Proof. 1. Since $x \leq y$ then by Proposition 4.4.5 there is a vertex v in $\mathrm{BG}(y)$ such that

$$v = ax + Z$$

where Z is orthogonal to x and $a \in \mathbb{K}^*$. However, it is possible that Z does interfere with x. With that in mind we decompose Z as

$$Z = Z_1 + Z_2$$

where $Z_1, Z_2 \in T(V)$. We choose Z_2 so that it doesn't interfere with x and every monomial m such that $(m \bullet Z_1) \neq 0$ satisfies $m < x$ and m is out of PBW order. In this way Z_1 contains all of the 'interfering' monomials in Z.

Then there is another vertex in $\mathrm{BG}(y)$

$$v' = ax + \big[Z_1\big] + Z_2$$

with $\big[Z_1\big]$ orthogonal to x. Then $X := \big[Z_1\big] + Z_2$ doesn't interfere with x.

2. Since X doesn't interfere with x, we obtain an injective mapping $\mathrm{BG}(x) \hookrightarrow \mathrm{BG}(y)$ defined by taking an edge e to e^X.

□

Definition 4.5.4. If $y \in \langle V \rangle$ and p is a path

$$\sum x_i \xrightarrow[r_{B_1\sigma_1 C_1}]{w_1} \cdots \xrightarrow[r_{B_n\sigma_n C_n}]{w_n} \sum z_i$$

then we define the *right stitch* of p by y to be the path:

$$\sum x_i y \xrightarrow[r_{B_1\sigma_1(C_1 y)}]{w_1} \cdots \xrightarrow[r_{B_n\sigma_n(C_n y)}]{w_n} \sum z_i y.$$

We write this path as $p \,\dot{|}\, y$. The *left stitch* $y \,\dot{|}\, p$ is defined analogously:

$$\sum y x_i \xrightarrow[r_{(yB_1)\sigma_1 C_1}]{w_1} \cdots \xrightarrow[r_{(yB_n)\sigma_n C}]{w_n} \sum y z_i.$$

Remark 4.5.5. Although we will not use the full structure here, stitching provides a $T(V)$-bimodule structure on the additive free group of paths in the Bergman graph.

Lemma 4.5.6. *1. For a path p and $y \in \langle V \rangle$ the following holds:*

$$\nu(p \,\dot{|}\, y) = \nu(p)\pi(y).$$

Likewise:

$$\nu(y \,\dot{|}\, p) = \pi(y)\,\nu(p).$$

2. *Let $X \in T(V)$ and $a \in \langle V \rangle$ be such that X doesn't interfere with a. If p is a path starting from a then the following holds:*

$$\nu(p^X) = \nu(p).$$

Proof. 1. We prove the right hand case, as the left hand case is completely analogous. From the definition of ν we only need check this on paths of length 1. Say the edge has weight w and reduction $r_{B\sigma C}$. Then $p \mathbin{¦} y$ has weight w and reduction $r_{B\sigma(Cy)}$. Therefore we have the equations:

$$\begin{aligned}\nu(p \mathbin{¦} y) &= w\pi(B) \otimes \rho_\sigma \otimes \pi(Cy) \\ &= (w\pi(B) \otimes \rho_\sigma \otimes \pi(C))\pi(y) \\ &= \nu(p)\pi(y).\end{aligned}$$

2. Again, we only need to consider paths of length one by the definition of ν. Translating an edge e by X changes the tail and head but does not change the weight or corresponding reduction of that edge because X doesn't interfere with $t(e)$. However, the definition of ν depends only on the corresponding reduction and the weight. Therefore $\nu(e^X) = \nu(e)$.

□

4.6 Diamonds in the Bergman Graph

We define a class of subgraphs of the Bergman graph that are precisely the structures that Bergman's *Diamond* Lemma refers to.

We would like it if every time a vertex a has two arrows emanating from it

$$r_1 := r_{B\sigma C} \text{ and } r_2 := r_{D\tau E},$$

these two arrows form the first two sides of a diamond. However, for reasons that will become clear, this would introduce far too many special cases and make any proofs laborious. With that in mind we make a slightly unnatural definition now, with the promise that it saves a lot of work later on. Therefore we define a diamond to be one of the subgraphs defined in the following four cases.

(1) $DW_\tau E \neq BW_\sigma C$.

  (1a) $r_2(r_1(a)) = r_1(r_2(a))$

  (1b) $r_2(r_1(a)) \neq r_1(r_2(a))$

(2) $DW_\tau E = BW_\sigma C$ and $a = bBW_\sigma C + \Theta$ where $b \in \mathbb{K}^*$ and Θ doesn't interfere with $BW_\sigma C$.

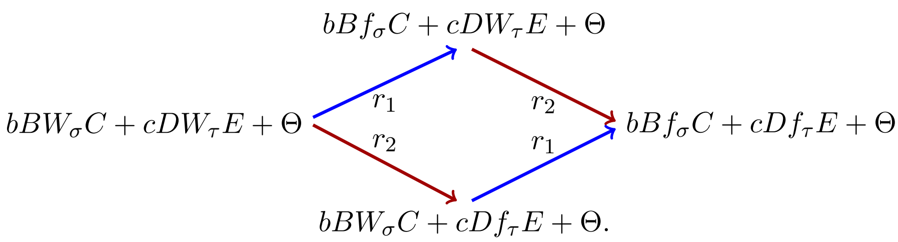
  (2a) $\deg(B) = \deg(D) \pm 1$. This case is called an *overlap ambiguity*.

(2b) $\deg(B) \neq \deg(D) \pm 1$.

We describe the subgraph that forms the diamond in each case.

(1) In both of these cases, $a = bBW_\sigma C + cDW_\tau E + \Theta$ where Θ is orthogonal to both $BW_\sigma C$ and $DW_\tau E$ and $b, c \in \mathbb{K}^*$. Let $m_1 = BW_\sigma C$ and $m_2 = DW_\tau E$. Since $m_1, m_2 \in \langle V \rangle$ and $m_1 \neq m_2$, we know $(m_1 \bullet m_2) = 0$.

(1a) The subgraph is precisely that formed by the edges corresponding to r_1 and r_2 in the equality: $r_2(r_1(a)) = r_1(r_2(a))$. That is the subgraph:



(1b) Since $r_2(r_1(a)) \neq r_1(r_2(a))$, we claim that either $m_2 \leq' m_1$ or $m_1 \leq' m_2$ (but not both since $m_1 \neq m_2$). Assume neither inequality holds for a contradiction. Then as in the previous case $r_1(a) = br_1(m_1) + cm_2 + \Theta$, and since $m_2 \not\leq' m_1$, we know that $r_2(r_1(a)) = br_1(m_1) + cr_2(m_2) + \Theta$. But by symmetry we have that $r_1(r_2(a)) = br_1(m_1) + cr_2(m_2) + \Theta = r_2(r_1(a))$, which contradicts the definition of case (1b).

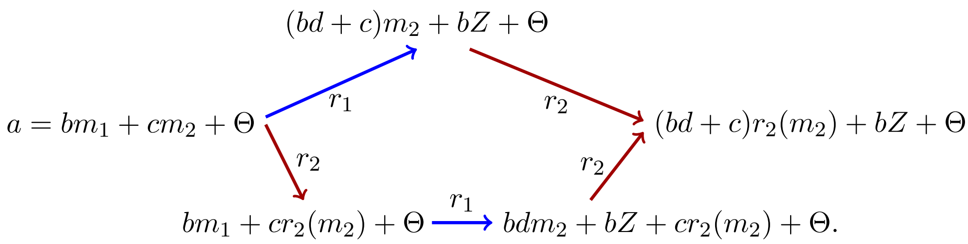
Then without loss of generality we may assume $m_2 \leq' m_1$. Let $r_1(m_1) = dm_2 + Z$, where Z is orthogonal to both m_1 and m_2, and $d \in \mathbb{K}^*$. Then the following holds:

$$\begin{aligned} r_2(r_1(a)) &= r_2(r_1(bm_1 + cm_2 + \Theta)) \\ &= r_2(bdm_2 + bZ + cm_2 + \Theta) \\ &= (bd + c)r_2(m_2) + bZ + \Theta. \end{aligned}$$

Alternatively, we also have:

$$\begin{aligned} r_2(r_1(r_2(a))) &= r_2(r_1(r_2(bm_1 + cm_2 + \Theta))) \\ &= r_2(r_1(bm_1 + cr_2(m_2) + \Theta)) \\ &= r_2(bdm_2 + bZ + cr_2(m_2) + \Theta) \text{ since } m_1 \not\leq' m_2 \\ &= (bd + c)r_2(m_2) + bZ + \Theta = r_2(r_1(a)). \end{aligned}$$

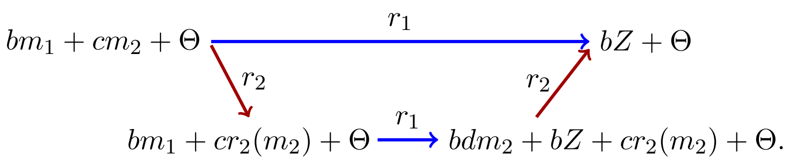
The diamond is the subgraph corresponding to this identity:



We note that there is an special case here, where $bd + c$ is actually 0. This means that

$$r_1(a) = r_2(r_1(r_2(a)))$$

and the graph is therefore slightly different:



$(2a)$ In this case we have a vertex of the form

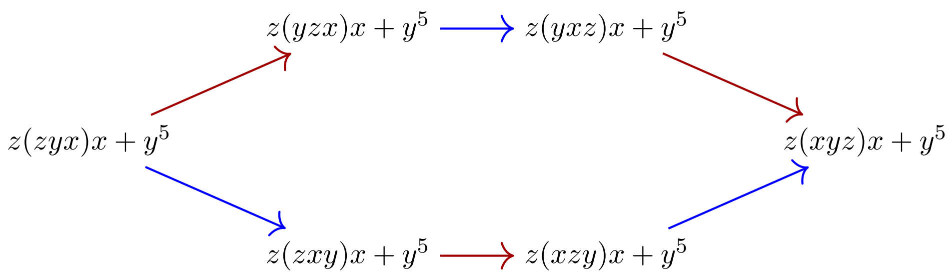
$$a = ba_1 \cdots a_i xyz a_{i+3} \cdots a_k + \Theta,$$

where $b \in \mathbb{K}^*$ and $\Theta \in T(V)$ does not interfere with $a - \Theta$. Further, r_1 and r_2 correspond to the first arrows in two simplification paths for xyz. Let these two paths be p_1 and p_2. Then the diamond is the subgraph made up by the paths:

$$(ba_1 \cdots a_i \mid p_1 \mid a_{i+3} \cdots a_k)^{\Theta} \text{ and } (ba_1 \cdots a_i \mid p_2 \mid a_{i+3} \cdots a_k)^{\Theta}.$$

These two paths converge at $ba_1 \cdots a_i [xyz] a_{i+3} \cdots a_k + \Theta$.

Example 4.6.1. Consider $A = \mathbb{K}[x, y, z]$ the commutative polynomial ring with the PBW basis $\{x^i y^j z^k\}$. An example of a diamond of case $(2a)$ is:



where in this case $a_1 \ldots a_i := z$, $a_{i+3} \ldots a_k := x$ and $\Theta := y^5$.

$(2b)$ Recall that $BW_\sigma C = DW_\tau E$. The vertex in this case is of the form

$$a = ba_1 \cdots a_i W_\sigma a_{i+2} \cdots a_j W_\tau a_{j+2} \cdots a_k + \Theta$$

for $b \in \mathbb{K}^*$, where $\Theta \in T(V)$ does not interfere with $a - \Theta$. Let $D_i \in \langle V \rangle$ and $d_i \in \mathbb{K}^*$ for $i \in I$ a finite index set be such that

$$a_1 \cdots a_i f_\sigma a_{i+2} \cdots a_j = \sum_{i \in I} d_i D_i.$$

Likewise let $C_j \in \langle V \rangle$ and $c_j \in \mathbb{K}^*$ for $j \in J$ a finite index set be such that:

$$a_{i+2} \cdots a_j f_\tau \cdots a_k = \sum_{j \in J} c_j C_j.$$

Note also that by definition, $E = a_{j+2} \cdots a_k$ and $B = a_1 \cdots a_i$. Then the diamond is the subgraph made up of the paths:

$$r_1 * \prod_{i \in I} (D_i \mathrel{\vdots} r_\tau \mathrel{\vdots} E) \text{ and } r_2 * \prod_{j \in J} (B \mathrel{\vdots} r_\sigma \mathrel{\vdots} C_j)$$

where we have used $*$ and $\prod$ to denote edge concatenation. Note that example 4.3.6 shows an example of this type of diamond.

These diamonds form (a subset of) the basic units about which authors such as Bergman [Ber78] and Newman [New42] have previously developed the combinatorial theory of reduction sequences. However, we do not have the extra case in which W_σ is a subword of W_τ, as all of the relations are homogeneous of the same degree. The condition that in case (2) the extra terms Θ do not interfere makes the reasoning considerably simpler in our setting, but the word diamond as used by Bergman in [Ber78] does not have this condition.

Definition 4.6.2. Two arrows e and f emanating from the same vertex are said to *start a diamond* if they are the first two arrows in a diamond.

Definition 4.6.3. Two paths p and q in the Bergman graph are said to *differ by a diamond* if all of their arrows are equal except for the sides of a diamond.

Two paths p_0 and p_m are said to *differ by diamonds* if there exists a sequence of paths $p_1, \ldots, p_{m-1}$ such that for each $1 \leq i \leq m$, p_i and p_{i-1} differ by a diamond.

Lemma 4.6.4. *If two simplification paths p and q differ by a diamond then*

$$\nu(p) - \nu(q) \in \operatorname{Ker}(k_2).$$

Proof. We carry out a case-by-case analysis for the four cases of diamond. In each case we use the notation introduced whilst defining the diamonds above.

$(1a)$ In this case $a = bm_1 + cm_2 + \Theta$. The two paths around the diamond correspond to $r_2 r_1$ and $r_1 r_2$. Since $m_1 \not\leq' m_2$ and $m_2 \not\leq' m_1$ we know that the weights associated with both edges corresponding to r_1 are the same, and likewise for those corresponding to r_2. That is:

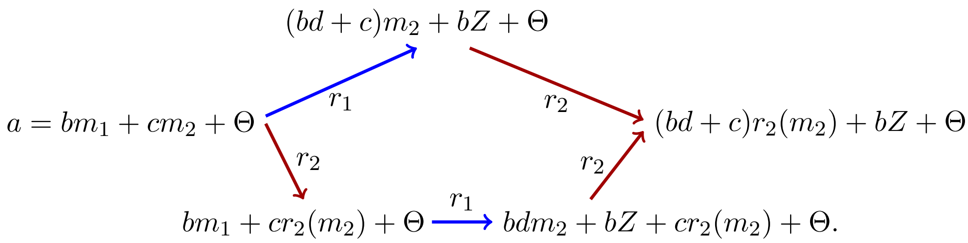
$$\nu(p) - \nu(q) = (bB \otimes \rho_\sigma \otimes C + cD \otimes \rho_\tau \otimes E) - (cD \otimes \rho_\tau \otimes E + bB \otimes \rho_\tau \otimes E) = 0.$$

$(1b)$ We have that $a = bm_1 + cm_2 + \Theta$. Recall that we also let $r_1(m_1) = dm_2 + Z$. We show that $\nu(p-q) = 0$ in either the general case or the case where $bd + c = 0$.

In the general case recall that:

$$r_2(r_1(r_2(a))) = (bd + c)r_2(m_2) + bZ + \Theta = r_2(r_1(a)),$$

corresponding to the graph:



Recall that $m_1 = BW_\sigma C$ and $m_2 = DW_\tau E$. Now, if we set p and q to be the bottom and top paths respectively we have that

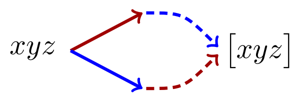
$$\begin{aligned}\nu(p) &= c(D \otimes \rho_\tau \otimes E) + b(B \otimes \rho_\sigma \otimes C) + bd(D \otimes \rho_\tau \otimes E)\\ &= b(B \otimes \rho_\sigma \otimes C) + (bd + c)(D \otimes \rho_\tau \otimes E)\\ &= \nu(q).\end{aligned}$$

In the case where $bd + c = 0$, then $r_2(r_1(r_2(a))) = r_1(a)$ and

$$\begin{aligned}\nu(p) &= c(D \otimes \rho_\tau \otimes E) + b(B \otimes \rho_\sigma \otimes C) + bd(D \otimes \rho_\tau \otimes E)\\ &= b(B \otimes \rho_\sigma \otimes C)\\ &= \nu(q).\end{aligned}$$

$(2a)$ Recall that we write $\big[Z\big]$ for the PBW reduced form of an element $Z \in T(V)$. We show that overlap ambiguities actually correspond to elements of K_3. Observe that the definition of the diamond in this case is a translation of a stitch of two simplification paths for xyz. By Lemma 4.5.6, the difference $\nu(p-q)$ is the same as the image under ν of the difference of these two simplification paths.

Thus it suffices to consider the following situation, where a red corresponds to the colour 0, blue corresponds with the colour 1 and a coloured dashed arrow corresponds to zero or more arrows with the first one being the colour shown.



Let p be the top path and q be the bottom path. By the definition of a path, if we take the sum $\sum_e (t(e) - h(e))$ over edges e in a path, we obtain an expression equal to the beginning of the path minus the end. Applying this to p and q we obtain two ways of writing the expression $xyz - [xyz]$. For p we obtain:

$$\sum c_i x_i r_i + \sum c'_j r'_j x'_j = xyz - [xyz], \tag{†}$$

where the $c_i, c'_i \in \mathbb{K}^*$ and the r_i, r'_i are relations. Since p is of length at least one, there is some c'_j that is nonzero. By symmetry we get another expression for the bottom path:

$$\sum d_i y_i s_i + \sum d'_j s'_j y'_j = xyz - [xyz], \tag{††}$$

where there is at least one d_i nonzero, all the $d_i, d'_j \in \mathbb{K}^*$ and s_i, s'_j are relations.

Taking the difference of the right hand sides of (†) and (††), we get $0 = xyz - [xyz] - (xyz - [xyz])$. However, by grouping together like terms in the difference of the left hand sides we get an expression of the form:

$$0 = \sum_j e_j z_j t_j - \sum_i e'_i t'_i z'_i \in T(V),$$

with both of the sums being nonzero, and the e_i's and e'_i's nonzero scalars. This implies that

$$\sum_j e_j |z_j| t_j | 1 = \sum_i e'_i |t'_i| z'_i | 1 \in K_3$$

and so

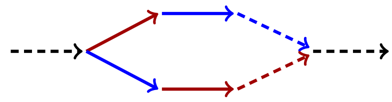
$$\nu(p-q) = \sum_i e'_i |t'_i| z'_i - \sum_j e_j z_j | t_j | 1 = k_3 (\sum e_j | z_j | t_j | 1)$$

by the definition of k_3. Therefore $k_2(\nu(p-q)) = 0$.

(2b) To start we consider the case of a monomial of the form

$$a_1 \cdots a_i W_\sigma a_{i+2} \cdots a_j W_\tau a_{j+2} \cdots a_k.$$

The graph in question will have the following appearance:



If p and q are the top and bottom paths respectively then $\nu(p-q)$ is as follows, recalling the definition of c_i, C_i, b_j and B_j:

$$\begin{aligned}\nu(p)-\nu(q) &= \pi(B)\otimes\rho_\sigma\otimes\pi(C)+\sum_i d_i\pi(D_i)\otimes\rho_\tau\otimes\pi(E)\\ &\qquad -\pi(D)\otimes\rho_\tau\otimes\pi(E)-\sum_j c_j\pi(B)\otimes\rho_\sigma\otimes\pi(C_j)\\ &= \pi(B)\otimes\rho_\sigma\otimes\pi(C)-\pi(B)\otimes\rho_\sigma\otimes\pi\left(\sum_j c_jC_j\right)\\ &\qquad -\pi(D)\otimes\rho_\tau\otimes\pi(E)+\pi\left(\sum_i d_iD_i\right)\otimes\rho_\tau\otimes\pi(E)\\ &= 0.\end{aligned}$$

The last line follows since $\left[C\right]=\left[\sum_j c_jC_j\right]$ and $\left[D\right]=\left[\sum_i d_iD_i\right]$ by definition.

To complete the proof observe that the general definition of the diamond is a translation of the above case, which by Lemma 4.5.6 has no effect on the image under ν. $\square$

Lemma 4.6.5. *Let p and q be two simplification paths for some element w that first branch from each other at a vertex a. Let e_1 and e_2 be respectively the arrows taken by p and q emanating from a. Then either e_1 and e_2 start a diamond, or there is a third arrow e_3 emanating from a such that e_1 and e_3 start a diamond and e_2 and e_3 start a diamond.*

Proof. If e_1 and e_2 do not start a diamond, then by the definition of diamonds they are in neither case (1) nor case (2). Therefore the vertex $a := t(e_1)$ at which the branching happens must be of the form $a = bm + \Theta$ where $b \in \mathbb{K}^*$ and $\Theta \in T(V)$ interferes with m, whilst e_1 and e_2 correspond two reductions r_1 and r_2 that act nontrivially on m.

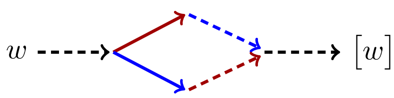
In this case, by definition there exists some monomial m' such that $(m' \bullet \Theta) \neq 0$, $m' \leq m$, and $m' \neq \left[m'\right]$. Therefore, there is a at least a third arrow emanating from a corresponding to a reduction of m', since this is not in PBW order. Choose any such arrow and call it e_3. Then e_1 and e_3 correspond to reductions on different monomials and so start a diamond of case (1). The same reasoning applies to e_2 and e_3.

$\square$

Lemma 4.6.6. *Any two distinct simplification paths for some element w differ by diamonds.*

Proof. Call the two paths p_1 and p_m. We reduce to the case that the first time p_1 and p_m differ from each other, the arrows that they take start a diamond. If this is not the case, call the arrows at which p_1 and p_m diverge e_1 and e_2 respectively. By Lemma 4.6.5 there is a third arrow e_3 emanating from a so that e_1 and e_3 start a diamond and e_2 and e_3 start a diamond. Choose any simplification path q for w which follows e_3. Then if we prove that p_1 and q differ by diamonds, and that p_m and q differ by diamonds, then we have proved that p_1 and p_m differ by diamonds.

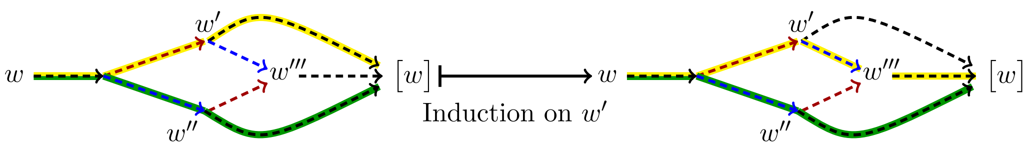
Therefore from here onwards we assume that the first branching of p_1 and p_m occurs along two arrows that start a diamond. The total number of diamonds in $\mathrm{BG}(w)$ is finite since the number of arrows emanating from each vertex is finite and the number of vertices is finite. We induct on the number of diamonds appearing in $\mathrm{BG}(w)$, say n. If $n = 1$ then we have a graph of the form:



where we use dashed lines to represent 0 or more arrows of any colour, and coloured dashed arrows for those within a diamond. Note that there can be no other branchings in the graph as this would imply the existence of another diamond by Lemma 4.6.5. Clearly p_1 and p_m differ by diamonds in this case.

Now for the inductive step. In the following we use $*$ to denote path concatenation. If we highlight the paths p_1 in yellow and p_m in green, the proof is very easy to visualise. The two paths split at the beginning of a diamond, and then possibly some where along this diamond the paths split off away from it, at w' and w'' say. Since the Bergman graphs of w and w' are nontrivial subgraphs of $\mathrm{BG}(w)$, we know that the number of diamonds in $\mathrm{BG}(w)$ and $\mathrm{BG}(w')$ must be smaller than that in $\mathrm{BG}(w)$. Therefore we may use the inductive hypothesis to fix this second splitting off. Anything in the following graphs highlighted in yellow is to be taken as a path that differ by diamonds from p_1, similarly for green highlighted paths and p_2.

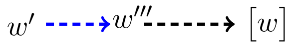
We firstly show the induction on w':



That is, if p is the path

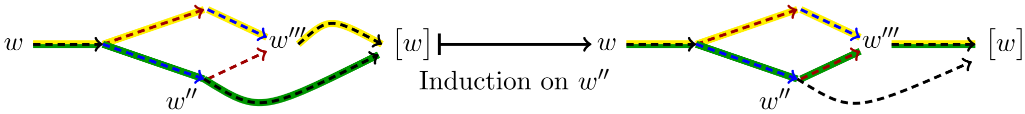
q is the path

and q' is the path



then by induction q and q' differ by diamonds, so that $p_1 = p * q$ and $p * q'$ differ by diamonds.

Likewise, we can also induct on w'':



After these manipulations, we are left with two paths that differ only by the initial diamond. This completes the proof.

□

4.7 The Main Theorem

We are now ready to define the map m_2. We have seen that ν defines a map from simplification paths to K_2, and that different simplification paths differ by elements of $\operatorname{Im} k_3$. It should be unsurprising then that m_2 will be somehow built up out of ν and a choice of simplification paths. Indeed, we shall see that if we choose a simplification path for every pair of monomials x and y then this suffices to define a map m_2 such that the relevant square with m_1 commutes. We introduce several pieces of notation before stating the theorem precisely.

Let $S \subseteq \langle V \rangle \times \langle V \rangle$ be the set of pairs (x, y) such that $x = [x]$ and $y = [y]$. Then S is $\mathbb{N}$-graded by $S_d = \{(x, y) \in S \mid \deg(x) + \deg(y) = d\}$ so that:

$$S = \coprod_{d=0}^{\infty} S_d.$$

Also let $F : S \to \mathcal{P}(\mathrm{Path}(G))$ be the map that takes a pair (x, y) to the set of simplification paths for xy. Then we define a set of bimodule maps:

$$\Gamma := \{\mu : B_2 \to \mathrm{Path}(G) \mid \mu(1 \otimes \pi(x) \otimes \pi(y) \otimes 1) \in F(x, y) \text{ for any } (x, y) \in S\}.$$

In other words, an element of Γ corresponds to a choice of simplification path for xy for each $(x, y) \in S$.

The aim of this section is to prove the following theorem.

Theorem 4.7.1. *For any $\mu \in \Gamma$, if $m_2 := \nu \circ \mu$ then the following diagram commutes:*

$$\begin{array}{ccc} B_2 & \xrightarrow{b_2} & B_1 \\ {\scriptstyle m_2}\downarrow & & \downarrow{\scriptstyle m_1} \\ K_2 & \xrightarrow{k_2} & K_1 \end{array}$$

We will approach this theorem in a slightly roundabout fashion. We will first prove that if the theorem holds for any element of Γ, then it holds for all elements of Γ. Then we will construct a (nonempty) subset of Γ for which the theorem holds. The argument will be by induction using the $\mathbb{N}$-grading of S. With that in mind we define the following set of statements parametrised by $d \in \mathbb{N}$.

$$(Q_d) : \text{If } (x, y) \in S_d \text{ and } q \in F(x, y) \text{ then } k_2 \circ \nu(q) = m_1 \otimes b_2(1 \otimes x \otimes y \otimes 1).$$

We note that Theorem 4.7.1 is equivalent to Q_d holding for all $d \in \mathbb{N}$.

We define a grading of S_d so that S is $\mathbb{N}^2$-graded.

$$S_{d,a} = \{(x, y) \in S_d \mid \mathrm{BG}(xy) \text{ has } a \text{ } \deg(x)\text{-coloured arrows.}\}.$$

So that:

$$S = \coprod_{d=0}^{\infty} \coprod_{a=0}^{\infty} S_{d,a}.$$

Then we define the following statement for any $d, a \in \mathbb{N}$:

$$(P_{d,a}) : \forall (x, y) \in S_{d,a}\ \exists p \in F(x, y) \text{ such that } k_2 \circ \nu(p) = m_1 \otimes b_2(1 \otimes x \otimes y \otimes 1).$$

Proposition 4.7.2. *For any $d \in \mathbb{N}$, $Q_d \iff$ for all $a \in \mathbb{N}$ $P_{d,a}$.*

Proof. $\Longrightarrow$ This direction is obvious as it is merely passing from the set of all simplification paths to an element of that set.

$\Longleftarrow$ If $(x, y) \in S$, we know that $(x, y) \in S_{d,a}$ for some a. Therefore we take the particular path p that exists by the assumption of $P_{d,a}$. Then for any path $q \in F(x, y)$ we know by Lemma 4.6.6 that p and q differ by diamonds, say by a sequence of paths $p_1 := p, p_2, \ldots, p_m := q$. Then we have:

$$k_2 \circ \nu(p - q) = \sum_{i=1}^{m-1} k_2(\nu(p_i - p_{i+1})) = 0.$$

Where the final equality holds by Lemma 4.6.4. □

Lemma 4.7.3.

1. *Q_0 holds.*
2. *$\forall d_0 \in \mathbb{N}\ P_{d_0,0}$*

Proof. Both of these statements hold because of the following fact: if $x, y \in \langle V \rangle$ are reduced monomials and $xy = \big[xy\big]$ then

$$\begin{aligned} m_1 \circ b_2(1|x|y|1) &= m_1(x|y|1 - 1|xy|1 + 1|x|y) \\ &= x\left(\sum_i y_1 \cdots y_{i-1}|y_i|y_{i+1} \cdots y_n\right) + \left(\sum_i x_1 \cdots x_{i-1}|x_i|x_{i+1} \cdots x_m\right) y \\ &\qquad - m_1(1 \otimes xy \otimes 1) \\ &= 0. \end{aligned}$$

Furthermore, if $(x, y) \in S_0$ or $(x, y) \in S_{d_0,0}$ then $xy = \big[xy\big]$ and so any simplification path p is the empty path. This implies that

$$k_2 \circ \nu(p) = 0 = m_1 \circ b_2(1 \otimes x \otimes y \otimes 1).$$

□

Proof of Theorem 4.7.1. It suffices to prove Q_d for all $d \in \mathbb{N}$, which we do by induction on d. The base case of Q_0 is the first half of Lemma 4.7.3. For the induction step we assume Q_d for all $d < d_0$ and prove $P_{d_0,a}$ for all $a \in \mathbb{N}$. This implies Q_{d_0} by Proposition 4.7.2.

The rest of the proof is therefore taken up with proving $P_{d_0,a}$ for all $a \in \mathbb{N}$. We do this by an induction, whose base case of $P_{d_0,0}$ is the second half of Lemma 4.7.3. We thus assume the following inductive hypotheses:

(a) Q_d holds for any $d < d_0$.

(b) $P_{d_0,a}$ holds for any $a < a_0$.

The aim is to prove that P_{d_0,a_0} holds.

With that in mind, take $(x,y) = (x_1 \cdots x_n, y_1 \cdots y_m) \in S_{d_0,a_0}$. We describe how to construct a simplification path q for xy for which the equality

$$k_2 \circ \nu(q) = m_1 \circ b_2(1 \otimes \pi(x) \otimes \pi(y) \otimes 1)$$

holds. Since ν is defined edgewise, we can calculate the image of this path under ν by taking the sum of the image of each phase, which we do alongside the construction. The construction follows three phases.

Phase 1. The first edge of q is of colour n by definition:

$$x_1 \cdots x_n y_1 \cdots y_m \to \sum_i c_i x_1 \cdots x_{n-1} f_1^i f_2^i y_2 \cdots y_m,$$

where $x_n y_1 - \sum_i c_i f_1^i f_2^i$ is a relation.

The image of this under ν is

$$\pi(x_1 \cdots x_{n-1}) \otimes (x_n y_1 - \sum_i c_i f_1^i f_2^i) \otimes \pi(y_2 \cdots y_m). \qquad (\Delta)$$

Phase 2. In this phase the path will avoid following any arrows of colour n. We do this by first reducing all of the words of the form $x_1 \cdots x_{n-1} f_1^i$, and then those of the form $f_2^i y_2 \cdots y_m$. With that in mind set $X := \sum_i c_i x_1 \cdots x_{n-1} f_1^i f_2^i y_2 \cdots y_m$.

Choose an i such that

$$X - c_i x_1 \cdots f_1^i f_2^i y_2 \cdots y_m$$

doesn't interfere with $c_i x_1 \cdots x_{n-1} f_1^i f_2^i y_2 \cdots y_m$. This is possible since it is equivalent to choosing a monomial that is minimal with respect to $\leq$ amongst the set of those appearing in X with nonzero coefficient.

Choose a simplification path p for $c_i x_1 \cdots f_1^i$. Then extend q by

$$(p \,\vdots\, f_2^i y_2 \cdots y_m)^{X - c_i x_1 \cdots f_1^i f_2^i y_2 \cdots y_m}.$$

This path ends at

$$Z := X - c_i x_1 \cdots x_{n-1} f_1^i f_2^i y_2 \cdots y_m + c_i \big[x_1 \cdots x_{n-1} f_1^i\big] f_2^i y_2 \cdots y_m.$$

Iterate the above procedure on Z until there are no arrows of colour less than n by which to extend q.

Now we repeat this phase but for arrows of colour greater than n. In this way we arrive at

$$Y := \sum c_i \big[x_1 \cdots x_{n-1} f_1^i\big] \big[f_2^i y_2 \cdots y_m\big].$$

The image of the path in this phase is calculated by taking p_i a simplification path for $x_1 \cdots f_1^i$ and q_i is a simplification path for $f_2^i \cdots y_m$. Then we have:

$$\sum_i c_i \, \nu(p_i)\pi\big(f_2^i y_2 \cdots y_m\big) + c_i \pi\big(\big[x_1 \cdots f_1^i\big]\big) \, \nu(q_i). \tag{Ξ}$$

We have used Lemma 4.5.6 in order to evaluate the image of the stitched and translated paths.

Phase 3. Set M_{ij} and N_{ij} to be monomials in PBW order and $d_{ij} \in \mathbb{K}^*$ such that:

$$Y = \sum c_i(\big[x_1 \cdots x_{n-1} f_1^i\big])(\big[f_2^i y_2 \cdots y_m\big]) = \sum_i c_i \sum_j d_{ij} M_{ij} N_{ij}.$$

We choose a monomial m such that $(Y \bullet m) \neq 0$ and $Y - m$ doesn't interfere with m. We know that m is of the form czw for a constant $c \in \mathbb{K}^*$ and a pair $(z, w) \in S_d$ with $\deg(z) = n$. Since $m \leq xy$, by Lemma 4.5.3 we know that $\mathrm{BG}(m)$ appears as a proper subgraph of $\mathrm{BG}(xy)$ rooted at Y. In particular, since the first arrow of q was of colour n, we know that $\mathrm{BG}(m)$ must have has strictly fewer n-coloured arrows than $\mathrm{BG}(xy)$. Therefore, $(z, w) \in S_{d,a}$ with $a < a_0$ so that we can apply the inductive hypothesis (b) and obtain a simplification path for m, say p_m, for which

$$k_2 \circ \nu(p_m) = m_1 \circ b_2(c \otimes z \otimes w \otimes 1).$$

Then extend q by p_m^{Y-m}. Iterate this over all of the terms in Y until q terminates at $\big[xy\big]$.

The image under ν of this phase is

$$\sum_m \nu(p_m), \tag{Ψ}$$

which by the induction hypothesis satisfies

$$\sum_m k_2 \circ \nu(p_m) = \sum_i c_i(\sum_j m_1 \circ b_2(d_{ij} \otimes \pi(M_{ij}) \otimes \pi(N_{ij}) \otimes 1)).$$

From here on we use labels in our equations to label the expression on the line that the label appears, rather than the equation as a whole. Applying k_2 to $\nu(q)$, we have the following equation:

$$\begin{aligned}
k_2 \circ \nu(q) &= k_2((\Delta) + (\Xi) + (\Psi)) \\
&= k_2(\pi(x_1 \cdots x_{n-1}) \otimes (x_n y_1 - \sum_i c_i f_1^i f_2^i) \otimes \pi(y_2 \cdots y_m)) \qquad \text{(Phase 1 } \dagger) \\
&\quad + \sum_i c_i k_2 \circ \nu(p_i) \pi\big(f_2^i y_2 \cdots y_m\big) + c_i \pi\big(\big[x_1 \cdots f_1^i\big]\big) k_2 \circ \nu(q_i) \quad \text{(Phase 2 } \dagger\dagger) \\
&\quad + \sum_i c_i (\sum_j m_1 \circ b_2(d_{ij} \otimes \pi(M_{ij}) \otimes \pi(N_{ij}) \otimes 1)). \qquad \text{(Phase 3 } \dagger\dagger\dagger)
\end{aligned}$$

Each path p_i of q_i that appears in the expression (††) is the simplification path for an element of S of the form $(x_1 \cdots x_{n-1}, f_1^i)$ or $(f_2^i, y_2 \cdots y_m)$. These have total degree strictly less than $n+m$ and so by induction hypothesis (a) we have the equality:

$$(\dagger\dagger) = \sum_i c_i m_1 \circ b_2(1 \otimes \pi(x_1 \cdots x_{n-1}) \otimes \pi\big(f_1^i\big) \otimes 1)\pi\big(f_2^i y_2 \cdots y_m\big) \tag{4.1}$$

$$+ \sum_i c_i \pi\big(x_1 \cdots f_1^i\big) m_1 \circ b_2(1 \otimes \pi\big(f_2^i\big) \otimes \pi(y_2 \cdots y_m) \otimes 1). \tag{4.2}$$

We can now use the definition of b_2 to obtain:

$$\begin{aligned}
(4.1) &= \sum_i c_i m_1(\pi(x_1 \cdots x_{n-1}) \otimes \pi\big(f_1^i\big) \otimes 1)\pi\big(f_2^i y_2 \cdots y_m\big) \\
&\quad + \sum_i c_i m_1(1 \otimes \pi(x_1 \cdots x_{n-1}) \otimes \pi\big(f_1^i\big))\pi\big(f_2^i y_2 \cdots y_m\big) \\
&\quad - \sum_i c_i m_1(1 \otimes \pi\big(x_1 \cdots x_{n-1} f_1^i\big) \otimes 1)\pi\big(f_2^i y_2 \cdots y_m\big).
\end{aligned} \tag{4.3}$$

By definition of M_{ij}, N_{ij}:

$$(4.3) = -\sum_i c_i \sum_j d_{ij} m_1(1 \otimes \pi(M_{ij}) \otimes \pi(N_{ij})).$$

Whilst on the other hand:

$$\begin{aligned}
(4.2) &= \sum_i c_i \pi\big(x_1 \cdots f_1^i\big) m_1(\pi\big(f_2^i\big) \otimes \pi(y_2 \cdots y_m) \otimes 1) \\
&\quad + \sum_i c_i \pi\big(x_1 \cdots f_1^i\big) m_1(1 \otimes \pi\big(f_2^i\big) \otimes \pi(y_2 \cdots y_m)) \\
&\quad - \sum_i c_i \pi\big(x_1 \cdots f_1^i\big) m_1(1 \otimes \pi\big(f_2^i y_2 \cdots y_m\big) \otimes 1).
\end{aligned} \tag{4.4}$$

Now by definition:

$$(4.4) = -\sum_i c_i \sum_j d_{ij} m_1(\pi(M_{ij}) \otimes \pi(N_{ij}) \otimes 1).$$

We now consider (†††). By using the definition of b_2 we obtain:

$$\begin{aligned}
(\dagger\dagger\dagger) &= \sum_i c_i \sum_j m_1(d_{ij} \otimes \pi(M_{ij}) \otimes \pi(N_{ij})) && (4.5)\\
&+ \sum_i c_i \sum_j m_1(d_{ij}\pi(M_{ij}) \otimes \pi(N_{ij}) \otimes 1) && (4.6)\\
&- \sum_i c_i \sum_j m_1(d_{ij} \otimes \pi(M_{ij}N_{ij}) \otimes 1).
\end{aligned}$$

Note that:

$$(4.5) + (4.3) = 0 \text{ and } (4.6) + (4.4) = 0.$$

After these calculations we can rewrite $k_2 \circ \nu(q)$ as:

$$\begin{aligned}
k_2 \circ \nu(q) = k_2(\pi(x_1 \cdots x_{n-1}) &\otimes (x_n y_1 - \sum_i c_i f_1^i f_2^i) \otimes \pi(y_2 \cdots y_m))\\
&\left.\begin{aligned}
&+ \sum_i c_i m_1(\pi(x_1 \cdots x_{n-1}) \otimes \pi(f_1^i) \otimes 1)\pi(f_2^i y_2 \cdots y_m)\\
&+ \sum_i c_i m_1(1 \otimes \pi(x_1 \cdots x_{n-1}) \otimes \pi(f_1^i))\pi(f_2^i y_2 \cdots y_m)
\end{aligned}\right\} \quad \text{(From (4.1))}\\
&\left.\begin{aligned}
&+ \sum_i c_i \pi(x_1 \cdots f_1^i) m_1(\pi(f_2^i) \otimes \pi(y_2 \cdots y_m) \otimes 1)\\
&+ \sum_i c_i \pi(x_1 \cdots f_1^i) m_1(1 \otimes \pi(f_2^i) \otimes \pi(y_2 \cdots y_m))
\end{aligned}\right\} \quad \text{(From (4.2))}\\
&- \sum_i c_i \sum_j m_1(d_{ij} \otimes \pi(M_{ij}N_{ij}) \otimes 1).
\end{aligned}$$

Call the right hand side of the above expression $(*)$.

We now calculate:

$$\begin{aligned}
m_1 \circ b_2(1 \otimes \pi(x) \otimes \pi(y) \otimes 1) &= m_1(\pi(x) \otimes \pi(y) \otimes 1) && (4.7)\\
&- m_1(1 \otimes \pi(xy) \otimes 1) && (4.8)\\
&+ m_1(1 \otimes \pi(x) \otimes \pi(y)). && (4.9)
\end{aligned}$$

Recall that by definition

$$\pi(xy) = \pi\big([xy]\big) = \pi\left(\sum_i c_i\left(\sum_j d_{ij}M_{ij}N_{ij}\right)\right),$$

so that:

$$(4.8) = -\sum_i c_i \sum_j m_1(d_{ij}\otimes\pi(M_{ij}N_{ij})\otimes 1).$$

Furthermore, by the definition of m_1 we have:

$$\begin{aligned}(4.7) = &\sum_i \big(c_i\pi\big(x_1\cdots f_1^i\big)m_1(\pi\big(f_2^i\big)\otimes\pi(y_2\cdots y_m)\otimes 1)\big)\\ &+\pi(x_1\cdots x_n)\otimes y_1\otimes\pi(y_2\cdots y_m)\end{aligned}$$

and

$$\begin{aligned}(4.9) = &\sum_i \big(c_i m_1(1\otimes\pi(x_1\cdots x_{n-1})\otimes\pi\big(f_1^i\big))\pi\big(f_2^i y_2\cdots y_m\big)\big)\\ &+\pi(x_1\cdots x_{n-1})\otimes x_n\otimes\pi(y_1\cdots y_m).\end{aligned}$$

We can therefore rewrite $(*)$ as:

$$\begin{aligned}k_2\circ\nu(q) = k_2&\left(\pi(x_1\cdots x_{n-1})\otimes(x_n y_1-\sum_i c_i f_1^i f_2^i)\otimes\pi(y_2\cdots y_m)\right)\\ &+\sum_i c_i m_1\left(\pi(x_1\cdots x_{n-1})\otimes\pi\big(f_1^i\big)\otimes 1\right)\pi\big(f_2^i y_2\cdots y_m\big)\\ &+(4.7)-\pi(x_1\cdots x_{n-1})\otimes x_n\otimes\pi(y_1\cdots y_m)\\ &+(4.9)-\pi(x_1\cdots x_n)\otimes y_1\otimes\pi(y_2\cdots y_m).\\ &+\sum_i c_i\pi\big(x_1\cdots f_1^i\big)m_1\left(1\otimes\pi\big(f_2^i\big)\otimes\pi(y_2\cdots y_m)\right)\\ &+(4.8).\end{aligned}$$

Now finally we use the definition of k_2 and so expand

$$(\dagger) = k_2\left(\pi(x_1\cdots x_{n-1})\otimes(x_ny_1-\sum_i c_if_1^if_2^i)\otimes\pi(y_2\cdots y_m)\right)$$

$$= \pi(x_1\cdots x_{n-1})\Big(\pi(x_n)\otimes y_1\otimes 1-\sum_i c_i\pi\big(f_1^i\big)m_1(1\otimes\pi\big(f_2^i\big)\otimes 1)$$
$$+1\otimes x_n\otimes\pi(()y_1)-\sum_i c_im_1(1\otimes\pi\big(f_1^i\big)\otimes\pi(()f_2^i))\Big)\pi(y_2\cdots y_m).$$

That is to say that $(*)$ may be rewritten:

$$\begin{aligned}
k_2\circ\nu(q) &= (4.7)+(4.8)+(4.9)\\
&\quad + k_2\left(\pi(x_1\cdots x_{n-1})\otimes(x_ny_1-\sum_i c_if_1^if_2^i)\otimes\pi(y_2\cdots y_m)\right)\\
&\quad - k_2\left(\pi(x_1\cdots x_{n-1})\otimes(x_ny_1-\sum_i c_if_1^if_2^i)\otimes\pi(y_2\cdots y_m)\right)\\
&= m_1\circ b_2(1\otimes\pi(x)\otimes\pi(y)\otimes 1)
\end{aligned}$$

This proves the induction step. $\square$

Chapter 5

Primary Obstructions to Infinitesimal Deformations

5.1 Introduction

In Chapter 3 we calculated the degree two component of the second Hochschild cohomology of two algebras. As discussed in that chapter, this is equivalent to calculating the set of isomorphism classes of infinitesimal deformations of these two algebras. However, it is generally a much harder question to establish whether any of these infinitesimal deformations integrate to formal deformations.

One possible first step in answering this question of integration is to calculate the subset of infinitesimal deformations which have vanishing primary obstruction. By work of Gerstenhaber (see Proposition 2.3.17), this is equivalent to calculating the subset of $\mathrm{HH}^2_2(C)$ of elements f such that $[f, f]$ is zero in cohomology. Recall from Definition 2.3.15 that for elements $f, g \in B^2$ the Gerstenhaber bracket is defined on pure tensors by

$$[f, g]\,(1|c_1|c_2|c_3|1) = f(1|g(1|c_1|c_2|1)|c_3|1) - f(1|c_1|g(1|c_2|c_3|1)|1) + g(1|f(1|c_1|c_2|1)|c_3|1) - g(1|c_1|f(1|c_2|c_3|1)|1)$$

In general, determing this set of infinitesimal deformations is difficult. Our method relies upon the map m_2 from Chapter 4 to reduce the problem to one that is amenable to a computer script. We first give an overview of the theory behind the calculation, before explaining the details of applying this theory to the two algebras discussed previously, A and A_q. Note that the implementation details of m_2 and the Gerstenhaber bracket can be found in Appendix B.

5.2 Calculations of Obstruction-Free Infinitesimal Deformations

Let C be a PBW algebra. Recall from Proposition 2.3.16 that the Gerstenhaber bracket defines a graded Lie algebra structure on B^n which descends to a commutative Lie bracket:

$$[-,-] : \mathrm{HH}_2^2 \otimes \mathrm{HH}_2^2 \to \mathrm{HH}_3^3 .$$

Furthermore, by Proposition 2.3.17 an infinitesimal deformation $f \in \mathrm{HH}_2^2$ has vanishing primary obstruction precisely when $[f, f]$ is a coboundary. The following lemma moves the question of obstructions from the bar complex to the Koszul complex.

Lemma 5.2.1. *Let $\phi_* : B_* \to K_*$ be a section of the inclusion $i_* : K_* \to B_*$. Let $f \in K^2$ be a cocycle such that its cohomology class in HH_2^2 is F. Then F has vanishing primary obstruction if and only if $i^3 \left[\phi^2(f), \phi^2(f)\right]$ is trivial in cohomology.*

Proof. This is simply unwinding definitions. In particular, since i_* and ϕ_* are quasi-isomorphisms they induce isomorphisms on the cohomology spaces and we may apply Proposition 2.3.17. □

The general question of finding a map ϕ_* that is a section of i_* is open and difficult. Common approaches to this problem are to work around it by using only the existence of the section (e.g. in [BG96]) or to find other bracket structures which induce the same Lie algebra structure on HH^* (see [NW14]).

Since the algebras A and A_q are not just Koszul but PBW, the problem of finding a map ϕ_* is more tractable. Our solution to this problem is the content of Chapter 4, which (partially) provides a family of choices for such a map ϕ_* in m_*.

From this point forwards we fix a choice of the map m_2. By Theorem 4.7.1 this can be done by choosing a simplification path of xy for every pair of monomials x, y.

Definition 5.2.2. Let K^n be $K^n(C)$. We define

$$[-,-] : K^2 \otimes K^2 \to K^3 \ \text{ by } \ [f, g] = i^3 \left[m^2(f), m^2(g)\right] .$$

With these preliminaries we turn our attention to the main question of the chapter. Let $\Delta = \{b_1, \ldots, b_n\} \subseteq K^2$ be an ordered set of cocycles such that the cohomology classes of the b_i form a basis of $\mathrm{HH}_2^2(C)$. We ask which elements of $\mathrm{sp}_{\mathbb{K}}(\Delta)$ have vanishing primary obstruction. Although this is a non-linear question, in the sense that the set of solutions need not be a vector space, we use linear algebra so far as possible in order to make the calculations tractable by computer. In the following Z^3 and B^3 are the spaces of Koszul 3-cocycles and 3-coboundaries respectively.

We start with a high level overview of the calculation. Our aim is to describe the set

$$\Lambda := \{ f \in \mathrm{sp}_{\mathbb{K}} \Delta \mid f \text{ has vanishing primary obstruction } \}.$$

Let $\pi : Z^3 \to \mathrm{HH}^3$ be the canonical projection map. It is a fundamental fact of Gerstenhaber's deformation theory (see Proposition 2.3.13) that $[f, f] \in Z^3$. Furthermore, by Proposition 2.3.17 we can write Λ in terms of the Gerstenhaber bracket:

$$\Lambda = \{ f \in \mathrm{sp}_{\mathbb{K}} \Delta \mid \pi([f, f]) = 0 \}.$$

We describe in detail how to find this set.

To start we factor the map $f \mapsto [f, f]$ into linear and non-linear factors. Let $\mathbb{K}^{\binom{n+1}{2}}$ have a chosen basis of

$$\{ v_{i,j} \mid 1 \leq i \leq j \leq n \}.$$

Furthermore, let $w_{i,j} = [b_i, b_j]$. We define the linear map $\lambda : \mathbb{K}^{\binom{n+1}{2}} \to Z^3$ by

$$\lambda(v_{i,j}) = w_{i,j}.$$

On the other hand, we define the polynomial map

$$p : \mathrm{sp}_{\mathbb{K}} \Delta \to \mathbb{K}^{\binom{n+1}{2}} \quad \sum_i a_i b_i \mapsto \sum_i a_i^2 v_{i,i} + \sum_{i<j} 2 a_i a_j v_{i,j}.$$

Lemma 5.2.3. *For $f \in \mathrm{sp}_{\mathbb{K}} \Delta$ we have*

$$[f, f] = \lambda \circ p(f).$$

Proof. This follows from the commutativity and bilinearity properties of the Gerstenhaber bracket. In particular, if $f = \sum_i a_i b_i$ then

$$\begin{aligned} [f, f] &= \left[\sum_i a_i b_i, \sum_j a_j b_j \right] = \sum_{i \neq j} a_i a_j \, [b_i, b_j] + \sum_i a_i^2 \, [b_i, b_i] \\ &= \sum_{j>i} 2 a_i a_j \, [b_i, b_j] + \sum_i a_i^2 \, [b_i, b_i] \\ &= \lambda \circ p(f). \end{aligned}$$

□

Therefore, Λ has another description as

$$\Lambda = \left\{ \sum_i a_i b_i \in \mathrm{sp}_{\mathbb{K}}(\Delta) \;\middle|\; \pi \circ \lambda \circ p(\sum_i a_i b_i) = 0 \right\}$$
$$= \left\{ \sum_i a_i b_i \in \mathrm{sp}_{\mathbb{K}}(\Delta) \;\middle|\; p(\sum_i a_i b_i) \in \ker(\pi \circ \lambda) \right\}.$$

Since $\ker(\pi \circ \lambda)$ is the kernel of a linear map, we can calculate a basis of for it using simple linear algebra. Let $m := \dim(B^3)$ and $Y := \{y_1, \ldots, y_m\}$ be a basis for B^3. Then define M to be the matrix

$$M := \left(w_{1,1} \mid \ldots \mid w_{n,n} \mid y_1 \mid \ldots \mid y_m\right).$$

Note that $\ker(M) \subseteq \mathbb{K}^{\binom{n+1}{2}+m}$. Let $\mu : \mathbb{K}^{\binom{n+1}{2}+m} \to \mathbb{K}^{\binom{n+1}{2}}$ be the canonical projection map onto the first $\binom{n+1}{2}$ components.

Lemma 5.2.4. *With π, λ, μ and M defined as above, we have the following equality of vector spaces*

$$\ker(\pi \circ \lambda) = \mu(\ker(M))$$

Proof. On the one hand, if $\sum_{i,j} c_{i,j} v_{i,j} \in \mu(\ker(M))$ with $c_{i,j} \in \mathbb{K}$, then there exist $d_1, \ldots, d_m \in \mathbb{K}$ such that

$$\sum_{i,j} c_{i,j} w_{i,j} + \sum_k d_k y_k = 0 \in B^3.$$

From this it follows that

$$\pi \circ \lambda(\sum_{i,j} c_{i,j} v_{i,j}) = \pi(\sum_{i,j} c_{i,j} w_{i,j}) = -\pi(\sum_k d_k y_k) = 0,$$

and so $\sum_{i,j} c_{i,j} v_{i,j} \in \ker(\pi \circ \lambda)$.

On the other hand, if $\sum_{i,j} e_{i,j} v_{i,j} \in \ker(\pi \circ \lambda)$ with $e_{i,j} \in \mathbb{K}$, then

$$\lambda(\sum_{i,j} e_{i,j} v_{i,j}) = \sum_{i,j} e_{i,j} w_{i,j} \in \ker(\pi) = B^3.$$

Therefore there exists $f_1, \ldots, f_m \in \mathbb{K}$ such that

$$\sum_{i,j} e_{i,j} w_{i,j} + \sum_k f_k y_k = 0.$$

In other words $\sum_{i,j} e_{i,j} v_{i,j} \in \mu(\ker(M))$. □

Therefore we may finally describe Λ as

$$\Lambda = \left\{ \sum_i a_i b_i \in \mathrm{sp}_{\mathbb{K}}(\Delta) \;\middle|\; p(\sum_i a_i b_i) \in \mu(\ker(M)) \right\}.$$

With this in mind, we calculate the space of $f \in \mathrm{sp}_{\mathbb{K}}(\Delta)$ with vanishing primary obstruction as follows.

1. For each pair $\left\{(i,j) \in \mathbb{N}^2 \;\middle|\; 1 \le i \le j \le n\right\}$ calculate $w_{i,j} = [b_i, b_j]$.

2. Choose a basis $\{y_1, \ldots, y_m\}$ of B^3 and form the matrix

$$M = \left(w_{1,1} \;\middle|\; \ldots \;\middle|\; w_{n,n} \;\middle|\; y_1 \;\middle|\; \ldots \;\middle|\; y_m \right).$$

3. Calculate the right kernel of M and choose a basis of $\mu(\ker(M)) \subseteq \mathbb{K}^{\binom{n+1}{2}}$ expressed in terms of the basis $\{v_{i,j}\}$ of $\mathbb{K}^{\binom{n+1}{2}}$.

4. Use the preceding step to deduce which f in $\mathrm{sp}_{\mathbb{K}}(\Delta)$ have $p(f)$ in $\mu(\mathrm{Ker}(M))$.

In the above calculation, steps 1-3 are entirely computer based. Most of the computing time is concentrated in step 1 as this involves the long calculation of paths in Bergman graphs in order to find the Gerstenhaber bracket using m^2. The rest of the computation consists mostly of Gaussian elimination which 'Sage' implements to be relatively fast. The output of the computer calculations is a basis of $\mu(\ker(M))$. After this basis is determined we reason by hand to compute the set Λ.

For example, we will see that in both of the cases of interest $C = A$ or A_q that $v_{1,1}$ appears in the output basis of $\mu(\ker(M))$. From the definition of p it follows that $p(a_1 b_1) \in \mu(\ker(M))$, and therefore $a_1 b_1 \in \Lambda$ and has vanishing primary obstruction. Note that in general determining elements of Λ is more complicated than this example.

In Section 5.3 we go through an example by hand in the context of the algebra A to make the above more concrete before explaining the computer code and discussing the output.

5.3 The Obstruction-Free Infinitesimal Deformations of A

We apply the calculations in Section 5.2 in the context of the algebra A. Recall that from Section 3.3 we know that we can choose $\Delta = \{b_1, \ldots, b_8\}$ to be the ordered set:

$$\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_3^2 \\ 0 \\ x_1x_4 \\ 0 \\ 0 \\ x_3x_4 \end{pmatrix} \begin{pmatrix} x_1^2 \\ 0 \\ 0 \\ x_2x_1 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \\ -x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x_2^2 \\ 0 \\ x_2x_4 \end{pmatrix}$$

Example 5.3.1. To understand the following calculations an example worked through by hand may be helpful. Recall that A has six relations

$$R = \left\{ \begin{matrix} r_1 := x_3x_1 - x_1x_3, & r_2 := x_4x_2 - x_2x_4, & r_3 := x_4x_1 - x_2x_3 \\ r_4 := x_1x_2 - x_2x_3, & r_5 := x_3x_2 - x_1x_4, & r_6 := x_4x_3 - x_1x_4 \end{matrix} \right\},$$

and four doubly defined relations

$$D := \left\{ \begin{matrix} d_1 := x_3r_4 + x_1(r_6 - r_5) = r_1x_2 - r_5x_3, & d_2 := x_4r_1 - x_1r_3 = r_6x_1 + (r_4 - r_3)x_3 \\ d_3 := x_4r_5 - x_1r_2 = r_6x_2 + (r_4 - r_3)x_4 & d_4 := x_4r_4 + x_2(r_6 - r_5) = r_3x_2 - r_2x_3 \end{matrix} \right\}.$$

We calculate $[f, f]$ for

$$f := \begin{pmatrix} x_3^2 \\ 0 \\ x_1x_4 \\ 0 \\ 0 \\ x_3x_4 \end{pmatrix} = b_2 \in \Delta.$$

Recall that under the vector notation, f is a function that maps r_2, r_4 and r_5 to zero and r_1, r_3 and r_6 to x_3^2, x_1x_4 and x_3x_4 respectively. Now, $[f, f]$ is a function defined on K_3 and so we write it as

$$[f, f] = i^3 \left[m^2(f), m^2(f)\right] = \begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{pmatrix}$$

using the vector notation introduce in Notation 3.2.1, so that $[f, f]$ sends d_i to Θ_i. We now calculate Θ_1. We remind the reader that we use the | symbol to represent $\otimes_{\mathbb{K}}$.

$$\begin{aligned}\Theta_1 = &\left[m^2(f), m^2(f)\right](d_1) = 2(m^2(f)\circ m^2(f))(d_1)\\ &= 2m^2(f)(1|m^2(f)(1|r_1|1)|x_2|1 - 1|m^2(f)(1|r_5|1)|x_3|1)\\ &\quad - 2m^2(f)(1|x_3|m^2(f)(1|r_4|1)|1 + 1|x_1|m^2(f)(1|r_6|1)|1 - 1|x_1|m^2(f)(1|r_5|1)|1).\end{aligned}$$

Recall that m_2 is a section of the inclusion map from $K_2 = A\otimes R\otimes A$ into B_2, and so acts trivially on relations. That is to say that for any relation r_i we have

$$m^2(f)(1|r_i|1) = f(m_2(1|r_i|1)) = f(1|r_i|1).$$

Therefore, the above continues as:

$$\begin{aligned}\Theta_1 &= 2\left(m^2(f)(1|x_3^2|x_2|1-0) - m^2(f)(0+1|x_1|x_3x_4|1-0)\right)\\ &= 2f\left(m_2(1|x_3^2|x_2|1) - m_2(1|x_1|x_3x_4|1)\right).\end{aligned}$$

The map m_2 is defined on a pure tensor $1|x|y|1$ by choosing a simplification path of xy in the graph $\mathrm{BG}(xy)$ and evaluating the function ν on this path (see Section 4.7 for details). In particular, since $x_1x_3x_4$ is in PBW order we can apply the reasoning of Lemma 4.7.3 and deduce that

$$m_2(1|x_1|x_3x_4|1) = 0.$$

As for $x_3^2x_2$, the Bergman graph is as follows:

$$x_3^2x_2 \xrightarrow{r_{x_3r_5}} x_3x_1x_4 \xrightarrow{r_{r_1x_4}} x_1x_3x_4.$$

Since there is a unique simplification path in this Bergman graph,

$$m_2(1|x_3^2|x_2|1) = x_3|r_5|1 + 1|r_1|x_4,$$

and so

$$\Theta_1 = 2f(x_3|r_5|1 + 1|r_1|x_4) = 2x_3^2x_4.$$

The rest of the calculations of the Θ_i's follows similarly and we obtain that

$$[f,f] = 2\begin{pmatrix} x_3^2x_4 \\ -x_1x_3x_4 \\ 0 \\ 0 \end{pmatrix}.$$

By comparison with the basis of $\mathrm{Im}(k^3)_2$ in Appendix A.1.3 we rewrite this as

$$[f,f] = 2\begin{pmatrix} x_3^2x_4 \\ -x_1x_3x_4 \\ 0 \\ 0 \end{pmatrix} = 2\begin{pmatrix} x_3^2x_4 \\ -x_1x_3x_4 \\ -x_3x_4^2 \\ x_1x_4^2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ x_3x_4^2 \\ -x_1x_4^2 \end{pmatrix}$$

where both of the vectors appearing on the right hand side are coboundaries. This allows us to conclude that $[f,f]$ is a coboundary, and therefore that f is an infinitesimal deformation with vanishing primary obstruction.

We have included all 36 of the Gerstenhaber brackets $[b_i, b_j]$ in Appendix A.1.4. By comparing the above with the ninth vector in that appendix, we confirm that our calculations agree with those of the computer.

5.3.1 Computer Script

We now step through the script and explain the computer calculations for Steps 1-3 in the procedure to calculate Λ. The script begins by building the list `W` of vectors $w_{i,j}$ by taking the Gerstenhaber bracket of pairs of elements in `Delta`.

```
W = []
for index1, vec1 in enumerate(Delta):
    for index2, vec2 in enumerate(Delta):
        if index2< index1:
            continue
        else:
            func = GerstenhaberBracket(vec1, vec2, KnBases[3])
            W.append(func)
W = [polygnomeVectorToSage(vec, 3, 3) for vec in W]
```

The defining code for the Gerstenhaber bracket can be found in Appendix B.4 and the Gerstenhaber brackets calculated in the preceding script are recorded in Appendix A.1.4.

Next, we construct Y to be a basis of the image of k^3. Since from Theorem 3.3.1 we know that $\dim(\mathrm{Im}(k^2)_2) = 22$ and $\dim((K^2)_2) = 60$, Y is a list of 38 vectors and can be found in Appendix A.1.3. This calculation is very similar to those in Section 3.3, so we don't explain it in detail.

```
matrixOfk_3Dual = [k_3Dual(vec, KnBases[3])
                                 for vec in K2DualBasis]
matrixOfk_3Dual = [polygnomeVectorToSage(vec, 3, 3)
                              for vec in matrixOfk_3Dual]
matrixOfk_3Dual = sage.matrix(K, matrixOfk_3Dual)
Y = matrixOfk_3Dual.row_space().basis()
```

We now combine `W` and `Y` into a matrix and use 'Sage' to calculate a basis of the kernel of that matrix. Note that in this context '+' means concatenation of lists. Also, we are using a list slice here to truncate a vector, which corresponds to taking the projection under the map μ. Note that $36 = \binom{8+1}{2}$ here is the number of vectors $[b_i, b_j]$ with $i \leq j$.

```
M = sage.Matrix(K, W + Y)

output = matrix.left_kernel().basis()
output = [vec[:36] for vec in output]
```

We present the output of this script which is a basis of the space $\mu(\ker(M))$.

$$\left\{\begin{array}{ccccc} v_{1,1}, & v_{1,2}, & v_{1,3}, & v_{1,4}, & v_{1,5}+v_{2,4}, \\ v_{1,6}+v_{3,4}, & v_{1,7}+v_{6,7}, & v_{1,8}+v_{6,8}, & v_{2,2}, & v_{2,3}, \\ v_{2,5}, & v_{2,6}+v_{3,5}, & v_{2,7}+v_{6,7}, & v_{2,8}+v_{6,8}, & v_{3,3}, \\ v_{3,6}, & v_{3,7}+v_{6,7}, & v_{3,8}+v_{6,8}, & v_{4,4}, & v_{4,5}, \\ v_{4,6}, & v_{4,7}-v_{6,7}, & v_{4,8}-v_{6,8}, & v_{5,5}, & v_{5,6}, \\ v_{5,7}-v_{6,7}, & v_{5,8}-v_{6,8}, & v_{6,6}, & v_{7,7}, & v_{7,8},\ v_{8,8} \end{array}\right\}$$

It remains to carry out Step 4 to determine Λ. The result of this calculation is the following theorem:

Theorem 5.3.2. *An element $\sum a_i b_i \in \mathrm{HH}_2^2(A)$ has a vanishing primary obstruction if and only if both of the following conditions hold:*

1.
$$\operatorname{rank}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \leq 1.$$

2. Either

(a) $a_7 = a_8 = 0$,

(b) $a_1 = a_4$, $a_2 = a_5$ *and* $a_3 = a_6$ *or*

(c) $a_1 + a_2 + a_3 = 0$.

Proof. Recalling the notation of Section 5.2, we need to find $\Lambda := p^{-1}(\mu(\ker(M)))$. Therefore we need to find conditions on the vector $\sum a_i b_i$ such that $\pi \circ \lambda \circ p(\sum a_i b_i)$ is 0.

In this case, by inspection

$$\begin{aligned} &u_1 = \pi \circ \lambda(v_{1,5}), \quad u_2 = \pi \circ \lambda(v_{1,6}), \quad u_3 = \pi \circ \lambda(v_{1,7}), \\ &u_4 = \pi \circ \lambda(v_{1,8}), \quad u_5 = \pi \circ \lambda(v_{2,6}) \end{aligned} \tag{5.1}$$

form a basis of $\mathrm{Im}(\pi \circ \lambda)$. With that basis in mind, π sends any $v_{i,j}$ to zero except those in (5.1) and:

$$\begin{array}{llll} \pi \circ \lambda(v_{2,4}) = -u_1, & \pi \circ \lambda(v_{3,4}) = -u_2, & \pi \circ \lambda(v_{3,5}) = -u_5, & \\ \pi \circ \lambda(v_{2,7}) = u_3, & \pi \circ \lambda(v_{3,7}) = u_3, & \pi \circ \lambda(v_{4,7}) = -u_3, & \\ \pi \circ \lambda(v_{5,7}) = -u_3, & \pi \circ \lambda(v_{6,7}) = -u_3, & \pi \circ \lambda(v_{2,8}) = u_4, & \\ \pi \circ \lambda(v_{3,8}) = u_4, & \pi \circ \lambda(v_{4,8}) = -u_4, & \pi \circ \lambda(v_{5,8}) = -u_4, & \pi \circ \lambda(v_{6,8}) = -u_4. \end{array}$$

Therefore we have that, with respect to the basis $\{u_i\}$

$$\pi \circ \lambda \circ p(\sum a_i b_i) = \begin{pmatrix} 2(a_1a_5 - a_2a_4) \\ 2(a_1a_6 - a_3a_4) \\ 2(a_1 + a_2 + a_3 - a_4 - a_5 - a_6)a_7 \\ 2(a_1 + a_2 + a_3 - a_4 - a_5 - a_6)a_8 \\ 2(a_2a_6 - a_3a_5) \end{pmatrix}.$$

This is zero precisely when the conditions in the statement of the proposition hold. □

Therefore the set of infinitesimal deformations with vanishing primary obstructions is a variety lying in $\mathrm{HH}^2_2 \cong \mathbb{K}^8$. The variety decomposes into the union of three components, namely:

$$V_g := \left\{(a_1, \ldots, a_8) \in \mathbb{K}^8 \;\middle|\; a_1 = a_4, a_2 = a_5 \text{ and } a_3 = a_6\right\},$$

$$V_q := \left\{(a_1, \ldots, a_8) \in \mathbb{K}^8 \;\middle|\; a_7 = a_8 = 0 \text{ and } \operatorname{rank}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \leq 1\right\}$$

and

$$V_u = \left\{(a_1, \ldots, a_8) \in \mathbb{K}^8 \;\middle|\; a_1 + a_2 + a_3 = 0 \text{ and } \operatorname{rank}\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} \leq 1\right\}.$$

From these definitions it is clear that V_g is a five dimensional vector subspace of $\mathbb{K}^8$, V_u is a five dimensional variety and V_q is a four dimensional determinental variety. We also list the intersections of these three varieties, all of which lie in $\mathbb{K}^8$.

- $V_g \cap V_q$ is a three dimensional vector space

$$\left\{(a_1, a_2, a_3, a_1, a_2, a_3, 0, 0) \in \mathbb{K}^8\right\}.$$

- $V_g \cap V_u$ is a four dimensional vector space

$$\{(a_1, a_2, a_3, a_1, a_2, a_3, a_7, a_8) \mid a_1 + a_2 + a_3 = 0\}.$$

- $V_q \cap V_u$ is a three dimensional (non-linear) variety

$$\{(a_1, a_2, a_3, a_4, a_5, a_6, 0, 0) \mid a_1 + a_2 + a_3 = a_4 + a_5 + a_6 = a_3 a_5 - a_2 a_6 = 0\}.$$

- $V_q \cap V_g \cap V_u$ is a two dimensional vector space

$$\{(a_1, a_2, a_3, a_1, a_2, a_3, 0, 0) \mid a_1 + a_2 + a_3 = 0\}.$$

We will see later that these varieties correspond to some very different deformations. For example, V_g will be shown to be the space of infinitesimals arising from deformations of A defined in terms of automorphisms of surfaces birational to $\mathbb{P}^1 \times \mathbb{P}^1$, and that all deformations in V_g integrate. Such examples of deformations were the topic of the paper [RS12]. All of these deformations are birationally commutative.

In contrast to this, V_q will be shown to contain the infinitesimal associated to the family of deformations A_q. None of these are birationally commutative, having a noncommutative rational function field of $\mathbb{K}_q(u, v)$. We are able to integrate most deformations in V_q. In contrast, we know little about those lying in V_u.

5.4 The Obstruction-Free Infinitesimal Deformations of A_q

We now turn our attention to A_q. We repeat the procedure outlined in Section 5.3 for A_q. By work in Section 3.4 we know that we can choose Δ to be the following set.

$$\left\{ \begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ qx_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \right\}$$

The 'Sage' script is almost identical to that in the preceding section so we will not go through it line by line. Note that since Δ is of size 4, we have truncated the vectors to the first $10 = \binom{4+1}{2}$ entries.

```
W = []
for index1, vec1 in enumerate(Delta):
    for index2, vec2 in enumerate(Delta):
        if index2< index1:
            continue
        else:
            func = GerstenhaberBracket(vec1, vec2,
                                             qKnBases[3])
            W.append(func)
W = [polygnomeVectorToSage(vec, 3, 3) for vec in W]


matrixOfk_3Dual = [k_3Dual(vec, qKnBases[3])
                            for vec in qK2DualBasis]
matrixOfk_3Dual = [polygnomeVectorToSage(vec, 3, 3)
                       for vec in matrixOfk_3Dual]
matrixOfk_3Dual = sage.matrix(K, matrixOfk_3Dual)
Y = matrixOfk_3Dual.row_space().basis()

M = sage.Matrix(K, W + Y)

output = matrix.left_kernel().basis()
output = [vec[:10] for vec in output]
```

The output of this script is the basis of $\mu(\ker(M)$.

$$\left\{ \begin{matrix} v_{1,1} & v_{1,2} & v_{1,3} & v_{1,4} & v_{2,2} \\ v_{2,3} & v_{2,4} & v_{3,3} & v_{3,4} & v_{4,4} \end{matrix} \right\}$$

Notice that this is the standard basis of $\mathbb{K}^{10}$.

Proposition 5.4.1. *All infinitesimal deformations of A_q have vanishing primary obstruction.*

Proof. Since $\mu(\ker(M)) = \mathbb{K}^{10}$, the condition $p(f) \in \mu(\ker(M))$ is trivial. Therefore the set Λ of infinitesimal deformations with vanishing primary obstruction is

$$p^{-1}(\mathbb{K}^{10}) = \mathrm{HH}_2^2(A_q).$$

□

If $\mathrm{HH}_3^3(A_q)$ were trivial then we could immediately deduce that all infinitesimal deformations integrate to formal deformations. However, we note briefly that $\mathrm{HH}_3^3(A_q) \neq 0$ by calculations similar to those in Section 3.3 so that this result is not a priori obvious. We shall return to this question in Chapter 7.

Chapter 6

Infinitesimal Deformations Arising From Automorphisms of Minimal Rational Surfaces

6.1 Introduction

In this chapter we discuss some of the infinitesimal deformations of the algebra A. Recall that if

$$E = \{t, ut, vt, uvt\} \subseteq \mathbb{K}(u, v)[t; \sigma]$$

then $A \cong \mathbb{K}\langle E\rangle$. In [RS12] it is shown that for a certain family $\{\tau\} \subseteq \operatorname{Aut}(\mathbb{K}(u, v))$ the set

$$E' = \{t, ut, vt, uvt\} \subseteq \mathbb{K}(u, v)[t; \sigma \circ \tau]$$

generates a family of deformations of A. The aim of this chapter is to investigate the infinitesimal structure of such deformations but generalised in two directions: firstly to the whole Lie algebra of the automorphism group of $\mathbb{P}^1 \times \mathbb{P}^1$ and secondly to Lie algebras of automorphism groups of other surfaces entirely.

In particular, we answer two questions.

1. What infinitesimal deformations arise due to the deformations of $Q_{\mathrm{gr}}(A)$ discussed in [RS12]?

2. Does expanding the set of automorphisms considered to those of other minimal rational surfaces increase the space of infinitesimal deformations that occur?

We show in Theorem 6.4.1 that the answer to the first question is precisely the set V_g. In Theorems 6.5.1 and 6.6.2 we show that the answer to the second question is no. This is a strong signal that in order to find new families of deformations of A, we need to concentrate on rings that are not birationally commutative.

Notation 6.1.1. In this chapter we change notation slightly to agree better with that in [RS12]. From this point onwards, in this chapter only, we write σ for the birational self map of $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$[x : y][z : w] \dashrightarrow [xz : yw][z : w]$$

On the other hand, the automorphism of $\mathbb{K}(u, v)$ previously referred to by σ will now be written σ^*. That is to say that σ^* is the automorphism of $\mathbb{K}(u, v)$ induced by choosing coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ of

$$u = \frac{x}{y} \text{ and } v = \frac{z}{w},$$

so that

$$\sigma^*(u) = uv \text{ and } \sigma^*(v) = v.$$

We hope this does not cause confusion.

Consider $D := \mathbb{K}(u, v)[t, t^{-1}; \sigma^*]$, the graded quotient ring of A. If G is a Lie group acting faithfully on $\mathbb{K}(u, v)$ and τ_s^* is a one-parameter subgroup inside G, i.e. τ_s^* is the exponential of a one-dimensional subspace of $\mathrm{Lie}(G)$, then $D(s) := \mathbb{K}(u, v)[t, t^{-1}; \sigma^* \circ \tau_s^*]$ is a one-parameter family of deformations of D. In particular, in a formal neighbourhood of $s = 0$, we obtain an associative multiplication F on the vector space $D \otimes_{\mathbb{K}} \mathbb{K}[[s]]$ given by the $\mathbb{K}[[s]]$-linear extension of

$$F(c, d) = cd + \sum_{i=1}^{\infty} F_i(c, d)s^i$$

for any c, d in D. In this case F_1 defines an infinitesimal deformation of D (see Section 2.3). In this chapter our aim is to examine the relationship of such deformations with infinitesimal deformations of A.

In particular, we define a linear map $\Phi : \mathrm{Lie}(G) \to \mathrm{HH}^2_2(D)$. Furthermore, we shall construct a linear map $\tilde{\Lambda} : \mathrm{HH}^2_2(A) \to \mathrm{HH}^2_2(D)$. In Section 6.2 we prove that there is a tractable method for testing if a given $L \in \mathrm{Lie}(G)$ satisfies $\Phi(L) \in \mathrm{Im}\,\tilde{\Lambda}$: evaluating $\Phi(L)$ on $R \subseteq D \otimes D$, where R is the set of relations of A, and testing if this lies in A. That is to say we develop a method for taking certain elements of $\mathrm{Lie}(G)$ and producing from them infinitesimal deformations of A.

Definition 6.1.2. If a vector L in $\mathrm{Lie}(G)$ satisfies $\Phi(L) \in \mathrm{Im}(\tilde{\Lambda})$ then we refer to it as an *admissible direction*.

We then apply this test in the following natural situation. If Y is a minimal surface birational to $\mathbb{P}^1 \times \mathbb{P}^1$, then Y is $\mathbb{P}^2$ or $\mathbb{F}_n$ for $n \neq 1$ [Bea96, Theorem V.10]. In each case, $\mathrm{Aut}(Y)$ is a Lie group which we can regard as subgroup of the plane Cremona group [DI09, Section 4], and we can consider the associated Lie algebra. Following a calculation detailed below, we determine which infinitesimal deformations of A arise from these Lie algebras.

6.2 Infinitesimal Deformations of a Localisation

In this section we construct a map $\tilde{\Lambda} : \mathrm{HH}^2_2(A) \to \mathrm{HH}^2_2(D)$ and show that there is a simple test for whether a given $f \in \mathrm{HH}^2_2(D)$ lies in $\mathrm{Im}(\tilde{\Lambda})$: check that $f(R) \subseteq A$. This is the foundation for the rest of this chapter, in which we apply this test in several related situations. The results of this section do not depend on the PBW property of A, nor on the fact that D is the graded quotient ring of A. For that reason we work with more general algebras C and E.

Let C be a Koszul $\mathbb{K}$-algebra that is a domain. Recall the definition of Ore sets and localisation from Section 2.2.1. Recall further that the category of right modules over the enveloping algebra $C^e = C \otimes C^{\mathrm{op}}$ is equivalent to the category of C-bimodules.

Notation 6.2.1. For elements in $r, s \in C^{\mathrm{op}}$ we will always write $r *_{\mathrm{op}} s$ to be the opposite multiplication and rs to be the element of the underlying vector space C, i.e. under the usual multiplication. As previously stated, we write tensor products over $\mathbb{K}$ as unadorned tensor products $\otimes$.

Let E be a localisation of C with respect to some (left and right) Ore set S. We want to compare the infinitesimal deformations of C with those of E. This is equivalent to making a comparison between second Hochschild cohomology groups, i.e. second Ext groups. In fact we find that in the case of Koszul algebras, there is a finite dimensional test on infinitesimal deformations of E to determine if they correspond to deformations of C.

First we need some basic module theoretic facts regarding the enveloping algebra. In the following we set T to be the set $S \otimes S$.

Proposition 6.2.2. *(i) If C is a $\mathbb{K}$-algebra and X a right (resp. left) Ore set with respect to some spanning set B of C with $X \subseteq B$, then X is a right (resp. left) Ore set for R.*

(ii) T is a right and left Ore set for C^e.

(iii) E^e is the localisation of C^e with respect to T on the right (or left). In particular, E^e is flat as a right or left C^e-module.

Before we prove Proposition 6.2.2 we state a useful lemma on localisation.

Lemma 6.2.3 ([MR01, Lemma 2.1.6])**.** *$E^e \cong (C^e)_T$ if and only if E^e satisfies the following universal property: if Z is a ring and $\phi : C^e \to Z$ is a ring homomorphism such that all elements of $\phi(T)$ are invertible in Z, then ϕ factors through E^e.*

Proof of Proposition 6.2.2. We prove each statement in the right hand case, the left hand case follows mutatis mutandis and it is clear due to the symmetry in the definitions that there are no special considerations depending on handedness.

(i) In order to prove the Ore condition, take $x \in X$ and a finite sum $r := \sum_i c_i b_i \in C$ for some $b_i \in B$ and $c_i \in \mathbb{K}$. By hypothesis we know that for all i there exists an $x_i \in X$ and $a_i \in B$ such that $b_i x_i = x a_i$.

Also, the set $x_1 B \cap \dots x_n B \cap X$ is nonempty for any $x_1, \dots, x_n \in X$. This fact is a very slight modification of the first part of [GW04, Lemma 4.21 (a)] where, in this case, we do not know that X is an Ore set for the whole ring. However the proof requires no changes. Therefore we may take $t \in x_1 B \cap \dots \cap x_n B \cap X$, and choose $r_i \in B$ for each i such that $x_i r_i = t \in X$.

Then the following holds:

$$\left(\sum_i c_i b_i\right) t = \sum_i c_i b_i x_i r_i = \sum_i c_i x a_i r_i = x\left(\sum_i c_i a_i r_i\right) \in rX \cap xR.$$

Since B is a spanning set we know that any $r \in C$ can be written as such a finite sum and so this completes the proof.

(ii) To prove the right Ore condition we take pure tensors $a \otimes b \in C^e$ and $s \otimes t \in T$. Then there exist $m, n \in C$ and $q, r \in S$ such that $aq = sm$ and $rb = nt$ by the two Ore conditions on S. Then the following holds:

$$(a \otimes b)(q \otimes r) = aq \otimes (b *_{op} r) = aq \otimes rb = sm \otimes nt = (s \otimes t)(m \otimes n).$$

This proves that T is a right Ore set with respect to the pure tensors. But T is itself a set of pure tensors, and the pure tensors span C^e, so we can apply part (i) and conclude that T is a right Ore set for C^e.

(iii) The second part of this statement is a well known consequence of the first part (see e.g. [MR01, Proposition 2.1.16 (ii)]), so it suffices to show the first part.

Since S is both a left and a right Ore set we can localise with respect to it on either side and get the same ring [MR01, Corollary 2.1.6 (ii)]. So the following holds:

$$E^{\mathrm{op}} = (C_S)^{\mathrm{op}} \cong ({}_S C)^{\mathrm{op}}.$$

This allows us to write elements of E^{op} as $s^{-1}a$ which makes the following considerably neater.

Let Z be a ring and $\phi : C^e \to Z$ be a map as in Lemma 6.2.3. Then we may define $\psi : E^e \to Z$ by extending the following $\mathbb{K}$-linearly:

$$\psi(as^{-1} \otimes t^{-1}b) = \phi(a \otimes b)\phi(s \otimes t)^{-1}.$$

We must show that ψ is a ring homomorphism. It suffices to check that ψ is multiplicative on pure tensors. To that aim, take two pure tensors in E^e, $as^{-1} \otimes t^{-1}b$ and $cq^{-1} \otimes r^{-1}d$. We know there exists some $x \otimes y \in T$ and $e \otimes f \in C^e$ such that:

$$(c \otimes d)(x \otimes y) = (s \otimes t)(e \otimes f) \tag{6.1}$$

by the Ore condition (ii). Furthermore by the definition of ψ the following holds:

$$\psi(as^{-1} \otimes t^{-1}b)\psi(cq^{-1} \otimes r^{-1}d) = \phi(a \otimes b)\phi(s \otimes t)^{-1}\phi(c \otimes d)\phi(q \otimes r)^{-1}. \tag{6.2}$$

Whereas, in E^e we have:

$$\begin{aligned}(as^{-1} \otimes t^{-1}b)(cq^{-1} \otimes r^{-1}d) &= (a \otimes b)(s^{-1} \otimes t^{-1})(c \otimes d)(q^{-1} \otimes r^{-1}) \\ &= (a \otimes b)(e \otimes f)(x^{-1} \otimes y^{-1})(q^{-1} \otimes r^{-1}) \\ &= ae(qx)^{-1} \otimes (yr)^{-1}fb. \end{aligned} \tag{6.3}$$

Acting on this by ψ and then using the fact that ϕ is a ring homomorphism gives us:

$$\begin{aligned}\psi(ae(qx)^{-1} \otimes (yr)^{-1}fb) &= \phi(ae \otimes fb)\phi(qx \otimes yr)^{-1} \\ &= \phi(a \otimes b)\phi(e \otimes f)\phi(x \otimes y)^{-1}\phi(q \otimes r)^{-1}. \end{aligned} \tag{6.4}$$

Finally using equation (6.1) we get that:

$$\begin{aligned} &\phi(c \otimes d)\phi(x \otimes y) = \phi(s \otimes t)\phi(e \otimes f) \\ \implies &\phi(e \otimes f)\phi(x \otimes y)^{-1} = \phi(s \otimes t)^{-1}\phi(c \otimes d). \end{aligned} \tag{6.5}$$

So that:

$$\begin{aligned}\psi(as^{-1} \otimes t^{-1}b)\psi(cq^{-1} \otimes r^{-1}d) &= \phi(a \otimes b)\phi(s \otimes t)^{-1}\phi(c \otimes d)\phi(q \otimes r)^{-1} \text{ by (6.2)} \\ &= \phi(a \otimes b)\phi(e \otimes f)\phi(x \otimes y)^{-1}\phi(q \otimes r)^{-1} \text{ by (6.5)} \\ &= \psi(ae(qx)^{-1} \otimes (yr)^{-1}fb) \text{ by (6.4)} \\ &= \psi\left((as^{-1} \otimes t^{-1}b)(cq^{-1} \otimes r^{-1}d)\right) \text{ by (6.3)}. \end{aligned}$$

This is the statement that ψ is multiplicative on pure tensors and is therefore a ring homomorphism as required.

If we set $\psi' : C^e \to E^e$ to be the obvious inclusion homomorphism it is immediate that $\phi = \psi \circ \psi'$. By Lemma 6.2.3,

$$E^e \cong (C^e)_T.$$

$\square$

Lemma 6.2.4. *As right E^e modules, $C \otimes_{C^e} E^e$ is isomorphic to E.*

Proof. We adopt the convention for writing pure tensor elements of $C \otimes_{C^e} E^e$ as $a|b \otimes d$, so that the bar represents the tensor over C^e.

We define a map $\lambda : C \otimes_{C^e} E^e \to E$ of E^e-modules by extending the following E^e-linearly:

$$a|c^{-1} \otimes e^{-1} \mapsto e^{-1}ac^{-1}.$$

Of course, since the $|$ is a tensor over C^e, any pure tensor is of this form since:

$$a|bs^{-1} \otimes t^{-1}c = cab|s^{-1} \otimes t^{-1}.$$

In order to show that λ is an isomorphism, we define its inverse. Take $d = ab^{-1} \in E$ and set: $\delta(d) = a|b^{-1} \otimes 1 \in C \otimes_{C^e} E^e$. If $ab^{-1} = st^{-1}$ there are some $x, y \in C$ such that $ay = sx$ and $by = tx$ by the definition of localisation, so that

$$\begin{aligned} s|t^{-1} \otimes 1 &= s|xx^{-1}t^{-1} \otimes 1 = sx|x^{-1}t^{-1} \otimes 1 \\ &= ay|y^{-1}b^{-1} \otimes 1 = a|b^{-1} \otimes 1 \end{aligned}$$

so δ is well defined.

We check that indeed $\delta = \lambda^{-1}$. Firstly, consider $a|b^{-1} \otimes t^{-1} \in C \otimes_{C^e} E^e$ and note that by the Ore condition we have elements $m \in C$, $n \in S$ such that:

$$tm = an \implies mn^{-1} = t^{-1}a,$$

where the second equality is in E. Then the following shows $\delta\lambda$ is equal to $\mathrm{id}_{C \otimes_{C^e} E^e}$:

$$\begin{aligned} a|b^{-1} \otimes t^{-1} &\overset{\lambda}{\mapsto} t^{-1}ab^{-1} = mn^{-1}b^{-1} \\ &\overset{\delta}{\mapsto} m|n^{-1}b^{-1} \otimes 1 = m|n^{-1}b^{-1} \otimes t^{-1}t \\ &= tm|n^{-1}b^{-1} \otimes t^{-1} = an|n^{-1}b^{-1} \otimes t^{-1} \\ &= a|b^{-1} \otimes t^{-1}. \end{aligned}$$

It is also the case that $\lambda\delta$ is equal to id_E since the following holds, for $d = ab^{-1} \in E$:

$$\lambda\delta(d) = \lambda\delta(ab^{-1}) = \lambda(a|b^{-1} \otimes 1) = ab^{-1} = d.$$

Therefore $\lambda = \delta^{-1}$, and λ is an isomorphism. □

Lemma 6.2.5. *For X a vector space, the free right C^e-module $C \otimes X \otimes C$ satisfies:*

$(C \otimes X \otimes C) \otimes_{C^e} E^e$ is naturally isomorphic to $E \otimes X \otimes E$ as right E^e-modules.

Here the right hand side module has multiplication

$$1 \otimes z \otimes 1(e \otimes e') = e' \otimes z \otimes e.$$

Proof. For a $\mathbb{K}$-algebra Δ and a vector space Γ, the equivalence of categories of C-bimodules and right C^e-modules is given by the following isomorphism of right Δ^e-modules:

$$\Gamma \otimes \Delta^e \cong \Delta \otimes \Gamma \otimes \Delta,$$

where the right hand side has module multiplication for $\gamma \in \Gamma$ and $\delta, \delta' \in \Delta$:

$$(1 \otimes \gamma \otimes 1)(\delta \otimes \delta') = \delta' \otimes \gamma \otimes \delta.$$

By definition, the free right C^e-module on X is $X \otimes C^e$. This lemma is simply then a reformulation of the fact that $X \otimes C^e \otimes_{C^e} E^e$ is naturally isomorphic to $X \otimes E^e$. □

Notation 6.2.6. Recall that C is a Koszul $\mathbb{K}$-algebra which is a domain and $E = {}_SC = C_S$ is the localisation of C at a left and right Ore set S. Let $K_* = C \otimes \overline{K}_* \otimes C$ be the Koszul complex of C, B_* be the bar resolution of C and $i_* : K_* \to B_*$ the natural inclusion (see Section 2.1 for definitions). Further, let

$$\phi_* : B_* \to K_*$$

be any section of i_*. Then let

$$I_* : E \otimes \overline{K}_* \otimes E \to E \otimes C^{\otimes *} \otimes E$$

be the value of the functor $- \otimes_{C^e} E^e$ at i_*, taking into account the isomorphism from Lemma 6.2.5. Likewise, let

$$\Phi_* : E \otimes C^{\otimes *} \otimes E \to E \otimes \overline{K}_* \otimes E$$

be the value of $- \otimes_{C^e} E^e$ at ϕ_*.

For any $n \in \mathbb{N}$, let

$$\psi_n : C^{\otimes n+2} \hookrightarrow C \otimes E^{\otimes n} \otimes C$$

be the map induced by the canonical localisation map $C \hookrightarrow E$. Then we write Ψ_* for the value of the functor $- \otimes_{C^e} E^e$ at ψ_*.

In the result below we will repeatedly make use of the Comparison Theorem [Wei94, Theorem 2.2.6].

Lemma 6.2.7. *The following is a commutative diagram in which the rows are resolutions of E as a right E^e-module:*

$$\begin{array}{ccc}
E^{\otimes *+2} & \longrightarrow & E \\
{\scriptstyle \Psi_*}\uparrow & & \| \\
E \otimes C^{\otimes *} \otimes E & \longrightarrow & E \\
{\scriptstyle I_*}\uparrow & & \| \\
E \otimes \overline{K}_* \otimes E & \longrightarrow & E.
\end{array}$$

Proof. We have two free resolutions of C as a C^e-module, the bar and Koszul resolutions. These are quasi-isomorphic under i_* by the Comparison Theorem and the following diagram commutes

$$\begin{array}{ccc}
C^{\otimes *+2} & \longrightarrow & C \\
{\scriptstyle i_*}\uparrow & & \| \\
K_* & \longrightarrow & C.
\end{array}$$

Part (iii) of proposition 6.2.2 implies that $- \otimes_{C^e} E^e$ is an exact functor from the category of right C^e-modules to the category of right E^e modules. Therefore we obtain two free resolutions of $C \otimes_{C^e} E^e$:

$$\begin{array}{ccc}
C^{\otimes *+2} \otimes_{C^e} E^e & \longrightarrow & C \otimes_{C^e} E^e \\
\uparrow & & \| \\
K_* \otimes_{C^e} E^e & \longrightarrow & C \otimes_{C^e} E^e.
\end{array}$$

By Lemma 6.2.4, $C \otimes_{C^e} E^e$ is isomorphic to E as an E^e module. Furthermore, we may apply Lemma 6.2.5 twice and obtain the following commutative diagram with exact rows:

$$\begin{array}{ccc} E \otimes C^{\otimes *} \otimes E & \longrightarrow & E \\ {\scriptstyle I_*}\uparrow & & \| \\ E \otimes \overline{K}_* \otimes E & \longrightarrow & E. \end{array}$$

Finally, by the Comparison Theorem the bar resolution of E itself is quasi-isomorphic to both of these resolutions. The chain map realising this is precisely Ψ_*, and so we obtain the diagram in the lemma. □

Notation 6.2.8. Let $\Lambda_n : \mathrm{Hom}_{C^e}(C^{\otimes n+2}, C) \to \mathrm{Hom}_{E^e}(E \otimes C^{\otimes n} \otimes E, E)$ be the map taking $f \in \mathrm{Hom}_{C^e}(C^{\otimes n+2}, C)$ to the map defined on pure tensors by

$$\Lambda_n(f)(e \otimes c_1 \otimes \ldots \otimes c_n \otimes e') = ef(1 \otimes c_1 \otimes \ldots \otimes c_n \otimes 1)e'.$$

We note that each Λ_n is an injection whose image is

$$\left\{ g \in \mathrm{Hom}(E \otimes C^{\otimes n} \otimes E, E) \;\middle|\; g(1 \otimes C^{\otimes n} \otimes 1) \subseteq C \right\},$$

Furthermore, by the definition of the boundary map δ_* on $E \otimes C^{\otimes n} \otimes E$ and the boundary map b_* on $C^{\otimes n+2}$, it follows that $\delta^*(\Lambda(f)) = \Lambda(\delta^* f)$, and so each Λ_n descends to map $\tilde{\Lambda} : \mathrm{HH}^n(A) \to \mathrm{HH}^n(D)$.

If Δ is a $\mathbb{K}$-vector space and Γ is a $\mathbb{K}$-algebra, we use without further comment the adjunction isomorphism

$$\mathrm{Hom}_{\mathbb{K}}(\Delta, \Gamma) \cong \mathrm{Hom}_{\Gamma^e}(\Gamma \otimes \Delta \otimes \Gamma, \Gamma).$$

Theorem 6.2.9. *Let C be Koszul and E a localisation of C with respect to a left and right Ore set. If $f \in Z^2(E)$ is a Hochschild 2-cocycle then its cohomology class $[f] \in \mathrm{HH}^2(E)$ is determined by its restriction to R, the relations of C. In particular, if $f(1 \otimes R \otimes 1) \subseteq C$ then $[f] \in \mathrm{Im}(\tilde{\Lambda})$. Thus f is cohomologous to some $g \in Z^2(E)$ such that $g(1 \otimes C \otimes C \otimes 1) \subseteq C$, i.e. f determines an isomorphism class of infinitesimal deformations of C.*

We do not expect a converse to Theorem 6.2.9 to hold true in general. We make an intuitive argument for this before proving the theorem itself. Let C' be a subalgebra of E with relations R', distinct from but isomorphic to C. Then it is entirely plausible that there will be a Hochschild 2-cocycle $f \in Z^2(E)$ satisfying $f(1 \otimes R' \otimes 1) \subseteq C'$ but $f(1 \otimes R \otimes 1) \not\subseteq C$. In this way we know by Theorem 6.2.9 that f determines an isomorphism class of infinitesimal deformations of $C' \cong C$ but $f(1 \otimes R \otimes 1) \not\subseteq C$.

Proof of Theorem 6.2.9. Consider the dual of the commutative diagram from Lemma 6.2.7. In the following diagram we write Hom for Hom_{E^e}.

$$\begin{array}{ccc}
\mathrm{Hom}(E^{\otimes *+2}, E) & \longleftarrow & E \\
\big\downarrow{\scriptstyle \Psi^*} & & \big\| \\
\mathrm{Hom}(E \otimes C^{\otimes *} \otimes E, E) & \longleftarrow & E \\
\big\downarrow{\scriptstyle I^*} & & \big\| \\
\mathrm{Hom}(E \otimes \overline{K}^* \otimes E, E) & \longleftarrow & E.
\end{array}$$

Let $f \in \mathrm{Hom}_{E^e}(E^{\otimes 4}, E)$ be a cocycle in the (dual) bar complex of E, i.e. f is in the top row of the diagram. We may consider the restriction to R, $f|_{(1\otimes R\otimes 1)}$. Note that since I_* and Ψ_* are inclusion morphisms, their duals are restriction morphisms so that

$$f|_{(1\otimes R\otimes 1)} = I^2(\Psi^2(f)).$$

If two functions have cohomologous restrictions then they must be cohomologous to each other since the maps I_* and Ψ_* are quasi-isomorphisms. This establishes the first part of the theorem.

Furthermore, the map

$$\Phi^* : \mathrm{Hom}(E \otimes \overline{K}_* \otimes E, E) \to \mathrm{Hom}(E \otimes C^{\otimes *} \otimes E, E)$$

has the property that if $f \in \mathrm{Hom}(E^{\otimes 4}, E)$ is a cocycle such that

$$I^* \circ \Psi^*(f)(1 \otimes R \otimes 1) \subseteq C$$

then $G = \Phi^* \circ I^* \circ \Psi^*(f)$ satisfies

$$G(1 \otimes C^{\otimes 2} \otimes 1) \subseteq C.$$

Additionally, by the Comparison Theorem there exists a section to Ψ_* and so there exists some cocycle $g \in \mathrm{Hom}(E^{\otimes 4}, E)$ such that $\Psi^*(g) = G$ with g being cohomologous to f. Finally, since $G \in \mathrm{Im}\,\Lambda$ there is some $h \in B^2(C)$ satisfying $\Lambda(h) = G$. Furthermore, since G is a cocycle it must be the case that $\delta^*(\Lambda(h)) = \Lambda(\delta^*(h)) = 0$ and since Λ is injective we can conclude that h is a Hochschild cocycle. In this way, f determines $[h]$ which is an isomorphism class of infinitesimal deformations of C.

□

6.3 Overview of Calculations

In this section we describe a general procedure for applying Theorem 6.2.9 in the context of families of deformations of $\mathrm{Q}_{\mathrm{gr}}(A)$ induced by automorphisms of minimal rational surfaces. Since the procedure can be described in the abstract, we do so here. In addition, we carry out some preliminary calculations that will be used in every case considered in the rest of the chapter.

Recall from Section 6.1 the definition of $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Recall further that A is the subalgebra of $D := \mathbb{K}(u, v)[t : \sigma^*]$ generated by

$$E := \{x_1 := t, x_2 := ut, x_3 := vt, x_4 := uvt\}.$$

For any surface Y birational to $\mathbb{P}^1 \times \mathbb{P}^1$, under b say, we get an induced birational self map of Y given by $\sigma_Y := b \circ \sigma \circ b^{-1}$.

Since b is a birational map, we know by basic birational geometry (see [Har77, Theorem 4.4]) that $\mathbb{K}(Y)$ and $\mathbb{K}(\mathbb{P}^1 \times \mathbb{P}^1)$ are isomorphic fields, with the pullback of b defining an isomorphism, b^*, between them. Under this identification, σ_Y^* is equal to σ^* as automorphisms of $\mathbb{K}(u, v)$.

Since we are interested in infinitesimal deformations of A, we consider a vector L in the Lie algebra associated with $\mathrm{Aut}(Y)$, which is a Lie group [DI09, Section 4]. By considering a one-dimensional subspace of vectors $\{sL\}$ we can apply the exponential map and obtain a one-parameter subgroup of automorphisms $\tau_s = \exp(sL)$.

As discussed in Section 6.1, such a τ_s defines a deformation of D and so we can use Theorem 6.2.9 in order to test whether it actually corresponds to an infinitesimal deformation of A. Write the deformed multiplication on D induced by τ_s as:

$$F(a, b) = ab + F_1(a, b)s + O(s^2).$$

Recall that F_1 is the *infinitesimal* of this deformation, and that it therefore satisfies

$$F_1(a, b) = \left.\frac{\partial(F(a, b))}{\partial s}\right|_{s=0}.$$

Theorem 6.2.9 implies that F induces an infinitesimal deformation of A if $F_1(R) \subseteq A$, where $R = \mathrm{sp}_{\mathbb{K}}\{r_i | 1 \leq i \leq 6\}$ is given by the six relations:

$$r_1 = x_3x_1 - x_1x_3, \quad r_2 = x_2x_4 - x_4x_2, \quad r_3 = x_4x_1 - x_2x_3,$$
$$r_4 = x_1x_2 - x_2x_3, \quad r_5 = x_3x_2 - x_1x_4, \quad r_6 = x_4x_3 - x_1x_4.$$

As in Notation 3.2.1 we write a function $f \in \mathrm{HH}^2_2(A)$ as a vector, with the ith component being the image of r_i under f. Since our algebra A is Koszul, the bar resolution is quasi-isomorphic to the Koszul resolution and so these 6 images determine the cohomology class of the map f.

For any Y and $L \in \mathrm{Lie}(\mathrm{Aut}(Y))$, we can calculate the associated F_1. We start by setting $\tau_s := \exp(sL)$ and defining $U(s) := \sigma^* \circ \tau_s^*(u)$ and $V(s) := \sigma^* \circ \tau_s^*(v)$, so that in $\mathbb{K}(u,v)[[s]][t;\sigma^* \circ \tau_s^*]$ we can write:

$$tv = V(s)t \text{ and } tu = U(s)t.$$

Since $\tau_0 = \mathrm{id}$ we must have that $U(0) = uv$ and $V(0) = v$. Then we calculate $U'(0)$ and $V'(0)$. Once we have these derivatives we can use basic differentiation rules to calculate F_1. Since the following calculations are simple applications of the product rule we show only the first in detail and record the rest in Appendix C.1.

$$\begin{aligned} F_1(r_1) &= \left.\frac{\partial(F(r_1))}{\partial s}\right|_{s=0} = \left.\frac{\partial(F(x_3x_1 - x_1x_3))}{\partial s}\right|_{s=0} \\ &= \left.\frac{\partial(vt^2 - tvt)}{\partial s}\right|_{s=0} = \left.\frac{\partial(vt^2 - V(s)t^2)}{\partial s}\right|_{s=0} \\ &= -\left.\frac{\partial(V(s)t^2)}{\partial s}\right|_{s=0} = -V'(0)t^2. \end{aligned}$$

The other calculations proceed along similar lines. We record the results in Table 6.1.

Relation r	$F_1(r)$
$x_3x_1 - x_1x_3$	$-V'(0)t^2$
$x_4x_2 - x_2x_4$	$-u^2vV'(0)t^2$
$x_4x_1 - x_2x_3$	$-uV'(0)t^2$
$x_1x_2 - x_2x_3$	$(U'(0) - uV'(0))t^2$
$x_3x_2 - x_1x_4$	$-uvV'(0)t^2$
$x_4x_3 - x_1x_4$	$-vU'(0)t^2$

Table 6.1: F_1 Applied to the Relations of A

Now that this leg work is done, all it takes is to calculate the different values of the functions in these formulae, and we get the relevant infinitesimals. So by Theorem 6.2.9, a sufficient condition for L to correspond to an infinitesimal deformation is for each of the above six expressions to lie in A_2. For example in the final row in Table 6.1 one must check that $-vU'(0)t^2$ is in the span of degree two monomials in the generators of A.

Finally, we label the ordered basis of $\mathrm{HH}^2_2(A)$ chosen in Section 3.3 by $b_1, \ldots, b_8$. Recall that this basis is

$$\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_3^2 \\ 0 \\ x_1x_4 \\ 0 \\ 0 \\ x_3x_4 \end{pmatrix} \begin{pmatrix} x_1^2 \\ 0 \\ 0 \\ x_2x_1 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \\ -x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x_2^2 \\ 0 \\ x_2x_4 \end{pmatrix}$$

6.4 Infinitesimals Arising from Automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

In this case we have $Y := X = \mathbb{P}^1 \times \mathbb{P}^1$ and $b : X \dashrightarrow X$ is the identity automorphism, so that $\sigma_Y = \sigma$. The automorphism group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to the wreath product $\mathrm{PGL}_2(\mathbb{K}) \wr S_2$ [DI09, Section 4.3], however we only need to consider the identity component. This component of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is isomorphic to $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ which means the associated Lie algebra is isomorphic to $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. So for L we take $M \times N \in \mathfrak{sl}_2 \times \mathfrak{sl}_2$. This case is studied using very different methods in [RS12], where the deformed algebras are analysed in their own right, rather than in terms of formal deformation theory.

Take a matrix $M \in \mathfrak{sl}_2$ and a formal deformation parameter s and consider

$$sM = s \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Then clearly:

$$\exp(sM) = \begin{pmatrix} 1+as & bs \\ cs & 1-as \end{pmatrix} + O(s^2),$$

and we get a similar formula for

$$N := \begin{pmatrix} d & e \\ f & -d \end{pmatrix}.$$

In the following we omit all terms which have power of s greater than 1, since this has no impact on the first order deformations and no impact on any derivatives appearing.

Define $\tau_s = \exp(sM, sN)$ so that:

$$\tau_s([x:y][z:w]) = [(1+as)x + bsy : csx + (1-as)y][(1+ds)z + esw : fsz + (1-ds)w].$$

We have the formula

$$\tau_s \circ \sigma([x:y][z:w]) = \\ [(1+as)xz + bsyw : csxz + (1-as)yw][(1+ds)z + esw : fsz + (1-ds)w].$$

From this we can calculate $U'(0)$ by the following method:

$$\begin{aligned} U(s) &= \sigma^* \circ \tau_s^*(u) \\ &= \frac{(1+as)xz + bsyw}{csxz + (1-as)yw} = \frac{(1+as)uv + bs}{csuv + (1-as)}. \end{aligned}$$

As a quick check on these calculations we note that $U(0) = uv$ as required. In order to calculate the derivative, let F be the numerator and G be the denominator. We have the following:

$$\begin{aligned} F(0) &= uv, \quad F'(0) = auv + b \\ G(0) &= 1, \quad G'(0) = cuv - a. \end{aligned}$$

So that

$$\begin{aligned} U'(0) &= \frac{G(0)F'(0) - F(0)G'(0)}{G(0)^2} \\ &= auv + b - uv(cuv - a) \\ &= b - cu^2v^2 + 2auv \end{aligned}$$

Likewise:

$$\begin{aligned} V(s) = \sigma^* \circ \tau_s^*(v) &= \frac{(1+ds)z + esw}{fsz + (1-ds)w} \\ &= \frac{(1+ds)v + es}{fsv + (1-ds)}. \end{aligned}$$

Note again this still makes some sense since $V(0) = v$. The derivative is somewhat simpler to calculate in this case:

$$\begin{aligned} V'(0) &= \frac{\partial}{\partial s}\left(\frac{(1+ds)v + es}{vfs + (1-ds)}\right)\Bigg|_{s=0} \\ &= \frac{(vfs + (1-ds))(dv+e) - ((1+ds)v + es)(vf - d)}{(vfs + (1-ds))^2}\Bigg|_{s=0} \\ &= 2dv + e - fv^2. \end{aligned}$$

So putting this together with the calculations before, Table 6.2 records the images of the relations under F_1.

Relation	Formula	Image Under F_1
$r_1 = x_3x_1 - x_1x_3$	$-V'(0)$	$(fv^2 - 2dv - e)t^2$
$r_2 = x_2x_4 - x_4x_2$	$-u^2vV'(0)$	$(fu^2v^3 - eu^2v - 2du^2v^2)t^2$
$r_3 = x_4x_1 - x_2x_3$	$-uV'(0)$	$(fuv^2 - 2duv - eu)t^2$
$r_4 = x_1x_2 - x_2x_3$	$U'(0) - uV'(0)$	$(b + 2auv - cu^2v^2 - eu - 2duv + fuv^2)t^2$
$r_5 = x_3x_2 - x_1x_4$	$-uvV'(0)$	$(fuv^3 - euv - 2duv^2)t^2$
$r_6 = x_4x_3 - x_1x_4$	$-vU'(0)$	$(cu^2v^3 - bv - 2auv^2)t^2$

Table 6.2: Images of the Six Relations of A Under F_1

Now we have the information necessary to check the *admissibility* of L. Recall that this means we must check if each of the terms in the right hand column of this table lies in

$$\begin{aligned} A_2 &= \mathrm{sp}_{\mathbb{K}}\left(\left\{u^iv^jt^2 \middle| 0 \le i \le 2, 0 \le j \le 3\right\} \setminus \{v^3t^2, u^2t^2\}\right) \\ &= \mathrm{sp}_{\mathbb{K}}\{x_2x_1, x_2^2, x_2x_3, x_2x_4, x_1^2, x_1x_3, x_1x_4, x_3^2, x_3x_4, x_4^2\}. \end{aligned} \tag{6.6}$$

It is immediate by comparing the right hand column of Table 6.2 with (6.6) that $F_1(r_i)$ is always an element of A_2. Therefore we can conclude that all of the directions are admissible and so we do in fact get infinitesimal deformations of A with any choice of L. In fact, the image of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ in $\mathrm{HH}^2(A)$ coincides precisely with the unobstructed component V_g from Theorem 5.3.2.

Theorem 6.4.1. *All elements in $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ are admissible. Furthermore, for any infinitesimal $f \in V_g \subseteq \mathrm{HH}^2_2(A)$ there exists a one-parameter subgroup $\{\tau_s\} \subseteq \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ such that the associated infinitesimal deformation of A is isomorphic to f.*

Proof. We have already shown that all elements are admissible. Identify $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ with $\mathbb{K}^6$ under the map :

$$\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix}\right) \leftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}.$$

From the right hand column of Table 6.2 we obtain a linear map $\Phi : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathrm{HH}^2_2(A)$

represented by left multiplication by the matrix

$$\mathbb{A} := \begin{pmatrix} 0 & 0 & 0 & -2x_1x_3 & -x_1^2 & x_3^2 \\ 0 & 0 & 0 & -2x_2x_4 & -x_2^2 & x_4^2 \\ 0 & 0 & 0 & -2x_2x_3 & -x_2x_1 & x_1x_4 \\ 2x_2x_3 & x_1^2 & -x_2x_4 & -2x_2x_3 & -x_2x_1 & x_1x_4 \\ 0 & 0 & 0 & -2x_1x_4 & -x_2x_3 & x_3x_4 \\ -2x_1x_4 & -x_1x_3 & x_4^2 & 0 & 0 & 0 \end{pmatrix}.$$

By referring to the basis of $\mathrm{HH}_2^2(A)$ calculated in Chapter 3 (see Theorem 3.3.1) we write the image of this map as an element of the vector space spanned by $\{b_1, \dots, b_8\}$.

$$\Phi\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \begin{pmatrix} d & e \\ f & -d \end{pmatrix}\right) = \\ -2d(b_1 + b_4) + f(b_2 + b_5) - e(b_3 + b_6) + bb_7 + cb_8 \in \mathrm{HH}_2^2(A). \quad (6.7)$$

Note that in (6.7) we have written the cohomology classes of the columns of the matrix $\mathbb{A}$, which therefore may differ from the columns up to a coboundary. This is a vector of the form $\sum a_i b_i$ satisfying $a_1 = a_4$, $a_2 = a_5$ and $a_3 = a_6$. By Theorem 5.3.2 this is precisely the defining equations of V_g. Furthermore, Φ is clearly surjective onto V_g. $\square$

Of course one notices that a is not present on the right hand side of equation (6.7), which corresponds to the fact that this direction induces a trivial deformation. As a check on our calculations we show this is indeed the case.

Lemma 6.4.2. *Let*

$$\tau = \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \in \mathrm{PGL}_2 \times \mathrm{PGL}_2,$$

then the subalgebra B of $\mathbb{K}(u,v)[t,t^{-1};\sigma^* \circ \tau^*]$ generated by E is isomorphic to A.

Proof. We prove this be defining an explicit isomorphism $\phi : A \to B$, given by

$$x_1 \mapsto x_1$$
$$x_2 \mapsto x_2$$
$$x_3 \mapsto a^2 x_3$$
$$x_4 \mapsto a^2 x_4$$

To see that ϕ is an isomorphism we note a few facts. Firstly, since $\sigma^* \circ \tau^*(u) = a^2uv$

and $\sigma^* \circ \tau^*(v) = v$, B has the following as a basis for its space of relations:

$$s_1 = x_3x_1 - x_1x_3, \quad s_2 = x_2x_4 - x_4x_2, \quad s_3 = x_4x_1 - x_2x_3,$$
$$s_4 = x_1x_2 - a^2x_2x_3, \quad s_5 = x_3x_2 - x_1x_4, \quad s_6 = a^2x_4x_3 - x_1x_4.$$

From this it is clear that B has the same PBW-basis as A with respect to the lexicographic ordering $x_2 < x_1 < x_3 < x_4$. One must simply check that ϕ is well defined and is bijective. Once we have checked it is well defined, then it is clear that ϕ is bijective since it is a bijection on the PBW-basis (up to a scalar multiple). Therefore we show ϕ is well defined.

With that aim, we evaluate ϕ on the relations of A and show that they are in the kernel:

$$\begin{aligned}
\phi(r_1) &= \phi(x_3x_1 - x_1x_3) = a^2x_3x_1 - a^2x_1x_3 = a^2s_1 = 0\\
\phi(r_2) &= \phi(x_2x_4 - x_4x_2) = a^2x_2x_4 - a^2x_4x_2 = a^2s_2\\
\phi(r_3) &= \phi(x_4x_1 - x_2x_3) = a^2x_4x_1 - a^2x_2x_3 = a^2s_3\\
\phi(r_4) &= \phi(x_1x_2 - x_2x_3) = x_1x_2 - a^2x_2 = s_4\\
\phi(r_5) &= \phi(x_3x_2 - x_1x_4) = a^2x_3x_2 - a^2x_1x_4 = a^2s_5\\
\phi(r_6) &= \phi(x_4x_3 - x_1x_4) = a^4x_4x_3 - a^2x_1x_4 = a^2s_6.
\end{aligned}$$

This concludes the proof, and so B is a trivial deformation of A. □

The infinitesimal associated to B is therefore also trivial; this infinitesimal is precisely the image in HH^2_2 of

$$L = \left(\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Remark 6.4.3. In [RS12], an automorphism $\tau \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ is discussed which defines an algebra $A(\tau)$ that has GK-dimension 3. For this reason we do not expect a flat family of deformations of A to be parameterised by the entire group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. However, the family discussed by Rogalski and Sierra are parameterised by a complement of the plane by a countable union of varieties which, in particular, contains the identity. Therefore we expect the associated infinitesimals to integrate to a formal deformation. This reasoning leads us to suspect that all elements of V_g integrate to formal deformations although we do not prove this in this thesis. Instead we concentrate on flat families with infinitesimals lying outside of V_g.

6.5 Infinitesimals Arising from Automorphisms of $\mathbb{P}^2$

We have seen that for every vector v in the variety $V_g \subset \mathrm{HH}^2_2(A)$ we can find an admissible direction $L \in \mathfrak{sl}_2 \times \mathfrak{sl}_2$ whose image under Φ is v. One might hope that by considering automorphisms of other surfaces one might find admissible directions whose images comprise V_u and V_q. In the following sections we show that this is not the case, and that all admissible directions have images lying in V_g.

6.5.1 Calculating $\sigma_{\mathbb{P}^2}$

We recall some basic facts about $\mathbb{P}^1 \times \mathbb{P}^1$. Under the Segre embedding (see e.g. [EH00, Section III.2.3]), we get an embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. However, we change coordinates slightly from the usual definition of the Segre embedding to ensure the formulae work out nicely. We note here that this has no effect on the properties of the map, it is purely a coordinate change. We define then:

$$S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \ \ [x : y][z : w] \mapsto [xw : xz : yw : yz].$$

This is an embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ as the quadric surface

$$\mathcal{Q} := \{[\alpha : \beta : \gamma : \delta] \mid \alpha\delta - \beta\gamma = 0\},$$

where our rational coordinates are now $u = \alpha/\gamma$ and $v = \beta/\alpha$. Now, if we take the projection from $[0 : 0 : 0 : 1]$ of this surface, we get dominant rational map, b^{-1}, to $Y := \mathbb{P}^2$. This map is given by

$$[\alpha : \beta : \gamma : \delta] \mapsto [\alpha : \beta : \gamma],$$

with birational inverse, b,

$$[\alpha : \beta : \gamma] \mapsto [\alpha^2 : \alpha\beta : \alpha\gamma : \beta\gamma].$$

Now to transfer σ across to $\mathbb{P}^2$. Firstly we transfer it to $\mathcal{Q}$, where it is simply the composition:

$$\begin{aligned}[xw : xz : yw : yz] &\mapsto [x : y][z : w] \\ &\dashrightarrow [xz : yw][z : w] \\ &\mapsto [xwz : xz^2 : yw^2 : ywz] = [vxw : vxz : yw : yz],\end{aligned}$$

which can be more neatly written as $[\alpha : \beta : \gamma : \delta] \dashrightarrow [\beta\alpha : \beta^2 : \gamma\alpha : \delta\alpha]$.

Then if $[\alpha : \beta : \gamma]$ is in $\mathbb{P}^2$, $\sigma_{\mathbb{P}^2}$ acts as the following composition:

$$[\alpha : \beta : \gamma] \dashmapsto [\alpha^2 : \alpha\beta : \alpha\gamma : \beta\gamma] \dashmapsto [\beta\alpha : \beta^2 : \alpha\gamma : \beta\gamma] \dashmapsto [\beta\alpha : \beta^2 : \alpha\gamma].$$

As a quick check that this is still the same σ^* on function fields, we note that σ^* sends $u = \alpha/\gamma$ and $v = \beta/\alpha$ to uv and v respectively, as required.

Now the automorphism group of $\mathbb{P}^2$ is of course PGL_3 which has Lie algebra $\mathfrak{sl}_3$. Take a matrix $L \in \mathfrak{sl}_3$ and a formal deformation parameter s and consider

$$sL = s\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a-e \end{pmatrix}.$$

Then we have

$$\tau_s := exp(sL) = \begin{pmatrix} (1+as) & bs & cs \\ ds & (1+es) & fs \\ gs & hs & (1-as-es) \end{pmatrix} + O(s^2).$$

This acts on $\mathbb{P}^2$ by the following formula, where we drop the $O(s^2)$ terms as before:

$$[\alpha : \beta : \gamma] \mapsto [(1+as)\alpha + bs\beta + cs\gamma : ds\alpha + (1+es)\beta + fs\gamma : gs\alpha + hs\beta + (1-as-es)\gamma].$$

So using the coordinates $u = \alpha/\gamma$ and $v = \beta/\alpha$, we note that $\beta/\gamma = uv$, and calculate the action of $\sigma^* \circ \tau_s^*$ on u and v. Firstly, we calculate the composition:

$$\begin{aligned} \tau_s \circ \sigma([\alpha, \beta, \gamma]) &= \tau_s([\beta\alpha : \beta^2 : \alpha\gamma]) \\ &= [(1+as)\beta\alpha + bs\beta^2 + cs\alpha\gamma : ds\alpha\beta + (1+es)\beta^2 + fs\alpha\gamma : \\ &\qquad gs\alpha\beta + hs\beta^2 + (1-as-es)\alpha\gamma]. \end{aligned}$$

We can easily check that indeed $U(0) = uv$ and $V(0) = v$, and then calculate the derivatives:

$$\begin{aligned} \left.\frac{\partial(U(s))}{\partial s}\right|_{s=0} &= \left(\frac{\partial}{\partial s}\right)\left.\left(\frac{[(1+as)\beta\alpha + bs\beta^2 + cs\alpha\gamma}{gs\alpha\beta + hs\beta^2 + (1-as-es)\alpha\gamma}\right)\right|_{s=0} \\ &= \left(\frac{\partial}{\partial s}\right)\left.\left(\frac{(1+as)uv + bsuv^2 + cs}{gsuv + hsuv^2 + (1-as-es)}\right)\right|_{s=0} \\ &= 2auv + euv + buv^2 - gu^2v^2 - hu^2v^3 + c. \end{aligned}$$

Similarly, we have that:

$$
\begin{aligned}
\left.\frac{\partial(V(s))}{\partial s}\right|_{s=0} &= \left(\frac{\partial}{\partial s}\right)\left.\left(\frac{ds\alpha\beta+(1+es)\beta^2+fs\alpha\gamma}{(1+as)\beta\alpha+bs\beta^2+cs\alpha\gamma}\right)\right|_{s=0}\\
&= \left(\frac{\partial}{\partial s}\right)\left.\left(\frac{dsuv+(1+es)uv^2+fs}{(1+as)uv+bsuv^2+cs}\right)\right|_{s=0}\\
&= d+ev+fu^{-1}v^{-1}-av-bv^2-cu^{-1}.
\end{aligned}
$$

We can see already that to be able to apply Theorem 6.2.9, f and c will both have to be 0. This is because $F_1(r_1) = -V'(0)$ and if these are not zero then $V'(0)t^2$ will not be an element of A_2 as required. Likewise, h must be 0 since $-vU'(0)t^2$ must be in A_2 (equation 6 in Section 6.3) and this would contain the term $hu^2v^4t^2$, which has a power of v that is too high for this to lie in A_2.

After setting $f = c = h = 0$, we record this data in Table 6.3 for ease of reading.

Relation	Formula	Image Under F_1
$x_3x_1 - x_1x_3$	$-V'(0)$	$(bv^2 + av - ev - d)t^2$
$x_2x_4 - x_4x_2$	$-u^2vV'(0)$	$(bu^2v^3 + au^2v^2 - eu^2v^2 - du^2v)t^2$
$x_4x_1 - x_2x_3$	$-uV'(0)$	$(buv^2 + auv - euv - du)t^2$
$x_1x_2 - x_2x_3$	$U'(0) - uV'(0)$	$(-gu^2v^2 + 2buv^2 + 3auv - du)t^2$
$x_3x_2 - x_1x_4$	$-uvV'(0)$	$(buv^3 + auv^2 - euv^2 - duv)t^2$
$x_4x_3 - x_1x_4$	$-vU'(0)$	$(gu^2v^3 - 2auv^2 - euv^2 - buv^3)t^2$

Table 6.3: Images of the Six Relations of A under F_1

Theorem 6.5.1. *Let Φ be the map from admissible directions to $\mathrm{HH}^2_2(A)$ determined by the deformations induced by τ_s. Every admissible direction in $\mathfrak{sl}_3$ is sent under Φ to an infinitesimal deformation lying in V_g. Furthermore, the image of these directions is a four dimensional subspace of V_g.*

Proof. We embed the space of admissible directions in $\mathfrak{sl}_3$ into $\mathbb{K}^5$ using the mapping

$$
\begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & 0 & -a-e \end{pmatrix} \leftrightarrow \begin{pmatrix} a \\ b \\ d \\ e \\ g \end{pmatrix}.
$$

The map Φ on admissible directions can be written in the form of a matrix:

$$\mathbb{A} := \begin{pmatrix} x_1x_3 & x_3^2 & -x_1^2 & -x_1x_3 & 0 \\ x_2x_4 & x_4^2 & -x_2^2 & -x_2x_4 & 0 \\ x_2x_3 & x_1x_4 & -x_2x_1 & -x_2x_3 & 0 \\ 3x_2x_3 & 2x_1x_4 & -x_2x_1 & 0 & -x_2x_4 \\ x_1x_4 & x_3x_4 & -x_2x_3 & -x_1x_4 & 0 \\ -x_1x_4 & x_3x_4 & 0 & -x_1x_4 & x_4^2 \end{pmatrix}.$$

We can write this map in terms of the chosen basis $\{b_1, \ldots, b_8\}$ as:

$$\Phi \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ g & 0 & -a-e \end{pmatrix} = (a-e)(b_1+b_4)+b(b_2+b_5)-d(b_3+b_6)+gb_8 \in \mathrm{HH}^2_2(A). \quad (6.8)$$

Note that in (6.8) we have written the cohomology classes of the columns of the matrix $\mathbb{A}$, which therefore may differ from the columns up to a coboundary. By Theorem 5.3.2, this image always lies in V_g and is a four dimensional space. □

From equation (6.8) one can observe that the kernel is spanned by the vector:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We note that the image under Φ in this case is one dimension smaller than that in the case of $\mathbb{P}^1 \times \mathbb{P}^1$.

6.5.2 Other Choices for the Map b

In the above calculation we have chosen a specific map b corresponding to blowing up the point $F = [0:1][1:0]$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and blowing down images of the two rulings through F. For any choice of point $p \in \mathbb{P}^1 \times \mathbb{P}^1$ we obtain a set of such birational maps M_p. M_p is a well understood set (see e.g. [Gat14, Remark 9.29]). The elements in this set differ from each other only by composition with an automorphism of $\mathbb{P}^2$. That is to say if $m_1, m_2 \in M_p$ then there exists $\beta \in \mathrm{Aut}(\mathbb{P}^2)$ such that $m_1 = \beta m_2$. For each $p \in \mathbb{P}^1 \times \mathbb{P}^1$ pick such a birational map b_p. We investigate the effect of each different possible choice of such a p on the image of admissible directions in HH^2_2.

Notation 6.5.2. We establish some notation for σ and $\mathbb{P}^1 \times \mathbb{P}^1$; this notation is in keeping with that introduced in [RS12]. For $[x:y][z:w]$ coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ we let

$$\begin{aligned} X &:= \mathbb{V}(x) = [0:1] \times \mathbb{P}^1, \quad Y := \mathbb{V}(y) = [1:0] \times \mathbb{P}^1, \\ Z &:= \mathbb{V}(z) = \mathbb{P}^1 \times [0:1], \quad W := \mathbb{V}(w) = \mathbb{P}^1 \times [1:0]. \end{aligned}$$

We also name the four points of intersection:

$$\begin{aligned} P &:= Z \cap X = [0:1][0:1], \quad Q := Z \cap Y = [1:0][0:1], \\ F &:= W \cap X = [0:1][1:0] \quad G := W \cap Y = [1:0][1:0]. \end{aligned}$$

Note then that the fundamental points of σ are precisely Q and F whilst G and P are the fundamental points of σ^{-1}.

To make the calculations easier to follow we distinguish the following cases of choice for investigation:

1. $p = Q$ the other fundamental point of σ.
2. $p = G$ or $p = P$ the two fundamental points of σ^{-1}.
3. $p \in X \setminus \{P, F\}$ or $p \in Y \setminus \{G, Q\}$.
4. $p \in Z \setminus \{P, Q\}$ or $p \in W \setminus \{F, G\}$.
5. $p \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus (X \cup Y \cup Z \cup W)$, i.e. p is a point off the "axes".

Proposition 6.5.3. *Let $\alpha : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the automorphism defined by*

$$\alpha([x:y][z:w]) = [y:x][w:z].$$

Then if $\alpha(p) = q$ then the image of the admissible directions arising from automorphisms of $\mathbb{P}^2$ under the map b_p is equal to that under b_q.

Before proving this proposition we state a useful lemma for Ore extensions.

Lemma 6.5.4 ([GW04, Lemma 1.11 and Exercise 1N])**.** *If R is a ring, f and g automorphisms of R then we have the following isomorphism of Ore extensions:*

$$R[t, t^{-1}; g] \cong R[t, t^{-1}; f \circ g \circ f^{-1}].$$

Proof of Proposition 6.5.3. Each choice of map $b_p \in \mathbb{P}^1 \times \mathbb{P}^1$ defines an injection

$$\iota_p : \mathrm{Aut}(\mathbb{P}^2) \hookrightarrow \mathrm{Aut}(\mathbb{K}(u, v)).$$

We claim that if $\alpha(p) = q$ then

$$\iota_p(\mathrm{Aut}(\mathbb{P}^2)) = \alpha^* \iota_q(\mathrm{Aut}(\mathbb{P}^2)) \alpha^*.$$

Since $\alpha(p) = q$, we know that $b_p\alpha$ is a birational map with unique fundamental point q. It follows that $b_p\alpha \in M_q$, and so there exists an automorphism $\beta \in \mathrm{Aut}(\mathbb{P}^2)$ satisfying

$$\beta b_q = b_p\alpha.$$

In particular $b_p\alpha b_q^{-1} = \beta$ is an automorphism of $\mathbb{P}^2$ and for $f \in \mathrm{Aut}(\mathbb{P}^2)$ we have

$$\iota_p(\beta^{-1} f \beta) = \alpha^* \iota_q(f)\alpha^*.$$

The claim follows immediately.

Therefore, if τ_s^* is some family of automorphisms arising from the choice of coordinates from b_p, then there exists some family ρ_s^* arising from the choice of coordinates from b_q such that:

$$\tau_s^* = \alpha^* \rho_s^* \alpha^*.$$

Note that since α is an involution, $(\alpha^*)^{-1} = \alpha^*$. Furthermore,

$$\alpha^* \sigma^* \alpha^* = \sigma^*.$$

Therefore, applying Lemma 6.5.4 we have the isomorphism

$$\begin{aligned} \mathbb{K}(u,v)[t,t^{-1};\sigma^* \circ \tau_s^*] &= \mathbb{K}(u,v)[t,t^{-1};\sigma^* \circ \alpha^* \circ \rho_s^* \circ \alpha^*] \\ &= \mathbb{K}(u,v)[t,t^{-1};\alpha^* \circ \sigma^* \circ \rho_s^* \circ \alpha^*] \\ &\cong \mathbb{K}(u,v)[t,t^{-1};\sigma^* \circ \rho_s^*]. \end{aligned}$$

In other words any infinitesimal deformation arising from studying b_q will also arise from studying b_p, and so the associated spaces of infinitesimal deformations will are equal.

□

By Proposition 6.5.3, in Cases 2, 3 and 4 we only need consider one of the two possible options. Furthermore, this proposition means that we have already considered Case 1 in the preceding work of Section 6.5.1.

We elaborate here on the final case in detail. The details of the calculations for the other cases can be found in Appendix C.2, whereas the results are recorded in Table 6.4 at the end of this section.

Case 5

In this case $p \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus (X \cup Y \cup Z \cup W)$, i.e. p is a point away from the four 'axes'. Choosing a point to blow up is equivalent to choosing a point on $\mathcal{Q}$ to project from. We use the coordinates $[A : B : C]$ on $\mathbb{P}^2$, $[\alpha : \beta : \gamma : \delta]$ for coordinates on $\mathbb{P}^3$ (in which

$\mathcal{Q}$ is embedded) and $[x:y][z:w]$ as coordinates for $\mathbb{P}^1 \times \mathbb{P}^1$. We will use the same τ_s as before, defined by:

$$\tau_s := \begin{pmatrix} (1+as) & bs & cs \\ ds & (1+es) & fs \\ gs & hs & (1-as-es) \end{pmatrix} + O(s^2).$$

We also point out that in order for an element of the Lie algebra to be inadmissible we need to find terms that do not lie in A. We can determine these terms simply by observing the powers of u and v appearing in $U'(0)$ and $V'(0)$.

If $p \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus (X \cup Y \cup Z \cup W)$ then we can write it as $p = [1:M][1:N]$ for some $M, N \in \mathbb{K}^*$. This corresponds to the point $[N:1:MN:M]$ on $\mathcal{Q}$ and so we project from this point.

This is then the map $[\alpha:\beta:\gamma:\delta] \dashrightarrow [\alpha - N\beta : \gamma - MN\beta : \delta - M\beta]$. Composing this with the Segre embedding gives us the map:

$$b_p : [x:y][z:w] \dashrightarrow [xw - Nxz : yw - MNxz : yz - Mxz].$$

Note that since

$$b_p^{-1} : [A, B, C] \dashrightarrow [A : B - NC][C : B - MA],$$

this map induces $u = \frac{A}{B-NC}$ and $v = \frac{C}{B-MA}$ as coordinates on $\mathbb{P}^2$. Therefore $\sigma_{\mathbb{P}^2}$ is the following composition:

$$\begin{aligned} \sigma_{\mathbb{P}^2} : [A:B;C] &\dashrightarrow [A : B - NC][C : B - MA] \\ &\overset{\sigma}{\dashrightarrow} [AC : (B-NC)(B-MA)][C : B - MA] \\ &\dashrightarrow [AC(B-MA) - NAC^2 : (B-NC)(B-MA)^2 - MNAC^2 : \\ &\quad (B-NC)(B-MA)C - MAC^2]. \end{aligned}$$

We note that

$$\begin{aligned} u &\mapsto \frac{AC(B-MA) - NAC^2}{(B-NC)(B-MA)^2 - MNAC^2 - N(B-NC)(B-MA)C + MNAC^2} \\ &= (\frac{A}{B-NC})(\frac{C}{B-MA}) = uv, \end{aligned}$$

and

$$\begin{aligned} v &\mapsto \frac{(B-NC)(B-MA)C - MAC^2}{(B-NC)(B-MA)^2 - MNAC^2 - MAC(B-MA) + MNAC^2} \\ &= \frac{C}{B-MA} = v. \end{aligned}$$

We record the following formulae which are easily derivable from the above:

$$\frac{C}{B} = \frac{Muv - v}{MNuv - i}, \ \frac{A}{B} = \frac{Nuv - u}{MNuv - 1}.$$

Then $\sigma^* \circ \tau_s^*$ has the following effect (up to degree 1) on v:

$$u \overset{\tau_s^*}{\longmapsto} \frac{(1+as)A + bsB + csC}{dsA + (1+es)B + fsC - N(gsA + hsB + (1-as-es)C)}$$
$$= \frac{(1+as)(Nuv-u) + bs(MNuv-1) + cs(Muv-v)}{\chi(s)}$$

where

$$\chi(s) = ds(Nuv-u) + (1+es)(MNuv-1) + fs(Muv-v) - \\ N(gs(Nuv-u) + hs(MNuv-1) + (1-as-es)(Muv-v)).$$

Under σ this is sent to $F(s)/G(s)$ where:

$$F(s) = (1+as)(Nuv^2 - uv) + bs(MNuv^2 - 1) + cs(Muv^2 - v),$$

and

$$G(s) = ds(Nuv^2 - uv) + (1+es)(MNuv^2 - 1) + fs(Muv^2 - v) - \\ N(gs(Nuv^2 - uv) + hs(MNuv^2 - 1) + (1-as-es)(Muv^2 - v)).$$

In preparation for using the quotient rule to differentiate this we calculate some intermediate values.

$$F(0) = Nuv^2 - uv, \ \ F'(0) = a(Nuv^2 - uv) + b(MNuv^2 - 1) + c(Muv^2 - v),$$

$$G(0) = NV - 1, \ G'(0) = d(Nuv^2 - uv) + e(MNuv^2 - 1) + f(Muv^2 - v) - \\ N(g(Nuv^2 - uv) - h(MNuv^2 - 1) + (a+e)M(uv^2 - v).$$

Applying these formula gives us the following result:

$$\begin{aligned}\partial_s(\sigma^* \circ \tau_s^*(u))|_{s=0} = auv + aNuv\frac{Muv^2 - v}{Nv - 1} + b\frac{MNuv - 1}{Nv - 1} + c\frac{Muv^2 - v}{Nv - 1} - du^2v^2 \\ - fuv\frac{Muv - v}{Nv - 1} - Nguv - Nhuv\frac{MNuv^2 - 1}{Nv - 1} \\ + eNuv\frac{Muv^2 - v}{Nv - 1} - euv\frac{MNuv^2 - 1}{Nv - 1}.\end{aligned}$$

Then because of the denominators appearing above, we must have that $a = b = c = f = h = 0$ in order for this to be admissible.

Turning then to the calculation with respect to v:

$$\begin{aligned} v &\overset{\tau_s^*}{\longmapsto} \frac{gsA + hsB + (1 - as - es)C}{dsA + (1 + es)B + fsC - M((1 + as)A + bsB + csC)} \\ &= \frac{gsA + (1 - es)C}{dsA + (1 + es)B - MA} \\ &= \frac{gs(Nuv - u) + (1 - es)(Muv - v)}{ds(Nuv - u) + (1 + es)(MNuv - 1) - M(Nuv - u)} \\ &\overset{\sigma}{\longmapsto} \frac{gs(Nuv^2 - uv) + (1 - es)(Muv^2 - v)}{ds(Nuv^2 - uv) + (1 + es)(MNuv^2 - 1) - M(Nuv^2 - uv)}\end{aligned}$$

Which has the following derivative:

$$\begin{aligned}\partial_s(\sigma^* \circ \tau_s^*(v))|_{s=0} = g\frac{Nuv^2 - uv}{Muv - 1} \\ - e\frac{(Muv - v)(Muv + MNuv^2 - 1)}{(Muv - 1)^2} \\ - d\frac{(Muv - v)(Nuv^2 - uv)}{(Muv - 1)^2}\end{aligned}$$

Similarly to above, just by looking at the denominators one can see that no cancellation will occur here and indeed it is required that e, d and g are all 0 for this to be an admissible direction. Therefore in this case the image of the admissible direction is the trivial deformation.

Conclusion

The results of the calculations above (and those in Appendix C.2) are recorded in Table 6.4.

Case	Dimension of Image in HH^2_2
1) $p = Q$	4
2) $p := G$ or $p = P$	4
3) $p \in X \setminus \{P, F\}$ or $p \in Y \setminus \{G, Q\}$	0
4)$p \in Z \setminus \{P, Q\}$ or $p \in W \setminus \{F, G\}$	1
5)$p \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus (X \cup Y \cup Z \cup W)$	0

Table 6.4: Dimension of the Image of the Admissible Directions for other choices of b

We can summarise the results in the following proposition.

Proposition 6.5.5. *For any choice of point $p \in \mathbb{P}^1 \times \mathbb{P}^1$ and birational map b_p, the set of admissible directions in $\mathfrak{sl}_3$ is sent under Φ to infinitesimal deformations of A lying in V_g.*

6.6 Infinitesimals Arising from Automorphisms of $\mathbb{F}_n$

In this section we turn to the higher Hirzebruch surfaces. We follow the procedure of Section 6.3 and choose a birational map $b : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{F}_n$. Since we found in Proposition 6.5.5 and Table 6.4 that the best choice of b was one that blows one of the fundamental points of σ we only consider one such map in this section. We take Proposition 6.5.5 as evidence that this will not reduce the space of infinitesimals we come across.

6.6.1 Calculating $\sigma_{\mathbb{F}_n}$

In what follows we assume $n \geq 2$.

Recall that a Hirzebruch surface, denoted $\mathbb{F}_n$, is the projective variety given as a subvariety of $\mathbb{P}^{n+3}$ by:

$$\mathbb{F}_n = \left\{ [x_0 : x_1 : \ldots : x_{n+1} : y_0 : y_1] \,\middle|\, \operatorname{rank} \begin{pmatrix} x_0 & x_1 & \ldots & x_n & y_0 \\ x_1 & x_2 & \ldots & x_{n+1} & y_1 \end{pmatrix} = 1 \right\}.$$

We first wish to calculate $\sigma_{\mathbb{F}_n}$. There is a natural map from $\mathbb{F}_n$ to $\mathcal{Q}$, the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ as a quadric surface. Since $\operatorname{rank} \begin{pmatrix} x_0 & x_1 & \ldots & x_n & y_0 \\ x_1 & x_2 & \ldots & x_{n+1} & y_1 \end{pmatrix} = 1$, this implies that the minor $\begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \end{pmatrix}$ has determinant 0. This means that:

$$[x_0 : x_1 : \ldots : y_1] \dashmapsto [x_0 : x_1 : y_0 : y_1].$$

is a birational map $\mathbb{F}_n \dashrightarrow \mathcal{Q}$, and so we can use this to transfer σ. We note that the requirement on the rank of the matrix is equivalent to the fact that the ratio from x_0 to x_1 is the same as the ratio from x_i to x_{i+1}. This allows us to define the inverse birational map as:

$$[x_0 : x_1 : y_0 : y_1] \dashmapsto [x_0 : x_1 : x_1^2/x_0 : x_1^3/x_0^2 : \ldots : y_0 : y_1].$$

We can now calculate $\sigma_{\mathbb{F}_n}$ as the following composition:

$$[x_0 : x_1 : \ldots : y_1] \dashmapsto [x_0 : x_1 : y_0 : y_1] \dashmapsto [x_1 : x_1^2/x_0 : y_0 : y_1] \dashmapsto [x_1 : x_1^2/x_0 : x_1^3/x_0^2 \ldots : y_1],$$

or more simply:

$$[x_0 : x_1 : \ldots : y_1] \dashmapsto [x_1 : x_2 : \ldots : x_{n+1} : x_{n+1}\frac{x_1}{x_0} : y_0 : y_1].$$

As a check on these calculations we verify that $v = x_1/x_0$ gets sent to $(\frac{x_1^2}{x_0})\frac{1}{x_1} = v$ and that $u = x_0/y_0$ gets sent to $\frac{x_1}{y_0} = \frac{x_1}{x_0}\frac{x_0}{y_0} = uv$ as required.

6.6.2 Automorphisms of $\mathbb{F}_n$ for $n \geq 2$

Firstly we recall some definitions, and then relate these to the Hirzebruch surfaces. The weighted projective space $\mathbb{P}(1,1,n)$ is defined similarly to $\mathbb{P}^2$, in that it is a quotient of $\mathbb{K}^3$. However, the equivalence relation is slightly different, in that for any nonzero $\lambda \in \mathbb{K}$

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda^n z).$$

In order to make it clear what kind of space a point is in, we write $[a, b, c]$ for a point in $\mathbb{P}(1,1,n)$ and $[a : b : c]$ for a point in $\mathbb{P}^2$. It is well known (e.g. [Dol82, Section 1.2.3]) that $\mathbb{F}_n$ is the blow up of $\mathbb{P}(1,1,n)$ at the unique singular point $p := [0,0,1]$. The following is well known to experts but we have not found a reference for it so we prove it here.

Lemma 6.6.1. *Let X be a projective surface with a unique singular point, p, and $\widetilde{X}$ the blow up at this point, such that $\widetilde{X}$ has a unique divisor of self-intersection $-n$ whose image under the blow up map is p. Then there is an isomorphism between* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(\widetilde{X})$ *induced by the blow up map.*

Proof. Let $\pi : \widetilde{X} \to X$ be the blow up map and consider $\alpha \in \mathrm{Aut}(X)$. Since p is the unique singular point of X, it must be the case that α fixes p. By the universal property of blow ups [Har77, Proposition 7.14] we know that there is a unique morphism $\widetilde{\alpha} : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \widetilde{\alpha} = \alpha \circ \pi$. The same reasoning applies to α^{-1}, which means we have a morphism $\widetilde{\alpha} \circ \widetilde{(\alpha^{-1})}$ which satisfies:

$$\pi \circ \widetilde{\alpha} \circ \widetilde{(\alpha^{-1})} = \alpha \circ \alpha^{-1} \circ \pi = 1 \circ \pi.$$

The universal property means that $\widetilde{\alpha} \circ \widetilde{(\alpha^{-1})}$ is unique, so that it must be the identity.

Therefore $\widetilde{\alpha}$ is an automorphism of $\widetilde{X}$.

On the other hand, any automorphism of $\widetilde{X}$ must fix the unique divisor of self-intersection $-n$. Therefore it also defines an automorphism of the blow down of this divisor, which is X. Therefore $\mathrm{Aut}(\widetilde{X}) \cong \mathrm{Aut}(X)$. □

In our setting, this lemma means that the group of automorphisms of $\mathbb{P}(1,1,n)$ is isomorphic to the group of automorphisms of $\mathbb{F}_n$. We need to describe the blow up explicitly though, which is much easier if we embed $\mathbb{P}(1,1,n)$ in projective space $\mathbb{P}^{n+1}$, which can be achieved by using a Veronese-type embedding:

$$[x,y,z] \mapsto [x^n : x^{n-1}y : \ldots : y^n : z].$$

Explicitly, this blow up is given by the following map: $\pi : \mathbb{F}_n \dashrightarrow \mathbb{P}(1,1,n)$ where a point $[x_0 : x_1 : \ldots : x_{n+1} : y_0 : y_1]$ is taken to the equivalence class of any nonzero vector in the row space of $\begin{pmatrix} x_0 & x_1 & \ldots & x_n & y_0 \\ x_1 & x_2 & \ldots & x_{n+1} & y_1 \end{pmatrix}$, which has rank one by definition. Now, at a generic point in $\mathbb{F}_n$ this map will be,

$$[x_0 : x_1 : \ldots : x_{n+1} : y_0 : y_1] \mapsto [x^n : x^{n-1}y : \ldots : y^n : z] \text{ where } x^{n-i}y^i = x_0 \left(\frac{x_1}{x_0}\right)^i,$$

and this image then is isomorphic to $\mathbb{P}(1,1,n)$ by using the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^n$ which is well defined since:

$$[\lambda x, \lambda y, \lambda^n z] \mapsto [\lambda^n x^n : \lambda^n x^{n-1}y : \ldots : \lambda^n y^n : \lambda^n z] = [x^n : x^{n-1}y : \ldots : y^n : z].$$

The fact that this morphism is an isomorphism follows from the fact that if w is an nth root of unity, then $[x,y,z] = [wx, wy, z]$ and so this map has an inverse represented by choosing an nth root of unity.

From [DI09, Theorem 4.10] we have that automorphisms of $\mathbb{P}(1,1,n)$ are given by

$$\rho : \ [x,y,z] \mapsto [ax+by, cx+dy, ez + \sum_{i=0}^{n} f_i x^{n-i} y^i],$$

where $a,b,c,d,e,f_i \in \mathbb{K}$ for every i, with the restrictions that e is nonzero and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible. We need to transfer such a map, τ say, up to $\mathbb{F}_n$. To do so we firstly take it across to $\mathbb{P}^{n+1}$ via the Veronese map above. This is calculated as the following composition:

$$[x^n : x^{n-1}y : \ldots : y^n : z] \mapsto [x, y, z] \mapsto [ax+by, cx+dy, ez + \sum_{i=0}^{n} f_i x^{n-i} y^i]$$
$$\mapsto [(ax+by)^n : (cx+dy)(ax+by)^{n-1} : \ldots : (cx+dy)^n : ez + \sum_i f_i x^{n-i} y^i].$$

For ease of reading we set $A_i := (cx+dy)^i(ax+by)^{n-i}$ for i from 1 to n and $A_{n+1} := ey_0 + \sum f_i x^{n-i}y^i$. Then we can transfer this back up to $\mathbb{F}_n$ by using the blow up map. If we take a point $[x_0 : \ldots : x_{n+1} : y_0 : y_1]$ and act on it by σ then we get the point $[x_1 : x_2 \ldots : x_{n+1}\frac{x_1}{x_0} : y_0 : y_1]$. We assume we are in the open set where x_0 is nonzero, then in the above equations we choose x and y so that

$$x^{n-i}y^i = x_0 \left(\frac{x_1}{x_0}\right)^{i+1}.$$

This means that under τ this point is sent to:

$$\tau \circ \sigma[x_0 : \ldots : x_{n+1} : y_0 : y_1] = [A_0 : A_1 : \ldots : A_n \frac{A_1}{A_0} : A_{n+1} : A_{n+1}\frac{A_1}{A_0}].$$

Now, the identity automorphism in this situation corresponds to setting

$$(a, b, c, d, e, f_0, \ldots, f_n) := (1, 0, 0, 1, 1, 0, \ldots, 0),$$

so that perturbing in one direction with parameter s corresponds to a choice of automorphism given by $(1+as, bs, cs, 1+ds, 1+es, sf_0, \ldots, sf_n)$. This gives us the following formulae for $U(s)$ and $V(s)$:

$$\begin{aligned} U(s) &= \frac{A_0}{A_{n+1}} \\ &= \frac{((1+as)x + bsy)^n}{(1+es)y_0 + \sum_i^n f_i x^{n-i} y^i} \end{aligned}$$

and

$$\begin{aligned} V(s) &= \frac{A_1}{A_0} \\ &= \frac{(csx + (1+ds)y)((1+as)x + bsy)^{n-1}}{((1+as)x + bsy)^n}. \end{aligned}$$

Taking each in turn, we have the following:

$$\begin{aligned} V(0) &= \frac{yx^{n-1}}{x^n} \\ &= \frac{uv^2}{uv} = v. \end{aligned}$$

As for the derivative, we start by multiplying top and bottom by $1/y_0$ so that everything is in terms of u's and v's. The numerator has a derivative at zero of:

$$cuv + duv^2 + (n-1)auv^2 + (n-1)buv^3,$$

whereas the denominator has derivative:

$$nauv + nbuv^2.$$

This then implies a derivative of:

$$\begin{aligned} V'(0) &= \frac{uv(cuv + duv^2 + (n-1)auv^2 + (n-1)buv^3) - uv^2(nauv + nbuv^2)}{(uv)^2} \\ &= c + dv - av - bv^2. \end{aligned}$$

Likewise we can calculate $U'(0)$ in a similar fashion. The derivative of the numerator at zero is

$$nauv + nbuv^2$$

and of the denominator is:

$$e + \sum f_i uv^{i+1},$$

which gives a derivative of:

$$nauv + nbuv^2 - euv - \sum_i^n f_i u^2 v^{i+2}.$$

We can now consider the admissibility of L. Since $-vU'(0)t^2$ will contain the terms $\sum_i f_i u^2 v^{i+3} t^2$, for this to be admissible we certainly need $f_i = 0$ for any i greater than 0, since no powers of v higher than 3 appear in A_2. These are the only restrictions on admissibility. For this reason we write f for f_0 and record the results in Table 6.5.

Relation	Image Under F_1
$x_3x_1 - x_1x_3$	$(av + bv^2 - dv - c)t^2$
$x_2x_4 - x_4x_2$	$(bu^2v^3 + au^2v^2 - du^2v^2 - cu^2v)t^2$
$x_4x_1 - x_2x_3$	$(buv^2 + auv - duv - cu)t^2$
$x_1x_2 - x_2x_3$	$((n+1)buv^2 + (n+1)auv - fu^2v^2 - duv - euv - cu)t^2$
$x_3x_2 - x_1x_4$	$(buv^3 + auv^2 - duv^2 - cuv)t^2$
$x_4x_3 - x_1x_4$	$(-nbuv^3 + fu^2v^3 - nauv^2 + euv^2)t^2$

Table 6.5: Images of the Six Relations of A under F_1

Theorem 6.6.2. *Let Φ be the map from admissible directions to $\mathrm{HH}^2_2(A)$ determined by the deformations induced by τ_s. All admissible directions of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{F}_n))$ are sent under to Φ to infinitesimal deformations lying in V_g. Furthermore, the image of the space of admissible directions is a four dimensional subspace of V_g which is independent of n.*

In fact we shall see that the image of the space of admissible directions is precisely the same four dimensional subspace as arose in the case of $\mathbb{P}^2$ in Theorem 6.5.1.

Proof. The Lie algebra of admissible vectors is spanned by vectors of the following form which we embed in $\mathbb{K}^6$:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix}\right) \leftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$$

Then the map Φ can be written as the following matrix:

$$\mathbb{A} := \begin{pmatrix} x_1x_3 & x_3^2 & -x_1^2 & -x_1x_3 & 0 & 0 \\ x_2x_4 & x_4^2 & -x_2^2 & -x_2x_4 & 0 & 0 \\ x_2x_3 & x_1x_4 & -x_2x_1 & -x_2x_3 & 0 & 0 \\ (n+1)x_2x_3 & (n+1)x_1x_4 & -x_2x_1 & -x_2x_3 & -x_2x_3 & -x_2x_4 \\ x_1x_4 & x_3x_4 & -x_2x_3 & -x_1x_4 & 0 & 0 \\ -nx_1x_4 & -nx_3x_4 & 0 & 0 & x_1x_4 & x_4^2 \end{pmatrix}.$$

We can write this in terms of the chosen basis of $\mathrm{HH}^2_2(A)$ as:

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e \\ f \end{pmatrix}\right) = (a-d)(b_1+b_4) + b(b_2+b_5) - c(b_3+b_6) + fb_8 \in \mathrm{HH}^2_2(A).$$

We can see by reference to Theorem 5.3.2 that the image of Φ lies in V_g, and it is obviously four dimensional. □

As one can see, the kernel of this map is spanned by

$$\left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right\}.$$

In this case then, the admissible space is just as large as in the $\mathbb{P}^1 \times \mathbb{P}^1$ case, but the kernel is one dimension larger. Again we find that the $\mathbb{P}^1 \times \mathbb{P}^1$ case includes all of the infinitesimal deformations that we find in this case.

6.7 Closing Remarks

The results of this chapter can be summarised in the following table:

Surface Y	Dimension of admissible space	Dimension of image in HH^2_2
$\mathbb{P}^1 \times \mathbb{P}^1$	6	5
$\mathbb{P}^2$	5	4
$\mathbb{F}_n,\ n \geq 2$	6	4

Interestingly, we have found that all of the infinitesimal deformations that are induced by Lie algebras of automorphisms of surfaces lie in the same variety in HH^2_2. This is not a priori obvious and may hint at some underlying structure. We have also seen the utility of Theorem 6.2.9 in reducing an otherwise difficult infinite dimensional problem to a tractable, finite dimensional one.

The takeaway result is that although we have considered a much larger set of deformations than in [RS12], we have found no new infinitesimal deformations of A. In order to find new deformations of A one must look away from the automorphisms of surfaces birational to $\mathbb{P}^1 \times \mathbb{P}^1$, and also away from V_g. We turn our attention to such deformations now.

Chapter 7

Deformations of A_q Arising from Quantum Analogues of Geometric Automorphisms

7.1 Introduction

In this chapter we examine the infinitesimal deformations of the algebra A_q and relate these back to deformations of A. We do this by mimicking the work on geometric automorphisms of $\mathbb{K}(u, v)$ (see Chapter 6) but in the context of A_q. Recall from Section 3.4 that $\sigma \in \mathrm{Aut}(\mathbb{K}_q(u, v))$ is the automorphism defined by

$$\sigma(u) = uv \text{ and } \sigma(v) = v.$$

Then in this chapter we wish to study deformations of

$$Q_{gr}(A_q) = \mathbb{K}_q(u, v)[t, t^{-1}; \sigma]$$

which arise from quantum analogues of geometric automorphisms of $\mathbb{K}(u, v)$.

The automorphism group of $\mathbb{K}_q(u, v)$ is in general not well understood (see e.g. [AC99] or [Fry14]). However, Alev and Dumas [AD95] have carried out a study of subgroups of automorphisms of $\mathbb{K}_q(u, v)$ that correspond precisely to 'quantised' automorphisms of $\mathbb{K}(u, v)$. For this reason we discuss this paper at length in Section 7.2 before applying their work in Section 7.3.

We find that these deformations correspond to a four dimensional space of infinitesimal deformations of A_q. Since by Theorem 3.4.3 $\mathrm{HH}^2_2(A_q)$ is four dimensional, these deformations have infinitesimals that comprises all of $\mathrm{HH}^2_2(A_q)$. Furthermore, taking the semi-classical limit $q \to 1$ we obtain a 2 dimensional space of infinitesimal deformations of the algebra A which lies in the set V_q (see Section 5.3).

7.2 A Discussion of a Paper of Alev and Dumas

Any definitions and propositions from this section are taken directly from [AD95]. This paper is written in French and we have translated any quoted material here. For this reason we do not include citations for every proposition and definition for this section.

7.2.1 Overview

The paper [AD95] is an examination of the automorphism group of a few skew fields of interest. The relevant portions of the paper concern $\mathbb{K}_q(u,v)$, the division ring of the quantum plane $\mathbb{K}_q[u,v]$. We discuss their results here as we wish to apply them in the context of the algebra A_q.

The main idea of the work is to quantise the structure of the Cremona group. Two subgroups of the Cremona group are singled out in particular. Following from this, quantum analogues of those subgroups are identified in the quantum Cremona group.

Definition 7.2.1. The *quantum Cremona group* is the group of automorphisms of the division ring $\mathbb{K}_q(u,v)$.

7.2.2 Subgroups of the Classical Cremona Group

We have already come across several subgroups of the Cremona group in Chapter 6. Alev and Dumas bring particular attention to one of the subgroups appearing in Chapter 6: the automorphism group of $\mathbb{P}^2$.

Definition 7.2.2. We define Z to be the subgroup of $\mathrm{Aut}(\mathbb{K}(u,v))$ isomorphic to $\mathrm{PGL}_3(\mathbb{K})$ where a matrix

$$M := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

acts by

$$M(u) = \frac{au+bv+c}{gu+hv+i} \text{ and } M(v) = \frac{du+ev+f}{gu+hv+i}.$$

Alev and Dumas also draw attention to a second subgroup of the Cremona group, which we have not considered before, although it contains the identity component of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$.

Definition 7.2.3. For any $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{K})$ and $\begin{pmatrix} a(u) & c(u) \\ b(u) & d(u) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{K}(u))$ we define an automorphism $\theta \in \mathrm{Aut}(\mathbb{K}(u,v))$ by

$$\theta(v) = \frac{a(u)v+b(u)}{c(u)v+d(u)} \text{ and } \theta(u) = \frac{\alpha u+\beta}{\gamma u+\delta}.$$

These automorphisms form a subgroup of $\mathrm{Aut}(\mathbb{K}(u,v))$ which we refer to as Y. Note that Y is precisely the subgroup of automorphisms θ of $\mathbb{K}(u,v)$ that satisfy

$$\theta(\mathbb{K}(u)) = \mathbb{K}(u).$$

7.2.3 Subgroups of the Quantum Cremona Group

We now turn our attention to the quantum analogues of the above groups.

Definition 7.2.4. For any $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $(\alpha,\beta) \in (\mathbb{K}^*)^2$ we define an automorphism ψ of $\mathbb{K}_q(u,v)$ by

$$\psi(v) = \alpha u^b v^a \text{ and } \psi(u) = \beta u^d v^c.$$

Automorphisms defined in this manner form a subgroup of $\mathrm{Aut}(\mathbb{K}_q(u,v))$, which we call H.

Note that by Proposition 1.6 of [AD96], this group is precisely the extension to $\mathbb{K}_q(u,v)$ of the automorphism group of the quantum torus $\mathbb{K}_q[u^{\pm 1}, v^{\pm 1}]$.

Alev and Dumas show that there is a subgroup of H which is the quantum analogue of $Z \cong \mathrm{PGL}_3(\mathbb{K})$.

Proposition 7.2.5. *[AD95, Proposition 1.5] Let $C \leq H$ be the subgroup of those automorphisms whose defining matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ lies in the cyclic subgroup generated by $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Then C is the subgroup of $\mathrm{Aut}(\mathbb{K}_q(u,v))$ of elements θ such that*

$$\theta(v) = UW^{-1} \text{ and } \theta(u) = VW^{-1},$$

with $U, V, W \in \mathbb{K}_q[u,v]$ nonzero elements of degree at most one.

Definition 7.2.6. For $\alpha \in \mathbb{K}^*$ and $f \in \mathbb{K}(u)^*$ we define an automorphism $\mu \in \mathrm{Aut}(\mathbb{K}_q(u,v))$ by

$$\mu(u) = \alpha u \text{ and } \mu(v) = f(u)v.$$

These automorphisms form a group which we call B^+. Likewise we define B^- to be the group of automorphisms of the form

$$\lambda(u) = ug(v) \text{ and } \lambda(v) = \beta v$$

for some $\beta \in \mathbb{K}^*$ and $g \in \mathbb{K}(v)^*$.

Let ω be the involution defined by $\omega(u) = u^{-1}$ and $\omega(v) = v^{-1}$. Then we define B to be the subgroup generated by the elements of B^+ along with ω.

The following proposition is the statement that B is a quantum analogue of Y.

Proposition 7.2.7. *[AD95, Proposition 1.4] B is equal to the subgroup of* $\mathrm{Aut}(\mathbb{K}_q(u,v))$ *consisting of θ such that the restriction of θ to $\mathbb{K}(u)$ is an automorphism of $\mathbb{K}(u)$.*

7.3 Infinitesimal Deformations of A_q Arising from the Quantum Cremona Group

We consider deformations of $\mathbb{K}_q(u,v)[t,t^{-1};\sigma]$ that arise by composing σ with a one-parameter subgroup $\{\tau_s\} \subseteq \mathrm{Aut}(\mathbb{K}_q(u,v))$. For examples of this kind of deformation in the birationally commutative setting see Chapter 6. By Theorem 6.2.9 we can test whether a deformation of $\mathbb{K}_q(u,v)[t,t^{-1};\sigma]$ corresponds to an infinitesimal deformation of A_q by verifying that the image under the infinitesimal of the relations R_q lies in A_q.

The τ_s we consider will also define automorphisms of $\mathbb{K}_{q'}(u,v)$ for any $q' \neq 0 \in \mathbb{K}$. For that reason we consider a general case of deformation to a family of the form

$$\mathbb{K}_{q'}(u,v)[t,t^{-1};\sigma \circ \tau_s],$$

where by an abuse of notation we write σ and τ_s for automorphisms of $\mathbb{K}_{q'}(u,v)$ with q' varying. In this family we write $q' = qe^{\lambda s}$ for some $\lambda \in \mathbb{K}$, so that up to first order we have the following equation:

$$vu = q(1+\lambda s)uv.$$

In this way we allow both q' and τ_s to vary with s and we will have a formal deformation in the variable s.

Taking a lead from Alev and Dumas, we consider τ_s as lying in one of the subgroups discussed in Section 7.2. We first consider the quantum analogue of $\mathrm{Aut}(\mathbb{P}^2)$ and find that τ_s in this case must be a map that scales u and v. The second case of the group B^+ is more complicated and we find every element of HH^2_2 occurs as an infinitesimal of such a deformation.

7.3.1 The Quantum Analogue of Y

We first note that if $\tau_s \in H$ is a one-parameter subgroup of automorphisms of $\mathbb{K}_q(u,v)$ then since τ_0 is the identity the corresponding matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ must be the identity matrix. Therefore, τ_s must be a family of automorphisms of the form:

$$\tau_s(v) = \alpha_s v \text{ and } \tau_s(u) = \beta_s u.$$

Automorphisms of this type also lie in B^+ and so we move to the more general case of $\tau_s \in B^+$.

From here onwards, we consider $\tau_s \in \mathrm{Aut}(\mathbb{K}_q(u,v))$ to be a one-parameter subgroup of automorphisms lying in the group B^+. That is to say that

$$\tau_s(u) = uf_s(v) \text{ and } \tau_s(v) = \alpha_s v,$$

where $f_s(v) \in \mathbb{K}(v)[[s]]^*$ and $\alpha_s \in \mathbb{K}[[s]]^*$ where * is used to denote the invertible elements in the rings. Note that this defines an automorphism of $\mathbb{K}_{q'}(u,v)$ where $q' = qe^{\lambda s}$.

There exist $X \in \mathbb{K}(v)$ and $a \in \mathbb{K}$ such that the following holds up to first order in s:

$$\sigma \circ \tau_s(u) = \sigma(uf_s(v)) = uvf_s(v) = u(v + Xs) \text{ and } \sigma \circ \tau_s(v) = \sigma(\alpha_s v) = (1 + as)v.$$

In particular, up to first order:

$$tu = u(v + Xs)t \text{ and } tv = (1 + as)vt.$$

We therefore look for conditions on X, λ and a for $\mathbb{K}_{q'}(u,v)[t, t^{-1}; \sigma \circ \tau_s]$ to define an infinitesimal deformation of A_q.

Recall that the relations of A_q are:

$$R_q = \left\{ \begin{array}{lll} r_1 := x_3x_1 - x_1x_3, & r_2 := x_4x_2 - qx_2x_4, & r_3 := x_4x_1 - x_2x_3 \\ r_4 := x_1x_2 - x_2x_3, & r_5 := x_3x_2 - qx_1x_4, & r_6 := x_4x_3 - x_1x_4 \end{array} \right\}.$$

The deformed multiplication is determined by a sequence of bilinear functions F_i so that

$$F(a,b) = \sum_i F_i(a,b)s^i.$$

We need to calculate $F_1(R_q)$ and determine if this lies in A_q.

Firstly, consider $r_4 = x_1x_2 - x_2x_3$. Then we obtain the following:

$$\begin{aligned} F_1(r_4) = F_1(tut - utvt) &= \frac{\sigma \circ \tau_s(u)t^2 - u\sigma \circ \tau_s(v)t^2}{s} \\ &= \frac{u(v + Xs)t^2 - uv(1 + as)t^2}{s} \\ &= uXt^2 - auvt^2. \end{aligned} \tag{7.1}$$

Recall that if uv^jt^2 is an element of $(A_q)_2$ then j must satisfy $0 \le j \le 3$. For this reason, since we require that (7.1) lies in $(A_q)_2$ it must be true that

$$X = b + cv + dv^2 + ev^3 \text{ for some } b, c, d, e \in \mathbb{K}.$$

Furthermore, if we now consider $r_6 = x_4x_3 - x_1x_4$ we have the following equations up to first order in s:

$$\begin{aligned} sF_1(r_6) = uvt * vt - t * uvt &= uv\sigma \circ \tau_s(v)t^2 - \sigma \circ \tau_s(uv)t^2 \\ &= uv^2(1+as)t^2 - u(v+sX)(1+as)vt^2 \\ &= -suvXt^2 \text{ up to first order in } s. \end{aligned} \tag{7.2}$$

For (7.2) to lie in $s(A_q)_2$ it must be that $e = 0$, and so

$$X = b + cv + dv^2, \tag{7.3}$$

for some $b, c, d \in \mathbb{K}$.

We omit here the remaining calculations of applying F_1 to the relations as they continue without further complication. They can be found in full in Appendix D.1, in which we have assumed (7.3). The results of these calculations are collected in Table 7.1.

Relation	Image Under F_1	Image as element of A_q
$x_3x_1 - x_1x_3$	$-avt^2$	$-ax_1x_3$
$x_2x_4 - qx_4x_2$	$(q\lambda u^2v^2 - qau^2v^2)t^2$	$q(\lambda - a)x_2x_4$
$x_4x_1 - x_2x_3$	$-auvt^2$	$-ax_2x_3$
$x_1x_2 - x_2x_3$	$(bu + cuv + duv^2 - auv)t^2$	$bx_2x_1 + (c-a)x_2x_3 + dx_1x_4$
$x_3x_2 - qx_1x_4$	$(q\lambda uv^2 - qauv^2)t^2$	$q(\lambda - a)x_1x_4$
$x_4x_3 - x_1x_4$	$(-buv - cuv^2 - duv^3)t^2$	$-bx_2x_3 - cx_1x_4 - \frac{d}{q}x_3x_4$

Table 7.1: Images of the Six Relations of A_q Under F_1

Recall the chosen basis for $\mathrm{HH}^2_2(A_q)$ which we print here.

$$\left\{ \begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ qx_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \right\}$$

We label this basis in order as $e_1, \dots, e_4$. In Appendix D.2 we have written the infinitesimal F_1 in terms of our chosen basis of $\mathrm{HH}^2_2(A_q)$ and coboundaries. From this expansion we can conclude that the cohomology class of the infinitesimal F_1 of A_q

associated to the deformation $\mathbb{K}_{q'}(u,v)[t,t^{-1};\sigma\circ\tau_s]$ is

$$[F_1] = a(-e_1 - qe_2) + (1-q)be_4 - \frac{(1-q)}{q}de_3 + q\lambda e_2. \qquad (7.4)$$

Note that the parameter c does not appear here, since in the expansion in Appendix D.2 c is a coefficient of a coboundary. This is analogous to behaviour we saw in Lemma 6.4.2 where functions that scaled u had trivial associated infinitesimals.

From Equation (7.4) it is clear that by varying a, b, d and λ we can find a deformation that corresponds to any chosen direction in $\mathrm{HH}_2^2(A_q)$. Thus we have proved:

Theorem 7.3.1. *For every isomorphism class of infinitesimal deformations L of A_q there exists a family of deformations of $Q_{gr}(A_q)$ such that the associated infinitesimal F_1 satisfies:*

$$[F_1|_{R_q}] = L.$$

7.3.2 Semi-classical Limits as Deformations of A

The preceding calculations have shown that for a general $q \notin \{0,1\}$ we can find deformations of $Q_{gr}(A_q)$ that correspond to any infinitesimal deformation of A_q. The work can be split into three steps.

(1) Choose a one-parameter subgroup $\{\tau_s\} \subseteq \mathrm{Aut}(\mathbb{K}_q(u,v))$, with each τ_s also defining an automorphism of $\mathbb{K}_{q'}(u,v)$.

(2) Calculate the infinitesimal deformation of A_q associated to the deformation

$$\mathbb{K}_{q'}(u,v)[t,t^{-1};\sigma\circ\tau_s] \text{ of } \mathrm{Q_{gr}}(A_q)$$

(3) Find the cohomology class associated to the infinitesimal calculated in Step (2).

Firstly, we note that the subgroup chosen in Section 7.3 also defines a one-parameter subgroup of $\mathrm{Aut}(\mathbb{K}(u,v))$. Secondly, the calculations in Section 7.3 corresponding to Step (2) were independent of the value of q so long as $q \neq 0$. Therefore, we can move to the semi-classical limit $q \to 1$ without changing the validity of the results in the first two steps.

In Step (3) however, we cannot simply substitute $q = 1$ into all of the equations as the cocycles and coboundaries of A and A_q are different. Therefore, we expand the infinitesimal associated to the deformation in this case in Appendix D.3. We obtain the fact that

$$[F_1] = a\left(\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}\right) + \lambda \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}$$

Therefore we have as kind of 'semi-classical analogue' of Theorem 7.3.1:

Proposition 7.3.2. *For every isomorphism class of infinitesimal deformations L of A lying in the space spanned by:*

$$\left\{\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}\right\}$$

there exists a family of deformations of $Q_{gr}(A)$ whose infinitesimal deformation F_1 satisfies:

$$[F_1|_R] = L.$$

Note that these infinitesimals lie in the set V_q (see Section 5.3 for definition), unlike those discussed in Chapter 6 which lie in V_g. In particular, A_q itself corresponds to varying λ but having the automorphism be the identity. In this way the family of deformations A_q of A corresponds to the infinitesimal

$$\begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \in \mathrm{HH}^2_2(A).$$

Chapter 8

A Family of Deformations of A with the PBW Property

In this chapter we define a new family of algebras that are deformations of the algebra A. This family gives rise to a 3 dimensional space of infinitesimal deformations that lie in the unobstructed component of $\mathrm{HH}^2_2(A)$ we have called V_q. The main theorem is Theorem 8.2.1, which states that algebras in this family are PBW. We also classify elements of this family up to birational equivalence. In particular, in Corollary 8.2.7 we show that these algebras can have the function skew field $\mathbb{K}_q(u,v)$, $\mathbb{K}(u,v)$ or $D_1(\mathbb{K})$ depending on the parameters.

The methods used to discover this family are described in Section 8.1. Since the Hochschild 2-cocycle space is large (22-dimensional), we applied a heuristic search strategy. If the reader prefers they may skip to Section 8.2 for the mathematical content of the chapter.

8.1 A Heuristic Search Approach to Finding Deformations

8.1.1 Overview

Our aim is to find families of deformations of A by deforming the set of relations R. Recall from Definition 1.1.1 that this set is

$$R = \left\{ \begin{array}{lll} r_1 := x_3x_1 - x_1x_3, & r_2 := x_4x_2 - x_2x_4, & r_3 := x_4x_1 - x_2x_3 \\ r_4 := x_1x_2 - x_2x_3, & r_5 := x_3x_2 - x_1x_4, & r_6 := x_4x_3 - x_1x_4 \end{array} \right\}.$$

Instead of approaching the completely general problem, we will restrict attention to deformations which satisfy the PBW property with respect to the deformed relations and the lexicographic ordering given by $x_2 < x_1 < x_3 < x_4$. We call this property P.

In order to do this we will use two tools. The first is Bergman's Diamond Lemma, which allows us to automate testing a set of relations for the PBW property. The second is the basis of the Koszul 2-cocycles calculated in Section 3.3, which will reduce our search space considerably.

Consider the set of PBW reduced monomials in A of degree 2:

$$Z = \left\{ \begin{array}{lllll} z_1 = x_2x_2, & z_2 = x_2x_1, & z_3 = x_2x_3, & z_4 = x_2x_4, & z_5 = x_1x_1, \\ z_6 = x_1x_3, & z_7 = x_1x_4, & z_8 = x_3x_3, & z_9 = x_3x_4, & z_{10} = x_4x_4 \end{array} \right\}.$$

We wish to choose elements $\{\sum_i a_{i,j} z_i\}_{j=1}^6 \subseteq \mathrm{sp}_{\mathbb{K}}(Z)$ so that if we choose deformed relations by setting for each $j \in [1, 6]$

$$r'_j = r_j - \sum_i a_{i,j} z_i$$

we obtain an algebra

$$A' := \frac{\mathbb{K}\langle V \rangle}{(\{r'_1, \ldots, r'_6\})}$$

with property P. One can make small deductions, for example that $a_{8,1} = 0$ since $x_3^2 \not< x_3x_1$. However, this leaves us with a nearly 60 dimensional problem. We make some arguments to reduce this.

Each choice of such an $a_{i,j}$ will define a family of algebras. If this is to be a deformation of A then certainly we expect the vector

$$\begin{pmatrix} \sum_i a_{i,1} z_i \\ \vdots \\ \sum_i a_{i,6} z_i \end{pmatrix}$$

to define a Koszul 2-cocycle, since this will be the restriction of the associated infinitesimal to R. This reduces the search space to a twenty-two dimensional vector space by Theorem 3.3.1.

Since we wish to study nontrivial deformations, we restrict our attention further to those elements of this space with nonzero cohomology class. If we label the 22 vectors of the basis recorded in Appendix A.1.1 as $v_1, \ldots, v_{22}$, then in particular the following are labelled $v_1, \ldots, v_8$:

$$\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_3^2 \\ 0 \\ x_1x_4 \\ 0 \\ 0 \\ x_3x_4 \end{pmatrix} \begin{pmatrix} x_1^2 \\ 0 \\ 0 \\ x_2x_1 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \\ -x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x_2^2 \\ 0 \\ x_2x_4 \end{pmatrix}$$

By inspection we can conclude that the coefficients of v_2 and v_5 must be zero because in the ordering we have chosen $x_3^2 \not< x_3x_1$ and $x_4^2 \not< x_4x_2$. For example, if the coefficient of v_2 is $b \neq 0$ then we have a relation

$$x_3x_1 - x_1x_3 - bx_3x_3.$$

But $x_3^2 \not< x_3x_1$ and so this algebra cannot have property P.

We consider each v_i in turn. Since it is not at all clear that by chance the choice of basis of HH_2^2 we have made is a particularly good one, we make an educated guess as to a set Ξ_i of coboundaries that will 'interact well' with v_i. That is to say that although v_1 itself may not yield an algebra with property P, perhaps if $x \in \Xi_1$ then $v_1 \pm x$ will. This is a heuristic process; the justification is that it works.

For example, for the vector v_1 we choose the following coboundaries as Ξ_1 as those we expect to yield deformations.

$$\left\{ v_{10} = \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -x_1x_4 \\ -x_1x_4 \end{pmatrix}, v_{13} = \begin{pmatrix} 0 \\ 0 \\ -x_2x_3 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \right\}.$$

We make the assertion that if there is a deformation arising from property P with associated cohomology class of v_1 then it will probably arise using a cocycle of the form $v_1 + c_1v_{10} + c_2v_{13}$ where each $c_i \in \{0, 1, -1\}$. To be clear, this is a heuristic argument, not a statement of fact.

Once we have found all of the cocycles of this form which define algebras with the property P we try adding them together. For example, we would try $a(v_1 + v_{10}) + cv_3$. In doing so we can reduce the number of cases to test substantially. We will see that this process does lead to a new family of algebras corresponding to a choice of infinitesimal lying in V_q (see Section 5.3).

8.1.2 Implementation

We describe the implementation of a program to find deformations of A with property P by applying the reasoning from Section 8.1.1. First we need some functionality to test if a cocycle generates an algebra with property P.

We firstly define two functions that reduce an overlap with respect to a set of relations in two different ways. If we have a degree 3 monomial xyz then `reduceRight` will firstly reduce the monomial yz whilst `reduceLeft` will start by reducing xy.

```
def reduceRight(mono, alg):
    firstStep = alg.reduce(mono[1] * mono[2])
    return alg.reduce(mono[0] * firstStep)

def reduceLeft(mono, alg):
    firstStep = alg.reduce(mono[0] * mono[1])
    return alg.reduce(firstStep * mono[2])
```

Given the PBW basis that we are searching for, there are four overlaps that must be tested in order to apply Bergman's Diamond Lemma:

$$\{x_4x_1x_2,\ x_4x_3x_1,\ x_4x_3x_2,\ x_3x_1x_2\}.$$

The following function `differenceOfOverlaps` reduces the overlaps in two separate ways and stores the difference in a list which it returns. In order to make this function more general we have an optional parameter `substitutionFunction` which can be used to give relations between the coefficients of the vectors. For example, we will use this to record the fact that we are looking for relations in which the four coefficients a, c, d and f satisfy

$$af - cd = 0,$$

since this is a defining equation of V_q. If no function is given as a `substitutionFunction` then it is set to be the identity function.

```
def differenceOfOverlaps(alg, substitutionFunction=None):
    if substitutionFunction is None:
        def substitutionFunction(x): return x

    overlaps = [x4 * x1 * x2,
                x4 * x3 * x1,
                x4 * x3 * x2,
                x3 * x1 * x2]
    answer = []
    for overlap in overlaps:
        right = reduceRight(overlap, alg)
        left = reduceLeft(overlap, alg)
        difference =substitutionFunction(right - left)
        answer.append(difference)
    return answer
```

We now give the example of testing for cocycles with nonzero coefficient of v_1. To start we build the list of guesses for cocycles that will lead to deformations with the property P.

```
guesses = [v1            ,
           v1 + v10      ,
           v1 - v10      ,
           v1       + v13,
           v1       - v13,
           v1 + v10 + v13,
           v1 + v10 - v13,
           v1 - v10 + v13,
           v1 - v10 - v13]

guesses = [vec * a for vec in guesses]
```

We now test each of these cocycles in turn and store those that pass the test. The result of the following script is a list, each element of which corresponds to a family of deformations of A that is PBW.

```
goodCocycles = []
for cocycle in guesses:
    deformedAlgebra = makeAlgebra(cocycle)
    differences = differenceOfOverlaps(deformedAlgebra)
    guessWorks = True
    for difference in differences:
        if difference != 0:
            guessWorks = False
            break

    if guessWorks:
        goodCocycles.append(cocycle)
```

The output of this script is the following set:

$$\left\{ \begin{pmatrix} ax_1x_3 \\ 0 \\ 0 \\ ax_2x_3 \\ ax_1x_4 \\ 0 \end{pmatrix}, \begin{pmatrix} ax_1x_3 \\ 0 \\ ax_2x_3 \\ ax_2x_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} ax_1x_3 \\ 0 \\ 0 \\ 0 \\ ax_1x_4 \\ ax_1x_4 \end{pmatrix} \right\}.$$

We have written scripts that do the same as the preceding scripts but for v_2, v_3 and v_6 which are almost identical. The choices of guesses of Ξ_i are recorded in Appendix E.1. Combining these results leads to the following cocycle as defining an algebra with property P as long as $af - cd = 0$:

$$\begin{pmatrix} cx_1^2 + ax_1x_3 \\ fx_2^2 + dx_2x_4 \\ fx_2x_1 + dx_2x_3 \\ 0 \\ cx_2x_3 + ax_1x_4 \\ cx_2x_3 + ax_1x_4 \end{pmatrix}$$

That is to say that the script implies this cocycle will define a PBW algebra with PBW basis $\{x_2^i x_1^j x_3^k x_4^l\}$. We verify this fact in Theorem 8.2.1. If one sets $a = d$ and $c = f$ then this family has an infinitesimal lying in V_g and we obtain algebras which integrate some of the deformations discussed in Chapter 6. If instead, one sets $c = f = 0$ then one recovers the family discussed in Chapter 7. We note that this cocycle is an element of V_q for any values of a, c, d and f satisfying $af - cd = 0$.

8.2 A Family of Deformations of A

In this section we discuss a new family of algebras which deform A and have associated infinitesimal deformations that lie in the unobstructed component V_q of $\mathrm{HH}^2_2(A)$. Let $V = \{x_1, x_2, x_3, x_4\}$.

Consider the algebra

$$A(a, c, d, f) := \frac{\mathbb{K}\langle V\rangle}{(R_{a,c,d,f})}$$

where $R_{a,c,d,f}$ is the set of relations

$$R_{a,c,d,f} = \left\{ \begin{array}{cc} r_1 := x_3x_1 - (1+a)x_1x_3 - cx_1^2, & r_2 := x_4x_2 - (1+d)x_2x_4 - fx_2^2, \\ r_3 := x_4x_1 - (1+d)x_2x_3 - fx_2x_1, & r_4 := x_1x_2 - x_2x_3, \\ r_5 := x_3x_2 - (1+a)x_1x_4 - cx_2x_3, & r_6 := x_4x_3 - (1+a)x_1x_4 - cx_2x_3 \end{array} \right\}.$$

We note that there that the cases $a = -1$ and $d = -1$ are degenerate in the sense that the properties of the algebras in these cases will be very different to the general case. For example, if $a = -1$ then the algebra is not a domain since

$$(x_3 - cx_1)x_1 = 0.$$

Theorem 8.2.1. *If $a \neq -1 \neq d$ then $A(a, c, d, f)$ is a PBW algebra with basis $\{x_2^i x_1^j x_3^k x_4^l\}$ if and only if $af - cd = 0$. In this case $A(a, c, d, f)$ has the Hilbert series of a commutative polynomial ring with four generators and specialises to A when $a = c = d = f = 0$.*

Proof. It is obvious that this algebra specialises to A by inspection of the relations. We show that $A(a, c, d, f)$ is PBW with the stated basis, which establishes the Hilbert series to be

$$\frac{1}{(1-p)^4}$$

as required.

We show that this algebra is PBW by using the Diamond Lemma (Theorem 2.1.12). With the ordering $x_2 < x_1 < x_3 < x_4$ we observe that the relations are all of the form

$$xy - \sum_i c_i f_i^1 f_i^2$$

where $x, y, f_i^1, f_i^2 \in V$ and $f_i^1 f_i^2 < xy$. Therefore we need only check overlap ambiguities are resolvable in order to establish the claimed basis. There are four overlap ambiguities:

$$\{x_4x_1x_2, x_3x_1x_2, x_4x_3x_2, x_4x_3x_1\}.$$

By repeatedly applying the relations, we obtain two distinct simplification paths for each overlap. Since these calculations are quite long, we include only the first and final vertices here; the full calculations can be found in Appendix E.2.

$$x_4x_1x_2 \qquad (c+f+cd)x_2^2x_3 + (1+a+d+ad)x_2x_1x_4$$

$$x_3x_1x_2 \qquad (2c+ac)x_2x_3^2 + (1+a)^2x_1^2x_4$$

$$(c^2+cf+acf)x_2^2x_3 + (c+a^2f+2af+ac+f)x_2x_1x_4 + (1+a)(1+d)x_2x_3x_4$$

$$x_4x_3x_2$$

$$(c^2+c^2d+cf)x_2^2x_3 + (c+f+af+acd+ac+cd)x_2x_1x_4 + (1+a)(1+d)x_2x_3x_4$$

$$(c^2+cf+acf)x_2x_1^2 + (c+a^2f+2af+ac+f)x_2x_1x_3 + (1+a)(1+d)x_2x_3^2$$

$$x_4x_3x_1$$

$$(c^2+c^2d+cf)x_2x_1^2 + (c+f+af+acd+ac+cd)x_2x_1x_3 + (1+a)(1+d)x_2x_3^2$$

In the first two cases we see that the overlaps are always resolvable for any values of a, c, d or f. However, in the second two cases the overlap is not resolvable unless the following holds:

$$(c^2d - acf)x_2^2x_3 + (cd - af + acd - a^2f)x_2x_1x_4 = 0$$

and

$$(c^2d - acf)x_2x_1^2 + (cd - af + acd - a^2f)x_2x_1x_3 = 0.$$

Both of these hold if and only if $cd - af = 0$. □

Corollary 8.2.2. *$A(a, c, d, f)$ is a flat family of algebras which deforms A over*

$$\frac{\mathbb{K}[a, c, d, f, \frac{1}{1+a}, \frac{1}{1+d}]}{(af - cd)}$$

Proof. This is a standard consequence of the Hilbert series being the same for each element of the family [Har77, Theorem III.9.9]. □

Corollary 8.2.3. *For any $a, c, d, f \in \mathbb{K}$ with $af - cd = 0$ and $a \neq -1 \neq d$, the algebra $A(a, c, d, f)$ is not noetherian.*

Proof. Note that none of the relations of $A(a, c, d, f)$ have x_3x_4 or x_3^2 appearing with nonzero coefficient. For this reason we may deduce that for any $n \in \mathbb{N}_0$ there is no monomial $a \in A$ that has PBW order $\lfloor a \rfloor$ satisfy $(\lfloor a \rfloor \bullet x_3^n x_4) \neq 0$, except $x_3^n x_4$ for itself. We may deduce from this that the equation

$$\sum_{i=0}^{n-1} x_3^i x_4 a_i = x_3^n x_4$$

has no solutions $a_0, \ldots, a_{n-1} \in A$. Therefore the right ideal generated by $\{x_3^n x_4\}_{n=0}^{\infty}$ is not finitely generated. □

8.2.1 The Function Skew Field of $A(a, c, d, f)$

One of the basic properties of a noncommutative projective surface is its function skew field (see Definition 2.2.8). In this section we find the function skew field of each algebra in the family of $A(a, c, d, f)$. From here onwards we assume $a \neq -1 \neq d$ and $af - cd = 0$.

Notation 8.2.4. Our convention for Ore extensions is that for a ring R and an automorphism $f \in \mathrm{Aut}(R)$, then for every $r \in R$ we have the following equality in the Ore extension $R[t; f]$:

$$tr = f(r)t.$$

Let D be the division ring of the Ore extension $\mathbb{K}[z][w; \alpha]$, where $\alpha \in \mathrm{Aut}(\mathbb{K}(z))$ is defined by

$$\alpha(z) = \frac{(1+a)}{(1+d)} z - \frac{(1+a)}{(1+d)} f + c.$$

Lemma 8.2.5. *The equations*

$$\beta(z) = \frac{z - c}{1 + a} \text{ and } \beta(w) = \frac{(z - f)}{(1 + d)} w$$

define an automorphism $\beta : D \to D$.

Proof. It suffices to confirm that $\beta(r) = 0$ for r the defining relation of D. By definition of an Ore extension,

$$r = wz - \alpha(z)w.$$

We evaluate β on each term of r in turn. Firstly,

$$\begin{aligned}
\beta(w)\beta(z) &= \frac{(z-f)}{(1+d)}w\frac{(z-c)}{(1+a)} = \frac{(z-f)}{(1+a)(1+d)}(wz-wc) \\
&= \frac{(z-f)}{(1+a)(1+d)}\left(\frac{(1+a)}{(1+d)}z - \frac{(1+a)}{(1+d)}f + c - c\right)w \\
&= \frac{(z-f)^2}{(1+d)^2}w.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\beta(\alpha(z))\beta(w) &= \left(\frac{(1+a)}{(1+d)}\beta(z) - \frac{(1+a)}{(1+d)}f + c\right)\beta(w) \\
&= \left(\frac{(z-c)}{(1+d)} - \frac{(1+a)}{(1+d)}f + c\right)\frac{(z-f)}{(1+d)}w \\
&= \frac{1}{(1+d)^2}(z-c-f-af+c+cd)(z-f)w \\
&= \frac{(z-f)^2}{(1+d)^2}w \text{ since } af = cd.
\end{aligned}$$

Therefore β extends to an automorphism of D. □

Proposition 8.2.6. *Consider the set*

$$F = \{y_1 := t,\, y_2 := wt,\, y_3 := zt,\, y_4 := zwt\} \subseteq D[t;\beta],$$

and let $T := \mathbb{K}\langle F\rangle$*. If* $a \neq -1 \neq d$ *and* $af - cd = 0$ *then we have an isomorphism of algebras*

$$A(a,c,d,f) \cong T$$

Proof. We prove this with an explicit isomorphism defined by

$$\phi : A(a,c,d,f) \to T \quad x_i \mapsto y_i.$$

We establish firstly that this defines an algebra homomorphism, which immediately implies that it is surjective. This is done by checking that the y_i satisfy the defining relations of $A(a,c,d,f)$. These calculations are recorded in Appendix E.3, but we include the simplest case here being $r_4 = x_1x_2 - x_2x_3$.

Firstly, we note that

$$\begin{aligned}
w(z-c) &= \left(\left(\frac{(1+a)}{(1+d)}z - \frac{(1+a)}{(1+d)}f + c\right)w - cw\right) \\
&= \frac{(1+a)}{(1+d)}(z-f)w \qquad (8.1)
\end{aligned}$$

In particular this implies that

$$w\beta(z) = \beta(w). \tag{8.2}$$

Now we can check that whilst

$$y_1y_2 = twt = \beta(w)t^2,$$

we also have that

$$y_2y_3 = wtzt = w\beta(z)t^2 = \beta(w)t^2 \text{ by (8.2).}$$

From this calculation, and those contained in Appendix E.3, we may conclude ϕ is a surjective algebra homomorphism. Therefore it remains only to verify that ϕ is injective. This is done by showing that the Hilbert series of T is the same as that of $A(a,c,d,f)$. Note that since ϕ is a surjective (graded) homomorphism, it must be the case that

$$\dim(A(a,c,d,f)_n) \geq \dim(T_n).$$

We establish that $\dim(T_n) \geq \dim(A_n)$. Since $\dim(A_n) = \dim(A(a,c,d,f)_n$ by Theorem 8.2.1, this implies that the Hilbert series of T and $A(a,c,d,f)$ are equal.

Recall that

$$E = \{e_1 := t,\, e_2 := ut,\, e_3 := vt,\, e_4 := uvt\} \subseteq \mathbb{K}(u,v)[t;\sigma]$$

generates an algebra isomorphic to A.

We define ∂_1 on monomials of A of the form $u^iv^jt^n$ by

$$\partial_1(u^iv^jt^n) = (i,j) \in \mathbb{N}^2.$$

Then by [RS12, Lemma 4.12 (1)] we know that we have the following equality:

$$|\{\partial_1(e_{i_1}\cdots e_{i_n}) \mid i_1,\ldots,i_n \in [1,4]\}| = \binom{n+3}{3} = \dim(A_n).$$

Likewise, we define ∂_2 on polynomials in T of the form $g(z)w^it^k$, where $g(z) \in \mathbb{K}[z]$, by

$$\partial_2(g(z)w^it^k) = (i, \deg(g)).$$

We claim for any $n \geq 1$ and $i_1,\ldots,i_n \in [1,4]$, that

$$\partial_1(e_{i_1}\cdots e_{i_n}) = \partial_2(y_{i_1}\cdots y_{i_n}).$$

We prove this by induction on n. The base case of $n = 1$ is trivial, and so we move to the inductive step.

Let $R = \mathbb{K}[z][w;\alpha][t;\beta]$. Then by [BGTV03, Corollary 3.3], R has a basis of the form $\{z^i w^j t^k\}$. Furthermore, by [BGTV03, Corollary 2.10 (2)], we have that if $X \in R^*$ and $\partial_2(X) = (\alpha, \beta)$ then if $y \in F$

$$\partial_2(Xy) = \partial_2(z^\beta w^\alpha t^k y). \tag{8.3}$$

The following are immediate consequences of the facts that $\partial_2(\alpha(z)) = (0,1) = \partial_2(\beta(z))$ and $\partial_2(\beta(w)) = (1,1)$:

$$\partial_2(w^s z^r) = (s,r),\ \partial_2(t^k z^r) = (0,r) \text{ and } \partial_2(t^k w^s) = (s, ks).$$

Together these imply that for $X \in T^*$

$$\partial_2(z^\beta w^\alpha X) = (\alpha, \beta) + \partial_2(X), \tag{8.4}$$

and that

$$\partial_2(t^k z^{c_2} w^{c_1} t) = \partial_2(t^k z^{c_2}) + \partial_2(t^k w^{c_1}) = (c_1, c_2 + kc_1). \tag{8.5}$$

For the induction then we assume that

$$(\alpha, \beta) = \partial_1(e_{i_1} \cdots e_{i_k}) = \partial_2(y_{i_1} \cdots y_{i_k})$$

and that $e_{i_{k+1}} = u^{c_1} v^{c_2} t$ whilst $y_{i_{k+1}} = z^{c_2} w^{c_1} t$. On the one hand:

$$\begin{aligned}\partial_1(e_{i_1} \cdots e_{i_{k+1}}) &= \partial_1(u^\alpha v^\beta t^k u^{c_1} v^{c_2} t) \\ &= \partial_1(u^{\alpha+c_1} v^{\beta+c_2+kc_1} t^{k+1}) = (\alpha + c_1, \beta + c_2 + kc_1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\partial_2(y_{i_1} \cdots y_{i_{k+1}}) &= \partial_2(z^\beta w^\alpha t^k z^{c_2} w^{c_1} t) \text{ by (8.3)} \\ &= (\alpha, \beta) + \partial_2(t^k z^{c_2} w^{c_1} t) \text{ by (8.4)} \\ &= (\alpha, \beta) + (c_1, c_2 + kc_1) \text{ by (8.5)} \\ &= \partial_1(e_{i_1} \cdots e_{i_{k+1}}).\end{aligned}$$

This proves the claim. Therefore we have an injection of sets from

$$\{\partial_1(e_{i_1} \cdots e_{i_n}) \mid i_1, \ldots, i_n \in [1,4]\}$$

into

$$S := \{\partial_2(y_{i_1} \cdots y_{i_n}) \mid i_1, \ldots, i_n \in [1,4]\}\,.$$

By definition of T_n, we have that $\dim(T_n) \geq |S|$, and this injection implies that $|S| \geq$

$\dim(A_n)$. This establishes that

$$\dim(T_n) \geq \dim(A_n) = \binom{n+3}{3}$$

and so $T_n \cong A(a,c,d,f)$. $\square$

We have as an immediate corollary:

Corollary 8.2.7. *The graded quotient ring $Q_{gr}(A(a,c,d,f))$ is $D[t,t^{-1};\beta]$ and D is the function skew field of $A(a,c,d,f)$.*

Proof. This follows since $z = y_3y_1^{-1}$ and $w = y_2y_1^{-1}$. $\square$

This classifies $A(a,c,d,f)$ up to birational equivalence. Furthermore, we obtain:

Corollary 8.2.8. *$A(a,c,d,f)$ is a domain.*

Proof. Since $R = \mathbb{K}[z][w;\alpha][t;\beta]$ is an iterated Ore extension of a domain, where both α and β are injective, [GW04, Excercise 2O] implies that R is a domain. Proposition 8.2.6 shows that $A(a,c,d,f)$ is isomorphic to a subalgebra of R and so must also be a domain. $\square$

We further show that, depending on the values of a, c, d and f, the function skew field D is isomorphic to the division ring of one of three algebras: the polynomial ring $\mathbb{K}[u,v]$, the quantum plane $\mathbb{K}_q[u,v]$ or the Weyl algebra

$$A_1(\mathbb{K}) = \frac{\mathbb{K}\langle u,v\rangle}{(vu-uv+1)}.$$

The division ring of $A_1(\mathbb{K})$ is written $D_1(\mathbb{K})$.

Proposition 8.2.9. *D is isomorphic to*

(a) $\mathbb{K}(u,v)$ if $a=d$ and $c=f$.

(b) $D_1(\mathbb{K})$ if $a=d=0$ and $c \neq f$.

(c) $\mathbb{K}_q(u,v)$ where $q = \frac{(1+d)}{(1+a)}$ if $a \neq d$.

Proof. (a) This follows immediately from the definition of α since if $a=d$ and $c=f$ then $\alpha(z) = z$.

(b) Let $u = \frac{zw}{f-c}$ and $v = w^{-1}$. Note that since $\alpha(z) = z-(f-c)$ we have

$$w\left(\frac{z}{f-c}\right) = \left(\frac{z}{f-c} - 1\right)w.$$

It follows that in the division ring

$$\frac{z}{f-c} = w^{-1}\frac{zw}{f-c} - 1. \tag{8.6}$$

Therefore we have that

$$vu = \frac{w^{-1}zw}{f-c} = \frac{z}{f-c} + 1 = uv + 1,$$

where the second equality follows from (8.6). By [GW04, Corollary 2.2], the Weyl algebra $A_1(\mathbb{K})$ is simple, and therefore u and v generate a subalgebra of D isomorphic to $A_1(\mathbb{K})$. Furthermore, the division ring of this subalgebra is the entire quotient ring D, which implies that D is the division ring $D_1(\mathbb{K})$.

(c) If $q = \frac{(1+d)}{(1+a)}$ then let $u := w$ and $v := (q^{-1}-1)z - q^{-1}f + c$. Then a calculation shows that

$$\begin{aligned} uv &= w\left((q^{-1}-1)z - q^{-1}f + c\right) \\ &= (q^{-1}-1)(q^{-1}z - q^{-1}f + c)w + (c - q^{-1}f)w \\ &= q^{-1}((q^{-1}-1)z - q^{-1}f + c)w = q^{-1}vu. \end{aligned}$$

Therefore we may define a ring homomorphism $\phi : \mathbb{K}_q[x, y] \to D$ by $\phi(x) = u$ and $\phi(y) = v$.

The set $\{u^i v^j\}$ is linearly independent because its set of leading terms is $\{z^i w^j\}$ which is linearly independent. Therefore, ϕ is an isomorphism onto its image. Finally, it follows from the definitions of u and v that the division ring that contains both of them must contain all of D and so $D \cong \mathbb{K}_q(u, v)$.

□

8.2.2 Closing Remarks

This chapter has shown that in spite of a large search space, a few small steps of informed guesswork allow new families of algebras to be discovered with some interesting properties. The family discussed here accounts for 3 out of 4 of the dimensions of V_q, and it seems plausible that this family is a specialisation of one that accounts for all of V_q. Since this hypothetical family could not be PBW, other techniques will have to be applied to answer this question.

We have established some of the basic properties of the new algebra $A(a, c, d, f)$ that we have discovered: it is a non-noetherian PBW domain of GK-dimension 4. Furthermore we have shown that it has $\mathbb{K}(u, v)$, $\mathbb{K}_q(u, v)$ or $D_1(\mathbb{K})$ as its function skew field, and given conditions on a, c, d and f for each case. There are now several questions which would be very interesting for further investigation. The most interesting of these are whether $A(a, c, d, f)$ is ever AS-Gorenstein, since none of the previously studied deformations of A have this property. It would also be fascinating if there are families of deformations of A that specialise to $A(a, c, d, f)$ for which the generic element has the noetherian property.

Appendices

Appendix A

Bases Relevant to Calculations on Hochschild Cohomology

Recall that both A and A_q are PBW algebras with respect to the lexicographic ordering on monomials induced by the ordering of generators

$$x_2 < x_1 < x_3 < x_4.$$

With this in mind we choose a basis for K^n in degree m. We start by taking monomials of degree m in order

$$m_1 := x_2^m < m_2 := x_2^{m-1}x_1 < m_3 \ldots < m_z := x_4^m$$

where $z = \binom{m+3}{3}$. Then the if $\overline{K_n}$ is k dimensional the basis is the following ordered set of vectors with k components (using notation as in Notation 3.2.1).

$$\left\{ \begin{pmatrix} m_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} m_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} m_z \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ m_1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ m_z \end{pmatrix} \right\}$$

A.1 Calculations For A

A.1.1 Basis of the Kernel of k^3

The content of this section is calculated using the 'Sage' script discussed in Section 3.3, being a basis for the space $\mathrm{Ker}(k^3)$. This space is 22 dimensional and is written using the vector notation defined in Notation 3.2.1. We have reordered the output so that the first eight vectors have cohomology classes that are a basis of the cohomology space (compare the following with the basis presented in Section A.1.2).

$$\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_3^2 \\ 0 \\ x_1x_4 \\ 0 \\ 0 \\ x_3x_4 \end{pmatrix} \begin{pmatrix} x_1^2 \\ 0 \\ 0 \\ x_2x_1 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \\ -x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x_2^2 \\ 0 \\ x_2x_4 \end{pmatrix}$$

$$\begin{pmatrix} -x_2x_3 + x_1x_4 \\ 0 \\ x_2x_4 \\ x_2^2 \\ -x_2x_4 \\ -x_2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -x_1x_4 \\ -x_1x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1x_4 \\ x_1x_4 \\ -x_3x_4 \\ -x_3x_4 \end{pmatrix} \begin{pmatrix} -x_1x_4 + x_3x_4 \\ 0 \\ x_4^2 \\ x_2x_4 \\ -x_4^2 \\ -x_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_3 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x_2x_3 + -x_1x_4 \\ -x_1x_3 \\ x_1^2 + -x_1x_3 \\ x_1x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_1x_4 + -x_3x_4 \\ -x_3^2 \\ x_1x_3 + -x_3^2 \\ x_3^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_1x_4 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_2x_1 + -x_2x_3 \\ 0 \\ -x_2^2 \\ -x_2^2 \\ x_2^2 \\ x_2x_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -x_2x_1 \\ -x_2x_1 \\ x_2x_3 \\ x_2x_3 \end{pmatrix} \begin{pmatrix} x_2x_3 + -x_1x_4 \\ 0 \\ -x_2x_4 \\ -x_2x_4 \\ x_2x_4 \\ x_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_1 \\ 0 \\ -x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x_2x_1 + x_2x_3 \\ x_1^2 \\ 0 \\ -x_1^2 \\ -x_1^2 + x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ -x_2x_3 + x_1x_4 \\ x_1x_3 \\ 0 \\ -x_1x_3 \\ -x_1x_3 + x_3^2 \end{pmatrix}$$

A.1.2 Basis of the Image of k^2

The 'Sage' script discussed in Section 3.3 output a basis for the space $\mathrm{Im}(k^2)$. This space is 14 dimensional, which combined with the basis in Section A.1.1 implies that $\mathrm{HH}_2^2(A)$ is an eight dimensional vector space.

$$\begin{pmatrix} -x_2x_3 + x_1x_4 \\ 0 \\ x_2x_4 \\ x_2^2 \\ -x_2x_4 \\ -x_2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -x_1x_4 \\ -x_1x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1x_4 \\ x_1x_4 \\ -x_3x_4 \\ -x_3x_4 \end{pmatrix} \begin{pmatrix} -x_1x_4 + x_3x_4 \\ 0 \\ x_4^2 \\ x_2x_4 \\ -x_4^2 \\ -x_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_3 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x_2x_3 + -x_1x_4 \\ -x_1x_3 \\ x_1^2 + -x_1x_3 \\ x_1x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_1x_4 + -x_3x_4 \\ -x_3^2 \\ x_1x_3 + -x_3^2 \\ x_3^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_1x_4 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_2x_1 + -x_2x_3 \\ 0 \\ -x_2^2 \\ -x_2^2 \\ x_2^2 \\ x_2x_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -x_2x_1 \\ -x_2x_1 \\ x_2x_3 \\ x_2x_3 \end{pmatrix} \begin{pmatrix} x_2x_3 + -x_1x_4 \\ 0 \\ -x_2x_4 \\ -x_2x_4 \\ x_2x_4 \\ x_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_1 \\ 0 \\ -x_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x_2x_1 + x_2x_3 \\ x_1^2 \\ 0 \\ -x_1^2 \\ -x_1^2 + x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ -x_2x_3 + x_1x_4 \\ x_1x_3 \\ 0 \\ -x_1x_3 \\ -x_1x_3 + x_3^2 \end{pmatrix}$$

A.1.3 Basis of the Image of k^3

The content of this section is a basis for $\mathrm{Im}(k^3)$, output using a 'Sage' script that is nearly identical to that discussed in Section 3.3. The relevance of this space is that it is used to detect when a Gerstenhaber bracket is a coboundary, which is discussed in detail in Section 5.3. $\mathrm{Im}(k^3)$ is a 38 dimensional vector space, which agrees with the result in Section A.1.1 since we know that

$$\dim(\mathrm{Im}(k^3)) = \dim(K^2) - \dim(\mathrm{Ker}(k^3)) = 60 - 22.$$

$$\begin{pmatrix} 0 \\ -x_1^2x_3 + x_1x_3^2 \\ x_1x_3x_4 \\ -x_1^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ -x_1x_3^2 + x_3^3 \\ x_3^2x_4 \\ -x_1x_3x_4 \end{pmatrix} \begin{pmatrix} -x_2^3 \\ x_2^2x_4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_2^2x_3 \\ x_2^2x_3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_2x_1x_4 \\ x_2x_1x_4 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -x_2^2x_4 \\ x_2x_4^2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_2x_3^2 \\ x_2x_1x_3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_1^2x_4 \\ x_2x_3^2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_2x_3x_4 \\ x_2x_3x_4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_1x_3x_4 \\ x_1^2x_4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -x_1x_4^2 \\ x_1x_4^2 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -x_2x_4^2 \\ x_4^3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_1x_4 \\ x_2^2x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_1x_3 \\ x_2x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_3^2 \\ x_2x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_3x_4 \\ x_2x_1x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_1^3 \\ x_1^2x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -x_1^2x_4 \\ x_1^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_1x_3^2 \\ x_3^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_1x_3x_4 \\ x_1x_3x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_1x_4^2 \\ x_2x_3x_4 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_3x_4 + -x_1x_4^2 \\ x_4^3 \\ -x_2x_4^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -x_1^2x_3 \\ x_1x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_1x_4 + -x_2x_3x_4 \\ x_2x_4^2 \\ -x_2^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ -x_1^3 + x_1^2x_3 \\ x_1^2x_4 \\ -x_2x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_3x_4^2 \\ -x_1x_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2^2x_3 \\ -x_2^2x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_2x_3x_4 \\ -x_2^2x_3 \\ -x_2^2x_4 \\ x_2^2x_4 \end{pmatrix} \begin{pmatrix} x_1x_4^2 \\ -x_2x_1x_4 \\ -x_2x_4^2 \\ x_2x_4^2 \end{pmatrix} \begin{pmatrix} x_1^2x_3 \\ -x_1^2x_3 \\ -x_1^2x_4 \\ x_2x_1x_3 \end{pmatrix} \begin{pmatrix} x_1x_3^2 \\ -x_1x_3^2 \\ -x_1x_3x_4 \\ x_2x_3^2 \end{pmatrix} \begin{pmatrix} x_3^3 \\ -x_3^3 \\ -x_3^2x_4 \\ x_1^2x_4 \end{pmatrix} \begin{pmatrix} x_3^2x_4 \\ -x_1x_3x_4 \\ -x_3x_4^2 \\ x_1x_4^2 \end{pmatrix}$$

$$\begin{pmatrix} x_3x_4^2 \\ -x_2x_3x_4 \\ -x_4^3 \\ x_4^3 \end{pmatrix} \begin{pmatrix} -x_1^3 + x_1^2x_3 \\ 0 \\ x_2x_1x_3 \\ -x_2x_1^2 \end{pmatrix} \begin{pmatrix} x_2x_1x_4 \\ -x_2^2x_1 \\ -x_2^3 \\ x_2^3 \end{pmatrix} \begin{pmatrix} x_2x_1x_3 \\ -x_2x_1^2 \\ -x_2^2x_3 \\ x_2^2x_1 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2x_3 + -x_2x_1x_4 \\ x_2^2x_4 \\ -x_2^3 \end{pmatrix}$$

A.1.4 Gerstenhaber Brackets of the Basis of $\mathrm{HH}^2_2(A)$

The content of this section is the set of vectors $[b_i, b_j]$ calculated in the script discussed in Section 5.3. Note the vectors lie in $K^3(A)$.

$$[b_1,b_1] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ [b_1,b_2] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ [b_1,b_3] = \begin{pmatrix} x_2x_3^2 \\ -x_2x_1x_3 \\ 0 \\ 0 \end{pmatrix}, \ [b_1,b_4] = \begin{pmatrix} 0 \\ 0 \\ x_2x_3x_4 \\ -x_2x_1x_4 \end{pmatrix},$$

$$[b_1,b_5] = \begin{pmatrix} -x_1x_3x_4 \\ 0 \\ 0 \\ -x_2x_3x_4 \end{pmatrix}, \ [b_1,b_6] = \begin{pmatrix} x_2x_3^2 \\ 0 \\ 2x_2x_1x_4 \\ -x_2^2x_3 \end{pmatrix}, \ [b_1,b_7] = \begin{pmatrix} 0 \\ -x_1^2x_3 \\ -x_1^2x_4 \\ 0 \end{pmatrix},$$

$$[b_1,b_8] = \begin{pmatrix} -x_2x_3x_4 \\ 0 \\ 0 \\ -x_2^2x_4 \end{pmatrix}, \ [b_2,b_2] = \begin{pmatrix} 2x_3^2x_4 \\ -2x_1x_3x_4 \\ 0 \\ 0 \end{pmatrix}, \ [b_2,b_3] = \begin{pmatrix} x_1^2x_4 \\ -x_2x_3^2 \\ 0 \\ 0 \end{pmatrix},$$

$$[b_2,b_4] = \begin{pmatrix} 0 \\ x_1^2x_4 \\ x_1x_4^2 \\ 0 \end{pmatrix}, \ [b_2,b_5] = \begin{pmatrix} -x_3^2x_4 \\ x_1x_3x_4 \\ x_3x_4^2 \\ -x_1x_4^2 \end{pmatrix}, \ [b_2,b_6] = \begin{pmatrix} x_1^2x_4 \\ x_2x_3^2 \\ x_2x_3x_4 \\ x_2x_1x_4 \end{pmatrix},$$

$$[b_2,b_7] = \begin{pmatrix} -2x_1x_3^2 \\ x_1^2x_3 \\ 0 \\ -2x_2x_3^2 + x_1^2x_4 \end{pmatrix}, \ [b_2,b_8] = \begin{pmatrix} 0 \\ -x_2x_3x_4 \\ 0 \\ -x_2^2x_4 \end{pmatrix}, \ [b_3,b_3] = \begin{pmatrix} 2x_2x_1x_3 \\ -2x_2x_1^2 \\ 0 \\ 0 \end{pmatrix},$$

$$[b_3,b_4] = \begin{pmatrix} -x_2x_3^2 \\ 0 \\ 0 \\ -x_2^2x_3 \end{pmatrix}, \ [b_3,b_5] = \begin{pmatrix} -2x_1^2x_4 \\ 0 \\ -x_2x_3x_4 \\ -x_2x_1x_4 \end{pmatrix}, \ [b_3,b_6] = \begin{pmatrix} 0 \\ 0 \\ x_2^2x_3 \\ -x_2^2x_1 \end{pmatrix},$$

$$[b_3,b_7] = \begin{pmatrix} -x_1^3 + x_1^2x_3 \\ -x_1^3 \\ -x_2x_3^2 + x_2x_1x_3 \\ -x_2x_1^2 \end{pmatrix}, \ [b_3,b_8] = \begin{pmatrix} -x_2^2x_3 \\ 0 \\ -x_2^2x_4 \\ 0 \end{pmatrix}, \ [b_4,b_4] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$[b_4,b_5]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_4,b_6]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_4,b_7]=\begin{pmatrix}x_1^2x_3\\0\\0\\x_2x_1x_3\end{pmatrix},\ [b_4,b_8]=\begin{pmatrix}x_2x_3x_4\\0\\0\\x_2^2x_4\end{pmatrix},$$

$$[b_5,b_5]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_5,b_6]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_5,b_7]=\begin{pmatrix}x_1x_3^2\\0\\0\\x_2x_3^2\end{pmatrix},\ [b_5,b_8]=\begin{pmatrix}x_1x_4^2\\0\\0\\x_2x_4^2\end{pmatrix},$$

$$[b_6,b_6]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_6,b_7]=\begin{pmatrix}x_1^2x_3\\0\\x_2x_1x_3\\0\end{pmatrix},\ [b_6,b_8]=\begin{pmatrix}2x_2x_1x_4+-x_2^2x_3\\0\\-x_2^2x_4\\2x_2^3\end{pmatrix},$$

$$[b_7,b_7]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_7,b_8]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_8,b_8]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

A.2 Calculations For A_q

A.2.1 Basis of the Kernel of k^3

The content of this section is calculated using the 'Sage' script discussed in Section 3.4, which gives a basis for the space $\mathrm{Ker}(k^3)$. This space is 18-dimensional and is written using the vector notation defined in Notation 3.2.1. We have reordered the output so that the first four vectors have cohomology classes that are a basis of the cohomology space. In order to see this please compare the following with the basis presented in Section A.2.2.

$$\begin{pmatrix}x_1x_3\\0\\0\\x_2x_3\\qx_1x_4\\0\end{pmatrix}\begin{pmatrix}0\\x_2^2\\0\\0\\x_2x_3\\0\end{pmatrix}\begin{pmatrix}0\\x_2x_4\\0\\0\\x_1x_4\\0\end{pmatrix}\begin{pmatrix}0\\x_4^2\\0\\0\\x_3x_4\\0\end{pmatrix}$$

$$\begin{pmatrix} -x_2x_3 + qx_1x_4 \\ 0 \\ qx_2x_4 \\ x_2^2 \\ -qx_2x_4 \\ -x_2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ -qx_2x_3 + qx_1x_4 \\ x_1x_3 \\ 0 \\ -qx_1x_3 \\ -x_1x_3 + x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_3 - qx_1x_4 \\ -x_1x_3 \\ x_1^2 - x_1x_3 \\ x_1x_3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x_2x_3 - qx_2x_1 \\ x_1^2 \\ 0 \\ -qx_1^2 \\ -x_1^2 + x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ x_1x_4 - qx_3x_4 \\ -x_3^2 \\ x_1x_3 - x_3^2 \\ x_3^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ (1-q)x_4^2 \\ -x_1x_4 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ x_1x_4 \\ qx_1x_4 \\ -qx_3x_4 \\ -x_3x_4 \end{pmatrix} \begin{pmatrix} 0 \\ (1-q)x_2^2 \\ x_2x_1 \\ 0 \\ -qx_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_1 \\ -x_2x_1 \\ x_2x_3 \\ x_2x_3 \end{pmatrix} \begin{pmatrix} x_2x_3 - x_1x_4 \\ 0 \\ -x_2x_4 \\ -x_2x_4 \\ qx_2x_4 \\ x_4^2 \end{pmatrix}$$

$$\begin{pmatrix} x_2x_1 - x_2x_3 \\ 0 \\ -x_d^2 \\ -x_2^2 \\ x_d^2 \\ qx_2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ 0 \\ -qx_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -qx_1x_4 \\ -x_1x_4 \end{pmatrix} \begin{pmatrix} x_3x_4 - x_1x_4 \\ 0 \\ x_4^2 \\ qx_2x_4 \\ -qx_4^2 \\ -x_4^2 \end{pmatrix}$$

A.2.2 Basis of the Image of k^2

We include the output of the 'Sage' script discussed in Section 3.4, being a basis for the space $\mathrm{Im}(k^2)$. This space is 14 dimensional, which combined with the basis in Section A.2.1 implies that $\mathrm{HH}_2^2(Aq)$ is an four dimensional vector space.

$$\begin{pmatrix} -x_2x_3 + qx_1x_4 \\ 0 \\ qx_2x_4 \\ x_2^2 \\ -qx_2x_4 \\ -x_2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ -qx_2x_3 + qx_1x_4 \\ x_1x_3 \\ 0 \\ -qx_1x_3 \\ -x_1x_3 + x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_3 - qx_1x_4 \\ -x_1x_3 \\ x_1^2 - x_1x_3 \\ x_1x_3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x_2x_3 - qx_2x_1 \\ x_1^2 \\ 0 \\ -qx_1^2 \\ -x_1^2 + x_1x_3 \end{pmatrix} \begin{pmatrix} 0 \\ x_1x_4 - qx_3x_4 \\ -x_3^2 \\ x_1x_3 - x_3^2 \\ x_3^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ (1-q)x_4^2 \\ -x_1x_4 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ x_1x_4 \\ qx_1x_4 \\ -qx_3x_4 \\ -x_3x_4 \end{pmatrix} \begin{pmatrix} 0 \\ (1-q)x_2^2 \\ x_2x_1 \\ 0 \\ -qx_2x_3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -x_2x_1 \\ -x_2x_1 \\ x_2x_3 \\ x_2x_3 \end{pmatrix} \begin{pmatrix} x_2x_3 - x_1x_4 \\ 0 \\ -x_2x_4 \\ -x_2x_4 \\ qx_2x_4 \\ x_4^2 \end{pmatrix}$$

$$\begin{pmatrix} x_2x_1 - x_2x_3 \\ 0 \\ -x_d^2 \\ -x_2^2 \\ x_d^2 \\ qx_2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ 0 \\ -qx_1x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -qx_1x_4 \\ -x_1x_4 \end{pmatrix} \begin{pmatrix} x_3x_4 - x_1x_4 \\ 0 \\ x_4^2 \\ qx_2x_4 \\ -qx_4^2 \\ -x_4^2 \end{pmatrix}$$

A.2.3 Basis of the Image of k^3

The content of this section is a basis for $\mathrm{Im}(k^3)$, output using a 'Sage' script that is nearly identical to that discussed in Section 3.4. The relevance of this space is that it is used to detect when a Gerstenhaber bracket is a coboundary, which is discussed in detail in Section 5.4. $\mathrm{Im}(k^3)$ is a 42 dimensional vector space, which agrees with the result in Section A.2.1 since we know that

$$\dim(\mathrm{Im}(k^3)) = \dim(K^2) - \dim(\mathrm{Ker}(k^3)) = 60 - 18.$$

$$\begin{pmatrix} x_2^3 \\ -q^2x_2^2x_4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} x_2^2x_3 \\ 0 \\ q^2x_2^2x_4 \\ 0 \end{pmatrix} \begin{pmatrix} x_2^2x_4 \\ -x_2x_4^2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} q^2x_2x_1x_3 \\ -qx_2x_1^2 \\ 0 \\ (-1+q^2)x_2^2x_1 \end{pmatrix} \begin{pmatrix} qx_2x_1x_4 \\ 0 \\ 0 \\ x_2^3 \end{pmatrix}$$

$$\begin{pmatrix} x_2x_3^2 \\ 0 \\ 0 \\ x_2^2x_3 \end{pmatrix} \begin{pmatrix} x_2x_3x_4 \\ 0 \\ 0 \\ x_2^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_3x_4^2 \\ -qx_1x_4^2 \end{pmatrix} \begin{pmatrix} -q^2x_2x_4^2 \\ x_4^3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} x_1^3 \\ 0 \\ 0 \\ x_2x_1^2 \end{pmatrix} \begin{pmatrix} x_1^2x_3 \\ 0 \\ 0 \\ x_2x_1x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ x_4^3 \\ -x_2^2x_4 \end{pmatrix} \begin{pmatrix} -qx_1^2x_4 \\ x_2x_3^2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} x_1x_3^2 \\ 0 \\ 0 \\ x_2x_3^2 \end{pmatrix} \begin{pmatrix} x_1x_3x_4 \\ 0 \\ 0 \\ x_2x_3x_4 \end{pmatrix} \begin{pmatrix} x_1x_4^2 \\ 0 \\ 0 \\ x_2x_4^2 \end{pmatrix} \begin{pmatrix} x_3^3 \\ 0 \\ 0 \\ x_1^2x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_3^2x_4 \\ -x_1x_3x_4 \\ 0 \\ (1-q^2)x_1x_4^2 \end{pmatrix} \begin{pmatrix} x_3x_4^2 \\ 0 \\ 0 \\ x_4^3 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2x_1 \\ x_2^3 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2^2x_3 \\ qx_2^2x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_1x_3 \\ 0 \\ x_2^2x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -qx_2x_1x_4 \\ 0 \\ -x_2^3 \end{pmatrix} \begin{pmatrix} 0 \\ x_2x_3x_4 \\ 0 \\ qx_2^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ x_1^3 \\ 0 \\ x_2x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ x_1^2x_3 \\ 0 \\ qx_1^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ x_1^2x_4 \\ 0 \\ qx_2x_3x_4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ x_1x_3^2 \\ 0 \\ qx_1x_3x_4 \end{pmatrix} \begin{pmatrix} 0 \\ x_1x_4^2 \\ 0 \\ q^2x_2x_4^2 \end{pmatrix} \begin{pmatrix} 0 \\ x_3^3 \\ qx_3^2x_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ qx_2^2x_3 \\ -x_2^2x_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_1x_3 \\ -x_2x_1x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -qx_2x_1x_4 \\ x_2^2x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3^2 \\ -x_2x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_2x_3x_4 \\ -x_2x_1x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ q^2x_2x_4^2 \\ -x_2^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1^3 \\ -x_1^2x_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ x_1^2x_3 \\ -x_1x_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1^2x_4 \\ -x_1^2x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1x_3^2 \\ -x_3^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1x_3x_4 \\ -x_1x_3x_4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ x_1x_4^2 \\ -x_2x_3x_4 \end{pmatrix}$$

A.2.4 Gerstenhaber Brackets of the Basis of $\mathrm{HH}_2^2(A_q)$

The content of this section is the set of vectors $[b_i, b_j]$ calculated in the script discussed in Section 5.4.

$$[b_1,b_1]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_1,b_2]=\begin{pmatrix}0\\0\\x_2x_3x_4\\-x_2x_1x_4\end{pmatrix},\ [b_1,b_3]=\begin{pmatrix}-x_1x_3x_4\\0\\0\\-x_2x_3x_4\end{pmatrix},$$

$$[b_1,b_4]=\begin{pmatrix}x_2x_3^2\\0\\2qx_2x_1x_4\\-x_2^2x_3\end{pmatrix},\ [b_2,b_2]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_2,b_3]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

$$[b_2,b_4]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_3,b_3]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_3,b_4]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ [b_4,b_4]=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

Appendix B

Polygnome Source Code

In this appendix we gather together relevant pieces of source code from 'Polygnome' and explain any particularly opaque segments. Polygnome is a Python software package written by the author and freely available online [Cam]. The package was designed for manipulating elements of PBW algebras and was specifically tailored to allow the map m_2 to be defined in it. After a review of the software at the time, none was found that would the allow low level interaction with reduction sequences that was required for this task.

Note that in this code a 'Decorator' called `bimoduleMapDecorator` is used. A decorator in Python modifies a function so that the same code does not have to be rewritten multiple times. In this case, the decorator takes a function defined on a generating set for a free bimodule over an algebra and modifies it to be defined on the whole bimodule. Note that the decorator takes as arguments the domain and codomain of the bimodule map.

B.1 Koszul Boundary Maps

The following code defines the boundary maps `k_1` and `k_3`. It also defines the dual map `k_3Dual`. For a in depth discussion of the following code and the definition of `k_2` please see Section 3.2.1.

The Map k_1

```
def k_1(tens, alg):
    freeAlgebra = algebra()
    K1 = tensorAlgebra([alg, freeAlgebra, alg])
    K0 = tensorAlgebra([alg, alg])

    @bimoduleMapDecorator(K1, K0)
    def k_1Inner(pT):
        assert isinstance(pT, pureTensor)
        generator = pT[1]
        return pureTensor([generator, 1])\
            - pureTensor([1, generator])
    return k_1Inner(tens)
```

The Maps k_3 and k^3

```
def k_3(tens, alg):
    freeAlgebra = algebra()
    K3 = K2 = tensorAlgebra([alg, freeAlgebra, alg])

    @bimoduleMapDecorator(K3, K2)
    def k_3Inner(pT):
        answer= tensor()
        dd = pT[1] #dd stands for doubly defined
        for generator, rel in dd.leftHandRepresentation:
            answer = answer + \
                pureTensor((generator, rel, 1)).clean()
        for rel, generator in dd.rightHandRepresentation:
            answer = answer \
                -pureTensor((1, rel, generator)).clean()
        return answer
    return k_3Inner(tens)

k_3Dual = koszulDualMap(k_3)
```

In this code segment we have used the function `koszulDualMap` which is defined in Section 3.2.1.

B.2 The Map m_2

We define a map $m_2 : B_2 \to K_2$ (as in Section 4.7) in the following code. We print here the code in full before discussing a few of the details.

```
1  def m_2(abcd, alg):
2      B2 = tensorAlgebra([alg]*4)
3      freeAlgebra = algebra()
4      K2 = tensorAlgebra([alg, freeAlgebra, alg])
5
6      @bimoduleMapDecorator(B2, K2)
7      def m_2Inner(PT):
8          assert isinstance(PT, pureTensor)
9          assert len(PT) == 4
10         PT = PT.clean()
11         xy = PT[1] * PT[2]
12         answer = tensor()
13         sequence = alg.makeReductionSequence(xy)
14         for reductionFunction, weight in sequence:
15             answer += PT.coefficient * weight * PT[0] \
16                 * pureTensor(\
17                         [reductionFunction.leftMonomial,
18                         reductionFunction.relation,
19                         reductionFunction.rightMonomial])\
20                 * PT[3]
21         return answer
22
23     return m_2Inner(abcd)
```

Most of the above code is self-explanatory. However, we draw attention to a few key features. Firstly, `makeReductionSequence` is a method defined for an algebra in 'Polygnome' that generates reduction sequences for any polynomial. Currently, the strategy for generating such a sequence is to iterate arbitrarily through the monomials and for each monomial find the left most pair of out of order generators. By Theorem 4.7.1, any choice of reduction sequence will define a chain map m_2 that is a section of the canonical inclusion $i_2 : K_2 \to B_2$.

Secondly, `reductionFunction` is a class that encapsulates a reduction function $r_{A\sigma B}$ for monomials A, B and σ a reduction (see Section 4.1). This class has fields `leftMonomial` and `rightMonomial` that store A and B respectively. This class also has a field `relation` that stores ρ_σ, the relation associated with σ. The lines 15-20 above correspond to applying the function ν (see Definition 4.3.4).

The Map m^2

Since most of the work in this thesis is in the setting of Hochschild cohomology, the map m_2 is rarely used directly, but instead we use its dual form. Therefore we include the defining code for `m_2Dual` which sends $f : K_2 \to A$ to the function $f \circ m_2 : B_2 \to A$.

```
def m_2Dual(func):
    def newFunc(tensor):
        return func(m_2(tensor, func.algebra))
    return newFunc
```

B.3 The Map i_*

The following code defines the injection $i_n : K_n \to B_n$ in the cases $n = 2$ and $n = 3$. The dual map i^3 is also defined below.

The Map i_2

```
def i_2(tens, alg):
    freeAlgebra = algebra()
    B2 = tensorAlgebra([alg] * 4)
    K2 = tensorAlgebra([alg, freeAlgebra, alg])

    @bimoduleMapDecorator(K2, B2)
    def i_2Inner(pT):
        answer = tensor()
        rel = pT[1]
        for term in rel.leadingMonomial:
            answer = answer \
                + term.coefficient \
                * pureTensor((1, term[0], term[1], 1))
        for term in rel.lowerOrderTerms:
            answer = answer \
                - term.coefficient \
                * pureTensor((1, term[0], term[1], 1))
        return answer
    return i_2Inner(tens)
```

The Map i_3

```
def i_3(tens, alg):
    freeAlgebra = algebra()
    B3 = tensorAlgebra([alg] * 5)
    K3 = tensorAlgebra([alg, freeAlgebra, alg])

    @bimoduleMapDecorator(K3, B3)
    def i_3Inner(pT):
        answer = tensor()
        dd = pT[1] # dd stands for doubly defined
        for generator, rel in dd.leftHandRepresentation:
            rightHandSide = generator \
                * i_2(pureTensor([1, rel, 1]), alg)

            answer = answer \
                + pureTensor(1).tensorProduct(rightHandSide)
        return answer
    return i_3Inner(tens)
```

The Map i^3

```
def i_3Dual(func, alg, basisOfK3):
    images= []
    for i in basisOfK3:
        images.append(func(i_3(i, alg)))
    return functionOnKn(alg, basisOfK3, images)
```

B.4 Gerstenhaber Bracket

We present the defining code for the Gerstenhaber bracket from $\mathrm{HH}_2^2 \times \mathrm{HH}_2^2$ to HH_3^3. Firstly, we define the circle products $\circ_i$ on B^2. Note that as in Definition 5.2.2 we define the Gerstenhaber bracket on $f, g \in K^n$ by

$$[f, g, =] \, i^3 \left(\left[m^2(f), m^2(g) \right] \right).$$

```
def o0(f,  g,  alg):
    B3 = tensorAlgebra([alg] * 5)

    @bimoduleMapDecorator(B3, alg)
    def local0(abcde):
        retVal = g(pureTensor([1, abcde[1], abcde[2], 1]))
        retVal = pureTensor(abcde[0]).tensorProduct(retVal)
        retVal = retVal.tensorProduct(abcde[3:])
        return f(retVal)
    return local0

def o1(f, g, alg):
    B3 = tensorAlgebra([alg] * 5)

    @bimoduleMapDecorator(B3, alg)
    def local0(abcde):
        retVal = g(pureTensor([1, abcde[2], abcde[3], 1]))
        retVal = abcde[:2].tensorProduct(retVal)
        retVal = retVal.tensorProduct(abcde[4])
        return f(retVal)
    return local0
```

In order to define the Gerstenhaber bracket we also need both `m_2Dual` (see Appendix B.2) and `i_3Dual` (see Appendix B.3.

With these preliminaries completed, the Gerstenhaber bracket is simple to define.

```
def o(f, g, alg):
    def local0(abcde):
        return o0(f, g, alg)(abcde)-o1(f, g, alg)(abcde)
    return local0

def GerstenhaberBracket(f, g, basisOfK3):
    alg = f.algebra
    f = m_2Dual(f)
    g = m_2Dual(g)

    def localBracket(abcde):
        return o(f, g, alg)(abcde)+o(g, f, alg)(abcde)

    return i_3Dual(localBracket, alg, basisOfK3)
```

Appendix C

Calculations Relevant to Deformations of A Arising from Geometric Automorphisms of $\mathbb{K}(u, v)$

C.1 Calculation of F_1 Applied to the Relations of A

We include here the full derivations of $F_1(r)$ for each relation of the algebra A. These are recorded in Section 6.3 in Table 6.1. Recall that in the following calculations $U(s)$ and $V(s)$ satisfy:

$$U(0) = uv \text{ and } V(0) = v.$$

1.

$$\begin{aligned} F_1(r_1) &= \left.\frac{\partial(F(r_1))}{\partial s}\right|_{s=0} = \left.\frac{\partial(F(x_3x_1 - x_1x_3))}{\partial s}\right|_{s=0} \\ &= \left.\frac{\partial(vt^2 - tvt)}{\partial s}\right|_{s=0} = \left.\frac{\partial(vt^2 - V(s)t^2)}{\partial s}\right|_{s=0} \\ &= -\left.\frac{\partial(V(s)t^2)}{\partial s}\right|_{s=0} = -V'(0)t^2 \end{aligned}$$

2.

$$
\begin{aligned}
F_1(r_2) &= \frac{\partial(F(x_4x_2 - x_2x_4)))}{\partial s}\bigg|_{s=0} = \frac{\partial(uvtut - utuvt)}{\partial s}\bigg|_{s=0} \\
&= \frac{\partial(uvU(s)t^2 - uU(s)V(s)t^2)}{\partial s}\bigg|_{s=0} \\
&= (uvU'(0) - uU'(0)V(0) - uU(0)V'(0))t^2 \\
&= -u^2vV'(0)t^2 \text{ where we have used the fact that } V(0) = v.
\end{aligned}
$$

3.

$$
\begin{aligned}
F_1(r_3) &= \frac{\partial(F(x_4x_1 - x_2x_3))}{\partial s}\bigg|_{s=0} = \frac{\partial(uvt^2 - utvt)}{\partial s}\bigg|_{s=0} \\
&= \frac{\partial(uvt^2 - uV(s)t^2)}{\partial s}\bigg|_{s=0} = -uV'(0)t^2
\end{aligned}
$$

4.

$$
\begin{aligned}
F_1(r_4) &= \frac{\partial(F(x_1x_2 - x_2x_3))}{\partial s}\bigg|_{s=0} = \frac{\partial(tut - utvt)}{\partial s}\bigg|_{s=0} \\
&= \frac{\partial(U(s)t^2 - uV(s)t^2)}{\partial s}\bigg|_{s=0} = (U'(0) - uV'(0))t^2
\end{aligned}
$$

5.

$$
\begin{aligned}
F_1(r_5) &= \frac{\partial(F(x_3x_2 - x_1x_4))}{\partial s}\bigg|_{s=0} = \frac{\partial(vtut - tuvt)}{\partial s}\bigg|_{s=0} \\
&= \frac{\partial(vU(s)t^2 - U(s)V(s)t^2)}{\partial s}\bigg|_{s=0} \\
&= (vU'(0) - U(0)V'(0) - U'(0)V(0))t^2 \\
&= -uvV'(0)t^2
\end{aligned}
$$

6.

$$
\begin{aligned}
F_1(r_6) &= \frac{\partial(F(x_4x_3 - x_1x_4))}{\partial s}\bigg|_{s=0} = \frac{\partial(uvtvt - tuvt)}{\partial s}\bigg|_{s=0} \\
&= \frac{\partial(uvV(s)t^2 - U(s)V(s)t^2)}{\partial s}\bigg|_{s=0} \\
&= (uvV'(0) - U'(0)V(0) - U(0)V'(0)))t^2 \\
&= -vU'(0)t^2
\end{aligned}
$$

C.2 Investigation of Other Choices for the Map b in the Case of $\mathbb{P}^2$

We include here the details of calculating the image of the admissible directions in the case of $\mathbb{P}^2$. See Section 6.5.2 background and in particular Notation 6.5.2 for the notation we use here. In each case, choosing a point to blow up is equivalent to choosing a point on $\mathcal{Q}$ to project from. In all cases we use the coordinates $[A : B : C]$ on $\mathbb{P}^2$, $[\alpha : \beta : \gamma : \delta]$ for coordinates on $\mathbb{P}^3$ (in which $\mathcal{Q}$ lives) and $[x : y][z : w]$ as coordinates for $\mathbb{P}^1 \times \mathbb{P}^1$. Of course, in every case we will use the same τ_s defined by:

$$\tau_s := \begin{pmatrix} (1+as) & bs & cs \\ ds & (1+es) & fs \\ gs & hs & (1-as-es) \end{pmatrix} + O(s^2).$$

We also point out that in order for an element of the Lie algebra to be inadmissible we need to find terms that do not lie in A. We can determine these terms simply by observing the powers of u and v appearing in $U'(0)$ and $V'(0)$. In Cases 2,3 and 4 we only carry out the calculations in one of two possible sub-cases since by Proposition 6.5.3 the answers in the two sub-cases are always equal.

Case 2) $p := G$ **or** $p = P$

$P = [0 : 1][0 : 1]$ is sent to $[0 : 0 : 1 : 0]$ on $\mathcal{Q}$ so we project from this point, which gives the map $[\alpha : \beta : \gamma : \delta] \dashrightarrow [\alpha : \beta : \delta]$. Composing this with the Segre embedding gives us the map:

$$[x : y][z : w] \dashrightarrow [xw : xz : yz].$$

We note then that $u = B/C$ and $v = B/A$ in these coordinates, so that $\sigma_{\mathbb{P}^2}$ is the following composition:

$$\begin{aligned} \sigma_{\mathbb{P}^2} : [A : B : C] &\dashrightarrow [B : C][B : A] \\ &\overset{\sigma}{\dashrightarrow} [B^2 : AC][B : A] \\ &\dashrightarrow [B^2A : B^3 : BAC] = [BA : B^2 : AC] \end{aligned}$$

We note that $u \mapsto \frac{B^2}{AC} = uv$ and $v \mapsto \frac{B^2}{BA} = v$ as required. Then $\sigma^* \circ \tau_s^*$ has the following effect (up to degree 1) on u:

$$\begin{aligned}
u \overset{\tau_s^*}{\longmapsto} & \frac{dsA + (1+es)B + fsC}{gsA + hsB + (1-as-es)C} \\
= & \frac{dsuv^{-1} + (1+es)u + fs}{gsuv^{-1} + hsu + (1-as-es)} \\
\overset{\sigma}{\mapsto} & \frac{dsu + (1+es)uv + fs}{gsu + hsv + (1-as-es)}.
\end{aligned}$$

This has the following derivative:

$$\begin{aligned}
\partial_s(\sigma^* \circ \tau_s^*(u))|_{s=0} &= (du + euv + f) - uv(gu + huv - a - e) \\
&= du + f - gu^2v - hu^2v^2 - auv
\end{aligned}$$

As for v we find instead:

$$\begin{aligned}
v \overset{\tau_s^*}{\longmapsto} & \frac{dsA + (1+es)B + fsC}{(1+as)A + bsB + csC} \\
= & \frac{dsuv^{-1} + (1+es)u + fs}{(1+as)uv^{-1} + bsu + cs} \\
\overset{\sigma}{\mapsto} & \frac{dsu + (1+es)uv + fs}{(1+as)u + bsuv + cs}
\end{aligned}$$

Which has the following derivative:

$$\begin{aligned}
\partial_s(\sigma^* \circ \tau_s^*(v))|_{s=0} &= \frac{u(du + euv + f) - uv(au + buv + c)}{u^2} \\
&= d + ev + fu^{-1} - av - bv^2 - cu^{-1}v
\end{aligned}$$

It is clear from these two, by comparison with Table 6.1 that for the direction in question to be admissible one must require $f = 0 = c$. Therefore the size of the admissible space one dimension bigger than for the fundamental points of σ. We note however that the obtained image in HH_2^2 is the same size and still lies in V_g.

Relation	Formula	Image Under F_1
$r_1 = x_3x_1 - x_1x_3$	$-V'(0)$	$(bv^2 + av - ev - d)t^2$
$r_2 = x_2x_4 - x_4x_2$	$-u^2vV'(0)$	$(bu^2v^3 + au^2v^2 - eu^2v^2 - du^2v)t^2$
$r_3 = x_4x_1 - x_2x_3$	$-uV'(0)$	$(buv^2 + auv - euv - du)t^2$
$r_4 = x_1x_2 - x_2x_3$	$U'(0) - uV'(0)$	$(-hu^2v^2 - gu^2v + buv^2 - euv)t^2$
$r_5 = x_3x_2 - x_1x_4$	$-uvV'(0)$	$(buv^3 + auv^2 - euv^2 - duv)t^2$
$r_6 = x_4x_3 - x_1x_4$	$-vU'(0)$	$(hu^2v^3 + gu^2v^2 + auv^2 - duv)t^2$

Case 3) $p \in X \setminus \{P, F\}$ **or** $p \in Y \setminus \{G, Q\}$

Let $p = [0 : 1][1 : M] \in X \setminus \{P, F\}$, where $M \in \mathbb{K}^*$. The point $[0 : 1][1 : M]$ corresponds to $[0 : 0 : M : 1]$ on $\mathcal{Q}$ so we project from this point. This is then the map $[\alpha : \beta : \gamma : \delta] \dashrightarrow [\alpha : \beta : \gamma - M\delta]$. Composing this with the Segre embedding gives us the map:

$$[x : y][z : w] \dashrightarrow [xw : xz : yw - Myz].$$

Then $u = \frac{A-MB}{C}$ and $v = B/A$ in these coordinates, so that $\sigma_{\mathbb{P}^2}$ is the following composition:

$$\begin{aligned}
\sigma_{\mathbb{P}^2} : [A : B; C] &\dashrightarrow [A - MB : C][B : A] \\
&\overset{\sigma}{\dashrightarrow} [AB - MB^2 : AC][B : A] \\
&\dashrightarrow [A^2B - MB^2A : AB^2 - MB^3 : A^2C - MBAC] \\
&= [A^2B : AB^2 : A^2C]
\end{aligned}$$

We note that

$$u \mapsto \frac{A^2B - MAB^2}{A^2C} = \left(\frac{A^2 - MAB}{AC}\right)\left(\frac{B}{A}\right) = uv$$

and

$$v \mapsto \frac{AB^2}{BA^2} = v$$

as required. Also, the following formula is very helpful in simplifying the following expression and is easily derivable from the above:

$$\frac{C}{A} = (1 - Mv)u^{-1}$$

Then $\sigma^* \circ \tau_s^*$ has the following effect (up to degree 1) on v:

$$\begin{aligned}
v &\overset{\tau_s^*}{\longmapsto} \frac{dsA + (1 + es)B + fsC}{(1 + as)A + bsB + csC} \\
&= \frac{ds + (1 + es)v + fs(1 - Mv)u^{-1}}{(1 + as) + bsv + cs(1 - Mv)u^{-1}} \\
&= \frac{dsu + (1 + es)uv + fs(1 - Mv)}{(1 + as)u + bsuv + cs(1 - Mv)} \\
&\overset{\sigma}{\mapsto} \frac{dsuv + (1 + es)uv^2 + fs(1 - Mv)}{(1 + as)uv + bsuv^2 + cs(1 - Mv)}
\end{aligned}$$

Which has the following derivative:

$$\partial_s(\sigma^* \circ \tau_s^*(v))|_{s=0} = \frac{uv(duv + euv^2 + f(1 - Mv)) - uv^2(auv + buv^2 + c(1 - Mv))}{u^2v^2}$$
$$= d + ev + f(1 - Mv)u^{-1}v^{-1} - av + buv^2 + c(1 - Mv)u^{-1}$$

From this formula we can deduce immediately that for this to be admissible we require f, c and b to be 0. As for u,

$$u \overset{\tau_s^*}{\longmapsto} \frac{(1 + as)A - M(dsA + (1 + es)B)}{gsA + hsB + (1 - as - es)C}$$
$$= \frac{(1 + as) - M(ds + (1 + es)v)}{gs + hsv + (1 - as - es)(1 - Mv)u^{-1}}$$
$$\overset{\sigma}{\mapsto} \frac{(1 + as) - M(ds + (1 + es)v)}{gs + hsv + (1 - as - es)(1 - Mv)u^{-1}v^{-1}}$$

In the following we set:

$$\chi = (a - Md - Mev - (1 - Mv)(g + hv + (-a - e)(1 - Mv)u^{-1}v^{-1})).$$

Then the derivative is as follows:

$$\partial_s(\sigma^* \circ \tau_s^*(u))|_{s=0} = \frac{(1 - Mv)u^{-1}v^{-1}\chi}{((1 - Mv)u^{-1}v^{-1})^2}$$
$$= \frac{a - Md - Mev - guv - huv^2 + (-a - e)(1 - Mv)}{(1 - Mv)u^{-1}v^{-1}}$$

Now from this formula we can see that in fact all of the terms must be zero, since they will be multiplied by the term $(1 - Mv)^{-1}$ and no such term lies in A.

Case 4) $p \in Z \setminus \{P, Q\}$ **or** $p \in W \setminus \{F, G\}$

If $p \in Z \setminus \{P, Q\}$ then we can write it as $p = [1 : M][0 : 1]$ for some $M \in \mathbb{K}^*$. $[1 : M][0 : 1]$ is sent to $[1 : 0 : M : 0]$ on $\mathcal{Q}$ so we project from this point. This is then the map $[\alpha : \beta : \gamma : \delta] \dashrightarrow [\beta : \gamma - M\alpha : \delta]$. Composing this with the Segre embedding gives us the map:

$$[x : y][z : w] \dashrightarrow [xz : yw - Mxw : yz].$$

Then that $u = \frac{A}{C}$ and $v = \frac{C-MA}{B}$ in these coordinates, so that $\sigma_{\mathbb{P}^2}$ is the following composition:

$$\begin{aligned}\sigma_{\mathbb{P}^2} : [A : B; C] &\dashrightarrow [A : C][C - MA : B] \\ &\overset{\sigma}{\dashrightarrow} [A(C - MA) : CB][C - MA : B] \\ &\dashrightarrow [A(C - MA)^2 : CB^2 - MAB(C - MA) : CB(C - MA)]\end{aligned}$$

We note that

$$u \mapsto \frac{A(C - MA)^2}{CB(C - MA)} = (\frac{A}{C})(\frac{C - MA}{B}) = uv$$

and

$$v \mapsto \frac{CB(C - MA) - MA(C - MA)^2}{CB^2 - MAB(C - MA)} = \frac{(CB - MA(C - MA))(C - MA)}{B(CB - MA(C - MA))}v$$

as required. Also, the following formula is very helpful in simplifying the following expression and is easily derivable from the above:

$$\frac{B}{C} = (1 - Mu)v^{-1}$$

Then $\sigma^* \circ \tau_s^*$ has the following effect (up to degree 1) on v:

$$\begin{aligned} v &\overset{\tau_s^*}{\longmapsto} \frac{gsA + hsB + (1 - as - es)C - M((1 + as)A + bsB + csC)}{dsA + (1 + es)B + fsC} \\ &= \frac{gsu + hs(1 - Mu)v^{-1} + (1 - as - es) - M((1 + as)u + bs(1 - Mu)v^{-1} + cs)}{dsu + (1 + es)(1 - Mu)v^{-1} + fs} \\ &= \frac{gsuv + hs(1 - Mu) + (1 - as - es)v - M((1 + as)uv + bs(1 - Mu) + csv)}{dsuv + (1 + es)(1 - Mu) + fsv} \\ &\overset{\sigma}{\mapsto} \frac{gsuv^2 + hs(1 - Muv) + (1 - as - es)v - M((1 + as)uv^2 + bs(1 - Muv) + csv)}{dsuv^2 + (1 + es)(1 - Muv) + fsv}\end{aligned}$$

We let $F(s)$ and $G(s)$ represent the numerator and denominator respectively. Then the following intermediate values will expedite the calculation of the derivative:

$$\begin{aligned}&F(0) = v(1 - Muv), \\ &F'(0) = guv^2 + h(1 - Muv) - av - ev - Mauv^2 - bM(Muv) + cvM \\ &G(0) = 1 - Muv,\ G'(0) = duv^2 + e(1 - Muv) + fv\end{aligned}$$

So the derivative is as follows:

$$\begin{aligned}
\partial_s(\sigma^* \circ \tau_s^*(v))|_{s=0} &= \frac{(1-Muv)F'(0) - v(1-Muv)G'(0)}{(1-Muv)^2} \\
&= \frac{1}{1-Muv}(F'(0) - vG'(0)) \\
&= g\frac{uv^2}{1-Muv} + h - e\frac{v(-2+Muv)}{1-Muv} - a\frac{v(1+Muv)}{1-Muv} \\
&- bM + c\frac{vM}{1-Muv} - d\frac{uv^3}{1-Muv} - f\frac{v^2}{1-Muv}
\end{aligned}$$

From this formula we can deduce immediately that for this to be admissible we require all of the parameters to be 0 except perhaps b and h.

As for the calculation on u,

$$\begin{aligned}
u &\overset{\tau_s^*}{\longmapsto} \frac{(1+as)A + bsB + csC}{gsA + hsB + (1-as-es)C} \\
&= \frac{A + bsB}{hsB + C} \\
&= \frac{u + bs(1-Mu)v^{-1}}{hs(1-Mu)v^{-1} + 1} \\
&\overset{\sigma}{\mapsto} \frac{uv + bs(1-Muv)v^{-1}}{hs(1-Muv)v^{-1} + 1}
\end{aligned}$$

Which has the following derivative:

$$\partial_s(\sigma^* \circ \tau_s^*(u))|_{s=0} = b(1-Muv)v^{-1} - h(1-Muv)u$$

Since we require that $F_1(x_1x_2 - x_2x_3) = (U'(0) - uV'(0)t^2) \in A$ (see Table 6.1), we can see that this implies that $b = 0$ as well. However, h is free to be set as one wishes. Whatever the value of h however, the infinitesimal deformations obtained lie in V_g.

Case 5)

See Section 6.5.2 for the calculations in this case.

Appendix D

Calculations Relevant to Deformations of A_q Arising from Quantised Geometric Automorphisms of $\mathbb{K}_q(u, v)$

D.1 Calculation of F_1 applied to the Relations of A_q

We include here the full derivations of F_1 applied to the relations of A_q. This supplements the work in Section 7.3.1, and the results are collected in Table 7.1.

1. Recall that $r_1 = x_3x_1 - x_1x_3$. Therefore we have the following:

$$\begin{aligned} F_1(r_1) = F_1(vt^2 - tvt) &= vt^2 - v(1+as)t^2 \\ &= -\, svat^2 \end{aligned}$$

2. Recall that $r_2 = x_4x_2 - qx_2x_4$ and that $\sigma \circ \tau_s(u) = uvf_s(v)$. Therefore we have the following:

$$\begin{aligned} F_1(r_2) &= F_1(uvtut - qutuvt) \\ &= \frac{uvuvf_s(v)t^2 - qu^2vf_s(v)(1+as)vt^2}{s}\bigg|_{s=0} \\ &= \frac{u^2q(1+\lambda s)v^2f_s(v)t^2 - qu^2(1+as)v^2f_s(v)t^2}{s}\bigg|_{s=0} \\ &= (q\lambda u^2v^2 - qau^2v^2)t^2 \end{aligned}$$

3. Recall that $r_3 = x_4x_1 - x_2x_3$. Therefore we have the following:

$$\begin{aligned} F_1(r_3) = F_1(uvt^2 - utvt) &= \frac{uvt^2 - uv(1+as)t^2}{s} \\ &= -\,auvt^2 \end{aligned}$$

4. See Section 7.3.1 for this case.

5. Recall that $r_5 = x_3x_2 - qx_1x_4$. Therefore we have the following:

$$\begin{aligned} F_1(r_5) = F_1(vtut - qtuvt) &= \frac{vuvf_s(v)t^2 - quvf_s(v)v(1+as)t^2}{s}\Bigg|_{s=0} \\ &= \frac{q(1+\lambda s)uv^2f_s(v) - q(1+as)uv^2f_s(v)t^2}{s}\Bigg|_{s=0} \\ &= (q\lambda uv^2 - qauv^2)t^2 \end{aligned}$$

6. See Section 7.3.1 for this case.

D.2 Calculation of the Cohomology Class of F_1 for A_q

We include here the infinitesimal F_1 corresponding to the deformations discussed in Section 7.3, using the vector notation defined in Notation 3.2.1. We have also expanded this vector to be written in the chosen basis of 2-cocycles from Appendix A.2.1; we have underlined vectors that are coboundaries.

$$
\begin{aligned}
& a\begin{pmatrix} -x_1x_3 \\ -qx_2x_4 \\ -x_2x_3 \\ -x_2x_3 \\ -qx_1x_4 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2x_1 \\ 0 \\ -x_2x_3 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \\ -x_1x_4 \end{pmatrix} + \frac{d}{q}\begin{pmatrix} 0 \\ 0 \\ 0 \\ qx_1x_4 \\ 0 \\ -x_3x_4 \end{pmatrix} + q\lambda\begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \\
&= a\left(-\begin{pmatrix} x_1x_3 \\ 0 \\ 0 \\ x_2x_3 \\ qx_1x_4 \\ 0 \end{pmatrix} - q\begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} - \underline{\begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ 0 \\ -qx_1x_4 \\ 0 \end{pmatrix}}\right) \\
&+ b\left(-\underline{\begin{pmatrix} 0 \\ 0 \\ -x_2x_1 \\ -x_2x_1 \\ x_2x_3 \\ x_2x_3 \end{pmatrix}} - \underline{\begin{pmatrix} 0 \\ (1-q)x_2^2 \\ x_2x_1 \\ 0 \\ -qx_2x_3 \\ 0 \end{pmatrix}} + (1-q)\begin{pmatrix} 0 \\ x_2^2 \\ 0 \\ 0 \\ x_2x_3 \\ 0 \end{pmatrix}\right) \\
&+ q\lambda\begin{pmatrix} 0 \\ x_2x_4 \\ 0 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} + c\left(\underline{\begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -qx_1x_4 \\ -x_1x_4 \end{pmatrix}} - \underline{\begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ 0 \\ -qx_1x_4 \\ 0 \end{pmatrix}}\right) \\
&+ \frac{d}{q}\left(\underline{\begin{pmatrix} 0 \\ 0 \\ x_1x_4 \\ qx_1x_4 \\ -qx_3x_4 \\ -x_3x_4 \end{pmatrix}} + \underline{\begin{pmatrix} 0 \\ (1-q)x_4^2 \\ -x_1x_4 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}} - (1-q)\begin{pmatrix} 0 \\ x_4^2 \\ 0 \\ 0 \\ x_3x_4 \\ 0 \end{pmatrix}\right)
\end{aligned}
$$

D.3 Calculation of the Cohomology Class of F_1 for A

We include here the infinitesimal F_1 corresponding to the deformations discussed in Section 7.3.2, using the vector notation defined in Notation 3.2.1. We have also expanded this vector to be written in the chosen basis of 2-cocycles from Appendix A.1.1;

we have underlined vectors that are coboundaries.

$$a\begin{pmatrix}-x_1x_3\\-x_2x_4\\-x_2x_3\\-x_2x_3\\-x_1x_4\\0\end{pmatrix}+b\begin{pmatrix}0\\0\\0\\x_2x_1\\0\\-x_2x_3\end{pmatrix}+c\begin{pmatrix}0\\0\\0\\x_2x_3\\0\\-x_1x_4\end{pmatrix}+d\begin{pmatrix}0\\0\\0\\x_1x_4\\0\\-x_3x_4\end{pmatrix}+\lambda\begin{pmatrix}0\\x_2x_4\\0\\0\\x_1x_4\\0\end{pmatrix}$$

$$=a\left(-\begin{pmatrix}x_1x_3\\0\\0\\x_2x_3\\x_1x_4\\0\end{pmatrix}-\begin{pmatrix}0\\x_2x_4\\0\\0\\x_1x_4\\0\end{pmatrix}-\underline{\begin{pmatrix}0\\0\\x_2x_3\\0\\-x_1x_4\\0\end{pmatrix}}\right)$$

$$+\lambda\begin{pmatrix}0\\x_2x_4\\0\\0\\x_1x_4\\0\end{pmatrix}+c\left(\underline{\begin{pmatrix}0\\0\\x_2x_3\\x_2x_3\\-x_1x_4\\-x_1x_4\end{pmatrix}}-\underline{\begin{pmatrix}0\\0\\x_2x_3\\0\\-x_1x_4\\0\end{pmatrix}}\right)$$

$$+d\left(\underline{\begin{pmatrix}0\\0\\x_1x_4\\x_1x_4\\-x_3x_4\\-x_3x_4\end{pmatrix}}+\underline{\begin{pmatrix}0\\0\\-x_1x_4\\0\\x_3x_4\\0\end{pmatrix}}\right)+b\left(-\underline{\begin{pmatrix}0\\0\\-x_2x_1\\-x_2x_1\\x_2x_3\\x_2x_3\end{pmatrix}}-\underline{\begin{pmatrix}0\\0\\x_2x_1\\0\\-x_2x_3\\0\end{pmatrix}}\right)$$

Appendix E

Calculations Relevant to a Family of PBW Deformations of A

E.1 Choices of the Sets Ξ_i

In this section we print our guesses for sets Ξ_i that will 'interact well' with the chosen basis elements of HH^2_2. For a discussion of the background to this please see Section 8.1.

$$\Xi_1 = \left\{ v_{10} = \begin{pmatrix} 0 \\ 0 \\ x_2x_3 \\ x_2x_3 \\ -x_1x_4 \\ -x_1x_4 \end{pmatrix}, v_{13} = \begin{pmatrix} 0 \\ 0 \\ -x_2x_3 \\ 0 \\ x_1x_4 \\ 0 \end{pmatrix} \right\} = \Xi_4$$

$$\Xi_3 = \left\{ v_{18} = \begin{pmatrix} 0 \\ 0 \\ -x_2x_1 \\ -x_2x_1 \\ x_2x_3 \\ x_2x_3 \end{pmatrix}, v_{20} = \begin{pmatrix} 0 \\ 0 \\ x_2x_1 \\ 0 \\ -x_2x_3 \\ 0 \end{pmatrix} \right\} = \Xi_6$$

E.2 Resolving Overlap Ambiguities for $A(a, c, d, f)$

In this section we include for each overlap ambiguity the two simplification paths that lead to their resolution. See Section 8.2 for the background and notation.

$$R_{a,c,d,f} = \left\{ \begin{array}{cc} r_1 := x_3x_1 - (1+a)x_1x_3 - cx_1^2, & r_2 := x_4x_2 - (1+d)x_2x_4 - fx_2^2, \\ r_3 := x_4x_1 - (1+d)x_2x_3 - fx_2x_1, & r_4 := x_1x_2 - x_2x_3, \\ r_5 := x_3x_2 - (1+a)x_1x_4 - cx_2x_3, & r_6 := x_4x_3 - (1+a)x_1x_4 - cx_2x_3 \end{array} \right\}.$$

Overlap 1) The overlap in this case is $x_4x_1x_2$. The right hand simplification path is:

$$x_4x_1x_2 \xrightarrow{r_{x_4r_4}} x_4x_2x_3 \xrightarrow{r_{r_2x_3}} fx_2^2x_3 + (1+d)x_2x_4x_3$$

$$\xrightarrow{r_{x_2r_6}} (c+f+cd)x_2^2x_3 + (1+a+d+ad)x_2x_1x_4$$

The left hand simplification path is:

$$x_4x_1x_2 \xrightarrow{r_{r_3x_2}} fx_2x_1x_2 + (1+d)x_2x_3x_2 \xrightarrow{r_{x_2r_4}} fx_2^2x_3 + (1+d)x_2x_3x_2$$

$$\xrightarrow{r_{x_2r_5}} (c+f+cd)x_2^2x_3 + (1+a+d+ad)x_2x_1x_4$$

Overlap 2) The overlap in this case is $x_3x_1x_2$. The right hand simplification path is:

$$x_3x_1x_2 \xrightarrow{r_{x_3r_4}} x_3x_2x_3 \xrightarrow{r_{r_5x_3}} cx_2x_3^2 + (1+a)x_1x_4x_3$$

$$\xrightarrow{r_{x_1r_6}} cx_2x_3^2 + (c+ac)x_1x_2x_3 + (1+2a+a^2)x_1^2x_4$$

$$\xrightarrow{r_{r_4x_3}} (2c+ac)x_2x_3^2 + (1+2a+a^2)x_1^2x_4$$

The left hand simplification path is:

$$x_3x_1x_2 \xrightarrow{r_{r_1x_2}} cx_1^2x_2 + (1+a)x_1x_3x_2 \xrightarrow{r_{x_1r_4}} cx_1x_2x_3 + (1+a)x_1x_3x_2$$

$$\xrightarrow{r_{r_4x_3}} cx_2x_3^2 + (1+a)x_1x_3x_2$$

$$\xrightarrow{r_{x_1r_5}} cx_2x_3^2 + (c+ac)x_1x_2x_3 + (1+2a+a^2)x_1^2x_4$$

$$\xrightarrow{r_{r_4x_3}} (2c+ac)x_2x_3^2 + (1+2a+a^2)x_1^2x_4$$

Overlap 3) The overlap in this case is $x_4x_3x_1$. The right hand simplification path is:

$$x_4x_3x_1$$
$$\downarrow r_{x_4r_1}$$
$$cx_4x_1^2 + (1+a)x_4x_1x_3$$
$$\downarrow r_{r_3x_1}$$
$$cfx_2x_1^2 + (c+cd)x_2x_3x_1 + (1+a)x_4x_1x_3$$
$$\downarrow r_{x_2r_1}$$
$$(c^2+c^2d+cf)x_2x_1^2 + (c+cd+ac+acd)x_2x_1x_3 + (1+a)x_4x_1x_3$$
$$\downarrow r_{r_3x_3}$$
$$(c^2+c^2d+cf)x_2x_1^2 + (c+f+af+acd+ac+cd)x_2x_1x_3 + (1+a+d+ad)x_2x_3^2$$

The left hand simplification path is:

$$x_4x_3x_1$$
$$\downarrow r_{r_6x_1}$$
$$cx_2x_3x_1 + (1+a)x_1x_4x_1$$
$$\downarrow r_{x_2r_1}$$
$$c^2x_2x_1^2 + (c+ac)x_2x_1x_3 + (1+a)x_1x_4x_1$$
$$\downarrow r_{x_1r_3}$$
$$c^2x_2x_1^2 + (c+ac)x_2x_1x_3 + (f+af)x_1x_2x_1 + (1+a+d+ad)x_1x_2x_3$$
$$\downarrow r_{r_4x_1}$$
$$c^2x_2x_1^2 + (c+ac)x_2x_1x_3 + (af+f)x_2x_3x_1 + (1+a+d+ad)x_1x_2x_3$$
$$\downarrow r_{x_2r_1}$$
$$(c^2+cf+acf)x_2x_1^2 + (c+a^2f+2af+ac+f)x_2x_1x_3 + (1+a+d+ad)x_1x_2x_3$$
$$\downarrow r_{r_4x_3}$$
$$(c^2+cf+acf)x_2x_1^2 + (c+a^2f+2af+ac+f)x_2x_1x_3 + (1+a+d+ad)x_2x_3^2$$

Overlap 4) The overlap in this case is $x_4x_3x_2$. The right hand simplification path is:

$$x_4x_3x_2$$
$$\downarrow r_{x_4r_5}$$
$$cx_4x_2x_3 + (1+a)x_4x_1x_4$$
$$\downarrow r_{r_2x_3}$$
$$cfx_2^2x_3 + (c+cd)x_2x_4x_3 + (1+a)x_4x_1x_4$$
$$\downarrow r_{x_2r_6}$$
$$(c^2+c^2d+cf)x_2^2x_3 + (c+cd+ac+acd)x_2x_1x_4 + (1+a)x_4x_1x_4$$
$$\downarrow r_{r_3x_4}$$
$$(c^2+c^2d+cf)x_2^2x_3 + (c+f+af+acd+ac+cd)x_2x_1x_4 + (1+a+d+ad)x_2x_3x_4$$

The left hand simplification path is:

$$x_4x_3x_2$$
$$\downarrow r_{r_6x_2}$$
$$cx_2x_3x_2 + (1+a)x_1x_4x_2$$
$$\downarrow r_{x_2r_5}$$
$$c^2x_2^2x_3 + (c+ac)x_2x_1x_4 + (1+a)x_1x_4x_2$$
$$\downarrow r_{x_1r_2}$$
$$c^2x_2^2x_3 + (c+ac)x_2x_1x_4 + (f+af)x_1x_2^2 + (1+a+d+ad)x_1x_2x_4$$
$$\downarrow r_{r_4x_2}$$
$$c^2x_2^2x_3 + (c+ac)x_2x_1x_4 + (af+f)x_2x_3x_2 + (1+a+d+ad)x_1x_2x_4$$
$$\downarrow r_{x_2r_5}$$
$$(c^2+cf+acf)x_2^2x_3 + (c+a^2f+2af+ac+f)x_2x_1x_4 + (1+a+d+ad)x_1x_2x_4$$
$$\downarrow r_{r_4x_4}$$
$$(c^2+cf+acf)x_2^2x_3 + (c+a^2f+2af+ac+f)x_2x_1x_4 + (1+a+d+ad)x_2x_3x_4$$

For an interpretation of these results please see the proof of Theorem 8.2.1.

E.3 Verifying that F Satisfies the Relations of $A(a, c, d, f)$

We include here the confirmation that the set F defined in Proposition 8.2.6 does satisfy the relations of $A(a, c, d, f)$. Recall that these relations are:

$$R_{a,c,d,f} = \left\{ \begin{array}{cc} r_1 := x_3x_1 - (1+a)x_1x_3 - cx_1^2, & r_2 := x_4x_2 - (1+d)x_2x_4 - fx_2^2, \\ r_3 := x_4x_1 - (1+d)x_2x_3 - fx_2x_1, & r_4 := x_1x_2 - x_2x_3, \\ r_5 := x_3x_2 - (1+a)x_1x_4 - cx_2x_3, & r_6 := x_4x_3 - (1+a)x_1x_4 - cx_2x_3 \end{array} \right\}.$$

1. Firstly,

$$y_3y_1 = zt^2.$$

   On the other hand, we have:

$$\begin{aligned} (1+a)y_1y_3 + cy_1^2 &= (1+a)tzt + ct^2 \\ &= (1+a)\frac{(z-c)}{(1+a)}t^2 + ct^2 = zt^2. \end{aligned}$$

2. On the one hand:

$$y_4y_2 = zwtwt = zw\beta(w)t^2.$$

   Whereas,

$$\begin{aligned} (1+d)y_2y_4 + fy_2^2 &= (1+d)wtzwt + fwtwt \\ &= (1+d)w\frac{(z-c)}{(1+a)}\beta(w)t^2 + fw\beta(w)t^2 \\ &= \frac{(1+d)}{(1+a)}\frac{(1+a)}{(1+d)}(z-f)w\beta(w)t^2 + fw\beta(w)t^2 \text{ by (8.1)} \\ &= zw\beta(w)t^2 \end{aligned}$$

3. Firstly,

$$y_4y_1 = zwt^2.$$

   On the other hand:

$$\begin{aligned} (1+d)y_2y_3 + fy_2y_1 &= (1+d)wtzt + fwt^2 \\ &= (1+d)w\frac{(z-c)}{(1+a)}t^2 + fwt^2 \\ &= \frac{(1+d)}{(1+a)}\frac{(1+a)}{(1+d)}(z-f)wt^2 \text{ by (8.1)} \\ &= zwt^2. \end{aligned}$$

4. Please see the proof of Proposition 8.2.6 for this relation.

5. Firstly,

$$y_3y_2 = ztwt = z\beta(w)t^2.$$

Whereas,

$$\begin{aligned}(1+a)y_1y_4 + cy_2y_3 &= (1+a)ztwt + cwtzt \\ &= (1+a)\frac{(z-c)}{(1+a)}\beta(w)t^2 + cw\frac{(z-c)}{(1+a)}t^2 \\ &= (z-c)\beta(w)t^2 + c\beta(w)t^2 \text{ by (8.2)} \\ &= z\beta(w)t^2. \end{aligned} \tag{E.1}$$

6.

$$y_4y_3 = zwtzt = zw\beta(z)t^2 = z\beta(w)t^2 \text{ by (8.2)}.$$

By (E.1) this equals $(1+a)y_1y_4 + cy_2y_3$ as required.

Therefore we may conclude that the y_i satisfy the defining relations of $A(a,c,d,f)$.

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