CENTRAL SERIES IN MCLAIN GROUPS

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Contents

<u>Chapter 1. Summary</u>

The groups studied in this thesis were introduced by D. H. McLain in his Ph. D. dissertation. By a McLain group we mean a group G derived as follows. Let V be a vector space, basis $\{e_\lambda : \lambda \in \Omega\}$ where Ω is a partially ordered set, over a field F. For $\lambda < \mu \in \Omega$, define an automorphism of V, $t(\lambda,\mu)$, by

$$t(\lambda,\mu)(e_\mu) = e_\mu + e_\lambda$$

$$t(\lambda,\mu)(e_\sigma) = e_\sigma \qquad \sigma \neq \mu$$

Let $G = gp\{t(\lambda,\mu) : \lambda < \mu \in \Omega\}$

In Chapter 2 we define the altitude and depth of an element α of Ω, and the separation between two elements α and β. Some properties of altitude, depth and separation are described, and results which will be needed in later chapters are given.

In Chapter 3 we study the upper central series of G and show that a necessary and sufficient condition for G to have a centre is that Ω has a minimal element $\underline{\omega}$ and a maximal element $\bar{\omega}$ with $\underline{\omega} < \bar{\omega}$. We find an expression for a general term of the upper central series, viz.

$$\zeta_\lambda(G) = gp\{t(\alpha,\beta) : alt\alpha + dep\beta < \lambda\}$$

and hence an expression for the central height of G. A necessary and sufficient condition for G to be a ZA group is also given.

In Chapter 4 we consider the lower central series of G and obtain an expression for a general term of the lower central series, viz.

$$\gamma_\lambda(G) = gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \lambda\}$$

Some results connecting the central height and central depth of G are given, and some partial results about the derived series of G are obtained. We show that if G is a ZA group then G is a ZD group, but give an example to show that the converse is not necessarily true.

Finally, in Chapter 5, we consider the central series of the semidirect product, V.G. The upper central series of V.G is completely characterised, and partial results are obtained for the lower central sreies and the derived series of the semidirect product.

Chapter 2. Definitions and preliminary results

Let V be a vector space, basis $\{e_\lambda : \lambda \in \Omega\}$, where Ω is a partially ordered set with order $<$, over a field F.

For $\lambda < \mu \in \Omega$, define $t(\lambda,\mu)$ by

$$t(\lambda,\mu)(e_\mu) = e_\mu + e_\lambda$$

$$t(\lambda,\mu)(e_\sigma) = e_\sigma \qquad \sigma \neq \mu$$

Let $G = gp\{t(\lambda,\mu) : \lambda,\mu \in \Omega \text{ and } \lambda < \mu\}$

G is usually written $Mc(\Omega,F)$, the McLain group of Ω over F.

Let G have identity 1 and V have identity 0.

We first give some results about expanding commutators.

Lemma 2.1 In any group K, if $x, x_1, \ldots x_n, y, y_1, \ldots y_r \in K$, then

$$[x_1 \ldots\ldots x_n, y] = [x_1,y]^{x_2 \ldots x_n} \ldots\ldots [x_n,y]$$

and $$[x,y_1 \ldots\ldots y_r] = [x,y_r][x,y_{r-1}]^{y_r} \ldots\ldots [x,y_1]^{y_2 \ldots y_r}$$

where $[x,y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy = x[x,y]$

Lemma 2.2 If $\alpha, \beta, \gamma, \delta \in \Omega$ with $\alpha < \beta$ and $\gamma < \delta$, then

$$[t(\alpha,\beta),t(\gamma,\delta)] = 1 \text{ if } \alpha \neq \delta \text{ and } \beta \neq \gamma$$

$$= t(\alpha,\delta) \text{ if } \beta = \gamma$$

$$= t(\gamma,\beta)^{-1} \text{ if } \alpha = \delta$$

We use the following definitions and conventions :-

$[x,y,z] = [[x,y],z]$
ω is the first infinite ordinal
$a \leq b$ means that a is less than b or a = b
$A \subseteq B$ means that A is contained in B or A = B
* denotes reverse ordering
$m\{a,b\}$ = maximum of a and b.

<u>Corollary 2.3</u> If $\alpha, \beta, \gamma, \delta \in \Omega$ with $\alpha < \beta$ and $\gamma < \delta$, then

$$[t(\alpha,\beta)^k, t(\gamma,\delta)^n] = 1 \text{ if } \alpha \neq \delta,\ \beta \neq \gamma$$
$$= t(\alpha,\delta)^{kn} \text{ if } \beta = \gamma$$
$$= t(\gamma,\beta)^{-kn} \text{ if } \alpha = \delta$$

The next result is proved in (10).

<u>Lemma 2.4</u> If Ω is a partially ordered set with order $<$, then there exists a total ordering $\leqslant$ of Ω where if $\lambda < \mu$ then $\lambda \leqslant \mu$. Call $<$ and $\leqslant$ compatible.

Fix a total ordering $\leqslant$ on Ω which is compatible with $<$. Since $\leqslant$ is an extension of $<$, this should not cause any confusion. We use this total order to obtain a canonical form for an element u of G.

<u>Lemma 2.5</u> Let $u = t(\lambda_1,\mu_1)^{m_1} \ldots\ldots t(\lambda_r,\mu_r)^{m_r} \in G$ where $\forall i\ m_i \neq 0$. Then u can be expressed in a canonical form $t(\alpha_1,\beta_1)^{n_1} \ldots\ldots t(\alpha_k,\beta_k)^{n_k}$ with $n_i \neq 0$ where the pairs (α_i,β_i) are distinct and if $1 \leqslant i < j \leqslant k$, then

$\alpha_i = \alpha_j \rightarrow \beta_i < \beta_j$ or β_i and β_j are not comparable and

$\beta_i = \beta_j \rightarrow \alpha_i > \alpha_j$ or α_i and α_j are not comparable.

<u>Proof</u> $u = t(\lambda_1,\mu_1)^{m_1} \ldots\ldots t(\lambda_r,\mu_r)^{m_r}$.

With the total order $\leqslant$, $\{\lambda_1,\mu_1,\ldots\ldots\ldots\lambda_r,\mu_r\}$ forms a finite totally ordered set, $\{\sigma_1 \leqslant \sigma_2 \leqslant \ldots\ldots\ldots \leqslant \sigma_n\}$ say.

For $1 \leqslant i,j \leqslant n$ and $\sigma_i \leqslant \sigma_j$ define the weight of (σ_i,σ_j), written $w(\sigma_i,\sigma_j)$ as follows:-

$w(\sigma_i,\sigma_j) = 0$ if $\forall k\ \sigma_k \leqslant \sigma_i$ or $\sigma_j \leqslant \sigma_k$

$w(\sigma_i,\sigma_j) = q$ if $\exists k$ such that $w(\sigma_i,\sigma_k) = q-1$ and

$w(\sigma_k,\sigma_j) = 0$

Order the pairs (σ_i,σ_j) as follows, using $<'$ for the order.

If $w(\sigma_i,\sigma_j) < w(\sigma_k,\sigma_m)$, then $(\sigma_i,\sigma_j) <' (\sigma_k,\sigma_m)$.

If $w(\sigma_i,\sigma_j) = w(\sigma_k,\sigma_m)$ and $\sigma_i \leq \sigma_k$ then $(\sigma_i,\sigma_j) <' (\sigma_k,\sigma_m)$.

Denote $\{(\sigma_{i_1},\sigma_{j_1}) <' \ldots\ldots\ldots\ldots <' (\sigma_{i_n},\sigma_{j_n})\}$ by $\{(\gamma_1,\delta_1) <' \ldots\ldots\ldots <' (\gamma_n,\delta_n)\}$.

If $t(\gamma_1,\delta_1) = t(\lambda_1,\mu_1)$ and $t(\gamma_1,\delta_1)$ does not occur anywhere else in the expression for u, or if $t(\gamma_1,\delta_1)$ does not occur at all in the expression for u, leave $t(\gamma_1,\delta_1)$ and consider $t(\gamma_2,\delta_2)$. If not, 'move' all occurrences of $t(\gamma_1,\delta_1)$ to the beginning of the expression for u, using the relation $ab = ba[a,b]$. The commutators $[a,b]$ obtained in this way will either be 1 or of the form $t(\lambda,\mu)$ with at least one of $\lambda < \gamma_1$ and $\mu > \delta_1$ holding, by corollary 2.3 i.e. $w(\lambda,\mu) \ngtr w(\gamma_1,\delta_1)$ and so $(\lambda,\mu) = (\gamma_j,\delta_j)$ for some $j > 1$.

Suppose $u = t(\alpha_1,\beta_1)^{n_1}\ldots t(\alpha_x,\beta_x)^{n_x} t(\lambda_m,\mu_m)^{r_m}\ldots t(\lambda_y,\mu_y)^{r_y}$ where $(\alpha_1,\beta_1) <' (\alpha_2,\beta_2) <' \ldots\ldots <' (\alpha_x,\beta_x)$.

Let $(\gamma_j,\delta_j) = (\lambda_i,\mu_i)$ for some i with $m \leq i \leq y$ be such that $(\alpha_x,\beta_x) <' (\gamma_j,\delta_j)$ and if $(\gamma_k,\delta_k) <' (\gamma_j,\delta_j)$ then $(\gamma_k,\delta_k) <' (\alpha_x,\beta_x)$ or $(\gamma_k,\delta_k) \neq (\lambda_z,\mu_z)$ $m \leq z \leq y$.

Move $t(\gamma_j,\delta_j)$ to the position immediately after $t(\alpha_x,\beta_x)^{n_x}$ using the relation $ab = ba[a,b]$. The commutators $[a,b]$ will either be 1 or of the form $t(\lambda,\mu)$ with at least one of $\lambda < \gamma_j$ or $\mu \ngtr \delta_j$ holding i.e. $w(\lambda,\mu) > w(\gamma_j,\delta_j)$ and so $(\lambda,\mu) = (\gamma_k,\delta_k)$ for some $k > j$.

In this way u can be put in the form stated in the lemma.

A similar result is proved in (4).

Once $\leq$ has been fixed, the canonical form for u obtained in the above manner is unique, but not otherwise. In what follows 'canonical form' means the canonical form obtained using $\leqslant$ as in the proof of lemma 2.5.

Note that u in canonical form is the identity iff no terms occur in it.

Let $u = t(\alpha_1, \beta_1)^{n_1} \ldots\ldots t(\alpha_k, \beta_k)^{n_k} \in G$ be in canonical form and let $\bar{\beta} \in \{\beta_1, \ldots\ldots, \beta_k\}$. Call $\bar{\alpha}$ a maximal element associated with $\bar{\beta}$ in u if $t(\bar{\alpha}, \bar{\beta})$ is one of the generators occurring in the canonical expression for u and if $t(\alpha_i, \bar{\beta})$ is one of the generators occurring in the canonical expression for u, then $\alpha_i < \bar{\alpha}$ or α_i and $\bar{\alpha}$ are not comparable.

Let $\underline{\alpha} \in \{\alpha_1, \ldots\ldots, \alpha_k\}$. Call $\underline{\beta}$ a minimal element associated with $\underline{\alpha}$ in u if $t(\underline{\alpha}, \underline{\beta})$ is one of the generators occurring in the canonical expression for u and if $t(\underline{\alpha}, \beta_j)$ is one of the generators occurring in the canonical expression for u, then $\beta_j > \underline{\beta}$ or β_j is not comparable with $\underline{\beta}$.

If Ω is totally ordered, then there is a unique maximal element associated with $\bar{\beta}$ and a unique minimal element associated with $\underline{\alpha}$ in u.

The next results introduce the natural sum of two ordinal numbers which will be needed in chapter 3.

<u>Lemma 2.6</u> Any ordinal $\rho \neq 0$ can be uniquely represented in the normal form

$$\rho = \omega^{\beta_1} y_1 + \omega^{\beta_2} y_2 + \ldots\ldots + \omega^{\beta_k} y_k$$

where ω is the first infinite ordinal, k, $y_1, \ldots\ldots y_k$ are finite ordinals $\neq 0$ and the exponents are ordinals satisfying $\beta_1 > \beta_2 > \ldots\ldots > \beta_k \geqslant 0$.

The above lemma is theorem 10 of chapter 3 §11 of (1).

Let σ, ρ be in normal form as in lemma 2.6,

$$\sigma = \sum_x \omega^{\beta_x} \gamma_x, \qquad \rho = \sum_x \omega^{\beta_x} \delta_x$$

Since each sum contains only a finite number of terms, the same exponents β_x can be used in both these cases, admitting a finite number of coefficients $\gamma_x = 0$ and $\delta_x = 0$. Then the natural sum of σ and ρ, written $\sigma \oplus \rho$ is defined by

$$\sigma \oplus \rho = \sum_x \omega^{\beta_x}(\gamma_x + \delta_x)$$

Note that $\sigma \oplus \rho = \rho \oplus \sigma$, although $\sigma \oplus \rho$ need not equal either of the ordered sums $\sigma + \rho$ and $\rho + \sigma$.

Lemma 2.7 If $\sigma < \rho$, then $\sigma \oplus \tau < \rho \oplus \tau$ for any ordinal τ.

Proof Let $\sigma = \sum_y \omega^{\beta_y} \gamma_y$, $\rho = \sum_y \omega^{\beta_y} \delta_y$ and $\tau = \sum_y \omega^{\beta_y} \lambda_y$.

Then, since each sum contains only a finite number of terms, the same exponents β_y can be used in all three cases, admitting a finite number of coefficients $\gamma_y = 0$, $\delta_y = 0$ and $\lambda_y = 0$.

Then
$$\sigma \oplus \tau = \sum_y \omega^{\beta_y}(\gamma_y + \lambda_y)$$

$$\rho \oplus \tau = \sum_y \omega^{\beta_y}(\delta_y + \lambda_y)$$

Since $\sigma < \rho$, $\exists$ k such that $\gamma_k < \delta_k$ and for $i < k$, $\gamma_i = \delta_i$.

So for some k, since γ_k, δ_k and λ_k are finite, $\gamma_k + \lambda_k < \delta_k + \lambda_k$ and, for $i < k$, $\gamma_i + \lambda_i = \delta_i + \lambda_i$.

Hence $\sigma \oplus \tau < \rho \oplus \tau$.

A chain C is a maximal totally ordered set.

Call α and $\beta \in \Omega$ end elements of a chain if $\alpha < \beta$ and $\forall \gamma \in \Omega$, $\alpha < \gamma$ or α and γ are incomparable and $\beta > \gamma$ or β and γ are incomparable.

Call a chain with end elements an interval.

Denote any interval with end elements α and β by $[\alpha,\beta]$. If Ω is partially ordered there may be several intervals $[\alpha,\beta]$, but if Ω is totally ordered $[\alpha,\beta]$ is unique.

If $\alpha < \beta$ and α, $\beta \in$ chain C, then the subinterval $[\alpha,\beta]$ is $\{\alpha,\beta,\gamma : \gamma \in C \text{ and } \alpha < \gamma < \beta\}$.

The length of a chain is the ordinal number of the order type of the chain, if this exists. Otherwise define the length of the chain to be ∞. By convention $\infty > \lambda$ for all ordinals λ and $\lambda + \infty = \infty + \lambda = \infty$.

The altitude, altα, of $\alpha \in \Omega$ is defined by

alt$\alpha = 0$ if $\forall\beta \in \Omega$, $\beta > \alpha$ or β and α are not comparable.

If every subchain containing α contains a minimum below α and all subchains $[\alpha_x, \alpha]$ are well ordered, then

$$\text{alt}\,\alpha = \lim_{\beta < \alpha \text{ and } \beta \neq \alpha} (\text{alt}\,\beta + 1)$$

Otherwise define altα to be ∞ and then alt$\alpha > \lambda$ for all ordinals λ.

The depth, depβ, of $\beta \in \Omega$ is defined by

dep$\beta = 0$ if $\forall \gamma \in \Omega$, $\gamma < \beta$ or γ and β are not comparable.
If every subchain containing β contains a maximum above β and all subchains $[\beta, \beta_y]^*$ are well ordered, then

$$\text{dep}\,\beta = \lim_{\gamma > \beta \text{ and } \gamma \neq \beta} (\text{dep}\,\gamma + 1)$$

Otherwise define depβ to be ∞ and then dep$\beta > \lambda$ for all ordinals λ.

For convenience, if altα is some ordinal μ, write alt$\alpha < \infty$, and if depβ is some ordinal ν, write dep$\beta < \infty$.

The following lemmas give some elementary properties of altitude and depth.

Lemma 2.8 (1) If $\text{alt}\,\alpha < \infty$ and $\delta < \alpha$, then $\text{alt}\,\delta < \infty$ and $\text{alt}\,\delta < \text{alt}\,\alpha$.

(2) If $\text{alt}\,\alpha = \infty$, then $\exists\theta < \alpha$ with $\text{alt}\,\theta = \infty$.

(3) If $\text{dep}\,\beta < \infty$ and $\gamma > \beta$, then $\text{dep}\,\gamma < \infty$ and $\text{dep}\,\gamma < \text{dep}\,\beta$.

(4) If $\text{dep}\,\beta = \infty$, then $\exists\phi > \beta$ with $\text{dep}\,\phi = \infty$.

Proof (1) If $\delta < \alpha$, then $\text{alt}\,\alpha = \lim_{\beta<\alpha}(\text{alt}\,\beta + 1) > \text{alt}\,\delta$.

Hence $\text{alt}\,\delta < \text{alt}\,\alpha < \infty$.

(2) Suppose that if $\delta < \alpha$ then $\text{alt}\,\delta < \infty$.

Then $\text{alt}\,\alpha = \lim_{\delta<\alpha}(\text{alt}\,\delta + 1) < \infty$, since $\text{alt}\,\delta + 1 < \infty\ \forall\delta < \alpha$.

So $\exists\theta < \alpha$ with $\text{alt}\,\theta = \infty$.

(3) Proof similar to (1).

(4) Proof similar to (2).

Lemma 2.9 (1) If $\text{alt}\,\alpha = \lambda$ and $\mu < \lambda$ then $\exists\gamma < \alpha$ with $\text{alt}\,\gamma = \mu$.

(2) If $\text{dep}\,\beta = \rho$ and $\sigma < \rho$ then $\exists\delta > \beta$ with $\text{dep}\,\delta = \rho$.

Proof (1) By induction.

Let $\text{alt}\,\alpha = 1$.

Since $\text{alt}\,\alpha \neq 0$, $\exists\beta < \alpha$ and $\text{alt}\,\beta < \text{alt}\,\alpha$ i.e. $\text{alt}\,\beta = 0$

So (1) is true when $\lambda = 1$.

Suppose that (1) holds when $\text{alt}\,\alpha = \lambda$.

Let $\text{alt}\,\alpha = \lambda + 1$.

Then $\exists\beta' < \alpha$ with $\text{alt}\,\beta' = \lambda$, since if $\text{alt}\,\beta < \lambda\ \forall\beta < \alpha$, then $\text{alt}\,\alpha = \lim_{\beta<\alpha}(\text{alt}\,\beta + 1) \leq \lambda$, which is impossible. Since $\text{alt}\,\beta' = \lambda$, the induction hypothesis can be applied to show that $\exists\gamma < \beta'$ (and so $< \alpha$) with $\text{alt}\,\gamma = \mu$.

Now let λ be a limit ordinal, and suppose (1) holds $\forall\mu < \lambda$.

Then $\exists\beta' < \alpha$ with $\mu \leq \mathrm{alt}\beta' < \lambda$, since if $\mathrm{alt}\beta < \mu \ \forall\beta < \alpha$, then

$\mathrm{alt}\,\alpha = \lim_{\beta<\alpha}(\mathrm{alt}\beta + 1) \leq \mu < \lambda$, which is impossible.

If $\mathrm{alt}\beta' = \mu$, there is nothing more to prove.

If $\mathrm{alt}\beta' > \mu$, then by the induction hypothesis $\exists y < \beta' < \alpha$ with $\mathrm{alt}y = \mu$.

Hence result.

(2) is proved similarly.

<u>Lemma 2.10</u> If $\mathrm{alt}\,\alpha = k$ ($<\omega$), then all chains $[\alpha_x, \alpha]$ have length $\leq k + 1$.

If $\mathrm{dep}\beta = k$, then all chains $[\beta, \beta_y]$ have length $\leq k + 1$.

<u>Proof</u> Let $\mathrm{alt}\,\alpha = k$, and let $[\alpha_0, \alpha]$ be a chain of length $k + 2$, say

$$\alpha_0 < \alpha_1 < \alpha_2 \ldots\ldots\ldots < \alpha_k < \alpha$$

Then $\mathrm{alt}\alpha_0 \geq 0 \rightarrow \mathrm{alt}\alpha_1 \geq 1 \rightarrow \ldots \mathrm{alt}\alpha_k \geq k \rightarrow \mathrm{alt}\alpha \geq k + 1$, which is a contradiction. So all chains $[\alpha_x, \alpha]$ have length $\leq k + 1$.

The result for depth is proved similarly.

<u>Corollary 2.11</u> If $\mathrm{alt}\,\alpha = k$ ($<\omega$), then $\exists$ a chain $[\alpha_x, \alpha]$ of length $k + 1$.

If $\mathrm{dep}\beta = k$, then $\exists$ a chain $[\beta, \beta_y]$ of length $k + 1$.

<u>Proof</u> By induction on k.

Let $k = 1$.

If $\mathrm{alt}\,\alpha = 1$, then $\forall\beta < \alpha$, $\mathrm{alt}\beta = 0$ i.e. β is minimal, and $\exists\beta' < \alpha$ with $\mathrm{alt}\beta' = 0$ i.e. $[\beta', \alpha]$ has length 2.

Suppose that the result holds for $k \leq n$. Let $\mathrm{alt}\,\alpha = n + 1$.

Then by lemma 2.9, $\exists\delta < \alpha$ with $\mathrm{alt}\,\delta = n$.

By the induction hypothesis, $\exists$ a chain $[\alpha_x, \delta]$ of length $n + 1$.

Then $[\alpha_x, \alpha] = [\alpha_x, \delta] \cup \{\alpha\}$ has length $n + 2$.

$[\alpha_x, \alpha] = [\alpha_x, \delta] \cup \{\alpha\}$ since if $\delta < \gamma < \alpha$, then alt$\gamma >$ alt$\delta = n$ and alt$\gamma <$ alt$\alpha = n + 1$, which is impossible.

The result for depth is proved similarly.

<u>Lemma 2.12</u> alt$\alpha = k$ iff all chains $[\alpha_x, \alpha]$ have length $\leqslant k + 1$ and $\exists$ a chain $[\alpha_0, \alpha]$ of length $k + 1$.

dep$\beta = k$ iff all chains $[\beta, \beta_y]$ have length $\leqslant k + 1$ and $\exists$ a chain $[\beta, \beta_0]$ of length $k + 1$.

<u>Proof</u> Only if:- by lemma 2.10 and corollary 2.11.

If:- by induction on k.

Suppose all chains $[\alpha_x, \alpha]$ have length 2 and $\exists$ a chain $[\alpha_0, \alpha]$ of length 2. Then α_0 is a minimal element and alt$\alpha_0 = 0$.
So alt$\alpha \geqslant 1$. If alt$\alpha > 1$, by corollary 2.11 there is a chain $[\alpha_x, \alpha]$ of length > 2, which is impossible. So alt$\alpha = 1$.

Suppose the result holds whenever $k \leqslant n$.
Let all chains $[\alpha_x, \alpha]$ have length $\leqslant n + 2$ and let $[\alpha_0, \alpha]$ be a chain of length $n + 2$. Let $\delta < \alpha$. Then any chain $[\alpha_x, \delta]$ can be extended to a chain $[\alpha_x, \alpha]$, and so any chain $[\alpha_x, \delta]$ has length $\leqslant n + 1$, and by the induction hypothesis alt$\delta \leqslant n$.
So if $\delta < \alpha$, then alt$\delta \leqslant n$. Since there is a chain $[\alpha_0, \alpha]$ of length $n + 2$, α has an immediate predecessor in that chain, γ say, and $[\alpha_0, \gamma]$ has length $n + 1$. So alt$\gamma = n$, and so alt$\alpha = n+1$.

The result for depth is proved similarly.

<u>Lemma 2.12(a)</u> If $[\alpha_x, \alpha]$ has order type $\lambda + 1$, then alt$\alpha \geqslant \lambda$

If $[\beta, \beta_y]^*$ has order type $\lambda + 1$, then dep$\beta \geqslant \lambda$, where * denotes reverse ordering.

Proof By induction.

$\text{alt}\alpha \geq 0 \; \forall \alpha \in \Omega$ so the result is true when $[\alpha_x, \alpha]$ has order type 1.

Suppose that if $[\alpha_x, \alpha]$ has order type $\lambda + 1$, then $\text{alt}\alpha \geq \lambda$.

Let $[\alpha_x, \alpha]$ have order type $\lambda + 2$.

Since $\lambda + 1$ is not a limit ordinal, α has an immediate predecessor in $[\alpha_x, \alpha]$ - call it β. By the induction hypothesis, $\text{alt}\beta \geq \lambda$ and so $\text{alt}\alpha \geq \lambda + 1$.

Let λ be a limit ordinal and suppose that $\forall \mu < \lambda$, if $[\alpha_x, \alpha]$ has order type $\mu + 1$, then $\text{alt}\alpha \geq \mu$. Let $[\alpha_x, \alpha]$ have order type $\lambda + 1$. Then $\forall \mu < \lambda$, $\exists \beta_\mu \in [\alpha_x, \alpha]$ such that $[\alpha_x, \beta_\mu]$ has order type $\mu + 1$, since it has a last element.

By the induction hypothesis, $\text{alt}\beta_\mu \geq \mu$.

$\forall \mu < \lambda$, $\text{alt}\alpha > \text{alt}\beta_\mu \geq \mu$ i.e. $\text{alt}\alpha > \mu$.

so $\text{alt}\alpha \geq \lambda$, and the result follows.

The result for depth is proved similarly

Lemma 2.13 If $\text{alt}\alpha < \lambda$ and $[t(\alpha,\beta), t(\gamma,\delta)] = t(\theta,\phi)^{\pm 1}$, then $\text{alt}\theta < \lambda$.

If $\text{dep}\beta < \mu$ and $[t(\alpha,\beta), t(\gamma,\delta)] = t(\theta,\phi)^{\pm 1}$, then $\text{dep}\phi < \mu$.

Proof By corollary 2.3, $[t(\alpha,\beta), t(\gamma,\delta)] = 1$

or $t(\alpha,\delta)$ if $\gamma = \beta$

or $t(\gamma,\beta)^{-1}$ if $\alpha = \delta$.

So $[t(\alpha,\beta), t(\gamma,\delta)] = t(\theta,\phi)^{\pm 1}$ means that $\theta = \alpha$ (and then $\gamma = \beta$) or $\theta = \gamma$ (and then $\alpha = \delta$)

If $\theta = \alpha$, then $\text{alt}\theta < \lambda$.

If $\theta = \gamma$, then $\gamma < \delta = \alpha$ and so by lemma 2.8(1) $\text{alt}\theta < \lambda$.

Hence result

The result for depth is proved similarly.

An ordered set S is well ordered if every non-empty subset of S has a first element. The empty set is also considered to be well ordered.

An equivalent definition of well ordering is given by

Lemma 2.14 An ordered set S is well ordered iff it has no subset of type ω^*, where * denotes reverse ordering.

Proof If S is well ordered, then every non-empty subset of S has a first element, and so no subset of S has order type ω^*.

Conversely, if no subset of S has order type ω^*, then every non-empty subset of S has a first element. For, if there were a non-empty subset without a first element, it would contain a sequence (s_k) such that, for any k, $s_{k+1} < s_k$ i.e. a sequence of type ω^*.

The above lemma is taken from (1).

Lemma 2.15 Let Ω be totally ordered. Then

(1) $\text{alt}\alpha = \lambda$ iff Ω has a minimal element $\underline{\omega}$ and $[\underline{\omega}, \alpha]$ has order type $\lambda + 1$.

(2) $\text{dep}\beta = \lambda$ iff Ω has a maximal element $\overline{\omega}$ and $[\beta, \overline{\omega}]^*$ has order type $\lambda + 1$.

Proof (1) By induction on λ.

Let Ω have a minimal element $\underline{\omega}$ and let $[\underline{\omega}, \alpha]$ have order type $\lambda+1$.

Let $\lambda = 0$. Then $[\underline{\omega}, \alpha] = \{\underline{\omega}\}$ and $\text{alt}\alpha = \text{alt}\underline{\omega} = 0$.

Suppose that whenever $\lambda \leqslant \mu$, if $[\underline{\omega}, \alpha]$ has order type $\lambda + 1$, then $\text{alt}\alpha = \lambda$. Let $[\underline{\omega}, \alpha]$ have order type $\mu + 2$.

If $\beta < \alpha$, $[\underline{\omega}, \beta]$ has order type $\leqslant \mu + 1$ and so $\text{alt}\beta \leqslant \mu$.

Since $\mu + 1$ is not a limit ordinal, α has an immediate predecessor

β' and $[\underline{\omega}, \beta']$ has order type $\mu + 1$. By the induction hypothesis $\mathrm{alt}\beta' = \mu$.

Hence $\mathrm{alt}\alpha = \lim_{\beta<\alpha}(\mathrm{alt}\beta + 1) = \mu + 1$.

Let λ be a limit ordinal and suppose that $\forall\mu < \lambda$ if $[\underline{\omega}, \alpha]$ has order type $\mu + 1$, then $\mathrm{alt}\alpha = \mu$. Let $[\underline{\omega}, \alpha]$ have order type $\lambda + 1$. $\forall\beta < \alpha$, let $[\underline{\omega}, \beta]$ have order type $\mu_\beta + 1$, since $[\underline{\omega}, \beta]$ has a last element. $\lim\mu_\beta = \lambda$ and by the induction hypothesis $\mathrm{alt}\beta = \mu_\beta$. $\mathrm{alt}\alpha = \lim_{\beta<\alpha}(\mathrm{alt}\beta + 1) = \lim_{\beta<\alpha}(\mu_\beta + 1) = \lambda$.

Conversely, let $\mathrm{alt}\alpha = \lambda$. Show by induction on λ that Ω has a minimal element $\underline{\omega}$ and that $[\underline{\omega}, \alpha]$ has order type $\lambda + 1$.

Let $\lambda = 0$. If $\mathrm{alt}\alpha = 0$, then α is a minimal element $\underline{\omega}$ say, and $\{\alpha\}$ has order type 1.

Suppose that if $\mathrm{alt}\alpha \leqslant \lambda$, then Ω has a minimal element $\underline{\omega}$ and $[\underline{\omega}, \alpha]$ has order type $\mathrm{alt}\alpha + 1$. Let $\mathrm{alt}\alpha = \lambda + 1$.

Then by lemma 2.9, $\exists\beta < \alpha$ with $\mathrm{alt}\beta = \lambda$. By the induction hypothesis, Ω has a minimal element $\underline{\omega}$ and $[\underline{\omega}, \beta]$ has order type $\lambda + 1$. β is the immediate predecessor of α, for if $\exists\gamma$ with $\beta < \gamma < \alpha$, then $\mathrm{alt}\gamma > \mathrm{alt}\beta$ i.e. $\mathrm{alt}\gamma > \lambda$ and $\mathrm{alt}\gamma < \mathrm{alt}\alpha$ i.e. $\mathrm{alt}\gamma < \lambda + 1$, which is impossible.

So $[\underline{\omega}, \alpha] = [\underline{\omega}, \beta] \cup \{\alpha\}$ has order type $\lambda + 2$.

Let λ be a limit ordinal and suppose that if $\mathrm{alt}\alpha = \mu\ (<\lambda)$, then Ω has a minimal element $\underline{\omega}$ and $[\underline{\omega}, \alpha]$ has order type $\mu + 1$.

Let $\mathrm{alt}\alpha = \lambda$. Then $\lim_{\beta<\alpha}(\mathrm{alt}\beta + 1) = \lambda$.

Since $\exists\beta < \alpha$, with $\mathrm{alt}\beta < \lambda$, Ω has a minimal element $\underline{\omega}$ by the induction hypothesis. $[\underline{\omega}, \alpha]$ has order type $\geqslant \lambda + 1$, since it has a last element and if $\mu < \lambda$, $\exists\beta_\mu < \alpha$ with $\mathrm{alt}\beta_\mu = \mu$.

By the induction hypothesis, $[\underline{\omega}, \beta_\mu]$ has order type $\mu + 1$.
Suppose that α has an immediate predecessor β i.e. $\exists\beta \in \Omega$ such that $\beta < \alpha$ and $\forall\gamma \in \Omega$ either $\gamma < \beta$ or $\alpha < \gamma$.
Then since $\text{alt}\beta < \text{alt}\alpha$, $\text{alt}\beta = \mu < \lambda$ and $[\underline{\omega}, \beta]$ has order type $\mu + 1$. $[\underline{\omega}, \alpha] = [\underline{\omega}, \beta] \cup \{\alpha\}$ has order type $\mu + 2$ and so by the first part of the proof $\text{alt}\alpha = \mu + 1$. This is a contradiction.
So α has no immediate predecessor and $[\underline{\omega}, \alpha]$ has order type $\lambda + 1$.

(2) is proved similarly.

<u>Lemma 2.16</u> If Ω is totally ordered, then for any ordinal λ, there is at most one element α of Ω with $\text{alt}\alpha = \lambda$ and at most one element β of Ω with $\text{dep}\beta = \lambda$.

<u>Proof</u> Suppose $\exists\alpha, \alpha' \in \Omega$ with $\text{alt}\alpha = \text{alt}\alpha' = \lambda$.
Since Ω is totally ordered α and α' are comparable.
If $\alpha < \alpha'$, then by lemma 2.8 $\text{alt}\alpha < \text{alt}\alpha'$, which is a contradiction.
If $\alpha' < \alpha$, then by lemma 2.8 $\text{alt}\alpha' < \text{alt}\alpha$, which is a contradiction.
Hence result.

The result for depth is proved similarly.

<u>Lemma 2.17</u> If every subchain containing α contains a minimum below α, then $\text{alt}\alpha < \infty$ and all subintervals $[\alpha_x, \alpha]$ are well ordered.

If every subchain containing β contains a maximum **above** β, then $\text{dep}\beta < \infty$ and all subintervals $[\beta, \beta_y]^*$ are well ordered.

<u>Proof</u> Suppose that every subchain containing α contains a minimum below α. Let C be a subchain containing α.
Then $\{\gamma : \gamma \in C \text{ and } \gamma < \alpha \text{ or } \gamma = \alpha\}$ contains a minimum below α and can be written $[\alpha_x, \alpha]$.

Suppose that there is a subinterval $[\alpha_x, \alpha]$ which is not well ordered i.e. $[\alpha_x, \alpha]$ has a subset of type ω^*, say $\{\alpha > \alpha_1 > \ldots\ldots > \alpha_n > \ldots\ldots\}$. This subset can be extended to a subchain of $[\alpha_x, \alpha]$ (by inserting elements between α_i and α_{i+1}) which does not have a minimal element. This is a contradiction. Hence $[\alpha_x, \alpha]$ is well ordered.

If alt$\alpha = \infty$, then by lemma 2.8 $\exists \alpha_1 < \alpha$ with alt$\alpha_1 = \infty$. Suppose that if alt$\alpha = \infty$, then $\exists \alpha_1 > \alpha_2 > \ldots\ldots > \alpha_n$ with alt$\alpha_i = \infty$. Then by lemma 2.8 $\exists \alpha_{n+1} < \alpha_n$ with alt$\alpha_{n+1} = \infty$. This subset $\alpha > \alpha_1 > \ldots\ldots > \alpha_n > \ldots\ldots$ can be extended to a subchain of Ω which does not have a minimal element by inserting elements between α_i and α_{i+1}. This is a contradiction. Hence alt$\alpha < \infty$.

The result for depth is proved similarly.

<u>Lemma 2.18</u> If alt$\alpha < \infty$, then every subchain containing α contains a minimum below α and all subchains $[\alpha_x, \alpha]$ are well ordered.

If dep$\beta < \infty$, then every subchain containing β contains a maximum above β and all subchains $[\beta, \beta_y]^*$ are well ordered.

<u>Proof</u> This is an immediate consequence of the definition.

Let alt$\alpha = 0$. Then α is a minimal element and the result holds. Suppose that the result holds when alt$\alpha < \mu$ and let alt$\alpha = \mu$. Let C be a subchain containing α. Either α is a minimal element for C (and then the result holds) or $\exists \beta \in C$ with $\beta < \alpha$. Since $\beta < \alpha$, alt$\beta < \mu$, and so C contains a minimum, α_x say, below β and hence below α.

So every subchain containing α contains a minimum below α and by lemma 2.17 all subchains $[\alpha_x, \alpha]$ are well ordered.

The second part of the lemma is proved similarly.

Lemma 2.19 If alt$\alpha = \infty$, then α belongs to a chain C with no least element and if dep$\beta = \infty$, then β belongs to a chain C' with no greatest element.

Proof Follows from lemma 2.17 and lemma 2.18.

Lemma 2.20 If Ω is partially ordered, then alt$\alpha < \infty$ and dep$\alpha < \infty$ $\forall \alpha \in \Omega$ iff all chains in Ω are finite.

Proof Let $\alpha \in \Omega$.

If all chains in Ω are finite, then each subchain containing α contains a minimum below α and a maximum above α.

All subintervals $[\alpha_x, \alpha]$ and $[\alpha, \alpha_y]^*$ are finite and therefore well ordered, and so by lemma 2.17 alt$\alpha < \infty$ and dep$\alpha < \infty$.

Now suppose that alt$\alpha < \infty$ and dep$\alpha < \infty$ $\forall \alpha \in \Omega$.

By definition every subchain containing α contains a maximum above α and a minimum below α.

Suppose that there is a subchain of type ω, $\alpha < \alpha_1 < \ldots < \alpha_n < \ldots$

Then this subchain does not contain a maximum above α contradicting the definition of depth. Similarly any subchain of type ω^* containing α will not contain a minimum below α contradicting the definition of altitude.

So all chains are finite.

Corollary 2.21 If alt$\alpha < \infty$ and dep$\alpha < \infty$ whenever α is neither a maximal nor a minimal element, then all chains in Ω are finite.

<u>Corollary 2.22</u> If Ω is totally ordered, then $\text{alt}\,\alpha < \infty$ and $\text{dep}\,\alpha < \infty\ \forall \alpha \in \Omega$ iff Ω is finite.

<u>Lemma 2.23</u> If $u \in \text{gp}\{t(\alpha,\beta) : \text{alt}\,\alpha \oplus \text{dep}\,\beta < \lambda\}$ and $u = t(\alpha_1,\beta_1)^{n_1} \cdots\cdots\cdots t(\alpha_k,\beta_k)^{n_k}$ then $\text{alt}\,\alpha_i \oplus \text{dep}\,\beta_i < \lambda\ \forall i$.

<u>Proof</u> If $\text{alt}\,\alpha \oplus \text{dep}\,\beta < \lambda$ and $\text{alt}\,\gamma \oplus \text{dep}\,\delta < \lambda$, then

if $[t(\alpha,\beta),t(\gamma,\delta)] = t(\theta,\phi)^{\pm 1}$ then $\text{alt}\,\theta \oplus \text{dep}\,\phi < \lambda$.

For if $[t(\alpha,\beta),t(\gamma,\delta)] = t(\theta,\phi)^{\pm 1}$ then either $\alpha = \theta$ and $\delta = \phi$ or $\gamma = \theta$ and $\beta = \phi$.

If $\alpha = \theta$ and $\delta = \phi$ then $\beta = \gamma$ and $\beta < \delta$.

So $\text{alt}\,\alpha \oplus \text{dep}\,\delta < \text{alt}\,\alpha \oplus \text{dep}\,\beta < \lambda$.

If $\gamma = \theta$ and $\beta = \phi$ then $\alpha = \delta$ and $\gamma < \alpha$.

So $\text{alt}\,\gamma \oplus \text{dep}\,\beta < \text{alt}\,\alpha \oplus \text{dep}\,\beta < \lambda$.

Since $u \in \text{gp}\{t(\alpha,\beta) : \text{alt}\,\alpha \oplus \text{dep}\,\beta < \lambda\}$, u can be expressed in the form $t(\rho_1,\sigma_1)^{m_1} \cdots\cdots t(\rho_r,\sigma_r)^{m_r}$ with $\text{alt}\,\rho_i \oplus \text{dep}\,\sigma_i < \lambda$.

u is put into canonical form by using the relation

$$t(\rho_i,\sigma_i)^{m_i} t(\rho_j,\sigma_j)^{m_j} = t(\rho_j,\sigma_j)^{m_j} t(\rho_i,\sigma_i)^{m_i} [t(\rho_i,\sigma_i)^{m_i}, t(\rho_j,\sigma_j)^{m_j}]$$

and by the remark at the beginning of the proof

$$[t(\rho_i,\sigma_i)^{m_i}, t(\rho_j,\sigma_j)^{m_j}] \in \{t(\alpha,\beta) : \text{alt}\,\alpha \oplus \text{dep}\,\beta < \lambda\}.$$

hence result.

For $\alpha < \beta \in \Omega$ define the separation of α and β, $\text{sep}(\alpha,\beta)$ by $\text{sep}(\alpha,\beta) = 1$ if $\forall \gamma \in \Omega$, $\gamma < \alpha$ or $\beta < \gamma$ or γ is not comparable with α and β.

If $\exists \gamma \in \Omega$ with $\alpha \lneq \gamma \lneq \beta$, then

$$\text{sep}(\alpha,\beta) = \lim_{\substack{\alpha < \gamma < \beta \\ \alpha \neq \gamma,\ \gamma \neq \beta}} \{m\{\text{sep}(\alpha,\gamma),\text{sep}(\gamma,\beta)\} + 1\}$$

If $\alpha < \beta$ and $\text{sep}(\alpha,\beta)$ cannot be defined, write $\text{sep}(\alpha,\beta) = \infty$.

If $\text{sep}(\alpha,\beta) = \infty$, then $\text{sep}(\alpha,\beta) > \lambda$ for any ordinal λ.

Note that separation is a order theoretic property.

An equivalent definition of separation is given by

$\text{sep}(\alpha,\beta) = 1$ if $\forall \gamma \in \Omega$, $\gamma < \alpha$ or $\beta < \gamma$ or γ is not comparable with α and β.

$\text{sep}(\alpha,\beta) = \mu + 1$ if $\forall \gamma \in \Omega$ with $\alpha < \gamma < \beta$, $\underset{\gamma\neq\beta}{\text{sep}}(\alpha,\gamma) \leq \mu$ and $\underset{\alpha\neq\gamma}{\text{sep}}(\gamma,\beta) \leq \mu$ and $\exists\gamma$ with $\alpha < \gamma < \beta$ and $\underset{\gamma\neq\beta}{\text{sep}}(\alpha,\gamma) = \mu$ or $\underset{\alpha\neq\gamma}{\text{sep}}(\gamma,\beta) = \mu$.

$\text{sep}(\alpha,\beta) = \lambda$(a limit ordinal) if $\forall \gamma \in \Omega$ with $\alpha < \gamma < \beta$ $\underset{\gamma\neq\beta}{\text{sep}}(\alpha,\gamma) < \lambda$ and $\underset{\alpha\neq\gamma}{\text{sep}}(\gamma,\beta) < \lambda$ and $\forall\mu < \lambda\ \exists\ \kappa_\mu, \delta_\mu$ with $\kappa_\mu < \delta_\mu$ and $\kappa_\mu, \delta_\mu \in [\alpha,\beta]$ and $\text{sep}(\kappa_\mu,\delta_\mu) = \mu$.

<u>Proof</u> Let $\lim_{\substack{\alpha<\gamma<\beta \\ \alpha\neq\gamma,\ \gamma\neq\beta}}\{m\{\text{sep}(\alpha,\gamma),\text{sep}(\gamma,\beta)\} + 1\} = \mu + 1$.

Then $\forall\gamma$ with $\alpha < \gamma < \beta$, $\text{sep}(\alpha,\gamma) \leq \mu$ and $\text{sep}(\gamma,\beta) \leq \mu$.

If $\text{sep}(\alpha,\gamma) < \mu$ and $\text{sep}(\gamma,\beta) < \mu$ $\forall\gamma$ with $\alpha < \gamma < \beta$, then $\lim_{\alpha<\gamma<\beta,\ \alpha\neq\gamma,\ \gamma\neq\beta}\{m\{\text{sep}(\alpha,\gamma),\text{sep}(\gamma,\beta)\} + 1\} \leq \mu$, which is impossible.

So $\exists\gamma$, $\alpha < \gamma < \beta$, with $\text{sep}(\alpha,\gamma) = \mu$ or $\text{sep}(\gamma,\beta) = \mu$.

Conversely, if $\forall\gamma$ with $\alpha < \gamma < \beta$, $\text{sep}(\alpha,\gamma) \leq \mu$ and $\text{sep}(\gamma,\beta) \leq \mu$ and $\exists\gamma$, $\alpha < \gamma < \beta$, with $\text{sep}(\alpha,\gamma) = \mu$ or $\text{sep}(\gamma,\beta) = \mu$, then $\lim_{\alpha<\gamma<\beta,\ \alpha\neq\gamma,\ \gamma\neq\beta}\{m\{\text{sep}(\alpha,\gamma),\text{sep}(\gamma,\beta)\} + 1\} = \mu + 1$.

Let λ be a limit ordinal and suppose that the two definitions are equivalent for all $\mu < \lambda$.

Let $\lim_{\alpha<\gamma<\beta,\ \alpha\neq\gamma,\ \gamma\neq\beta}\{m\{\text{sep}(\alpha,\gamma),\text{sep}(\gamma,\beta)\} + 1\} = \lambda$.

Since $\lim_{\alpha<\gamma<\beta,\ \alpha\neq\gamma,\ \gamma\neq\beta}\{m\{\text{sep}(\alpha,\gamma),\text{sep}(\gamma,\beta)\} + 1\} = \lambda > \mu$ $\exists\gamma$, $\alpha < \gamma < \beta$ with $\text{sep}(\alpha,\gamma) > \mu$ or $\text{sep}(\gamma,\beta) > \mu$. Suppose w.l.o.g. $\text{sep}(\alpha,\gamma) > \mu$.

Let $\mathrm{sep}(\alpha, \gamma) = \mu + \rho < \lambda$.

If $\mu + \rho$ is a limit ordinal then by hypothesis there is a pair κ_μ, δ_μ with $\kappa_\mu < \delta_\mu$ and $\kappa_\mu, \delta_\mu \in [\alpha, \gamma]$ and $\mathrm{sep}(\kappa_\mu, \delta_\mu) = \mu$.

Since $\gamma < \beta$, $\kappa_\mu, \delta_\mu \in [\alpha, \beta]$ and there is nothing more to prove.

If $\mu + \rho$ is not a limit ordinal, find a pair κ_μ, δ_μ of the required form by induction on ρ.

If $\mathrm{sep}(\alpha, \gamma) = \mu + 1$, then by definition there is a δ, $\alpha < \delta < \gamma$, with $\mathrm{sep}(\alpha, \delta) = \mu$ or $\mathrm{sep}(\delta, \gamma) = \mu$.

Suppose that a pair κ_μ, δ_μ can be found whenever $\rho \leqslant \sigma$.

Let $\mathrm{sep}(\alpha, \gamma) = \mu + \sigma + 1$. Then by definition there is a δ, $\alpha < \delta < \gamma$, with $\mathrm{sep}(\alpha, \delta) = \mu + \sigma$ or $\mathrm{sep}(\delta, \gamma) = \mu + \sigma$.

By the induction hypothesis applied to (α, δ) or (δ, γ), a pair κ_μ, δ_μ of the required form can be found.

Finally suppose that $\forall \gamma$ with $\alpha < \gamma < \beta$, $\mathrm{sep}(\alpha, \gamma) < \lambda$ and $\mathrm{sep}(\gamma, \beta) < \lambda$ and $\forall \mu < \lambda\ \exists \kappa_\mu, \delta_\mu$ with $\kappa_\mu < \delta_\mu$ and $\kappa_\mu, \delta_\mu \in [\alpha, \beta]$ and $\mathrm{sep}(\kappa_\mu, \delta_\mu) = \mu$.

Since $\alpha \leqslant \kappa_\mu$, $\mathrm{sep}(\alpha, \delta_\mu) \geqslant \mu$ and since $\delta_\mu \leqslant \beta$, $\mathrm{sep}(\delta_\mu, \beta) \geqslant \mu$.

So $\forall \mu < \lambda$, $\exists \kappa_\mu, \delta_\mu$ with $\mathrm{sep}(\alpha, \delta_\mu) \geqslant \mu$ and $\mathrm{sep}(\kappa_\mu, \beta) \geqslant \mu$.

Also, $\forall \gamma$ with $\alpha < \gamma < \beta$, $\mathrm{sep}(\alpha, \gamma) < \lambda$ and $\mathrm{sep}(\gamma, \beta) < \lambda$.

So $\lim_{\alpha<\gamma<\beta,\ \alpha\neq\gamma,\ \gamma\neq\beta}\{m\{\mathrm{sep}(\alpha, \gamma), \mathrm{sep}(\gamma, \beta)\} + 1\} = \lambda$.

The following lemmas give some properties of separation and some results connecting separation, altitude and depth.

Lemma 2.24 If $\alpha < \gamma < \beta$ and $\mathrm{sep}(\alpha, \beta) < \infty$, then $\mathrm{sep}(\alpha, \gamma) < \infty$ and $\mathrm{sep}(\gamma, \beta) < \infty$.

Proof Let $\mathrm{sep}(\alpha, \beta) = \lambda$.

Then, by definition, if $\alpha < \gamma < \beta$, $\mathrm{sep}(\alpha, \gamma) < \lambda$ and $\mathrm{sep}(\gamma, \beta) < \lambda$.

<u>Corollary</u> If $\alpha \leqslant \gamma < \delta \leqslant \beta$ and $\operatorname{sep}(\alpha,\beta) < \infty$, then $\operatorname{sep}(\gamma,\delta) < \infty$.

<u>Lemma 2.25</u> If $\operatorname{sep}(\alpha,\beta) = \infty$, then there is a δ with $\alpha < \delta < \beta$ and $\operatorname{sep}(\alpha,\delta) = \infty$ or $\operatorname{sep}(\delta,\beta) = \infty$.

<u>Proof</u> Suppose that for all δ with $\alpha < \delta < \beta$, $\operatorname{sep}(\alpha,\delta) < \infty$ and $\operatorname{sep}(\delta,\beta) < \infty$.

Then $\lim_{\alpha<\delta<\beta,\ \alpha\neq\delta,\ \delta\neq\beta} \{ \max\{\operatorname{sep}(\alpha,\delta), \operatorname{sep}(\delta,\beta)\} + 1 \} < \infty$.

Hence result.

<u>Lemma 2.26</u> Let Ω be partially ordered and let $\alpha < \beta \in \Omega$. If $[\alpha,\beta]$ is infinite, then it has a proper subinterval which is infinite.

<u>Proof</u> Let $[\alpha,\beta]$ be infinite and let $\sigma \in [\alpha,\beta]$ with $\alpha < \sigma < \beta$.

If $\lambda \in [\alpha,\beta]$, then $\lambda \leqslant \sigma$ or $\lambda \geqslant \sigma$, since $[\alpha,\beta]$ is a totally ordered set. Hence $\lambda \in [\alpha,\sigma]$ or $\lambda \in [\sigma,\beta]$.

So $[\alpha,\beta] = [\alpha,\sigma] \cup [\sigma,\beta]$.

If $[\alpha,\sigma]$ is finite and $[\sigma,\beta]$ is finite, then their union must be finite i.e. $[\alpha,\beta]$ must be finite.

So either $[\alpha,\sigma]$ is infinite or $[\sigma,\beta]$ is infinite.

<u>Lemma 2.27</u> If $\operatorname{sep}(\alpha,\beta)$ is finite, then all chains joining α and β are finite. In particular, if $\operatorname{sep}(\alpha,\beta) = n$, then there is a chain of length $n + 1$ joining α and β, and all chains joining α and β have length $\leqslant n + 1$.

<u>Proof</u> Let $\operatorname{sep}(\alpha,\beta) = n$.

Suppose that there is a chain $[\alpha,\beta]_{n+2}$ of length $\geqslant n + 2$.

Then there exists a set of elements, $\{ \alpha < \alpha_1 < \alpha_2 < \ldots\ldots < \alpha_n < \beta\}$ say, belonging to $[\alpha,\beta]_{n+2}$.

$\operatorname{Sep}(\alpha,\alpha_1) \geqslant 1 \rightarrow \operatorname{sep}(\alpha,\alpha_2) \geqslant 2 \rightarrow \ldots\ldots \rightarrow \operatorname{sep}(\alpha,\alpha_n) \geqslant n \rightarrow \operatorname{sep}(\alpha,\beta) \geqslant n+1$.

This is a contradiction.

So all chains joining α and β have length $< n + 2$.

Use induction on n to show that there is a chain $[\alpha,\beta]$ of length $n + 1$.

Let $sep(\alpha,\beta) = 1$. Then $[\alpha,\beta]$ has length 2 as no element of lies between α and β.

Suppose that if $sep(\alpha',\beta') = p$ then there is a chain $[\alpha,\beta]$ of length $p + 1$ and let $sep(\alpha,\beta) = p + 1$.

Then there is a δ with $\alpha < \delta < \beta$ and $sep(\alpha,\delta) = p$ or there is a γ with $\alpha < \gamma < \beta$ and $sep(\gamma,\beta) = p$.

Suppose w.l.o.g. that there is a δ, $\alpha < \delta < \beta$ with $sep(\alpha,\delta) = p$.

Then there is a chain $[\alpha,\delta]$ of length $p + 1$ which can be extended to a chain $[\alpha,\beta]$ of length $\geq p + 2$.

By the first part of the proof, any chain joining α and β has length $\leq p + 2$.

So there is a chain $[\alpha,\beta]$ of length $p + 2$ and the result follows.

<u>Corollary 2.28</u> If Ω is totally ordered, then $sep(\alpha,\beta) = n$ iff $[\alpha,\beta]$ has length $n + 1$.

<u>Lemma 2.29</u> If Ω is totally ordered, then for all α,β in Ω with $\alpha < \beta$, $sep(\alpha,\beta) < \omega$ or $sep(\alpha,\beta) = \infty$.

<u>Proof</u> If $sep(\alpha,\beta) > \omega$, then $\exists \alpha_1,\beta_1$ with $sep(\alpha_1,\beta_1) = \omega$.

So suppose that there is a pair α,β in Ω with $sep(\alpha,\beta) = \omega$.

Then if $\alpha < \delta < \beta$, $sep(\alpha,\delta)$ is finite and $sep(\delta,\beta)$ is finite.

Let $sep(\alpha,\delta) = n$ and $sep(\delta,\beta) = m$.

By corollary 2.28, $[\alpha,\delta]$ has length $n + 1$ and $[\delta,\beta]$ has length $m + 1$.

So $[\alpha,\beta] = [\alpha,\delta] \cup [\delta,\beta]$ has length $n + m + 1$, and by corollary 2.28 $sep(\alpha,\beta) = n + m < \omega$. This is a contradiction. So if $\alpha < \beta$, $sep(\alpha,\beta) < \omega$ or $sep(\alpha,\beta) = \infty$.

<u>Lemma 2.30</u> If all chains joining α and β are finite, then $sep(\alpha,\beta) < \infty$.

<u>Proof</u> Suppose that all chains joining α and β are finite.

Let $sep(\alpha,\beta) = \infty$.

By lemma 2.25, $\exists\delta_1$, $\alpha < \delta_1 < \beta$ such that $sep(\alpha,\delta_1) = \infty$ or $sep(\delta_1,\beta) = \infty$.

If $\exists\delta_1,\ldots,\delta_n$ belonging to some chain $[\alpha,\beta]$ such that for each δ_i, $\exists\gamma_i \in \{\alpha,\delta_1,\ldots,\delta_{i-1},\beta\}$ with $sep(\delta_i,\gamma_i) = \infty$ or $sep(\gamma_i,\delta_i) = \infty$ then there is a $\delta_{n+1} \in [\alpha,\beta]$ and a $\gamma_{n+1} \in \{\alpha,\delta_1,\ldots,\delta_n,\beta\}$ such that $sep(\delta_{n+1},\gamma_{n+1}) = \infty$ or $sep(\gamma_{n+1},\delta_{n+1}) = \infty$.

So there is an infinite sequence $\{\delta_i\} \in [\alpha,\beta]$ i.e. there is an infinite chain joining α and β.

<u>Corollary 2.31</u> If all chains in Ω are order isomorphic to a subset of the integers, then $sep(\alpha,\beta) < \infty \; \forall\alpha,\beta \in \Omega$ with $\alpha < \beta$.

<u>Proof</u> Let $\alpha,\beta \in \Omega$ with $\alpha < \beta$.

Then any chain $[\alpha,\beta]$ cannot have order type ω since it has a last element and cannot have order type ω^* since it has a first element. So any chain $[\alpha,\beta]$ is finite and by lemma 2.30 $sep(\alpha,\beta) < \infty$.

<u>Lemma 2.32</u> $sep(\alpha,\beta) < \infty \; \forall\alpha,\beta \in \Omega$ with $\alpha < \beta$ iff all subsets of Ω which are chains are order isomorphic to a subset of the integers.

<u>Proof</u> If:- follows from corollary 2.31

Only if:- Let $sep(\alpha,\beta) < \infty \ \forall \alpha,\beta \in \Omega$ with $\alpha < \beta$.

Suppose that there is a subset of type $\omega^* + \omega + 1$ which is a chain, say $\ldots\ldots < \alpha_{-n} < \alpha_{-n+1} < \ldots\ldots < \alpha_1 < \alpha_2 < \ldots \quad \ldots < \beta$.

Then $\forall i$ $[\alpha_i,\beta]$ has order type $\omega + 1$.

By lemma 2.27 $sep(\alpha_i,\beta) \geqslant \omega$.

Suppose that $sep(\alpha_i,\beta) = \omega$.

Then $sep(\alpha_{i+1},\beta) < \omega$, say $sep(\alpha_{i+1},\beta) = n$.

By lemma 2.27 $[\alpha_{i+1},\beta]$ has order type $\leqslant n + 1$, and so $[\alpha_i,\beta]$ has order type $\leqslant n + 2$, which is impossible.

Suppose that $\forall i$, if $[\alpha_i,\beta]$ has order type $\omega+1$, then $sep(\alpha_i,\beta) \geqslant \mu$ $(< \infty)$.

Let $sep(\alpha_k,\beta) = \mu$. As $\alpha_k \lneq \alpha_{k+1}$, $sep(\alpha_{k+1},\beta) < \mu$ although $[\alpha_{k+1},\beta]$ has order type $\omega+1$. This is a contradiction.

Hence result.

<u>Lemma 2.33</u> If there is a chain $[\alpha,\beta]$ of length $n + 1$, then $sep(\alpha,\beta) \geqslant n$. In particular, if any longest chain joining α and β has length $n + 1$, then $sep(\alpha,\beta) = n$.

<u>Proof</u> By induction on n.

The result holds if $n = 1$.

Suppose that the result holds whenever $n \leqslant k - 1$, and suppose that there is a chain $[\alpha,\beta]$ of length $k + 1$.

Since $[\alpha,\beta]$ is finite, α has an immediate successor, γ say, and $[\gamma,\beta]$ has length k. So by the induction hypothesis $sep(\gamma,\beta) \geqslant k-1$ and $sep(\alpha,\beta) > sep(\gamma,\beta) \geqslant k - 1$ i.e. $sep(\alpha,\beta) \geqslant k$.

If any longest chain $[\alpha,\beta]$ has length $k + 1$, then for any δ with $\alpha < \delta < \beta$, any chain $[\alpha,\delta]$ has length $\leqslant k$ and any chain $[\delta,\beta]$ has length $\leqslant k$.

By the induction hypothesis, if $\alpha < \delta < \beta$, then $sep(\alpha,\delta) \leqslant k - 1$

and $\text{sep}(\delta,\beta) \leq k - 1$.

Hence $\text{sep}(\alpha,\beta) \leq k$.

<u>Lemma 2.34</u> $\text{Sep}(\alpha,\beta) = n$ iff any longest chain joining α and β has length $n + 1$.

<u>Proof</u> If :- follows from lemma 2.33.

Only if :- follows from lemma 2.27.

<u>Lemma 2.35</u> If $\text{sep}(\alpha,\beta) = k$ $(<\omega)$ and $i + j = k$ with $i,j > 0$, then there is a γ with $\alpha < \gamma < \beta$ and $\text{sep}(\alpha,\gamma) = i$ and $\text{sep}(\gamma,\beta) = j$.

<u>Proof</u> By lemma 2.27 there is a chain $[\alpha,\beta]$ of length $k + 1$. So $\exists\gamma \in [\alpha,\beta]$ such that $[\alpha,\gamma]$ has length $i + 1$ and $[\gamma,\beta]$ has length $j + 1$. So by lemma 2.33 $\text{sep}(\alpha,\gamma) \geq i$ and $\text{sep}(\gamma,\beta) \geq j$. Suppose that $\text{sep}(\alpha,\gamma) > i$, say $\text{sep}(\alpha,\gamma) = m$.

Then by lemma 2.27 there is a chain $[\alpha,\gamma]$ of length $m + 1$ and so there is a chain $[\alpha,\beta] = [\alpha,\gamma] \cup [\gamma,\beta]$ of length $m + j + 1$, i.e. a chain $[\alpha,\beta]$ of length $> k + 1$, which is a contradiction. So $\text{sep}(\alpha,\gamma) = i$ and a similar proof shows that $\text{sep}(\gamma,\beta) = j$.

<u>Lemma 2.36</u> Let $u = [g_1,h_1]^{n_1}\ldots\ldots[g_k,h_k]^{n_k} \in G$ where each g_i is of the form $g_i = t(\alpha_{i_1},\beta_{i_1})^{n_{i_1}}\ldots\ldots t(\alpha_{i_r},\beta_{i_r})^{n_{i_r}}$, with for $1 \leq j \leq r$, $\text{sep}(\alpha_{i_j},\beta_{i_j}) \geq \rho$. Then when the commutators are expanded to give u in the form $t(\lambda_1,\mu_1)^{m_1}\ldots.t(\lambda_x,\mu_x)^{m_x}$, for $1 \leq j \leq x$, $\text{sep}(\lambda_j,\mu_j) \geq \rho + 1$ or $= \infty$.

<u>Proof</u> Let $h_i = t(\gamma_1,\delta_1)^{k_1}\ldots\ldots t(\gamma_y,\delta_y)^{k_y}$.

For convenience, write $g_i = t(\alpha_1,\beta_1)^{n_1}\ldots\ldots t(\alpha_r,\beta_r)^{n_r}$.

Then, by lemma 2.1,

$[g_i,h_i] = [t(\alpha_1,\beta_1)^{n_1}\ldots\ldots t(\alpha_r,\beta_r)^{n_r},h_i]$

$$= [t(\alpha_1,\beta_1)^{n_1},h_i]^{t(\alpha_2,\beta_2)^{n_2}\ldots t(\alpha_r,\beta_r)^{n_r}}\ldots[t(\alpha_r,\beta_r)^{n_r},h_i]$$

$[t(\alpha_j,\beta_j)^{n_j},h_i] = [t(\alpha_j,\beta_j)^{n_j},t(\gamma_1,\delta_1)^{k_1}\ldots\ldots t(\gamma_y,\delta_y)^{k_y}]$

$$= [t(\alpha_j,\beta_j)^{n_j},t(\gamma_y,\delta_y)^{k_y}]\ldots[t(\alpha_j,\beta_j)^{n_j},t(\gamma_1,\delta_1)^{k_1}]^{t(\gamma_2,\delta_2)^{k_2}\ldots t(\gamma_y,\delta_y)^{k_y}}$$

So $[g_i,h_i]$ is a product of terms of the form

$$[t(\alpha_j,\beta_j)^{n_j},t(\gamma_i,\delta_i)^{k_i}]^{t(\gamma_{i+1},\delta_{i+1})^{k_{i+1}}\ldots\ldots t(\alpha_r,\beta_r)^{n_r}}.$$

If $\alpha_j \neq \delta_i$ and $\beta_j \neq \gamma_i$, then by corollary 2.3 this is 1.

If $\beta_j = \gamma_i$, then the expression becomes (using corollary 2.3)

$$\left(t(\alpha_j,\delta_i)^{n_jk_i}\right)^{t(\gamma_{i+1},\delta_{i+1})^{k_{i+1}}\ldots\ldots t(\alpha_r,\beta_r)^{n_r}}$$

$$= t(\alpha_j,\delta_i)^{n_jk_i}[t(\alpha_j,\delta_i)^{n_jk_i},t(\gamma_{i+1},\delta_{i+1})^{k_{i+1}}\ldots t(\alpha_r,\beta_r)^{n_r}]$$

Since $\beta_j < \delta_i$, $sep(\alpha_j,\delta_i) > sep(\alpha_j,\beta_j) \geqslant \rho$.

So $sep(\alpha_j,\delta_i) \geqslant \rho + 1$ or $sep(\alpha_j,\delta_i) = \infty$.

Any terms obtained by expanding the commutator

$$[t(\alpha_j,\delta_i)^{n_jk_i},t(\gamma_{i+1},\delta_{i+1})^{k_{i+1}}\ldots\ldots t(\alpha_r,\beta_r)^{n_r}]$$

will be of the form $t(\lambda,\mu)^n$ with $\lambda \leqslant \alpha_j$ and $\mu \geqslant \delta_i$ $(>\beta_j)$ with at least one strict inequality. So $sep(\lambda,\mu) > sep(\alpha_j,\beta_j) \geqslant \rho$ i.e. $sep(\lambda,\mu) \geqslant \rho + 1$ or $sep(\lambda,\mu) = \infty$.

A similar argument can be used if $\alpha_j = \delta_i$.

Hence, for each i, $[g_i,h_i]$ is a product of $t(\lambda,\mu)$s of the required form and so u is of the form stated in the lemma.

<u>Lemma 2.37</u> If $sep(\alpha,\beta) = \rho(\leqslant\omega)$, then $dep\,\alpha \geqslant \rho$ and $alt\,\beta \geqslant \rho$.

<u>Proof</u> By induction on ρ.

If $\text{sep}(\alpha,\beta) = 1$, then $\alpha < \beta$ and so $\text{alt}\beta \geq 1$ and $\text{dep}\alpha \geq 1$.

Suppose that the lemma is true for $\rho \leq n$.

Let $\rho = n + 1$. Then by lemma 2.27 there is a chain of length $n + 2$ joining α and β, $[\alpha,\beta]$ say. Since the chain is finite, β has an immediate predecessor in this chain, δ say, and there is a chain of length $n + 1$ joining α and δ. So by lemma 2.33 $\text{sep}(\alpha,\delta) = n$ (since any chain $[\alpha,\delta]$ has length at most $n + 1$) and by the induction hypothesis, $\text{alt}\delta \geq n$.

Since $\beta > \delta$, $\text{alt}\beta > \text{alt}\delta$ i.e. $\text{alt}\beta \geq n + 1$.

Similarly, since $[\alpha,\beta]$ is finite, α has an immediate successor in the chain, γ say, and by applying lemma 2.33 and using the induction hypothesis, $\text{dep}\gamma \geq n$ and so $\text{dep}\alpha \geq n + 1$.

Let $\text{sep}(\alpha,\beta) = \omega$. Then if $n < \omega, \exists\gamma_n, \delta_n$ with $\alpha \leq \gamma_n < \delta_n \leq \beta$ such that $\text{sep}(\gamma_n,\delta_n) = n$.

By the induction hypothesis, $\text{alt}\delta_n \geq n$ and $\text{dep}\gamma_n \geq n$.

So $\forall n < \omega$, $\exists\delta_n \leq \beta$ such that $\text{alt}\delta_n \geq n$ i.e. $\text{alt}\beta \geq \omega$

and $\exists\gamma_n \geq \alpha$ such that $\text{dep}\gamma_n \geq n$ i.e. $\text{dep}\alpha \geq \omega$.

Corollary Let $\rho \leq \omega$. If $\text{sep}(\alpha,\beta) \geq \rho$, then $\text{dep}\alpha \geq \rho$ and $\text{alt}\beta \geq \rho$.

Lemma 2.38 Let $n < \omega$.

If $\text{alt}\alpha = n$, then $\forall\beta < \alpha$, $\text{sep}(\beta,\alpha) \leq n$.

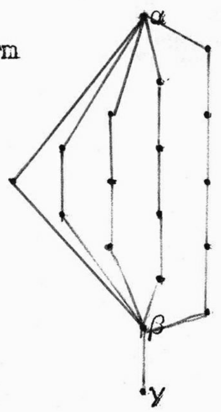
If $\text{dep}\gamma = n$, then $\forall\delta > \gamma$, $\text{sep}(\gamma,\delta) \leq n$.

Proof If $\text{alt}\alpha = n$, then by lemma 2.10 all chains $[\alpha_\pi,\alpha]$ have length $\leq n + 1$. If $\beta < \alpha$, then all chains $[\beta,\alpha]$ have length $\leq n + 1$ and so by lemma 2.33, $\text{sep}(\beta,\alpha) \leq n$.

The second part of the lemma is proved similarly.

Lemma 2.38 is not true in general if alt$\alpha \geqslant \omega$.
Let Ω be an ordered set of the following form. There is a pair of elements α and β with $\beta < \alpha$ and for any $n < \omega$ there is an interval $[\beta, \alpha]$ of length n. α is the ma imal element of Ω and there is a $\gamma < \beta$ which is a minimal element. No element of Ω lies between γ and β.

Ω is of the form



etc.

Then alt$\alpha = \omega$ but sep$(\gamma, \alpha) = \omega + 1$ since sep$(\beta, \alpha) = \omega$.

<u>Lemma 2.39</u> Let $n < \omega$.

If altα = n, then $\forall m \leqslant n$, $\exists \beta_m < \alpha$ with sep$(\beta_m, \alpha) = m$.

If depγ = n, then $\forall m \leqslant n$, $\exists \delta_m > \gamma$ with sep$(\gamma, \delta_m) = m$.

<u>Proof</u> Let altα = n.
By lemma 2.38, if $\beta < \alpha$ then sep$(\beta, \alpha) \leqslant n$.
By lemma 2.12, there is a chain $[\alpha_x, \alpha]$ of length n + 1.
By lemma 2.33, sep$(\alpha_x, \alpha) \geqslant n$ and since $\alpha_x < \alpha$ sep$(\alpha_x, \alpha) \leqslant n$.
So there is an $\alpha_x < \alpha$ with sep$(\alpha_x, \alpha) = n$.

Suppose that if $j + 1 \leqslant i \leqslant n$, then there is a $\beta_i < \alpha$ with sep$(\beta_i, \alpha) = i$. If $\beta_{j+1} < \beta < \alpha$, then sep$(\beta, \alpha) <$ sep(β_{j+1}, α).
So if $\beta_{j+1} < \beta < \alpha$, then sep$(\beta, \alpha) \leqslant j$.
By lemma 2.34, there is a chain $[\beta_{j+1}, \alpha]$ of length j + 2.
Since $[\beta_{j+1}, \alpha]$ is finite, β_{j+1} has an immediate successor,

β_j say, and $[\beta_j, \alpha]$ has length $j + 1$.

By lemma 2.33, $\text{sep}(\beta_j, \alpha) \geq j$ and since $\beta_{j+1} < \beta_j < \alpha$, $\text{sep}(\beta_j, \alpha) \leq j$.

So $\text{sep}(\beta_j, \alpha) = j$ and the result follows by induction.

The result for depth is proved similarly.

Lemma 2.40 Let $k < \omega$.

If $\text{alt}\beta = k$ and $i + j = k$ with $j \neq 0$, then there is an $\alpha < \beta$ with $\text{alt}\alpha = i$ and $\text{sep}(\alpha, \beta) = j$.

If $\text{dep}\gamma = k$ and $i + j = k$ with $j \neq 0$, then there is a $\delta > \gamma$ with $\text{dep}\delta = i$ and $\text{sep}(\gamma, \delta) = j$.

Proof Let $k = 1$.

If $\text{alt}\beta = 1$, then there is an $\alpha < \beta$ with $\text{alt}\alpha = 0$ and $\text{sep}(\alpha, \beta) = 1$. So the result holds when $k = 1$.

Suppose that the result holds whenever $k \leq n$.

Let $\text{alt}\beta = n + 1$ and let $i + j = n + 1$.

By lemma 2.12, there is a chain $[\beta_0, \beta]$ of length $n + 2$.

Since $[\beta_0, \beta]$ is finite, β has an immediate predecessor, α say, and $[\beta_0, \alpha]$ has length $n + 1$. Also any chain $[\beta_x, \alpha]$ has length $\leq n + 1$ (if not, there would be a chain $[\beta_x, \beta]$ of length $> n + 2$) and so by lemma 2.12, $\text{alt}\alpha = n$.

$\text{Sep}(\alpha, \beta) = 1$ for if there is a γ with $\alpha < \gamma < \beta$, then $[\beta_0, \alpha]$ can be extended to a chain $[\beta_0, \beta]$ of length $> n + 2$.

So if $i = n$ and $j = 1$ the result holds

If $i < n$, then by the induction hypothesis there is a $\gamma < \alpha$ with $\text{alt}\gamma = i$ and $\text{sep}(\gamma, \alpha) = j - 1$.

By lemma 2.34, there is a chain $[\gamma, \alpha]$ of length j and any longest chain $[\gamma, \alpha]$ has length j.

Since α is an immediate predecessor of β, any longest chain $[\gamma, \alpha]$ can be extended to a longest chain $[\gamma, \beta]$ of length $j + 1$

and so by lemma 2.34 $sep(\gamma,\beta) = j$.

The result for depth is proved similarly.

Corollary 2.41 Let $k < \omega$.

If $alt\beta \geq k$ and $i + j = k$ with $j > 0$, then there is an $\alpha < \beta$ with $alt\alpha \geq i$ and $sep(\alpha,\beta) \geq j$.

If $dep\gamma \geq k$ and $i + j = k$ with $j > 0$, then there is a $\delta > \gamma$ with $dep\delta \geq i$ and $sep(\gamma,\delta) \geq j$.

Proof If $alt\beta \geq k$, then by lemma 2.9 $\exists\delta \leq \beta$ with $alt\delta = k$.
By lemma 2.40, $\exists\alpha < \delta$ with $alt\alpha = i$ and $sep(\alpha,\delta) = j$.
So $alt\alpha = i$ and $sep(\alpha,\beta) \geq j$.
Hence result.

The result for depth is proved similarly.

Lemma 2.42 Let $\lambda \geq \omega$.

If $sep(\alpha,\beta) \geq \lambda$ and $sep(\beta,\gamma) = n$ $(<\omega)$, then $sep(\alpha,\gamma) \geq \lambda + n$.

Proof By induction on n.
If $sep(\beta,\gamma) = 1$, then $sep(\alpha,\gamma) > sep(\alpha,\beta) \geq \lambda$ i.e. $sep(\alpha,\gamma) \geq \lambda+1$.
Suppose that the result is true for $n \leq k$.
Let $sep(\beta,\gamma) = k + 1$.
Then since $sep(\beta,\gamma)$ is finite , there is a $\delta < \gamma$ with $sep(\beta,\delta) = k$.
By the induction hypothesis, $sep(\alpha,\delta) \geq \lambda + k$.
So, since $sep(\alpha,\gamma) > sep(\alpha,\delta)$, $sep(\alpha,\gamma) \geq \lambda + k + 1$.
Hence result.

Corollary 2.43 Let $\lambda \geq \omega$.

If $sep(\alpha,\beta) \geq \lambda$ and $sep(\beta,\gamma) \geq \omega$, then $sep(\alpha,\gamma) \geq \lambda + \omega$.

Proof If $sep(\beta,\gamma) > \omega$, then $\exists\rho < \gamma$ with $sep(\beta,\rho) = \omega$, or $\exists\sigma > \beta$ with $sep(\sigma,\gamma) = \omega$. So suppose w.l.o.g. $sep(\beta,\gamma) = \omega$.

Since $\text{sep}(\beta, \gamma) = \omega$, $\forall n < \omega\ \exists \kappa_n, \delta_n$ with $\beta \leqslant \kappa_n < \delta_n \leqslant \gamma$ and

$\text{sep}(\kappa_n, \delta_n) = n$.

Since $\text{sep}(\alpha, \beta) \geqslant \lambda$ and $\beta \leqslant \kappa_n$, $\text{sep}(\alpha, \kappa_n) \geqslant \lambda$.

By lemma 2.42, $\text{sep}(\alpha, \delta_n) \geqslant \lambda + n$.

So, since $\text{sep}(\alpha, \gamma) \geqslant \text{sep}(\alpha, \delta_n) \geqslant \lambda + n\ \forall n$, $\text{sep}(\alpha, \gamma) \geqslant \lambda + \omega$.

<u>Lemma 2.44</u> If $u \in \text{gp}\{t(\alpha, \beta) : \text{sep}(\alpha, \beta) \geqslant \mu\}$ and if in its canonical form $u = t(\alpha_1, \beta_1)^{n_1} \ldots\ldots t(\alpha_k, \beta_k)^{n_k}$, then $\text{sep}(\alpha_i, \beta_i) \geqslant \mu$ for $1 \leqslant i \leqslant k$.

<u>Proof</u> If $\text{sep}(\alpha, \beta) \geqslant \mu$ and $\text{sep}(\gamma, \delta) \geqslant \mu$, then

$[t(\alpha, \beta), t(\gamma, \delta)] = 1$ or is of the form $t(\theta, \phi)^{\pm 1}$ with $\text{sep}(\theta, \phi) \geqslant \mu$.

For $[t(\alpha, \beta), t(\gamma, \delta)] = 1$ or $t(\alpha, \delta)$ where $\alpha < \beta = \gamma < \delta$ or

$t(\gamma, \beta)^{-1}$ where $\gamma < \delta = \alpha < \beta$ by corollary 2.3.

If $\beta = \gamma$, then $\beta < \delta$ and $\text{sep}(\alpha, \delta) > \text{sep}(\alpha, \beta) \geqslant \mu$, or $\text{sep}(\alpha, \delta) = \text{sep}(\alpha, \beta) = \infty$.

If $\alpha = \delta$, then $\gamma < \alpha$ and $\text{sep}(\gamma, \beta) > \text{sep}(\alpha, \beta) \geqslant \mu$, or $\text{sep}(\gamma, \beta) = \text{sep}(\alpha, \beta) = \infty$.

Since $u \in \text{gp}\{t(\alpha, \beta) : \text{sep}(\alpha, \beta) \geqslant \mu\}$, u can be expressed in the form $t(\theta_1, \phi_1)^{m_1} \ldots\ldots t(\theta_r, \phi_r)^{m_r}$ with $\text{sep}(\theta_i, \phi_i) \geqslant \mu$.

u is put in its canonical form by using the relation

$$t(\theta_i, \phi_i)^{m_i} t(\theta_j, \phi_j)^{m_j} = t(\theta_j, \phi_j)^{m_j} t(\theta_i, \phi_i)^{m_i} [t(\theta_i, \phi_i)^{m_i}, t(\theta_j, \phi_j)^{m_j}]$$

and by the remark at the beginning of the proof

$[t(\theta_i, \phi_i)^{m_i}, t(\theta_j, \phi_j)^{m_j}] \in \text{gp}\{t(\alpha, \beta) : \text{sep}(\alpha, \beta) \geqslant \mu\}$.

Hence result.

Chapter 3. The upper central series of G

Denote a term of the upper central series of G by $\zeta_\mu(G)$.

$$\zeta_0(G) = 1, \quad \frac{\zeta_{\mu+1}(G)}{\zeta_\mu(G)} = \zeta_1\left(\frac{G}{\zeta_\mu(G)}\right)$$ and if λ is a limit ordinal, $\zeta_\lambda(G) = \bigcup_{\mu<\lambda} \zeta_\mu(G)$.

G has central height h if $\zeta_h(G) \supsetneq \zeta_\mu(G)\ \forall \mu < h$ and $\zeta_{h+1}(G) = \zeta_h(G)$.

*

We first prove a lemma about expanding commutators which will be needed for the proofs of theorems 3.3 and 3.5.

<u>Lemma 3.1</u> Let $u = t(\alpha_1,\beta_1)^{n_1} \ldots . t(\alpha_k,\beta_k)^{n_k}$ be in canonical form and let $\beta \in \{\beta_1, \ldots\ldots, \beta_k\}$. Let $\bar{\alpha}$ be a maximal element associated with β in u. If there is a $\gamma > \beta$, then $[u, t(\beta,\gamma)]$ is a product containing one and only one term $t(\bar{\alpha},\gamma)^n$ $(\neq 1)$ in its canonical form where $n = n_i$ for some i.

Let $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$ and let $\underline{\beta}$ be a minimal element associated with α in u. If there is a $\delta < \alpha$, then $[u, t(\delta,\alpha)]$ contains one and only one term $t(\delta,\underline{\beta})^n$ $(\neq 1)$ in its canonical form where $n = n_i$ for some i.

<u>Proof</u> Let u, $\bar{\alpha}$, β be as stated in the lemma and let $\gamma > \beta$. Then, by lemma 2.1,

$$[u,t(\beta,\gamma)] = [t(\alpha_1,\beta_1)^{n_1} \ldots\ldots . t(\alpha_k,\beta_k)^{n_k}, t(\beta,\gamma)]$$

$$= [t(\alpha_1,\beta_1)^{n_1}, t(\beta,\gamma)]^{t(\alpha_2,\beta_2)^{n_2} \ldots t(\alpha_k,\beta_k)^{n_k}} \ldots\ldots\ldots\ldots . [t(\alpha_k,\beta_k)^{n_k}, t(\beta,\gamma)].$$

By corollary 2.3, if $(\alpha_i,\beta_i) \neq (\bar{\alpha},\beta)$, then

$$[t(\alpha_i,\beta_i)^{n_i}, t(\beta,\gamma)] = 1 \text{ if } \beta_i \neq \beta \text{ and } \alpha_i \neq \gamma$$

* For any group H, $\zeta_1(H)$ is the centre of H.

H is nilpotent if H has a normal series $H = A_0 \supset \cdots \supset A_r = 1$, r finite, where A_{i-1}/A_i is in the centre of H/A_i for $i = 1, \ldots, r$.

H is locally nilpotent if all finitely generated subgroups of H are nilpotent.

or $t(\alpha_i,\gamma)^{n_i}$ if $\beta_i = \beta$ and then $\alpha_i < \bar{\alpha}$ or not comparable with $\bar{\alpha}$ or $t(\beta,\beta_i)^{-n_i}$ if $\gamma = \alpha_i$ and then $\beta_i > \gamma$.

So $[u,t(\beta,\gamma)]$ is a product of terms of the form

$$\left(t(\alpha_i,\gamma)^{n_i}\right)^{t(\alpha_{i+1},\beta_{i+1})^{n_{i+1}}\ldots t(\alpha_k,\beta_k)^{n_k}}$$ with $\alpha_i < \bar{\alpha}$ or not comparable with $\bar{\alpha}$ and

$$\left(t(\beta,\beta_j)^{-n_j}\right)^{t(\alpha_{j+1},\beta_{j+1})^{n_{j+1}}\ldots t(\alpha_k,\beta_k)^{n_k}}$$ with $\gamma < \beta_j$ and

$$\left(t(\bar{\alpha},\gamma)^{n}\right)^{t(\alpha_x,\beta_x)^{n_x}\ldots t(\alpha_k,\beta_k)^{n_k}}$$ which occurs once and only once since u is in canonical form.

Since $\left(t(\bar{\alpha},\gamma)^{n}\right)^{t(\alpha_x,\beta_x)^{n_x}\ldots\ldots t(\alpha_k,\beta_k)^{n_k}}$

$= t(\bar{\alpha},\gamma)^{n}[t(\bar{\alpha},\gamma)^{n},t(\alpha_x,\beta_x)^{n_x}\ldots\ldots t(\alpha_k,\beta_k)^{n_k}]$, $t(\bar{\alpha},\gamma)^{n}$ does occur in the expression for $[u,t(\beta,\gamma)]$.

$$\left(t(\alpha_i,\gamma)^{n_i}\right)^{t(\alpha_{i+1},\beta_{i+1})^{n_{i+1}}\ldots\ldots t(\alpha_k,\beta_k)^{n_k}}$$

$$= t(\alpha_i,\gamma)^{n_i}[t(\alpha_i,\gamma)^{n_i},t(\alpha_{i+1},\beta_{i+1})^{n_{i+1}}\ldots\ldots t(\alpha_k,\beta_k)^{n_k}]$$

The commutator is either 1 or a product of terms of the form $t(\rho,\sigma)^{m}$ where $\rho \leqslant \alpha_i$, $\sigma \geqslant \gamma$ and at least one inequality is strict. So terms arising from the commutator cannot cancel $t(\bar{\alpha},\gamma)^{n}$ when $[u,t(\beta,\gamma)]$ is reduced to canonical form.

Similarly, $\left(t(\beta,\beta_j)^{-n_j}\right)^{t(\alpha_{j+1},\beta_{j+1})^{n_{j+1}}\ldots\ldots t(\alpha_k,\beta_k)^{n_k}}$ can be expanded as a product of $t(\beta,\beta_j)^{-n_j}$ with terms $t(\xi,\eta)^{r}$ where $\xi \leqslant \beta$ and $\eta \geqslant \beta_j$ and at least one inequality is strict. Since $\beta < \gamma < \beta_j$, none of these pairs (ξ,η) can contain γ and so they cannot cancel $t(\bar{\alpha},\gamma)^{n}$ when $[u,t(\beta,\gamma)]$ is reduced to canonical form.

Hence, in its canonical form, $[u, t(\beta, \gamma)]$ contains one and only one term $t(\overline{\alpha}, \gamma)^n$.

The second part of the lemma is proved similarly.

Lemma 3.2 If α belongs to a subchain C which has no least element or β belongs to a subchain C' which has no greatest element, then $t(\alpha,\beta) \notin \zeta_\lambda(G)$ for any λ.

Proof By induction on λ.

If α belongs to a subchain C which has no least element, then $\exists \delta < \alpha$ which also is a member of C.

By lemma 2.2, $[t(\delta,\alpha), t(\alpha,\beta)] = t(\delta,\beta)$ i.e. $t(\alpha,\beta) \notin \zeta_1(G)$.

Let λ be a limit ordinal.

Suppose that $\forall \mu < \lambda$, if α is a member of a subchain C with no least element, then $\forall \beta > \alpha$, $t(\alpha,\beta) \notin \zeta_\mu(G)$.

Then $t(\alpha,\beta) \notin \bigcup_{\mu<\lambda} \zeta_\mu(G) = \zeta_\lambda(G)$

Suppose that $\forall \alpha \in \Omega$ if α is a member of a subchain C which has no least element, then $\forall \beta > \alpha$, $t(\alpha,\beta) \notin \zeta_\mu(G)$.

Let α be a member of a subchain C with no least element.

Let $\beta > \alpha$. Then $t(\alpha,\beta) \in \zeta_{\mu+1}(G)$ iff $[t(\alpha,\beta), t(\xi,\eta)] \in \zeta_\mu(G)$ for all $\xi < \eta \in \Omega$. Since C has no least element, $\exists \gamma \in C$ such that $\gamma < \alpha$.

Then $[t(\gamma,\alpha), t(\alpha,\beta)] = t(\gamma,\beta) \notin \zeta_\mu(G)$ since γ satisfies the conditions of the induction hypothesis, i.e. $t(\alpha,\beta) \notin \zeta_{\mu+1}(G)$.

Hence result.

A similar induction proof gives the result when β belongs to a subchain C' which has no greatest element.

The next theorem gives a necessary and sufficient condition for G to have a centre. We then use this to find an expression for a general term of the upper central series of G by induction. This theorem is Theorem 4.1(b) of (6).

Theorem 3.3 G has a centre iff $\exists \alpha, \beta \in \Omega$ with $\alpha < \beta$ and $\text{alt}\alpha + \text{dep}\beta = 0$.

$$\zeta_1(G) = \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha + \text{dep}\beta = 0\}.$$

Proof If $\text{alt}\alpha + \text{dep}\beta = 0$, then $\text{alt}\alpha = \text{dep}\beta = 0$.

Let $t(\lambda,\mu) \in \{t(\alpha,\beta) : \text{alt}\alpha + \text{dep}\beta = 0\}$.

Then $\forall \gamma \in \Omega$ with $\gamma \neq \lambda$ or μ, $\lambda < \gamma$ or $\gamma < \mu$ or γ is not comparable with λ or μ.

By lemma 2.2, $[t(\lambda,\mu),t(\gamma,\delta)] = 1 \ \forall \gamma,\delta \in \Omega$ i.e. $t(\lambda,\mu) \in \zeta_1(G)$.

So $\text{gp}\{t(\alpha,\beta) : \text{alt}\alpha + \text{dep}\beta = 0\} \subset \zeta_1(G)$.

Let $u = t(\alpha_1,\beta_1)^{n_1} \ldots\ldots t(\alpha_k,\beta_k)^{n_k} \in \zeta_1(G)$ be in canonical form. Then $\forall \gamma,\delta \in \Omega$ with $\gamma < \delta$, $[u,t(\gamma,\delta)] = 1$.

Suppose w.l.o.g. that all pairs (α_i,β_i) with $t(\alpha_i,\beta_i)$ a generator occurring in the canonical form for u satisfy $\text{alt}\alpha_i + \text{dep}\beta_i > 0$. (If $\text{alt}\alpha_i + \text{dep}\beta_i = 0$, then $t(\alpha_i,\beta_i)$ can be taken out as it belongs to $\text{gp}\{t(\alpha,\beta) : \text{alt}\alpha + \text{dep}\beta = 0\}$)

Then, for all such pairs (α_i,β_i), $\exists \delta_i < \alpha_i$ or $\exists \gamma_i > \beta_i$.

Let $\beta_i \in \{\beta_1,\ldots\ldots,\beta_k\}$ and let $\gamma_i > \beta_i$.

Let $\bar{\alpha}_i$ be a maximal element associated with β_i in u.

Then, by lemma 3.1, $[u,t(\beta_i,\gamma_i)]$ contains a term $t(\bar{\alpha}_i,\gamma_i)^{n_i}$ which is not the identity and which cannot be cancelled.

So $u \notin \zeta_1(G)$, which is a contradiction.

Hence there is no $\gamma_i > \beta_i$ i.e. $\text{dep}\beta_i = 0$.

Now let $\alpha_j \in \{\alpha_1,\ldots\ldots,\alpha_k\}$ and let $\delta_j < \alpha_j$.

Let $\underline{\beta}_j$ be a minimal element associated with α_j in u.

Then, by lemma 3.1, $[u,t(\delta_j,\alpha_j)]$ contains a term $t(\delta_j,\beta_j)^{-n_j}$ which is not the identity and which cannot be cancelled.

So $u \notin \zeta_1(G)$, which is a contradiction.

Hence there is no $\delta_j < \alpha_j$ i.e. $\text{alt}\alpha_j = 0$.

So $u \in \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha + \text{dep}\beta = 0\}$.

Hence result.

<u>Corollary 3.4</u> If Ω is totally ordered, then G has a centre iff Ω has a maximal element $\overline{\omega}$ and a minimal element $\underline{\omega}$, and then $\zeta_1(G) = \text{gp}\{t(\underline{\omega},\overline{\omega})\}$.

<u>Proof</u> If Ω is totally ordered, then Ω will have at most one maximal element and at most one minimal element, i.e. there will be at most one pair (α,β) with $\text{alt}\alpha + \text{dep}\beta = 0$.

<u>Theorem 3.5</u> $\zeta_\lambda(G) = \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \lambda\}$.

<u>Proof</u> By induction.

The theorem is true if $\lambda = 1$ by theorem 3.3.

Let λ be a limit ordinal and suppose that $\forall \mu < \lambda$,

$\zeta_\mu(G) = \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \mu\}$.

$$\zeta_\lambda(G) = \bigcup_{\mu<\lambda}\zeta_\mu(G) = \bigcup_{\mu<\lambda}\text{gp}\{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \mu\}$$
$$\subset \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \lambda\}.$$

If $t(\gamma,\delta) \in \{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \lambda\}$, then $\text{alt}\gamma \oplus \text{dep}\delta = \mu$ for some $\mu < \lambda$, and so $t(\gamma,\delta) \in \zeta_{\mu+1}(G)$.

Hence $t(\gamma,\delta) \in \bigcup_{\mu<\lambda} \zeta_\mu(G) = \zeta_\lambda(G)$.

So $\zeta_\lambda(G) = \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \lambda\}$.

Suppose that $\zeta_\mu(G) = \text{gp}\{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \mu\}$.

Let $t(\gamma,\delta) \in \{t(\alpha,\beta) : \text{alt}\alpha \oplus \text{dep}\beta < \mu + 1\}$.

By lemma 2.2, $\forall \theta,\varphi \in \Omega$ with $\theta < \varphi$, $[t(\gamma,\delta),t(\theta,\phi)] = 1$ if $\gamma \neq \phi$ and $\theta \neq \delta$.

So if $\gamma \neq \phi$ and $\theta \neq \delta$, then $[t(\gamma,\delta),t(\theta,\phi)] = 1 \in \zeta_{\mu}(G)$.

If $\gamma = \phi$, then $[t(\gamma,\delta),t(\theta,\phi)] = t(\theta,\delta)^{-1}$, where $\theta < \phi = \gamma$.

By lemma 2.8, $\mathrm{alt}\,\theta < \mathrm{alt}\,\gamma$ and by lemma 2.7,

$\mathrm{alt}\,\theta \oplus \mathrm{dep}\,\delta < \mathrm{alt}\,\gamma \oplus \mathrm{dep}\,\delta < \mu + 1$.

So $\mathrm{alt}\,\theta \oplus \mathrm{dep}\,\delta < \mu$ i.e. $[t(\gamma,\delta),t(\theta,\phi)] = t(\theta,\delta)^{-1} \in \zeta_{\mu}(G)$.

If $\theta = \delta$, then $[t(\gamma,\delta),t(\theta,\phi)] = t(\gamma,\phi)$, where $\phi > \theta = \delta$.

By lemma 2.8, $\mathrm{dep}\,\phi < \mathrm{dep}\,\delta$ and by lemma 2.7,

$\mathrm{alt}\,\gamma \oplus \mathrm{dep}\,\phi < \mathrm{alt}\,\gamma \oplus \mathrm{dep}\,\delta < \mu + 1$.

So $\mathrm{alt}\,\gamma \oplus \mathrm{dep}\,\phi < \mu$ i.e. $[t(\gamma,\delta),t(\theta,\phi)] = t(\gamma,\phi) \in \zeta_{\mu}(G)$.

So $\mathrm{gp}\{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta < \mu + 1\} \subset \zeta_{\mu+1}(G)$.

To complete the induction proof it is necessary to show that if $u \in \zeta_{\mu+1}(G)$, then $u \in \mathrm{gp}\{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta < \mu + 1\}$.

Let $u = t(\alpha_1,\beta_1)^{n_1} \ldots\ldots\ldots t(\alpha_k,\beta_k)^{n_k} \in \zeta_{\mu+1}(G)$ be in canonical form.
Suppose w.l.o.g. that all pairs (α_i,β_i) where $t(\alpha_i,\beta_i)$ is one of the generators occurring in the canonical form for u satisfy $\mathrm{alt}\,\alpha_i \oplus \mathrm{dep}\,\beta_i > \mu$. (If $\mathrm{alt}\,\alpha_i \oplus \mathrm{dep}\,\beta_i \leqslant \mu$, then $t(\alpha_i,\beta_i)$ can be taken out since it belongs to $\mathrm{gp}\{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta < \mu + 1\}$)
Since $u \in \zeta_{\mu+1}(G)$, $\forall \gamma,\delta \in \Omega$ with $\gamma < \delta$, $[u,t(\gamma,\delta)] \in \zeta_{\mu}(G)$.
Suppose that $\mathrm{alt}\,\alpha_i = 0$ for some i. Let $\underline{\beta}_i$ be a minimal element associated with α_i in u. Then, since $\mathrm{alt}\,\alpha_i \oplus \mathrm{dep}\,\underline{\beta}_i > \mu$, $\mathrm{dep}\,\underline{\beta}_i > \mu$ and so by lemma 2.9 $\exists \theta > \underline{\beta}_i$ with $\mathrm{dep}\,\theta \geqslant \mu$.
By lemma 3.1, $[u,t(\underline{\beta}_i,\theta)]$ contains a term $t(\alpha_i,\theta)^{n_i}$ which is not the identity and which cannot be cancelled.
$\mathrm{Dep}\,\theta \oplus \mathrm{alt}\,\alpha_i \geqslant \mu$ and so $u \notin \zeta_{\mu+1}(G)$, which is a contradiction.
So $\mathrm{alt}\,\alpha_i \neq 0$, $1 \leqslant i \leqslant k$.
Similarly, $\mathrm{dep}\,\beta_i \neq 0$, $1 \leqslant i \leqslant k$.

Now assume that $\mathrm{alt}\,\alpha_i > 0$, $\mathrm{dep}\,\beta_i > 0$ and $\mathrm{alt}\,\alpha_i \oplus \mathrm{dep}\,\beta_i > \mu$.
Let $\beta_j \in \{\beta_1,\ldots,\beta_k\}$ and let $\bar{\alpha}_j$ be a maximal element associated

with βj in u. If $\theta < \bar{\alpha} j$, then $[u, t(\theta, \bar{\alpha} j)]$ contains a term $t(\theta, \beta j)^{-nj}$ which is not the identity and which cannot be cancelled and so, since $u \in \zeta_{\mu+1}(G)$, $t(\theta, \beta j) \in \zeta_\mu(G)$.

Let $\text{alt}\bar{\alpha} j = \sum_{i=1}^{k} \omega^{\kappa i} n'_i$, $\text{dep}\beta j = \sum_{i=1}^{k} \omega^{\kappa i} m_i$ and $\mu = \sum_{i=1}^{k} \omega^{\kappa i} r_i$.

Then $\text{alt}\bar{\alpha} j \oplus \text{dep}\beta j = \sum_{i=1}^{k} \omega^{\kappa i}(n'_i + m_i) > \sum_{i=1}^{k} \omega^{\kappa i} r_i = \mu$.

So $\exists x \leqslant k$ with $n'_i + m_i = r_i$ $(i < x)$ and $n'_x + m_x > r_x$.

If $n'_x > 0$, let $\rho = \sum_{i=1}^{x-1} \omega^{\kappa i} n'_i + \omega^{\kappa x}(n'_x - 1) + \sum_{i=x+1}^{k} \omega^{\kappa i}(r_i + 1)$.

Then $\rho < \text{alt}\bar{\alpha} j$ and $\rho \oplus \text{dep}\beta j \geqslant \mu$.

By lemma 2.9, $\exists \theta < \bar{\alpha} j$ with $\text{alt}\,\theta = \rho$.

So $t(\theta, \beta j) \in \zeta_\mu(G)$ and $\text{alt}\,\theta \oplus \text{dep}\beta j \geqslant \mu$, which is a contradiction.

Now assume $n'_x = 0$. Then $m_x > 0$.

Either βj is a minimal element associated with $\bar{\alpha} j$ in u or there is a $\underline{\beta} j < \beta j$ which is a minimal element associated with $\bar{\alpha} j$ in u. If βj is a minimal element associated with $\bar{\alpha} j$ in u, then by lemma 3.1 if $\phi > \beta j$, $[u, t(\beta j, \phi)]$ contains a term $t(\bar{\alpha} j, \phi)^{nj}$ which is not the identity and which cannot be cancelled.

So if $\phi > \beta j$, $t(\bar{\alpha} j, \phi) \in \zeta_\mu(G)$.

Let $\sigma = \sum_{i=1}^{x-1} \omega^{\kappa i} m_i + \omega^{\kappa x}(m_x - 1) + \sum_{i=x+1}^{k} \omega^{\kappa i}(r_i + 1)$.

Then $\sigma < \text{dep}\beta j$ and $\text{alt}\bar{\alpha} j \oplus \sigma \geqslant \mu$.

By lemma 2.9, $\exists \phi > \beta j$ with $\text{dep}\,\phi = \sigma$.

Then $t(\bar{\alpha} j, \phi) \in \zeta_\mu(G)$ but $\text{alt}\bar{\alpha} j \oplus \text{dep}\,\phi \geqslant \mu$ which is a contradiction.

Finally, if there is a $\underline{\beta} j < \beta j$ which is a minimal element associated with $\bar{\alpha} j$ in u, then by applying lemma 3.1

to $[u, t(\underline{\beta}j, \beta j)]$, we get $t(\bar{\alpha}j, \beta j) \in \zeta_\mu(G)$ which is a contradiction.

So $u \in gp\{t(\alpha,\beta) : \text{alt}\,\alpha + \text{dep}\,\beta < \mu + 1\}$.

Hence result.

Some of the corollaries which follow are in (6), but they are included for the sake of completeness as the methods of proof are in general different.

Corollary 3.6 If Ω is finite, then G is nilpotent.
This is contained in Corollary 4 of Chapter 4 of (6).

Corollary 3.7 If Ω is totally ordered and $|\Omega| = n$, then G is nilpotent of class $n - 1$.
This is contained in Corollary 4 of Chapter 4 of (6).

Corollary 3.8 G is nilpotent of class n iff any longest chain in Ω has length $n + 1$.

Proof If any longest chain in Ω has length $n + 1$ and $\alpha < \beta \in \Omega$, then α and β belong to some chain C of length $< n + 2$.
So $\text{alt}\,\alpha + \text{dep}\,\beta < n$ for all pairs α, β with $\alpha < \beta \in \Omega$, i.e. $\zeta_n(G) = \zeta_{n+1}(G)$.
Since there is a chain of length n+1, $\exists\, \alpha, \beta \in \Omega$ with $\alpha < \beta$ and $\text{alt}\,\alpha + \text{dep}\,\beta = n - 1$ i.e. $\zeta_n(G) \neq \zeta_{n-1}(G)$
Now suppose that $\zeta_n(G) = G \supsetneq \zeta_{n-1}(G)$. Then for all pairs α, β with $\alpha < \beta \in \Omega$,
$t(\alpha,\beta) \in gp\{t(\alpha,\beta) : \text{alt}\,\alpha + \text{dep}\,\beta < n\}$
So $\forall \alpha, \beta \in \Omega$ $\text{alt}\,\alpha < \infty$ $\text{dep}\,\beta < \infty$ and by lemma 2.20 all chains in Ω are finite. Suppose that there is a chain $[\alpha,\beta]$ of length $n + 2$ - any chain can be written in this form since all chains are finite. By lemma 2.12(a), $\text{alt}\,\beta \geq n + 1$ and $t(\alpha,\beta) \notin \zeta_n(G)$ which is a contradiction. So any longest chain in Ω has length $< n + 2$. Since there is a pair α, β with $\alpha < \beta$ and

$\mathrm{alt}\alpha + \mathrm{dep}\beta = n - 1$, there must be a chain of length $n + 1$.

Hence result.

This is Corollary 3 of Chapter 4 of (6).

Corollary 3.10 G is locally nilpotent.

Proof Let F be a finitely generated subgroup of G, say

$$F = gp\{t(\alpha_1,\beta_1),\ldots\ldots,t(\alpha_n,\beta_n)\}.$$

Then $\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\}$ is a finite partially ordered set, Ω' say. So any longest chain in Ω' is finite - of length $k + 1$ say - and by corollary 3.8, F is nilpotent.

So G is locally nilpotent.

This is Corollary 5 of Chapter 4 of (6).

Corollary 3.11 For every ordinal λ, there is a partially ordered set Ω such that $G = Mc(\Omega,F)$, where F is some field, has central height λ.

Proof Suppose that λ is not a limit ordinal, $\lambda = \mu + 1$ say. Let Ω be an interval of order type $\lambda + 2$, $[\underline{\omega},\alpha]$ say.

Then by lemma 2.15, $\mathrm{alt}\alpha = \mu + 1$ and if $\beta < \alpha$, $\mathrm{alt}\beta \leq \mu$.

So if β is not a maximal element, $\mathrm{alt}\beta \leq \mu$.

Since $\mu + 1$ is not a limit ordinal, α has an immediate predecessor, γ, and $\mathrm{alt}\gamma = \mu$. $\mathrm{Dep}\alpha = 0$ and so there is a pair α,γ with $\mathrm{alt}\gamma + \mathrm{dep}\alpha = \mu$, and $G = Mc(\Omega,F)$ has central height $\mu + 1$.

Let λ be a limit ordinal and let $[\underline{\omega},\beta] = \Omega$ be an interval of order type $\lambda + 1$. If $\alpha < \beta$, then $\mathrm{alt}\alpha < \lambda$. Also if $\mu < \lambda$, there is an interval $[\underline{\omega},\alpha_\mu]$ of order type $\mu + 1$. So if $\mu < \lambda$, there is an $\alpha_\mu < \beta$ with $\mathrm{alt}\alpha_\mu = \mu$.

So $G = Mc(\Omega,F)$ has central height λ.

This is Theorem 4.4 of (6).

Corollary 3.9 Let G have central height h.

Then $h = \lim \{\text{alt}\alpha \oplus \text{dep}\beta + 1\}$ where $\text{alt}\alpha < \infty$ and $\text{dep}\beta < \infty$ and $\alpha < \beta \in \Omega$.

Lemma 3.12 If Ω is totally ordered, then $[\zeta_{\mu+1}(G), G] = \zeta_\mu(G)$ where $\mu < h$, the central height of G.

Proof By definition, $[\zeta_{\mu+1}(G), G] \subset \zeta_\mu(G)$.

If $h = 0$, the result is trivially true.

If $h > 0$, then by corollary 3.4, Ω has a maximal element $\overline{\omega}$ and a minimal element $\underline{\omega}$.

Let $t(\alpha,\beta)$ be a generator of $\zeta_\mu(G)$.

By lemma 2.17, $[\underline{\omega}, \alpha]$ and $[\beta, \overline{\omega}]^*$ are well ordered.

Since $\zeta_{\mu+1}(G) \neq \zeta_\mu(G)$, at least one of these intervals can be extended to a longer well ordered interval. So either α has an immediate successor, γ say, or β has an immediate predecessor, δ.

If α has an immediate successor γ, then $\text{alt}\gamma = \text{alt}\alpha + 1$, and $[\underline{\omega}, \gamma]$ is well ordered. Then $\text{alt}\gamma \oplus \text{dep}\beta < \mu + 1$ and $t(\alpha,\beta) = [t(\alpha,\gamma), t(\gamma,\beta)] \in [\zeta_{\mu+1}(G), G]$.

A similar argument gives the result if β has an immediate predecessor.

G is a ZA group of ZA length λ if $\zeta_\lambda(G) = G$ and $\zeta_\mu(G) \subsetneq G$ for all $\mu < \lambda$.

Corollary 3.13 G is a ZA group iff all chains in Ω are finite.

Proof Suppose that all chains in Ω are finite.

Then by lemma 2.20, if $\alpha \in \Omega$, $\text{alt}\alpha < \infty$ and $\text{dep}\alpha < \infty$.

So for all pairs $\alpha, \beta \in \Omega$ with $\alpha < \beta$, $\text{alt}\alpha < \infty$ and $\text{dep}\beta < \infty$.

So for all pairs α, β with $\alpha < \beta$, $\text{alt}\alpha \oplus \text{dep}\beta < \infty$.

Let $\lambda = \lim_{\alpha<\beta} \{\mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta + 1\}$.

If $t(\alpha,\beta)$ is a generator of G, then $\mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta < \lambda$.

So $G = \zeta_\lambda(G) = \mathrm{gp}\{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta < \lambda\}$.

So G is a $Z\Lambda$ group.

Now suppose that G is a $Z\Lambda$ group, $G = \zeta_\mu(G)$ say.

Let $\alpha,\beta \in \Omega$ with $\alpha < \beta$.

Since $t(\alpha,\beta) \in \zeta_\mu(G)$, $\mathrm{alt}\,\alpha < \infty$ and $\mathrm{dep}\,\beta < \infty$.

So $\mathrm{alt}\,\alpha < \infty$ whenever α is not a maximal element and $\mathrm{dep}\,\beta < \infty$ whenever β is not a minimal element.

By corollary 2.21, all chains in Ω are finite.

Hence result.

Chapter 4. The lower central series of G and some results about the derived series of G

Denote a term of the lower central series of G by $\gamma_\mu(G)$. Then $\gamma_1(G) = G$, $\gamma_{\mu+1}(G) = [\gamma_\mu(G), G]$ and if λ is a limit ordinal

$\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\mu(G)$.

G has central depth d if $\gamma_d(G) < \gamma_\mu(G)$ for all $\mu < d$, $\gamma_{d+1}(G) = \gamma_d(G)$ and $d \geq \omega$ or if $\gamma_{d+1}(G) < \gamma_\mu(G)$ for all $\mu < d + 1$, $\gamma_{d+2}(G) = \gamma_{d+1}(G)$ and $d < \omega$.

Lemma 4.1 If $\text{sep}(\alpha,\beta) = \infty$, then $t(\alpha,\beta) \in \gamma_\lambda(G)$ for every ordinal λ.

Proof By induction.

If $\text{sep}(\alpha,\beta) = \infty$, then $t(\alpha,\beta) \in \gamma_1(G) = G$.

Let λ be a limit ordinal and suppose that if $\mu < \lambda$, then $t(\alpha,\beta) \in \gamma_\mu(G)$.

Then $t(\alpha,\beta) \in \bigcap_{\mu<\lambda} \gamma_\mu(G) = \gamma_\lambda(G)$.

Finally, suppose that if $\text{sep}(\alpha,\beta) = \infty$, then $t(\alpha,\beta) \in \gamma_\mu(G)$.

If $\text{sep}(\alpha,\beta) = \infty$, then by lemma 2.25 there is a δ with $\alpha < \delta < \beta$ and $\text{sep}(\alpha,\delta) = \infty$ or $\text{sep}(\delta,\beta) = \infty$.

So $t(\alpha,\beta) = [t(\alpha,\delta), t(\delta,\beta)] \in [\gamma_\mu(G), G] = \gamma_{\mu+1}(G)$.

Hence result.

Theorem 4.2 $\gamma_\lambda(G) = \text{gp}\{t(\alpha,\beta) : \text{sep}(\alpha,\beta) \geq \lambda\}$.

Proof By induction.

By lemma 4.1, if $\text{sep}(\alpha,\beta) = \infty$, then $t(\alpha,\beta) \in \gamma_\lambda(G)$.

$\gamma_1(G) = G = \text{gp}\{t(\alpha,\beta) : \text{sep}(\alpha,\beta) \geq 1\}$.

Let λ be a limit ordinal.

$$\gamma_\lambda(G) = \bigcap_{\mu<\lambda} \gamma_\mu(G)$$

$$= \bigcap_{\mu<\lambda} gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \mu\}$$

$$\supset gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \mu \ \forall \mu < \lambda\}$$

$$= gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \lambda\}.$$

Let $g = t(\alpha_1,\beta_1)^{n_1} \ldots\ldots t(\alpha_k,\beta_k)^{n_k} \in \gamma_\lambda(G)$ be in canonical form.

Then if $\mu < \lambda$, $g \in \gamma_\mu(G)$ and by lemma 2.44, $sep(\alpha_i,\beta_i) \geq \mu$.

So $\forall \mu < \lambda$, $sep(\alpha_i,\beta_i) \geq \mu$ i.e. $sep(\alpha_i,\beta_i) \geq \lambda$.

So $g \in gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \lambda\}$.

Suppose that $\gamma_\mu(G)$ is of the form stated in the theorem.

Let $t(\rho,\sigma) \in \{t(\alpha,\beta) : sep(\alpha,\beta) \geq \mu + 1\}$.

Then there is a δ with $\rho < \delta < \sigma$ such that $sep(\rho,\delta) \geq \mu$ or $sep(\delta,\sigma) \geq \mu$.

Hence $t(\rho,\sigma) = [t(\rho,\delta),t(\delta,\sigma)] \in [\gamma_\mu(G),G] = \gamma_{\mu+1}(G)$.

So $gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \mu + 1\} \subset \gamma_{\mu+1}(G)$.

Let $u = [g_1,h_1]^{n_1} \ldots\ldots [g_k,h_k]^{n_k} \in \gamma_{\mu+1}(G)$ where for $1 \leq i \leq k$, $g_i \in \gamma_\mu(G)$.

By lemma 2.36, when the commutators are expanded to give u in the form $t(\lambda_1,\mu_1)^{m_1} \ldots\ldots t(\lambda_r,\mu_r)^{m_r}$, for $1 \leq j \leq r$, $sep(\lambda_j,\mu_j) \geq \mu + 1$ or $sep(\lambda_j,\mu_j) = \infty$.

So $\gamma_{\mu+1}(G) = gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \mu + 1\}$.

Hence result.

The next result is given in (6), but it is included for the sake of completeness.

Corollary 4.3 $G = G'$ (the derived group of G) iff Ω is everywhere dense i.e. given $\alpha < \beta \in \Omega$, $\exists \delta$ with $\alpha < \delta < \beta$.

Proof Let Ω be everywhere dense and let $t(\alpha,\beta)$ be a generator of G. Then there is a δ lying between α and β and $t(\alpha,\beta) = [t(\alpha,\delta),t(\delta,\beta)] \in G'$.

So $G = G'$.

Suppose that $G = G'$ and let $\alpha < \beta \in \Omega$ with $sep(\alpha,\beta) = 1$. Since $t(\alpha,\beta) \in G$, $t(\alpha,\beta) \in G'$ and so $t(\alpha,\beta)$ can be expressed in the form $[t(\alpha_1,\beta_1),t(\gamma_1,\delta_1)]^{n_1}\ldots\ldots[t(\alpha_k,\beta_k),t(\gamma_k,\delta_k)]^{n_k}$.

By lemma 2.2, $[t(\alpha_i,\beta_i),t(\gamma_i,\delta_i)] = 1$ or $t(\alpha_i,\delta_i)$ with $\alpha_i < \beta_i = \gamma_i < \delta_i$ or $t(\gamma_i,\beta_i)^{-1}$ with $\gamma_i < \delta_i = \alpha_i < \beta_i$.

So each $[t(\alpha_i,\beta_i),t(\gamma_i,\delta_i)]^{n_i}$ is 1 or of the form $t(\lambda_i,\mu_i)^{\pm n_i}$ where there is a τ_i with $\lambda_i < \tau_i < \mu_i$.

So $(\alpha,\beta) \neq (\lambda_i,\mu_i)$ for any i.

$t(\alpha,\beta) = t(\lambda_1,\mu_1)^{m_1}\ldots\ldots t(\lambda_r,\mu_r)^{m_r} \in gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq 2\}$.

By lemma 2.44, since the canonical form of $t(\alpha,\beta)$ is $t(\alpha,\beta)$, $sep(\alpha,\beta) \geq 2$ which is a contradiction.

So Ω is everywhere dense.

Corollary 4.4 If Ω is totally ordered, then d, the central depth of G, $\leq \omega$.

Proof Follows from lemma 2.29

Corollary 4.5 If Ω is totally ordered, h = central height of G and d = central depth of G, then

(1) if $h \geq \omega$, then $d = \omega$.

(2) if $h < \omega$, then $d \geq \left[\frac{h-1}{2}\right]$ where $[x]$ is the

largest integer $\leqslant x$.

Proof (1) By corollary 4.4, $d \leqslant \omega$.

Since $h \geqslant \omega$, $\lim_{\alpha<\beta} \{\text{alt}\alpha \oplus \text{dep}\beta + 1\} \geqslant \omega$ where $\text{alt}\alpha < \infty$ and $\text{dep}\beta < \infty$.

So if $n < \omega$, $\exists \alpha_n\ \beta_n \in \Omega$ with $\text{alt}\alpha_n + \text{dep}\beta_n = n$.

Since $\text{alt}\alpha_n + \text{dep}\beta_n = n$, either $\text{alt}\alpha_n \geqslant \left[\frac{n}{2}\right]$ or $\text{dep}\beta_n \geqslant \left[\frac{n}{2}\right]$.

So if $n < \omega$, $\exists \gamma_n \in \Omega$ with $\text{alt}\gamma_n \geqslant \left[\frac{n}{2}\right]$ or $\text{dep}\gamma_n \geqslant \left[\frac{n}{2}\right]$.

By lemma 2.39, $\exists \delta_n \in \Omega$ such that $\text{sep}(\gamma_n, \delta_n) = \left[\frac{n}{2}\right]$ or $\text{sep}(\delta_n, \gamma_n) = \left[\frac{n}{2}\right]$.

So if $m < \omega$, there is a pair λ_m, μ_m with $\text{sep}(\lambda_m, \mu_m) = m = \left[\frac{2m}{2}\right]$.

So, by theorem 4.2, if $m < \omega$, $\gamma_m(G) \neq \gamma_{m+1}(G)$.

Hence $d = \omega$.

(2) Let $h = n$.

Then $\zeta_{n+1}(G) = \zeta_n(G)$.

So if $\alpha < \beta \in \Omega$, $\text{alt}\alpha + \text{dep}\beta < n$ and there is at least one pair α, β with $\text{alt}\alpha + \text{dep}\beta = n - 1$.

Either $\text{alt}\alpha \geqslant \left[\frac{n-1}{2}\right]$ or $\text{dep}\beta \geqslant \left[\frac{n-1}{2}\right]$.

So by lemma 2.39, $\exists \gamma \in \Omega$ with $\text{sep}(\gamma, \alpha) = \left[\frac{n-1}{2}\right]$ or $\text{sep}(\beta, \gamma) = \left[\frac{n-1}{2}\right]$.

Let $\left[\frac{n-1}{2}\right] = m$.

Then $\gamma_m(G) \neq \gamma_{m+1}(G)$.

Hence $d \geqslant m = \left[\frac{n-1}{2}\right] = \left[\frac{h-1}{2}\right]$.

Examples

(1) Let Ω be an interval $[\alpha,\beta]$ of order type $\omega 2 + 1$.

Then G has central height $\omega 2$.

Since Ω is totally ordered, if $\lambda < \mu$, $\text{sep}(\lambda,\mu) < \omega$ or $\text{sep}(\lambda,\mu) = \infty$ by lemma 2.29.

Since $[\alpha,\beta]$ has order type $\omega 2 + 1$, if $n < \omega$ there is an $\alpha_n > \alpha$ with $\text{alt}\,\alpha_n = n$.

By lemma 2.39, $\exists \delta_n < \alpha_n$ with $\text{sep}(\delta_n, \alpha_n) = n$.

So G has central depth ω.

(2) Let Ω_1 be an interval $[\gamma,\alpha]$ of length n.

Let Ω_2 be a totally ordered set which has no maximal element and no minimal element.

Let $\Omega_3 = \{\beta\}$.

Let Ω consist of Ω_1 followed by Ω_2 followed by Ω_3.

Then $\text{alt}\,\alpha = n - 1$, $\text{dep}\,\beta = 0$ and if $\lambda > \alpha$, $\text{alt}\,\lambda = \infty$ and if $\mu < \beta$, $\text{dep}\,\mu = \infty$.

So G has central height n.

$\text{Sep}(\gamma,\alpha) = n - 1$ and so the central depth of G $\geq n - 1$.

Corollary 4.6 If Ω is partially ordered, then

(1) If $h \geq \omega$, then $d \geq \omega$

(2) If $h < \omega$, then $d \geq \left\lceil \frac{h-1}{2} \right\rceil$.

Corollary 4.7 If λ is any ordinal and $\mu \leq \omega$, then

$$[\gamma_\lambda(G), \gamma_\mu(G)] \subset \gamma_{\lambda+\mu}(G).$$

Proof By lemma 2.42 and corollary 2.43.

Corollary 4.7 need not be true if $\mu > \omega$, for consider the following example.

Let $\alpha \in \Omega$ be the only minimal element of Ω, and let $\beta > \alpha$ be such that for any $n < \omega$, there is a chain $[\alpha,\beta]$ of length n. It is easy to see that $\text{sep}(\alpha,\beta) = \omega$.

Let $\gamma > \beta$ with no element of Ω lying between γ and β.

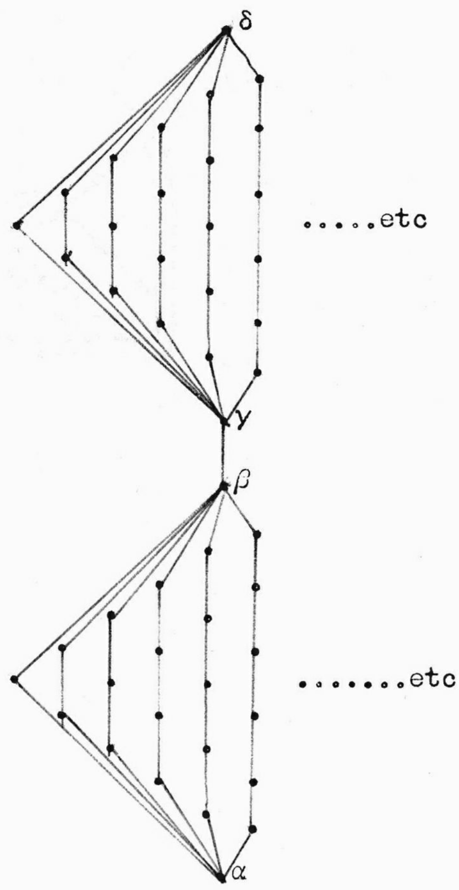
Let $\delta > \gamma$ be such that for any $n < \omega$, there is a chain $[\gamma,\delta]$ of length n and let δ be the only maximal element of Ω.

It is easy to see that $\text{sep}(\gamma,\delta) = \omega$ and $\text{sep}(\beta,\delta) = \omega + 1$.

However $\text{sep}(\alpha,\delta) = \omega + \omega$, for if $\alpha < \sigma < \delta$, then $\text{sep}(\alpha,\sigma) < \omega+\omega$ and $\text{sep}(\sigma,\delta) < \omega + \omega$.

$t(\alpha,\beta) \in \gamma_{\omega}(G)$, $t(\beta,\delta) \in \gamma_{\omega+1}(G)$ but $t(\alpha,\delta) \in \gamma_{\omega 2}(G) - \gamma_{\omega 2+1}(G)$.

$\Omega =$



Denote a term of the derived series of G by $\delta_\mu(G)$. Then

$$\delta_0(G) = G,$$

$$\delta_{\mu+1}(G) = [\delta_\mu(G), \delta_\mu(G)], \text{ and}$$

if λ is a limit ordinal, $\delta_\lambda(G) = \bigcap_{\mu<\lambda} \delta_\mu(G)$.

<u>Theorem 4.8</u> $\delta_n(G) = \gamma_{2^n}(G) = gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq 2^n\}$.

<u>Proof</u> By induction.

$\delta_0(G) = G = \gamma_1(G) = \gamma_{2^0}(G)$.

It is proved in (3) (theorem 3.4 (iii)) that $\delta_n(G) \subset \gamma_{2^n}(G)$ for any group G.

Suppose that $\delta_k(G) = \gamma_{2^k}(G) = gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq 2^k\}$.

Let $t(\alpha,\beta) \in \{t(\alpha,\beta) : sep(\alpha,\beta) \geq 2^{k+1}\}$.

Then $sep(\alpha,\beta) \geq 2^{k+1}$ and by lemma 2.35, there is a γ with $\alpha < \gamma < \beta$ and $sep(\alpha,\gamma) \geq 2^k$ and $sep(\gamma,\beta) \geq 2^k$.

Then $t(\alpha,\beta) = [t(\alpha,\gamma),t(\gamma,\beta)] \in [\delta_k(G), \delta_k(G)] = \delta_{k+1}(G)$.

So $\gamma_{2^{k+1}}(G) \subset \delta_{k+1}(G)$.

Hence result.

<u>Corollary 4.9</u> $\delta_\omega(G) = \gamma_\omega(G) = gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \omega\}$.

<u>Proof</u> $\delta_\omega(G) = \bigcap_{n<\omega} \delta_n(G)$

$$= \bigcap_{n<\omega} \gamma_{2^n}(G)$$

$$\supset \bigcap_{k<\omega} \gamma_k(G) = \gamma_\omega(G).$$

Let $g \in \delta_\omega(G)$.

If $n < \omega$, then $g \in \gamma_{2^n}(G)$.

Let $m < \omega$. Then $\exists k$ such that $2^k > m$.

$g \in \gamma_{2^k}(G) \subset \gamma_m(G)$.

So $g \in \gamma_m(G)\ \forall m < \omega$.

Hence $g \in \bigcap_{m<\omega} \gamma_m(G) = \gamma_\omega(G)$.

Corollary 4.10 $\delta_{\omega+1}(G) \subset \gamma_{\omega 2}(G)$.

Proof $\delta_{\omega+1}(G) = [\delta_\omega(G), \delta_\omega(G)]$

$= [\gamma_\omega(G), \gamma_\omega(G)]$ by corollary 4.9

$\subset \gamma_{\omega 2}(G)$ by corollary 4.7.

G is a ZD group of ZD length λ if $\gamma_\lambda(G) = \{1\}$, $\gamma_\mu(G) \neq \{1\}$ for $\mu < \lambda$ and $\lambda \geq \omega$ or $\gamma_{\lambda+1}(G) = \{1\}$, $\gamma_\mu(G) \neq \{1\}$ for $\mu < \lambda + 1$ and $\lambda < \omega$.

Theorem 4.11 G is a ZD group iff all chains in Ω are order isomorphic to a subset of the integers.

Proof Let G be a ZD group of ZD length λ.

Then $\gamma_{\lambda+1}(G) = \{1\}$ - if $\lambda \geq \omega$, $\gamma_\lambda(G)$ is also $\{1\}$.

So $gp\{t(\alpha,\beta) : sep(\alpha,\beta) \geq \lambda + 1\} = \{1\}$.

So if $\alpha < \beta$, $sep(\alpha,\beta) < \infty$.

By lemma 2.32, all chains in Ω are order isomorphic to a subset of the integers.

Now suppose that all chains in Ω are order isomorphic to a subset of the integers.

Then by lemma 2.32, $sep(\alpha,\beta) < \infty\ \forall \alpha,\beta \in \Omega$ with $\alpha < \beta$.

Let $\mu = \lim_{\alpha<\beta}\{sep(\alpha,\beta) + 1\}$.

Then $\gamma_\mu(G) = \{1\}$ and G is a ZD group.

Corollary 4.12 If G is a ZA group, then G is a ZD group.

Proof By corollary 3.13, if G is a ZA group, then all chains in Ω are finite.

By theorem 4.11, if all chains in Ω are finite, then G is a ZD group.

The converse to corollary 4.12 is not true, as the following example shows.

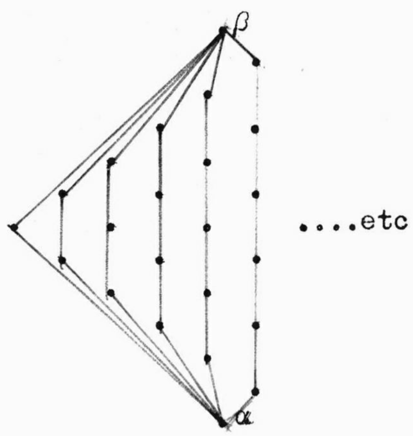
Let $\Omega = N$ (the natural numbers) with the usual order. Then Ω has no maximal element and so by theorem 3.3, G does not have a non-trivial centre i.e. G is not a ZA group. However, if $m,n \in N$ with $m < n$, then $sep(m,n) = n - m$. So separation can be defined for all pairs $m,n \in N$.
Then $\gamma_\omega(G) = gp\{t(i,j) : sep(i,j) \geq \omega\} = \{1\}$.
So G is a ZD group of ZD length ω.

If G is a McLain group with ZA length λ and ZD length μ, it is possible for $\lambda = \mu$, $\lambda > \mu$ or $\lambda < \mu$ as the following examples show.

<u>1. $\lambda = \mu$</u>

Let Ω consist of two elements α and β with $\alpha < \beta$ and chains of all finite lengths joining α and β.

$\Omega =$



Then $sep(\alpha,\beta) = \omega$ and if $(\gamma,\delta) \neq (\alpha,\beta)$, then $sep(\gamma,\delta) < \omega$.

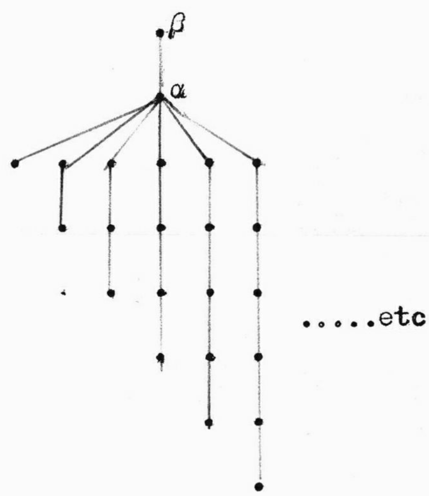
So G has ZD length ω.

If $\gamma \in \Omega$ and $\alpha < \gamma < \beta$, then altγ is finite and depγ is finite. So for all $\gamma, \delta \in \Omega$ with $\gamma < \delta$, altγ ⊕ depδ is finite. For each n, there is a γ_n with altγ_n = n. So G has ZA length ω.

2. $\lambda > \mu$

Let Ω consist of two elements α and β with $\alpha < \beta$ and no element of Ω lying between them, together with chains of all finite lengths below α

Ω =



Then alt$\alpha = \omega$, dep$\beta = 0$ and if $\gamma < \alpha$, then altγ is finite and depγ is finite. So G is a ZA group of ZA length $\omega + 1$.

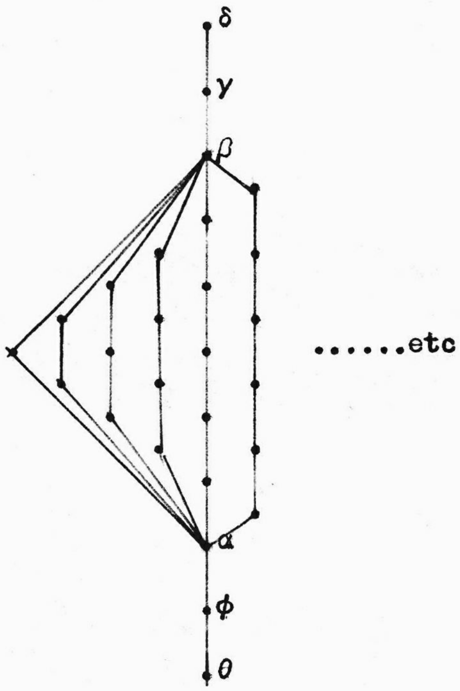
If $\gamma < \delta \in \Omega$, then sep$(\gamma, \delta)$ is finite. Also, if $n < \omega$, there is a pair (γ_n, δ_n) with sep(γ_n, δ_n) = n. So G is a ZD group of ZD length ω.

3. $\lambda < \mu$

Let Ω consist of two elements α and β with $\alpha < \beta$ and chains of all finite lengths joining α and β. Let there be a chain $\beta < \gamma < \delta$ above β and a chain $\theta < \phi < \alpha$ below α. Let δ

be the only maximal element and θ the only minimal element.

$\Omega =$



Then $\text{alt}\beta = \omega$, $\text{alt}\gamma = \omega + 1$, $\text{dep}\alpha = \omega$ and $\text{dep}\phi = \omega + 1$.
So $\forall \rho, \sigma \in \Omega$, $\text{alt}\rho \oplus \text{dep}\sigma \leqslant \omega + 1$ and G is a ZA group of ZΛ length $\omega + 2$.

Sep$(\alpha, \beta) = \omega$ and so sep$(\theta, \delta) = \omega + 4$. If $\rho, \sigma \in \Omega$, sep$(\rho, \sigma) \leqslant \omega + 4$ and so G is a ZD group of ZD length $\omega + 4$.

Chapter 5. The central series of the semidirect product, V.G

Let H = V.G. Then H is the set of elements (g,v) with g a member of G and v a member of V, where multiplication is defined by

$$(g,v)(h,w) = (gh, h(v)+w).$$

The set of elements (1,v) forms a normal subgroup of H which is isomorphic to V, and the set of elements (g,0) forms a subgroup isomorphic to G.

$H/V \simeq G.$

Where no confusion can arise, (1,v)(g,0) is sometimes written vg, (g,0)(1,v) is sometimes written gv and h(v) is sometimes written v^h.

We first consider the upper central series of H.

Lemma 5.1 Let $\alpha \in \Omega$. Then $\mathrm{alt}\,\alpha = \theta$ iff $t(\lambda,\mu)(e_\alpha) = e_\alpha$ for all $\lambda, \mu \in \Omega$ with $\lambda < \mu$.

Corollary $\mathrm{Alt}\,\alpha = 0$ iff $g(e_\alpha) = e_\alpha \ \forall g \in G$.

Lemma 5.2 $\zeta_\lambda(H) \cap G \subset \zeta_\lambda(G)$.

Lemma 5.3 $\zeta_\lambda(H) \cap V = \mathrm{gp}\{ e_\alpha : \mathrm{alt}\,\alpha < \lambda\}$.

Proof By induction.

V is abelian and so $v \in \zeta_1(H) \cap V$ iff

$$[(1,v),(g,0)] = (1,0) \ \forall g \in G$$

i.e. iff $g(v) - v = 0 \ \forall g \in G$

i.e. iff $g(v) = v \ \forall g \in G$.

Let $v = \sum_{i=1}^{k} n_i e_{\alpha_i}$ where if $i < j$, then $\alpha_i < \alpha_j$ or α_i and α_j are not comparable.

If $\text{alt}\alpha_j \neq 0$, then there is a $\gamma < \alpha_j$.

Then $t(\gamma, \alpha_j)(v) = v + n_j e_\gamma \neq v$.

So $v \in \zeta_1(H) \cap V$ iff $v \in gp\{ e_\alpha : \text{alt}\alpha = 0\}$.

So $\zeta_1(H) \cap V = gp\{ e_\alpha : \text{alt}\alpha < 1\}$.

Let λ be a limit ordinal.

$$\begin{aligned} \text{Then } \zeta_\lambda(H) \cap V &= \Big(\bigcup_{\mu<\lambda} \zeta_\mu(H) \Big) \cap V \\ &= \bigcup_{\mu<\lambda} \big(\zeta_\mu(H) \cap V \big) \\ &= \bigcup_{\mu<\lambda} gp\{ e_\alpha : \text{alt}\alpha < \mu\} \\ &= gp\{ e_\alpha : \text{alt}\alpha < \lambda\}. \end{aligned}$$

Finally suppose that $\zeta_\mu(H) \cap V = gp\{ e_\alpha : \text{alt}\alpha < \mu\}$.

Let $e_\beta \in \{ e_\alpha : \text{alt}\alpha < \mu + 1\}$.

Then $[(1,e_\beta),(t(\phi,\theta),0)] = t(\phi,\theta)(e_\beta) - e_\beta = 0$ if $\beta \neq \theta$ or e_ϕ where $\phi < \beta$ if $\beta = \theta$ and then $\text{alt}\phi < \mu$.

Hence $gp\{ e_\alpha : \text{alt}\alpha < \mu + 1\} \subset \zeta_{\mu+1}(H) \cap V$.

Let $v = \sum_{i=1}^{k} n_i e_{\alpha_i} \in \zeta_{\mu+1}(H) \cap V$ where the α_i are distinct.

Suppose that $\text{alt}\alpha_i \geq \mu + 1$ $(1 \leq i \leq k)$ - otherwise e_{α_i} can be left out as it belongs to $gp\{ e_\alpha : \text{alt}\alpha < \mu + 1\}$.

If $\text{alt}\alpha_i \geq \mu + 1$, then there is a $\beta < \alpha_i$ with $\text{alt}\beta \geq \mu$.

$$\begin{aligned} \text{Then } [(1,v),(t(\beta,\alpha_i),0)] &= t(\beta,\alpha_i)(v) - v \\ &= v + n_i e_\beta - v \\ &= n_i e_\beta \text{ and } \text{alt}\beta \geq \mu. \end{aligned}$$

$e_\beta \notin \zeta_\mu(H) \cap V$ and so $v \notin \zeta_{\mu+1}(H) \cap V$.

Hence result.

Theorem 5.4 $\zeta_\mu(H) = (\zeta_\mu(H) \cap V).(\zeta_\mu(H) \cap G)$.

Proof Proved in (9).

Lemma 5.5 Let $g = t(\alpha_1,\beta_1)^{n_1}.....t(\alpha_k,\beta_k)^{n_k}$ be in canonical form and let $\operatorname{alt}\alpha_i < \gamma$ $(1 \leq i \leq k)$. Let $h \in G$ be in canonical form and let $[g,h] = t(\lambda_1,\mu_1)^{m_1}.....t(\lambda_r,\mu_r)^{m_r}$ in its canonical form. Then $\operatorname{alt}\lambda_i < \gamma (1 \leq i \leq r)$.

Proof Let $h = t(\kappa_1,\eta_1)^{r_1}.....t(\kappa_n,\eta_n)^{r_n}$.

$$[g,h] = [t(\alpha_1,\beta_1)^{n_1},h]^{t(\alpha_2,\beta_2)^{n_2}...t(\alpha_k,\beta_k)^{n_k}} ....[t(\alpha_k,\beta_k)^{n_k},h].$$

$[t(\alpha_i,\beta_i)^{n_i},h] =$

$$[t(\alpha_i,\beta_i)^{n_i},t(\kappa_n,\eta_n)^{r_n}]..[t(\alpha_i,\beta_i)^{n_i},t(\kappa_1,\eta_1)^{r_1}]^{t(\kappa_2,\eta_2)^{r_2}..t(\kappa_n,\eta_n)^{r_n}}.$$

So $[g,h]$ is a product of terms of the form

$$[t(\alpha_i,\beta_i)^{n_i},t(\kappa_j,\eta_j)^{r_j}]^{t(\kappa_{j+1},\eta_{j+1})^{r_{j+1}}.....t(\alpha_k,\beta_k)^{n_k}}.$$

By corollary 2.3, $[t(\alpha_i,\beta_i)^{n_i},t(\kappa_j,\eta_j)^{r_j}]$ is 1 or $t(\alpha_i,\eta_j)^{n_i r_j}$ if $\beta_i = \kappa_j$ or $t(\kappa_j,\beta_i)^{-n_i r_j}$ if $\alpha_i = \eta_j$ and then $\kappa_j < \alpha_i$.

So $[g,h]$ is a product of terms of the form

$$\left(t(\theta_{ij},\phi_{ij})^{z_{ij}}\right)^{t(\kappa_{j+1},\eta_{j+1})^{r_{j+1}}.......t(\alpha_k,\beta_k)^{n_k}}, \text{ with } \theta_{ij} \leq \alpha_i$$

i.e. $\operatorname{alt}\theta_{ij} < \gamma$.

Since $x^y = x[x,y]$, the conjugates of $t(\theta_{ij},\phi_{ij})^{z_{ij}}$ can be expanded as a product of commutators of $t(\theta_{ij},\phi_{ij})^{z_{ij}}$ and by corollary 2.3, these commutators will either be 1 or $t(\nu_{ij},\delta_{ij})^{y_{ij}}$ with $\nu_{ij} \leq \theta_{ij}$ i.e. $\operatorname{alt}\nu_{ij} < \gamma$.

So $[g,h] = t(\nu_1,\delta_1)^{x_1}....t(\nu_y,\delta_y)^{x_y}$ with $\operatorname{alt}\nu_i < \gamma$.

Since an element of G is reduced to its canonical form by using products of commutators, it is easy to see by using lemma 2.13 that when $[g,h]$ is put in its canonical form $t(\lambda_1,\mu_1)^{m_1}.....t(\lambda_r,\mu_r)^{m_r}$, $\text{alt}\lambda_i < \gamma$.

Lemma 5.6 For $n < \omega$, $\zeta_n(H) \cap G = \zeta_{n-1}(G)$.

Proof By induction.

$\zeta_n(H) \cap G \subset \zeta_n(G)$.

$g \in \zeta_1(H) \cap G$ iff $g(v) = v \ \forall v \in V$ and $g \in \zeta_1(G)$.

So if $g \in \zeta_1(H) \cap G$, then $g = 1$ since g is an automorphism of V.

So $\zeta_1(H) \cap G = \zeta_0(G)$.

Suppose that $\zeta_n(H) \cap G = \zeta_{n-1}(G)$.

Then $g \in \zeta_{n+1}(H) \cap G$ iff $[g,h] \in \zeta_n(H) \cap G$ for all h in G and $[(1,v),(g,0)] \in \zeta_n(H) \cap V$ for all v in V.

So if $g \in \zeta_{n+1}(H) \cap G$, then $[g,h] \in \zeta_{n-1}(G)$ i.e. $g \in \zeta_n(G)$.

Conversely, let $g \in \zeta_n(G) = t(\alpha_1,\beta_1)^{n_1}.....t(\alpha_k,\beta_k)^{n_k}$ in its canonical form where $\text{alt}\alpha_i + \text{dep}\beta_i < n$.

Let $v = \sum_{i=1}^{r} n_i e_{\gamma_i}$ where the γ_i are distinct.

Then $g(v) = v + \sum_{i=1}^{p} n_j n_{j'} e_{\alpha_j}$ where $\text{alt}\alpha_j < n$ and $n_{j'}$ is a sum of n_i's, and $p \geq 0$.

So $[(1,v),(g,0)] = g(v) - v \in \text{gp}\{ e_\alpha : \text{alt}\alpha < n\}$

$= \zeta_n(H) \cap V$ by lemma 5.3.

If $g \in \zeta_n(G)$, then $g \in \zeta_{n+1}(H) \cap G$.

Hence $\zeta_{n+1}(H) \cap G = \zeta_n(G)$.

Corollary $\zeta_\omega(H) \cap G = \zeta_\omega(G)$.

Proof $\zeta_\omega(H) \cap G = (\bigcup_{n<\omega} \zeta_n(H)) \cap G$

$$= \bigcup_{n<\omega} (\zeta_n(H) \cap G)$$

$$= \bigcup_{n<\omega} \zeta_{n-1}(G)$$

$$= \zeta_\omega(G).$$

Lemma 5.7 If λ ($\geq\omega$) is a limit ordinal and $n \geq 0$, then

(1) $\zeta_\lambda(H) \cap G = \zeta_\lambda(G)$

(2) $\zeta_{\lambda+n+1}(H) \cap G = gp\{ \zeta_{\lambda+n}(G) \cup \{t(\alpha,\beta) : alt\alpha \oplus dep\beta = \lambda+n$ and $alt\alpha < \lambda\} \}$.

Proof By induction on $\lambda+n$.

The induction starts by lemma 5.6 and its corollary.

Let λ be a limit ordinal and suppose that the theorem is true for all $\mu < \lambda$. Then

$$\zeta_\lambda(H) \cap G = (\bigcup_{\mu<\lambda} \zeta_\mu(H)) \cap G$$

$$= \bigcup_{\mu<\lambda} (\zeta_\mu(H) \cap G)$$

$$= \bigcup (\zeta_{\nu+n}(G) \cup gp\{t(\alpha,\beta) : alt\alpha \oplus dep\beta = \nu + n \text{ and } alt\alpha < \nu\})$$

where $n < \omega$ and $\nu \leq \lambda$, ν limit ordinal

$$= \bigcup_{\mu<\lambda} \zeta_\mu(G)$$

$$= \zeta_\lambda(G).$$

This proves (1).

We now show that (2) is true when $n = 0$.

Let λ be a limit ordinal.

$\zeta_\lambda(H) \cap G = \zeta_\lambda(G) \subset \zeta_{\lambda+1}(H) \cap G \subset \zeta_{\lambda+1}(G)$ by lemma 5.2.

Let $g = t(\alpha_1,\beta_1)^{n_1}\ldots\ldots t(\alpha_k,\beta_k)^{n_k} \in \zeta_{\lambda+1}(G)$ be in canonical form and suppose that $alt\alpha_j = \lambda$ for some $j \leq k$.

Then $[(1,e_{\beta_j}),(g,0)] = n_j e_{\beta_j}$ (+ other terms $n_i e_{\alpha_i}$ perhaps with $\alpha_i \neq \alpha_j$), but $n_j e_{\alpha_j} \notin \zeta_\lambda(H) \cap V$.

So if $g \in \zeta_{\lambda+1}(H) \cap G$, then $\mathrm{alt}\,\alpha_i < \lambda$ for $1 \leqslant i \leqslant k$.

Now suppose that $\mathrm{alt}\,\alpha_i < \lambda$ for $1 \leqslant i \leqslant k$.

Since $g \in \zeta_{\lambda+1}(G)$, $\forall h \in G$, $[g,h] \in \zeta_\lambda(G) = \zeta_\lambda(H) \cap G$.

Let $v \in V = \sum_{i=1}^{n} n_i e_{\gamma_i}$ with the γ_i distinct.

Then $[(1,v),(g,0)] = g(v) - v = 0$ if $\beta_i \neq \gamma_j$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$

or $\sum_{j=1}^{r} k_j e_{\alpha_j'}$ where $\alpha_j' = \alpha_i$ for some i.

So $[(1,v),(g,0)] = 0$ or $\sum_{j=1}^{r} k_j e_{\alpha_j'}$ with $\mathrm{alt}\,\alpha_j' < \lambda$

$$\in \mathrm{gp}\{\, e_\alpha : \mathrm{alt}\,\alpha < \lambda\} = \zeta_\lambda(H) \cap V.$$

Hence $g \in \zeta_{\lambda+1}(H) \cap G$.

So $\zeta_{\lambda+1}(H) \cap G = \mathrm{gp}\{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta \leqslant \lambda$ and $\mathrm{alt}\,\alpha < \lambda\}$

$= \mathrm{gp}\{\, \zeta_\lambda(G) \cup \{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta = \lambda$ and $\mathrm{alt}\,\alpha < \lambda\}\}$.

So (2) is true when $n = 0$.

Now let $n > 0$ and suppose that (2) is true for $m < n$.

Then $\zeta_{\lambda+n}(H) \cap G = \mathrm{gp}\{\, \zeta_{\lambda+n-1}(G) \cup \{t(\alpha,\beta) : \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta = \lambda+n-1$

and $\mathrm{alt}\,\alpha < \lambda\}\}$.

By lemma 5.2, $\zeta_{\lambda+n+1}(H) \cap G \subset \zeta_{\lambda+n+1}(G)$.

First, we show that $\zeta_{\lambda+n}(G) \subset \zeta_{\lambda+n+1}(H) \cap G$.

Let $g = t(\alpha_1,\beta_1)^{n_1} \ldots\ldots t(\alpha_k,\beta_k)^{n_k} \in \zeta_{\lambda+n}(G)$ be in canonical form.

Then if h belongs to G, $[g,h] \in \zeta_{\lambda+n-1}(G) \subset \zeta_{\lambda+n}(H) \cap G$.

Let $v = \sum_{j=1}^{n} n_j e_{\gamma_j} \in V$ where the γ_j are distinct.

Then $[(1,v),(g,0)] = g(v) - v = 0$ or is of the form

$\sum_{j=1}^{r} k_j e_{\alpha j'}$ where each $\alpha j' = \alpha i$ for some i i.e. $\text{alt}\alpha j' < \lambda + n$.

So $[(1,v),(g,0)] \in gp\{ e_\alpha : \text{alt}\alpha < \lambda + n\} = \zeta_{\lambda+n}(H) \cap V$.

So $\zeta_{\lambda+n}(G) \subset \zeta_{\lambda+n+1}(H) \cap G$.

Now let $g = t(\gamma_1, \delta_1)^{m_1} \ldots t(\gamma_k, \delta_k)^{m_k} \in \zeta_{\lambda+n+1}(G)$ be in canonical form and suppose that $g \notin \zeta_{\lambda+n}(G)$.

W.l.o.g. suppose that $\text{alt}\gamma_i \oplus \text{dep}\delta_i = \lambda + n$, $1 \leq i \leq k$.

If there is a pair (γ_j, δ_j) with $\text{alt}\gamma_j = \lambda + n$, then

$[(1,e_{\delta_j}),(g,0)] = n_j e_{\gamma_j}$ (+ perhaps some terms $m_i e_{\gamma_i}$ with $\gamma_i \neq \gamma_j$)

but $m_j e_{\gamma_j} \notin \zeta_{\lambda+n}(H) \cap V$.

So $\text{alt}\gamma_i < \lambda + n$, $1 \leq i \leq k$.

If $\text{alt}\gamma_i \geq \lambda$ for some i, then $\text{dep}\delta_i$ is finite and > 0 and so by lemma 2.9, there is a $\theta > \delta_i$ with $\text{dep}\theta = \text{dep}\delta_i - 1$.

Suppose that γ_i is a maximal element associated with δ_i in g (if not, γ_i' which is maximal can be chosen such that $\text{alt}\gamma_i' \geq \lambda$).

Then, by lemma 3.1, $[g,t(\delta_i,\theta)]$ contains a term $t(\gamma_i,\theta)^{m_i}$ in its canonical form which cannot be cancelled and is not the identity.

Since $\text{alt}\gamma_i \oplus \text{dep}\theta = \lambda + n - 1$ and $\text{alt}\gamma_i \geq \lambda$,

$[g,t(\delta_i,\theta)] \notin \zeta_{\lambda+n}(H) \cap G$.

So $\text{alt}\gamma_i < \lambda$, $1 \leq i \leq k$.

Let $v = \sum_{i=1}^{r} n_i e_{\alpha i} \in V$ where the αi are distinct.

$[(1,v),(g,0)] = g(v) - v = 0$ if $\alpha i \neq \delta_j$, $1 \leq i \leq r$, $1 \leq j \leq k$,

or is of the form $\sum_{j=1}^{n} k_j e_{\gamma j'}$ where $\gamma j' = \gamma i$ for some i, i.e. $\text{alt}\gamma j' < \lambda$.

So $[(1,v),(g,0)] \in \zeta_{\lambda+n}(H) \cap V$.

Let $h \in G$ be in canonical form, and let

$[g,h] = t(\alpha_1,\beta_1)^{n_1} \ldots\ldots t(\alpha_k,\beta_k)^{n_k}$ in its canonical form.

Since $g \in \zeta_{\lambda+n+1}(G)$, $[g,h] \in \zeta_{\lambda+n}(G)$ i.e. $\mathrm{alt}\,\alpha_i \oplus \mathrm{dep}\,\beta_i < \lambda + n$.

By lemma 5.5, $\mathrm{alt}\,\alpha_i < \lambda$, $1 \leqslant i \leqslant k$.

So $[g,h] \in \zeta_{\lambda+n}(H) \cap G$.

Hence $\zeta_{\lambda+n+1}(H) \cap G = \mathrm{gp}\{\ \zeta_{\lambda+n}(G) \cup \{t(\alpha,\beta)\ :\ \mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta = \lambda + n$

$\text{and } \mathrm{alt}\,\alpha < \lambda\}\ \}$.

Hence result.

<u>Theorem 5.8</u> If λ $(\geqslant\omega)$ is a limit ordinal and $0 < n < \omega$, then

$$\zeta_\lambda(H) = \mathrm{gp}\{\ e_\alpha\ :\ \mathrm{alt}\,\alpha < \lambda\}.\zeta_\lambda(G)$$

$$\zeta_n(H) = \mathrm{gp}\{\ e_\alpha\ :\ \mathrm{alt}\,\alpha < n\}.\zeta_{n-1}(G)$$

$$\zeta_{\lambda+n}(H) = \mathrm{gp}\{\ e_\alpha\ :\ \mathrm{alt}\,\alpha < \lambda + n\}.\mathrm{gp}\{\ \zeta_{\lambda+n-1}(G) \cup \{t(\alpha,\beta)\ :$$

$$\mathrm{alt}\,\alpha \oplus \mathrm{dep}\,\beta = \lambda + n - 1 \text{ and } \mathrm{alt}\,\alpha < \lambda\}\ \}.$$

<u>Proof</u> Follows from lemmas 5.3, 5.6 and 5.7 and theorem 5.4.

<u>Corollary</u> If the central height of G is a limit ordinal or 0, then the central height of H $\geqslant$ central height of G.

Otherwise, the central height of H $>$ central height of G.

We now consider the lower central series of H and obtain a partial characterisation of it.

<u>Lemma 5.9</u> $\gamma_\lambda(H) = (\ \gamma_\lambda(H) \cap V).\gamma_\lambda(G)$ for every ordinal λ.

<u>Proof</u> By induction.

The lemma is true if $\lambda = 1$.

Let λ be a limit ordinal.

$$\gamma_\lambda(H) = \bigcap_{\mu<\lambda} \gamma_\mu(H)$$

$$= \bigcap_{\mu<\lambda} (\gamma_\mu(H) \cap V).\gamma_\mu(G)$$

$$= \bigcap_{\mu<\lambda} (\gamma_\mu(H) \cap V). \bigcap_{\mu<\lambda} \gamma_\mu(G)$$

$$= (\gamma_\lambda(H) \cap V).\gamma_\lambda(G).$$

Suppose that $\gamma_\mu(H) = (\gamma_\mu(H) \cap V).\gamma_\mu(G)$.

Denote $\gamma_\mu(H) \cap V$ by V_μ and $\gamma_\mu(G)$ by G_μ.

It is clear that $\gamma_{\mu+1}(H) \cap V.\gamma_{\mu+1}(G) \subset \gamma_{\mu+1}(H)$.

$\gamma_{\mu+1}(H) = [\gamma_\mu(H),H] = [V_\mu.G_\mu,V.G]$.

Choose a set of generators of $\gamma_{\mu+1}(H)$ of the form $[v_\mu g_\mu, vg]$

with $v_\mu \in V_\mu$, $g_\mu \in G_\mu$, $v \in V$ and $g \in G$. **Let** ab be one of these generators. Then

$$ab = [v_\mu g_\mu, vg]$$

$$= [v_\mu, vg]^{g_\mu}[g_\mu, vg]$$

$$= [v_\mu, g]^{g_\mu}[v_\mu, v]^{gg_\mu}[g_\mu, g][g_\mu, v]^{g}.$$

V_μ, V are both normal in H and G_μ is normal in G.

$$ab = [v_\mu^{g_\mu}, g^{g_\mu}][v_\mu, v]^{gg_\mu}[g_\mu, g][g_\mu^{g}, v^{g}]$$

$v_\mu^{g_\mu} \in V_\mu$, $v^g \in V$, $g_\mu^{g} \in G$ since V_μ and V are normal in H

and G_μ is normal in G.

Hence $[v_\mu^{g_\mu}, g^{g_\mu}] \in V$ since V is normal in H

$[v_\mu, v]^{gg_\mu} \in V$ since V is normal in H

$[g_\mu^{g}, v^{g}] \in V$ since V is normal in H.

and $[g_\mu, g] \in \gamma_{\mu+1}(G)$.

Since $v_\mu, g_\mu \in \gamma_\mu(H)$, $[v_\mu^{g_\mu}, g^{g_\mu}]$, $[v_\mu, v]^{gg_\mu}$, $[g_\mu^{g}, v^{g}] \in \gamma_{\mu+1}(H)$.

Hence $ab \in (\gamma_{\mu+1}(H) \cap V).\gamma_{\mu+1}(G)$.

Hence result.

<u>Lemma 5.10</u> If $\text{dep}\,\alpha = \infty$, then $e_\alpha \in \gamma_\lambda(H) \cap V \ \forall \lambda$.

<u>Proof</u> By induction.

$e_\alpha \in \gamma_1(H) \cap V = V$.

Let λ be a limit ordinal.

If $e_\alpha \in \gamma_\mu(H) \cap V \ \forall \mu < \lambda$, then $e_\alpha \in \bigcap_{\mu<\lambda} \gamma_\mu(H) \cap V = \gamma_\lambda(H) \cap V$.

Suppose that if $\text{dep}\,\sigma = \infty$, then $e_\sigma \in \gamma_\mu(H) \cap V$.

If $\text{dep}\,\alpha = \infty$, then by lemma 2.8, $\exists \beta > \alpha$ with $\text{dep}\,\beta = \infty$ and so

$e_\beta \in \gamma_\mu(H) \cap V$.

$e_\alpha = [e_\beta, t(\alpha,\beta)] \in [\gamma_\mu(H), H] = \gamma_{\mu+1}(H)$.

Hence $e_\alpha \in \gamma_{\mu+1}(H) \cap V$.

<u>Lemma 5.11</u> If $\text{dep}\,\alpha \geq \mu$, then $t(\rho,\sigma)(e_\alpha) \in \text{gp}\{ e_\beta : \text{dep}\,\beta \geq \mu\}$.

<u>Proof</u> $t(\rho,\sigma)(e_\alpha) = e_\alpha$ if $\sigma \neq \alpha$ and then $\text{dep}\,\alpha \geq \mu$

$= e_\alpha + e_\rho$ if $\sigma = \alpha$.

Since $\rho < \sigma = \alpha$, $\text{dep}\,\rho > \text{dep}\,\alpha \geq \mu$ or $\text{dep}\,\rho = \text{dep}\,\alpha = \infty$.

<u>Corollary</u> If $\text{dep}\,\alpha \geq \mu$, then $[e_\alpha, t(\rho,\sigma)] = t(\rho,\sigma)(e_\alpha) - e_\alpha$

$\in \text{gp}\{ e_\beta : \text{dep}\,\beta \geq \mu + 1\}$.

<u>Proof</u> If $t(\rho,\sigma)(e_\alpha) = e_\alpha$, then $t(\rho,\sigma)(e_\alpha) - e_\alpha = 0$.

If $t(\rho,\sigma)(e_\alpha) = e_\alpha + e_\rho$, then $[e_\alpha, t(\rho,\sigma)] = e_\rho$.

Since $\rho < \sigma = \alpha$, $\text{dep}\rho > \text{dep}\alpha \geqslant \mu$ or $\text{dep}\rho = \text{dep}\alpha = \infty$ i.e. $\text{dep}\rho \geqslant \mu$.

<u>Lemma 5.12</u> If $v = \sum_{i=1}^{n} a_i e_{\alpha_i}$ with $a_i \in F$ and $\text{dep}\alpha_i \geqslant \mu$, $1 \leqslant i \leqslant n$,

then $\forall g \in G$, $g(v) = \sum_{i=1}^{k} b_i e_{\beta_i}$ with $b_i \in F$ and $\text{dep}\beta_i \geqslant \mu$, $1 \leqslant i \leqslant k$.

<u>Proof</u> Follows from lemma 5.11.

<u>Corollary 5.13</u> If $v = \sum_{i=1}^{n} a_i e_{\alpha_i}$, $a_i \in F$ and $\text{dep}\alpha_i \geqslant \mu$, $1 \leqslant i \leqslant n$,

then $\forall g \in G$, $[v,g] = g(v) - v = \sum_{i=1}^{n} c_i e_{\gamma_i}$ with $c_i \in F$ and

$\text{dep}\gamma_i \geqslant \mu + 1$, $1 \leqslant i \leqslant n$.

<u>Proof</u> Follows from the corollary to lemma 5.11.

<u>Lemma 5.14</u> Let $x = [wh, vg]$ with $w, v \in V$, $g \in G$ and $h \in \gamma_{n+1}(G)$.
If $w \in gp\{ e_\alpha : \text{dep}\alpha \geqslant n\}$, then $x \in gp\{ e_\alpha : \text{dep}\alpha \geqslant n+1\}.\gamma_{n+2}(G)$.

<u>Proof</u> $x = [wh, vg]$

$$= [w,g]^h [w,v]^{gh} [h,g][h,v]^g.$$

Since V is abelian, $[w,v] = 0$ and so $[w,v]^{gh} = 0$.

Hence $x = [w^h, g^h][h,g][h,v]^g$

$$= [w^h, g^h][h,g][h,v][h,v,g].$$

Since $w \in gp\{ e_\alpha : \text{dep}\alpha \geqslant n\}$, by lemma 5.12,

$w^h \in gp\{ e_\alpha : \text{dep}\alpha \geqslant n\}$.

By corollary 5.13, $[w^h, g^h] \in gp\{ e_\alpha : \text{dep}\alpha \geqslant n + 1\}$.

Since $h \in \gamma_{n+1}(G)$, $[h,g] \in \gamma_{n+2}(G)$.

h can be expressed in the form $t(\alpha_1, \beta_1)^{n_1} \ldots\ldots\ldots t(\alpha_k, \beta_k)^{n_k}$

with $\operatorname{sep}(\alpha_i, \beta_i) \geq n + 1$.

Hence, by the corollary to lemma 2.37, $\operatorname{dep}\alpha_i \geq n + 1$, $1 \leq i \leq k$.

Let $v = \sum_{i=1}^{n} d_i e_{\delta_i}$ with $d_i \in F$.

$[h,v] = [v,h]^{-1} = \sum_{j=1}^{r} f_j e_{\sigma_j}$ with $\operatorname{dep}\sigma_j \geq n + 1$, since

$h(e_{\delta_j}) = e_{\delta_j}$ or $e_{\delta_j} + \sum_{i=1}^{q} n_i e_{\alpha_i}$ with $\operatorname{dep}\alpha_i \geq n + 1$.

So $[h,v] \in \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq n + 1\}$.

Finally, $[h,v,g] = [[h,v],g] \in \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq n + 1\}$ by corollary 5.13.

Hence result.

<u>Lemma 5.15</u> If $n < \omega$, then $\gamma_{n+1}(H) \cap V = \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq n\}$.

<u>Proof</u> The lemma is true if $n = 0$.

Suppose that $\gamma_k(H) \cap V = \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq k - 1\}$.

Let $x \in \gamma_{k+1}(H) = (\gamma_{k+1}(H) \cap V).\gamma_{k+1}(G)$ by lemma 5.9.

Then x can be expressed in the form

$x = [w_1h_1, v_1g_1]^{n_1} \ldots [w_kh_k, v_kg_k]^{n_k}$ with $w_i, v_i \in V$, $g_i \in G$, $h_i \in \gamma_k(G)$ and $w_i \in \gamma_k(H) \cap V = \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq k - 1\}$ $\forall i$.

By lemma 5.14, $[w_ih_i, v_ig_i] \in \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq k\}.\gamma_{k+1}(G)$.

So $x \in \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq k\}.\gamma_{k+1}(G)$.

So $\gamma_{k+1}(H) \cap V \subset \operatorname{gp}\{ e_\alpha : \operatorname{dep}\alpha \geq k\}$.

Let $e_\sigma \in \{ e_\alpha : \operatorname{dep}\alpha \geq k\}$.

By lemma 2.39, $\exists \rho > \sigma$ with $\operatorname{sep}(\sigma, \rho) = k$.

Then $t(\sigma, \rho) \in \gamma_k(G) = \gamma_k(H) \cap G$.

So $e_\sigma = [e_\rho, t(\sigma,\rho)] \in [H, \gamma_K(H)] = \gamma_{K+1}(H)$.

So $gp\{ e_\alpha : dep\alpha \geq k\} \subset \gamma_{K+1}(H) \cap V$.

Hence result.

<u>Corollary 5.16</u> $\gamma_\omega(H) \cap V = gp\{ e_\alpha : dep\alpha \geq \omega\}$.

<u>Proof</u> $\gamma_\omega(H) \cap V = (\bigcap_{n<\omega} \gamma_n(H)) \cap V$

$$= \bigcap_{n<\omega} \gamma_n(H) \cap V$$

$$= \bigcap_{n<\omega} gp\{ e_\alpha : dep\alpha \geq n - 1\}$$

$$= gp\{ e_\alpha : dep\alpha \geq \omega\}.$$

<u>Theorem 5.17</u> If $n < \omega$, then $\gamma_{n+1}(H) = gp\{ e_\alpha : dep\alpha \geq n\}.\gamma_{n+1}(G)$.

$$\gamma_\omega(H) = gp\{ e_\alpha : dep\alpha \geq \omega\}.\gamma_\omega(G).$$

<u>Proof</u> Follows from lemmas 5.9 and 5.15 and corollary 5.16.

<u>Lemma 5.18</u> Let $\mu \geq \omega$. Then

$$gp\{ e_\alpha : dep\alpha \geq \mu\} \subset \gamma_\mu(H) \cap V.$$

<u>Proof</u> By induction on μ.

The induction starts by theorem 5.17.

Let λ be a limit ordinal and let $dep\alpha \geq \lambda$.

Then $\forall\mu < \lambda$, $dep\alpha > \mu$ and $e_\alpha \in gp\{ e_\beta : dep\beta \geq \mu\}$.

$\forall\mu < \lambda$, $e_\alpha \in \gamma_\mu(H) \cap V$ and so

$$e_\alpha \in \bigcap_{\mu<\lambda} \gamma_\mu(H) \cap V = \gamma_\lambda(H) \cap V.$$

Suppose that the result is true whenever $\mu \leq \lambda$.

Let $dep\alpha \geq \lambda + 1$.

Then there is a $\beta > \alpha$ with $dep\beta \geq \lambda$ and $e_\beta \in \gamma_\lambda(H) \cap V$.

$e_\alpha = [e_\beta, t(\alpha,\beta)] \in [\gamma_\lambda(H), H] = \gamma_{\lambda+1}(H).$

The following results, which are extensions of lemmas 5.11, 5.12 and 5.14, are needed to extend the results about the lower central series of H.

<u>Lemma 5.19</u> Let $\mu \geqslant \omega$. If $e_\alpha \in \{ e_\beta$: dep$\beta \geqslant \omega$ and there is a $\gamma > \beta$ with sep$(\beta,\gamma) \geqslant \mu\} = V_\mu$, then $\forall \rho, \sigma \in \Omega$ with $\rho < \sigma$,

$t(\rho,\sigma)(e_\alpha) \in$ gp$\{ e_\beta$: dep$\beta \geqslant \omega$ and there is a $\gamma > \beta$, sep$(\beta,\gamma) \geqslant \mu\}$.

i.e. V_μ is normal in H.

<u>Proof</u> $t(\rho,\sigma)(e_\alpha) = e_\alpha$ if $\sigma \neq \alpha$ and then the result is obvious.

$= e_\alpha + e_\rho$ if $\sigma = \alpha$ and then dep$\rho >$ dep$\alpha \geqslant \omega$, or depρ = dep$\alpha = \infty$.

Since $\rho < \alpha$ and there is a γ with sep$(\alpha,\gamma) \geqslant \mu$, sep$(\rho,\gamma) \geqslant \mu$.

Hence result.

<u>Corollary</u> Let $\mu \geqslant \omega$. If $e_\alpha \in \{ e_\beta$: dep$\beta \geqslant \omega$ and there is a $\gamma > \beta$ with sep$(\beta,\gamma) \geqslant \mu\}$, then $\forall \rho, \sigma \in \Omega$ with $\rho < \sigma$,

$[e_\alpha, t(\rho,\sigma)] \in$ gp$\{ e_\beta$: dep$\beta \geqslant \omega$ and there is a $\gamma > \beta$, sep$(\beta,\gamma) \geqslant \mu+1\}$.

<u>Proof</u> $[e_\alpha, t(\rho,\sigma)] = t(\rho,\sigma)(e_\alpha) - e_\alpha$.

If $t(\rho,\sigma)(e_\alpha) = e_\alpha$, then $[e_\alpha, t(\rho,\sigma)] = 0$.

If $t(\rho,\sigma)(e_\alpha) = e_\alpha + e_\rho$, then $[e_\alpha, t(\rho,\sigma)] = e_\rho$.

Since $\rho < \alpha$, dep$\rho \geqslant \omega$.

There is a $\gamma > \alpha$ with sep$(\alpha,\gamma) \geqslant \mu$ and so sep$(\rho,\gamma) \geqslant \mu + 1$.

<u>Lemma 5.20</u> Let $\mu \geqslant \omega$ and let $v = \sum_{i=1}^{n} a_i e_{\alpha_i}$ with $a_i \in F$ and

$e_{\alpha i} \in \{ e_\beta$: dep$\beta \geq \omega$ and there is a $\gamma > \beta$ with sep$(\beta,\gamma) \geq \mu\} = V_\mu$.

Then $\forall g \in G$, $g(v) = \sum_{i=1}^{k} b_i e_{\beta i}$ with $b_i \in F$ and

$e_{\beta i} \in gp\{ e_\beta$: dep$\beta \geq \omega$ and there is a $\gamma > \beta$ with sep$(\beta,\gamma) \geq \mu\}$.

i.e. V_μ is normal in H.

Proof Follows from lemma 5.19

Corollary 5.21 Let $\mu \geq \omega$ and let $v = \sum_{i=1}^{n} a_i e_{\alpha i}$ with $a_i \in F$ and

$e_{\alpha i} \in \{ e_\beta$: dep$\beta \geq \omega$ and there is a $\gamma > \beta$ with sep$(\beta,\gamma) \geq \mu\}$.

Then $\forall g \in G$, $[v,g] = \sum_{i=1}^{n} c_i e_{\gamma i}$ with $c_i \in F$ and

$e_{\gamma i} \in \{ e_\beta$: dep$\beta \geq \omega$ and there is a $\gamma > \beta$ with sep$(\beta,\gamma) \geq \mu + 1\}$.

Proof Follows from the corollary to lemma 5.19 and lemma 5.19.

Lemma 5.22 Let $\mu \geq \omega$ and let $x = [wh, vg]$ with $w,v \in V$, $g \in G$, and $h \in \gamma_{\mu+1}(G)$. If $w \in V_\mu$, where V_μ is the subgroup defined in lemma 5.19, then $x \in V_{\mu+1}.\gamma_{\mu+2}(G)$.

Proof $x = [wh,vg] = [w,g]^h[w,v]^{gh}[h,g][h,v]^g$.

Since V is abelian, $[w,v] = 0$ and so $[w,v]^{gh} = 0$.

So $x = [w^h,g^h][h,g][h,v]^g$

$= [w^h,g^h][h,g][h,v][h,v,g]$.

Since $w \in V_\mu$, by lemma 5.19, $w^h \in V_\mu$.

By corollary 5.21, $[w^h,g^h] \in V_{\mu+1}$.

Since $h \in \gamma_{\mu+1}(G)$, $[h,g] \in \gamma_{\mu+2}(G)$.

h can be expressed in the form $t(\alpha_1,\beta_1)^{n_1}\ldots\ldots t(\alpha_k,\beta_k)^{n_k}$ with $sep(\alpha_i,\beta_i) \geq \mu + 1$. By the corollary to lemma 2.37, $dep\alpha_i \geq \omega$.

Let $v = \sum_{i=1}^{n} d_i e_{\delta_i}$.

$h(e_{\delta_j}) = e_{\delta_j}$ if $\delta_j \neq \beta_i$, $1 \leq i \leq k$ and $1 \leq j \leq n$,

or $e_{\delta_j} + \sum_{i=1}^{s} n_i e_{\alpha_i}$ where $\alpha_i = \alpha_m$ for some $m \in \{1,\ldots,k\}$.

$Dep\alpha_i \geq \omega$ and so $\sum_{i=1}^{s} n_i e_{\alpha_i} \in V_{\mu+1}$.

So $[e_{\delta_j},h] \in V_{\mu+1}$ and so $[h,v] \in V_{\mu+1}$.

Finally, since $[h,v] \in V_{\mu+1}$, by corollary 5.21, $[h,v,g] \in V_{\mu+1}$.

Hence result.

Lemma 5.23 Let $\omega 2 > \mu \geq \omega$. Then $\gamma_{\mu+1}(H) \cap V = V_\mu$ where

$V_\mu = gp\{ e_\alpha : dep\alpha \geq \mu + 1$ or $dep\alpha \geq \omega$ and there is a $\beta > \alpha$ with $sep(\alpha,\beta) \geq \mu\}$.

Proof By induction.

Let $x \in \gamma_{\omega+1}(H) = (\gamma_{\omega+1}(H) \cap V).\gamma_{\omega+1}(G)$ by lemma 5.9.

Then x can be expressed in the form $[w_1h_1,v_1g_1]^{n_1}\ldots[w_kh_k,v_kg_k]^{n_k}$ with $v_i \in V$, $g_i \in G$, $h_i \in \gamma_\omega(G)$ and $w_i \in \gamma_\omega(H) \cap V$.

$[w_ih_i,v_ig_i] = [w_i^{h_i},g_i^{h_i}][h_i,g_i][h_i,v_i][h_i,v_i,g_i]$ since $[w_i,v_i] = 0$.

$w_i \in gp\{ e_\alpha : dep\alpha \geq \omega \}$ and so by lemma 5.12, $w_i^{h_i} \in gp\{e_\alpha : dep\alpha \geq \omega\}$.

By corollary 5.13, $[w_i^{h_i},g_i^{h_i}] \in gp\{ e_\alpha : dep\alpha \geq \omega + 1\}$.

Since $h_i \in \gamma_\omega(G)$, $[h_i,g_i] \in \gamma_{\omega+1}(G)$.

h_i can be expressed in the form $t(\alpha_{i1},\beta_{i1})^{n_{i1}}\ldots t(\alpha_{ir},\beta_{ir})^{n_{ir}}$ with $sep(\alpha_{ij},\beta_{ij}) \geq \omega$. Since $sep(\alpha_{ij},\beta_{ij}) \geq \omega$, $dep\alpha_{ij} \geq \omega$.

For if $\text{dep}\,\alpha_{ij} = n$ $(<\omega)$, then by lemma 2.38, $\text{sep}(\alpha_{ij},\beta_{ij}) \leq n$.

Let $v = \sum_{j=1}^{n} d_{ij} e_{\delta_{ij}}$.

By an argument similar to that used in the proof of lemma 5.22, $[h_i, v_i] \in V_\omega$.

By corollary 5.21, $[h_i, v_i, g_i] \in V_\omega$.

So $\gamma_{\omega+1}(H) \cap V \subset V_\omega$.

Let $\text{dep}\,\alpha \geq \omega + 1$.

Then there is a $\delta > \alpha$ with $\text{dep}\,\delta \geq \omega$ and so $e_\delta \in \gamma_\omega(H) \cap V$.

$e_\alpha = [e_\delta, t(\alpha,\delta)] \in [\gamma_\omega(H), H] = \gamma_{\omega+1}(H)$.

Let $\text{dep}\,\alpha \geq \omega$ and let $\text{sep}(\alpha,\beta) \geq \omega$. $t(\alpha,\beta) \in \gamma_\omega(G) = \gamma_\omega(H) \cap G$.

Then $e_\alpha = [e_\beta, t(\alpha,\beta)] \in [H, \gamma_\omega(H)] = \gamma_{\omega+1}(H)$.

So the lemma is true when $\mu = \omega$.

Now suppose that $\gamma_{\mu+1}(H) \cap V$ is of the form stated in the lemma. Let $x \in \gamma_{\mu+2}(H) = (\gamma_{\mu+2}(H) \cap V).\gamma_{\mu+2}(G)$ by lemma 5.9.

Then x can be expressed in the form $[w_1h_1, v_1g_1]^{n_1} \ldots [w_kh_k, v_kg_k]^{n_k}$ with $v_i \in V$, $g_i \in G$, $h_i \in \gamma_{\mu+1}(G)$ and $w_i \in \gamma_{\mu+1}(H) \cap V$.

By lemmas 5.13 and 5.22, $x \in V_{\mu+1}.\gamma_{\mu+2}(G)$.

So $\gamma_{\mu+2}(H) \cap V \subset V_{\mu+1}$.

Let $\text{dep}\,\alpha \geq \mu + 2$. Then there is a $\delta > \alpha$ with $\text{dep}\,\delta \geq \mu + 1$ and so $e_\delta \in \gamma_{\mu+1}(H) \cap V$.

$e_\alpha = [e_\delta, t(\alpha,\delta)] \in \gamma_{\mu+2}(H)$.

Let $\text{dep}\,\alpha \geq \omega$ and let $\text{sep}(\alpha,\beta) \geq \mu + 1$. $t(\alpha,\beta) \in \gamma_{\mu+1}(G)$.

Then $e_\alpha = [e_\beta, t(\alpha,\beta)] \in \gamma_{\mu+2}(H)$.

Hence result.

Corollary 5.24 $\gamma_{\omega 2}(H) \cap V = gp\{ e_\alpha : dep\alpha \geq \omega 2$ or $dep\alpha \geq \omega$ and $\forall \mu < \omega 2$, there is a $\beta_\mu > \alpha$ with $sep(\alpha, \beta_\mu) = \mu\}$.

Finally, we consider the derived series of H.

Lemma 5.25 $\delta_\lambda(H) = (\delta_\lambda(H) \cap V).\delta_\lambda(G)$.

Proof By induction.

The lemma is true if $\lambda = 0$.

Let λ be a limit ordinal.

$$\delta_\lambda(H) = \bigcap_{\mu<\lambda} \delta_\mu(H)$$

$$= \bigcap_{\mu<\lambda} (\delta_\mu(H) \cap V).\delta_\mu(G)$$

$$= \bigcap_{\mu<\lambda} (\delta_\mu(H) \cap V). \bigcap_{\mu<\lambda} \delta_\mu(G)$$

$$= (\delta_\lambda(H) \cap V).\delta_\lambda(G).$$

Suppose that $\delta_\mu(H) = (\delta_\mu(H) \cap V).\delta_\mu(G)$

Call $\delta_\mu(H) \cap V$ V_μ and $\delta_\mu(G)$ G_μ.

Clearly $(\delta_{\mu+1}(H) \cap V).\delta_{\mu+1}(G) \subset \delta_{\mu+1}(H)$.

$\delta_{\mu+1}(H) = [\delta_\mu(H), \delta_\mu(H)] = [V_\mu G_\mu, V_\mu G_\mu]$.

Choose a set of generators of $\delta_{\mu+1}(H)$ of the form $[vg,wh]$ where $v,w \in V_\mu$ and $g,h \in G_\mu$. Let x be one of these generators.

Then $x = [vg,wh]$

$$= [v,wh]^g[g,wh]$$

$$= [v,h]^g[v,w]^{hg}[g,h][g,w]^h.$$

V_μ is normal in H and G_μ is normal in G.

$x = [v^g,h^g][v,w]^{hg}[g,h][g^h,w^h]$.

v^g and w^h belong to V_μ and h^g and g^h belong to G_μ.

$[g,h] \in [G_\mu, G_\mu] = \delta_{\mu+1}(G)$.

$[v^h, h^g]$, $[v,w]^{hg}$ and $[g^h, w^h]$ belong to $[\delta_\mu(H), \delta_\mu(H)] = \delta_{\mu+1}(H)$.

Also, $[v^h, h^g]$, $[v,w]^{hg}$ and $[g^h, w^h]$ belong to V as V is a normal subgroup of H.

So $x \in (\delta_{\mu+1}(H) \cap V).\delta_{\mu+1}(G)$.

Hence result.

<u>Theorem 5.26</u> $\delta_n(H) = \gamma_{2^n}(H)$.

<u>Proof</u> $\delta_n(H) = (\delta_n(H) \cap V).\delta_n(G)$ by lemma 5.25

$= (\delta_n(H) \cap V).\gamma_{2^n}(G)$ by theorem 4.8.

By theorem 3.4(iii) in (3),

$\delta_n(H) \subset \gamma_{2^n}(H) = (\gamma_{2^n}(H) \cap V).\gamma_{2^n}(G)$.

So it is only necessary to show that

$\gamma_{2^n}(H) \cap V \subset \delta_n(H) \cap V$.

The proof is by induction.

It is true if $n = 0$.

Suppose that $\gamma_{2^n}(H) \cap V \subset \delta_n(H) \cap V$.

By lemma 5.15, $gp\{ e_\alpha : dep\alpha \geq 2^n - 1\} \subset \delta_n(H) \cap V$.

Let $e_\sigma \in \{ e_\alpha : dep\alpha \geq 2^{n+1} - 1\}$.

By corollary 2.41, $\exists \rho$ with $dep\rho \geq 2^n - 1$ and $sep(\sigma, \rho) \geq 2^n$.

Then $t(\sigma, \rho) \in \gamma_{2^n}(G) = \delta_n(G)$ and $e_\rho \in \delta_n(H) \cap V$.

So $e_\sigma = [e_\rho, t(\sigma,\rho)] \in \delta_{n+1}(H) \cap V$.

Hence $\gamma_{2^{n+1}}(H) \cap V \subset \delta_{n+1}(H) \cap V$.

Hence result.

<u>Corollary 5.27</u> $\delta_\omega(H) = \gamma_\omega(H)$.

<u>Proof</u> $\delta_\omega(H) = \bigcap_{n<\omega} \delta_n(H)$

$= \bigcap_{n<\omega} \gamma_{2^n}(H)$

$= \gamma_\omega(H)$ by reasoning similar to that used in the proof of corollary 4.9.

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