

The thesis includes the following published papers:

1. On the dynamics of an Electron.
Phil. Mag. p 977 1928
2. On the electromagnetic field of an electron.
Phil. Mag p.425 1929
3. Hamilton's Principle and the Field equations of radiation.
Phil. Mag p 568. 1930
4. Electromagnetic phenomena in a uniform gravitational field.
Proc. Roy. Soc. E. p.71. 1931
5. Wave equations of an electron in a real form
Phil. Mag. p 834 1932.

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P r e f a c e.

The application of the General Theory of Relativity to electromagnetic phenomena proceeds along several lines.

1. Maxwell's electromagnetic equations and the four laws of conservation are presented in a general covariant form, and it is considered that such equations express the electromagnetic laws in a gravitational field.

2. Einstein's space ($G_{\mu\nu} = 0$) is extended so as to include ~~not~~ only gravitation but also the electromagnetic field.

Attempts have been made to include the electromagnetic field by making use of a non-Riemannian space (Weyl, Biddington), by increasing the number of dimensions of space to five (Kaluza), or by associating certain properties of space with electromagnetic laws (Einstein)

3. With the appearance of the New Quantum Mechanics a series of attempts has been made to present Schrödinger's

and Dirac's equations as an ordinary wave or Maxwell's equations in a five dimensional space.

The present thesis deals with the application of the General Theory of Relativity to problems mentioned under the headings 1 and 3.

The conception of curved space is applied to four, eight and five dimensions.

The following is a concise account of the content of this work.

Chapter 1. The radiation of an electron in a uniform gravitational field is evaluated.

Although the real gravitational field is non uniform, nevertheless the uniform field has certain important mathematical and physical properties.

The fundamental form of such space can be transformed into a Galilean one, and hence a solution of Maxwell's equations is obtained by a mere transformation of coordinates.

This space considerably differs from a Galilean one, and, hence, it is more suitable to test the validity of foundations of the General Theory of Relativity than the ordinary gravitational field, which only slightly deviates from Galilean space.

The evaluation of radiation from a freely moving electron is based on purely relativistic principles, namely on the general covariant form of Poynting's law of conservation of energy, and on the null-geodesic propagation of light; the obtained result is Larmor's value for radiation. This shows that the Principle of Equivalence is not only valid for mechanical but also for electromagnetic phenomena.

Chapter 2. According to Schrödinger motion (with Relativistic) of n electrons is represented in a $3n+1$ dimensional space. The present chapter deals with the application of the General Theory of Relativity

to an eight dimensional space.

It is shown that the motion of two electrons can be represented in an eight dimensional curved space, and the tracks of the two particles are geodesics of such space.

It is further shown that the wave equation of such curved eight dimensional space is Schrödinger's wave equation for two particles.

Chapter 3. A historical account is given of the origin and development of the idea of five dimensional space, and it is then applied to Dirac's equations

These equations are derived from Hamilton's Principle, and it appears that they can be considered as Maxwell's equations in a five dimensional space, or, what comes to the same thing, Maxwell's equations are a four dimensional projection of the wave equations

The wave functions consist of an antisymmetric

tensor of the second rank in a five-dimensional space, consisting of 10 (6+4) components, four of which are complex, and of two scalar functions.

The usual wave equations (Schrödinger's and Dirac's) and their solutions are imaginary. The five-dimensional equations are represented in a real form, and solved by real functions.

These real equations clear up the well-known difficulty connected with the imaginary equations, and pointed out by Dirac, namely that the same equations are valid for positive and negative particles; this is due to the imaginary terms. As our equations are real this difficulty does not arise in our case. The equations and their solutions are different for positive and negative particles.

Chapter I

Radiation of an electron in a uniform gravitational field.

Includes:

Electromagnetic Phenomena in a Uniform
Gravitational field.

Proc. Roy. Soc. E p71 1931

XI.—Electromagnetic Phenomena in a Uniform Gravitational Field.

By D. Meksyn, Ph.D., Mathematical Department, Edinburgh University. Communicated by Professor E. T. Whittaker, F.R.S.

(MS. received March 2, 1931. Read May 4, 1931.)

§ 1. INTRODUCTION AND SUMMARY.

IN two recent papers Professor E. T. Whittaker* has solved the electromagnetic equations for the case of a uniform gravitational field. The fundamental tensor associated with such a field makes the Riemannian tensor vanish, since such a field can be transformed away by a suitable choice of co-ordinates. This property enables us to find the electromagnetic field in a uniform gravitational field without solving Maxwell's equations, but by a mere transformation of co-ordinates.

As is known from Differential Geometry, one surface is applicable to another by bending without stretching, if both have the same Gaussian curvature. Analytically this has the following interpretation: let the squares of the elements of length for the two surfaces be $g_{\mu\nu}dx_\mu dx_\nu$ and $g'_{\mu\nu}dx'_\mu dx'_\nu$, then a transformation of co-ordinates can be found

$$x = f(x') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which will satisfy the following equation:

$$g_{\mu\nu}dx_\mu dx_\nu = g'_{\mu\nu}dx'_\mu dx'_\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

i.e. if the values of x given by (1) are inserted in the left side of (2) we obtain the right side of the same equation.

The condition that the Riemannian tensors of the two surfaces shall be equal is not only necessary, but also sufficient for the existence of such transformations.

Now the Riemannian tensor is equal to zero for Euclidean space, and, as we have pointed out, the same tensor also vanishes for a uniform gravitational field, hence it is possible to deduce a kind of Lorentz's transformations which connect these two spaces; and, if we have solved an electromagnetic problem for Euclidean space, we can obtain a solution of a corresponding problem for a uniform gravitational field by a mere transformation of co-ordinates.

We evaluate in the present paper the vector potential and the electromagnetic field of an electron moving freely in a uniform gravitational field.

* Proc. Roy. Soc., A, 116, p. 720 ; A, 120, p. 1.

It appears that an electron radiates energy at the rate of $\frac{2}{3} \frac{e^2 g^2}{c^3}$, which is Larmor's value.

It may appear at first that the solution of the electromagnetic equations for the uniform gravitational field is of little value since this is not the "natural" gravitational field. This is, however, not the case. The solution of this problem provides us with a good test of the principle of equivalence and the idea of curved space.

The idea of curved space is borne out only for mechanical phenomena and for the case of space slightly differing from Euclidean. The question, therefore, arises whether this principle is applicable only for the case of a "natural" field and for mechanical phenomena, or it is valid for electromagnetic phenomena and for every conceivable gravitational space.

The space of uniform gravitation, considered as a whole, is anything but the "natural space." It is bounded by a plane, say $x = -a$; light emitted from any point at a finite distance from the boundary will never reach the boundary, which is also impenetrable for material bodies.

The electromagnetic equations can be solved rigorously, and in evaluating the radiation we have to carry out an integration with respect to the whole space.

The result of these calculations is Larmor's value for radiation; and, what is important, the rate of radiation is expressed, not through the acceleration of an electron, but through the metrical properties of space.

§ 2. THE TRANSFORMATION OF CO-ORDINATES.

The fundamental form for a uniform gravitational field is *

$$ds^2 = (1 + 2\omega x)dt^2 - \frac{dx^2}{1 + 2\omega x} - dy^2 - dz^2 \left. \vphantom{ds^2} \right\} \omega = \frac{g}{c^2} \quad (3)$$

where g is the gravitational acceleration and $c=1$ is the velocity of light.

Let us find the transformation of this space into a Euclidean one.

We have to solve the equation

$$dt_0^2 - dx_0^2 - dy_0^2 - dz_0^2 = (1 + 2\omega x)dt^2 - \frac{dx^2}{1 + 2\omega x} - dy^2 - dz^2 \quad (4)$$

We assume

$$y_0 = y; z_0 = z \quad (5)$$

and from (4) and (5) we obtain

$$\left. \begin{aligned} d(t_0 - x_0) &= \lambda \left[\sqrt{1 + 2\omega x} dt - \frac{dx}{\sqrt{1 + 2\omega x}} \right] \\ d(t_0 + x_0) &= \frac{1}{\lambda} \left[\sqrt{1 + 2\omega x} dt + \frac{dx}{\sqrt{1 + 2\omega x}} \right] \end{aligned} \right\} \quad (6)$$

* Proc. Roy. Soc., A, 116, p. 722.

The condition that the right side of (6) is a total differential gives us two equations:

$$\left. \begin{aligned} \frac{\partial}{\partial x} \left[\lambda \sqrt{1+2\omega x} \right] &= - \frac{\partial}{\partial t} \left[\frac{\lambda}{\sqrt{1+2\omega x}} \right] \\ \frac{\partial}{\partial x} \left[\frac{\sqrt{1+2\omega x}}{\lambda} \right] &= \frac{\partial}{\partial t} \left[\frac{1}{\lambda \sqrt{1+2\omega x}} \right] \end{aligned} \right\} \dots \dots (7)$$

which are to be satisfied by the same function λ .

We know *a priori* that these equations have a common solution.

Without going into further details we give here the required transformations. They are

$$\left. \begin{aligned} x_0 &= \frac{\sqrt{1+2\omega x}}{\omega} \cosh \omega t - \frac{1}{\omega} \\ t_0 &= \frac{\sqrt{1+2\omega x}}{\omega} \sinh \omega t \end{aligned} \right\} \dots \dots \dots (8)$$

It is easily seen that, if ω tends to zero, $x_0 = x$ and $t_0 = t$; ω is a very small quantity, the second and higher powers of which can be neglected, and, again, to this order of approximation $x_0 = x$ and $t_0 = t$.

In (8) $\omega t = v$ is the velocity of the system (x_0, t_0) ; the latter started from rest and has been in motion relative to (x, t) during the time t .

Differentiating (8) we obtain

$$\left. \begin{aligned} dx_0 &= \frac{\cosh \omega t}{\sqrt{1+2\omega x}} dx + \sqrt{1+2\omega x} \sinh \omega t \cdot dt \\ dt_0 &= \frac{\sinh \omega t}{\sqrt{1+2\omega x}} dx + \sqrt{1+2\omega x} \cosh \omega t \cdot dt \end{aligned} \right\} \dots \dots (9)$$

By a direct substitution of (9) in (4) we can confirm the validity of the transformations (8).

From (9) we find:

$$\left. \begin{aligned} \frac{\partial x_0}{\partial x} &= \frac{\cosh \omega t}{\sqrt{1+2\omega x}} & \frac{\partial x_0}{\partial t} &= \sqrt{1+2\omega x} \cdot \sinh \omega t \\ \frac{\partial t_0}{\partial x} &= \frac{\sinh \omega t}{\sqrt{1+2\omega x}} & \frac{\partial t_0}{\partial t} &= \sqrt{1+2\omega x} \cdot \cosh \omega t \end{aligned} \right\} \dots \dots (10)$$

We can solve (8), express x, t as a function of x_0, t_0 , and find the inverse differential coefficients:

$$\left. \begin{aligned} \frac{\partial x}{\partial x_0} &= \sqrt{1+2\omega x} \cosh \omega t & \frac{\partial x}{\partial t_0} &= - \sqrt{1+2\omega x} \sinh \omega t \\ \frac{\partial t}{\partial x_0} &= \frac{-\sinh \omega t}{\sqrt{1+2\omega x}} & \frac{\partial t}{\partial t_0} &= \frac{\cosh \omega t}{\sqrt{1+2\omega x}} \end{aligned} \right\} \dots \dots (11)$$

§ 3. THE ELECTROMAGNETIC FIELD.

We can now easily find the vector potential and the electromagnetic field of an electron moving freely in a uniform gravitational field, and observed from a system at rest.

Let the electrostatic potential and the charge of an electron at rest in Euclidean space be

$$K_4^0(x^0, y^0, z^0); I_4^0 = \rho^0(x^0, y^0, z^0)$$

where

$$\nabla_{x_0, y_0, z_0}^2 K_4^0 = -\rho^0. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Using the transformations for covariant tensors

$$\left. \begin{aligned} A_\mu &= \frac{\partial x_a^0}{\partial x_\mu} A_a^0 \\ A_{\mu\nu} &= \frac{\partial x_a^0}{\partial x_\mu} \frac{\partial x_\beta^0}{\partial x_\nu} A_{a\beta}^0 \end{aligned} \right\} . \quad . \quad . \quad . \quad . \quad (13)$$

we easily find from (10) and (13) the vector potential, the stream vector, and the electromagnetic field, as measured by an observer at rest, for an electron moving freely from rest in a gravitational field.

They are as follows:

$$\left. \begin{aligned} K_1 &= \frac{\sinh \omega t}{\sqrt{1+2\omega x}} K_4^0(x_0, y_0, z_0) = \frac{\sinh \omega t}{\sqrt{1+2\omega x}} K_4^0 \left(\frac{\sqrt{1+2\omega x}}{\omega} \cosh \omega t - \frac{1}{\omega}, y, z \right) \\ K_4 &= \sqrt{1+2\omega x} \cosh \omega t \cdot K_4^0 = \sqrt{1+2\omega x} \cosh \omega t \cdot K_4^0 \left(\frac{\sqrt{1+2\omega x}}{\omega} \cosh \omega t - \frac{1}{\omega}, y, z \right) \end{aligned} \right\} (14)$$

and

$$I_1 = \frac{\sinh \omega t}{\sqrt{1+2\omega x}} \cdot I_4^0; \quad I_4 = \sqrt{1+2\omega x} \cdot \cosh \omega t \cdot I_4^0 \quad . \quad . \quad . \quad (15)$$

The electromagnetic tensor (covariant) is

$$\left. \begin{aligned} X &= X^0 & Y &= \sqrt{1+2\omega x} (Y^0 \cosh \omega t - \gamma^0 \sinh \omega t) \\ & & Z &= \sqrt{1+2\omega x} (Z^0 \cosh \omega t + \beta^0 \sinh \omega t) \\ \alpha &= \alpha_0 & \beta &= \frac{\beta^0 \cosh \omega t + Z^0 \sinh \omega t}{\sqrt{1+2\omega x}} \\ & & \gamma &= \frac{\gamma^0 \cosh \omega t - Y^0 \sinh \omega t}{\sqrt{1+2\omega x}} \end{aligned} \right\} . \quad (16)$$

For a contravariant electromagnetic tensor the expressions are as follows:

$$\left. \begin{aligned} X = X^0 & \quad Y = \frac{Y^0 \cosh \omega t - \gamma^0 \sinh \omega t}{\sqrt{1 + 2\omega x}} \\ & \quad Z = \frac{Z^0 \cosh \omega t + \beta^0 \sinh \omega t}{\sqrt{1 + 2\omega x}} \\ a = \alpha_0 & \quad \beta = \sqrt{1 + 2\omega x} (\beta^0 \cosh \omega t + Z^0 \sinh \omega t) \\ & \quad \gamma = \sqrt{1 + 2\omega x} (\gamma^0 \cosh \omega t - Z^0 \sinh \omega t) \end{aligned} \right\} \quad (17)$$

where $X^0, Y^0, Z^0, \alpha^0, \beta^0, \gamma^0$ is the electromagnetic tensor in Euclidean space.

For the case of an electron moving freely from rest in a gravitational field $\alpha^0 = \beta^0 = \gamma^0 = 0$.

Of course the expressions (14) and (15) satisfy the equations for the vector potential in the General Theory of Relativity (they are given below).

§ 4. RADIATION.

In the classical electrodynamics the rate of radiation is given by the divergence of Poynting's vector:

$$\int \operatorname{div} P \cdot dx \, dy \, dz \quad \dots \quad (18)$$

Let us calculate this vector in our case.

We make use of the mixed tensor E_{μ}^{ν} . The equation of motion is

$$E_{\mu, \nu}^{\nu} = h_{\mu} \quad \dots \quad (19)$$

where h_{μ} is the electromagnetic force. In our case $h_{\mu} = 0$.

Now

$$E_{\mu, \nu}^{\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x_{\nu}} (E_{\mu}^{\nu} \sqrt{-g}) - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x_{\mu}} E^{\alpha\beta} \quad \dots \quad (20)$$

For the equation of energy $\mu = 4$, and as $g_{\alpha\beta}$ is independent at t , the last term vanishes; also $\sqrt{-g} = 1$, hence

$$E_{4, \nu}^{\nu} = \frac{\partial E_4^{\nu}}{\partial x_{\nu}} = 0 \quad \dots \quad (21)$$

the first three terms in (21) represent the divergence of Poynting's vector.

The tensor E_{μ}^{ν} can be evaluated either from

$$E_{\mu}^{\nu} = -F^{\nu\alpha} F_{\mu\alpha} + \frac{1}{4} g_{\mu}^{\nu} F^{\alpha\beta} F_{\alpha\beta}$$

or by transformation of co-ordinates

$$E_{\mu}^{\nu} = \frac{\partial x_{\alpha}^0}{\partial x_{\nu}} \cdot \frac{\partial x_{\nu}}{\partial x_{\beta}^0} E_{\alpha}^{\beta 0}.$$

The expressions obtained are

$$\left. \begin{aligned} E_4^1 &= -(1 + 2\omega x) \left[\frac{Y_0^2 + Z_0^2 + \beta_0^2 + \gamma_0^2}{2} \sinh 2\omega t + (\beta_0 Z_0 - \gamma_0 Y_0) \cosh 2\omega t \right] \\ E_4^2 &= \sqrt{1 + 2\omega x} [(X_0 Y_0 + \alpha_0 \beta_0) \sinh \omega t - (\gamma_0 X_0 - \alpha_0 Z_0) \cosh \omega t] \\ E_4^3 &= \sqrt{1 + 2\omega x} [(X_0 Z_0 + \alpha_0 \gamma_0) \sinh \omega t - (\alpha_0 Y_0 - \beta_0 X_0) \cosh \omega t] \end{aligned} \right\} \quad (22)$$

In our case $\alpha_0 = \beta_0 = \gamma_0 = 0$ and X_0, Y_0, Z_0 are the expressions for the electrostatic force of an electron at rest.

The equation (18) becomes

$$\int (E_4^1 \cos \xi + E_4^2 \cos \eta + E_4^3 \cos \zeta) dS \quad (23)$$

The evaluation of this integral (the details are given in the next paragraph) leads to the value

$$\frac{2}{3} \frac{e^2 g^2}{c^3} \quad (24)$$

for the rate of radiation due to motion of an electron.

§ 5. EVALUATION OF RADIATION.

In order to evaluate the integral (23) we have to express the time as a function of the space co-ordinates of the field point.

The electromagnetic field emitted by the electron, which is at the origin of co-ordinates, propagates along the null geodesic of the space and reaches the field point.

The null geodesic for a uniform gravitational field is given in Whittaker's paper.* The equations are:

$$\left. \begin{aligned} \frac{(1 + 2\omega x)^{\frac{1}{2}} \cosh (\omega t + \lambda)}{\omega} &= \alpha \\ y + \frac{(1 + 2\omega x)^{\frac{1}{2}} \sin \mu \sinh (\omega t + \lambda)}{\omega} &= \beta \\ z + \frac{(1 + 2\omega x)^{\frac{1}{2}} \cos \mu \sinh (\omega t + \lambda)}{\omega} &= \gamma \end{aligned} \right\} \quad (25)$$

where $\alpha, \beta, \gamma, \lambda, \mu$ are arbitrary constants.

Writing down the condition that the geodesic passes through the points $\bar{x}, \bar{y}, \bar{z}, \bar{t}$ and x, y, z, t we obtain the required equation for our geodesic

$$(1 + 2\omega x)^{\frac{1}{2}} (1 + 2\omega \bar{x})^{\frac{1}{2}} \cosh \omega(t - \bar{t}) = 1 + \omega x + \omega \bar{x} + \frac{\omega^2}{2} \{(y - \bar{y})^2 + (z - \bar{z})^2\} \quad (26)$$

* *Proc. Roy. Soc., A*, 120, p. 6.

The electron is at the point (0, 0, 0, 0), hence (26) becomes

$$(1 + 2\omega x)^{\frac{1}{2}} \cosh \omega t = 1 + \omega x + \frac{\omega^2}{2}(y^2 + z^2) \quad . \quad . \quad . \quad (27)$$

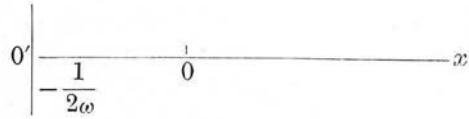
The space of gravitation is

bounded by the plane $x = -\frac{1}{2\omega}$.

We integrate (23) over the space

bounded by the plane $x = -\frac{1}{2\omega}$ and a hemisphere of infinite radius with a

centre at the point $x = -\frac{1}{2\omega}, 0, 0$.



In the second integration the last two terms in (23) are of a lower order of magnitude than the first one, and in the first integration they do not appear at all; we therefore give here only the transformation of the first term.

We have from (22) and (8)

$$E_4^1 = \frac{-(1 + 2\omega x)e^2(y^2 + z^2) \sinh 2\omega t}{2 \left\{ \left[\frac{\sqrt{1 + 2\omega x \cosh \omega t} - 1}{\omega} \right]^2 + y^2 + z^2 \right\}^3} \quad . \quad . \quad . \quad (28)$$

We transfer the origin of co-ordinates to the point $x = -\frac{1}{2\omega}, 0, 0$, and denote

$$x' = x - \frac{1}{2\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

also

$$\sinh 2\omega t = 2 \cosh \omega t \sinh \omega t = 2 \cosh \omega t \sqrt{\cosh^2 \omega t - 1} \quad . \quad . \quad . \quad (30)$$

We find now from (27), (28), (29), and (30)

$$E_4^1 = - \frac{e^2(y^2 + z^2) \left[\frac{1}{2} + \omega x' + \frac{\omega^2}{2}(y^2 + z^2) \right] \sqrt{\left[\frac{1}{2} + \omega x' + \frac{\omega^2}{2}(y^2 + z^2) \right]^2 - 2\omega x'}}{\left\{ \left[x' - \frac{1}{2\omega} + \frac{\omega}{2}(y^2 + z^2) \right]^2 + y^2 + z^2 \right\}^3} \quad (31)$$

(a) We integrate over half a sphere of radius $r \rightarrow \infty$ with a centre at the point 0'. We transform (31) to spherical co-ordinates:

$$\left. \begin{aligned} x' &= r \cos \theta \\ y &= r \sin \theta \cos \phi \\ z &= r \sin \theta \sin \phi \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

the limits of integration are for θ from 0 to $\frac{\pi}{2}$, and for ϕ from 0 to 2π .

Now $\cos \xi = \frac{\omega'}{r}$ and $dS = r^2 \sin \theta d\theta d\phi$, hence

$$\int E_4^1 \cos \xi dS = -e^2 \int \frac{\cos \theta \cdot r^2 \sin^2 \theta \cdot \left[\frac{1}{2} + \omega r \cos \theta + \frac{\omega^2}{2} r^2 \sin^2 \theta \right] \sqrt{\left[\frac{1}{2} + \omega r \cos \theta + \frac{\omega^2}{2} r^2 \sin^2 \theta \right]^2 - 2\omega r \cos \theta \cdot r^2 \sin \theta d\theta d\phi}}{\left\{ \left[r \cos \theta - \frac{1}{2\omega} + \frac{\omega}{2} r^2 \sin^2 \theta \right]^2 + r^2 \sin^2 \theta \right\}^3}$$

As r tends to infinity we can omit in the above expression all terms of the lower order of magnitude, and obtain after easy simplifications:

$$\int E_4^1 \cos \xi dS \approx \frac{-e^2}{4\omega^2 r^4} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\phi=0}^{\phi=2\pi} \frac{\cos \theta \sin \theta d\theta d\phi}{\left[\sin^2 \theta + \frac{4}{\omega r} \cos \theta \right]^3} \approx -\frac{\pi e^2}{2r^2} \quad (33)$$

or this expression is equal to zero for $r = \infty$.

(b) We integrate (23) along the plane $x' = 0$; for this case $\cos \xi = -1$, $\cos \eta = \cos \zeta = 0$. Putting in (31) $x' = 0$ we obtain

$$\int E_4^1 \cos \xi dS = \int_{-\infty}^{+\infty} \int \frac{e^2 (y^2 + z^2) \left[\frac{1}{2} + \frac{\omega^2}{2} (y^2 + z^2) \right]^2}{\left\{ \left[-\frac{1}{2\omega} + \frac{\omega}{2} (y^2 + z^2) \right]^2 + y^2 + z^2 \right\}^3} dx dy \quad (34)$$

We transform (34) to polar co-ordinates:

$$y = r \cos \phi, \quad z = r \sin \phi.$$

The denominator in (34) is equal to $\left[\frac{1}{2\omega} + \frac{\omega}{2} (y^2 + z^2) \right]^6$, and we obtain from (34)

$$\int E_4^1 \cos \xi dS = 2^5 e^2 \omega^6 \int_0^{2\pi} \int_0^{\infty} \frac{r^3 dr d\phi}{(1 + \omega^2 y^2)^5} = \frac{8\pi}{3} e^2 \omega^2 \quad (35)$$

In order to express the radiation in the usual units we multiply (35) by $\frac{c}{4\pi}$ and obtain

$$\frac{2}{3} \frac{g^2 e^2}{c^3}$$

where c is the velocity of light and g the acceleration of gravity.

§ 6. THE ELECTROMAGNETIC EQUATIONS.

We consider briefly the electromagnetic equations in the General Theory of Relativity. If K_μ I_μ are the vector potential and the stream

vector respectively, the equations for K_μ will be

$$\left. \begin{aligned} g^{\alpha\beta}K_{\mu, \alpha\beta} &= I_\mu - G_\mu^\epsilon K_\epsilon \\ K^\mu_{, \mu} &= 0 \end{aligned} \right\} \dots \dots \dots (36)$$

where G_μ^ϵ is the Riemannian contracted tensor. In our case $G_\mu^\epsilon = 0$.

The evaluation of (36) is simple; for the case of a uniform gravitational field it was given in Whittaker's paper.*

The obtained equations are

$$\left. \begin{aligned} \frac{1}{1+2\omega x} \frac{\partial^2 K_1}{\partial t^2} - \frac{2\omega}{(1+2\omega x)^2} \frac{\partial K_4}{\partial t} - 4\omega \frac{\partial K_1}{\partial x} - (1+2\omega x) \frac{\partial^2 K_1}{\partial x^2} - \frac{\partial^2 K_1}{\partial y^2} - \frac{\partial^2 K_1}{\partial z^2} &= I_1 \\ \frac{1}{1+2\omega x} \frac{\partial^2 K_2}{\partial t^2} - 2\omega \frac{\partial K_2}{\partial x} - (1+2\omega x) \frac{\partial^2 K_2}{\partial x^2} - \frac{\partial^2 K_2}{\partial y^2} - \frac{\partial^2 K_2}{\partial z^2} &= I_2 \\ \frac{1}{1+2\omega x} \frac{\partial^2 K_3}{\partial t^2} - 2\omega \frac{\partial K_3}{\partial x} - (1+2\omega x) \frac{\partial^2 K_3}{\partial x^2} - \frac{\partial^2 K_3}{\partial y^2} - \frac{\partial^2 K_3}{\partial z^2} &= I_3 \\ \frac{1}{1+2\omega x} \frac{\partial^2 K_4}{\partial t^2} - 2\omega \frac{\partial K_4}{\partial t} - (1+2\omega x) \frac{\partial^2 K_4}{\partial x^2} - \frac{\partial^2 K_4}{\partial y^2} - \frac{\partial^2 K_4}{\partial z^2} &= I_4 \end{aligned} \right\} (37)$$

Also the equation $K^\mu_{, \mu} = 0$ or $G^{\mu\nu} K_{\mu, \nu} = 0$ becomes

$$(1+2\omega x) \frac{\partial K_1}{\partial x} + 2\omega K_1 + \frac{\partial K_2}{\partial y} + \frac{\partial K_3}{\partial z} - \frac{1}{1+2\omega x} \frac{\partial K_4}{\partial t} = 0 \dots \dots (38)$$

For an electrostatic field Euclidean space the vector potential and stream vector are equal to

$$\left. \begin{aligned} K_\mu &= (-F, -G, -H, \phi) & F = G = H = 0 \\ I_1 = I_2 = I_3 &= 0 & I_4 = \rho \end{aligned} \right\} \dots \dots (39)$$

and

$$\nabla^2 K_4^0 = -\rho_0 \dots \dots \dots (40)$$

hence for an electron moving freely from rest in a gravitational field the stream vector is given by (15).

It can be shown by direct substitution in (37) and (38) that the vector potential is given in the present case by (14), where K_4^0 satisfies (40) and the electron is at the origin of co-ordinates.

* *Proc. Roy. Soc., A*, 116, p. 723.

§7. On the evaluation of a null geodesic

The null geodesic (27) can also be found in the following way.

The track of a ray of light in Galilean space (if the ray starts from the point $0, 0, 0, 0$) is

$$x_0^2 + y_0^2 + z_0^2 = t_0^2$$

if the velocity of light is equal to 1.

Inserting the values of x_0, t_0 from (8) we obtain

$$y^2 + z^2 + \left\{ \frac{\sqrt{1+2\omega c}}{\omega} \operatorname{cosh} \omega t - \frac{1}{\omega} \right\}^2 = \left(\frac{1+2\omega c}{\omega^2} \right) \operatorname{sinh}^2 \omega t$$

or

$$\sqrt{1+2\omega c} \operatorname{cosh} \omega t = \frac{\omega^2}{2} (y^2 + z^2) + 1 + \omega c$$

which is equation (27).

Chapter II.

Dynamics of an electron in a
manydimensional space.

Includes:

1. On the dynamics of an electron.
Phil. Mag. p 977 1928
2. On the electromagnetic field of an electron.
Phil. Mag p 425 1929.

A. Motion of two electrons.

in an eight dimensional space.

On the Dynamics of an Electron.
By D. MEKSYN *.

I. SPACE AND TIME IN PHYSICS.

§ 1. *The General Principle of Relativity and Motion of Electrons.*

AN attempt to apply the General Principle of Relativity to the motion of Electrons is met at the outset with insurmountable difficulties. It appears as if the problem is inherently self-contradictory. As a matter of fact, the track of an electron in an electromagnetic field is a curvilinear one; on the other hand the space, as follows from the law of motion of an electron, is not curved (unless a gravitational field exists); hence the motion cannot be a free one, as it should be according to the General Principle of Relativity.

It seems that this contradiction is intimately connected with the conception of Space and Time in the Principle of Relativity.

§ 2. *Space and Time in Geometry and in the Special Theory of Relativity.*

Space and Time are considered in the Special Theory of Relativity and in Geometry to be a physical Entity *sui generis*, which imposes its metrical laws upon solid bodies located in it. So, for instance, we could conceive two spaces (and times) which are similar in every respect, except that the dimensions of the first are, say, only half of the second

* Communicated by H. T. Flint, D.Sc.

one. If we transfer a solid body from the second space into the first one, the latter will cause the body to shrink and take only half the size that it had previously. Such a change in dimensions of Space can be brought about, according to the Special Theory of Relativity, if we set a body in motion.

Until the appearance of the Principle of Relativity it was usually held that space had three dimensions, and its metric was given by Laws of Geometry. The Special Principle of Relativity has discarded the conception of a three-dimensional space and one-dimensional time, but has accepted the idea of Geometry that space is a kind of objective physical Entity which imposes its metrical laws upon solid bodies; therefore the Theory of Relativity has drawn the conclusion that, because it follows from Lorentz's transformations that the space dimensions of a fourfold system decrease if the latter is set in motion, a solid physical body which belongs to a moving system will shrink in its space dimensions.

§ 3. *Geometrical or Rigid Space.*

What, however, is meant by Space in Geometry is as follows:—There exist solid physical bodies which possess some metrical properties; they have finite dimensions (length, surface, volume); they do not change their dimensions if transferred from one part of space into another.

Now, it appears that it is possible to account by deduction for all metrical properties of solid bodies, if we construct an appropriate system of reference, a "space," in such a way that all metrical properties of solid bodies (length, surface, volume) are properties of the space itself, and every element of length is expressed by some definite (quadratic) function of three appropriately chosen directions in space.

Hence, rigid bodies do not change their size if transferred from one part of Geometrical Space into another, not because the latter is homogeneous, but conversely the Geometrical Space is homogeneous for bodies which do not change their dimensions by such transfer.

It does, of course, not follow that there is no objective space, or that this space has no metrical properties of its own. (We do not touch upon this question here; what it is necessary to notice is that the Geometrical Space, which gives us the metrical properties of solid bodies, is something quite different from the objective space (if we admit its existence).)

§ 4. *Four-dimensional or Kinematical Space.*

The idea of Space in Geometry will become clearer if we turn our attention to classical kinematics. It was built up upon different lines than Geometry.

In Geometry, as we have seen, all metrical properties of solid bodies are properties of the space itself.

In classical kinematics, side by side with metrical properties of Space, it was necessary to introduce some quantities which do not belong to the space (velocity and acceleration). The classical kinematics has lacked the unity of design which characterized Geometry.

The achievement of the Special Principle of Relativity was to restore this unity. The four-dimensional space is constructed in such a way that all kinematical elements of motion appear as metrical forms of the space itself; so velocity is the tangent of the angle between time axes, and acceleration is the radius of curvature of the four-track.

§ 5. *Dynamical Space.*

If we turn our attention to the dynamics of the Theory of Relativity we notice that here again the above-mentioned unity is not maintained; the fundamental quantities of dynamics, momentum, and Energy appear not as metrical properties of the Space itself, but, so to say, as properties brought into the space.

It is clear, therefore, that the natural generalization of the Special Theory of Relativity is to construct a dynamical space in which momentum and energy would be properties belonging to the Space itself.

§ 6. *Electricity and Gravitation.*

Before proceeding further we must bear in mind the divergence between an Electromagnetic field and a field of Gravitation. An isolated mass-point possesses a field of Gravitation, and it is usually held that an isolated charged mass-point has also an Electromagnetic field. We think, however, that the latter appears only if there are at least two electrons at a finite distance.

Hence, in the corresponding fundamental form this must be brought out by appearance of cross-terms of different spaces rather than by a change in the fundamental form of every space separately.

II. THE FUNDAMENTAL FORM OF DYNAMICAL SPACE.

§ 7. Lagrange's Equations in the Special Theory of Relativity.

The idea of a dynamical space is not a new one. Such space is used in Lagrange's equations. Lagrange's space, however, is built up in conformity with the old kinematics, i. e. in it the time coordinate is separated from Space. This ought to be modified in such a way that time shall appear symmetrically with Space; this is necessary to bring it into agreement with the Special Theory of Relativity.

Lagrange's equations are :

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_k} - \frac{\partial T}{\partial x_k} = Q_k. \quad \dots \quad (1)$$

In the classical Mechanics T is the kinetic Energy; but in the Theory of Relativity T is the so-called kinetic Potential, a quantity which has neither a dynamical nor a metrical meaning.

The first step to be made is to bring Lagrange's equations in the Special Theory of Relativity into such a form that it shall contain only metrical quantities.

For one material point this is easily done. If the fundamental form is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \quad \dots \quad (2)$$

where t is chosen in light seconds. Lagrange's equations become in a variational form

$$\delta \int_{\tau_0}^{\tau_1} (m ds + V d\tau) = 0, \quad \dots \quad (3)$$

where V is the potential Energy. The Geometrical interpretation of (3) is that the four-track is a geodesic line on some surface $V=0$. If V disappears the track is a geodesic line of a free space.

In the case of several material points we consider the fundamental form

$$ds^2 = \sum_{k=1}^{k=n} m_k (dt_k^2 - dx_k^2 - dy_k^2 - dz_k^2), \quad \dots \quad (4)$$

which has $4n$ dimensions. It is easily seen that Lagrange's equations become

$$\delta \int_{\tau_0}^{\tau_1} \left(ds + \frac{1}{\sqrt{\sum_k m_k}} V d\tau \right) = 0; \quad \dots \quad (5)$$

see (2)

as a matter of fact, from (5) we obtain

$$m_k \frac{d}{d\tau} \left(\frac{dx_k}{ds} \right) + \frac{1}{\sqrt{\sum_k m_k}} \frac{\partial V}{\partial x_k} = 0. \quad \dots (6)$$

There are in all $4n - 1$ independent equations and only $3n$ given space projections of force, so that we are left with $n - 1$ time projections which we can dispose.

We define them from the following $n - 1$ equations :

$$dt_k^2 - dx_k^2 - dy_k^2 - dz_k^2 = dt_l^2 - dx_l^2 - dy_l^2 - dz_l^2; \dots (7) \quad \sum dt^2$$

they are evidently equivalent to

$$X_k dx_k + Y_k dy_k + Z_k dz_k - T_k dt_k = 0, \quad \dots (7a)$$

where X_k, Y_k, Z_k, T_k are the projections of the force.

From (7) the denominator in (6) becomes

$$ds = \sqrt{\sum_{l=1}^{l=n} m_l \cdot (dt_k^2 - dx_k^2 - dy_k^2 - dz_k^2)},$$

or inserting (ds) in (6) we obtain

$$m_k \frac{d}{d\tau} \left(\frac{dx_k}{\sqrt{dt_k^2 - dx_k^2 - dy_k^2 - dz_k^2}} \right) + \frac{\partial V}{\partial x_k} = 0,$$

which is the law of motion.

The expression $\sqrt{\sum m_k}$ in (6) appears to be rather disappointing, as it has had hitherto no dynamical meaning; it need not, however, trouble us, because it will later disappear in the rigorous laws of motion.

What we have to notice is, that for several material points the law of motion is a geodesic line of a $4n$ dimensional space.

§ 8. *The Electromagnetic Field and the Form of Space.*

We turn our attention to the case of an Electromagnetic Force. Lagrange's equations for an electron are

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}_k} (T - M) - \frac{\partial}{\partial x_k} (T - M) = 0, \quad \dots (8)$$

where T is the kinetic Energy,

$$M = e_1(\phi - u_1 F - v_1 G - w_1 H), \quad \dots (9)$$

$-\phi$ is the scalar, F, G, H the vector potential of Force, e_1 the charge, and u_1, v_1, w_1 the velocity of the electron.

To find now the fundamental form of space, we must bear in mind two points :

- (1) In (8) the kinetic and potential Energies appear perfectly symmetrically.
- (2) Lagrange's expression (1) for the equations of Motion becomes a geodesic if T is a quadratic function of x_k .

Hence our task will be achieved if we transform (9) into a quadratic expression.

Suppose that the external field is produced by one moving electron whose coordinates, charge, and velocity are

$$x_2, y_2, z_2, t_2, e_2, u_2, v_2, w_2.$$

Then (9) becomes

$$M = \frac{e_1 e_2}{r} (1 - u_1 u_2 - v_1 v_2 - w_1 w_2), \dots (10)$$

where

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

This expression for M suggests at once the expression for the fundamental form of space in this case. It is

$$\begin{aligned} ds^2 = & m(dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2) \\ & + \frac{e_1 e_2}{r} (dx_4 dx_8 - dx_1 dx_5 - dx_2 dx_6 - dx_3 dx_7) \\ & + M(dx_8^2 - dx_5^2 - dx_6^2 - dx_7^2). \dots (11) \end{aligned}$$

In the general case where the external force is produced by $n-1$ electrons, *i. e.* in the case of the n electron problem, if we denote the mass, charge, and coordinates of the k th electron respectively by

$$m_k, e_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, \dots (12)$$

we obtain the following expression for the fundamental form

$$\begin{aligned} ds^2 = & g_{\mu\nu} dx_\mu dx_\nu \\ = & \sum_{k=1}^{k=n} m_k (dx_{k+4}^2 - dx_{k+1}^2 - dx_{k+2}^2 - dx_{k+3}^2) \\ & + \sum_{\substack{k,l=1 \\ k,l=n}} \phi_{kl} (dx_{k+4} dx_{l+4} \\ & - dx_{k+1} dx_{l+1} - dx_{k+2} dx_{l+2} - dx_{k+3} dx_{l+3}), \dots (13) \end{aligned}$$

where

$$\left. \begin{aligned} k \neq l \text{ and } \phi_{kl} &= \frac{e_k e_l}{r_{kl}} \\ r_{kl} &= \sqrt{(x_{k+1} - x_{l+1})^2 + (x_{k+2} - x_{l+2})^2 + (x_{k+3} - x_{l+3})^2} \end{aligned} \right\}, \quad (14)$$

or

$$\phi_{kl} = \frac{e_k e_l}{c^2 r_{kl}}$$

if c is the velocity of light.

The Equation of Motion will be

$$\delta \int ds = 0, \quad \dots \dots \dots (15)$$

or a geodesic line in $4n$ dimensional space, which we proceed to discuss.

III. THE TWO-ELECTRON PROBLEM.

§ 9. The Form of Space.

We consider the problem of 2 electrons. The fundamental form for this particular case is

Handwritten notes:
 $2 dx_4^2$
 $= dx_8^2$
 See (29)

$$\begin{aligned} ds^2 &= m(dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2) \\ &+ \frac{eE}{r} (dx_4 dx_8 - dx_1 dx_5 - dx_2 dx_6 - dx_3 dx_7) \\ &+ M(dx_8^2 - dx_5^2 - dx_6^2 - dx_7^2) \dots \dots \dots (16) \end{aligned}$$

A glance at this expression shows us that it satisfies the physical conditions of an electromagnetic Field. The eight-dimensional space is split up into two four-dimensional spaces which are Euclidean, but taken as a whole this space is not Euclidean because of the cross terms. Hence an electron will not move in a straight line.

§ 10. The Fundamental Tensor.

The discriminant g is equal to

$$g = \begin{vmatrix} g_{11} & \cdot & \cdot & \cdot & g_{15} & \cdot & \cdot & \cdot \\ \cdot & g_{22} & \cdot & \cdot & \cdot & g_{26} & \cdot & \cdot \\ \cdot & \cdot & g_{33} & \cdot & \cdot & \cdot & g_{37} & \cdot \\ \cdot & \cdot & \cdot & g_{44} & \cdot & \cdot & \cdot & g_{48} \\ g_{51} & \cdot & \cdot & \cdot & g_{55} & \cdot & \cdot & \cdot \\ \cdot & g_{62} & \cdot & \cdot & \cdot & g_{66} & \cdot & \cdot \\ \cdot & \cdot & g_{73} & \cdot & \cdot & \cdot & g_{77} & \cdot \\ \cdot & \cdot & \cdot & g_{84} & \cdot & \cdot & \cdot & g_{88} \end{vmatrix} \dots (17)$$

and is split up into four separate determinants.

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Or

$$g = (g_{11}g_{55} - g_{15}^2)(g_{22}g_{66} - g_{26}^2)(g_{33}g_{77} - g_{37}^2)(g_{44}g_{88} - g_{48}^2), \quad (18)$$

and the contravariant tensor $g^{\mu\nu}$ is equal to

$$\left. \begin{aligned} g^{11} &= \frac{g_{55}}{g_{11}g_{55} - g_{15}^2} \dots, & g^{55} &= \frac{g_{11}}{g_{11}g_{55} - g_{15}^2}, \\ g^{15} &= \frac{-g_{15}}{g_{11}g_{55} - g_{15}^2} \dots, & g^{48} &= \frac{-g_{48}}{g_{44}g_{88} - g_{48}^2}, \end{aligned} \right\} \dots \quad (19)$$

in our particular case

$$\left. \begin{aligned} g_{11} &= g_{22} = g_{33} = -g_{44} = -m, \\ g_{55} &= g_{66} = g_{77} = -g_{88} = -M, \\ g_{15} &= g_{26} = g_{37} = -g_{48} = -\frac{eE}{r}, \end{aligned} \right\} \dots \quad (20)$$

where

$$m, e, x_1, x_2, x_3, x_4, \text{ and } M, E, x_5, x_6, x_7, x_8$$

relate respectively to the Electron and Proton.

§ 11. *The Equation of Motion* is, as we have seen, a geodesic line of this space, or

$$\frac{d^2 x_\alpha}{ds^2} + \{\mu\nu\alpha\} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} = 0. \dots \quad (21)$$

In the general case of an eight-dimensional space, there are 288 three-index symbols, but in our case the number reduces to 136, and, if $g_{\mu\nu}$ does not depend upon time, to 102.

The only symbols which do not vanish are

$$\left. \begin{aligned} \{\alpha\alpha', \alpha\} &= g^{\alpha\alpha'} \frac{\partial g_{\alpha\alpha'}}{\partial x_\alpha}, \\ \{\alpha\nu, \alpha\} &= \frac{1}{2} g^{\alpha\alpha'} \frac{\partial g_{\alpha\alpha'}}{\partial x_\nu} \quad (\text{but } \{\alpha\alpha', \alpha\} = 0), \\ \{\mu\mu', \alpha\} &= -\frac{1}{2} g^{\alpha\alpha'} \frac{\partial g_{\mu\mu'}}{\partial x_\alpha} - \frac{1}{2} g^{\alpha\alpha'} \frac{\partial g_{\mu\mu'}}{\partial x_{\alpha'}}, \\ \{\mu\alpha', \alpha\} &= \frac{1}{2} g^{\alpha\alpha'} \frac{\partial g_{\alpha'a}}{\partial x_\mu}, \end{aligned} \right\} \quad (22)$$

where $(\alpha' - \alpha) = 4$.

There are in all 8 equations ; the first four give us the motion of the electron ; the last four, of the proton.

If we write out the equations (21) and insert the values of the 3-index terms, we obtain the following expressions :—

$$\left. \begin{aligned}
 \frac{d^2x_1}{ds^2} + g^{11} \left(\frac{\partial g_{51}}{\partial x_2} V_5 - \frac{\partial g_{26}}{\partial x_1} V_6 \right) V_2 + g^{11} \left(\frac{\partial g_{51}}{\partial x_3} V_5 - \frac{\partial g_{37}}{\partial x_1} V_7 \right) V_3 \\
 + g^{11} \left(\frac{\partial g_{51}}{\partial x_4} V_5 - \frac{\partial g_{48}}{\partial x_1} V_8 \right) V_4 + g^{15} \left(\frac{\partial g_{51}}{\partial x_7} V_1 - \frac{\partial g_{37}}{\partial x_5} V_3 \right) V_7 \\
 + g^{15} \left(\frac{\partial g_{15}}{\partial x_6} V_1 - \frac{\partial g_{26}}{\partial x_5} V_2 \right) V_6 + g^{15} \left(\frac{\partial g_{15}}{\partial x_8} V_1 - \frac{\partial g_{48}}{\partial x_5} V_4 \right) V_8 \\
 + g^{15} \left(\frac{\partial g_{15}}{\partial x_1} V_1 + \frac{\partial g_{15}}{\partial x_2} V_2 + \frac{\partial g_{15}}{\partial x_3} V_3 + \frac{\partial g_{15}}{\partial x_4} V_4 \right) V_1 \\
 + g^{11} \left(\frac{\partial g_{15}}{\partial x_5} V_5 + \frac{\partial g_{15}}{\partial x_6} V_6 + \frac{\partial g_{15}}{\partial x_7} V_7 + \frac{\partial g_{15}}{\partial x_8} V_8 \right) V_5 = 0, \\
 \dots \dots \dots \\
 \frac{d^2x_4}{ds^2} + g^{44} \left(\frac{\partial g_{84}}{\partial x_1} V_8 - \frac{\partial g_{15}}{\partial x_4} V_5 \right) V_1 + g^{44} \left(\frac{\partial g_{84}}{\partial x_2} V_8 - \frac{\partial g_{26}}{\partial x_4} V_6 \right) V_2 \\
 + g^{44} \left(\frac{\partial g_{84}}{\partial x_3} V_8 - \frac{\partial g_{37}}{\partial x_4} V_7 \right) V_3 + g^{48} \left(\frac{\partial g_{48}}{\partial x_5} V_4 - \frac{\partial g_{15}}{\partial x_8} V_1 \right) V_5 \\
 + g^{48} \left(\frac{\partial g_{48}}{\partial x_6} V_4 - \frac{\partial g_{26}}{\partial x_8} V_2 \right) + g^{48} \left(\frac{\partial g_{48}}{\partial x_7} V_4 - \frac{\partial g_{37}}{\partial x_8} V_3 \right) V_7 \\
 + g^{48} \left(\frac{\partial g_{48}}{\partial x_1} V_1 + \frac{\partial g_{48}}{\partial x_2} V_2 + \frac{\partial g_{48}}{\partial x_3} V_3 + \frac{\partial g_{48}}{\partial x_4} V_4 \right) V_4 \\
 + g^{44} \left(\frac{\partial g_{84}}{\partial x_5} V_5 + \frac{\partial g_{84}}{\partial x_6} V_6 + \frac{\partial g_{84}}{\partial x_7} V_7 + \frac{\partial g_{84}}{\partial x_8} V_8 \right) V_8 = 0.
 \end{aligned} \right\} \dots \dots \dots (23)$$

§ 12. The equations (23) could be brought in a form which will at once show their relation to the usual equations of Motion of an electron.

Let F, G, H, -φ be the vector potential ; then the electromagnetic force is

$$\left. \begin{aligned}
 \alpha &= \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z} \dots, \\
 X &= -\frac{\partial F}{\partial t} - \frac{\partial \phi}{\partial x} \dots
 \end{aligned} \right\} \dots \dots \dots (24)$$

From the values of g_{15} , g_{26} , we see that (except for a constant factor)

$$\left. \begin{aligned} g_{15} V_5 &= -eF_p, & g_{26} V_6 &= -eG_p, & g_{37} V_7 &= -eH_p, \\ & & & & g_{48} V_7 &= +e\phi_p, \\ g_{51} V_1 &= -EF_e, & g_{62} V_2 &= -EG_e, & g_{73} V_3 &= -EH_e, \\ & & & & g_{84} V_4 &= +E\phi_e, \end{aligned} \right\} \dots (25)$$

where the subscripts p and e denote that it is the vector-potential due to the proton (and, hence, acting on the electron) and the electron respectively.

We also denote the velocity of the electron and proton :

$$V_1 V_2 V_3 V_4 = V_e, \quad V_5 V_6 V_7 V_8 = V_p. \dots (26)$$

If we make use of (24) and (25), the equations (23) become

$$\left. \begin{aligned} \frac{d^2 x_1}{ds^2} &= -g^{11} e [X_p V_4 + Y_p V_2 - \beta_p V_3] \\ &\quad - g^{15} E [X_e V_8 + Y_e V_6 - \beta_e V_7] \\ &\quad - g^{11} \left(\frac{\partial g_{15}}{\partial x_5} V_5 + \frac{\partial g_{15}}{\partial x_6} V_6 + \frac{\partial g_{15}}{\partial x_7} V_7 + \frac{\partial g_{15}}{\partial x_8} V_8 \right) V_5 \\ &\quad - g^{15} \left(\frac{\partial g_{15}}{\partial x_1} V_1 + \frac{\partial g_{15}}{\partial x_2} V_2 + \frac{\partial g_{15}}{\partial x_3} V_3 + \frac{\partial g_{15}}{\partial x_4} V_4 \right) V_1, \\ \frac{d^2 x_4}{ds^2} &= g^{44} e [X_p V_1 + Y_p V_2 + Z_p V_3] \\ &\quad + g^{48} E [X_e V_5 + Y_e V_6 + Z_e V_7] \\ &\quad - g^{44} \left(\frac{\partial g_{84}}{\partial x_5} V_5 + \frac{\partial g_{84}}{\partial x_6} V_6 + \frac{\partial g_{84}}{\partial x_7} V_7 + \frac{\partial g_{84}}{\partial x_8} V_8 \right) V_8 \\ &\quad - g^{48} \left(\frac{\partial g_{48}}{\partial x_1} V_1 + \frac{\partial g_{48}}{\partial x_2} V_2 + \frac{\partial g_{48}}{\partial x_3} V_3 + \frac{\partial g_{48}}{\partial x_4} V_4 \right) V_4. \end{aligned} \right\} (27)$$

1. Let us consider the case of a steady electromagnetic field. Or we assume that $M \rightarrow \infty$ and the velocity of the proton is

$$V_5, V_6, V_7 \rightarrow 0.$$

Then

$$g^{11} = -\frac{1}{m}, \quad g^{44} = +\frac{1}{m}, \quad g^{15} = g^{26} = g^{37} = g^{48} = 0,$$

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+ meaning

2. In the general case, as is seen from (27), the force consists of four parts. Two of them are due to the direct action of the proton, and the other two forces to the reaction on the proton, due to the electromagnetic force of the electron.

The first one is the usual Lorentz's force, the second one is some new force ; it is equal to

$$-g^{11}(\text{grad } g_{15} \cdot V_p) V_5 - g^{15}(\text{grad } g_{12} \cdot V_e) V_1.$$

3. From the equations of motion (21) a remarkable consequence follows concerning the number of degrees of freedom of an electron.

As is known,

$$\frac{dx_\alpha}{ds} \left\{ \frac{d^2 x^\alpha}{ds^2} + \{ \mu\nu\alpha \} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right\} = 0, \quad \dots \quad (31)$$

or there are only $4n-1$ independent equations for n electrons. If we assume that the proton has three degrees of freedom, then we are left with $4n-4$ equations for $n-1$ electrons; or an electron has four degrees of freedom.

The fourth degree of freedom is in the direction of time, or, better, it is the kinetic energy, which is not equal to $m \frac{dt}{d\tau}$, as the Special Theory of Relativity gives, but has to be found from the equations of Motion.

Our assumption (29) about the connexion between different times is only an approximate one. We must choose one time (of the proton) as an independent variable, and find all others from the equations of motion.

4. The so-called vector potential of Electrodynamics is the covariant velocity in the dynamical space.

For instance,

$$\frac{dx_5}{ds} = -m \frac{dx^5}{ds} - g_{15} \frac{dx^1}{ds},$$

which, for the particular case where the proton is at rest, is proportional to the vector potential of the electron

$$\frac{dx_5}{ds} = EF_e.$$

24 = g_{51} V_1 + ...
25 = g_{52} V_2 + ...
26 = g_{53} V_3 + ...
27 = g_{54} V_4 + ...
28 = g_{55} V_5 + ...
29 = g_{56} V_6 + ...

§ 13. The Wave Equation

$$g^{\mu\nu} \phi_{\mu\nu} = g^{\mu\nu} \left(\frac{\partial^2 \phi}{\partial x_\mu \partial x_\nu} + \{ \mu\nu, \alpha \} \frac{\partial \phi}{\partial x_\alpha} \right) = 0 \quad \dots \quad (32)$$

is rather complicated. We consider here the simple instance, when $g_{15} = g_{26} = g_{37} = 0$, or of a purely electric field. For

this case $\phi_{\mu\nu} = \frac{\partial^2 \phi}{\partial x_\mu \partial x_\nu}$, except for the expression

$$\phi_{48} = \frac{\partial^2 \phi}{\partial x_4 \partial x_8} + \frac{1}{2} \left[g^{11} \frac{\partial g_{48}}{\partial x_1} \frac{\partial \phi}{\partial x_1} + g^{22} \frac{\partial g_{48}}{\partial x_2} \frac{\partial \phi}{\partial x_2} + g^{33} \frac{\partial g_{48}}{\partial x_3} \frac{\partial \phi}{\partial x_3} + g^{55} \frac{\partial g_{48}}{\partial x_5} \frac{\partial \phi}{\partial x_5} + g^{66} \frac{\partial g_{48}}{\partial x_6} \frac{\partial \phi}{\partial x_6} + g^{77} \frac{\partial g_{48}}{\partial x_7} \frac{\partial \phi}{\partial x_7} \right]. \quad (33)$$

We neglect, however, the cross terms in (33), because from the expression of ϕ , which we shall take, the second term for the Hydrogen Atom is only $\sim 10^{-10}$ of the first one.

So that the wave equation becomes

$$g^{11} \frac{\partial^2 \phi}{\partial x_1^2} + \dots + g^{44} \frac{\partial^2 \phi}{\partial x_4^2} + g^{55} \frac{\partial^2 \phi}{\partial x_5^2} + \dots + g^{88} \frac{\partial^2 \phi}{\partial x_8^2} + 2g^{84} \frac{\partial^2 \phi}{\partial x_4 \partial x_8} = 0. \quad (34)$$

Inserting in (34) the values of the tensor $g^{\mu\nu}$, we obtain after easy transformations

$$\frac{1}{m} \left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \right) + \frac{1}{M} \left(\frac{\partial^2 \phi}{\partial x_5^2} + \frac{\partial^2 \phi}{\partial x_6^2} + \frac{\partial^2 \phi}{\partial x_7^2} \right) + \left\{ -\frac{1}{m} \frac{\partial^2 \phi}{\partial x_4^2} - \frac{1}{M} \frac{\partial^2 \phi}{\partial x_8^2} + \frac{2V}{Mm} \frac{\partial^2 \phi}{\partial x_4 \partial x_8} \right\} = 0, \quad (35)$$

where $V = -\frac{Ee}{r}$ is the electrostatic potential.

This can be brought (under special assumptions) to Schrödinger's wave equations. Let us take for ϕ the expression

$$\phi \equiv \psi \cdot e^{\frac{2\pi}{h} \{ (m+E_1)x_4 + (M+E_2)x_8 \} i}, \quad \dots \quad (36)$$

or we assume in conformity with de Broglie's ideas that *not only Energy due to motion of an electron passes a frequency $\frac{E}{h}$, but also Energy due to the rest-mass has a frequency*

For the Hydrogen Atom $\frac{E_1}{m} = V^2 \approx 10^{-5}$; hence E_1^2, E_2^2 could be neglected in comparison with mE_1, ME_2 , and E_1 ,

must be considered as separate ? E1, E2 defined

E_2 in comparison with m and M . Under such approximation we easily obtain from (35) (36) the equation for ψ :

$$\frac{1}{m} \left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} \right) + \frac{1}{M} \left(\frac{\partial^2 \psi}{\partial x_5^2} + \frac{\partial^2 \psi}{\partial x_6^2} + \frac{\partial^2 \psi}{\partial x_7^2} \right) + \frac{8\pi^2}{h^2} \left[\frac{M+m}{2} + (E_1 + E_2) - V \right] \psi = 0. \quad (37)$$

We can now transform (37) as follows :—Let us change the variables :

$$\left. \begin{aligned} x &= x_1 - x_5 & (m+M)\xi &= mx_1 + Mx_5, \\ y &= x_2 - x_6 & (m+M)\eta &= mx_2 + Mx_6, \\ z &= x_3 - x_3 & (m+M)\zeta &= mx_3 + Mx_7. \end{aligned} \right\} \dots (38)$$

Using (38), we obtain from (37) :

$$\frac{1}{m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + \frac{1}{m+M} \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \zeta^2} \right) + \frac{8\pi^2}{h^2} \left[\frac{M+m}{2} + (E_1 + E_2) - V \right] \psi = 0, \quad \mu = \frac{mM}{m+M} \dots (39)$$

Let us assume for ψ :

$$\psi = f(x, y, z) g(\xi, \eta, \zeta); \dots (40)$$

hence

$$\frac{1}{\mu} g \nabla_{xyz}^2 f + \frac{1}{M+m} f \nabla_{\xi, \eta, \zeta}^2 g + \frac{8\pi^2}{h^2} \times \left[\frac{M+m}{2} + (E_1 + E_2) - V \right] g = 0. \dots (41)$$

Let $\frac{1}{m+M} \nabla^2 g = -\frac{4\pi^2}{h^2} g. \dots (42)$

From (41) and (42) we obtain

$$\frac{1}{\mu} \nabla^2 f + \frac{8\pi^2}{h^2} [(E_1 + E_2) - V] f = 0, \dots (43)$$

which is Schrödinger's equation.

The radiation function is

$$\psi = \psi e^{\frac{2\pi}{h} (M+m+E_1+E_2)it}, \dots (44)$$

B. On an electromagnetic field of
an electron.

The idea underlying the above given treatment of an electron in a manydimensional space is as follows.

A isolated mass point possesses a gravitational force. This can be tested by sending a ray of light through its space; light will be deflected.

An isolated charged particle does not possess what is usually considered to be an electromagnetic force. The latter appears only if there are at least two charged particles at a finite distance from each other.

This section is an attempt to explain the electromagnetic force from such a point of view.

On the Electromagnetic Field of an Electron.—The Electron as a Gravitational Phenomenon. By D. MEKSYN*.

§ 1. *The Properties of an Electron and the Special Theory of Relativity.*

ACCORDING to the present Electrodynamics, an Electron is an atom of charge in its own Electromagnetic Field; the latter could be represented as a system of stresses self-balanced in the whole field and resolved into a force acting on the Electron itself.

We make an attempt to discard this picture of an Electron, and we assume that an Electron represents the same entity as a neutral mass, with the only difference that, whereas matter, or energy, is located in a particle in a very small region, in an Electron it is spread all over the space according to the law $\frac{A}{r^2}$, or, what turns out to be the same, that an Electron is a Field of Gravitation whose potential is not $\frac{m}{r}$, but $\frac{m}{r^2}$; there are no stresses in the Field of an Electron †.

It is true that it is now accepted in some quarters that the stresses have no physical reality, and are only convenient mathematical conceptions. It makes, however, no difference

* Communicated by H. T. Flint, D.Sc.

† The Electron is assumed to have a special localization about a point from which r is measured, and the law of extension applies outside this central localization.

whether we consider them to be real or not, if only we accept the consequences which follow from this conception, as, for instance, the existence of an Electromagnetic Force, the connexion between the mass and Energy etc.

The "abolition" of stresses clears up from the outset some of the difficulties connected with the theory of Electrons.

1. If there are no stresses (or the stresses are mathematically equal to zero), an Electron does not possess an Electric Field, and hence it is unnecessary to "explain" the existence of an Electron. An Electron is a substance which is in equilibrium just as a mass particle.

2. From the dynamics of the Theory of Relativity it follows that mass and Energy are connected by

$$m = \frac{W^0}{c^2} \dots \dots \dots (1)$$

This is, however, not borne out by the Electron Theory, because, according to the latter, the momentum of an Electron is equal* to

$$G = \frac{4W^0u}{3c\sqrt{c^2-u^2}}; \dots \dots \dots (2)$$

hence the mass is equal to

$$m = \frac{4W^0}{3c^2} \dots \dots \dots (3)$$

If, however, we discard the stresses, the momentum will be

$$G = \frac{uW^0}{c\sqrt{c^2-u^2}} \dots \dots \dots (4)$$

and the mass

$$m = \frac{W^0}{c^2}, \dots \dots \dots (5)$$

or in complete agreement with dynamics.

3. The Law of Energy for an Electron is not satisfied for an accelerated Motion.

If we assume that the mass of an Electron is wholly electromagnetic, its momentum will be

$$G_x = \frac{4W^0u}{3c^2\sqrt{1-\frac{u^2}{c^2}}}, \dots \dots \dots (6)$$

* M. v. Laue, 'Die Relativitätstheorie,' 1921, p. 134.

and Energy

$$W = \frac{c^2 + \frac{1}{3}u^2}{c^2 \sqrt{1 - \frac{u^2}{c^2}}} W^0. \quad \dots \quad (7)$$

The Equation of Energy is

$$u \frac{dG_z}{dt} - \frac{dW}{dt} = \frac{d}{dt} \frac{\sqrt{1 - \frac{u^2}{c^2}}}{3} W^0; \quad \dots \quad (8)$$

or in order to preserve the Law of Energy, it is necessary to admit the existence of some additional mechanical Energy of value* :

$$\frac{\sqrt{1 - \frac{u^2}{c^2}}}{3} W^0. \quad \dots \quad (9)$$

In our case this difficulty also disappears because the Momentum and Energy are equal to

$$\dagger G = \frac{u W^0}{c^2 \sqrt{1 - \frac{u^2}{c^2}}}, \quad W = \frac{W^0}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad \dots \quad (10)$$

and hence the Law of Energy is satisfied.

§ 2. The Gravitational Field of an Electron.

1. Classical Theory.—According to the classical theory, the Gravitational Potential of continuous matter is given by

$$\nabla^2 \phi = -4\pi k\rho, \quad \dots \quad (11)$$

where ρ is the density of matter and k the constant of Gravitation.

In our case

$$\rho = \frac{e^2}{8\pi c^2 r^4}; \quad \dots \quad (12)$$

hence

$$\nabla^2 \phi = - \frac{ke^2}{2c^2 r^4}$$

and

$$\phi = - \frac{ke^2}{4c^2 r^2}. \quad \dots \quad (13)$$

* M. v. Laue, *ibid.* p. 226.

2. *The Theory of Relativity.*—The Gravitational potential could be directly obtained from Einstein's solution for Gravitational waves*.

We give here a somewhat different solution for a particular case in which

$$ds^2 = -(1 + 2\Omega)(dx^2 + dy^2 + dz^2) + (1 - 2\Omega) dt^2, \quad (14)$$

where Ω does not depend upon time.

The Equations of Gravitation are

$$G_{\mu\nu} = -8\pi k(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \quad \dots \quad (15)$$

where $T_{\mu\nu}$ is the Energy tensor and

$$T = g^{\mu\nu}T_{\mu\nu} \quad \dots \quad (16)$$

Now, neglecting the terms of the order Ω^2 , we have

$$G_{\mu\nu} = \frac{1}{2}g^{\sigma\rho} \left(\frac{\partial^2 g_{\mu\nu}}{\partial x_\rho \partial x_\sigma} + \frac{\partial^2 g_{\rho\sigma}}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 g_{\mu\sigma}}{\partial x_\rho \partial x_\nu} - \frac{\partial^2 g_{\nu\sigma}}{\partial x_\mu \partial x_\rho} \right), \quad (17)$$

which for our case become †

$$G_{\mu\nu} = \frac{1}{2}g^{\sigma\rho} \frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma \partial x_\rho} = -\frac{1}{2}\nabla^2 g_{\mu\nu} = \nabla^2 \Omega, \quad \dots \quad (18)$$

and the Law of Gravitation is

$$\nabla^2 \Omega = -8\pi k(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T); \quad \dots \quad (19)$$

in our case all $T_{\mu\nu} = 0$ except

$$T_{44} = g_{44}T_4^4 = \frac{e^2}{8\pi r^4 c^2} \quad \dots \quad (20)$$

and

$$T = g_{\mu\nu}T^{\mu\nu} = T_4^4 = \frac{e^2}{8\pi r^4 c^2} \quad \dots \quad (21)$$

Now the right side of (19) is equal for $\mu = \nu = 1, 2, 3$

$$-8\pi k \left(+ \frac{1}{2} \frac{e^2}{8\pi r^4 c^2} \right) = - \frac{ke^2}{2r^4 c^2}, \quad \dots \quad (22)$$

and for $\mu = \nu = 4$

$$-8\pi k \left(\frac{e^2}{8\pi r^4 c^2} - \frac{1}{2} \frac{e^2}{8\pi r^4 c^2} \right) = - \frac{1}{2} \frac{ke^2}{r^4 c^2}, \quad \dots \quad (23)$$

or the same as (22).

* A. Einstein, 'Über Gravitationswellen,' Berlin, Sitzungsberichte, 1918, p. 154.

† A. S. Eddington, 'The Mathematical Theory of Relativity,' 1924, p. 102.

Our assumption as to the value of $T_{\mu\nu}$ is consistent with Einstein's Law of Gravitation.

The Equation of Gravitation becomes

$$\nabla^2 \Omega = -\frac{1}{2} \frac{ke^2}{r^4 c^2}$$

and

$$\Omega = -\frac{ke^2}{4r^2 c^2}, \dots \dots \dots (24)$$

or the same as in the classical theory.

If we compare this solution with Nordström's and Jeffery's*, we notice an important discrepancy: our solution is only half of Jeffery's, which is equal to

$$\Omega = -\frac{1}{2} \frac{e^2}{r^2}$$

(if allowance is made for the factor $\frac{1}{4\pi}$ in the expression of the Electromagnetic Energy).

This is due to the value of the Energy Tensor $T_{\mu\nu}$, which was taken by Jeffery under the assumption of existence of stresses. We neglect here the ordinary gravitational contribution to $r: \frac{m}{r}$ which arises from the central localization.

§ 3. Relative and Invariant Mass.

If we accept Maxwell's stresses, a well-known difficulty arises in the explanation of the mass of an Electron †.

We have to discriminate between the relative mass T_{44} and the invariant mass $g_{\mu\nu} T^{\mu\nu}$.

Now it appears that the latter, for Maxwell's stresses, is equal to zero, and hence some additional assumption becomes necessary in order to explain the equality of relative and invariant mass for an Electron at rest.

If we discard the stresses, the invariant mass $g_{\mu\nu} T^{\mu\nu}$ becomes equal for an Electron at rest to its relative mass.

§ 4. Two Electrons in Space and the Electromagnetic Force.

We have seen that an Electron does not possess an electromagnetic field; the latter appears only if we have two or more electrons in space, and is wholly due to the increase (or decrease) of energy of space above (or below) the sum of energies of the two Electrons.

* A. S. Eddington, *ibid.* p. 185.

† *Id. ibid.* p. 183.

To define the laws of the electromagnetic force we must find the energy of two Electrons.

It is clear that, as both energies are spread over the space, some interaction may arise between them.

We know that the gravitational field of two material points and its energy are found (to the first approximation) by superposition of the two fields and their energies. Hence, if we translate the energy of an Electron as an equivalent energy of a gravitational field of matter, we can make some inferences about the laws of an electromagnetic field by applying the same superposition.

This interpretation can be carried out as follows:—

The energy density of an Electron is equal to

$$E = \frac{e^2}{8\pi r^4} = \frac{1}{8\pi} \left[\left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right], \quad \left. \begin{array}{l} \text{where} \\ V = \frac{e}{r}. \end{array} \right\} \quad (25)$$

By Green's transformation we have

$$\iiint \nabla \nabla^2 V \, dx \, dy \, dz + \iiint \left\{ \left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right\} \times dx \, dy \, dz + \iint V \frac{dV}{dn} \, dS = 0. \quad (26)$$

The last term vanishes at the boundary, and as $V = \frac{e}{r}$, the first term is equal to

$$\iiint \nabla \nabla^2 V \, dx \, dy \, dz = -4\pi \iiint \nabla \rho \, dx \, dy \, dz = -4\pi e V, \quad (27)$$

whence

$$E = \frac{1}{2} e V. \quad \dots \dots \dots (28)$$

Hence we come to the conclusion that the energy of an Electron is equal to the energy of a mass e in a gravitational field V , or, from (25), the energy at every point of the field is proportional to the square of its equivalent gravitational Force.

The latter is what we usually call the Electrostatic force of an Electron.

Now, if we have two Electrons to which we ascribe electric forces

$$X_1, Y_1, Z_1, \quad X_2, Y_2, Z_2,$$

the resultant force will be, according to the theory of gravitation,

$$X_1 + X_2, \quad Y_1 + Y_2, \quad Z_1 + Z_2,$$

and their common energy

$$\begin{aligned} & \frac{1}{8\pi} \iiint [(X_1 + X_2)^2 + (Y_1 + Y_2)^2 + (Z_1 + Z_2)^2] dx dy dz \\ &= \frac{1}{8\pi} \iiint (X_1^2 + Y_1^2 + Z_1^2) dx dy dz \\ &+ \frac{1}{4\pi} \iiint (X_1 X_2 + Y_1 Y_2 + Z_1 Z_2) dx dy dz \\ &+ \frac{1}{8\pi} \iiint (X_2^2 + Y_2^2 + Z_2^2) dx dy dz. \dots \dots (29) \end{aligned}$$

In (29) the first and the last term represent the energies of the two Electrons, and the middle term the energy of their interaction.

Of this energy

$$W = \frac{1}{4\pi} \iiint (X_1 X_2 + Y_1 Y_2 + Z_1 Z_2) dx dy dz \dots (30)$$

is due the so-called ponderomotive force of the field. If $W=0$, there is no mechanical action between two such Electrons.

The derivation of (29) cannot be considered as rigorous. We take the expression of W as one which has to be confirmed *a posteriori* rather than found by deduction.

For the general case of two electromagnetic fields we assume that the extra energy is, as in the case of two electrostatic fields, equal to the scalar product of the six vectors of force, *i. e.*

$$W = \iiint [(X_1 X_2 + Y_1 Y_2 + Z_1 Z_2) - (\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2)] \times dy dx dz \dots (31)$$

where XYZ , $\alpha\beta\gamma$ are the electric and magnetic forces.

§ 5. An Electron in an Electrostatic Field.

The extra energy is

$$W = \frac{1}{4\pi} \iiint \left(\frac{\partial V_1}{\partial x} \frac{\partial V_2}{\partial x} + \frac{\partial V_1}{\partial y} \frac{\partial V_2}{\partial y} + \frac{\partial V_1}{\partial z} \frac{\partial V_2}{\partial z} \right) dx dy dz,$$

or integrating by parts and omitting the surface integral,

$$W = - \frac{1}{4\pi} \iiint V_2 \nabla^2 V_1 dx dy dz = e_1 V_2.$$

§ 6. *An Electron in a Magnetic Field.*

The extra energy W (31) vanishes identically (because every term is equal to 0) ; hence there is no mechanical interaction between an electrostatic and magnetostatic field.

§ 7. *An Electron in Motion in an Electromagnetic Field (two Electromagnetic Fields).*

The energy integral is equal to

$$W = \frac{1}{4\pi} \iiint [(X_1 X_2 + Y_1 Y_2 + Z_1 Z_2) - (\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2)] \times dx dy dz.$$

We express the Electromagnetic force by means of a vector and scalar potential, and obtain :

$$\begin{aligned} W = \frac{1}{4\pi} \iiint \left\{ \left[\left(-\frac{\partial \psi_1}{\partial x} - \frac{\partial F_1}{\partial t} \right) \left(-\frac{\partial \psi_2}{\partial x} - \frac{\partial F_2}{\partial t} \right) \right. \right. \\ + \left(-\frac{\partial \psi_1}{\partial y} - \frac{\partial G_1}{\partial t} \right) \left(-\frac{\partial \psi_2}{\partial y} - \frac{\partial G_2}{\partial t} \right) \\ + \left. \left(-\frac{\partial \psi_1}{\partial z} - \frac{\partial H_1}{\partial t} \right) \left(-\frac{\partial \psi_2}{\partial z} - \frac{\partial H_2}{\partial t} \right) \right] \\ - \left[\left(\frac{\partial H_1}{\partial y} - \frac{\partial G_1}{\partial z} \right) \left(\frac{\partial H_2}{\partial y} - \frac{\partial G_2}{\partial z} \right) \right. \\ + \left(\frac{\partial F_1}{\partial z} - \frac{\partial H_1}{\partial x} \right) \left(\frac{\partial F_2}{\partial z} - \frac{\partial H_2}{\partial x} \right) \\ \left. \left. + \left(\frac{\partial G_1}{\partial x} - \frac{\partial F_1}{\partial y} \right) \left(\frac{\partial G_2}{\partial x} - \frac{\partial F_2}{\partial y} \right) \right] \right\} dx dy dz. \dots (32) \end{aligned}$$

We integrate (32) by parts, and obtain a volume integral :

$$\begin{aligned} W_1 = -\frac{1}{4\pi} \iiint \left\{ \psi_2 \left[\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial t \partial y} + \frac{\partial^2 \psi_1}{\partial t^2} + \frac{\partial^2 H_1}{\partial t \partial z} + \frac{\partial^2 F_1}{\partial t \partial x} \right] \right. \\ + F_2 \left[\frac{\partial^2 \psi_1}{\partial x \partial t} + \frac{\partial^2 F_1}{\partial t^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 H_1}{\partial x \partial z} + \frac{\partial^2 G_1}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} \right] \\ + G_2 \left[\frac{\partial^2 \psi_1}{\partial y \partial t} + \frac{\partial^2 G_1}{\partial t^2} + \frac{\partial^2 H_1}{\partial y \partial z} - \frac{\partial^2 G_1}{\partial z^2} - \frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y \partial x} \right] \\ \left. + H_2 \left[\frac{\partial^2 \psi_1}{\partial z \partial t} + \frac{\partial^2 H_1}{\partial t^2} - \frac{\partial^2 H_1}{\partial y^2} + \frac{\partial^2 G_1}{\partial z \partial y} + \frac{\partial^2 F_1}{\partial z \partial x} - \frac{\partial^2 H_1}{\partial x^2} \right] \right\} \\ \times dx dy dz. \end{aligned}$$

Making use of the equation :

$$\frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} + \frac{\partial \psi_1}{\partial t} = 0,$$

we obtain

$$W_1 = -\frac{1}{4\pi} \iiint \left\{ \psi_2 \left(\nabla^2 \psi_1 - \frac{\partial^2 \psi_1}{\partial t^2} \right) + F_2 \left(\frac{\partial^2 F_1}{\partial t^2} - \nabla^2 F_1 \right) \right. \\ \left. + G_2 \left(\frac{\partial^2 G_1}{\partial t^2} - \nabla^2 G_1 \right) + H_2 \left(\frac{\partial^2 H_1}{\partial t^2} - \nabla^2 H_1 \right) \right\} dx dy dz.$$

From the electron theory we have

$$\nabla^2 \psi_1 - \frac{\partial^2 \psi_1}{\partial t^2} = -4\pi \rho_1,$$

$$\nabla^2 F_1 - \frac{\partial^2 F_1}{\partial t^2} = -4\pi \rho_1 \frac{u}{c};$$

.

hence

$$W_1 = \iiint \rho_1 \left(\psi_2 - \frac{u F_2}{c} - \frac{v G_2}{c} - \frac{w H_2}{c} \right) dx dy dz \\ = e_1 \left(\psi_2 - \frac{u F_2}{c} - \frac{v G_2}{c} - \frac{w H_2}{c} \right) (33)$$

This is the expression used for the potential energy in Lagrange's equations, and is obtained from Lorentz's electromagnetic force.

The remaining parts of W are

$$W_2 = -\frac{1}{4\pi} \iiint \left[\frac{\partial}{\partial x} (\psi_2 X_1 + G_2 \gamma_1 - H_2 \beta_2) \right. \\ \left. + \frac{\partial}{\partial y} (\psi_2 Y_1 - F_2 \gamma_1 + H_2 \alpha_1) \right. \\ \left. + \frac{\partial}{\partial z} (\psi_2 Z_1 + F_2 \beta_1 - G_2 \alpha_1) \right] dx dy dz \\ - \frac{1}{4\pi} \iiint \frac{\partial}{\partial t} (F_2 X_1 + G_2 Y_1 + H_2 Z_1) dx dy dz.$$

The first integral is a surface one, and vanishes if the expressions in brackets are of the order of $\frac{1}{r^3}$.

The last integral is a volume one, and it represents some additional energy not accounted for by Lorentz's Force.

Chapter III.

Wave equations of an electron
in a five dimensional form.

Includes:

1. Hamilton's Principle and the Field
Equations of radiation. Phil. Mag p 568, 1930
2. Wave Equations of an electron in a
real form. Phil. Mag p 834 1932

A. Introduction

The Five dimensional space;
its origin and main applications.

§. 1. The origin of the conception of five-dimensional space; the geodesic motion of an electron, and union of gravitation and electricity.

The conception of five-dimensional space was advanced for the first time by Kaluza, in order to bring about the welding of gravitation and the electromagnetic field.

According to the General Principle of Relativity a particle in a gravitational field moves, as in the case of free space, along a geodesic; or its motion is a "free" one in a curved space.

In the General Theory of Relativity the law of gravitation is represented by a geometrical property of the space, described by vanishing of

Riemannian contracted tensor.

The reduction of motion under influence of a field of force to a free motion is one of the finest achievements of Einstein's theory of gravitation, and an attempt was soon made to bring into the same scheme the electromagnetic field also.

As the ten components of the fundamental tensor of the fourdimensional space are allotted to gravitation, it is clear that, in order to describe by the same tensor the electromagnetic field also, it is necessary to enlarge somehow the framework of Einstein's space.

It was accordingly suggested by Kaluza that this end could be achieved by increasing the number of dimensions from four to five.

The number of components of the fundamental tensor of the five-dimensional space is fifteen, and, as ten are allotted to gravitation, five independent components remain at our disposal to represent the electromagnetic force; on the other hand, it is known that the latter is described by four components of the vector potential. It is clear, therefore, that such a space might include also the electromagnetic force.

Also, Einstein's gravitational tensor $G_{\mu\nu}$ has in this case fifteen components, five of which remain at our disposal; as a matter of fact, four out of these five tensors give us the first set of Maxwell's equations.

The main result of Kaluza's work, which

soon was developed, and put forward independently in different countries is as follows.

a. The Law of motion.

The fundamental tensor of the five-dimensional space is

$$g^{ik} = g^{ik}, \quad g^{i0} = -\beta \psi^i, \quad g^{00} = 1 + \beta^2 \psi_k \psi^k$$

$$g_{ik} = g_{ik} + \beta^2 \psi_i \psi_k; \quad g_{i0} = \beta \psi_i \quad g_{00} = 1 \quad (1)$$

$$i, k = 1, 2, 3, 4$$

where g is the fundamental tensor of gravitation, and ψ_k is the vector potential; also

$$\beta = \sqrt{2\alpha} \quad (2)$$

where α is Einstein's gravitational constant.

the square of an element of length is

$$d\sigma^2 = g_{\alpha\beta} dx_\alpha dx_\beta \quad (\alpha, \beta = 0, 1, 2, 3, 4) \quad (3)$$



which can also be written

$$d\sigma^2 = d\vartheta^2 + ds^2$$

where

(4)

$$ds^2 = g_{ik} dx^i dx^k; \quad d\vartheta = dx^0 + \beta \varphi_k dx^k$$

and the law of motion of a charged particle in a gravitational and electromagnetic field is given by

$$\delta \int d\sigma = 0 \quad (5)$$

b. The electromagnetic equations.

Let R_{ik} ($i, k = 1, 2, 3, 4$) be the Riemannian contracted tensor and

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R \quad (6)$$

$$R = g^{ik} R_{ik}$$

Einstein's gravitational equations are

$$G^{ik} = -\kappa (T^{ik} + S^{ik}) \quad (7)$$

where T and S are the material and electromagnetic energy tensors respectively.

Let also

$$\begin{aligned} \Delta \text{div}_\beta \Gamma &= \Gamma_{\beta, \alpha}^{\alpha} \quad (\beta, \alpha = 0, 1, 2, 3, 4) \\ \text{div}_i G &= G_{i, k}^k \quad (i, k = 1, 2, 3, 4) \end{aligned} \quad (8)$$

denote respectively the five and four dimensional divergence.

Then, as Kaluza, Klein and others have shown, Einstein's gravitational, and Maxwell's electromagnetic equations are given by

$$\begin{aligned} \Gamma^{ik} &= -\alpha \Theta^{ik} \\ \Gamma_0^k &= -\alpha \Theta_0^k \quad (k = 1, 2, 3, 4) \end{aligned} \quad (9)$$

where Γ is a five dimensional tensor corresponding to

G in four dimensions, and

$$\Theta^{ik} = T^{ik}, \quad \Theta_0^k = \frac{1}{\beta} s^k \quad (10)$$

(i, k = 1, 2, 3, 4)

where s^k is the stream vector.

The first set of equations in (9) is equivalent to (7), and the second set to Maxwell's equations in the General theory of Relativity, which are

$$\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} F^{ik})}{\partial x_k} = S^i \quad (11)$$

(i, k = 1, 2, 3, 4)

The fifth equation has not yet found an application in physics.

According to Klein the fifth component of momentum is equal to

$$p_0 = \frac{e}{\beta c} ;$$

the fifth coordinate conjugate to this momentum is associated with charge.

London has suggested that the fifth coordinate is a conjugate to the moment of momentum $\frac{h}{2\pi}$, due to the spin of an electron.

According to this interpretation, the fifth dimension is associated with the fourth degree of freedom of an electron.

§ 2. The track of an electron as a null geodesic

Although the five-dimensional conception was put forward in order to bring about the welding of gravitation and electromagnetism, it achieved much more in its further development.

Kaluza's result was that the track of an electron is a geodesic; soon, however, Klein and others, have shown that the track of an electron is not only an ordinary geodesic, but a null geodesic.

To this end it is only necessary to assume in the fundamental tensor that

$$\beta = \frac{h}{imc} \quad (13)$$

This result is of certain importance, particularly in connection with the conceptions of wave mechanics.

According to these, a particle is associated with a wave, and the complete study of motion of an electron requires the introduction of a wave equation

Now, the track of a ray of light is a null geodesic, and, if an electron is also a wave, its track ought to be a null geodesic.

This appears to be the case, but only in five-dimensional space. It suggests therefore that this space may be a natural system of reference for radiation phenomena.

§ 3. Schrödinger's Equation

With the advance of wave mechanics attempts have been made to express Schrödinger's equation as a wave equation of five dimensional space.

Klein and others have succeeded in achieving this end.

The equation obtained is

$$\square U = \frac{4\pi^2 m}{h^2} U$$

$$\square U = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{\alpha\beta} \frac{\partial U}{\partial x^\beta} \right) \quad (14)$$

$$(\alpha, \beta = 0, 1, 2, 3, 4)$$

The equation (14) is however not completely similar to the corresponding four dimensional wave equation of light propagation, because of the term on the right hand side.

The next step was, therefore to get rid of this term. This was achieved independently by several writers.

It appears that the same fundamental tensor, which makes the track of an electron a null geodesic enables us to represent Schrödinger's equation as the usual wave equation in four-dimensional space, namely:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{\alpha\beta} \frac{\partial \psi}{\partial x^\beta} \right) = 0$$

B. Derivation of the

Wave Equations.

Hamilton's Principle and the Field Equations of Radiation.
By D. MEKSYN.

§ 1. *Summary.*

THE problem of finding from Hamilton's Principle the most general field laws for an antisymmetric tensor of the second rank in five dimensions is solved.

The tensor has 10 (6+4) components four of which are complex, and two scalar functions are introduced as a result of the variational problem ; in all there are sixteen functions. The sixteen equations obtained are those of radiation.

For the case of free motion and, to the first approximation, for an external electromagnetic field these sixteen equations can be combined into eight (C. G. Darwin's equations)†. For the case of an electromagnetic field these equations are presented in a general tensor form, and the well-known operators

$$p_1 = \frac{h}{2\pi i} \frac{\partial}{\partial x} + \frac{e}{c} A_1 \dots$$

appear quite naturally as terms in contravariant differentiation.

It appears that the five-dimensional continuum represents a natural system of reference for radiation phenomena.

§ 2. *The Method of Solution.*

We have to solve the problem of finding the most general field equations for a particular tensor in space, which follow from Hamilton's Principle.

The method of solution is purely formal, and is equally well applied to 3, 4, and 5 dimensions.

We describe the field in all cases by an antisymmetric tensor of the second rank. For 3, 4, and 5 dimensions we obtain the electrostatic, electromagnetic, and the radiation field equations.

We solve the problem for an antisymmetric tensor of the second rank, because we know that for 3 and 4 dimensions this tensor represents some existing physical state, and, hence, it is natural to inquire whether this is also the case for 5 dimensions. The equations obtained represent a formal generalization of Maxwell's ones.

† Proc. Roy. Soc. A, cxviii.p. 654.

In so far as the solution of the least action problem is concerned, we have to bear in mind the following: if the quantities in the Hamiltonian are differentials, the variational problem can be solved directly (as in dynamics), otherwise these quantities ought to be represented by means of differential coefficients of some other quantities, because without such representation the variational method cannot be applied.

Of course the equations obtained depend, to some extent, upon the form of this representation. We try therefore to find the most general form of representation of an anti-symmetric tensor of the second rank, with the only limitations that these expressions must not conflict with the law of transformation of the particular space.

§ 3. *Application to Three Dimensions.*

In order to make these considerations clear we give here the solution of this problem for the case of three dimensions.

The field is described by a three-vector $E(E_x, E_y, E_z)$. The Hamiltonian is

$$W = \frac{1}{2} \int E^2 dx dy dz. \quad \dots \dots (1)$$

Now Stokes has proved* that a three-vector can be represented by means of scalar and vector potentials $\phi, V(V_x, V_y, V_z)$ as follows:—

$$E = \text{grad } \phi + \text{rot } V, \quad \dots \dots (2)$$

under the condition that

$$\text{div } V = 0. \quad \dots \dots (3)$$

As a matter of fact ϕ and V are found from

$$\nabla^2 \phi = \text{div } E; \quad \nabla^2 V = -\text{rot } E.$$

The field equations are obtained from $\delta W = 0$, using (2) and (3).

We have

$$\int \left(\delta \frac{E^2}{2} + \mu \delta \text{div } V \right) dx dy dz = 0. \quad \dots \dots (4)$$

Using (2), and integrating by parts, we easily find that (4) is equivalent to

$$-\int \{ \text{div } E \cdot \delta \phi + ([\text{grad } \mu - \text{rot } E] \cdot \delta V) \} dx dy dz = 0,$$

* A. E. H. Love, 'A Treatise on the Mathematical Theory of Elasticity,' p. 47 (1920).

and the field equations are

$$\left. \begin{aligned} \operatorname{div} \mathbf{E} &= 0, \\ \operatorname{rot} \mathbf{E} &= \operatorname{grad} \mu. \end{aligned} \right\} \dots \dots \dots (5)$$

It can be shown that all four quantities \mathbf{E} and μ satisfy Laplace's equation.

In (5) there appear, except for the electrostatic vector \mathbf{E} , also a new quantity μ . It will be shown in a separate paper that this quantity represents the potential of a hydrostatic pressure, which is necessary for stability of an electron.

§ 4. *Some Metrical Properties of Five-dimensional Space.*

We give here for convenience' sake a few well-known laws of the five-dimensional space*.

The transformations for the space-time coordinates do not depend upon the fifth dimension, and are the same as in the Theory of Relativity. The transformation of the fifth dimension is merely

$$x_5' = x_5. \dots \dots \dots (6)$$

If we apply these rules to an antisymmetric tensor of the second rank

$$\Gamma^{\nu\mu} = \frac{\partial x_\mu'}{\partial x_\alpha} \frac{\partial x_\nu'}{\partial x_\beta} \Gamma^{\alpha\beta}, \dots \dots \dots (7)$$

we find that the six space-time components are transformed as an antisymmetric tensor of the second rank in the Theory of Relativity, and the four components associated with the fifth dimension

$$\Gamma^{\nu\mu 5} = \frac{\partial x_\mu'}{\partial x_\alpha} \frac{\partial x_\nu'}{\partial x_\beta} \Gamma^{\alpha\beta} = \frac{\partial x_\mu'}{\partial x_\alpha} \Gamma^{\alpha 5} \dots \dots \dots (8)$$

are transformed as a four-vector.

The fundamental tensor is

$$\left. \begin{aligned} \gamma^{ik} &= g^{ik}, & \gamma^{i5} &= -\beta\phi^i, & \gamma^{55} &= 1 + \beta^2\phi_k\phi^k, \\ \gamma_{ik} &= g_{ik} + \beta^2\phi_i\phi_k, & \gamma_{i5} &= \beta\phi_i, & \gamma_{55} &= 1 \end{aligned} \right\} \quad (i, k = 1, 2, 3, 4), \dots \dots \dots (9)$$

where g_{ik} is the gravitational tensor and ϕ_i is the vector-potential of an external electromagnetic field.

* O. Klein, *Zeits. f. Phys.* xlv. p. 189 (1928).

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For β we take the value

$$\beta = \frac{e}{imc^2} \dots \dots \dots (10)$$

For this value of β the track of an electron becomes, as Fock* and Fisher have shown, a null geodesic, and Schrödinger's wave equation appears to be the usual wave equation in this space.

§ 5. *Representation of an Antisymmetric Tensor of the Second Rank by means of Two Vectors.*

The method of derivation of the field equations in the case of five dimensions is similar to the classical one, and is based upon representation of an antisymmetric tensor of the second rank by means of differential coefficients of two five-vectors. As we have pointed out, such representation is necessary in order to solve the variational problem.

Let us find out under what conditions this is possible.

Let $F_{\alpha\beta}$ be an antisymmetric tensor, $k_1 \dots k_5$ and $l_1 \dots l_5$ two five-vectors. $F_{\alpha\beta}$ can be expressed as follows:—

$$F_{\alpha\beta} = \frac{\partial k_\alpha}{\partial x_\beta} - \frac{\partial k_\beta}{\partial x_\alpha} + \frac{\partial l_\delta}{\partial x_\gamma} - \frac{\partial l_\gamma}{\partial x_\delta}, \dots \dots (11)$$

$$\alpha\beta\gamma\delta = (1, 2, 3, 4).$$

The signs in the six equations (11) correspond to an odd number of permutations in the series $\alpha\beta\gamma\delta$. Also

$$F_{\alpha 5} = \frac{\partial k_\alpha}{\partial x_5} - \frac{\partial k_5}{\partial x_\alpha}; \quad G_{\alpha 5} = \frac{\partial l_\alpha}{\partial x_5} - \frac{\partial l_5}{\partial x_\alpha}, \dots (11a)$$

$$\alpha = (1, 2, 3, 4).$$

It appears that, except for $F_{\alpha\beta}$, new quantities $G_{15} \dots G_{45}$ have to be introduced; we can thus consider the fifth dimensional components of $F_{\alpha\beta}$ as complex.

The origin of these quantities is as follows:—In order to justify our form of representation of the tensor $F_{\alpha\beta}$ by means of two vectors k and l , we have to prove that from a given tensor $F_{\alpha\beta}$ we can always discover the vectors k, l (see equations (12)).

In the case of 3 and 4 dimensions k and l satisfy the usual wave equation, and we may expect that this will hold good for the present case.

* *Zeits. f. Phys.* xxxix. p. 22 (1926); J. K. Fisher, *Proc. Roy. Soc. A*, cxxiii. p. 489 (1929).

From the simple algebra of evaluating the equations (12) it follows that, unless we make use of additional quantities $G_{15\dots}$, we are unable to obtain the required equations for k and l .

We have now to prove that if $F_{\alpha\beta}$ and $G_{15\dots}$ are given, the two vectors k and l can be found.

The following expressions are easily obtained :—

$$\left. \begin{aligned} \frac{\partial F_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x_\beta} + \frac{\partial G_{\delta 5}}{\partial x_\delta} &= \square^2 l_\delta - \frac{\partial L}{\partial x_\delta}, \\ \frac{\partial G_{\alpha 5}}{\partial x_\alpha} &= -\square^2 l_5 + \frac{\partial L}{\partial x_5}, \\ \alpha\beta\gamma\delta &= (1, 2, 3, 4). \end{aligned} \right\} \quad (12)$$

The law of composition of the four equations which follow from the first expression in (12) is simple ; the values α, β, γ are any three from 1, 2, 3, 4 taken in order ; the sign in (12) corresponding to an even number of permutations in the series $\alpha\beta\gamma\delta$. Also

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} = \square^2 k_\alpha - \frac{\partial K}{\partial x_\alpha}, \quad \dots \dots (12a)$$

$\alpha, \beta = 1, 2, 3, 4, 5.$

In all the equations the expression must be summed with respect to those indices which occur twice.

In the equations (12) and (12a)

$$\left. \begin{aligned} K &= \text{div } k, \\ L &= \text{div } l, \\ \square^2 &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} + \frac{\partial^2}{\partial x_5^2}. \end{aligned} \right\} \quad (13)$$

We assume

$$K=0, \quad L=0, \quad \dots \dots (14)$$

and the two vectors $k_1\dots k_5, l_1\dots l_5$ can be evaluated from (12) and (12a).

It is clear that not all components of the tensors F and G are independent; the two antisymmetric tensors have fourteen components, whereas we have only eight independent quantities to express them. Hence the components of the antisymmetric tensor must satisfy six additional conditions. They are easily found, and are given later (equations (19)).

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§ 6. *The Variational Problem and the Field Equations.*

The Hamiltonian has in this case the following expression:—

$$W = \frac{1}{2} \int (F_{14}^2 + F_{24}^2 + F_{34}^2 + F_{23}^2 + F_{31}^2 + F_{12}^2 + F_{15}^2 + F_{25}^2 + F_{35}^2 + F_{45}^2 + G_{15}^2 + G_{25}^2 + G_{35}^2 + G_{45}^2) dx_1 dx_2 dx_3 dx_4 dx_5. \quad (15)$$

The field equations are obtained from the condition

$$\delta W = 0. \quad (16)$$

In the evaluation of (16) we have to make use of the equations (11), (11*a*), and (14) after the latter have been multiplied by indeterminate factors μ and λ respectively; the Hamiltonian becomes

$$\int \delta \left(\frac{F_{14}^2 + \dots + G_{45}^2}{2} \right) + \mu \delta K + \lambda \delta L dx_1 \dots dx_5 = 0. \quad (16a)$$

Integrating by parts (16*a*) we easily find the required equations. They are as follows:—

$$\frac{\partial F^{\alpha\beta}}{\partial x_\beta} + \frac{\partial \mu}{\partial x_\alpha} = 0, \quad (17)$$

$\alpha, \beta = 1, 2, 3, 4, 5;$

also

$$\left. \begin{aligned} \frac{\partial F^{\alpha\beta}}{\partial x_\gamma} + \frac{\partial F^{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F^{\gamma\alpha}}{\partial x_\beta} + \frac{\partial G^{\delta\delta}}{\partial x_5} + \frac{\partial \lambda}{\partial x_\delta} &= 0, \\ \alpha\beta\gamma\delta &= (1, 2, 3, 4), \\ \frac{\partial G^{\delta\delta}}{\partial x_i} + \frac{\partial \lambda}{\partial x_5} &= 0, \quad i = 1, 2, 3, 4, 5. \end{aligned} \right\} \quad (18)$$

The first expression in (18) comprises four equations; the signs correspond to an even number of permutations in the series $\alpha\beta\gamma\delta$.

As we have pointed out, there are six additional conditions to be imposed upon the components of the antisymmetric tensor. It is easily verified from (11) and (11*a*) that these conditions are

$$\frac{\partial F_{\alpha\delta}}{\partial x_\beta} - \frac{\partial F_{\beta\delta}}{\partial x_\alpha} + \frac{\partial G_{\delta\delta}}{\partial x_\gamma} - \frac{\partial G_{\gamma\delta}}{\partial x_\delta} - \frac{\partial F_{\alpha\beta}}{\partial x_5} = 0, \quad (19)$$

$\alpha, \beta, \gamma, \delta = (1, 2, 3, 4).$

These are six equations, and the signs in (19) correspond to an odd number of permutations in the series $\alpha\beta\gamma\delta$.

In order to eliminate imaginary quantities from the equations (17-19), and bring them into the same form as

Maxwell's equations, we have merely to bear in mind that the following quantities are purely imaginary:—

$$x_4, F_{15}, F_{25}, F_{35}, G_{45}, F_{12}, F_{31}, F_{23}, \mu. \dots (20)$$

The equations obtained (17-19) can be easily brought into J. M. Whittaker's equations *; his six-vector and two four-vectors correspond to our antisymmetric tensor with fourteen components. Whittaker's way of deriving these equations is different from ours. He assumes eight equations, as given, and derives the other eight ones from Hamilton's principle.

§ 7. We show now that the system of sixteen equations (17-19) can be easily brought into C. G. Darwin's form of Dirac's equations.

We consider the case when there is no electromagnetic or gravitational field. For that case we need not distinguish between covariant and contravariant tensors.

We combine the first three equations of (17) with (19) (for F_{14}, F_{24}, F_{34}) multiplied by $i = \sqrt{-1}$ respectively, and combine the fourth with the fifth equations (17) multiplied also by i ; the same procedure we adopt with respect to equations (18) and (19) (for F_{23}, F_{13}, F_{12}). The result is

$$\left. \begin{aligned} & \left(\frac{\partial}{\partial x_4} + i \frac{\partial}{\partial x_5} \right) (F_{14} - iF_{15}) + \frac{\partial}{\partial x_2} (F_{12} - iG_{35}) \\ & \quad - \frac{\partial}{\partial x_3} (F_{31} - iG_{25}) + \frac{\partial}{\partial x_1} (\mu + iF_{45}) = 0, \\ & \left(\frac{\partial}{\partial x_4} + i \frac{\partial}{\partial x_5} \right) (F_{24} - iF_{25}) - \frac{\partial}{\partial x_1} (F_{12} - iG_{35}) \\ & \quad + \frac{\partial}{\partial x_3} (F_{23} - iG_{15}) + \frac{\partial}{\partial x_2} (\mu + iF_{45}) = 0, \\ & \left(\frac{\partial}{\partial x_4} + i \frac{\partial}{\partial x_5} \right) (F_{34} - iF_{35}) + \frac{\partial}{\partial x_1} (F_{31} - iG_{25}) \\ & \quad - \frac{\partial}{\partial x_2} (F_{23} - iG_{15}) + \frac{\partial}{\partial x_2} (\mu + iF_{45}) = 0, \\ & \left(\frac{\partial}{\partial x_4} - i \frac{\partial}{\partial x_5} \right) (\mu + iF_{45}) - \frac{\partial}{\partial x_1} (F_{14} - iF_{15}) \\ & \quad - \frac{\partial}{\partial x_2} (F_{24} - iF_{25}) - \frac{\partial}{\partial x_3} (F_{34} - iF_{35}) = 0. \end{aligned} \right\} \dots (21)$$

These are exactly Dirac's equations in Darwin's form.

* Proc. Roy. Soc. A, cxxi. p. 543.

Now, a contravariant differentiation with respect to x_k ($k=1, 2, 3, 4$) is, to the first approximation, equivalent to the operator $\frac{h}{2\pi i c} \frac{\partial}{\partial x_k} + \frac{e}{c} \varphi_k$. We see that this operator appears quite naturally in our equations

To present (18) in a tensor form we make use of the same procedure as in the case of the second set of Maxwell's equations in the Special Theory of Relativity. We introduce instead of the tensor $F_{\alpha\beta}^*$ an equivalent tensor $\frac{G^{\alpha\delta}}{\sqrt{g}}$ in the following manner:

$$\left. \begin{array}{l} \text{Instead of} \\ \text{we take} \end{array} \right\} \begin{array}{l} F_{43}, F_{24}, F_{32}, F_{41}, F_{13}, F_{31} \\ G^{12}, G^{13}, G^{14}, G^{23}, G^{24}, G^{34} \end{array} \quad (28)$$

(We consider the case when there is no gravitational field and, hence $g=1$.) We complete thus the missing terms of the tensor $G^{\alpha\beta}$

The equations (18) become

$$g^{\alpha\beta}{}_{,\beta} + \lambda^\alpha = 0. \quad (29)$$

$$(\alpha, \beta = 1-5)$$

and the five equations (29) can be represented in the same form as (27)

Let us transform the set of equations (19). The simplest assumption is (say, for the first equation (19))

$$F_{15,2} - F_{25,1} + (G^4_5)^3 - (G^3_5)^4 - F_{12,5} = 0 \quad (30)$$

where $(G^4_5)^3 = g^{3\alpha} (G^4_5)_\alpha$ means the contravariant differentiation of G^4_5 .

Expressing F_{12} by means of F^{12} , when the gravitational field is absent

$$F^{12} = g^{1\alpha} g^{2\beta} F_{\alpha\beta} = F_{12} + g^{15} F_{52} + g^{25} F_{15}$$

and combining this expression with (30) we obtain

$$(F_{15})^2 - (F_{25})^1 + (G^4_5)^3 - (G^3_5)^4 - F^{12}{}_{,5} = 0 \quad (31)$$

or here again the operators $\frac{h}{2\pi i} \frac{\partial}{\partial x_k} + \frac{e}{c} \psi_k$ appear as in (27).

For the case when there is no gravitation

$$G^c_s = g^{cd} G_{ds} = g^{ci} G_{is} = G_{cs}$$

$$c = 1-4 \quad d = 1-5$$

or the operators are performed upon the same functions as in (27) and (30)

Our equations in a general tensor form are, thus, equivalent to the first order of approximation to Dirac's equations for the case of an external electromagnetic force.

We have, however to bear in mind that the association (28) does not possess a general covariance; it is based upon the following proposition.

In a space of n dimensions there exists an anti-symmetric tensor

$$\frac{\varepsilon^{\alpha\beta\gamma\dots\delta}}{\sqrt{-g}}$$

where the values of $\varepsilon^{\alpha\beta\gamma\dots\delta}$ are $-1, 0, +1$

If $T^{\alpha\beta}$ is an antisymmetric tensor of the second rank, then we have the tensor equation

$$T^{\alpha\beta} = \frac{\varepsilon^{\alpha\beta\gamma\dots\delta}}{\sqrt{-g}} A_{\gamma\dots\delta} \quad (32)$$

where $A_{\gamma\dots\delta}$ is an antisymmetric tensor of rank $n-2$

Now, as A and ε are both antisymmetric and ε has the values -1 and $+1$ the right hand side is equal to const. $\frac{A_{\gamma\dots\delta}}{\sqrt{-g}}$.

We can thus associate antisymmetric tensors in the following way.

$$T^{\alpha\beta} \equiv \frac{A_{\gamma\dots\delta}}{\sqrt{-g}}$$

or, generally speaking: in a space of n dimensions with

an antisymmetric tensor of the rank m can be associated
an antisymmetric tensor of the rank $n-m$.

In the case of four dimensions we obtain (23). In
the case of five dimensions an antisymmetric tensor of
the second rank ought to be associated with a tensor
of rank 3, if we are to adhere to the general
covariance.

We have therefore to assume, in order to retain the
associations (23), that the fifth dimension remains invariant,
and only the space time coordinates possess a general
covariance.

One has, however, to bear in mind that the
invariance of the fifth dimension is not peculiar to our
representation of the wave equations, but it is stipulated
in all the applications of the five dimensional space.

C. Real Solutions of the

Wave Equations.

LXXVII. *Wave Equations of an Electron in a real form.*
By D. MEKSYN, *Ph.D.*, *Mathematical Department, Edinburgh University* *.

§1. *Introduction and Summary.*

AS it is known the wave equations of an electron are imaginary expressions ; so are their solutions. It seems as if imaginary quantities are inherent in the wave mechanics.

In the present paper an attempt is made to present the wave equations and their solutions in a real form.

In a recent publication † I have derived Dirac's equations in a five-dimensional invariant form, and these equations, slightly modified for the case of an external electromagnetic field, may be presented in a real five-dimensional form.

It is proved that in the case when Dirac's equations have imaginary periodic solutions of the form $\psi e^{\frac{2\pi i}{h} Wt}$ our equations have a solution of the form

$$\lambda \cos \frac{2\pi}{h} (mcx_5 - Wt) + \mu \sin \frac{2\pi}{h} (mcx_5 - Wt), \quad (1)$$

where λ and μ are real functions.

Evaluating the energy tensor we obtain the characteristic difference of energies at transitions—namely, we get the following expression :

$$A \cos \frac{2\pi}{h} (W_1 - W_2)t + B \sin \frac{2\pi}{h} (W_1 - W_2)t, \quad (2)$$

* Communicated by the Author.

† *Phil. Mag.* [7] ix. p. 568 (1930).

where A and B are functions of the space variables; the five-dimensional term mcx_5 drops out from the energy tensor.

The real equations solve the difficulty mentioned by Dirac*, namely, that the same relativity wave equations are valid for particles of opposite charges; this is due to the imaginary terms in these equations.

As our equations are real, this difficulty does not arise in our case; the equations and their solutions are different for positive and negative particles.

It seems that the five-dimensional space represents a natural system of reference for the wave equations.

§2. Equations for Free Motion.

The equations are derived from the Hamiltonian Principle (Phil. Mag. *ibid.*). The field is described by an anti-symmetric tensor of the second rank in five dimensions. The tensor has ten components, four of which are complex, and two scalar functions are introduced as a result of the variational problem, in all sixteen equations.

The equations, thus obtained, are analogous to Maxwell's, the latter being merely a four-dimensional projection of the former. They are

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} + \frac{\partial \mu}{\partial x_\alpha} = 0, \quad \alpha, \beta = 1, 2, 3, 4, 5 \dots \quad (3)$$

and

$$\left. \begin{aligned} \frac{\partial F_{\sigma\beta}}{\partial x_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x_\beta} + \frac{\partial G_{\delta 5}}{\partial x_\delta} + \frac{\partial \lambda}{\partial x_\delta} = 0, \\ \frac{\partial G_{5i}}{\partial x_i} + \frac{\partial \lambda}{\partial x_5} = 0, \end{aligned} \right\} \quad (4)$$

$$\alpha, \beta, \gamma, \delta = 1-4, \quad i = 1-5.$$

The expressions (3) and (4) comprise ten equations, and the six remaining are

$$\frac{\partial F_{\alpha\beta}}{\partial x_5} = \frac{\partial F_{\alpha 5}}{\partial x_\beta} - \frac{\partial F_{\beta 5}}{\partial x_\alpha} + \frac{\partial G_{\delta 5}}{\partial x_\gamma} - \frac{\partial G_{\gamma 5}}{\partial x_\delta}, \quad \dots \quad (5)$$

$$\alpha, \beta, \gamma, \delta = 1-4.$$

In order to make these equations real we assume that the following components are purely imaginary:—

$$F_{12}, F_{31}, F_{12}, \mu, F_{15}, F_{25}, F_{35}, G_{45}, x_4 = ict.$$

* Proc. Roy. Soc. A, cxvii. p. 610 (1928).

If we substitute iF_{12}, \dots in (3), (4), (5) instead of F_{12}, \dots we obtain sixteen real equations. They can, however, be combined into eight real equations. It appears that for our purpose eight functions suffice.

We denote

$$\left. \begin{aligned} F_{14} + F_{15} = \phi_1, & \quad F_{24} + F_{25} = \phi_2, \\ & \quad F_{34} + F_{35} = \phi_3, \quad \lambda + G_{45} = -\phi_4, \\ F_{23} - G_{15} = \phi_5, & \quad F_{31} - G_{25} = \phi_6, \\ & \quad F_{12} - G_{35} = \phi_7, \quad \mu + F_{45} = \phi_8, \end{aligned} \right\} \quad (6)$$

and combining in two's the equations (3), (4), and (5) we easily obtain

$$\left. \begin{aligned} \left(-\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_1 + \frac{\partial\phi_7}{\partial x_2} - \frac{\partial\phi_6}{\partial x_3} + \frac{\partial\phi_8}{\partial x_1} &= 0, \\ \left(-\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_2 - \frac{\partial\phi_7}{\partial x_1} + \frac{\partial\phi_5}{\partial x_3} + \frac{\partial\phi_8}{\partial x_2} &= 0, \\ \left(-\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_3 + \frac{\partial\phi_6}{\partial x_1} - \frac{\partial\phi_5}{\partial x_2} + \frac{\partial\phi_8}{\partial x_3} &= 0, \\ \left(-\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_4 - \frac{\partial\phi_5}{\partial x_1} - \frac{\partial\phi_6}{\partial x_2} - \frac{\partial\phi_7}{\partial x_3} &= 0, \\ \left(\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_5 + \frac{\partial\phi_3}{\partial x_2} - \frac{\partial\phi_2}{\partial x_3} + \frac{\partial\phi_4}{\partial x_1} &= 0, \\ \left(\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_6 - \frac{\partial\phi_3}{\partial x_1} + \frac{\partial\phi_1}{\partial x_3} + \frac{\partial\phi_4}{\partial x_2} &= 0, \\ \left(\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_7 + \frac{\partial\phi_2}{\partial x_1} - \frac{\partial\phi_1}{\partial x_2} + \frac{\partial\phi_4}{\partial x_3} &= 0, \\ \left(\frac{\partial}{c\partial t} + \frac{\partial}{\partial x_5}\right)\phi_8 - \frac{\partial\phi_1}{\partial x_1} - \frac{\partial\phi_2}{\partial x_2} - \frac{\partial\phi_3}{\partial x_3} &= 0, \end{aligned} \right\} \quad (7)$$

These equations can be reduced to Darwin's*.

§3. Equations for the Case of an External Electromagnetic Field.

We now consider the case of an external electromagnetic field. For a free electron the equation of momenta is

$$\frac{E^2}{c^2} - p_1^2 - p_2^2 - p_3^2 = m^2c^2. \dots \dots \dots (8)$$

We assume that $mc = p_5$ is the momentum associated with the fifth dimension.

* Proc. Roy. Soc. A, cxviii. p. 658.

The wave function becomes accordingly

$$\psi \sim e^{\frac{2\pi i}{h}(p_1x_1 + p_2x_2 + p_3x_3 + p_5x_5 - Et)}, \dots \quad (9)$$

where E is positive.

Suppose now that we have an external electromagnetic field. In that case we know that instead of $\frac{\partial}{c\partial t}$ and $\frac{\partial}{\partial x_r}$ we have to substitute in our equations the expressions

$$\frac{\partial}{c\partial t} - \frac{2\pi i eV}{h \cdot c} \quad \text{and} \quad \frac{\partial}{\partial x_r} + \frac{2\pi i e}{h \cdot c} A_r.$$

We substitute, however, instead of $\frac{\partial}{c\partial t}$ and $\frac{\partial}{\partial x_r}$, the following forms (for the case of an electron) :

$$\frac{\partial}{c\partial t} - \frac{eV}{mc^2} \frac{\partial}{\partial x_5}, \quad \frac{\partial}{\partial x_r} + \frac{eA_r}{mc^2} \frac{\partial}{\partial x_5}, \quad r=1, 2, 3.. \quad (10)$$

The equations (7) remain real ; the wave functions enter all terms through their differential coefficients of the first order. We can combine the eight equations (7) into Dirac's four equations. We shall have then to solve them by imaginary quantities.

§4. *The Law of Conservation.*

Multiplying (7) by ϕ_1, ϕ_2, \dots and combining the eight equations we easily find

$$\left. \begin{aligned} \rho &= -\frac{1}{c} \left[\frac{\phi_1^2 + \phi_2^2 + \dots \phi_4^2 + \phi_5^2 + \dots \phi_8^2}{2} \right], \\ j_1 &= [\phi_1\phi_8 - \phi_2\phi_7 + \phi_3\phi_6 - \phi_4\phi_5], \\ j_2 &= [\phi_1\phi_7 - \phi_3\phi_5 + \phi_2\phi_8 - \phi_4\phi_6], \\ j_3 &= [\phi_2\phi_5 - \phi_1\phi_6 + \phi_3\phi_8 - \phi_4\phi_7], \\ j_5 &= \frac{eV}{mc^2} \rho + \frac{e}{mc^2} [A_1j_1 + A_2j_2 + A_3j_3] + H, \end{aligned} \right\} \quad (11)$$

where

$$H = \left[\frac{\phi_1^2 + \dots + \phi_4^2 - \phi_5^2 - \dots - \phi_8^2}{2} \right]. \quad (12)$$

The law of "conservation" becomes

$$\frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{\partial \rho}{\partial t} + \frac{\partial j_5}{\partial x_5} = 0. \quad (13)$$

In order that conservation shall take place it is necessary that the last term in (13) shall vanish, or

$$\frac{\partial j_5}{\partial x_5} = 0,$$

or, what comes to the same thing, that the current densities ρ, j_1, j_2, j_3 shall be independent of x_5 .

The fifth component in (11) is rather complicated. For the case of a free electron, if ρ is equal to the energy E , the current is equal to p_1, p_2, p_3 and $j_5 = mc$, where $p_1 \dots$ are the momenta of a moving electron, and m is the invariant mass; thus in this case the fifth component has a simple physical interpretation.

§5. Comparison with Dirac's Equations.

We find now the connexion between the functions ϕ and Dirac's ψ .

Dirac's equations are

$$\left. \begin{aligned} (p_0 + mc)\psi_1 + (p_1 - ip_2)\psi_4 + p_3\psi_3 &= 0, \\ (p_0 + mc)\psi_2 + (p_1 + ip_2)\psi_3 - p_3\psi_4 &= 0, \\ (p_0 - mc)\psi_3 + (p_1 - ip_2)\psi_2 + p_3\psi_1 &= 0, \\ (p_0 - mc)\psi_4 + (p_2 + ip_2)\psi_1 - p_3\psi_2 &= 0, \end{aligned} \right\} \quad (14)$$

where

$$p_0 = -\frac{h}{2\pi i} \frac{\partial}{c \partial t} + \frac{e}{c} V; \quad p_1 = \frac{h}{2\pi i} \frac{\partial}{\partial x_1} + \frac{e}{c} A_1. \quad (15)$$

Multiplying the first equation (7) by i , and subtracting from the second, we find that the obtained expression is equivalent to the second equation in (14) (in our expressions $\frac{\partial}{\partial x_5}$ appears instead of mc). Performing similar transformations with the remaining equations (7), and comparing the obtained results with (14), we arrive at the following connexion between the functions ϕ and ψ :-

$$\left. \begin{aligned} \psi_1 &= \phi_4 - i\phi_3, & \psi_2 &= \phi_2 - i\phi_1, \\ \psi_3 &= -\phi_1 - i\phi_2, & \psi_4 &= -\phi_5 - i\phi_6 \end{aligned} \right\} \quad (16)$$

§6. Solution of the Equations.

We shall now prove that in all cases, when Dirac's equations have imaginary periodic solutions, the real equations have real periodic solutions.

We consider the general case of an external electromagnetic field, and assume that the periodic solutions depend upon time in the following way:—

$$\phi \sim \frac{\sin \alpha}{\cos \alpha}, \quad \alpha = \frac{2\pi}{h}(mcx_3 - Wt). \quad \dots (17)$$

A glance at the equations (7) and (10) shows that in the general case they cannot be satisfied by a simple sine or cosine solution, but must be a combination of both.

We assume accordingly for the ϕ 's the following expressions:—

$$\left. \begin{aligned} \phi_1 &= -\lambda_2 \sin \alpha - \mu_2 \cos \alpha, & \phi_5 &= -\lambda_4 \cos \alpha + \mu_4 \sin \alpha, \\ \phi_2 &= \lambda_2 \cos \alpha - \mu_2 \sin \alpha, & \phi_6 &= -\lambda_4 \sin \alpha - \mu_4 \cos \alpha, \\ \phi_3 &= -\lambda_1 \sin \alpha - \mu_1 \cos \alpha, & \phi_7 &= -\lambda_3 \cos \alpha + \mu_3 \sin \alpha, \\ \phi_4 &= \lambda_1 \cos \alpha - \mu_1 \sin \alpha, & \phi_8 &= -\lambda_3 \sin \alpha - \mu_3 \cos \alpha, \end{aligned} \right\} \dots (18)$$

Inserting these values of ϕ in (7), and equating to zero separately the cofactors of $\sin \alpha$ and $\cos \alpha$, we obtain sixteen equations for the eight functions λ and μ ; there are, however, only eight different equations, and they can be combined into four, which are found to be Dirac's equations.

Thus we obtain the following connexion between Dirac's functions ψ and the λ and μ :

$$\psi_k = \lambda + i\mu_k, \quad k=1-4, \dots (19)$$

or the λ 's and μ 's are the real and imaginary parts of the ψ 's.

§7. The Energy Tensor.

We evaluate now the energy tensor, and show that we obtain the characteristic difference of energies at transitions. This is reached in wave mechanics by making use of imaginary quantities.

Let us consider only two terms in the expression ρ in (11), as the other terms give similar results.

For the case of transitions the terms $\phi_1^2 + \phi_2^2$ in ρ become

$$\phi_1 \phi_1^1 + \phi_2 \phi_2^1, \dots (20)$$

where ϕ and ϕ^1 belong to different states.

Inserting in (20) the values of the ϕ 's from (18) we easily obtain

$$\begin{aligned} \phi_1\phi_1^1 + \phi_2\phi_2^1 &= (\lambda_2\lambda_2^1 + \mu_2\mu_2^1) \cos(\alpha - \alpha^1) \\ &\quad + (\lambda_2\mu_2^1 - \mu_2\lambda_2^1) \sin(\alpha - \alpha^1) \\ \alpha^1 &= \frac{2\pi}{h}(mcx_5 - W^1t), \dots \dots \dots (21) \end{aligned}$$

and hence

$$\alpha - \alpha^1 = \frac{2\pi}{h}(W^1 - W)t. \dots \dots \dots (22)$$

This is the correct expression for the energies of transitions. As we see, the five-dimensional term mcx_5 drops out from the energy tensor.

It can be easily shown that our energy tensor is identical with Dirac's.

§8. *The Hydrogen Atom.*

We consider now the case of the hydrogen atom, in order to give an instance of a certain peculiarity inherent in these equations.

It is known that a law of conservation can be derived from the imaginary wave equations, and every solution of these equations satisfies the law of conservation.

Now the position appears to be somewhat different in the case of the real equations. A law of conservation follows also from these equations; it is, however, expressed in a five-dimensional form, which represents a conservation in the usual sense only if the fifth term vanishes.

We have therefore to discard all solutions which do not satisfy the condition

$$\frac{\partial j_5}{\partial x_5} = 0. \dots \dots \dots (23)$$

Let us now show the bearing of these considerations upon the solution of the problem of the hydrogen atom.

In that case $A_1 = A_2 = A_3 = 0$, and a glance at the equations (7) shows that they can be satisfied by assuming

$$\left. \begin{aligned} \phi_1, \phi_2, \phi_3, \phi_4 &\sim \sin \frac{2\pi}{h}(mcx_5 - Wt), \\ \phi_5, \phi_6, \phi_7, \phi_8 &\sim \cos \frac{2\pi}{h}(mcx_5 - Wt). \end{aligned} \right\} \dots \dots \dots (24)$$

Now, evaluating the energy tensor for this solution we obtain that

$$j_5 \sim \sin \frac{2\pi}{h} (mcx_5 - W_1 t) \cos \frac{2\pi}{h} (mcx_5 - W_2 t). \quad (25)$$

Thus the fifth dimension does not drop out; we do not obtain the difference of energies at transitions.

As in this case, however, $\frac{\partial j_5}{\partial x_5} \neq 0$, we have to conclude that the solution (24) does not represent a possible physical state.

Let us now give briefly the correct solution for this case. Dirac's equations have been rigorously solved for the hydrogen atom by Prof. Darwin*. We make use of Darwin's results.

The functions ψ 's have the following expressions (Darwin's equations 7.4) :—

$$\left. \begin{aligned} \psi_1 &= -iF_k R_{k+1}^u [\cos u\phi + i \sin u\phi], \\ \psi_2 &= -iF_k R_{k+1}^{u+1} [\cos (u+1)\phi + i \sin (u+1)\phi], \\ \psi_3 &= (k+u+1)G_k R_k^u [\cos u\phi + i \sin u\phi], \\ \psi_4 &= (-k+u)G_k R_k^{u+1} [\cos (u+1)\phi + i \sin (u+1)\phi], \end{aligned} \right\} (26)$$

where F_k and G_k are certain functions of r , and

$$R_k^u = (k-u)! \sin^u \theta \left(\frac{d}{d \cdot \cos \theta} \right)^{k+u} \frac{(\cos^2 \theta - 1)^k}{2^k \cdot k!}.$$

From (26) and (19) we obtain the values of λ and μ , and inserting the obtained expressions of λ and μ in (18) we get the required solutions of our equations.

So, for instance,

$$\lambda_1 = F_k R_{k+1}^u \sin u\phi, \quad \mu_1 = F_k R_{k+1}^u \cos u\phi;$$

hence

$$\begin{aligned} \phi_3 &= F_k R_{k+1}^u \cos \left[u\phi + \frac{2\pi}{h} (mcx_5 - Wt) \right], \\ \phi_4 &= F_k R_{k+1}^u \sin \left[u\phi + \frac{2\pi}{h} (mcx_5 - Wt) \right]. \end{aligned}$$

§9. The Wave Equations for Positive Particles.

Two difficulties, as Dirac † has pointed out, are inherent in the relativity wave equations.

* Proc. Roy. Soc. A, cxviii. p. 658.

† Proc. Roy. Soc. A, cxvii. p. 610.

(a) They have twice as many solutions as appear to be necessary. The solutions can be schematically represented as

$$\left. \begin{aligned} (1) \psi e^{-\frac{2\pi i}{h} Wt}, & \quad \bar{\psi} e^{\frac{2\pi i}{h} Wt}, \\ (2) \psi e^{\frac{2\pi i}{h} Wt}, & \quad \bar{\psi} e^{-\frac{2\pi i}{h} Wt}, \end{aligned} \right\} \dots (27)$$

where ψ stands for all four functions and $\bar{\psi}$ is the conjugate of ψ .

The second set of solutions is associated with negative kinetic energy.

The appearance of negative energy is inherent in any relativity theory. It appears also in the classical theory, but there the kinetic energy changes continuously, and, as it is initially positive, it cannot become negative.

In the quantum theory discontinuous transitions are possible, and it is therefore not easy to separate these two sets of solutions.

Schrödinger*, however, has recently suggested how to separate the positive and negative solutions.

(b) The relativity imaginary equations have another difficulty; they are valid both for electrons and protons. It can be shown that the conjugates of (14) are equivalent to the same equations where e is changed into $-e$.

Let us elucidate this point by considering the conjugate of the fourth equation (14). It is equal to

$$\left(-\frac{h}{2\pi i c} \frac{\partial}{\partial t} - \frac{e}{c} V + mc \right) \bar{\psi}_4 + \left[\left(\frac{h}{2\pi i} \frac{\partial}{\partial x} - \frac{e}{c} A_1 \right) - i \left(\frac{h}{2\pi i} \frac{\partial}{\partial y} - \frac{e}{c} A_2 \right) \right] \bar{\psi}_1 - \left(\frac{h}{2\pi i} \frac{\partial}{\partial z} - \frac{e}{c} A_3 \right) \bar{\psi}_2 = 0. \quad (28)$$

We now consider the first equation for a positive particle; we have to change e into $-e$, and we obtain

$$\left(-\frac{h}{2\pi i c} \frac{\partial}{\partial t} - \frac{e}{c} V + mc \right) \psi_1^+ + \left[\left(\frac{h}{2\pi i} \frac{\partial}{\partial x} - \frac{e}{c} A_1 \right) - i \left(\frac{h}{2\pi i} \frac{\partial}{\partial y} - \frac{e}{c} A_2 \right) \right] \psi_4^+ + \left(\frac{h}{2\pi i} \frac{\partial}{\partial z} - \frac{e}{c} A_3 \right) \psi_3^+ = 0. \quad (29)$$

Comparing (29) with (28) we find that

$$\begin{aligned} & \psi_1^+, \psi_2^+, \psi_3^+, \psi_4^+ \\ \text{are equivalent to} & \quad \bar{\psi}_4, -\bar{\psi}_3, -\bar{\psi}_2, \bar{\psi}_1, \end{aligned}$$

* *Berliner Berichte*, p. 63 (1931).

or the equations for the proton are equivalent to the conjugate equations for an electron.

(c) Let us now consider the case of the real equations. Instead of Dirac's four complex functions we have eight real ones, with the corresponding positive and negative solutions :—

$$\left. \begin{aligned} \phi &\sim \frac{\sin}{\cos} \left[\omega + \frac{2\pi}{h} (mcx_5 - Wt), \right] \\ \phi &\sim \frac{\sin}{\cos} \left[\omega + \frac{2\pi}{h} (mcx_5 + Wt), \right] \end{aligned} \right\} \dots (30)$$

and

$$\left. \begin{aligned} \phi &\sim \frac{\sin}{\cos} \left[\omega - \frac{2\pi}{h} (mcx_5 - Wt), \right] \\ \phi &\sim \frac{\sin}{\cos} \left[\omega - \frac{2\pi}{h} (mcx_5 + Wt), \right] \end{aligned} \right\} \dots (31)$$

where (30) relates to an electron and (31) to a proton; ω is a function of the space variables. We see that not only the equations but also the solutions are different for electrons and protons.

Dirac* has suggested that the negative solutions of an electron could be associated with a proton.

As we see from (30) and (31) this interpretation of negative solutions cannot be justified in our case because the negative solutions of (30) have a different form from (31).

To pass from (30) to (31) we have to change the sign not of W , but of m ; this, however, is equivalent in our equations to changing the sign of e .

§10. On the Meaning of the Fifth Dimension.

It is known that the idea of the fifth dimension was advanced by Kaluza in order to bring about the welding of gravitation and the electromagnetic field; but this conception appears to be very useful in wave-mechanics; the latter gives us a definite interpretation of the fifth coordinate.

An easy way of introducing the fifth dimension is to follow the same procedure as in the case of four dimensions.

As it is known the four-dimensional principle of relativity is based upon the law of constancy of velocity of light

* Proc. Roy. Soc. A, cxxvi. p. 360 (1930).

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propagation. If c is the velocity of light, the transformation of coordinates has to leave invariant the expression

$$\frac{ds}{dt} = c, \quad \dots \dots \dots (32)$$

or

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0; \quad \dots \dots \dots (33)$$

this leads to Lorentz's transformations.

In the case of five dimensions we start from the fundamental equation of the quantum theory,

$$h\nu = E. \quad \dots \dots \dots (34)$$

As de Broglie has pointed out, the transformation of this equation for moving systems does not conform with the usual principle of relativity; this led him to the wave conception of an electron.

Let us give a metrical representation of the equation (34). We have

$$h\nu = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0 c^2 dt}{\sqrt{dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}}}. \quad \dots (35)$$

Let

$$\frac{m_0 c^2}{h} = \nu_0 = \frac{1}{T_0}, \quad \nu = \frac{1}{T}; \quad \dots \dots (36)$$

we find from (35)

$$c^2 dt^2 - dx^2 - dy^2 - dz^2 - dx_5^2 = 0, \quad \dots \dots (37)$$

where

$$dx_5 = \frac{cT}{T_0} dt, \quad \dots \dots \dots (38)$$

or Planck's frequency condition, applied to an electron in four dimensions, can be considered as a "wave" in a five-dimensional space. From (38) we see that the fifth coordinate describes the periodic phenomenon associated with energy. From (37) we obtain the law of transformation for the fifth coordinate; it is invariant with respect to Lorentz's transformations.

I wish to express my thanks to Prof. Darwin for invaluable criticism of this paper and for many useful suggestions.

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