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T h e s i s

ON THE DIVERGENCE DIFFICULTY OF QUANTIZED FIELD THEORIES AND THE RIGOROUS TREATMENT OF RADIATION REACTION

with related additional papers

Submitted by

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C O N T E N T S

(with summaries)

THESIS

On the divergence difficulty of quantized field theories and the rigorous treatment of radiation reaction

(By an orthodox application of the perturbation theory to the general case of a quantized field, it is shown that the divergence difficulty hitherto encountered arises from a faulty application of the expansion method. The difficulty disappears if the degeneracy of the unperturbed system is properly treated by the method of secular perturbation. Physically, it is shown that this amounts to a rigorous treatment of the radiation reaction)

ADDITIONAL PAPERS

Group H

Anomalous scattering of mesons (with Heitler)

On the production of mesons by proton-proton collisions (with Heitler)

On the production of mesons by light quanta and related processes (with Hamilton)

On the cascade production of mesons

Theory of cosmic-ray mesons (with Hamilton and Heitler)

(By using Heitler's and Wilson's theory of radiation damping as a working hypothesis, the effects of the radiation reaction are examined for various processes involving the scattering and production of mesons. The physical consequences are compared with cosmic-ray measurements. It is thus concluded that the theory of radiation damping, though hypothetical, will be a preliminary treatment of the radiation reaction. This conclusion is now verified by the THESIS)

Group B

Quantum mechanics of fields. I. Pure fields (with Born)

Statistical Mechanics of fields and the 'apeiron' (with Born)

Quantum mechanics of fields. II. Statistics of pure fields (with Born)

Quantum mechanics of fields. III. Electromagnetic field and electron field in interaction (with Born)

(A new method of field quantization is developed, which leads to a generalization of Heisenberg and Pauli's usual method. The new method admits an arbitrary distribution of apeirons, while the usual method admits only the uniform distribution of apeirons. The apeiron distribution can be chosen so as to overcome the divergence difficulty, but this procedure appears at the moment arbitrary and speculative. Accordding to the THESIS, such generalization may not be necessary)

Group P

The divergence difficulty of quantized field theories

(A short note contains the fundamental ideas of the THESIS)

On the divergence difficulty of quantized field theories and the rigorous treatment of radiation reaction

By H. W. Peng, University of Edinburgh

By an orthodox application of the perturbation theory to the general case of a quantized field, it is shown that the divergence difficulty hitherto encountered arises from a faulty application of the expansion method. The difficulty disappears if the degeneracy of the unperturbed system is properly treated by the method of secular perturbation. Physically, it is shown that this amounts to a rigorous treatment of the radiation reaction.

The existence of elementary particles (photons, electrons, etc.) in integral numbers has been successfully explained by the application of quantum mechanics for the treatment of fields (Maxwell's field, Dirac(1927), Jordan and Pauli(1928); Dirac's field, Jordan and Wigner(1928); etc.). The interaction of the elementary particles is usually introduced by coupling the corresponding fields, for which a general description will be given in §1 below. The effects (self-energies and cross-sections) due to the interaction have been hitherto treated by the expansion method - a power series expansion with respect to the interaction constants. All terms of the resultant expansions for the self-energies and the cross-sections, excepting the term of the lowest degree in each expansion, are found to diverge. This difficulty has led many physicists to doubt either the usual formulation of the field theory or the usual procedure of quantization (which is generally known as that of Heisenberg and Pauli(1929)). But it will be shown in §2 below that for any quantized field, the system is degenerate in the sense of the perturbation theory and hence the expansion method must give way to the method of secular perturbation. Therefore no premature departure from the usual formulation of the quantized field theories will be considered in this paper. On the other hand, the mathematical treatment of secular perturbation which will be developed in §§3-5 will have to be used for the derivation of the self-energies and cross-sections in any new formulation of the quantized field theories.

In §3, the usual method of secular perturbation for degenerate systems will be presented in a form which can easily be adapted to the case of continuous spectrum. The general solution of the secular problem will be given in §§4-5, where it will be seen that most of the stationary states of the secular problem describe collisions of the elementary particles. For the calculation of the cross-sections for the collisions, one has to solve some integral equations which differ from those proposed by Heitler(1941) and Wilson(1941) in their theory of radiation damping merely by not neglecting or omitting the self-energies. (Their treatment is only preliminary and is integrated in the present treatment). The present treatment, which in fact is just the orthodox perturbation treatment and can be continued to any desired order of approximation, can therefore be

described physically as a <u>rigorous</u> treatment of the radiation reaction.
The self-energies are also obtained from the stationary
states of the secular problem, and are influenced by
the radiation reaction. As will be demonstrated in
§6 with the help of a simple example, the present
treatment leads to no divergence difficulty where the
usual treatment does.

1. General description of the interaction of elementary particles by means of a quantized field

All elementary particles can be shown to arise from the corresponding field thoeries by a definite procedure called 'quantization' (cf. Dirac(1935)Chap.XIII, and Pauli(1941)PartII). To account for the interaction between elementary particles of different nature (e.g. photons and electrons), the various fields are combined into an <u>inseparable whole</u> by including coupling terms between the different fields (e.g. Maxwell's field and Dirac's field). In the usual theory, particles of the same nature (e.g. two electrons) interact only indirectly via their interaction with particles of other nature(e.g. photons) in accordance with a similar idea in the classical theory, namely the action through a medium. In what follows, I shall always consider the whole field, including the interactions.

Experimentally, all phenomena concerning the interaction of the elementary particles are invariably studied in a volume which (being at least of atomic dimensions) is practically infinite in comparison with any effective volume of the elementary particles (as derived from the observed cross-sections of their collisions). Hence, for simplicity, one may suppose the whole field be periodic, the basis being an <u>arbitrarily</u> large 'fundamental' cube of sides L, because this does not affect the asymptotical results for large values of L.

The fundamental property of a field is, in the classical theory, the localization of the energy, momentum, and charge. This arises from the fact that, as the consequences of the field equations, an energy-momentum-stress tensor $T_\mu^{\ \nu}$ and a charge-current vector j^ν exist at every point $\underline{x} = (x^1, x^2, x^3)$ and time $t = x^0$ which obey the equations of continuity

$$\sum_{\nu=0}^{3} \frac{\partial T_\mu^{\ \nu}}{\partial x^\nu} = 0 \quad (\mu = 0, 1, 2, 3), \quad \sum_{\nu=0}^{3} \frac{\partial j^\nu}{\partial x^\nu} = 0. \tag{1.1}$$

By integrating these equations over the fundamental cube, one obtains the conservation laws of the total energy H, the total momentum $\underline{G} = (G_1, G_2, G_3)$, and the total charge Q contained in the cube

$$H = \iiint T_4^{\ 4} dx^1 dx^2 dx^3, \quad G_1 = \iiint T_1^{\ 4} dx^1 dx^2 dx^3, \ \ldots, \quad Q = \iiint j^4 dx^1 dx^2 dx^3 \tag{1.2}$$

which are therefore constant in time. If the field equations can be derived from a variational principle, it can be shown that $T_1^{\ 4}$, $T_2^{\ 4}$, $T_3^{\ 4}$ and j^4 are always bilinear expressions in the field quantities while $T_4^{\ 4}$ contains besides bilinear terms also terms of higher degree (Pauli(1941)Part I, §2). This is also the case if some field equation has to be regarded as supplementary to the fundamental equations, e.g. in the case of to the variational equations, e.g. in the case of Maxwell's field and Dirac's field in interaction (Born and Peng(1944) §3).

Let H^0, H^1, ... denote the parts of H which arise from the bilinear, trilinear, ... terms of $T_4{}^4$ respectively,

$$H^0 = \iiint (\text{bilinear terms of } T_4{}^4)\, dx^1 dx^2 dx^3, \tag{1.3}$$

$$H^1 = \iiint (\text{trilinear terms of } T_4{}^4)\, dx^1 dx^2 dx^3, \quad \ldots \tag{1.4}$$

In general, any field quantity (say f) is complex; real field quantities may be considered as special cases. By a three-dimensional Fourier transformation of all the field quantities,

$$f(\underline{x},t) = L^{-3/2} \sum_k f_k(t) e^{i\underline{k}\cdot\underline{x}}, \quad f^*(\underline{x},t) = L^{-3/2} \sum_k f_k^*(t) e^{-i\underline{k}\cdot\underline{x}}, \tag{1.5}$$

where the summation of the wave-vectors $\underline{k}$ extends over the lattice points of the infinite k-lattice (a cubic lattice of the lattice constant L^{-1}), the volume integration in H^0, $\underline{G}$, and Q is transformed into the lattice summation of the wave-vectors. The contribution from each wave-vector $\underline{k}$, being a Hermitian bilinear form of the corresponding f_k's and f_k^*'s of all the field quantities, can further be reduced to the canonical form by a linear transformation. One thus obtains

$$H^0 = \sum_k \sum_\sigma \varepsilon_{k\sigma} a_{k\sigma} a_{k\sigma}^*, \quad \underline{G} = \sum_k \sum_\sigma \underline{p}_{k\sigma} a_{k\sigma} a_{k\sigma}^*, \quad Q = \sum_k \sum_\sigma e_\sigma a_{k\sigma} a_{k\sigma}^*, \tag{1.6}$$

where there are as many $a_{k\sigma}$ and $a_{k\sigma}^*$, distinguished by the index σ, as the number of linearly independent

f_k's and f_k^*'s. The normalization factor $L^{-3/2}$ has been introduced in (1.5) to destroy the factor L^3 arising from the volume integration so that H^0, $\underline{G}$, and Q contain no factor in L. Since each additional field quantity brings in a factor $L^{-3/2}$, one has for $H^1, H^2, \ldots$

$$H^1 = L^{-3/2}(\text{trilinear form of all } a_{k\sigma} \text{ and } a_{k\sigma}^*), \tag{1.7}$$

$$H^2 = L^{-3}(\text{quadralinear form of all } a_{k\sigma} \text{ and } a_{k\sigma}^*), \ldots \tag{1.8}$$

where the a's and a*'s of different $k\sigma$ are intermingled in some definite way. In the usual theory, H^1 is linear in such natural constants as the elementary charge or the mesonic charge of a nucleon, while H^2 is bilinear in these interaction constants, and so on. The series of terms of increasingly higher degree in the a's and a*'s,

$$H = H^0 + H^1 + H^2 + \cdots, \tag{1.9}$$

can thus be regarded as an expansion of H with respect to the interaction constants.

In the quantum thoery, the field quantities are replaced by q-number quantities obeying non-commutative abgebra, while the four co-ordinates $\underline{x}$, t are retained as c-number quantities. The Fourier transformation (1.5) can be used for q-number field quantities $f(\underline{x},t)$ and q-number Fourier amplitudes $f_k(t)$ where the wave-vectors $\underline{k}$ are retained like $\underline{x}$ as c-number quantities. The $a_{k\sigma}$ and $a_{k\sigma}^*$, which are the linearly independent combinations of all the $f_k(t)$ and $f_k^*(t)$, are now q-number quantities. The total energy H of the system —

namely the whole field in the fundamental cube — plays the rôle of the Hamiltonian. The quantum conditions between the a's and the a*'s are determined by the condition that the quantum-mechanical equations of motion in the Heisenberg representation, viz.

$$\frac{\hbar}{i}\frac{\partial a_{k\sigma}}{\partial t} = H a_{k\sigma} - a_{k\sigma} H, \quad \frac{\hbar}{i}\frac{\partial a^*_{k\sigma}}{\partial t} = H a^*_{k\sigma} - a^*_{k\sigma} H, \tag{1.10}$$

should be formally the same as those ~~what are~~ obtained from the field equations in the classical theory. (The asterisk, when used with a q-number quantity, denotes the adjoint). In the linear transformation from the f_k and f^*_k to the $a_{k\sigma}$ and $a^*_{k\sigma}$ it is convenient to introduce a factor $\hbar^{-1/2}$ so that the quantum conditions between the a's and the a*'s contain no longer $\hbar$, viz.

$$a^*_{k\sigma} a^*_{k'\sigma'} \mp a^*_{k'\sigma'} a^*_{k\sigma} = a_{k\sigma} a_{k'\sigma'} \mp a_{k'\sigma'} a_{k\sigma} = 0,$$
$$a^*_{k\sigma} a_{k'\sigma'} \mp a_{k'\sigma'} a^*_{k\sigma} = \delta_{kk'}\delta_{\sigma\sigma'}. \tag{1.11}$$

The + sign (anti-commutation laws) is used only if <u>both</u> $a_{k\sigma}$ (or its adjoint) and $a_{k'\sigma'}$ (or its adjoint) are derived from field quantities which transform for the Lorentz transformation like spinors. The - sign (commutation laws), on the other hand, is used where at least one of these quantities is derived from field quantities which transform for the Lorentz transformation like tensors (including vectors and scalars).

As consequences of the quantum conditions, the q-number quantities $a_{k\sigma} a^*_{k\sigma}$ for various $k\sigma$ all commute. Their eigenvalues $N_{k\sigma}$ are independently 0,1,2,... in the case of commutation laws being used, or 0 and 1 in the case of anti-commutation laws being used. Since G_1, G_2, G_3, Q and H^0 of (1.6) are linear combinations of the commuting quantities $a_{k\sigma} a^*_{k\sigma}$ with c-number coefficients, they all commute. In fact, $\underline{G}$ and Q commute with the Hamiltonian H, which expresses the conservation laws of the total momentum and charge in quantum theory. In the following, the total momentum and charge of the system will be assumed to be known.

After having obtained the quantum conditons in the Heisenberg representation, it is practical to change to the Schroedinger representation in which all q-number quantities are represented by linear operators constant in time (Dirac(1935) §§31-32). The operand Ψ, called the wave function, describes the quantum <u>state</u> of the system and varies with time according to Schroedinger's wave equation in time‡

$$-\frac{\hbar}{i}\frac{\partial \Psi}{\partial t} = H\Psi \quad \text{where } \Psi \text{ is short for } \Psi(N_{k\sigma};t). \qquad (1.12)$$

‡The treatment described in the text is the usual short-cut way of applying Heisenberg and Pauli's method of quantization in the k-representation. The complete treatment which includes also the quantum-mechanical equations of motion in space can easily be reduced to the above simple treatment (cf. Born and Peng(1944) §4). In the complete treatment, the wave function $\Psi(N_{k\sigma};\underline{x},t)$ satisfies the wave equations in time and space,

$$-\frac{\hbar}{i}\frac{\partial\Psi(N_{k\sigma};\underline{x},t)}{\partial t} = H\Psi(N_{k\sigma};\underline{x},t), \quad \frac{\hbar}{i}\operatorname{grad}\Psi(N_{k\sigma};\underline{x},t) = \underline{G}\,\Psi(N_{k\sigma};\underline{x},t).$$

If the total momentum is known, say $\underline{G}'$, the complete treatment reduces to the simple treatment by putting

$$\Psi(N_{k\sigma};\underline{x},t) = L^{-3/2}\, e^{i\underline{G}'\cdot\underline{x}/\hbar}\;\Psi(N_{k\sigma};t).$$

The eigenvalues $N_{k\sigma}$ of $a_{k\sigma}a^*_{k\sigma}$ for all $k\sigma$ are adopted as the arguments of Ψ on which all $a_{k\sigma}$ and $a^*_{k\sigma}$ operate. For a set of basic functions of all $N_{k\sigma}$ one can use the simultaneous eigenfunctions of all $a_{k\sigma}a^*_{k\sigma}$, viz.

$$\varphi^{(0)}_n(N_{k\sigma}) = \prod_k \prod_\sigma \delta_{n_{k\sigma} N_{k\sigma}}, \tag{1.13}$$

where n denotes in short a set of quantum numbers $\{n_{k\sigma}\}$ with all $k\sigma$. These functions, for all n, form a complete system of linearly independent functions of the N's, and are orthonormal:

$$(\varphi^{(0)}_n, \varphi^{(0)}_m) \equiv \sum_{N_{k\sigma}} \varphi^{(0)*}_n(N_{k\sigma}) \varphi^{(0)}_m(N_{k\sigma}) = \prod_k \prod_\sigma \delta_{n_{k\sigma} m_{k\sigma}} \equiv \delta_{nm}. \tag{1.14}$$

As the zero-order approximation, it is convenient to introduce an idealized system, which is described by the simpler Hamiltonian H^0, with the same $\underline{G}$ and Q as before, see (1.6). Evidently, the functions (1.13) form a set of eigenfunctions for this idealized system.

$$H^0 \varphi^{(0)}_n = E^{(0)}_n \varphi^{(0)}_n, \quad \underline{G} \varphi^{(0)}_n = \underline{G}_n \varphi^{(0)}_n, \quad Q \varphi^{(0)}_n = Q_n \varphi^{(0)}_n \tag{1.15}$$

where $\varphi^{(0)}_n$ is short for $\varphi^{(0)}_n(N_{k\sigma})$, with the eigenvalues

$$E^{(0)}_n = \sum_k \sum_\sigma \varepsilon_{k\sigma} n_{k\sigma}, \quad \underline{G}_n = \sum_k \sum_\sigma \underline{p}_{k\sigma} n_{k\sigma}, \quad Q_n = \sum_k \sum_\sigma e_\sigma n_{k\sigma}. \tag{1.16}$$

The solution $\Phi^{(0)}$, short for $\Phi^{(0)}(N_{k\sigma}; t)$, of Schroedinger's wave equation for the idealized system is simple; the general solution being a superposition of the stationary-state solutions $\Phi^{(0)}_n$ with arbitrary constants.

$$-\frac{\hbar}{i} \frac{\partial \Phi^{(0)}}{\partial t} = H^0 \Phi^{(0)}, \quad \Phi^{(0)}_n = e^{-iE^{(0)}_n t/\hbar} \varphi^{(0)}_n(N_{k\sigma}). \tag{1.17}$$

The peculiar form of the eigenvalues (1.16) suggests the usual interpretation that, for the state $\varphi_n^{(0)}$ characterized by the set of quantum numbers $\{n_{k\sigma}\}$, there are $n_{k\sigma}$ quanta of the kind $k\sigma$ within the fundamental cube, each quantum of the kind $k\sigma$ possessing the energy $\varepsilon_{k\sigma}$, the momentum $\underline{p}_{k\sigma}$, and the charge e_σ. The index σ combines the description of the nature of the quantum (electron-like, photon-like, etc.) with its spin orientations ($2s + 1$ orientations for a quantum of spin s) and electric charge (positive, negative or neutral according to $e_\sigma = +e, -e$, or 0), while the index k describes the momentum of the quantum ($\underline{p}_{k\sigma} = \pm\hbar\underline{k}$ if $e_\sigma = \pm e$, $\underline{p}_{k\sigma} = \hbar\underline{k}$ if $e_\sigma = 0$). The energy $\varepsilon_{k\sigma}$ of the quantum, of the form $+\sqrt{c^2 p_{k\sigma}^2 + m_\sigma^2 c^4}$, is positive and independent of the spin orientations of the quantum and the directions of the momentum of the quantum. (It depends, however, on the nature of the quantum through the rest-energy $m_\sigma c^2$ of the quantum). It follows from this interpretation that, excepting only the vacuum-like state with all $n_{k\sigma} = 0$, one can also specify the state $\varphi_n^{(0)}$ by merely the numbers $n_1, n_2, \ldots n_K$ of the quanta of various kinds $(1,2,\ldots K)$ which really occur (i.e. $n_{k\sigma} \neq 0$ for $k\sigma = 1,2,\ldots K$), if at the same time the σ's and the $\underline{k}$'s of these quanta are given, say $\sigma_1, \sigma_2, \ldots \sigma_K$ and $\underline{k}_1, \ldots \underline{k}_{K-1}$ ($\underline{k}_K$ can then be inferred from the total momentum).

It must be remembered that the quanta of the system H^0 represent an idealization of the real elementary particles as observed in nature. The latter are described by the Hamiltonian H, including the interaction terms H^1, H^2, etc. In these terms, $a_{k\sigma}$ and $a^*_{k\sigma}$ do not occur in the simple combination $a_{k\sigma}a^*_{k\sigma}$. The effect of $a_{k\sigma}$ or $a^*_{k\sigma}$ operating on a function of all the N's is, apart from a numerical factor depending on the N's, to increase or decrease the argument $N_{k\sigma}$ by unity (the result vanishes if $N_{k\sigma}+1$ or $N_{k\sigma}-1$ is no longer an eigenvalue of $a_{k\sigma}a^*_{k\sigma}$). The effect of $a_{k\sigma}$ or $a^*_{k\sigma}$ operating on a basic function $\varphi^{(0)}_n$ is thus to produce, if the result does not vanish, a numerical multiple of a new basic function which corresponds to one more or one less quantum of the kind $k\sigma$ in the fundamental cube. From this, the effect of H^1, H^2, etc. on a function of all the N's can be deduced.

Any state of affairs concerning the elementary particles is described by a corresponding solution of the wave equation (1.12) for H, (not (1.17) for the quanta). In practice, only ~~steady~~ effects (of stationary processes) are studied, such as the cross-sections for the collisions of the elementary particles, and also their self-energies. Hence, throughout this paper, <u>only stationary</u>-state (or quasi-stationary-state) solutions of (1.12) are considered. These are of the form

$$\Psi_a = e^{-iE_a t/\hbar}\,\psi_a(N_{k\sigma}) \tag{1.18}$$

where the ψ_a and the E_a are the eigenfunctions and the eigenvalues of the Hamiltonian H,

$$H\psi_a = \psi_a E_a\,, \qquad H = H^0 + H^1 + H^2 + \cdots\,. \tag{1.19}$$

As the Ψ_a are functions of all the N's, they can be expressed, when normalized, with the help of a unitary transformation of the basic functions $\{\varphi_n^{(0)}\}$, say

$$\Psi_a = \sum_m \varphi_m^{(0)} (\varphi_m^{(0)} \cdot \Psi_a) . \tag{1.20}$$

Substitute this into (1.19) and develop $H\varphi_m^{(0)}$ in the basis $\{\varphi_n^{(0)}\}$. Then (1.19) is transformed into the system of algebraic equations

$$\sum_m (\varphi_n^{(0)} \cdot H \varphi_m^{(0)}) (\varphi_m^{(0)} \cdot \Psi_a) = (\varphi_n^{(0)} \cdot \Psi_a) E_a , \tag{1.21}$$

$$(\varphi_n^{(0)} \cdot H \varphi_m^{(0)}) = (\varphi_n^{(0)} \cdot H^0 \varphi_m^{(0)}) + (\varphi_n^{(0)} \cdot H^1 \varphi_m^{(0)}) + (\varphi_n^{(0)} \cdot H^2 \varphi_m^{(0)}) + \cdots , \tag{1.22}$$

with, according to (1.15) and (1.14),

$$(\varphi_n^{(0)} \cdot H^0 \varphi_m^{(0)}) = \delta_{nm} E_m^{(0)} . \tag{1.23}$$

The solution of (1.21) can be successively approximated by the perturbation method, by using (1.23) as the zero-order approximation.

For physical interpretation, it is sometimes convenient toanalyze the stationary-state solution (1.18) of the actual system into a superposition of the stationary-state solutions (1.17) of the idealized system, say

$$\Psi_a = \sum_m \Phi_m^{(0)} (\Phi_m^{(0)} \cdot \Psi_a) , \tag{1.24}$$

with the coefficients $(\Phi_m^{(0)} \cdot \Psi_a)$ varying with time,

$$(\Phi_m^{(0)} \cdot \Psi_a) = e^{i(E_m^{(0)} - E_a)t/\hbar} (\varphi_m^{(0)} \cdot \Psi_a) . \tag{1.25}$$

Substitute (1.24) into Schroedinger's wave equation

(1.12) and use (1.17). Then (1.12) is transformed into the system of ordinary differential equations

$$-\frac{\hbar}{i}\frac{d(\Phi_n^{(0)}\cdot\Psi_a)}{dt} = \sum_m (\Phi_n^{(0)}\cdot[H^1+H^2+\cdots]\Phi_m^{(0)})(\Phi_m^{(0)}\cdot\Psi_a)$$

$$= \sum_m e^{i(E_n^{(0)}-E_m^{(0)})t/\hbar}\left\{(\varphi_n^{(0)}\cdot H^1\varphi_m^{(0)})+(\varphi_n^{(0)}\cdot H^2\varphi_m^{(0)})+\cdots\right\}(\Phi_m^{(0)}\cdot\Psi_a). \tag{1.26}$$

The perturbation treatment for the successive approximation to the solution of (1.26) in the time-representation is of course mathematically connected with that of (1.21) in the energy-representation.

2. Degeneracy — the failure of the expansion method

In order to explain the failure of the expansion method, it will be shown that the unperturbed system H^0 is degenerate. For this purpose it is only necessary to consider the states $\varphi_n^{(0)}$ possessing equal eigenvalues for the total momentum $\underline{G}$ and equal eigenvalues for the total charge Q, because the states possessing different eigenvalues for $\underline{G}$ and/or Q fall into different non-combining classes.

The vacuum-like state, which belongs only to the class of zero total momentum and zero total charge, is of course non-degenerate. The states which correspond to but one kind of quanta in the fundamental cube (K=1) are specified by n_1 and σ_1 only. These are in general spin-degenerate (meaning that different states with different spin orientations of the quanta exist and possess precisely equal energy-values) and can be partitioned into a number of mutually exclusive groups of rigorously degenerate states.

For the states which are specified by $n_1, n_2, \ldots n_K$; $\sigma_1, \sigma_2, \ldots \sigma_K$ and $\underline{k}_1, \ldots \underline{k}_{K-1}$ with $K \geq 2$, since the k's can vary almost continuously (namely by units of L^{-1}), the energy-values form an almost continuous spectrum for each specification of the n's and σ's. In such circumstances, all the states $\varphi_m^{(0)}$ with $E_m^{(0)}$ lying between $E_n^{(0)} - \frac{1}{2}\Delta E_n^{(0)}$ and $E_n^{(0)} + \frac{1}{2}\Delta E_n^{(0)}$ can be said to be approximately degenerate with respect to the central state $\varphi_n^{(0)}$ within the allowance $\Delta E_n^{(0)}$. For sufficiently large values of L, the allowance $\Delta E_n^{(0)}$ (which is always large compared to the spacing $\hbar c L^{-1}$ of the almost continuous energy-values) can be made arbitrarily small, and hence the approximate degeneracy becomes almost rigorous. The totality of such states (including the state $\varphi_n^{(0)}$ itself) will be denoted by $\deg^{(0)} n$. The notation $m \varepsilon \deg^{(0)} n$ and $m \bar{\varepsilon} \deg^{(0)} n$ will be used to denote that $\varphi_m^{(0)}$ is or is not almost degenerate with respect to $\varphi_n^{(0)}$. Note that the total number of states contained in the group $\deg^{(0)} n$ increases with L. Further, it is impossible to partition the states of an almost continuous spectrum into a number of mutually exclusive groups of almost rigorously degenerate states. (In fact, the group $\deg^{(0)} n$ partly overlaps with the group $\deg^{(0)} m$ if $m \varepsilon \deg^{(0)} n$).

The notation $\deg^{(0)} n$ may also be used if the state $\varphi_n^{(0)}$ corresponds to but one kind of quanta in the cube (K=1), then $\deg^{(0)} m$ coincides with $\deg^{(0)} n$ if

$m \varepsilon \deg^{(0)} n$. For a non-degenerate state $\varphi_n^{(0)}$, $\deg^{(0)} n$ consists of only the state $\varphi_n^{(0)}$ itself.

The degeneracy of the states of the unperturbed system H^0 usually plays no rôle for the first-order perturbation, because $(\varphi_m^{(0)} \cdot H^1 \varphi_n^{(0)})$ usually vanishes for $m \varepsilon \deg^{(0)} n$. (Since H^1 is trilinear in the operators $a_{k\sigma}$ and $a^*_{k\sigma}$, $(\varphi_m^{(0)} \cdot H^1 \varphi_n^{(0)})$ differs from zero only if $n_{k\sigma} = m_{k\sigma} \pm 1$ simultaneously for three kinds of quanta, two of which are usually of the same nature. Then it is kinematically impossible to satisfy the conservation laws of both the momentum and the energy). The case of the electrostatic interaction between Dirac's field (or electrons) and the longitudinal part of Maxwell's field (or the longitudinal quanta of Fermi(1932)) is, however, exceptional. (Here the rest mass of the longitudinal quanta vanishes identically, and both conservation laws can be satisfied to any desired accuracy if the momentum and hence the energy of the longitudinal quanta concerned is arbitrarily small). But, in any case, as it will appear immediately, the degeneracy of the states of H^0 cannot be ignored from the second-order perturbation onwards.

The failure of the expansion method can now be demonstrated by applying the important theorem well-known in the perturbation theory for degenerate systems (cf. Born, Heisenberg and Jordan(1926), Schroedinger (1926), Born and Jordan(1930)Chap.V), which in these works is expressed in the energy-representation, (1.19) or (1.21).

Theorem If some states of the unperturbed system are degenerate, whether rigorously degenerate or almost so, then the eigenfunctions ψ_a of the perturbed system cannot be expanded straightforwardly with respect to the perturbation by a power series of the form,

$$\psi_a = \varphi_a^{(0)} + \psi_a^1 + \psi_a^2 + \cdots, \tag{2.1}$$

where ψ_a^1 contains the first power of the interaction constant as a factor, ψ_a^2 contains the second power of the interaction constant as a factor, and so on.

Proof This theorem can easily be proved by contradiction, as follows. If the expansion (2.1) were possible, (1.21) and (1.22) show that E_a can be similarly expanded, so (1.21) becomes, by using also (1.23),

$$\begin{aligned}&\sum_m \left\{\delta_{nm} E_m^{(0)} + (\varphi_n^{(0)} \cdot H^1 \varphi_m^{(0)}) + (\varphi_n^{(0)} \cdot H^2 \varphi_m^{(0)}) + \cdots\right\}\left\{\delta_{ma} + (\varphi_m^{(0)} \cdot \psi_a^1) + (\varphi_m^{(0)} \cdot \psi_a^2) + \cdots\right\} \\ &= \left\{\delta_{na} + (\varphi_n^{(0)} \cdot \psi_a^1) + (\varphi_n^{(0)} \cdot \psi_a^2) + \cdots\right\}\left\{E_a^{(0)} + E_a^1 + E_a^2 + \cdots\right\}.\end{aligned} \tag{2.2}$$

The zero-order approximation to (2.2) is of course an identity. The first approximation to (2.2) then leads to

$$(\varphi_n^{(0)} \cdot H^1 \varphi_a^{(0)}) = \delta_{na} E_a^1 + (E_a^{(0)} - E_n^{(0)})(\varphi_n^{(0)} \cdot \psi_a^1), \tag{2.3}$$

which gives rise to the condition of solubility

$$(\varphi_n^{(0)} \cdot H^1 \varphi_a^{(0)}) = \delta_{na} E_a^1 \quad \text{for } n \varepsilon \deg^{(0)} a. \tag{2.4}$$

Of interest are two cases. (i) $(\varphi_n^{(0)} \cdot H^1 \varphi_a^{(0)}) \neq 0$ for $n \varepsilon \deg^{(0)} a$ and $n \neq a$. Then the condition of solubility (2.4) is not satisfied, and hence the theorem is proved by contradiction. This is the case where Fermi's longitudinal quanta of arbitrarily small momentum and energy are concerned. (ii) $(\varphi_n^{(0)} \cdot H^1 \varphi_a^{(0)}) = 0$ for n ε $\deg^{(0)}$a. This represents the general case in quantized field theories. The condition of solubility (2.4) is then satisfied by $E_a^1 = 0$, and the solution of (2.3) gives

$$(\varphi_n^{(0)} \cdot \psi_a^1) = \frac{(\varphi_n^{(0)} \cdot H^1 \varphi_a^{(0)})}{E_a^{(0)} - E_n^{(0)}} \quad \text{for } n \,\bar{\varepsilon}\, \deg^{(0)} a. \tag{2.5}$$

But then the second approximation to (2.2) leads to a contradiction. It leads to, with the help of (2.5),

$$(\varphi_n^{(0)} \cdot H^2 \varphi_a^{(0)}) + \sum_{m \,\bar{\varepsilon}\, \deg^{(0)} a} \frac{(\varphi_n^{(0)} \cdot H^1 \varphi_m^{(0)})(\varphi_m^{(0)} \cdot H^1 \varphi_a^{(0)})}{E_a^{(0)} - E_m^{(0)}} = \delta_{na} E_a^2 + (E_a^{(0)} - E_n^{(0)})(\varphi_n^{(0)} \cdot \psi_a^2); \tag{2.6}$$

but the <u>condition of solubility</u> for (2.6), namely that, for n ε $\deg^{(0)}$a, the left-hand-side of (2.6) should be diagonal, is actually not satisfied. (In fact, for n ε $\deg^{(0)}$ a and $n \neq a$, the left-hand-side of (2.6) is usually called the transition matrix element from the state $\varphi_a^{(0)}$ to the state $\varphi_n^{(0)}$). Hence the expansion (2.1) fails.

As a consequence, (2.6) cannot be used for the calculation of E_a^2 by putting n = a, even if the resultant expression for the self-energy were not divergent.

The impossibility of the expansion (2.1) in the energy-representation implies, by (1.18), that it is impossible to expand the Ψ_a in the form

$$\Psi_a = \Psi_a^{(0)} + \Psi_a^1 + \Psi_a^2 + \cdots . \tag{2.7}$$

Hence, in the time-representation, it is impossible to integrate the system (1.26) of differential equations by the usual expansion method, in which (1.26) is decomposed, by using (2.7), into

$$-\frac{\hbar}{i}\frac{d(\Phi_n^{(0)}\cdot\Psi_a^1)}{dt} = \sum_m e^{i(E_n^{(0)}-E_m^{(0)})t/\hbar}(\varphi_n^{(0)}\cdot H^1\varphi_m^{(0)})\delta_{ma}, \tag{2.8}$$

$$-\frac{\hbar}{i}\frac{d(\Phi_n^{(0)}\cdot\Psi_a^2)}{dt} = \sum_m e^{i(E_n^{(0)}-E_m^{(0)})t/\hbar}\{(\varphi_n^{(0)}\cdot H^1\varphi_m^{(0)})(\Phi_m^{(0)}\cdot\Psi_a^1)+(\varphi_n^{(0)}\cdot H^2\varphi_m^{(0)})\delta_{ma}\}, \tag{2.9}$$

. . . .

The cross-sections thus obtained are therefore not justified even if the contributions from the terms of higher degree were not divergent.

The failure of the expansion method for <u>any</u> <u>degenerate</u> system is due to the faulty assumption, (2.1), that the eigenfunctions of H should equal approximately those of H^0, because the eigenfunctions for a degenerate system can be chosen arbitrarily to some extent. In such circumstances, the degeneracy of the unperturbed system has <u>first</u> to be treated by the method of secular perturbation. Such consideration is familiar wherever the perturbation method is applied, usually in the energy-representation, to problems of atomic and molecular structures. In dealing with field theories and the interaction of elementary particles, the treatment of the degeneracy has been hitherto overlooked, probably because the problem is usually presented in the time-representation.

3. The method of secular perturbation

In accordance with the perturbation theory developed by the authors quoted above, the arbitrariness of choosing the eigenfunctions $\varphi_n^{(0)}$ of the unperturbed degenerate system has to be treated by the method of secular perturbation. The following formulation of this method is designed to cover both the states which are rigorously degenerate and the states of the almost continuous spectrum which are almost so. The essential point of this method is the following. If the eigenfunctions of the perturbed system can be expanded, to the r-th approximation, in a basis $\{\varphi_a^{(r)}\}$ say, which are degenerate, then a suitable change of basis from $\{\varphi_a^{(r)}\}$ to $\{\varphi_a^{(r+1)}\}$ must be made in order that the eigenfunctions can be expanded in the new basis to the r+1-th approximation. In atomic and molecular problems, the basis for the first or the second approximation is usually non-degenerate, in which case the degeneracy is said to be removed by the secular change of basis. This is, however, not the case in quantized field theories where the degeneracy persists in all approximations. Hence the method of secular perturbation has to be continued indefinitely, unless the perturbation method is discontinued at a certain stage in favour of other approximation methods. ~~being used.~~

Let $\psi_a^{(r)}$ and $E_a^{(r)}$ denote the r-th approximation to the actual eigenfunctions ψ_a and eigenvalues E_a, so

$$H^{(r)}\psi_a^{(r)} = \psi_a^{(r)}E_a^{(r)}, \quad H^{(r)} = H^0 + H^1 + \cdots + H^r, \tag{3.1}$$

where quantities of higher degrees in the interaction constants than the r-th are to be neglected. It will be shown that there exists a secular problem for the r-th approximation, of the form

$$K^{(r)} \varphi_a^{(r)} = \varphi_a^{(r)} E_a^{(r)}, \quad K^{(r)} = H^0 + K^{(r)1} + \cdots + K^{(r)r}, \tag{3.2}$$

which supplies a basis $\{\varphi_a^{(r)}\}$ in which the $\psi_a^{(r)}$ of (3.1) can be expanded, just to the r-th approximation, say

$$\psi_a^{(r)} = \varphi_a^{(r)} + \psi_a^{(r)1} + \cdots + \psi_a^{(r)r}. \tag{3.3}$$

$K^{(r)}$ may be regarded as the Hamiltonian of a secular system which is nearer to the actual system than the unperturbed system H^0. The difference of $H^{(r)}$ and $K^{(r)}$ will be called the non-secular perturbation $J^{(r)}$,

$$J^{(r)} = H^{(r)} - K^{(r)} = J^{(r)1} + \cdots + J^{(r)r},$$
$$J^{(r)j} = H^j - K^{(r)j} \qquad (j = 1, \cdots r). \tag{3.4}$$

For the r-th approximation, the expansion (3.3) of $\psi_a^{(r)}$ is in fact arranged according to the order of magnitude of the non-secular perturbations $J^{(r)j}$ $(j = 1, \cdots r)$, while the secular perturbation $K^{(r)1} + \cdots + K^{(r)r}$ is not regarded as small. (Note that the superscript (r) with the brackets indicates the order of approximations r = 0, 1, 2, ...; while the superscript j without the brackets indicates the degree of the interaction constants, j = 1, .. r. Further, the letter φ is used in connection with the secular problems, while the letter ψ is used in connection with the actual problems).

For the first approximation, r = 1, it is well-known, in the case where all states can be partitioned into mutually exclusive groups of rigorously degenerate states, that the secular perturbation $K^{(1)1}$ is such that its matrix elements between two degenerate states are the same as those of H^1 while its matrix elements between states which are not degenerate with respect to each other vanish. That is

$$\left.\begin{aligned} (\varphi_c^{(0)} \cdot K^{(1)1} \varphi_b^{(0)}) &= (\varphi_c^{(0)} \cdot H^1 \varphi_b^{(0)}) \quad \text{for } c \,\varepsilon\, deg^{(0)} b, \\ (\varphi_c^{(0)} \cdot K^{(1)1} \varphi_b^{(0)}) &= 0 \quad \text{for } c \,\bar{\varepsilon}\, deg^{(0)} b. \end{aligned}\right\} \tag{3.5}$$

In this case, the matrix $(\varphi_c^{(0)} \cdot K^{(1)1} \varphi_b^{(0)})$ consists of step-matrices (which correspond to the partition of the states into mutually exclusive groups of degenerate states). The secular change of basis from $\{\varphi_a^{(0)}\}$ to $\{\varphi_a^{(1)}\}$ represents a unitary transformation,

$$\varphi_a^{(1)} = \sum_b \varphi_b^{(0)} (\varphi_b^{(0)} \cdot \varphi_a^{(1)}) , \tag{3.6}$$

where the transformation matrix $(\varphi_b^{(0)} \cdot \varphi_a^{(1)})$ also consists of step-matrices. That is

$$\left.\begin{aligned} (\varphi_b^{(0)} \cdot \varphi_a^{(1)}) &\neq 0 \quad \text{for } b \,\varepsilon\, deg^{(0)} a, \\ (\varphi_b^{(0)} \cdot \varphi_a^{(1)}) &= 0 \quad \text{for } b \,\bar{\varepsilon}\, deg^{(0)} a. \end{aligned}\right\} \tag{3.7}$$

The secular problem (3.2) for r = 1, now written in the basis $\{\varphi_c^{(0)}\}$, can then be partitioned accordingly

$$\sum_{b \,\varepsilon\, deg^{(0)} a} (\varphi_c^{(0)} \cdot K^{(1)} \varphi_b^{(0)})(\varphi_b^{(0)} \cdot \varphi_a^{(1)}) = (\varphi_c^{(0)} \cdot \varphi_a^{(1)}) E_a^{(1)}, \quad (c \,\varepsilon\, deg^{(0)} a) \tag{3.8}$$

with

$$(\varphi_c^{(0)} \cdot K^{(1)} \varphi_b^{(0)}) = \delta_{cb} E_b^{(0)} + (\varphi_c^{(0)} \cdot K^{(1)1} \varphi_b^{(0)}). \tag{3.9}$$

These equations, (3.5) - (3.9), can also be used in the case of the almost rigorous degeneracy for the states of the almost continuous spectrum, provided that each of the states is considered in turn, and, for the determination of $\varphi_a^{(1)}$ and $E_a^{(1)}$ for a particular state, the corresponding group $\deg^{(0)}a$ of states which are degenerate with respect to $\varphi_a^{(0)}$ within an arbitrarily small allowance $\Delta E_a^{(0)}$ is used. But, since it is impossible to partition the states of the almost continuous spectrum into mutually exclusive groups of degenerate states, the matrix $(\varphi_c^{(0)} \cdot K^{(1)1} \varphi_b^{(0)})$ now consists of some kind of band placed along the diagonal, outside of which the elements rapidly diminish and practically vanish. Since the number of states in the group $\deg^{(0)}a$ is large (and increases with L), the band covers a large number of diagonals on both sides of the principal diagonal (and even more so for larger values of L). The matrix $(\varphi_b^{(0)} \cdot \varphi_a^{(1)})$ is of a similar structure.

The choice (3.5) of the secular perturbation $K^{(1)1}$ follows essentially from the condition that, in the new basis $\{\varphi_b^{(1)}\}$, the eigenfunctions $\psi_a^{(1)}$ can be expanded to the first approximation. That is, in accordance with (3.3),

$$\psi_a^{(1)} = \sum_b \varphi_b^{(1)} (\varphi_b^{(1)} \cdot \psi_a^{(1)}); \quad (\varphi_b^{(1)} \cdot \psi_a^{(1)}) = \delta_{ba} + (\varphi_b^{(1)} \cdot \psi_a^{(1)1}). \tag{3.10}$$

In order to derive (3.5), let us regard $H^{(1)}$ as the

Hamiltonian of the perturbed system, $K^{(1)}$ as that of the unperturbed system, and treat the non-secular perturbation $J^{(1)} = J^{(1)1}$ by the expansion method. By (3.10), the eigenvalue problem (3.1) can be written in the form

$$(K^{(1)}+J^{(1)})\sum_b \varphi_b^{(1)}(\varphi_b^{(1)}\cdot\psi_a^{(1)}) = \sum_c \varphi_c^{(1)}(\varphi_c^{(1)}\cdot\psi_a^{(1)})\,E_a^{(1)}, \tag{3.11}$$

which, by using (3.2) and developing $J^{(1)}\varphi_b^{(1)}$ in the basis $\{\varphi_c^{(1)}\}$, becomes

$$\sum_b (\varphi_c^{(1)}\cdot J^{(1)}\varphi_b^{(1)})(\varphi_b^{(1)}\cdot\psi_a^{(1)}) = (E_a^{(1)}-E_c^{(1)})(\varphi_c^{(1)}\cdot\psi_a^{(1)}). \tag{3.12}$$

On the right-hand-side of (3.12), since the leading term δ_{ca} of $(\varphi_c^{(1)}\cdot\psi_a^{(1)})$ contributes nothing, the $(E_a^{(1)}-E_c^{(1)})$ may be approximated by $(E_a^{(0)}-E_c^{(0)})$.
By using the expansions (3.3) and (3.4), (3.12) gives

$$(\varphi_c^{(1)}\cdot J^{(1)1}\varphi_a^{(1)}) = (E_a^{(0)}-E_c^{(0)})(\varphi_c^{(1)}\cdot\psi_a^{(1)1}). \tag{3.13}$$

The condition of solubility for (3.13) demands

$$(\varphi_c^{(1)}\cdot J^{(1)1}\varphi_a^{(1)}) = 0 \quad \text{for } c\,\varepsilon\,\deg^{(0)} a, \text{ i.e.}$$

$$(\varphi_c^{(1)}\cdot K^{(1)1}\varphi_a^{(1)}) = (\varphi_c^{(1)}\cdot H^1\varphi_a^{(1)}) \quad \text{for } c\,\varepsilon\,\deg^{(0)} a. \tag{3.14}$$

When expressed in the basis $\{\varphi_a^{(0)}\}$, this gives the first part of (3.5). The second part of (3.5) has been chosen for simplicity and definiteness, but it is in principle arbitrary.

The majority of $E_a^{(1)}$, as determined by solving (3.8), will form an almost continuous spectrum. The majority of $\varphi_a^{(1)}$ are therefore degenerate. For the second approximation, r = 2, a further secular change of basis from $\{\varphi_a^{(1)}\}$ to $\{\varphi_a^{(2)}\}$ is then necessary, say

$$\varphi_a^{(2)} = \sum_b \varphi_b^{(1)}(\varphi_b^{(1)} \cdot \varphi_a^{(2)}), \tag{3.15}$$

where

$$\left.\begin{aligned} (\varphi_b^{(1)} \cdot \varphi_a^{(2)}) &\neq 0 \quad \text{for } b \,\varepsilon\, \deg^{(1)} a, \\ (\varphi_b^{(1)} \cdot \varphi_a^{(2)}) &= 0 \quad \text{for } b \,\bar{\varepsilon}\, \deg^{(1)} a. \end{aligned}\right\} \tag{3.16}$$

Here $\deg^{(1)} a$ denotes a group of states $\varphi_c^{(1)}$ with $E_c^{(1)}$ lying within an allowance $\Delta E_a^{(1)}$ around the central value $E_a^{(1)}$. The secular perturbation for the determination of the new basis $\{\varphi_a^{(2)}\}$ can be derived similarly by treating the non-secular perturbation $J^{(2)}$ for the second approximation by the expansion method, as follows. In the basis $\{\varphi_c^{(2)}\}$, (3.1) becomes, by (3.2),

$$\sum_b (\varphi_c^{(2)} \cdot J^{(2)} \varphi_b^{(2)})(\varphi_b^{(2)} \cdot \psi_a^{(2)}) = (E_a^{(2)} - E_c^{(2)})(\varphi_c^{(2)} \cdot \psi_a^{(2)}). \tag{3.17}$$

By using the expansions (3.3) and (3.4), and approximating the $(E_a^{(2)} - E_c^{(2)})$ by $(E_a^{(1)} - E_c^{(1)})$, (3.17) gives, for the first-order and the second-order quantities,

$$(\varphi_c^{(2)} \cdot J^{(2)1} \varphi_a^{(2)}) = (E_a^{(1)} - E_c^{(1)})(\varphi_c^{(2)} \cdot \psi_a^{(2)1}), \tag{3.18}$$

$$\begin{aligned} &(\varphi_c^{(2)} \cdot J^{(2)2} \varphi_a^{(2)}) + \sum_b (\varphi_c^{(2)} \cdot J^{(2)1} \varphi_b^{(2)})(\varphi_b^{(2)} \cdot \psi_a^{(2)1}) \\ &= (E_a^{(1)} - E_c^{(1)})(\varphi_c^{(2)} \cdot \psi_a^{(2)2}). \end{aligned} \tag{3.19}$$

The condition of solubility of (3.18) determines $K^{(2)1}$, the first-order secular perturbation in the second approximation, viz.

$$(\varphi_c^{(2)} \cdot J^{(2)1} \varphi_a^{(2)}) = 0 \quad \text{for} \quad c \in deg^{(1)} a, \text{ i.e.}$$

$$(\varphi_c^{(2)} \cdot K^{(2)1} \varphi_a^{(2)}) = (\varphi_c^{(2)} \cdot H^1 \varphi_a^{(2)}) \quad \text{for } c \in deg^{(1)} a. \tag{3.20}$$

For simplicity and definiteness, take

$$\begin{aligned}&(\varphi_c^{(2)} \cdot K^{(2)1} \varphi_a^{(2)}) = 0 \quad \text{for} \quad c \; \bar{\varepsilon} \; deg^{(1)} a, \text{ so}\\ &(\varphi_c^{(2)} \cdot J^{(2)1} \varphi_a^{(2)}) = (\varphi_c^{(2)} \cdot H^1 \varphi_a^{(2)}) \text{ for } c \; \bar{\varepsilon} \; deg^{(1)} a.\end{aligned} \tag{3.21}$$

The solution of (3.18) then gives

$$(\varphi_c^{(2)} \cdot \psi_a^{(2)1}) = \frac{(\varphi_c^{(2)} \cdot H^1 \varphi_a^{(2)})}{E_a^{(1)} - E_c^{(1)}} \quad \text{for} \quad c \; \bar{\varepsilon} \; deg^{(1)} a, \tag{3.22}$$

and, for simplicity and definiteness, take

$$(\varphi_c^{(2)} \cdot \psi_a^{(2)1}) = 0 \quad \text{for} \quad c \in deg^{(1)} a. \tag{3.23}$$

With the help of these relations, the condition of solubility for (3.19), namely

$$(\varphi_c^{(2)} \cdot J^{(2)2} \varphi_a^{(2)}) + \sum_b (\varphi_c^{(2)} \cdot J^{(2)1} \varphi_b^{(2)})(\varphi_b^{(2)} \cdot \psi_a^{(2)1}) = 0 \text{ for } c \in deg^{(1)} a, \tag{3.24}$$

determines the second-order secular perturbation $K^{(2)2}$,

$$\begin{aligned}(\varphi_c^{(2)} \cdot K^{(2)2} \varphi_a^{(2)}) &= (\varphi_c^{(2)} \cdot H^2 \varphi_a^{(2)})\\ &+ \sum_{b \; \bar{\varepsilon} \; deg^{(1)} a} \frac{(\varphi_c^{(2)} \cdot H^1 \varphi_b^{(2)})(\varphi_b^{(2)} \cdot H^1 \varphi_a^{(2)})}{E_a^{(1)} - E_b^{(1)}} \quad \text{for } c \in deg^{(1)} a.\end{aligned} \tag{3.25}$$

For definiteness, take

$$\begin{aligned}&(\varphi_c^{(2)} \cdot K^{(2)2} \varphi_a^{(2)}) = 0 \quad \text{for} \quad c \; \bar{\varepsilon} \; deg^{(1)} a, \quad \text{so}\\ &(\varphi_c^{(2)} \cdot J^{(2)2} \varphi_a^{(2)}) = (\varphi_c^{(2)} \cdot H^2 \varphi_a^{(2)}) \text{ for } c \; \bar{\varepsilon} \; deg^{(1)} a.\end{aligned} \tag{3.26}$$

The solution of (3.19) then gives

$$(\varphi_c^{(2)} \cdot \psi_a^{(2)2}) = \frac{1}{E_a^{(1)} - E_c^{(1)}} \Big\{ (\varphi_c^{(2)} \cdot H^2 \varphi_a^{(2)}) + \sum_{b \,\bar{\varepsilon}\, \deg^{(1)} a} \frac{(\varphi_c^{(2)} \cdot H^1 \varphi_b^{(2)})(\varphi_b^{(2)} \cdot H^1 \varphi_a^{(2)})}{E_a^{(1)} - E_b^{(1)}} \Big\} \quad \text{for } c \,\bar{\varepsilon}\, \deg^{(1)} a. \tag{3.27}$$

The normalization condition

$$\sum_b (\psi_c^{(2)} \cdot \varphi_b^{(2)})(\varphi_b^{(2)} \cdot \psi_a^{(2)}) = (\psi_c^{(2)} \cdot \psi_a^{(2)}) = \delta_{ca} \tag{3.28}$$

leads to

$$(\psi_c^{(2)2} \cdot \varphi_a^{(2)}) + (\varphi_c^{(2)} \cdot \psi_a^{(2)2}) + \sum_b (\psi_c^{(2)1} \cdot \varphi_b^{(2)})(\varphi_b^{(2)} \cdot \psi_a^{(2)1}) = 0 \tag{3.29}$$

which somewhat restricts the choice of $(\varphi_c^{(2)} \cdot \psi_a^{(2)2})$ for $c \,\varepsilon\, \deg^{(1)} a$.

From (3.15) and (3.16) follow, for any continuous function $f(E_b^{(1)})$ and sufficiently small allowance $\Delta E_a^{(1)}$ used in the definition of $\deg^{(1)} a$,

$$\sum_{b \,\varepsilon\, \deg^{(1)} a} f(E_b^{(1)})\, \varphi_b^{(1)} (\varphi_b^{(1)} \cdot \varphi_a^{(2)}) = f(E_a^{(1)})\, \varphi_a^{(2)} \tag{3.30}$$

and conversely, as the functions $\{\varphi_b^{(1)}\}$ are orthonormal,

$$\sum_{a \,\varepsilon\, \deg^{(1)} b} f(E_a^{(1)})\, \varphi_a^{(2)} (\varphi_a^{(2)} \cdot \varphi_b^{(1)}) = f(E_b^{(1)})\, \varphi_b^{(1)} . \tag{3.31}$$

Hence the secular problem for the determination of $\varphi_a^{(2)}$ and $E_a^{(2)}$ can be written in the basis $\{\varphi_c^{(1)}\}$ in the form

$$\sum_{b \,\varepsilon\, \deg^{(1)} a} (\varphi_c^{(1)} \cdot K^{(2)} \varphi_b^{(1)})(\varphi_b^{(1)} \cdot \varphi_a^{(2)}) = (\varphi_c^{(1)} \cdot \varphi_a^{(2)})\, E_a^{(2)}, \quad (c \,\varepsilon\, \deg^{(1)} a) \tag{3.32}$$

where, by using the secular problem for $\varphi_b^{(1)}$ and $E_b^{(1)}$,

$$\begin{aligned}(\varphi_c^{(1)} \cdot K^{(2)} \varphi_b^{(1)}) &= (\varphi_c^{(1)} \cdot K^{(1)} \varphi_b^{(1)}) + (\varphi_c^{(1)} \cdot [K^{(2)} - K^{(1)}]\, \varphi_b^{(1)}) \\ &= \delta_{cb} E_b^{(1)} + (\varphi_c^{(1)} \cdot [K^{(2)1} - K^{(1)1}]\, \varphi_b^{(1)}) + (\varphi_c^{(1)} \cdot K^{(2)2} \varphi_b^{(1)}) .\end{aligned} \tag{3.33}$$

By transforming (3.20) into the basis $\{\varphi_b^{(1)}\}$ and comparing the result with (3.14), one sees that the correction of the first-order secular perturbation arises only from the difference of $\deg^{(1)}b$ and $\deg^{(0)}b$:

$$(\varphi_c^{(1)}\cdot[K^{(2)1}-K^{(1)1}]\varphi_b^{(1)})=\begin{cases}(\varphi_c^{(1)}\cdot H^1\varphi_b^{(1)}) & \text{if } c\,\varepsilon\,\deg^{(1)}b \text{ but } c\,\bar{\varepsilon}\,\deg^{(0)}b,\\ -(\varphi_c^{(1)}\cdot H^1\varphi_b^{(1)}) & \text{if } c\,\varepsilon\,\deg^{(0)}b \text{ but } c\,\bar{\varepsilon}\,\deg^{(1)}b,\\ 0 & \text{if otherwise.}\end{cases} \tag{3.34}$$

By a change of basis, (3.25) and (3.26) become

$$(\varphi_c^{(1)}\cdot K^{(2)2}\varphi_b^{(1)}) = (\varphi_c^{(1)}\cdot H^2\varphi_b^{(1)}) + \sum_{a'\,\bar{\varepsilon}\,\deg^{(1)}b}\frac{\sum_{b'\varepsilon\deg^{(1)}a'}(\varphi_c^{(1)}\cdot H^1\varphi_{b'}^{(1)})(\varphi_{b'}^{(1)}\cdot\varphi_{a'}^{(2)})\sum_{c'\varepsilon\deg^{(1)}a'}(\varphi_{a'}^{(2)}\cdot\varphi_{c'}^{(1)})(\varphi_{c'}^{(1)}\cdot H^1\varphi_b^{(1)})}{E_b^{(1)}-E_{a'}^{(1)}} \tag{3.35}$$

for $c\ \varepsilon\ \deg^{(1)}b$, and

$$(\varphi_c^{(1)}\cdot K^{(2)2}\varphi_b^{(1)}) = 0 \quad \text{for } c\ \bar{\varepsilon}\ \deg^{(1)}b. \tag{3.36}$$

In the numerator of the fraction term of (3.35), note the occurence of the factors arising from the secular change of basis. If the states which are included in the summation over a' can be partitioned into mutually exclusive groups of degenerate states, these factors can be eliminated by using (3.31) and the orthogonality condition of the basis $\{\varphi_{b'}^{(1)}\}$. On the other hand, if the states to be summed form an almost continuous spectrum and thus cannot be partitioned into mutually exclusive groups of degenerate states, these factors remain and, as will be demonstrated later in §6, help to render the summation convergent.

The derivation of the secular perturbation for the higher approximations is similar. In quantized field theories, the self-energies arise only at the second, the fourth, ..., approximations. Then every two successive approximations, such as the first and the second, the third and the fourth, ..., can be dealt with in one step.

In the time-representation, the method of secular perturbation assumes the following form. Let

$$\Phi_a^{(r)} = e^{-iE_a^{(r)}t/\hbar}\varphi_a^{(r)}; \quad \Psi_a^{(r)} = e^{-iE_a^{(r)}t/\hbar}\psi_a^{(r)},$$
$$\Psi_a^{(r)j} = e^{-iE_a^{(r)}t/\hbar}\psi_a^{(r)j} \quad (j = 1, \dots r; \quad r = 1, 2, \dots) \tag{3.37}$$

For successive approximations, one has to solve (1.26) by the successive change of variables, (r = 1,2,...)

$$(\Phi_n^{(0)} \cdot \Psi_a) = \sum_m \dots \sum_b (\Phi_n^{(0)} \cdot \Phi_m^{(1)}) \dots (\Phi_c^{(r-1)} \cdot \Phi_b^{(r)})(\Phi_b^{(r)} \cdot \Psi_a) \tag{3.38}$$

with

$$(\Phi_c^{(r-1)} \cdot \Phi_b^{(r)}) = e^{i(E_b^{(r)} - E_c^{(r-1)})t/\hbar}(\varphi_c^{(r-1)} \cdot \varphi_b^{(r)}). \tag{3.39}$$

The condition that the r-th approximation to Ψ_a can be expanded in the form

$$\Psi_a^{(r)} = \Phi_a^{(r)} + \Psi_a^{(r)1} + \dots + \Psi_a^{(r)r} \tag{3.40}$$

determines the secular problem

$$-\frac{\hbar}{i}\frac{\partial \Phi_a^{(r)}}{\partial t} = K^{(r)}\Phi_a^{(r)}. \tag{3.41}$$

The explicit expression for $K^{(r)}$ can also be derived by applying the expansion method for the treatment of the non-secular perturbation in the time-representation.

From (3.41) follows the differential equations for the secular change of basis for the successive approximations, r = 1, 2, ...

$$-\frac{\hbar}{i}\frac{d(\Phi_c^{(r-1)}\cdot\Phi_a^{(r)})}{dt}=\sum_b(\Phi_c^{(r-1)}\cdot[K^{(r)}-K^{(r-1)}]\,\Phi_b^{(r-1)})(\Phi_b^{(r-1)}\cdot\Phi_a^{(r)}) \quad (3.42)$$

For r = 1 or 2, (3.42) is equivalent to (3.8) or (3.32), as is easily verified by using (3.39).

4. Asymptotical solution of the secular problem for large values of L — Regular case

The first-order secular perturbation generally vanishes in quantized field theories. The only exceptional case occurs where a Fermi's longitudinal quantum of arbitrarily small energy ε (of the order of $\hbar c L^{-1}$) is emitted or absorbed by (say) an electron-like quantum. As the corresponding matrix element varies as $\varepsilon^{-\frac{1}{2}}$, this physically exceptional case represents also a mathematically singular case. Since the usual divergence difficulty originates from the quanta of arbitrarily large energies and is independent of the nature of the quanta, it is convenient, for the present general investigation, to ignore altogether the existence of Fermi's longitudinal quanta of arbitrarily small energies, and thus avoids any special complication peculiar to this singular case alone. The method of solution of the secular problem in the regular case will be demonstrated in details by considering the second approximation. This method has immediate physical application, for example, in dealing with the interaction between the nucleon field and the meson field.

Suppose that the first-order secular perturbation vanishes, so

$$\varphi_a^{(1)} = \varphi_a^{(0)}, \quad E_a^{(1)} = E_a^{(0)}, \quad deg^{(1)} a = deg^{(0)} a. \tag{4.1}$$

The second-order secular problem (3.32) becomes

$$\sum_{b \varepsilon deg^{(0)} a} (\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)})(\varphi_b^{(0)} \cdot \varphi_a^{(2)}) = (\varphi_c^{(0)} \cdot \varphi_a^{(2)}) E_a^{(2)}, \quad (c \ \varepsilon \ deg^{(0)} a), \tag{4.2}$$

where, by (3.33) and (3.35),

$$(\varphi_c^{(0)} \cdot K^{(2)} \varphi_c^{(0)}) = E_c^{(0)} + (\varphi_c^{(0)} \cdot K^{(2)2} \varphi_c^{(0)}) = E_c^{(0)} + (\varphi_c^{(0)} \cdot H^2 \varphi_c^{(0)}) + \sum_{a' \bar{\varepsilon} deg^{(0)} c} \frac{\sum_{b' \varepsilon deg^{(0)} a'} (\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot \varphi_{a'}^{(2)}) \sum_{c' \varepsilon deg^{(0)} a'} (\varphi_{a'}^{(2)} \cdot \varphi_{c'}^{(0)})(\varphi_{c'}^{(0)} \cdot H^1 \varphi_c^{(0)})}{E_c^{(0)} - E_{a'}^{(0)}}, \tag{4.3}$$

$$(\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)}) = (\varphi_c^{(0)} \cdot H^2 \varphi_b^{(0)}) + \sum_{a' \bar{\varepsilon} deg^{(0)} b} \frac{\sum_{b' \varepsilon deg^{(0)} a'} (\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot \varphi_{a'}^{(2)}) \sum_{c' \varepsilon deg^{(0)} a'} (\varphi_{a'}^{(2)} \cdot \varphi_{c'}^{(0)})(\varphi_{c'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{a'}^{(0)}}, \quad (c \ \varepsilon \ deg^{(0)} b, \ c \neq b). \tag{4.4}$$

The diagonal elements and the non-diagonal elements are here written out separately because the respective fraction terms are of different orders in L. In fact, the non-diagonal elements (4.4) can be approximated by the familiar expression (known in the expansion method as the matrix element for the transition)

$$(\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)}) = (\varphi_c^{(0)} \cdot H^2 \varphi_b^{(0)}) + \sum_{b'} \frac{(\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{b'}^{(0)}}, \quad (c \ \varepsilon \ deg^{(0)} b, \ c \neq b) \tag{4.5}$$

where the summation automatically covers only the group of intermediate states (for which the summand differs from zero). This approximation follows from the fact that, in the circumstances, the intermediate states are finite in number, possess different energy-values apart from spin-degeneracy, and hence can be regarded as the central states of a few mutually exclusive groups of degenerate states, say $deg^{(0)}1$, $deg^{(0)}2$, ... $deg^{(0)}f$. By using (3.31), the main contribution to the fraction terms of (4.4) is easily seen to be

$$\left\{\sum_{a'\varepsilon deg^{(0)}1} + \sum_{a'\varepsilon deg^{(0)}2} + \cdots + \sum_{a'\varepsilon deg^{(0)}f}\right\} \frac{\sum_{b'\varepsilon deg^{(0)}a'} (\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot \varphi_{a'}^{(2)}) \sum_{c'\varepsilon deg^{(0)}a'} (\varphi_{a'}^{(2)} \cdot \varphi_{c'}^{(0)})(\varphi_{c'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{a'}^{(0)}}$$

$$= \left\{\sum_{b'\varepsilon deg^{(0)}1} + \sum_{b'\varepsilon deg^{(0)}2} + \cdots + \sum_{b'\varepsilon deg^{(0)}f}\right\} \frac{(\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{b'}^{(0)}} \qquad (4.6)$$

$$= \sum_{b'} \frac{(\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{b'}^{(0)}} \; ;$$

because for each of these mutually exclusive groups, say $deg^{(0)}i$ (i = 1, 2, .. f), one has, by a change of the order of summation, and by using (3.31) together with the fact that the fundtions $\{\varphi_{b'}^{(1)}\}$ are orthonormal,

$$\sum_{a'\varepsilon deg^{(0)}i} \frac{\sum_{b'\varepsilon deg^{(0)}a'} (\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot \varphi_{a'}^{(2)}) \sum_{c'\varepsilon deg^{(0)}a'} (\varphi_{a'}^{(2)} \cdot \varphi_{c'}^{(0)})(\varphi_{c'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{a'}^{(0)}}$$

$$= \sum_{b'\varepsilon deg^{(0)}i} \sum_{c'\varepsilon deg^{(0)}i} \frac{(\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})\, \delta_{b'c'}\, (\varphi_{c'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{b'}^{(0)}} \qquad (4.7)$$

$$= \sum_{b'\varepsilon deg^{(0)}i} \frac{(\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot H^1 \varphi_b^{(0)})}{E_b^{(0)} - E_{b'}^{(0)}} \, .$$

The same consideration shows that the contribution due to any other group of degenerate states vanishes. By (4.5) and (1.7), (1.8), the non-diagonal elements $(\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)})$, (c = b), are of the order of L^{-3}. On the other hand, all states $\varphi_{b'}^{(0)}$ for which $(\varphi_{b'}^{(0)} \cdot H^1 \varphi_c^{(0)}) \neq 0$ contribute to the fraction terms of (4.3). There are a few such states (arising from $\varphi_c^{(0)}$ by the virtual absorption of some quanta) which can be treated as above, with contributions of the order of L^{-3}. But the main contribution arises from the large number of states (arising from $\varphi_c^{(0)}$ by the virtual emission of some quanta) which cannot be partitioned into mutually exclusive groups of degenerate states, as their energy-values form an almost continuous spectrum. The number of such states varies as L^3, so their total contribution to (4.3) is of the order of L^0 (i.e. independent of L); but the actual evaluation of this contribution has to be postponed until the secular change of basis for these intermediate states of the almost continuous spectrum has been found (cf. (4.28) below).

For large values of L, let a state, say $\varphi_b^{(0)}$ be ~~also~~ denoted ~~also~~ by $\varphi_B^{(0)}(E_b^{(0)})$. Here the energy-value is indicated separately; while the capital letter used as a suffix symbolizes all the remaining detailed information which is needed for the specification of the state. The specification of the number and the nature of the quanta and their spin orientations are described by discrete variables, while the specification of the directions of the momenta of these quanta and the proportion in which the total energy is shared by these quanta are described by continuous variables. For the solution of (4.2), it is sufficient to restrict the group $\deg^{(0)}a$ to contain only such states $\varphi_b^{(0)}$ for which $(\varphi_b^{(0)} \cdot K^{(2)} \varphi_a^{(0)}) \neq 0$. Let $\rho_B^{(0)}(E_b^{(0)})\, dE_b^{(0)}$ denote the number of such states with their energy-values lying in the interval $dE_b^{(0)}$ around the value $E_b^{(0)}$ of the almost continuous spectrum which further conform to the detailed specification B. Then for any continuous fucntion $f(\varphi_b^{(0)})$ of $\varphi_b^{(0)}$, one has

$$\sum_{b \varepsilon \deg^{(0)}a} f(\varphi_b^{(0)}) = \sum_B \int_{E_a^{(0)} - \frac{1}{2}\Delta E_a^{(0)}}^{E_a^{(0)} + \frac{1}{2}\Delta E_a^{(0)}} f(\varphi_B^{(0)}(E_b^{(0)}))\, \rho_B^{(0)}(E_b^{(0)})\, dE_b^{(0)} . \qquad (4.8)$$

$\rho_B^{(0)}(E_b^{(0)})$ contains a factor L^3. It contains also the differentials of the continuous variables describing the detailed specification B, and the summation over B implies the integration with respect to these continuous variables.

<u>Theorem</u>. The asymptotical solution of (4.2) for large values of L is of the form

$$(\varphi_a^{(0)} \cdot \varphi_a^{(2)}) = 1 , \qquad (4.9)$$

$$(\varphi_b^{(0)} \cdot \varphi_a^{(2)}) = \left\{ \frac{P}{E_a^{(0)} - E_b^{(0)}} - i\pi\delta(E_a^{(0)} - E_b^{(0)}) \right\}_L (\varphi_b^{(0)} \cdot S^{(2)} \varphi_a^{(0)}),$$

$$(b \varepsilon \deg^{(0)}a, \quad b \neq a) . \qquad (4.10)$$

Here $\left\{ \frac{P}{E_a^{(0)} - E_b^{(0)}} - i\pi\delta(E_a^{(0)} - E_b^{(0)}) \right\}_L$ denotes‡ a continuous

‡ The notation is such that the 'limit' of this function as $L \to \infty$ will be denoted by the 'improper function' $\frac{P}{E_a^{(0)} - E_b^{(0)}} - i\pi\delta(E_a^{(0)} - E_b^{(0)})$ where the symbol P signifies the 'principal value' of the reciprocal function while δ denotes Dirac's delta function (Dirac (1935), p. 77).

function of $E_b^{(0)}$ for arbitrarily large (yet finite) values of L. Its chief properties are the following:

$$(E_a^{(0)}-E_b^{(0)})\left\{\frac{P}{E_a^{(0)}-E_b^{(0)}}-i\pi\delta(E_a^{(0)}-E_b^{(0)})\right\}_L = 1 \quad ; \quad (b\neq a) \tag{4.11}$$

$$\int_{E_a^{(0)}-\frac{1}{2}\delta E_a^{(0)}}^{E_a^{(0)}+\frac{1}{2}\delta E_a^{(0)}} \left\{\frac{P}{E_a^{(0)}-E_b^{(0)}}-i\pi\delta(E_a^{(0)}-E_b^{(0)})\right\}_L f(E_b^{(0)})\,dE_b^{(0)} = -i\pi f(E_a^{(0)}) \tag{4.12}$$

for any continuous function $f(E_b^{(0)})$ of $E_b^{(0)}$ and for any arbitrarily small $\delta E_a^{(0)}$. (Nevertheless, $\delta E_a^{(0)}$ is large in comparison with the spacing $\hbar c L^{-1}$ of the almost continuous energy-values. It needs not necessarily be the same as the allowance $\Delta E_a^{(0)}$ used in the definition of $\deg^{(0)} a$). The other factor of (4.10), namely $(\varphi_b^{(0)}\cdot S^{(2)}\varphi_a^{(0)}) = (\varphi_B^{(0)}(E_b^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)}))$ which is defined only for $b \varepsilon \deg^{(0)} a$ and $b\neq a$, is to be determined from the following system of integral equations (the integration over the continuous variables describing the detailed specification B is hidden under the summation sign)

$$\begin{aligned}(\varphi_C^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)})) &= (\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_A^{(0)}(E_a^{(0)})) \\ &-i\pi\sum_{B\neq C,A}(\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_B^{(0)}(E_a^{(0)}))\,\rho_B^{(0)}(E_a^{(0)})\,(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)})), \\ &\qquad (C\neq A).\end{aligned} \tag{4.13}$$

Here the allowance $\Delta E_a^{(0)}$ has been made arbitrarily small. The kernel $(\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_B^{(0)}(E_a^{(0)}))$, $(B\neq C)$, is of bounded variation with respect to the variables describing the detailed specification B, and so is the unique solution of (4.13). Since $\rho_B^{(0)}(E_a^{(0)})$ contains the factor L^3, and $(\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_B^{(0)}(E_a^{(0)}))$, $(B\neq C)$ contains the factor L^{-3}, (4.13) shows that $(\varphi_B^{(0)}(E_a^{(0)}) \cdot S^{(2)} \varphi_A^{(0)}(E_a^{(0)}))$, $(B \neq A)$, contains the factor L^{-3}.

In order to verify the solution (4.9) and (4.10), substitute these into (4.2). For c = a, (4.2) then becomes

$$E_a^{(2)} = (\varphi_a^{(0)} \cdot K^{(2)} \varphi_a^{(0)}) + \sum_{b \varepsilon deg^{(0)} a}^{b \neq a} (\varphi_a^{(0)} \cdot K^{(2)} \varphi_b^{(0)}) \left\{ \frac{P}{E_a^{(0)} - E_b^{(0)}} - i\pi \delta(E_a^{(0)} - E_b^{(0)}) \right\}_L (\varphi_b^{(0)} \cdot S^{(2)} \varphi_a^{(0)}). \tag{4.14}$$

By applying (4.8) and then (4.12), this gives for $E_a^{(2)}$,

$$E_a^{(2)} = (\varphi_a^{(0)} \cdot K^{(2)} \varphi_a^{(0)}) - i\pi \sum_{B \neq A} (\varphi_A^{(0)}(E_a^{(0)}) \cdot K^{(2)} \varphi_B^{(0)}(E_b^{(0)})) \rho_B^{(0)}(E_b^{(0)}) (\varphi_B^{(0)}(E_b^{(0)}) \cdot S^{(2)} \varphi_A^{(0)}(E_a^{(0)})) \tag{4.15}$$

The $i\pi$-term is of the order of L^{-3} and can be neglected in comparison with $(\varphi_a^{(0)} \cdot K^{(2)} \varphi_a^{(0)})$. ~~Now~~ Further, for $c \neq a$, (4.2) can be written in the form

$$(\varphi_c^{(0)} \cdot K^{(2)} \varphi_a^{(0)})(\varphi_a^{(0)} \cdot \varphi_a^{(2)}) + \sum_{b \varepsilon deg^{(0)} a}^{b \neq c, a} (\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)})(\varphi_b^{(0)} \cdot \varphi_a^{(2)})$$
$$= (\varphi_c^{(0)} \cdot \varphi_a^{(2)}) \left\{ E_a^{(2)} - (\varphi_c^{(0)} \cdot K^{(2)} \varphi_c^{(0)}) \right\}, \quad (c \varepsilon deg^{(0)} a, \ c \neq a). \tag{4.16}$$

In view of the fact that $(\varphi_c^{(0)} \cdot \varphi_a^{(2)})$ for $c \neq a$ contains $(\varphi_c^{(0)} \cdot S^{(2)} \varphi_a^{(0)})$, $(c \neq a)$, which is already of the order of L^{-3}, the coefficient on the right-hand-side of (4.16) can be replaced by $E_a^{(2)} - E_c^{(2)}$, on account of (4.15). To be consistent with the approximation made above (following (3.17)) in the derivation of the matrix elements for the secular perturbation, however, this coefficient is to be approximated by $E_a^{(1)} - E_c^{(1)}$ (that is, by (4.1), $E_a^{(0)} - E_c^{(0)}$). Thus (4.16) becomes, by substitution from (4.9) and (4.10),

$$(\varphi_c^{(0)} \cdot K^{(2)} \varphi_a^{(0)}) + \sum_{b \varepsilon \deg^{(0)} a}^{b \neq c, a} (\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)}) \left\{ \frac{P}{E_a^{(0)} - E_b^{(0)}} - i\pi \delta(E_a^{(0)} - E_b^{(0)}) \right\}_L (\varphi_b^{(0)} \cdot S^{(2)} \varphi_a^{(0)})$$

$$= (E_a^{(0)} - E_c^{(0)}) \left\{ \frac{P}{E_a^{(0)} - E_c^{(0)}} - i\pi \delta(E_a^{(0)} - E_c^{(0)}) \right\}_L (\varphi_c^{(0)} \cdot S^{(2)} \varphi_a^{(0)}) , \tag{4.17}$$

$$(c \, \varepsilon \deg^{(0)} a , \quad c \neq a).$$

By using (4.8), (4.11), (4.12), and making the allowance $\Delta E_a^{(0)}$ arbitrarily small, (4.17) becomes (4.13). Thus the asymptotical solution (4.9) and (4.10) is verified for any state $\varphi_a^{(2)}$ which is derived from the corresponding group $\deg^{(0)} a$ of the almost continuous spectrum.

The complex conjugate of (4.9) and (4.10) are

$$(\varphi_a^{(2)} \cdot \varphi_a^{(0)}) = 1 , \tag{4.18}$$

$$(\varphi_a^{(2)} \cdot \varphi_b^{(0)}) = \left\{ \frac{P}{E_a^{(0)} - E_b^{(0)}} + i\pi \delta(E_a^{(0)} - E_b^{(0)}) \right\}_L (S^{(2)} \varphi_a^{(0)} \cdot \varphi_b^{(0)}) ,$$

$$(b \, \varepsilon \deg^{(0)} a , \quad b \neq a). \tag{4.19}$$

For two different states, consider

$$(\varphi_c^{(2)} \cdot \varphi_a^{(2)}) = (\varphi_c^{(2)} \cdot \varphi_a^{(0)}) + (\varphi_c^{(0)} \cdot \varphi_a^{(2)}) + \sum_{b \neq c, a} (\varphi_c^{(2)} \cdot \varphi_b^{(0)})(\varphi_b^{(0)} \cdot \varphi_a^{(2)}) , \quad (c \neq a). \tag{4.20}$$

Of course, (4.20) vanishes if $\deg^{(0)} a$ and $\deg^{(0)} c$ do not overlap; it will be shown that (4.20) also vanishes if $c \, \varepsilon \deg^{(0)} a$, so the two groups $\deg^{(0)} a$ and $\deg^{(0)} c$ do overlap. By substituing (4.9), (4.10), (4.18) and (4.19) into (4.20), (4.20) becomes

$$(\varphi_c^{(2)}\cdot\varphi_a^{(2)}) = \left\{\frac{P}{E_c^{(0)}-E_a^{(0)}} + i\pi\delta(E_c^{(0)}-E_a^{(0)})\right\}_L (S^{(2)}\varphi_c^{(0)}\cdot\varphi_a^{(0)})$$

$$+\left\{\frac{P}{E_a^{(0)}-E_c^{(0)}} - i\pi\delta(E_a^{(0)}-E_c^{(0)})\right\}_L (\varphi_c^{(0)}\cdot S^{(2)}\varphi_a^{(0)})$$

$$+\sum_{b\,\varepsilon\,\deg^{(0)}c\,.\,\deg^{(0)}a}^{b\neq c,a} \left\{\frac{P}{E_c^{(0)}-E_b^{(0)}} + i\pi\delta(E_c^{(0)}-E_b^{(0)})\right\}_L (S^{(2)}\varphi_c^{(0)}\cdot\varphi_b^{(0)}) \times \tag{4.21}$$

$$\left\{\frac{P}{E_a^{(0)}-E_b^{(0)}} - i\pi\delta(E_a^{(0)}-E_b^{(0)})\right\}_L (\varphi_b^{(0)}\cdot S^{(2)}\varphi_a^{(0)}),$$

$$(c\ \varepsilon\ \deg^{(0)}a\ ,\quad c\neq a).$$

The summation indicated by $b\ \varepsilon\ \deg^{(0)}c.\deg^{(0)}a$ extends over the states in common to the two groups $\deg^{(0)}c$ and $\deg^{(0)}a$. By (4.8) and (4.12), the main contribution to this sum arises only from those states $\varphi_b^{(0)}$ with $E_b^{(0)}$ lying within an arbitrarily small interval $\delta E_a^{(0)}$ around the value $E_a^{(0)}$ and also those with $E_b^{(0)}$ within an arbitrarily small interval $\delta E_c^{(0)}$ around the value $E_c^{(0)}$. Hence (4.21) becomes

$$(\varphi_c^{(2)}\cdot\varphi_a^{(2)}) = \left\{\frac{P}{E_c^{(0)}-E_a^{(0)}} + i\pi\delta(E_c^{(0)}-E_a^{(0)})\right\}_L \Big\{(S^{(2)}\varphi_c^{(0)}\cdot\varphi_a^{(0)}) -$$

$$(\varphi_c^{(0)}\cdot S^{(2)}\varphi_a^{(0)}) - i\pi\sum_{B\neq C,A}(S^{(2)}\varphi_c^{(0)}\cdot\varphi_B^{(0)}(E_a^{(0)}))\rho_B^{(0)}(E_a^{(0)})(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_a^{(0)}) \tag{4.22}$$

$$-i\pi\sum_{B\neq C,A}(S^{(2)}\varphi_c^{(0)}\cdot\varphi_B^{(0)}(E_c^{(0)}))\rho_B^{(0)}(E_c^{(0)})(\varphi_B^{(0)}(E_c^{(0)})\cdot S^{(2)}\varphi_a^{(0)})\Big\}$$

$$(c\ \varepsilon\ \deg^{(0)}a\ ,\quad c\neq a).$$

Now, by (4.5), $(\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_A^{(0)}(E_a^{(0)}))$ is self-adjoint. The adjoint equation of (4.13) can then be written in the form

$$(S^{(2)}\varphi_C^{(0)}(E_a^{(0)})\cdot\varphi_A^{(0)}(E_a^{(0)})) = (\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_A^{(0)}(E_a^{(0)})) \tag{4.23}$$

$$+ i\pi\sum_{B\neq C,A}(S^{(2)}\varphi_C^{(0)}(E_a^{(0)})\cdot\varphi_B^{(0)}(E_a^{(0)}))\rho_B^{(0)}(E_a^{(0)})(\varphi_B^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_A^{(0)}(E_a^{(0)})),$$

$$(C\neq A).$$

From (4.23) and (4.13) follows

$$\begin{aligned}
&\left(S^{(2)}\varphi_C^{(0)}(E_a^{(0)})\cdot\varphi_A^{(0)}(E_a^{(0)})\right) - \left(\varphi_C^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)})\right)\\
&= i\pi \sum_{B\neq C,A}\Big\{\left(S^{(2)}\varphi_C^{(0)}(E_a^{(0)})\cdot\varphi_B^{(0)}(E_a^{(0)})\right)\rho_B^{(0)}(E_a^{(0)})\left(\varphi_B^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_A^{(0)}(E_a^{(0)})\right)\\
&\qquad +\left(\varphi_C^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_B^{(0)}(E_a^{(0)})\right)\rho_B^{(0)}(E_a^{(0)})\left(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)})\right)\Big\} \qquad (4.24)\\
&= 2i\pi \sum_{B\neq C,A}\left(S^{(2)}\varphi_C^{(0)}(E_a^{(0)})\cdot\varphi_B^{(0)}(E_a^{(0)})\right)\rho_B^{(0)}(E_a^{(0)})\left(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)})\right), \quad (C\neq A)
\end{aligned}$$

(the last step being effected by using (4.13) and (4.23) again, with B for C). Since for large values of L, the allowance $\Delta E_a^{(0)}$ can be made arbitrarily small, (4.24) shows that the right-hand-side of (4.22) vanishes. The $\varphi_a^{(2)}$'s are therefore mutually orthogonal.

From (4.9), (4.10), (4.18) and (4.19) follows also

$$\begin{aligned}
\left(\varphi_a^{(2)}\cdot\varphi_a^{(2)}\right) &= 1 + \sum_{b\,\varepsilon\,\deg^{(0)}a}^{b\neq a}\left|\left\{\frac{P}{E_a^{(0)}-E_b^{(0)}} - i\pi\delta(E_a^{(0)}-E_b^{(0)})\right\}_L\right|^2\left|\left(\varphi_b^{(0)}\cdot S^{(2)}\varphi_a^{(0)}\right)\right|^2\\
&= 1 + \int_{E_a^{(0)}-\frac{1}{2}\Delta E_a^{(0)}}^{E_a^{(0)}+\frac{1}{2}\Delta E_a^{(0)}}\left|\left\{\frac{P}{E_a^{(0)}-E_b^{(0)}} - i\pi\delta(E_a^{(0)}-E_b^{(0)})\right\}_L\right|^2\sum_{B\neq A}\left|\left(\varphi_B^{(0)}(E_b^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)})\right)\right|^2\rho_B^{(0)}(E_b^{(0)})\,dE_b^{(0)} \qquad (4.25)
\end{aligned}$$

the last step being effected by using (4.8). It will be shown later, (5.15) ÷ (5.17), that, for large values of L, the integral of (4.25) can be neglected in comparison with unity. Hence the $\varphi_a^{(2)}$'s are also normalized.

If the group $\deg^{(0)}a$ contains no other state $\varphi_b^{(0)}$ for which $(\varphi_b^{(0)}\cdot K^{(2)}\varphi_a^{(0)}) \neq 0$ than the state $\varphi_a^{(0)}$ itself, then the solution of the secular problem (4.2) is ~~then~~ trivial. Namely,

$$(\varphi_a^{(0)} \cdot \varphi_a^{(2)}) = 1\,, \text{ i.e. } \varphi_a^{(2)} = \varphi_a^{(0)}\ ; \quad \text{also } E_a^{(2)} = (\varphi_a^{(0)} \cdot K^{(2)} \varphi_a^{(0)}). \tag{4.26}$$

For the calculation of the emrgy-values by (4.15) or (4.26), the diagonal matrix element of $K^{(2)}$ can be evaluated as follows. Consider the numerator of the fraction term of (4.3). For an intermediate state $\varphi_{a'}^{(2)}$ which is derived from the group $\deg^{(0)} a'$ of states $\varphi_{b'}^{(0)}$ of the almost continuous spectrum, one has, by (4.9), (4.10), and applying (4.8) and (4.12),

$$\sum_{b' \varepsilon \deg^{(0)} a'} (\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot \varphi_{a'}^{(2)}) = (\varphi_c^{(0)} \cdot H^1 \varphi_{a'}^{(0)}) + \sum_{\substack{b' \neq a' \\ b' \varepsilon \deg^{(0)} a'}} (\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)}) \left\{ \frac{P}{E_{a'}^{(0)} - E_{b'}^{(0)}} - i\pi\delta(E_{a'}^{(0)} - E_{b'}^{(0)}) \right\}_L (\varphi_{b'}^{(0)} \cdot S^{(2)} \varphi_{a'}^{(0)}) \tag{4.27}$$

$$= (\varphi_c^{(0)} \cdot H^1 \varphi_{a'}^{(0)}) - i\pi \sum_{B' \neq A'} (\varphi_c^{(0)} \cdot H^1 \varphi_{B'}^{(0)}(E_{a'}^{(0)})) \rho_{B'}^{(0)}(E_{a'}^{(0)}) (\varphi_{B'}^{(0)}(E_{a'}^{(0)}) \cdot S^{(2)} \varphi_{A'}^{(0)}(E_{a'}^{(0)})).$$

By (4.27) and also its conjugate complex, (4.3) becomes

$$(\varphi_c^{(0)} \cdot K^{(2)} \varphi_c^{(0)}) = E_c^{(0)} + (\varphi_c^{(0)} \cdot H^2 \varphi_c^{(0)}) + \sum_{b'} \frac{(\varphi_c^{(0)} \cdot H^1 \varphi_{b'}^{(0)})(\varphi_{b'}^{(0)} \cdot H^1 \varphi_c^{(0)})}{E_c^{(0)} - E_{b'}^{(0)}}$$
$$+ \sum_{a' \bar{\varepsilon} \deg^{(0)} c} \frac{\left| (\varphi_c^{(0)} \cdot H^1 \varphi_{a'}^{(0)}) - i\pi \sum_{B' \neq A'} (\varphi_c^{(0)} \cdot H^1 \varphi_{B'}^{(0)}(E_{a'}^{(0)})) \rho_{B'}^{(0)}(E_{a'}^{(0)}) (\varphi_{B'}^{(0)}(E_{a'}^{(0)}) \cdot S^{(2)} \varphi_{A'}^{(0)}(E_{a'}^{(0)})) \right|^2}{E_c^{(0)} - E_{a'}^{(0)}}. \tag{4.28}$$

Here the first summation contains only a few intermediate states (arising from $\varphi_c^{(0)}$ by virtual absorption of some quanta), while the second summation extends over a large number of intermediate states (arising from $\varphi_c^{(0)}$ by virtual emission of some quanta). For large values of L, the first sum is of the order of L^{-3} and can be neglected in comparison with the second sum which is independent of L.

5. Physical interpretation of the secular change of basis for the states of the almost continuous spectrum

Consider the eigenfunction $\varphi_a^{(2)}$ of the secular problem. By (3.15), (4.1), (4.9) and (4.10), it reads

$$\varphi_a^{(2)} = \varphi_a^{(0)} + \sum_{b \varepsilon deg^{(0)} a}^{b \neq a} \varphi_b^{(0)} \left\{ \frac{P}{E_a^{(0)} - E_b^{(0)}} - i\pi\delta(E_a^{(0)} - E_b^{(0)}) \right\}_L (\varphi_b^{(0)} \cdot S^{(2)} \varphi_a^{(0)}) \tag{5.1}$$

The physical interpretation of this particular superposition of the states $\varphi_b^{(0)}$ of the almost continuous spectrum (b ε deg$^{(0)}$ a) can best be explained in the language of wave mechanics. In wave mechanics, each state $\varphi_b^{(0)}$ of the idealized system corresponds to a wave propagation in the configuration space of the quanta, the frequency of the wave propagation being given by $E_b^{(0)}/h$. The superposition (5.1) then represents a <u>pencil</u> of wave propagations, which is composed mainly of an <u>incident</u> wave $\varphi_a^{(0)}$ supported by the appropriate <u>scattered</u> waves $\varphi_b^{(0)}$ of approximately the same frequency, the amplitude $(\varphi_b^{(0)} \cdot S^{(2)} \varphi_a^{(0)})$, $(b \neq a)$, of each scattered wave being determined by the interaction $(\varphi_c^{(0)} \cdot K^{(2)} \varphi_b^{(0)})$, $(c \neq b)$, among all the waves in accordance with (4.13). This interpretation is suggested, in the special case where all the states $\varphi_b^{(0)}$ considered correspond to only two particles in the fundamental cube, by a comparison with the familiar wave-mechanical theory of atomic collisions. For example, let the rest mass of one of the particles be practically infinite (fixed center), and let the sides L of the fundamental cube tend to infinity. Because of the interaction, the stationary-state solution $\varphi_a^{(2)}$ is of the form (cf. Mott and Massey (1933) Chap. II),

$$\varphi_a^{(2)} = e^{i\underline{k}_a\cdot\underline{r}} + r^{-1}e^{ik_a r}\,S(\vartheta,\varphi) \tag{5.2}$$

for large values of the distance r from the fixed center.‡ In this particular case, Dirac (1935, p.199) has shown, by a direct Fourier transformation, that the scattered wave $r^{-1}e^{ik_a r}S(\vartheta,\varphi)$ can be resolved as a superposition of wave propagations

$$\varphi_b^{(0)} = e^{i\underline{k}_b\cdot\underline{r}} \tag{5.3}$$

by means of the particular combination of Dirac's delta function and the principle-valued reciprocal function as given by the expression within the braces of (5.1).

In order to show that the above interpretation is valid in the general case, it is convenient to work in the time-representation. By (4.1) and (3.37), the secular problem (3.42) reads, for r = 2,

$$-\frac{\hbar}{i}\frac{d(\Phi_c^{(0)}\cdot\Phi_a^{(2)})}{dt} = (\varphi_c^{(0)}\cdot K^{(2)2}\varphi_c^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)}) + \sum_{b\,\varepsilon\,\deg^{(0)}a}^{b\neq c} e^{i(E_c^{(0)}-E_b^{(0)})t/\hbar}(\varphi_c^{(0)}\cdot K^{(2)}\varphi_b^{(0)})(\Phi_b^{(0)}\cdot\Phi_a^{(2)}) \tag{5.4}$$

where, by (3.2), $K^{(2)2} = K^{(2)} - H^0$.

‡This holds only for the regular case. In the case of Coulomb interaction, the phase of the incident wave has to be modified according to Gordon (cf. Mott and Massey (1933) Chap. III). By a Fourier transformation, it is easily seen that such complication due to the long range of the Coulomb interaction is connected with Fermi's longitudinal quanta of arbitrarily small energy and momentum. As mentioned in the beginning of the last section, such complication will not be considered in this paper.

The asymptotical solution which is equivalent to that ~~given by (4.9)~~ given by (4.9) and (4.10) now assumes the form

$$(\Phi_a^{(0)} \cdot \Phi_a^{(2)}) = e^{i(E_a^{(0)} - E_a^{(2)})t/\hbar}, \tag{5.5}$$

$$(\Phi_b^{(0)} \cdot \Phi_a^{(2)}) = e^{i(E_a^{(0)} - E_a^{(2)})t/\hbar} \frac{e^{i(E_b^{(0)} - E_a^{(0)})t/\hbar} - 1}{E_a^{(0)} - E_b^{(0)}} (\varphi_b^{(0)} \cdot S^{(2)} \varphi_a^{(0)}), \tag{5.6}$$

$$(b \varepsilon \deg^{(0)} a, \quad b \neq a),$$

which are valid only for the following range of values for t

$$\frac{1}{\gamma_a^{(2)}} >> t >> \left| \frac{\hbar}{E_b^{(0)} - E_a^{(0)}} \right|, \quad (b \varepsilon \deg^{(0)} a, \ b \neq a). \tag{5.7}$$

Here, the lower bound for t is of the order of L/c and the condition $t >> L/c$ is of course necessary in order that the scattered wave reaches its stationary state. The upper bound for t is necessiated by the decay of the amplitude of the incident wave, the decay constant $\gamma_a^{(2)}$ being related to the imaginary part of $E_a^{(2)}$ as follows (compare (5.5)).

$$\gamma_a^{(2)} = -\frac{2}{\hbar} \mathcal{J}(E_a^{(2)})$$

$$= \frac{2\pi}{\hbar} \mathcal{R} \left\{ \sum_{B \neq A} (\varphi_A^{(0)}(E_a^{(0)}) \cdot K^{(2)} \varphi_B^{(0)}(E_a^{(0)})) \rho_B^{(0)}(E_a^{(0)}) (\varphi_B^{(0)}(E_a^{(0)}) \cdot S^{(2)} \varphi_A^{(0)}(E_a^{(0)})) \right\} \text{ by (4.15)}, \tag{5.8}$$

$$= \frac{2\pi}{\hbar} \sum_{B \neq A} \rho_B^{(0)}(E_a^{(0)}) \left| (\varphi_B^{(0)}(E_a^{(0)}) \cdot S^{(2)} \varphi_A^{(0)}(E_a^{(0)})) \right|^2 \text{ by (4.13)}.$$

Since $\rho_B^{(0)}(E_a^{(0)})$ contains a factor L^3, and, for $B \neq A$, $|(\varphi_B^{(0)}(E_a^{(0)}) \cdot S^{(2)} \varphi_A^{(0)}(E_a^{(0)}))|^2$ contains a factor L^{-6}, (5.8) shows that $\gamma_a^{(2)}$ contains a factor L^{-3}. The upper bound for t contains therefore a factor L^3. For large values of L, (5.7) can be satisfied without difficulty (say, by letting t contain a factor L^2).

The solution (5.5) and (5.6) in the time-representation can be verified directly by substituting the solution into (5.4). For $c = a$, (5.4) then becomes, (the common factor exp $i(E_a^{(0)}-E_a^{(2)})t/\hbar$ being removed throughout)

$$E_a^{(2)} - E_a^{(0)} = (\varphi_a^{(0)}\cdot K^{(2)2}\varphi_a^{(0)}) + \sum_{b\varepsilon \deg^{(0)}a}^{b\neq a} (\varphi_a^{(0)}\cdot K^{(2)}\varphi_b^{(0)}) \frac{1-e^{i(E_a^{(0)}-E_b^{(0)})t/\hbar}}{E_a^{(0)}-E_b^{(0)}} (\varphi_b^{(0)}\cdot S^{(2)}\varphi_a^{(0)}). \tag{5.9}$$

The sum can be transformed by (4.8) into an integral, which can then be evaluated by the following formula

$$\int_{E_a^{(0)}-\frac{1}{2}\Delta E_a^{(0)}}^{E_a^{(0)}+\frac{1}{2}\Delta E_a^{(0)}} \frac{1-e^{i(E_a^{(0)}-E_b^{(0)})t/\hbar}}{E_a^{(0)}-E_b^{(0)}} f(E_b^{(0)})\,dE_b^{(0)} = -i\pi f(E_a^{(0)}). \tag{5.10}$$

(5.10) holds for any continuous function $f(E_b^{(0)})$ of $E_b^{(0)}$ and for any arbitrarily small allowance $\Delta E_a^{(0)}$, provided that t satisfies (5.7) and hence $t\,\Delta E_a^{(0)}/\hbar > |t(E_b^{(0)}-E_a^{(0)})/\hbar| \gg 1$, $(b\varepsilon \deg^{(0)}a,\ b\neq a)$. Thus (5.9) gives the same expression as (4.15) for $E_a^{(2)}$. Further, for $c \neq a$, (5.4) becomes, by differentiating (5.6),

$$(E_a^{(2)}-E_a^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)}) + e^{i(E_c^{(0)}-E_a^{(2)})t/\hbar}(\varphi_c^{(0)}\cdot S^{(2)}\varphi_a^{(0)})$$
$$= (\varphi_c^{(0)}\cdot K^{(2)2}\varphi_c^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)}) + e^{i(E_c^{(0)}-E_a^{(0)})t/\hbar}(\varphi_c^{(0)}\cdot K^{(2)}\varphi_c^{(0)})(\Phi_a^{(0)}\cdot\Phi_a^{(2)}) \tag{5.11}$$
$$+\sum_{b\varepsilon \deg^{(0)}a}^{b\neq c,a} e^{i(E_c^{(0)}-E_b^{(0)})t/\hbar}(\varphi_c^{(0)}\cdot K^{(2)}\varphi_b^{(0)})(\Phi_b^{(0)}\cdot\Phi_a^{(2)})$$
$$(c\,\varepsilon \deg^{(0)}a,\ c\neq a).$$

To be consistent with the approximation made above (following (3.17)) in the derivation of the matrix elements for the secular perturbation, and by using

(4.1), $(E_a^{(2)}-E_c^{(2)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)})$ can be approximated by $(E_a^{(0)}-E_c^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)})$ for $c\,\varepsilon\,\deg^{(0)}a$ and $c\neq a$. Hence

$$\begin{aligned}(E_a^{(2)}-E_a^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)}) &= (E_c^{(2)}-E_c^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)}) \\ &= (\varphi_c^{(0)}\cdot K^{(2)2}\varphi_c^{(0)})(\Phi_c^{(0)}\cdot\Phi_a^{(2)}), \quad (c\,\varepsilon\,\deg^{(0)}a,\ c\neq a),\end{aligned} \tag{5.12}$$

the last step being effected by using (4.15), seeing that, for $c\neq a$, $(\Phi_c^{(0)}\cdot\Phi_a^{(2)})$ contains already the factor $(\varphi_c^{(0)}\cdot S^{(2)}\varphi_a^{(0)})$ which is of the order of L^{-3}. Now substract (5.12) from (5.11), introduce (5.5) and (5.6), and remove the common factor $\exp i(E_c^{(0)}-E_a^{(2)})t/\hbar$ throughout. By using (4.8) and (5.10), the result

$$\begin{aligned}(\varphi_c^{(0)}\cdot S^{(2)}\varphi_a^{(0)}) &= (\varphi_c^{(0)}\cdot K^{(2)}\varphi_a^{(0)}) + \\ &\sum_{b\,\varepsilon\,\deg^{(0)}a}^{b\neq c,a}(\varphi_c^{(0)}\cdot K^{(2)}\varphi_b^{(0)})\frac{1-e^{i(E_a^{(0)}-E_b^{(0)})t/\hbar}}{E_a^{(0)}-E_b^{(0)}}(\varphi_b^{(0)}\cdot S^{(2)}\varphi_a^{(0)}), \\ &(c\,\varepsilon\,\deg^{(0)}a,\ c\neq a)\end{aligned} \tag{5.13}$$

is easily seen to be the same as (4.13). Thus the solution (5.5) and (5.6) is verified for the range (5.7) of values of t.

The solution (5.5) and (5.6) are connected with the solution (4.9) and (4.10) by (3.39), that is

$$(\Phi_c^{(0)}\cdot\Phi_a^{(2)}) = e^{i(E_c^{(0)}-E_a^{(2)})t/\hbar}(\varphi_c^{(0)}\cdot\varphi_a^{(2)}) \tag{5.14}$$

Hence, for the range (5.7) of values of t, (or for t containing a factor L^2, say)

$$\left\{\frac{P}{E_a^{(0)}-E_b^{(0)}}-i\pi\delta(E_a^{(0)}-E_b^{(0)})\right\}_L = \frac{1-e^{i(E_a^{(0)}-E_b^{(0)})t/\hbar}}{E_a^{(0)}-E_b^{(0)}}, \tag{5.15}$$

$$(b\,\varepsilon\,\deg^{(0)}a,\quad b\neq a).$$

With the help of (5.15), the integral of (4.25) can be evaluated by the formula

$$\int_{E_a^{(0)}-\frac{1}{2}\Delta E_a^{(0)}}^{E_a^{(0)}+\frac{1}{2}\Delta E_a^{(0)}} \left|\frac{1-e^{i(E_a^{(0)}-E_b^{(0)})t/\hbar}}{E_a^{(0)}-E_b^{(0)}}\right|^2 f(E_b^{(0)})\,dE_b^{(0)} = \frac{2\pi t}{\hbar} f(E_a^{(0)}) \qquad (5.16)$$

which is valid under the same conditions as those for (5.10). By (5.8) and (5.7), the normalization condition (4.25) of the $\varphi_a^{(2)}$ is easily verified for large values of L,

$$(\varphi_a^{(2)}\cdot\varphi_a^{(2)}) = 1 + \gamma_a^{(2)}t \rightarrow 1\,. \qquad (5.17)$$

Strictly speaking, for finite values of L, $\Phi_a^{(2)}$ represents a quasi-stationary state of the secular problem. It can be represented by a probability distribution over the stationary states $\Phi_b^{(0)}$ of the idealized system, namely, by (5.5) and (5.6),

$$|(\Phi_a^{(0)}\cdot\Phi_a^{(2)})|^2 = e^{-\gamma_a^{(2)}t} = 1-\gamma_a^{(2)}t\,, \qquad (5.18)$$

$$\sum_{b\varepsilon\,\deg_B^{(0)} a} |(\Phi_b^{(0)}\cdot\Phi_a^{(2)})|^2 = \int_{E_a^{(0)}-\frac{1}{2}\Delta E_a^{(0)}}^{E_a^{(0)}+\frac{1}{2}\Delta E_a^{(0)}} \left|\frac{1-e^{i(E_a^{(0)}-E_b^{(0)})t/\hbar}}{E_a^{(0)}-E_b^{(0)}}\right|^2 |(\varphi_B^{(0)}(E_b^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)}))|^2 \rho_B^{(0)}(E_b^{(0)})\,dE_b^{(0)} \qquad (5.19)$$

$$= \frac{2\pi t}{\hbar}\rho_B^{(0)}(E_a^{(0)})\,|(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)}))|^2\,, \qquad (B\neq A),$$

which are valid only for the range (5.7) of values of t. As indicated by $\deg_B^{(0)} a$, (5.19) gives the total probability over all states $\Phi_b^{(0)}$ within an arbitrarily small allowance $\Delta E_a^{(0)}$ around $E_a^{(0)}$ which all conform to the detailed specification B. By differentiation with respect to t, it is seen that the rate of decay (namely $\gamma_a^{(2)}$) of $|(\Phi_a^{(0)}\cdot\Phi_a^{(2)})|^2$ equals, by (5.8), the sum of the rate of growth (say $\gamma^{(2)}_{BA}(E^{(0)}_a)$) for all groups $\deg^{(0)}_B a$, ~~(5.19)~~

$$\gamma^{(2)}_{BA}(E^{(0)}_a) = \frac{2\pi}{\hbar}\rho^{(0)}_B(E^{(0)}_a)\left|\left(\varphi^{(0)}_B(E^{(0)}_a)\cdot S^{(2)}\varphi^{(0)}_A(E^{(0)}_a)\right)\right|^2, \quad (B \neq A). \tag{5.19}$$

Similar analysis can be made for the quasi-stationary state $\Psi^{(2)}_a$ of the actual system

$$\Psi^{(2)}_a = e^{-iE^{(2)}_a t/\hbar}\left\{\varphi^{(2)}_a + \sum_c \varphi^{(2)}_c(\varphi^{(2)}_c\cdot\psi^{(2)1}_a) + \sum_c \varphi^{(2)}_c(\varphi^{(2)}_c\cdot\psi^{(2)2}_a)\right\}. \tag{5.20}$$

The probability distribution over the stationary states of the idealized system now extends to states of quite different energy-values, $c\ \bar{\varepsilon}\ \deg^{(0)} a$, (see (3.22) and (3.27)), which however does not grow with time. For $c\ \varepsilon\ \deg^{(0)} a$, (3.29) shows, with the help of (3.22), that $(\varphi^{(2)}_a\cdot\psi^{(2)2}_a)$ is independent of L,

$$(\psi^{(2)2}_a\cdot\varphi^{(2)}_a) + (\varphi^{(2)}_a\cdot\psi^{(2)2}_a) = -\sum_{b\,\bar{\varepsilon}\,\deg^{(0)}a}\frac{(\varphi^{(2)}_a\cdot H^1\varphi^{(2)}_b)(\varphi^{(2)}_b\cdot H^1\varphi^{(2)}_a)}{(E^{(0)}_a - E^{(0)}_b)^2}; \tag{5.21}$$

while $(\varphi^{(2)}_c\cdot\psi^{(2)2}_a)$ for $c\ \varepsilon\ \deg^{(0)} a$ and $c \neq a$ is of the order of L^{-3}. (This distinction is similar to what has been discussed above in connection with the diagonal <u>versus</u> the non-diagonal matrix elements of $K^{(2)}$). Hence for large values of L,

$$\left|(\Phi^{(0)}_c\cdot\Psi^{(2)}_a)\right|^2 = \left\{1 - \sum_{b\,\bar{\varepsilon}\,\deg^{(0)}a}\frac{(\varphi^{(2)}_a\cdot H^1\varphi^{(2)}_b)(\varphi^{(2)}_b\cdot H^1\varphi^{(2)}_a)}{(E^{(0)}_a - E^{(0)}_b)^2}\right\}\left|(\Phi^{(0)}_c\cdot\Phi^{(2)}_a)\right|^2, \quad (c\ \varepsilon\ \deg^{(0)} a), \tag{5.22}$$

which shows that the variation in time of the relative probabilities $|(\Phi_c^{(0)} \cdot \Psi_a^{(2)})|^2$ for c ε $\deg^{(0)}$a is the same as that of $|(\Phi_c^{(0)} \cdot \Phi_a^{(2)})|^2$. Now, on the one hand, from the theoretical point of view, one may regard the whole field (including the interactions) within the fundamental cube as a mechanical system, the state of affairs within which is described by the quasi-stationary state $\Psi_a^{(2)}$. On the other hand, from the experimental point of view, the fundamental cube represents roughly the region of space where different elementary particles collide, while outside this region the interaction may be neglected and so the free motion of these particles are described by the stationary states $\Phi_c^{(0)}$. The above analysis of the variation in time of the probability distribution $|(\Phi_c^{(0)} \cdot \Psi_a^{(2)})|^2$ shows that $\Phi_a^{(2)}$ may be used instead of $\Psi_a^{(2)}$ for the description of such collision problems. This justifies the physical interpretation of the secular eigenfunctions $\varphi_a^{(2)}$ stated in the beginning of this section. The $\gamma_{BA}^{(2)}(E_a^{(0)})$ of (5.19) thus gives th~~e~~he transition probabilities per unit time for the transition from $\varphi_a^{(0)}$ to the group $\deg_B^{(0)}$ a. The transition probabilities can be converted to the cross-sections in the usual way, which will then be independent of L.

In practice, the calculation of the cross-sections for collision problems involves the solution of the system of integral equations (4.13). These integral equations have already been obtained independently by Heitler (1941) and Wilson (1941), but their derivation did not claim to be mathematically rigorous, nor was the divergence difficulty removed. (In fact, they used the expansion method which makes no clear distinction between the secular and the non-secular perturbation and is, as shown in §2, bound to fail). Heitler (1941) has also shown, by a comparison with the classical electrodynamics in the special case of the non-relativistic scattering of light by an electron, that the cross-sections calculated by (5.19) and (4.13) includes the effect due to the reaction of the radiation (to the first order only), while that calculated by the usual expansion method corresponds to the neglection of this effect. This finding leads to a convenient interpretation of the secular change of basis for the states of almost continuous spectrum in the present treatment of collision problems by the method of secular perturbation. Namely, by the secular change of basis, which has to be done mathematically, one takes account of the reaction of the radiation. The method of secular perturbation can be continued to higher approximations, involving successive secular change of basis. By so doing, one takes account of the radiation reaction of higher order. Similar interpretation can also be given to the secular change of basis of the states of the almost continuous spectrum where such states act only as intermediate states.

6. The effects of the radiation reaction — finite self-energies and cross-sections

It will be shown in this section that the usual divergence difficulty automatically disappears if the interaction is treated by the method of secular perturbation. For simplicity, this will be demonstrated by means of an example, which deals with the interaction between the meson field and the nucleon field. (A nucleon is either a neutron or a proton)

Suppose the total charge of the system is +e, and the total momentum of the system vanishes. Let $\varphi_p^{(0)}$ denote the state which corresponds to a proton-like quantum at rest in the fundamental cube; let $\varphi_a^{(0)}$ denote the state which corresponds to a meson-like quantum of charge +e, and momentum $\underline{p}_a$, together with a neutron-like quantum of equal but opposite momentum. The energy-values of these states are

$$E_p^{(0)} = m_p c^2, \quad E_a^{(0)} = \sqrt{m_N^2 c^4 + c^2 p_a^2} + \sqrt{m_M^2 c^4 + c^2 p_a^2} \tag{6.1}$$

where m_p, m_N and m_M denote the rest masses of the proton-like, neutron-like, and meson-like quanta respectively. For simplicity, assume $m_p = m_N$ which is practically infinite, and neglect the dependence of the matrix elements $(\varphi_p^{(0)} \cdot H^1 \varphi_a^{(0)})$ on the spin orientations and the directions of the momenta of the quanta, so

$$\left.\begin{aligned} &E_a^{(0)} - E_P^{(0)} = \varepsilon_a \,, \quad \left(\varepsilon_a = \sqrt{m_M^2 c^4 + c^2 p_a^2}\right) \\ &(\varphi_P^{(0)} \cdot H^1 \varphi_a^{(0)}) = H(\varepsilon_a)\,, \text{ say}, \end{aligned}\right\} \tag{6.2}$$

which all depend only on the energy ε_a of the meson-like quantum. With this simplification, the kernel of the integral equation (4.13) reduces to a constant,

$$\begin{aligned} (\varphi_B^{(0)}(E_a^{(0)}) \cdot K^{(2)} \varphi_A^{(0)}(E_a^{(0)})) &= \frac{(\varphi_B^{(0)}(E_a^{(0)}) \cdot H^1 \varphi_P^{(0)})(\varphi_P^{(0)} \cdot H^1 \varphi_A^{(0)}(E_a^{(0)}))}{E_a^{(0)} - E_P^{(0)}} \\ &= \frac{|H(\varepsilon_a)|^2}{\varepsilon_a} = K(\varepsilon_a) \text{ say}, \quad (B \neq A). \end{aligned} \tag{6.3}$$

The density $\rho_B^{(0)}(E_a^{(0)})$ of states is given by

$$\rho_B^{(0)}(E_a^{(0)}) = L^3 c^{-2} h^{-3} p_a \varepsilon_a \, d\Omega_B \tag{6.4}$$

where $d\Omega_B$ denotes an element of solid angle around the direction of the momentum of the meson-like quantum (say). The solution of (4.13) is then trivial,

$$(\varphi_B^{(0)}(E_a^{(0)}) \cdot S^{(2)} \varphi_A^{(0)}(E_a^{(0)})) = \frac{K(\varepsilon_a)}{1 + i\pi K(\varepsilon_a)\rho(\varepsilon_a)} = S(\varepsilon_a) \text{ say}, \quad (B \neq A) \tag{6.5}$$

with

$$\rho(\varepsilon_a) = \sum_B \rho_B^{(0)}(E_a^{(0)}) = L^3 c^{-2} h^{-3} p_a \varepsilon_a 4\pi . \tag{6.6}$$

The transition probability (5.19) for the scattering of mesons of energy ε_a by a nucleon is, by (6.5),

$$\gamma_{BA}^{(2)}(E_a^{(0)}) = 2\pi \rho_B^{(0)}(E_a^{(0)}) \left| \frac{K(\varepsilon_a)}{1 + i\pi K(\varepsilon_a)\rho(\varepsilon_a)} \right|^2, \quad (B \neq A), \tag{6.7}$$

which differs from the usual expression given by the expansion method, namely

$$2\pi\rho_B^{(0)}(E_a^{(0)})\left|\left(\varphi_B^{(0)}(E_a^{(0)})\cdot K^{(2)}\varphi_A^{(0)}(E_a^{(0)})\right)\right|^2 = 2\pi\rho_B^{(0)}(E_a^{(0)})[K(\varepsilon_a)]^2, \quad (B\neq A) \tag{6.8}$$

by the factor $\{1+[\pi K(\varepsilon_a)\rho(\varepsilon_a)]^2\}^{-1}$. In the usual meson theory, for large values of ε_a, one has ($\propto$ meaning 'varies as'; g denoting the interaction constant)

$$H(\varepsilon_a) \propto L^{-\frac{3}{2}} g\, \varepsilon_a^{\frac{1}{2}} \text{ so } K(\varepsilon_a)\rho(\varepsilon_a) \propto g^2 \varepsilon_a^2 \tag{6.9}$$

by (6.3) and (6.6). The effect of the radiation reaction, as indicated by the appearance of the factor $\{1+[\pi K(\varepsilon_a)\rho(\varepsilon_a)]^2\}^{-1}$ in (6.7), is thus a reduction of the cross-sections for the scattering of mesons at high energies of the mesons. (For more accurate calculations of the scattering cross-sections, see Ma(1943), Heitler and Peng (1943), Ma and Hsüeh (1944); but the results are qualitatively similar to (6.7) discussed above. The effect of the radiation reaction makes the cross-sections a few hundred times smaller and is favoured by comparison with measurements on the scattering of cosmic-ray mesons by atomic nuclei (Heitler and Peng (1942)).

Consider now the self-energy of a proton, which, for the second approximation, is given by (cf.(4.28))

$$E_p^{(2)} - E_p^{(0)} = \sum_a \frac{\left|(\varphi_p^{(0)}\cdot H^1\varphi_a^{(0)}) - i\pi\sum_{B\neq A}(\varphi_p^{(0)}\cdot H^1\varphi_B^{(0)}(E_a^{(0)}))\rho_B^{(0)}(E_a^{(0)})(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)}))\right|^2}{E_p^{(0)} - E_a^{(0)}}. \tag{6.10}$$

From (6.2), (6.5) and (6.6) follows

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$$(\varphi_p^{(0)}\cdot H^1\varphi_a^{(0)}) - i\pi\sum_{B\neq A}(\varphi_p^{(0)}\cdot H^1\varphi_B^{(0)}(E_a^{(0)}))\rho_B^{(0)}(E_a^{(0)})(\varphi_B^{(0)}(E_a^{(0)})\cdot S^{(2)}\varphi_A^{(0)}(E_a^{(0)}))$$

$$= H(\varepsilon_a) - i\pi H(\varepsilon_a)\rho(\varepsilon_a)S(\varepsilon_a) = \frac{H(\varepsilon_a)}{1+i\pi K(\varepsilon_a)\rho(\varepsilon_a)}. \tag{6.11}$$

Hence (6.10) becomes

$$E_p^{(2)} - E_p^{(0)} = -\sum_a \left|\frac{H(\varepsilon_a)}{1+i\pi K(\varepsilon_a)\rho(\varepsilon_a)}\right|^2 \frac{1}{\varepsilon_a},$$

$$= -\int_{m_M c^2}^{\infty} \frac{|H(\varepsilon_a)|^2}{\varepsilon_a}\,\frac{\rho(\varepsilon_a)\,d\varepsilon_a}{1+[\pi K(\varepsilon_a)\rho(\varepsilon_a)]^2}, \tag{6.12}$$

by (4.8) and (6.7),

$$= -\int_{m_M c^2}^{\infty} \frac{K(\varepsilon_a)\rho(\varepsilon_a)}{1+[\pi K(\varepsilon_a)\rho(\varepsilon_a)]^2}\,d\varepsilon_a \quad \text{by (6.4)}.$$

This differs from the expression(which is usually taken to be the self-energy in the expansion method)

$$\sum_a \frac{|(\varphi_p^{(0)}\cdot H^1\varphi_a^{(0)})|^2}{E_p^{(0)} - E_a^{(0)}} = -\int_{m_M c^2}^{\infty}\frac{|H(\varepsilon_a)|^2}{\varepsilon_a}\rho(\varepsilon_a)\,d\varepsilon_a \left(= -\int_{m_M c^2}^{\infty} K(\varepsilon_a)\rho(\varepsilon_a)\,d\varepsilon_a\right) \tag{6.13}$$

again by the factor $\{1+[\pi K(\varepsilon_a)\rho(\varepsilon_a)]^2\}^{-1}$ in the integrand. This factor arises from the secular change of basis of the intermediate states, and may be interpreted as the effect of radiation reaction. For the usual meson theory, by (6.9), the integral (6.12) for the self-energy converges like $\int^{\infty}\frac{d\varepsilon_a}{\varepsilon_a^2}$ in the method of secular perturbation, while the integral (6.13) diverges ~~like~~ $\int^{\infty}\varepsilon_a^2\,d\varepsilon_a$ in the expansion method. The mathematical origin of the divergence difficulty arises from the fact that the factor

factor $\left\{1+[\pi K(\varepsilon_a)\rho(\varepsilon_a)]^2\right\}^{-1}$, though it assumes the value unity for vanishing interaction (g = 0), cannot be expanded as a power series in the interaction constant g for <u>all</u> values of ε_a.

For the calculation of the cross-sections for the scattering of mesons by a nucleon to the fourth approximation, say, the fourth-order secular perturbation $(\varphi_c^{(0)}\cdot K^{(4)}\varphi_b^{(0)})$, $(c\,\varepsilon\,deg^{(2)}b)$ can be derived by applying the expansion method in the basis $\{\varphi_b^{(4)}\}$ and then changing the basis, say to $\{\varphi_b^{(0)}\}$. (Compare (3.25) and (3.35) for an illustration). The non-diagonal element of $K^{(4)}$ contains in particular the sum (note that, for the non-degenerate state p, $\varphi_p^{(4)}=\varphi_p^{(2)}=\varphi_p^{(0)}$)

$$\sum_{b'\varepsilon\, deg^{(2)}b}\frac{(\varphi_c^{(0)}\cdot H^1\varphi_p^{(0)})\sum_{a'\varepsilon\, deg^{(2)}b'}(\varphi_p^{(0)}\cdot H^1\varphi_{a'}^{(0)})(\varphi_{a'}^{(0)}\cdot\varphi_{b'}^{(4)})\sum_{c'\varepsilon\, deg^{(2)}b'}(\varphi_{b'}^{(4)}\cdot\varphi_{c'}^{(0)})(\varphi_{c'}^{(0)}\cdot H^1\varphi_p^{(0)})(\varphi_p^{(0)}\cdot H^1\varphi_b^{(0)})}{(E_b^{(2)}-E_p^{(2)})\quad(E_b^{(2)}-E_{b'}^{(2)})\quad(E_b^{(2)}-E_p^{(2)})}\qquad (c\neq b). \tag{6.14}$$

In the expansion method, the secular change of basis in the numerator is neglected (also $E_a^{(0)}$ is used for $E_a^{(2)}$ which is not serious), resulting a divergent expression. In the method of secular perturbation, the non-diagonal elements (6.14) are needed for the determination of the secular eigenfunctions $\varphi_a^{(4)}$ for the fourth approximation, and for this purpose, (6.14) may be approximated by

$$\sum_{b'\varepsilon\, deg^{(2)}b}\frac{(\varphi_c^{(0)}\cdot H^1\varphi_p^{(0)})\sum_{a'\varepsilon\, deg^{(0)}b'}(\varphi_p^{(0)}\cdot H^1\varphi_{a'}^{(0)})(\varphi_{a'}^{(0)}\cdot\varphi_{b'}^{(2)})\sum_{c'\varepsilon\, deg^{(0)}b'}(\varphi_{b'}^{(2)}\cdot\varphi_{c'}^{(0)})(\varphi_{c'}^{(0)}\cdot H^1\varphi_p^{(0)})(\varphi_p^{(0)}\cdot H^1\varphi_b^{(0)})}{(E_b^{(2)}-E_p^{(2)})\quad(E_b^{(2)}-E_{b'}^{(2)})\quad(E_b^{(2)}-E_p^{(2)})},\qquad (c\neq b). \tag{6.15}$$

The approximation consists of replacing $\varphi_{b'}^{(4)}$ by $\varphi_{b'}^{(2)}$ which corresponds to the replacement of $\varphi_{b'}^{(2)}$ by $\varphi_{b'}^{(0)}$ ~~that~~ used above from (4.4) to (4.5) in the case of the second approximation. Because of the effect due to the secular change of basis from $\varphi^{(0)}_{a'}$ to $\varphi^{(2)}_{b'}$ for the intermediate states, (6.15), like (6.12), converges. Hence $(\varphi^{(0)}_c \cdot K^{(4)} \varphi^{(0)}_b)$, $(c \,\varepsilon\, \deg^{(2)} b,\ c \neq b)$ is of bounded variation, and the asymptotical solution of the secular problem can be obtained similarly. The transition probabilities $\gamma^{(4)}_{BA}(E^{(2)}_a)$ can be calculated from the corresponding $(\varphi^{(0)}_b \cdot S^{(4)} \varphi^{(0)}_a)$, $(b \,\varepsilon\, \deg^{(2)} a,\ b \neq a)$ without any divergence difficulty.

The self-energy for the fourth approximation can then be calculated with the help of $(\varphi^{(0)}_b \cdot S^{(4)} \varphi^{(0)}_a)$, $(b \,\varepsilon\, \deg^{(2)} a,\ b \neq a)$.

I wish to express my great indebtness to Ma and Hsüeh(1944). Without their new derivation of Heitler's and Wilson's integral equations in the energy-representation with the help of Dirac's delta function combined with the principle-valued reciprocal function, the above task to link up Heitler's and Wilson's preliminary ~~theory~~ treatment of radiation reaction with the rigorous perturbation theory for a system of almost continuous spectrum would have been more difficult.

I am also very much obliged to Professor Max Born for his continuous encouragement.

References

Born, Heisenberg and Jordan (1926) Zs. f. Physik, <u>35</u>, 557.

Born and Jordan (1930) 'Quantenmechanik', Springer, Berlin.

Born and Peng (1944) Proc.Roy.Soc.Edinburgh, <u>62</u>, 127.

Dirac (1927) Proc.Roy.Soc. <u>114</u>, 243, 710.

Dirac (1935) 'Quantum Mechanics', Oxford, 2nd edition.

Fermi (1932) Rev. Mod. Phys., <u>4</u>, 87.

Heisenberg and Pauli (1929) Zs. f. Physik, <u>56</u>, 1; <u>59</u>, 168.

Heitler (1941) Proc.Camb.Phil.Soc., <u>37</u>, 291.

Heitler and Peng (1942) Phys. Rev., <u>62</u>, 81.

Heitler and Peng (1943) Proc.Roy.Irish Acad., <u>49</u>, 112.

Jordan and Pauli (1928) Zs. f. Physik, <u>47</u>, 151.

Jordan and Wigner (1928) Zs. f. Physil, <u>47</u>, 631.

Ma (1943) Proc.Camb.Phil.Soc., <u>39</u>, 168.

Ma and Hsüch (1944) Proc.Camb.Phil.Soc., <u>40</u>, 167.

Mott and Massey (1933) 'Atomic Collisions', Oxford.

Pauli (1941) Rev. Mod. Phys., <u>13</u>, 203.

Schroedinger (1926) Ann. d. Physik, <u>80</u>, 437.

Wilson (1941) Proc.Camb.Phil.Soc., <u>37</u>, 301.

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Anomalous Scattering of Mesons

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May 22, 1942

IN two recent papers in this journal Code and Shutt[1] have described measurements of the anomalous scattering of mesons due to their interaction with nuclear particles. The cross section obtained was of the order of magnitude of 0.6×10^{-27} cm² per nuclear particle, for a selected sample of mesons having an average energy of 0.8×10^9 ev in Code's measurements. The energy was not measured in Shutt's experiment. We wish to point out that these results like previous measurements by Wilson[2] are in reasonable agreement with the meson theory of nuclear forces if we assume a spin 1 or 0 (pseudoscalar mesons) for the meson. We need not make any further *ad hoc* assumptions. The apparent discrepancy of earlier calculations[3] with the experiments is due to the fact that the usual expansion method, under the assumption that the meson-nucleus interaction is small, breaks down at high energies. The reason is that the coupling of the meson with the charge and spin degrees of freedom of the nuclear particles becomes increasingly strong at high energies.[4] It has now been shown by Wilson and by one of us[5] that a more exact quantum mechanical treatment of the scattering problem is possible which differs from the usual expansion method essentially by the inclusion of *damping*. The cross section thus obtained is in units of

$$(\hbar/\mu c)^2 = 5\times 10^{-26}\ \mathrm{cm}^2 \quad (\mu = \text{meson mass})$$

$$\sigma = 4\pi f'^2 \frac{G^2 p^4}{\epsilon^2(1+\kappa^2)}, \quad \kappa = G^2 p^3/\epsilon, \quad G^2 = g^2 + 2f^2 + f'^2, \tag{1}$$

where p, ϵ are the momentum and energy of the meson in units of the rest energy μc^2, and g, f, f' are the ordinary coupling constants for longitudinal, transverse, and pseudoscalar mesons, divided by $(\hbar c)^{\frac{1}{2}}$. (The primary meson is assumed to be pseudoscalar.) Some of these constants may be zero, but we believe that the best account of all experimental facts is obtained if they are all different from zero. All three are then of the order of magnitude $g^2 \sim f^2 \sim f'^2 \sim 1/10$. For energies of 0.8×10^9 ev σ becomes of the order of magnitude of 1.8×10^{-27} cm² which—in view of the scanty experimental material and our insufficient knowledge of the constants g, f, f'—may be considered as in reasonable agreement with the measurements quoted above. It must also be remembered that (1) is derived for an infinitely heavy nuclear particle and that the motion of the heavy particle will probably diminish the cross section by a factor 2 or so as ϵ approaches the value Mc^2.

Formula (1) has also been derived recently by Fierz[6] by a semiclassical treatment of the charge degree of freedom. It is also very similar to the cross section for a particle scattered by a classical magnetic dipole field.[7] This is not very surprising in view of the great similarity between the formalisms describing the spin and the charge.

Attempts have previously been made to explain the small experimental value of the anomalous scattering by introducing the hypothesis that the proton-neutron can exist in excited charge and spin states by which the inertia of the charge and spin degrees of freedom would be increased.[8] While this possibility has to be kept in mind we believe now that so far no sufficient foundation for such an hypothesis exists.

The quantum theory of damping can be developed in quite a general way. After the diverging self energies and other similar diverging integrals have purposely and systematically been omitted, a new set of equations can be obtained which is free from any singularities and differs from what is obtained by the usual expansion method by the inclusion of damping. The theory makes no assumption about the strength of the coupling. For a number of examples [compare, for instance, (5)] it can be seen that this damping corresponds to that part of the classical damping which is independent of the size and structure of the particle, but, of course, there is not always a classical analogy. The theory can, without difficulty, also be applied to the multiple processes ("explosions") occurring as a consequence of the meson theory and the results turn out reasonably in every respect. These questions are dealt with in a paper to appear shortly in the *Proceedings of the Cambridge Philosophical Society*.

[1] F. L. Code, Phys. Rev. 59, 229 (1941); R. P. Shutt, Phys. Rev. 61, 6 (1942).
[2] Wilson, Proc. Roy. Soc. 174, 73 (1940).
[3] Heitler, Proc. Roy. Soc. 166, 529 (1938).
[4] The case of "strong coupling" has recently been studied by Wentzel [Helv. Phys. Acta. 13, 269 (1940)] and by J. R. Oppenheimer and J. Schwinger, Phys. Rev. 60, 150 (1941).
[5] Heitler, Proc. Camb. Phil. Soc. 37, 291 (1941); Wilson, Proc. Camb. Phil. Soc. 37, 301 (1941).
[6] Fierz, Helv. Phys. Acta. 14, 257 (1941).
[7] Bhabha, Nature 145, 819 (1940); Proc. Ind. Acad. Sci. 11, 247 (1940); Proc. Roy. Soc. 178, 314 (1941); Bhabha and Corben, Proc. Roy. Soc. 178, 273 (1941).
[8] Heitler, Nature 145, 29 (1940); Bhabha, Proc. Ind. Acad. Sci. 11, 347 (1940); 13, 9 (1941); Heitler and Ma, Proc. Roy. Soc. 176, 368 (1940).

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ON THE PRODUCTION OF MESONS BY PROTON-PROTON COLLISIONS

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VII.

ON THE PRODUCTION OF MESONS BY PROTON-PROTON COLLISIONS.

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(From the Dublin Institute for Advanced Studies).

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Introduction.

According to the experimental evidence it is most likely that the mesons observed in cosmic radiation owe their origin to primary protons entering the atmosphere from the universe. This suggests that they are created according to the process

$$P + P \rightarrow P + N + Y^{+} \quad \text{or} \quad P + N \rightarrow P + P + Y^{-} \text{ etc.}$$

which is the meson analogue to the emission of Bremsstrahlung when a fast charged particle is deflected in the Coulomb field of another particle. Collision processes involving mesons can, however, only be treated adequately if proper account is taken of the reaction forces. For this purpose we shall use the theory of radiation damping proposed recently by us.[1] However, an attempt at calculating the rate of meson production by the above process directly on grounds of this theory would meet with exceedingly great mathematical difficulties. Instead a method due to Williams and v. Weizsäcker[2] can be used which has proved to be very valuable and useful in the electromagnetic case: We consider a fast proton colliding with a proton or neutron at rest. The field of the fast proton—in our case its meson field—is equivalent to a superposition of plane waves describing free mesons, provided that the velocity of the proton is very nearly equal to c, i.e. its energy large compared with Mc^2. This field can simply be obtained from the static meson field by a Lorentz-transformation and subsequent Fourier expansion (part I). Thus the field of a fast proton is virtually equivalent to a number of free mesons with a certain energy distribution. These mesons then act on the proton or neutron at rest and can, for instance, be scattered by the latter particle. To obtain the cross section for the production of a meson of given energy during the collision we simply have to multiply the cross section for

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[1] Heitler and Peng, Proc. Camb. Phil. Soc., 38, 296, 1942.

[2] Williams, Kgl. Dank. Vid. Selsk., 13, 4, 1935. v. Weizsäcker, Z. f. Phys., 88, 612, 1934.

scattering by the number of equivalent mesons occurring in the above energy spectrum taking, of course, duly into account the conservation laws. The only way in which the damping then occurs is in the cross section for scattering which can easily be obtained on grounds of our theory (part II).

In the following we shall call a "nucleon" a particle of protonic mass if it may either be a proton or a neutron. We shall put $\hbar = c = \mu$ (meson mass) $= 1$, thus expressing all energies in units μc^2, and all cross sections in units of $(\hbar/\mu c)^2 = 4{\cdot}3 \times 10^{-26}$ cm^2.

We shall use the meson theory in the form proposed by Møller and Rosenfeld,[3] which no doubt gives the most satisfactory account of the nuclear forces. We thus assume that charged and neutral, pseudoscalar and vector mesons exist. Part I is valid for an arbitrary choice of the various coupling constants, thus allowing the meson field to be described by an arbitrary superposition of charged and neutral pseudoscalar and vector fields. Only for the calculation of the scattering cross sections and for the final results we assume the connection of these constants to be that of Møller and Rosenfeld's theory. In order to see, however, in what way the results are modified if a different form of the meson theory is used, we shall also derive the scattering cross section assuming that only charged mesons exist.

PART I.

The Meson-Spectrum of a Moving Nucleon.

Our first task is to find the number of virtual quanta of the meson field associated with a moving nucleon which are effective for the production of mesons during a collision with a second nucleon originally at rest. For this purpose we shall follow closely the analogous treatment of the electromagnetic field by v. Weizsäcker and Williams.[2]

Let the path of the moving nucleon be ANX and the position of the second nucleon at rest be B. *Confining ourselves to collisions during which the transfer of momentum is small compared to the momentum of the moving nucleon* we can take ANX to be a straight line and use it for the x-axis. We take the line BN which meets ANX at N at right angles to be the z-axis, and denote the distance BN of closest approach by b. For the origin of time we choose the moment of closest approach when the moving nucleon passes N. We shall want the mesonic field at B at the time t produced by the moving nucleon, which at this time t is at A, say, and moving with a speed v.

It is convenient to denote the field at B at t by either primed quantities or unprimed quantities according to whether they refer to an

[3] Møller and Rosenfeld, Kgl. Dansk. Vid. Selsk., 17, 8, 1940.

observer moving with A or at rest with B, respectively. The connection between the two sets of coordinates is:

$$x' = \xi(x - vt), \quad y' = y, \quad z' = z, \quad t' = \xi(t - vx) \tag{1}$$

where

$$\xi = (1 - v^2)^{-\frac{1}{2}} \tag{2}$$

Referring to the observer moving with the moving nucleon the meson field satisfies the following equations with a static source at the origin:

(*a*) for the charged vector meson field

$$\left.\begin{aligned} \mathbf{F}' &= -\operatorname{grad}' V' - \frac{\partial}{\partial t'}\mathbf{U}' \\ \mathbf{G}' &= \operatorname{curl}'\mathbf{U}' + 4\pi f\,\Pi^\dagger \sigma\,\delta(\mathbf{r}') \\ \operatorname{curl}'\mathbf{G} &= -\mathbf{U}' + \frac{\partial}{\partial t'}\mathbf{F}' \\ \operatorname{div}'\mathbf{F}' &= -V' - 4\pi g\,\Pi^\dagger\,\delta(\mathbf{r}') \end{aligned}\right\} \tag{3 a}$$

(*b*) for the charged pseudoscalar meson field (Ψ' = pseudoscalar)

$$\left.\begin{aligned} \Phi' &= \frac{\partial}{\partial t'}\Psi' \\ \Gamma' &= -\operatorname{grad}'\Psi' - 4\pi f'\,\Pi^\dagger \sigma\,\delta(\mathbf{r}') \\ \operatorname{div}'\Gamma' + \frac{\partial}{\partial t'}\Phi' &= -\Psi' \end{aligned}\right\} \tag{3 b}$$

Since the operators Π $\Pi^\dagger$, σ are fixed operators (not to be affected by a Lorentz transformation) we have not affixed any primes to them.

The field at B at t referring to the moving observer is given by the static solution of (3a), (3b),[4]

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[4] The "static field" (4) is the solution which is usually used in the theory of nuclear forces. In fact (4) is not an exact solution of the quantized field equations, the operators Π, $\Pi^\dagger$, σ being treated as time-independent. More important still is the fact that (4) does not really describe the field of a free proton or neutron. It can easily be seen that for this solution the charge content of the field is zero, the field thus consisting of equal numbers of positive and negative mesons. The particle described by (4) is in fact a particle which is half proton, half neutron and has total charge 1/2. Exact solutions, referring to particles with charge 0 or 1, are not known, nor can they be known at the present state of the theory, because of the well-known divergence difficulties. It is not difficult to see (by a semi-classical treatment of the Π and Π^+ operators) that the solutions describing a free proton or neutron are in fact those semi-static solutions which vibrate with frequency $\nu > 0$ but small compared with 1. As functions of r these solutions differ only very slightly from (4). We shall therefore use (4), especially since for our purpose positive and negative mesons behave exactly the same, but we must not be surprised to find that the meson-current accompanying the moving nucleon is zero.

$$V' = -g\,\Pi^\dagger \frac{e^{-r'}}{r'}, \qquad \mathbf{U}' = f\,\Pi^\dagger\,[\sigma \mathbf{r}']\,\frac{d}{r'\,dr'}\,\frac{e^{-r'}}{r'}$$

$$\mathbf{F}' = g\,\Pi^\dagger\,\mathbf{r}'\,\frac{d}{r'\,dr'}\,\frac{e^{-r'}}{r'}$$

$$\mathbf{G}' = -f\,\Pi^\dagger\,(\sigma\mathbf{r}')\,\mathbf{r}'\left(\frac{d}{r'\,dr'}\right)^2\frac{e^{-r'}}{r'} - f\,\Pi^\dagger\,\sigma\,\frac{d}{r'\,dr'}\,\frac{e^{-r'}}{r'} + f\,\Pi^\dagger\,\sigma\,\frac{e^{-r'}}{r'} \tag{4}$$

$$\Psi' = f'\,\Pi^\dagger\,(\sigma\mathbf{r}')\,\frac{d}{r'\,dr'}\,\frac{e^{-r'}}{r'}, \qquad \Phi' = 0$$

$$\Gamma' = -4\pi f'\,\Pi^\dagger\,\sigma\,\delta(\mathbf{r}') - f'\,\Pi^\dagger\,\sigma\,\frac{d}{r'\,dr'}\,\frac{e^{-r'}}{r'} - f'\,\Pi^\dagger\,(\sigma\mathbf{r}')\,\mathbf{r}'\left(\frac{d}{r'\,dr'}\right)^2\frac{e^{-r'}}{r'}$$

For $\mathbf{r}'$ we have to insert the position vector of B in the moving system of reference, i.e.

$$\mathbf{r}' = (-\,\xi v t, \quad 0, \quad b) \tag{5}$$

according to (1). For r' we have to insert the positive square root of $b^2 + \xi^2 v^2 t^2$, and $\frac{1}{r'}\frac{d}{dr'}$, can be replaced by $\frac{1}{b}\frac{\partial}{\partial b}$. With the abbreviation

$$f_n = (\xi\, v\, t)^n\,\frac{e^{-\sqrt{b^2+\xi^2 v^2 t^2}}}{\sqrt{b^2+\xi^2 v^2 t^2}} \qquad n = 0,\ 1,\ 2 \tag{6}$$

we obtain from (4) for the field at B at t:

$$\begin{aligned}
V' &= -\,g\,\Pi^\dagger f_0, \qquad U_x' = f\,\Pi^\dagger\,\sigma_y\,\frac{\partial}{\partial b}\,f_0 \\
U_y' &= -\,f\,\Pi^\dagger\left(\sigma_z\,\frac{\partial}{b\,\partial b}\,f_1 + \sigma_x\,\frac{\partial}{\partial b}\,f_0\right) \\
U_z' &= f\,\Pi^\dagger\,\sigma_y\,\frac{\partial}{b\,\partial b}\,f_1
\end{aligned} \tag{7}$$

$$F_x' = -\,g\,\Pi^\dagger\,\frac{\partial}{b\,\partial b}\,f_1, \qquad F_y' = 0, \qquad F_z' = g\,\Pi^\dagger\,\frac{\partial}{\partial b}\,f_0$$

$$G_x' = f\,\Pi^\dagger\,\sigma_x\left(f_0 - \frac{\partial}{b\,\partial b}\,f_0 - \frac{\partial}{b\,\partial b}\,\frac{\partial}{b\,\partial b}\,f_2\right) + f\,\Pi^\dagger\,\sigma_z\,\frac{\partial}{\partial b}\,\frac{\partial}{b\,\partial b}\,f_1$$

$$G_y' = f\,\Pi^\dagger\,\sigma_y\left(f_0 - \frac{\partial}{b\,\partial b}\,f_0\right)$$

$$G_z' = f\,\Pi^\dagger\,\sigma_z\left(f_0 - \frac{\partial}{b\,\partial b}\,f_0 - b\,\frac{\partial}{\partial b}\,\frac{\partial}{b\,\partial b}\,f_0\right) + f\,\Pi^\dagger\,\sigma_x\,\frac{\partial}{\partial b}\,\frac{\partial}{b\,\partial b}\,f_1$$

$$\Psi' = -f'\Pi^\dagger\sigma_x\frac{\partial}{b\partial b}f_1 + f'\Pi^\dagger\sigma_z\frac{\partial}{\partial b}f_0, \qquad \Phi' = 0$$

$$\Gamma_x' = -f'\Pi^\dagger\sigma_x\left(\frac{\partial}{b\partial b}f_0 + \frac{\partial}{b\partial b}\frac{\partial}{b\partial b}f_2\right) + f'\Pi^\dagger\sigma_z\frac{\partial}{\partial b}\frac{\partial}{b\partial b}f_1$$

$$\Gamma_y' = -f'\Pi^\dagger\sigma_y\frac{\partial}{b\partial b}f_0$$

$$\Gamma_z' = -f'\Pi^\dagger\sigma_z\left(\frac{\partial}{b\partial b}f_0 + b\frac{\partial}{\partial b}\frac{\partial}{b\partial b}f_0\right) + f'\Pi^\dagger\sigma_x\frac{\partial}{\partial b}\frac{\partial}{b\partial b}f_1$$

(7) refers still to the moving observer. To obtain the meson field at B at t produced by a moving nucleon referring to the observer at rest with B we need only apply the Lorentz transformation (1):

$$\begin{aligned} V &= \xi(V' + vU_x'), & U_x &= \xi(U_x' + vV') \\ G_y &= \xi(G_y' - vF_z'), & G_z &= \xi(G_z' + vF_y') \\ F_y &= \xi(F_y' + vG_z'), & F_z &= \xi(F_z' - vG_y') \\ \Gamma_x &= \xi(\Gamma_x' + v\Phi'), & \Phi &= \xi(\Phi' + v\Gamma_x') \end{aligned} \tag{8}$$

The other field components are invariant.

(8) expresses the field at the position B of the second nucleon at the time t. It is a function of t and the distance b of closest approach, and also of the speed v of the incident nucleon.

We now try to represent the *field at B at t due to the incident nucleon as nearly as possible by an incident meson pulse built up by a superposition of free plane meson waves of all frequencies.* The meson waves satisfy the homogeneous field equations without source:

$$\begin{aligned} \mathbf{F} &= -\operatorname{grad} V - \frac{\partial}{\partial t}\mathbf{U}, & \mathbf{G} &= \operatorname{curl}\mathbf{U} \\ \operatorname{curl}\mathbf{G} - \frac{\partial}{\partial t}\mathbf{F} &= -\mathbf{U}, & \operatorname{div}\mathbf{F} &= -V \end{aligned} \tag{9}$$

$$\Phi = \frac{\partial}{\partial t}\Psi, \qquad \Gamma = -\operatorname{grad}\Psi, \qquad \operatorname{div}\Gamma + \frac{\partial}{\partial t}\Phi = -\Psi$$

referring from now on always to the observer at rest with B. For a typical plane wave solution of (9) with the propagation vector $\mathbf{k}$ in the x-direction and the frequency ϵ,

$$\epsilon^2 = 1 + k^2 \tag{10}$$

the various field quantities are interconnected by:

$$
\begin{aligned}
&F_x = +\frac{1}{k} JV, &&F_y = +\frac{\varepsilon}{k} G_z, &&F_z = -\frac{\varepsilon}{k} G_y \\
&U_x = +\frac{\varepsilon}{k} V, &&U_y = +\frac{1}{k} JG_z, &&U_z = -\frac{1}{k} JG_y \\
&\Phi = +\frac{\varepsilon}{k}\Gamma_x, &&\Psi = -\frac{1}{k} J\Gamma_x && \\
&G_x = \Gamma_y = \Gamma_z = 0 && &&
\end{aligned}
\tag{11}
$$

J stands for the operator of advancing the phase by $\pi/2$, i.e.

$$J\cos(\epsilon t - kx) = \sin(\epsilon t - kx), \quad J\sin(\epsilon t - kx) = -\cos(\epsilon t - kx)$$

(11) shows that for a plane wave of a given k there are still four independent field components, e.g. V, G_y, G_z, Γ_x which can be chosen arbitrarily. This means that there are 4 independent polarizations, namely, the three polarizations (one longitudinal, V, and two transverse G_y, G_z) of the vector-meson field and one polarization of the pseudoscalar meson field (Γ_x). Other plane wave solutions of (9) propagating along the y- or the z-direction can be obtained from (11) by cyclic permutations of the subscripts x, y, z, but will not occur in the following.

In view of the interdependence of the various field components for a plane meson wave it is impossible to match the meson field of a moving nucleon by free meson pulses exactly for all the field components. In the case, however, of the incident nucleon having an energy large compared with its rest energy its field can very nearly be represented by a meson-pulse (11) for all those partial waves for which $k \gg 1$. In the following we assume therefore[5]:

$$\xi \gg 1, \qquad k \gg 1 \tag{12}$$

The field (8) can then be represented by a superposition of waves (11) which all go in the forward direction. If (12) is satisfied, (8) and (11) both reduce to

$$U_x = +V, \quad F_y = +G_z, \quad F_z = -G_y, \quad \Phi = +\Gamma_x \tag{13}$$

while all other field components are small. On the other hand, k cannot be too large, for we have confined ourselves from the beginning to the case

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[5] If we wish to represent the Lorentz-transformed Coulomb field by a light pulse there is another case for which this can be done approximately, namely, the case $v \ll 1$. (cf. Williams, *l.c.*). In the meson case it turns out, however, that for $v \ll 1$ the other conditions which must be satisfied ((14) and conservation of energy) leave hardly any room for the application of the method. The reason is that the proton mass is only 10 times bigger than the meson mass. It is therefore not possible to apply the method for $v < 1$.

that the transfer of momentum during the collision is smaller than the momentum of the incident nucleon, i.e.

$$| k | \ll \xi\, M v \sim \xi\, M \tag{14}$$

M being the rest mass of the incident nucleon. If (14) is not satisfied the motion of the incident nucleon relative to the second nucleon would no longer be a uniform motion along a straight line as was assumed above, because the recoil accompanying the production of a meson then would distort the path of the nucleon.

We shall now represent the field (7), (8) of the moving nucleon by meson pulses (13). This is easily accomplished by resolving the field into Fourier integrals. According to (8) and (7) the field at B at t of the moving nucleon depends on t only through f_0, f_1, f_2. f_n is either an odd or an even function of t and can be expressed therefore as either a sine or a cosine integral:

$$f_n(t) = \frac{1}{\pi}\int_0^\infty g_n(\varepsilon) \begin{matrix}\sin\\ \cos\end{matrix}\, \varepsilon t \, d\varepsilon \qquad \left(n = \begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right) \tag{15}$$

$g_n(\varepsilon)$ is then given by

$$g_n(\varepsilon) = \int_{-\infty}^{+\infty} f_n(t) \begin{matrix}\sin\\ \cos\end{matrix}\, \varepsilon t\, dt \tag{16}$$

Substitute (6) into (16) and change the variables:

$$\xi v t = b \sinh(\phi + i a), \qquad \tan a = \frac{\varepsilon}{\xi} \tag{17}$$

(16) then becomes

$$\begin{matrix} i\,\xi\, g_n(\varepsilon) \\ \xi\, g_n(\varepsilon) \end{matrix} = \int_{-\infty - ia}^{+\infty - ia} b^n \sinh^n(\phi + ia)\, e^{-b \sec a \cosh \phi}\, d\phi \qquad \left(n = \begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right) \tag{18}$$

The integral (18) can be expressed by Hankel-functions. We deform the path of integration into the real axis and make use of the well-known integral representation of the K_0 function

$$\int_0^\infty e^{-z \cosh \phi}\, d\phi = K_0(z) = i\,\frac{\pi}{2}\, H_0^{(1)}(iz) \tag{19}$$

as well as its derivatives K'_0, K''_0, obtained by differentiating (19) with respect to the argument z. Then (18) is evaluated in terms of the K-functions of argument z,

$$z = b \sec a = b\sqrt{1 + (\varepsilon/\xi)^2}\,, \tag{20}$$

as follows:—

$$
\begin{aligned}
g_0(\varepsilon) &= \frac{2}{\xi} K_0(z) \\
g_1(\varepsilon) &= -\frac{2}{\xi} b \sin a\, K_0'(z) \\
g_2(\varepsilon) &= \frac{2}{\xi} b^2 (\cos^2 a - \sin^2 a) K''_0(z) - \frac{2}{\xi} b^2 \cos^2 a\, K_0(z)
\end{aligned}
\tag{21}
$$

If we insert (21) into (15), then (15) into (7) and then (7) into (8), we arrive at the Fourier resolution of the field of the moving nucleon.

It is convenient to put

$$- K_0' = K_1 \tag{22}$$

so that K_1 as well as K_0 assume positive real values for positive real values of the argument z. The Bessel equation for K_0 can then be written as

$$- (z K_1)' = z K_0 . \tag{23}$$

According to (8) and (7) the G, V, Γ components of the field contain f_0, f_1, f_2, linearly in the combinations: f_0, $\frac{\partial}{\partial b} f_0$, $\frac{\partial}{b \partial b}\left(f_0 + \frac{\partial}{b \partial b} f_2\right)$, $\frac{\partial}{\partial b} \frac{\partial}{b \partial b} f_1$ and $\left(b \frac{\partial}{\partial b} + 1\right) \frac{\partial}{b \partial b} f_0 = \frac{\partial^2}{\partial b^2} f_0$. The Fourier resolutions for these combinations can be considerably simplified by means of (22) and (23), remembering the definition (20) of z. Combining (8), (7), (15) and (21) we give the results of the Fourier resolution for the G, V, Γ components of the field at B at t of the moving nucleon:

$$G_x = 0 . \tag{24a}$$

$$
\begin{aligned}
G_y = {} & \frac{2}{\pi} f \Pi^\dagger \sigma_y \int_0^\infty \left(K_0 + \sec^2 a \frac{K_1}{z}\right) \cos \varepsilon t \, d\varepsilon + \\
& + \frac{2}{\pi} g \Pi^\dagger \int_0^\infty K_1 \sec a \cos \varepsilon t \, d\varepsilon
\end{aligned}
\tag{24b}
$$

$$
\begin{aligned}
G_z = {} & \frac{2}{\pi} f \Pi^\dagger \sigma_z \int_0^\infty (K_0 + \sec^2 a\, K_1') \cos \varepsilon t \, d\varepsilon + \\
& + \frac{2}{\pi} g \Pi^\dagger \int_0^\infty K_1 \tan a \sec a \sin \varepsilon t \, d\varepsilon
\end{aligned}
\tag{24c}
$$

$$
\begin{aligned}
V = {} & - \frac{2}{\pi} f \Pi^\dagger \sigma_y \int_0^\infty K_1 \sec a \cos \varepsilon t \, d\varepsilon - \\
& - \frac{2}{\pi} g \Pi^\dagger \int_0^\infty K_0 \cos \varepsilon t \, d\varepsilon
\end{aligned}
\tag{24d}
$$

$$\Gamma_x = \frac{2}{\pi} f' \Pi^\dagger \sigma_x \int_0^\infty K_0 \tan^2 a \cos \varepsilon t \, d\varepsilon +$$

$$+ \frac{2}{\pi} f' \Pi^\dagger \sigma_z \int_0^\infty K_1 \tan a \sec a \sin \varepsilon t \, d\varepsilon \tag{24e}$$

$$\Gamma_y = \Gamma_z = 0 \tag{24f}$$

(24) is equivalent to a meson pulse propagating along the direction x of the incident nucleon. The Poynting vector for this pulse is

$$S_x = \frac{1}{4\pi} \{ F_y^* G_z - F_z^* G_y + V^* U_x + \Phi^* \Gamma_x + adj \} \tag{25}$$

which describes the flow of momentum passing through B at t. Since we are dealing with collisions for which the transfer of momentum is small, we can neglect the motion of the second nucleon during the collision and thus obtain from (25) the *total flow* of momentum passing through B, during the whole collision process:

$$I_x = \int_{b_{min}}^\infty b \, db \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} S_x \, dt \tag{26}$$

Only the direction of incidence is fixed, the impact parameter BN assumes with equal probability all positions in the plane passing B orthogonal to the direction of incidence.

We have treated the motion of the incident nucleon by classical mechanics and assumed the other nucleon to be initially at rest. This requires a limitation of the impact parameter b at low values. An estimate of b_{min} can be obtained in the following way (due to Williams and v. Weizsäcker). The application of the very concept of an impact parameter requires that b must be larger than the extension of the wave packets of both nucleons in the direction perpendicular to the direction of incidence. The relative momentum in this direction was assumed to be zero and cannot be allowed to assume values larger than Mc, say, for two reasons:

If this momentum were larger than Mc the velocity of the nucleon would be comparable with c and no definite Lorentz-system could be affixed to the nucleon "at rest." Secondly, if the uncertainty of the momentum is larger than Mc an unknown number of nucleon pairs could be created making the number of nucleons indefinite. Hence, by the uncertainty relation, we have to put:

$$b_{min} = \frac{1}{M} \text{ (in our units)}.$$

Williams and v. Weizsäcker have shown that in all collision processes which can be treated by exact methods, the contribution from smaller impact parameters can be neglected. The same may be assumed to be the case here, although, in the meson case, there is so far no example which could be treated exactly and with which our results could be compared. Fortunately our results will be seen to depend very little on the value b_{min} within any reasonable limits (otherwise we could hardly trust this method very far).

Inserting (24), (25) into (26) all cross products vanish either by the integration over t or by the integration over θ (the latter, because of the explicit occurrence of σ_y, σ_z). By using the Fourier formulae (15), (16) again, the integration over t of (26) is transformed into an integration over the frequencies. The integration over b can be changed by (20) into one over z; and it happens then that all the products containing the K-functions can be integrated directly using (22) and (23). In this way we can express (26) in the following form

$$I_x = \int_0^\infty k\, q(\epsilon)\, d\epsilon \tag{27}$$

and *interpret* $q(\epsilon)$ *as the number of virtual quanta of the meson field accompanying the moving nucleon.* These mesons will be effective for the production of mesons during the collision of the moving nucleon with a second nucleon originally at rest. We see that virtual mesons of all polarizations occur, the transverse, longitudinal and pseudoscalar mesons arising from the FG, VU, $\Phi\Gamma$—terms of (25) respectively. The corresponding numbers of virtual quanta work out, from (24) to (27), to be the following:

$$\begin{aligned}
q_{tr}(\epsilon)\, d\epsilon &= \frac{d\epsilon}{\pi\epsilon} f^2 \left\{(\cos^2\alpha + \tan^2\alpha \sin^2\alpha) A + 2\sec^2\alpha\, K_1^2 + \tan^2\alpha\, B\right\} \\
&\quad + \frac{d\epsilon}{\pi\epsilon} g^2 B \\
q_{long}(\epsilon)\, d\epsilon &= \frac{d\epsilon}{\pi\epsilon} \{f^2 B + g^2 \cos^2\alpha\, A\} \qquad\qquad (28) \\
q_{ps}(\epsilon)\, d\epsilon &= \frac{d\epsilon}{\pi\epsilon} f'^2 \tan^2\alpha\, \{\sin^2\alpha\, A + B\} \\
A &= z^2(K_1^2 - K_0^2), \qquad B = z^2(K_0^2 - K_1^2) + 2zK_0K_1
\end{aligned}$$

The argument z of the K-functions here comes from the lower limit of integration of (26), i.e.

$$z = b_{min}\sec\alpha = \frac{1}{M}\sqrt{1 + (\epsilon/\xi)^2} \tag{29}$$

z ranges from $\frac{1}{M} = 0{\cdot}1$ to approximately one (for $\epsilon \sim \xi M$ = energy of the incident particle). (29) can be used to eliminate the trigometric functions of a in (28).

The number referring to the transverse mesons includes mesons of both transverse polarizations.

When neutral mesons are to be included a neutral meson field will also be associated with the moving nucleon. The corresponding numbers of virtual neutral quanta are given by (28), (30) with the coupling constants f^2, g^2, f'^2 now replaced by $f_0{}^2$, $g_0{}^2$, $f'_0{}^2$ (say) of the neutral meson field. In the theory of Møller and Rosenfeld these are related with f^2, g^2, f'^2 by:

$$\begin{aligned} f_0^2 = f'^2_0 = \tfrac{1}{2}f^2 = \tfrac{1}{2}f'^2 &= 0{\cdot}065 \\ g_0^2 = \tfrac{1}{2}g^2 &= 0{\cdot}027. \end{aligned} \tag{30}$$

The numerical values are taken from the theory of nuclear forces, assuming the rest mass of the meson = 1/10 of that of a nucleon. For numerical calculations the K_0 and K_1 used in (29), (31) can be found from the tables in Watson's work on Bessel functions. Tables for Hankel functions may also be used, cf. (19).

Finally, we give a table for the functions occurring in (28), and defined by:

$$\begin{aligned} q_{tr} &= \frac{d\varepsilon}{\pi\varepsilon}\left(f^2 D_{tr} + g^2 C\right) \\ q_{long} &= \frac{d\varepsilon}{\pi\varepsilon}\left(f^2 C + g^2 D_{long}\right) \\ q_{ps} &= \frac{d\varepsilon}{\pi\varepsilon} f'^2 D_{ps} \end{aligned} \tag{31}$$

TABLE I.

z	D_{tr}	D_{long}	D_{ps}	C
0·1	195	0·91	0	3·9
0·2	193	0·20	9·3	2·5
0·4	179	0·035	29	1·4
0·6	165	0·011	42	0·82
0·8	144	0·0042	49	0·51
1	122	0·0018	50	0·32

The g^2 term wherever it occurs is negligible; the number of longitudinal mesons is also very small. Far the largest part of the field is due to transverse mesons, their number depends little on z and therefore also on the value for b_{min}. The number of pseudoscalar mesons is not negligible but zero for small z (small ϵ) and depends not very strongly on b_{min} either. From (31) and Table I it is clear that the largest contributions to q are proportional to f^2, f'^2 thus arising from the *strong coupling of the meson field with the spin of the nucleon.*

PART II.

The Scattering of a Meson by a Nucleon.

In the earlier work on this subject it was assumed that only charged mesons, but no neutrettos (neutral mesons) exist. Their very occurrence causes some alteration in the formulae for scattering even of charged mesons. In this section we derive the cross-sections for scattering for all cases, including also that of extreme relativistic energies $>> M$, which has so far not been treated.[6] We denote the field quantities and the coupling constants referring to neutrettos by a suffix zero.

The Hamiltonian for the interaction of a nucleon with the charged and neutral, vector and pseudoscalar meson field is, according to Kemmer, Mϕller and Rosenfeld:

$$H_{int} = [\ ] + [\ ]^* + [\ ]_0 . \tag{1}$$

$$\begin{aligned} [\ ] = & \; g\,\Pi\,\{\operatorname{div}\psi + \rho_1(\sigma, \dot{\psi} - \phi)\} + \\ & + f\,\Pi\,\{\rho_3(\sigma \operatorname{curl}\phi) - \rho_2(\sigma, \dot{\phi} + \psi) - \\ & - g'\,\Pi\,\rho_2\Psi + f'\,\Pi\,\{(\sigma \operatorname{grad}\Psi) + \rho_1\dot{\Psi}\} \end{aligned} \tag{2}$$

denoting now conveniently the longitudinal, transverse, pseudoscalar wave functions by ψ, ϕ, Ψ respectively.

σ denotes the spin of the nucleon and ρ_1, ρ_2, ρ_3 are the Dirac matrices occurring in the Hamiltonian of the free nucleon (momentum P, rest mass M)

$$H_{nucl.} = \rho_1(\sigma\,\mathbf{P}) + \rho_3 M \tag{3}$$

The matrices ρ_1, ρ_2, ρ_3 satisfy the same algebraic relations as the Pauli spin matrices σ_x, σ_y, σ_z. $[\ ]^*$ and $[\ ]_0$ are obtained from $[\ ]$ by passing to the adjoint or by adding the suffix zero to Π, g, g', f, f' and to the field quantities respectively. Π, $\Pi^\dagger$ are operators for the transition of the nucleon from the proton-state to the neutron-state and vice versa. Π_0 allows no transition between these states, but is plus or minus one according to whether the nucleon is in the neutron or in the proton-state. Following the Pauli Weisskopf or the Heisenberg-

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[6] With the exception of a paper by S. T. Ma, Proc. Camb. Phil. Soc. in the press.

Pauli quantization for the charged or neutral meson field respectively we obtain from the interaction given above the following matrix elements of H_{int} needed for our present calculation :—

$$\text{The matrix elements for the} \begin{bmatrix} \text{absorption by a neutron of a positive} \\ \text{emission } \;,,\; \text{ neutron } \;,,\; \text{ negative} \\ \text{absorption } \;,,\; \text{ proton } \;,,\; \text{ negative} \\ \text{emission } \;,,\; \text{ proton } \;,,\; \text{ positive} \end{bmatrix} \begin{pmatrix} \text{longitudinal meson} \\ \text{transverse } \;,, \\ \text{pseudoscalar } \;,, \end{pmatrix}$$

$$= \sqrt{\frac{2\pi}{\varepsilon}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{pmatrix} g\{p - \varepsilon\rho_1(\sigma \mathbf{i})\} \\ f\{\rho_3(\sigma[\mathbf{p}\mathbf{j}]) + \rho_2\varepsilon(\sigma\mathbf{j})\} \\ f'\{(\sigma\mathbf{p}) - \rho_1\varepsilon\} \end{pmatrix} + \sqrt{\frac{2\pi}{\varepsilon}} \begin{bmatrix} -i \\ +i \\ -i \\ +i \end{bmatrix} \begin{pmatrix} f\rho_2(\sigma\mathbf{i}) \\ g\rho_2(\sigma\mathbf{j}) \\ g'\rho_2 \end{pmatrix} \tag{4}$$

$$\text{The matrix element for the} \begin{bmatrix} \text{absorption by a neutron of a neutral} \\ \text{emission } \;,,\; \text{ neutron } \;,,\; \;,, \\ \text{absorption } \;,,\; \text{ proton } \;,,\; \;,, \\ \text{emission } \;,,\; \text{ proton } \;,,\; \;,, \end{bmatrix} \begin{pmatrix} \text{longitudinal meson} \\ \text{transverse } \;,, \\ \text{pseudoscalar } \;,, \end{pmatrix}$$

$$= \sqrt{\frac{2\pi}{\varepsilon}} \begin{bmatrix} +i \\ -i \\ -i \\ +i \end{bmatrix} \begin{pmatrix} g_0\{p - \varepsilon\rho_1(\sigma \mathbf{i})\} \\ f_0\{\rho_3(\sigma[\mathbf{p}\mathbf{j}]) + \rho_2\varepsilon(\sigma\mathbf{j}\} \\ f'_0\{(\sigma\mathbf{p}) - \rho_1\varepsilon\} \end{pmatrix} + \sqrt{\frac{2\pi}{\varepsilon}} \begin{bmatrix} -1 \\ -1 \\ +1 \\ +1 \end{bmatrix} \begin{pmatrix} f_0\rho_2(\sigma\mathbf{i}) \\ g_0\rho_1(\sigma\mathbf{j}) \\ g'_0\rho_2 \end{pmatrix} \tag{4°}$$

$\mathbf{p}$, ϵ denote the momentum, energy of the meson concerned, $\mathbf{j}$ denotes the unit polarization vector for transverse mesons. The polarization vector $\mathbf{i}$ of the longitudinal meson is taken parallel to the momentum vector which is $\pm$ the wave vector for $\pm$ mesons. In either (4) or (4°) there are 4 entries in the square brackets which can be combined with the 3 entries in the round brackets, making altogether 12 cases each involving charged or neutral mesons. Each individual matrix element is the product of one factor from [ ] and one factor from ( ). It is understood that on the right-hand side of (4), (4°), the matrix elements of the operators ρ, σ (formed with the wave function of the nucleon) are always to be taken for the transition in question. The matrix elements for the absorption by a neutron of a negative meson and other cases where the charge of the system is not conserved vanish.

From these elementary matrix elements for absorption and emission we shall form the "compound matrix elements" of second order for scattering. Only the compound matrix elements for scattering are needed for the calculation of the scattering cross-section. We shall confine ourselves to two limiting cases where the energy of the meson to be scattered is either very small or very large compared with the rest mass M of the nucleon. Furthermore, we shall carry out all calculations for two different forms of the meson theory: (i) no neutrettos exist (so-called "charged theory"), i.e. $g_0^2 = g'^2_0 = f_0^2 = f'^2_0 = 0$; (ii) neutretto's exist, the coupling constants g_0^2, f_0^2, etc. are half of those for charged mesons (g^2, f^2, etc.), as is required in order that the *P–P* forces should be equal to the *P–N* forces (so-called "symmetrical theory"). Following Møller and Rosenfeld we assume in both cases $f^2 = f'^2$. In the "symmetrical theory" the values of the constants are given by (I) (30), but are somewhat different in the "charged theory." The final formulae which are only valid for one of these two forms of the theory are labelled "sym. Th." and "ch. Th." respectively, formulae without such labels are valid in either case.

The compound matrix elements for scattering are

$$H_{AB} = \sum_i \frac{H_{Ai} H_{iB}}{E_B - E_i} \tag{5}$$

A, B refer to two states of the same energy arising from each other by scattering, H_{Ai}, etc., are the matrix elements given by (4), (4°). In order to calculate the cross-section for scattering we have to solve the following radiation equations which include the damping (cf. ref. 1):

$$U_{AB} = H_{AB} + i\pi H_{AC} \rho_C U_{CB} \tag{6}$$

where, in the last term, the sum over all states C of the same energy as A, B is to be taken and ρ_C is the number of states per energy interval $d E_C$. The sum over C includes the summation over all directions of polarization and integration over all scattering angles. The cross-section for scattering is then given by

$$\Phi = \frac{\varepsilon}{p} 2\pi \rho_A \mid U_{AB} \mid^2 . \tag{7}$$

(i) *N.R. Non-relativistic case* $\epsilon \ll M$.

For the scattering of a meson of low energy ϵ by a nucleon the recoil of the latter may be entirely neglected. The energy of the secondary meson is then also ϵ. Taking the nucleon at rest we can simplify the matrix elements (4) and (4°) by putting $\rho_1 = \rho_2 = 0$, $\rho_3 = +1$ and thus treating the spin of the nucleon by the non-relativistic Pauli theory. The incident meson may be positive, neutral, or negative and this may be scattered by a proton or a neutron. During the scattering the charge of the meson may or may not be exchanged with that of the nucleon. Since the total charge of the system is conserved by the scattering, it is convenient to consider cases of different total charge separately.

CASE: *N.R. (2, −1) i.e. total charge of the system = 2 or −1.*

The system consists of a proton (neutron) together with a positive (negative) meson. For the calculation of H_{AB} (5) we note that there is only one intermediate state i in which both the primary and the secondary meson appear together with the nucleon. From (4) we obtain:

$$H_{AB} = -2\pi \frac{p^2}{\varepsilon^2} K_B K_A \tag{8}$$

where

$$K = g, \quad f(\sigma[\mathbf{p}\,\mathbf{j}]), \quad f'(\sigma\,\mathbf{i}) \tag{9}$$

according to whether the polarization of the meson in the states labelled A or B is longitudinal, transverse or pseudoscalar respectively. $(\mathbf{i} = \mathbf{p}/p)$.

To solve (6) we put

$$U_{AB} = -2\pi \frac{p^2}{\varepsilon^2} \{X_{AB} K_B K_A + Y_{AB} K_A K_B\}. \tag{10}$$

If one of the suffixes A, B refers to a longitudinal meson, K_A and K_B commute, and we can then put $Y_{AB} = 0$. Inserting (10) into (6) and carrying out the summation over the two transverse and one pseudoscalar polarizations on the right-hand side we can compare the coefficients of $K_B K_A$ and $K_A K_B$ on both sides, and equate them. We thus obtain a system of linear equations for the determination of the coefficients X_{AB}, Y_{AB} of (10):

$$\begin{aligned}
X_{gg} &= 1 - i\frac{p^3}{\varepsilon}(g^2 X_{gg} + 3f^2 X_{fg}) \\
X_{fg} &= 1 - i\frac{p^3}{\varepsilon}(g^2 X_{gg} - f^2 X_{fg}) \\
X_{gf} &= 1 - i\frac{p^3}{\varepsilon}(g^2 X_{gf} - f^2 X_{ff}) - i\frac{p^3}{\varepsilon} 3f^2 Y_{ff} \\
X_{ff} &= 1 - i\frac{p^3}{\varepsilon} 2f^2 X_{ff} \\
Y_{ff} &= -i\frac{p^3}{\varepsilon}(g^2 X_{gf} + f^2 X_{ff}) + i\frac{p^3}{\varepsilon} f^2 Y_{ff}.
\end{aligned} \tag{11}$$

Here the suffix f refers to either transverse or pseudoscalar mesons while g refers to a longitudinal meson. The exact solution of (11) is:

$$X_{ff} = \frac{1}{1 + 2i\tau}, \quad X_{fg} = X_{gf} = \frac{1}{1 + i\sigma - i\tau + 4\sigma\tau}$$

$$Y_{ff} = \frac{4\sigma\tau - i\sigma - i\tau}{(1 + 2i\tau)(1 + i\sigma - i\tau + 4\sigma\tau)}, \quad X_{gg} = \frac{1 - 4i\tau}{1 + i\sigma - i\tau + 4\sigma\tau} \tag{12}$$

with

$$\sigma = g^2 \frac{p^3}{\varepsilon}, \quad \tau = f^2 \frac{p^3}{\varepsilon}.$$

From (10), we find the cross-section for the scattering by (7). Integrating over all directions of the secondary meson we have for the integrated cross-section for scattering:

$$\Phi_{gg} = 4\pi \frac{p^4}{\varepsilon^2} g^4 \, | X_{gg} |^2, \quad \Phi_{gf} = \Phi_{fg} = 4\pi \frac{p^4}{\varepsilon^2} g^2 f^2 \, | X_{gf} |^2$$

$$\Phi_{ff} = 4\pi \frac{p^4}{\varepsilon^2} f^4 \{ | X_{ff} |^2 + | Y_{ff} |^2 - \tfrac{1}{3} (X_{ff} Y_{ff}^* + X_{ff}^* Y_{ff}) \}. \quad (13)$$

Here f refers to one particular tranverse or pseudoscalar polarization. For large values of ϵ, i.e. $g^2\epsilon^2$, $f^2\epsilon^2 \gg 1$ (13) simplifies into:

$$\Phi_{gg} = \frac{4\pi}{\varepsilon^2}, \quad \Phi_{gf} = \Phi_{fg} = \frac{\pi}{4 f^2 g^2 \varepsilon^6}, \quad \Phi_{ff} = \frac{4\pi}{3\varepsilon^2}. \quad (14)$$

In (13), (14) no summation has yet been carried out over the polarization of the secondary meson. For large ϵ Φ_{fg} is of a much smaller order of magnitude than Φ_{ff}, in other words a pseudoscalar meson is almost always scattered into a transverse or pseudoscalar meson, but not into a longitudinal meson. The occurrence of this *selection rule* is entirely due to the *damping* and is one of the nicest features of this theory. We shall find even more stringent selection rules below.

In the case (2, −1) neutrettos appear nowhere during the calculation. The results are valid for both the charged and symmetrical theories.

N.R. (1, 0). Total charge of the system = 1 or 0.

The system may consist of a positive meson (or negative meson) together with a neutron (or proton), and also, when the neutral mesons are included, it may consist of a proton (or a neutron) together with a neutral meson. In both the "charged theory" and the "symmetrical theory," the matrix element for the scattering of a charged meson into another charged meson is:

$$H_{AB} = 2\pi \frac{p^2}{\varepsilon^2} K_A K_B. \quad (15)$$

This differs from (8) by the different order of the K-factors. In the "charged theory" the solution of the radiation equation (6) with matrix element (15) is simply:

$$U_{AB} = 2\pi \frac{p^2}{\varepsilon^2} \frac{K_A K_B}{1 + i\sigma + 3i\tau}. \quad (16)$$

The corresponding cross-section for scattering is:

$$\Phi_{AB} = 4\pi \frac{p^4}{\varepsilon^2} \frac{K_A^2 K_B^2}{1 + (\sigma + 3\tau)^2} \qquad \text{(Ch. Th.)} \ (17)$$

K_A^2, K_B^2 denote, according to (9), simply g^2 or f^2 according to the polarization referred to being longitudinal or otherwise. For large values of ϵ, $g^2\epsilon^2$, $f^2\epsilon^2 \gg 1$, (17) becomes

$$\Phi_{gg} = \frac{4\pi}{\varepsilon^2} \frac{g^4}{(g^2 + 3f^2)^2}, \quad \Phi_{gf} = \Phi_{fg} = \frac{4\pi}{\varepsilon^2} \frac{f^2 g^2}{(g^2 + 3f^2)^2}$$

$$\Phi_{ff} = \frac{4\pi}{\varepsilon^2} \frac{f^4}{(g^2 + 3f^2)^2} \qquad \text{(Ch. Th.)} \ (18)$$

(17) and (18) are the formulae previously obtained by Wilson and Heitler[7]. They are only valid in the "charged theory" and only for the scattering of a Y^+ (Y^-) by a proton (neutron).

In the symmetrical theory also such matrix elements H_{AB} occur where one or both suffixes refer to neutral mesons. In these cases there are always two intermediate states where either no mesons or both the primary and secondary mesons appear together with the nucleon. The matrix elements are obtained from (4) and (4°) and, together with (15), may be comprised in the following form:

$$H_{AB} = 2\pi \frac{p^2}{\varepsilon^2} \{C_{AB} K_A K_B + D_{AB} K_B K_A\}. \qquad (19)$$

Here C_{AB}, D_{AB} are numerical coefficients depending on the charge and polarizations of the secondary and primary mesons. As a generalization of (9) we now have

$$K_A = g, \ f(\sigma[\mathbf{p}\mathbf{j}]), \ f(\sigma \mathbf{i}), \ g_0, \ f_0(\sigma[\mathbf{p}\mathbf{j}]), \ f_0(\sigma \mathbf{i}) \qquad (20)$$

according to whether A refers to a charged longitudinal, charged transverse, charged pseudoscalar, neutral longitudinal, neutral transverse, or neutral pseudoscalar meson. The numerical values of C_{AB}, D_{AB} are the same when one suffix refers to either the transverse or the pseudoscalar meson. They are summarized in the following matrices:

$$C = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{|l} g \\ f \\ f_0 \\ g_0 \end{array} \qquad (21)$$

[7] Heitler, Proc. Cam. Phil. Soc., 37, 291, 1941; Wilson, ibid., 37, 301, 1941.

where the columns and rows are arranged, as indicated by the coupling constants on the right-hand side in the order: charged longitudinal, charged transverse or pseudoscalar, neutral transverse or pseudoscalar and neutral longitudinal.

To solve the radiation equation (6) we put for U_{AB} :

$$U_{AB} = 2\pi \frac{p^2}{\varepsilon^2} \left\{ X_{AB} K_A K_B + Y_{AB} K_B K_A \right\} \qquad \text{(sym. Th.)} \quad (22)$$

X_{AB}, Y_{AB} have to be determined in a similar way as in the case N.R. (2, −1). Note that when one of the indices A, B refers to a longitudinal meson, $K_B K_A = K_A K_B$. In this case we have to put $Y_{AB} = 0$. so that the first and fourth rows and columns of the matrix Y are, like in D, zero. Inserting (19) and (22) into (6) and summing on the right-hand side over the transverse and pseudoscalar polarizations we obtain, by comparing the coefficients of $K_A K_B$ and $K_B K_A$ on both sides, the following linear equations which we write in the form of matrix equations:

$$\begin{aligned} X &= C + i(C - \tfrac{1}{3}D) LX - i\tfrac{1}{3}(C - D)Y \\ Y &= D + i\tfrac{2}{3} DLY \end{aligned} \qquad (23)$$

Here L denotes the diagonal matrix, putting now $f^2 = 2 f_0^2$, etc.:

$$L = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & 3\tau & 0 & 0 \\ 0 & 0 & \frac{3}{2}\tau & 0 \\ 0 & 0 & 0 & \frac{\sigma}{2} \end{pmatrix} \qquad (23')$$

The equation for Y is easily solved,

$$Y = \frac{1}{1 - i\tau} \frac{1}{1 + 2i\tau} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i\tau & 1 & 0 \\ 0 & 1 & 2i\tau - 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(sym. Th.)} \quad (24)$$

Here again

$$\begin{aligned} \tau &= f^2 p^3/\varepsilon = 2 f_0^2 p^3/\varepsilon \\ \sigma &= g^2 p^3/\varepsilon = 2 g_0^2 p^3/\varepsilon \end{aligned} \qquad (25)$$

Substituting (24) into (23) we obtain the equation for X

$$\begin{pmatrix} 1-i\sigma & -3i\tau & -3i\tau & -i\sigma \\ -i\sigma & 1-3i\tau & -i\tau & -i\sigma \\ -2i\sigma & -2i\tau & 1-2i\tau & 0 \\ -2i\sigma & -6i\tau & 0 & 1 \end{pmatrix} X =$$

$$= \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} - \frac{i\tau}{(1-i\tau)(1+2i\tau)} \begin{pmatrix} 0 & 1+i\tau & 2i\tau & 0 \\ 0 & i\tau & 1 & 0 \\ 0 & 1 & 2i\tau-1 & 0 \\ 0 & 2i\tau & 2 & 0 \end{pmatrix} \qquad \text{(sym. Th.)} \quad (26)$$

The exact solution of this four-row-four-column matrix equation is rather complicated. In practice one may solve (26) by numerical means for a few values of ϵ. For large values of ϵ, $g^2\epsilon^2$, $f^2\epsilon^2 \gg 1$ the solution is

$$X = \begin{pmatrix} i/\sigma & 0 & 0 & 0 \\ 0 & i/2\tau & 0 & 0 \\ 0 & 0 & i/\tau & 0 \\ 0 & 0 & 0 & 2i/\sigma \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i/2\tau & 0 & 0 \\ 0 & 0 & i/\tau & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(sym. Th.)} \quad (27)$$

where places marked by zeros contain terms proportional to high negative powers of ϵ (at least ϵ^{-6}). The cross-sections are also written in matrix form:

$$\Phi = \frac{4\pi}{\varepsilon^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{vmatrix} g \\ f \\ f_0 \\ g_0 \end{vmatrix} \qquad \text{(sym. Th.)} \quad (28)$$

Again (28) gives the scattering cross-section from one primary polarization into one single secondary polarization. Thus the second and third rows and columns have actually to be understood as submatrices: for instance:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

in (28) instead of 1/3 the 3 rows and columns referring to the 2 transverse and the pseudoscalar polarizations. Asymptotically ($f^2\epsilon^2 \gg 1$) Φ is the same in the N.R. (2, −1) and N.R. (1, 0) cases. The total cross-section for scattering of a meson of any polarization into all other polarizations is always

$$\Phi = \frac{4\pi}{\epsilon^2} \qquad \text{(sym. Th.)} \quad (29)$$

(29) is larger by a factor $(g^2 + 3f^2)/f^2$ than the corresponding cross-section (18) in the "charged theory." This factor is about 3. (18) was found to agree fairly well[8] with the scattering experiments. Using (29), the agreement would not be quite so good. Considering, however, that (29) is an asymtotic formula valid for $f^2\epsilon^2 \gg 1$ and $\epsilon \ll M$ (there is indeed not a great margin left by these two conditions) and that the scattering experiments are not very accurate yet we do not think that (29) is really in serious disagreement with experiments.

(28) shows the same selection rules as for the (2, −1) case (no transition $g \rightleftarrows j$) and in addition a selection rule for the charge: *The charge of the meson is practically conserved during the scattering, no transformation of charged mesons into neutrettos taking place.* This, however, holds only for $f^2\epsilon^2 \gg 1$. The latter selection rule is of great importance in view of the fact that this transformation has never been observed experimentally. The theory of damping explains this fact, which can therefore not be used to prove that neutrettos do not exist.

Finally, we give the total cross-sections (into *all* secondary polarizations) for scattering of a pseudoscalar meson for *low energies,* as obtained by a numerical solution of (26), including also the cross-section for transformation into a neutretto. The latter is large only for energies less than $\epsilon = 4$, for higher energies this transformation is forbidden by the selection rules. For the coupling constants the values part I (30) are used.

TABLE II.—*Scattering cross-sections.* (sym. Th.).

ϵ	1·5	2	3	5
$Y^+ + N \rightarrow N + Y^+$	0·41	1·02	0·63	0·46
$Y^+ + P \rightarrow P + Y^+$	0·48	1·14	1·16	0·50
$Y^+ + N \rightarrow P + Y^0$	0·20	0·56	0·28	0·028

[8] Heitler and Peng, Phys. Rev., 62, 81, 1942.

(ii) *E.R. Extreme relativistic case* $\epsilon \gg M$.

For the scattering of a meson of high energy by a nucleon, say, at rest, it is convenient to perform the calculation for this cross-section in a moving Lorentz frame of reference where the meson and nucleon have equal opposite momenta, $\mathbf{P} = -\mathbf{p}$. The cross-section for the case where the nucleon is originally at rest can then be obtained by a Lorentz-transformation. We consider again the two cases for the total charge of the system being 2 (or -1) and 1 (or 0) separately:

E.R. (2, -1). *Total charge of the system* $= 2$ *or* -1.

In the calculation of the compound matrix element for scattering (5) we note that, as in the non-relativistic case, there is only one intermediate state where both the primary and the secondary mesons appear together with the nucleon. Since the total momentum of the system is zero and conserved, the momentum of the nucleon in the intermediate state is $\mathbf{P}_i = -(\mathbf{p}_A + \mathbf{p}_B)$ if $\mathbf{p}_A$, $\mathbf{p}_B$ are the momenta of the meson before and after the scattering $(|p_A| = |p_B|)$. The energy of the nucleon in the intermediate state may be positive or negative. From the matrix elements (4) we obtain the compound matrix element for scattering (5), putting now $p = \epsilon$ and keeping only terms of the highest power of ϵ, after the summation over the two signs of energy of the nucleon in the intermediate state,

$$H_{AB} = - 2\pi\varepsilon \left(\Psi_A{}^* \frac{K_B(1 - \rho_1(\sigma \mathbf{i}_B))\rho_1(\sigma, -\mathbf{p}_A - \mathbf{p}_B)(1 - \rho_1(\sigma \mathbf{i}_A)) K_A}{(\mathbf{p}_A + \mathbf{p}_B)^2} \Psi_B \right) \tag{30}$$

Ψ_A is the wave function of the nucleon with the space dependent part already split off. A, B include the description of the energy and momentum of the nucleon. Ψ_B satisfies

$$\rho_1(\sigma \mathbf{P}_B)\Psi_B = E_B \Psi_B \quad \text{or} \quad \rho_1(\sigma \mathbf{i}_B)\Psi_B = -\Psi_B \tag{31}$$

where E_B is the energy of the nucleon and in the E.R. case $E_B = |\mathbf{P}_B| = |-\mathbf{p}_B|$. Similarly

$$\Psi_A{}^* \rho_1(\sigma \mathbf{i}_A) = -\Psi_A{}^* \tag{31'}$$

K now stands for:

$$K = g, \qquad f\rho_3(\sigma[\mathbf{i}\,\mathbf{j}]), \qquad -f\rho_1 \tag{32}$$

for the three polarizations respectively.

(30) can be greatly simplified by the help of (31), (31′) by changing the order of the factors in (30) and making use of the commutation relations of the ρ's and σ's. The result is, arranged in matrix form as in the N.R. case:

$$H_{AB} = -4\pi \Big\{ C_{AB} (\Psi_A{}^* K_A K_B \Psi_B) + $$
$$+ f^2 D_{AB} \left(\Psi_A{}^* \frac{(\mathbf{j}_A + i\rho_1[\mathbf{i}_A \mathbf{j}_A], \mathbf{j}_B - i\rho_1[\mathbf{i}_B \mathbf{j}_B])}{1 + (\mathbf{i}_A \mathbf{i}_B)} \Psi_B \right) \Big\} \quad (33)$$

where

$$C_{AB} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad D_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{|l} g \\ f \\ f' \end{array} \quad (34)$$

The rows and columns refer to longitudinal transverse and pseudoscalar polarizations respectively in the order indicated by the coupling constants on the right-hand side. If A B do not both refer to transverse polarizations the second term of (33) is zero.

To solve the radiation equations (6) we put:

$$U_{AB} = -4\pi \Big\{ X_{AB} (\Psi_A{}^* K_A K_B \Psi_B) + $$
$$+ f^2 D_{AB} Y \big(\Psi_A{}^* (\mathbf{j}_A + i\rho_1[\mathbf{i}_A \mathbf{j}_A], \mathbf{j}_B - i\rho_1[\mathbf{i}_B \mathbf{j}_B]) \Psi_B \big) \Big\} \quad (35)$$

X_{AB} will be constants which can be arranged in matrix form similar to (34) whilst Y will now be a function of the angle between $\mathbf{i}_A$, $\mathbf{i}_B$ which occurs in the second term of (33). We insert (35) and (33) into (6). In the term $i\pi\rho_C H_{AC} U_{CB}$ of (6) it is easy to see that all cross terms containing products $C_{AC} D_{CB} Y$ or $D_{AC} X_{CB}$ vanish after the summation over both transverse polarizations $\mathbf{j}_C$. We thus obtain two separate equations for X_{AB} and Y. The equation for X_{AB} is a matrix equation:

$$X = C - iCLX, \quad L = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 2\tau & 0 \\ 0 & 0 & \tau \end{pmatrix} \quad (36)$$

$$\sigma = g^2 \varepsilon^2, \qquad \tau = f^2 \varepsilon^2$$

For Y we obtain, after some calculation, which we shall not give here in detail (see ref. 9), the following integral equation:

$$Y(\gamma) = \frac{1}{1+\gamma} - 4\,i\,\tau \int \frac{d\,\Omega_C}{4\pi} \frac{1}{1+\beta}\, Y(\alpha) \left\{ \left(\frac{1+\alpha+\beta+\gamma}{1+\gamma} \right)^3 - \right.$$

$$\left. - \frac{3}{2} \frac{1+\alpha+\beta+\gamma}{1+\gamma} \frac{(1+\alpha)(1+\beta)}{1+\gamma} \right\} \tag{37}$$

$$\alpha = (\mathbf{i}_B\, \mathbf{i}_C)\,, \qquad \beta = (\mathbf{i}_C\, \mathbf{i}_A)\,, \qquad \gamma = (\mathbf{i}_A\, \mathbf{i}_B)$$

The integration is to be performed over all directions of $\mathbf{i}_C$.

The solution of (36) is a diagonal matrix:

$$X = \begin{pmatrix} \frac{1}{i\,\sigma} & 0 & 0 \\ 0 & \frac{1}{2\,i\,\tau} & 0 \\ 0 & 0 & \frac{1}{i\,\tau} \end{pmatrix} \begin{array}{|l} g \\ f \\ f' \end{array} \tag{38}$$

The integral equation for Y (37) is a special case of a more general equation solved in the following paper by Hamilton and Peng.[9] We give the solution without calculation. $Y(\gamma)$ is expanded into a series:

$$Y(\gamma) = \sum_{n=0}^{\infty} (2n+1)\, y_n\, P_n(\gamma) \tag{39}$$

where the P_n are spherical harmonics and y_n are coefficients to be determined from:

$$z_n = \left(1 - \frac{3}{2}\frac{n+1}{2n+3}\right) y_n + \frac{3}{2}\frac{n+3}{2n+5}\, y_{n+1} + \frac{3}{2}\frac{n+1}{2n+3}\, y_{n+2} +$$

$$+ \left(1 - \frac{3}{2}\frac{n+3}{2n+5}\right) y_{n+3} = \frac{2\,(-1)^n}{(n+1)(n+2)(n+3) + 8\,i\,\tau\,(-1)^n} \tag{40}$$

We now form the cross-section (7). Except if both the primary and the secondary mesons are transverse only X_{AB} occurs. Inserting (38) into (35) we find from (7) very easily in the usual way:

$$\Phi_{AB} = \frac{8\,\pi}{\varepsilon^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda + \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{|l} g \\ f \\ f' \end{array} \tag{41}$$

which again gives the cross-section for scattering from one primary polarization into one secondary polarization. If both mesons are

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[9] Hamilton and Peng, Proc. Roy. Irish Acad. in the press.

transverse there is a contribution to Φ arising from $|X_{AB}|^2$ which gives the term 1/4 in the matrix (41). In addition there will be terms arising from Y, which in (41) are denoted by λ. To calculate λ we note that again all cross products XY vanish after the summation over the two transverse polarizations j_A. The calculation of the additional part of the cross-section arising from $|Y|^2$ is also given in ref. 9. The result is, when averaged over the final polarization:

$$\bar{\lambda} = 2f^4\tau^2 \cdot \sum_{n=0}^{\infty} (n+2)\,|z_n|^2 \tag{42}$$

with z_n given by (40). We estimate the order of magnitude of $\bar{\lambda}$. We introduce instead of n the variable x defined by $n^3 = 8\tau x^3$. If τ is large, which is always here the case, the summation (42) over n can be replaced by an integral over x:

$$\bar{\lambda} = \tfrac{1}{2} f^4 \tau^{2/3} \int_0^\infty \frac{x\,dx}{1+x^6} = \frac{\pi}{6\sqrt{3}} f^4 \tau^{2/3} \tag{43}$$

$\bar{\lambda}$ is therefore approximately: $\bar{\lambda} \simeq 0{\cdot}3\, f^{16/3}\, \varepsilon^{4/3}$. Although it increases with ϵ it is much smaller than the term 1/4 arising from X except for extremely high energies. Using for f^2 the value $0{\cdot}13$ the energy at which $\bar{\lambda}$ becomes comparable with the main term 1/4 is $\varepsilon \sim 60$. (Remember that this is still in the Lorentz-system where the meson and nucleon collide with equal momenta.) For all smaller energies $\bar{\lambda}$ can be neglected.

As to the angular dependence of the scattering cross-section: The terms arising from X_{AB} are all independent of the scattering angle. Only that part which gives rise to the λ-term depends on angle. Therefore for all energies for which λ can be neglected the scattering is independent of the scattering angle in this Lorentz-system.

The selection rule for the polarization is, in the E.R. case, more stringent than in the N.R. case. The pseudoscalar polarization is now separated from the transverse polarizations so that there is practically no chance for the polarization of a meson to be changed by scattering. The two transverse polarizations may, however, change into each other.

From (41) we see that the total cross-section for the scattering of a particular primary polarization to all secondary polarizations of a meson is

$$\Phi = \frac{8\pi}{\varepsilon^2} \qquad \text{(primary polarization longitudinal or pseudoscalar)}$$

$$\Phi = \frac{8\pi}{\varepsilon^2}\left(\tfrac{1}{2} + 2\bar{\lambda}\right) \qquad \text{(primary polarization transverse)} \tag{44}$$

These results hold in both the charged and the symmetrical theory.

E.R. (1, 0). Total charge 1 or 0.

The calculation for this case, in the symmetrical theory, is very similar to that of the (2, −1) case, the only difference being that neutrettos also play an essential rôle, because the system may also consist of a neutretto and a proton, etc. If the result is to be represented in matrix form as above we obtain a matrix with 6 rows and columns. We give the result without calculation:

$$\Phi_{AB} = \frac{8\pi}{\epsilon^2}\begin{array}{c|c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}+\lambda_1 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 & \frac{1}{4}+\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{matrix} g \\ f \\ f' \\ f_0' \\ f_0 \\ g_0 \end{matrix} \end{array} \qquad \text{(sym. Th.)} \quad (45)$$

The attribution of the rows and columns to the various polarizations and the charge of the meson is indicated by the corresponding coupling constants on the right-hand side. The λ's are again of the same order of magnitude as λ equ. (43), but have different numerical factors, all of them, however, of the order of magnitude one. For $\epsilon < 60$ the λ's can all be neglected and the selection rules are very stringent: No transformation into neutrettos and no change of polarization (except between the two transverse polarizations) takes place. At very high energies, however, a charged transverse meson can be transformed into a transverse neutretto again. The total cross-sections are given by (44) with a correspondingly different $\bar{\lambda}$.

The calculation for the "charged theory" is very similar to that of the N.R. (1, 0) case. The result is:

$$\Phi_{gg} = \frac{8\pi}{\epsilon^2}\frac{g^4}{(g^2+3f^2)^2}, \qquad \Phi_{ff} = \frac{8\pi}{\epsilon^2}\frac{f^4}{(g^2+3f^2)^2}$$

$$\Phi_{gf} = \Phi_{fg} = \frac{8\pi}{\epsilon^2}\frac{f^2 g^2}{(g^2+3f^2)^2}. \qquad \text{(ch. Th.)} \quad (46)$$

There are no selection rules in this case.

All the previous formulae for the E.R. case hold in a Lorentz-system where the meson and nucleon collide with equal but opposite momenta. We now transform them into a Lorentz-system in which the nucleon is initially at rest. The energy of the scattered meson will then depend strongly upon the angle of scattering. The law of scattering can very easily be found for that part of the cross-section, which in the first Lorentz-system is independent of angle, i.e. for the part *not* included in the λ-contribution. Henceforward, we shall assume that $\epsilon < 60$

and therefore all λ's negligible. Let ϵ, θ be the energy and angle of scattering in the Lorentz-system where the nucleon moves, and $\bar{\varepsilon}$, $\bar{\varepsilon}'$, θ' the energies before and after the scattering and the scattering angle in the Lorentz-system where the nucleon is initially at rest. Then we have:

$$\bar{\varepsilon} = 2\epsilon^2/M, \qquad \cos\theta = \frac{2\bar{\varepsilon}'}{\bar{\varepsilon}}$$

$$\cos\tilde{\theta} = 1 + \frac{M}{\bar{\varepsilon}} - \frac{M}{\bar{\varepsilon}'} \tag{47}$$

by a simple Lorentz-transformation. Since in the first Lorentz-system the differential cross-sections are all proportional to $d\cos\theta$ the probability for the scattered meson to take up an energy between $\bar{\varepsilon}'$ and $\bar{\varepsilon}' + d\bar{\varepsilon}'$ $(0 < \bar{\varepsilon}' < \bar{\varepsilon})$ is simply proportional to $d\bar{\varepsilon}'/\bar{\varepsilon}$. The cross-section itself is invariant against this Lorentz-transformation, it is therefore obtained just by expressing ϵ by $\bar{\varepsilon}$ and $d\cos\theta$ by $d\bar{\varepsilon}'/\bar{\varepsilon}$. Thus the differential cross-sections for the scattered meson to take an energy in the energy interval $d\bar{\varepsilon}'$ are:

$$\Phi\, d\bar{\varepsilon}' = \frac{16\pi}{\bar{\varepsilon} M}\frac{d\bar{\varepsilon}'}{\bar{\varepsilon}} \qquad \text{(long. or pseudosc.)}$$

$$\Phi\, d\bar{\varepsilon}' = \frac{8\pi}{\bar{\varepsilon} M}\frac{d\bar{\varepsilon}'}{\bar{\varepsilon}} \qquad \text{(transv.)} \tag{48}$$

(48) holds for the (2, − 1)-case and in the sym. Th. also for the (1, 0)-case.

For the charged theory in the (1, 0)-case the formulae are slightly different:

$$\Phi\, d\bar{\varepsilon}' = \frac{16\pi}{\bar{\varepsilon} M}\frac{g^2}{(g^2 + 3f^2)^2}\frac{d\bar{\varepsilon}'}{\bar{\varepsilon}} \qquad \text{(longitudinal)}$$

$$\Phi\, d\bar{\varepsilon}' = \frac{16\pi}{\bar{\varepsilon} M}\frac{f^2}{(g^2 + 3f^2)^2}\frac{d\bar{\varepsilon}'}{\bar{\varepsilon}} \qquad \text{(transverse or pseudoscalar)}$$

(ch. Th.) (49)

(48) or (49) gives the scattering cross-section of a meson from the indicated primary polarization to all possible scattered polarizations.

According to (47) the angle $\tilde{\theta}$ is very small if $\bar{\varepsilon}' \simeq \bar{\varepsilon}$, but is large if $\bar{\varepsilon}' << \epsilon$, as is to be expected. For large $\tilde{\theta}$ almost the whole energy $\bar{\varepsilon}$ is taken up by the nucleon which receives a big recoil energy.

(48) is valid only as long as the λ's can be neglected. This is true if $\epsilon < 60$, or now, expressed in terms of $\bar{\varepsilon}$, if $\bar{\varepsilon} < \frac{2}{M}(60)^2 = 720$. This *limit* is *very high* ($\sim 7{\cdot}10^{10}$ e.v.), and for practically all energies occurring in cosmic radiation the λ's can be neglected. The condition $\epsilon >> M$ for the E.R. case becomes, in terms of $\bar{\varepsilon}$, $\bar{\varepsilon} >> 2M$.

PART III.

THE PRODUCTION OF MESONS BY COLLISIONS OF TWO NUCLEONS AND THE ENERGY LOSS OF A NUCLEON.

We now combine the results obtained in the previous parts I and II. According to I a fast moving nucleon can be considered as being equivalent to a free meson pulse moving in the same direction. This pulse contains mesons of various energies ϵ and of various polarizations, mainly transverse and pseudoscalar, which may be charged or neutral. The production of a meson of certain specified charge, energy, and polarization during the collision of two nucleons appears, in the treatment of Williams and v. Weizsäcker, as a meson scattered by the second nucleon from the equivalent mesons of the moving nucleon. The condition that the meson field of the moving nucleon can be replaced by a pulse of equivalent mesons has been obtained in I, equations (12) and (14), i.e.

$$1 \ll \epsilon \ll E\,, \qquad E \gg M\,, \tag{1}$$

E being the energy of the moving nucleon and ϵ, that of the equivalent meson. The energy distribution of the equivalent mesons is given by I (31) and table 1. The equivalent charged mesons replacing the field of a moving proton (neutron) are to be regarded as positive (negative).

The cross-section for the production of a meson in a collision between two nucleons is now obtained by forming the product of the scattering cross-section multiplied by the number of equivalent mesons in the energy range $d\epsilon$. There are, however, two points to be taken into account: (i) If ϵ is the energy of the equivalent meson then ϵ is equal to the sum of the energy of the meson produced plus the energy transferred to the nucleon initially at rest. In the N.R. case the latter is negligible, but in the E.R. case the energy is on the average shared in comparable parts by the meson produced and the recoil nucleon. (ii) For the production of a meson of given polarization equivalent mesons of several polarizations may contribute, according to whether the selection rules allow the polarization to change during the scattering or not. Accordingly, to obtain the production of mesons of given polarization, we have to sum over the polarizations of the equivalent mesons, as the case may be.

For the production of mesons of energies $\epsilon < M$ but not too small ($f^2\epsilon^2 \gg 1$) we can use the asymptotic formulae for the scattering cross-section given in Part II, for the N.R. case. In the symmetrical theory the scattering cross-sections for the cases $(2, -1)$ and $(1, 0)$ are then the same. We denote for instance the cross-section for the production of longitudinal neutrettos by Φ^{0}_{long} and that for the production of charged transverse meson Φ_{tr}, etc. From I (31) (the g^2-terms are neglected) and II (14), II (28) we obtain the cross-section for the production of a meson within the energy range $d\epsilon$:

$$\Phi_{long}\, d\varepsilon = 2\Phi^{0}_{long}\, d\varepsilon = \frac{d\varepsilon}{\varepsilon^3}\, 4 f^2\, C$$

$$\tfrac{1}{2}\,\Phi_{tr}\, d\varepsilon = \Phi_{ps}\, d\varepsilon = \Phi^{0}_{tr}\, d\varepsilon = 2\Phi^{0}_{ps}\, d\varepsilon = \frac{d\varepsilon}{\varepsilon^3}\, \tfrac{4}{3} f^2\, (D_{tr} + D_{ps})$$

$$(f^{-1} << \varepsilon << M)\,. \qquad \text{(sym. Th.)} \quad (2)$$

C and the D's are functions of $z = \frac{1}{M}\sqrt{1 + \varepsilon^2 M^2 / E^2}$. In the charged theory the scattering cross-sections for the cases (2, −1) and (1, 0) are different, given respectively by II (14) and II (18). We have then to distinguish between $P - P$ and $N - N$ collisions on the one hand and $P - N$ and $N - P$ collisions on the other hand. The cross-sections for the production of mesons of various polarizations are:

$$\left.\begin{aligned} \Phi_{long}\, d\varepsilon &= \frac{d\varepsilon}{\varepsilon^3}\, 4 f^2\, C \\ \tfrac{1}{2}\,\Phi_{tr}\, d\varepsilon &= \Phi_{ps}\, d\varepsilon = \frac{d\varepsilon}{\varepsilon^3}\, \tfrac{4}{3} f^2\, (D_{tr} + D_{ps}) \end{aligned}\right\} \text{(P–P and N–N collis.)}$$

$$\left.\begin{aligned} \Phi_{long}\, d\varepsilon &= \frac{d\varepsilon}{\varepsilon^3}\, \frac{4 f^4 g^2}{(g^2 + 3f^2)^2}\, (D_{tr} + D_{ps}) \\ \tfrac{1}{2}\,\Phi_{tr}\, d\varepsilon &= \Phi_{ps}\, d\varepsilon = \frac{d\varepsilon}{\varepsilon^3}\, \frac{4 f^6}{(g^2 + 3f^2)^2}\, (D_{tr} + D_{ps}) \end{aligned}\right\} \text{(P–N collis.)}$$

$$(f^{-1} << \varepsilon << M)\,. \qquad \text{(ch. Th.)} \quad (3)$$

For $P - N$ collisions a term $g^2 C$ has been neglected in comparison with $f^2 D_{tr}$. The sign of the charge of the meson produced, for both cases (2) and (3), is determined by the rule that it is always the *fast nucleon* that changes its charge by the collision. This is so because the production of a meson is due to the scattering of one of the equivalent mesons of the fast moving nucleon by the nucleon at rest, which does not change its charge. It is always the fast moving nucleon which loses the meson. This, however, is only true as long as the selection rule for the charge is valid, otherwise the fast moving nucleon may lose an equivalent neutretto which may be transformed into a charged particle.

For the production of mesons of energies lower than that considered above ($\epsilon \sim 1/f$) we have to use rather complicated formulae for the scattering cross-section. There is then no selection rule of the charge or polarization for the scattering. Each scattering cross-section possesses a maximum somewhere between $\epsilon = 2$ and $\epsilon = 3$ (cf. Table II).

The maximum is of the same order of magnitude for the scattering of mesons of various charges and polarizations. Since the spectrum of the equivalent mesons varies in this region slowly it is certain that the cross-section for the production of mesons reaches its maximum somewhere between $\epsilon = 2$ and $\epsilon = 3$, i.e. in the neighbourhood of $\epsilon = 1/f$. The cross-section for production then falls to zero for $\epsilon \to 1$.

For the production of mesons of energy $\epsilon' \gg 2M$ we have to use the scattering formulae II (48) and II (49); the $\sim$ used there to distinguish the Lorentz-system can now be dropped. A meson of energy ϵ' is produced by all the equivalent mesons of energies $\epsilon > \epsilon'$ the balance $E' = \epsilon - \epsilon'$ of the energy is transferred to the recoil nucleon. Thus from I (31) and II (48), II (49) we obtain the following cross-sections:

$$\Phi_{long}\, d\varepsilon' = 2\,\Phi^0_{long}\, d\varepsilon' = \frac{16 f^2}{M}\, d\varepsilon' \int_{\epsilon'} C\, \frac{d\varepsilon}{\varepsilon^3}$$

$$\Phi_{ps}\, d\varepsilon' = 2\,\Phi^0_{ps}\, d\varepsilon' = \frac{16 f^2}{M}\, d\varepsilon' \int_{\epsilon'} D_{ps}\, \frac{d\varepsilon}{\varepsilon^3}$$

$$\Phi_{tr}\, d\varepsilon' = 2\,\Phi^0_{tr}\, d\varepsilon' = \frac{8 f^2}{M}\, d\varepsilon' \int_{\epsilon'} D_{tr}\, \frac{d\epsilon}{\varepsilon^3}$$

$$(M \ll \varepsilon' \ll E). \qquad \text{(sym. Th.)} \quad (4)$$

In the "charged theory" we have again to distinguish between $P - P$ or $N - N$ collisions and $P - N$ or $N - P$ collisions. The cross-sections are respectively:

$$\left.\begin{aligned} \Phi_{long}\, d\varepsilon' &= \frac{16 f^2}{M}\, d\varepsilon' \int_{\epsilon'} C\, \frac{d\varepsilon}{\varepsilon^3} \\ \Phi_{ps}\, d\varepsilon' &= \frac{16 f^2}{M}\, d\varepsilon' \int_{\epsilon'} D_{ps}\, \frac{d\varepsilon}{\varepsilon^3} \\ \Phi_{tr}\, d\varepsilon' &= \frac{8 f^2}{M}\, d\varepsilon' \int_{\epsilon'} D_{tr}\, \frac{d\varepsilon}{\varepsilon^3}. \end{aligned}\right\} \begin{array}{c} (P\text{-}P \text{ or } N\text{-}N \\ \text{collisions.}) \\ \text{(ch. Th.)} \quad (5a) \end{array}$$

$$\left.\begin{aligned} \Phi_{long}\, d\varepsilon' &= \frac{16 f^2}{M}\, \frac{g^2 f^2}{(g^2 + 3f^2)^2}\, d\varepsilon' \int_{\epsilon'} (D_{tr} + D_{ps})\, \frac{d\varepsilon}{\varepsilon^3} \\ \Phi_{ps}\, d\varepsilon' = \tfrac{1}{2}\,\Phi_{tr}\, d\varepsilon' &= \frac{16 f^2}{M}\, \frac{f^4}{(g^2 + 3f^2)^2}\, d\varepsilon' \int_{\epsilon'} (D_{tr} + D_{ps})\, \frac{d\varepsilon}{\varepsilon^3} \end{aligned}\right\} \begin{array}{c} (P\text{-}N \text{ or } N\text{-}P \\ \text{collisions.}) \\ \text{(ch. Th.)} \quad (5b) \end{array}$$

$$(M \ll \varepsilon' \ll E).$$

We may put the upper limit of the integration in these integrals $\epsilon = E$. Since the main contribution to these integrals arises in the neighbourhood of the lower limit of integration the error made by assuming the spectrum I (31) to be valid for $\varepsilon \sim E$ is negligible if $\epsilon' \ll E$. The λ-terms of the scattering cross-sections have been omitted in (4) and (5). These equations hold therefore only for $\epsilon' < 720$.

For every meson or neutretto produced there is a recoil nucleon. We denote the energy of the recoil nucleon by E'. The differential cross-section for the production of a recoil nucleon accompanying the production of mesons and neutrettos of all polarizations is therefore

$$\Phi_{E'} dE' = \frac{24 f^2}{M} dE' \int_{E'} (D_{ps} + \tfrac{1}{2} D_{tr}) \frac{d\epsilon}{\epsilon^3} \qquad (6)$$

(sym. Th.)

$$(M \ll E' \ll E) .$$

For practical purposes the D's and C can very well be approximated by some simple functions or average values. They depend essentially on ϵ / E. We may use (compare Table I):

$$D_{ps} = 50 \sqrt{\frac{\epsilon}{E}} , \qquad D_{tr} = 165 , \qquad C = 0$$

$$D_{ps} + \frac{1}{2} D_{tr} = 115 , \qquad D_{ps} + D_{tr} = 200 \qquad (7)$$

Hence the cross-sections for production of a transverse or pseudoscalar meson of energy ϵ' becomes approximately, putting the upper limit equal to E and dropping the primes of ϵ:

$$2 \phi^0_{ps} d\epsilon = \Phi_{ps} d\epsilon = d\epsilon \frac{16 f^2}{M} \frac{2}{3} 50 \left(\frac{1}{\sqrt{E}} \frac{1}{\epsilon^{3/2}} - \frac{1}{E^2} \right) \qquad (8)$$

$$2 \phi^0_{tr} d\epsilon = \Phi_{tr} d\epsilon = d\epsilon \frac{8 f^2}{M} \frac{165}{2} \left(\frac{1}{\epsilon^2} - \frac{1}{E^2} \right)$$

$$(\epsilon \gg M)$$

(sym. Th.)

and

$$\phi^0_{tr} d\epsilon = 2 \phi^0_{ps} d\epsilon = \frac{1}{2} \Phi_{tr} d\epsilon = \Phi_{ps} d\epsilon = d\epsilon \frac{4}{3} f^2 \, 200 \frac{1}{\epsilon^3} \qquad (8')$$

$$(\epsilon \ll M)$$

(sym. Th.)

$\Phi_{long}\, d\epsilon$ is always small.

Thus, in a collision of two nucleons, mesons are produced with an energy spectrum of the form $d\epsilon / \epsilon^3$ if $\epsilon \ll M$, and $d\epsilon / \epsilon^2$ or $d\epsilon / \epsilon^{3/2}$ if $\epsilon \gg M$. There are also recoil nucleons with the same energy spectrum dE' / E'^2 (according to (16)) for $E' > M$.

We calculate the total cross-section for the production of a meson of any kind by a nucleon with energy $E \gg M$. We may use (8) and (8′) approximately for $\epsilon < M$ and $\epsilon > M$ respectively (instead of for $\epsilon << M$, $\epsilon >> M$) and (8′) also for $\epsilon > 1/f$ instead of $\epsilon >> 1/f$. The largest contribution arises from (8′) and is:

$$\Phi \simeq 400 f^4 \simeq 6.8 \tag{9}$$

(9) *is a very large cross-section.* (Remember that the units used are $(\hbar/\mu c)^2 = 4{\cdot}3 \times 10^{-26}$ cm.^{2}). A fast nucleon only travels a distance of 5·5 cm. H_2O before it produces a meson. Most of the mesons produced have, however, only a small energy, of the order of magnitude $1/f$. *Therefore, a fast nucleon will produce many mesons* before it is stopped. There are, however, also fast mesons produced, in small numbers though, but most of the energy lost by the nucleon is contained in these fast mesons. We can define a cross-section for energy loss

$$\Phi_{en.\,loss} = \Sigma \frac{1}{E} \int (\epsilon + E')\, \Phi \, d\epsilon \tag{10}$$

where the Σ is to be extended over all polarizations and charges of the mesons produced. $\epsilon + E'$ is the total energy lost by the nucleon including that given to the recoil nucleon. We find from (8), (8′), using the values I (30) for the constant f:

$$\Phi_{en.\,loss} = \frac{1}{E}\left(40 + 36 \log \frac{E}{M}\right) \tag{11}$$

(sym. Th.)

The second logarithmic term arises from mesons with energy $> M$, and is, of course, the bigger one. The energy loss also is very great. For instance, a fast nucleon with $E >> M$ loses per cm. H_2O an energy (measured in $\mu c^2 \sim 10^8$ e.v.)

$$-\frac{dE}{dx} = NE\,\Phi_{en.\,loss} = 1.08 + 0.97 \log \frac{E}{M} \tag{12}$$

(sym. Th.)

(N = the number of nucleons per cm.^{3})

which is more than 10^8 e.v. per cm. H_2O.

In the "charged theory" the energy loss is considerably smaller, firstly because no neutrettos are produced, secondly, because the cross-section for scattering is smaller in the case of $P-N$ and $N-P$ collisions. Taking the average for $P-N$ and $P-P$ collisions and using the values for the coupling constants

$$g^2 = 0.17, \qquad f^2 = f'^2 = 0.11 \tag{13}$$

(ch. Th.)

(which give the right values for the nuclear forces)

$$\Phi_{en.\,loss} = \frac{1}{E}\left(12 + 14 \log \frac{E}{M}\right) \tag{14}$$

(ch. Th.)

The application of this theory to the production of mesons in cosmic radiation will be published elsewhere.[10]

Finally, there are a few points, chiefly of theoretical interest, to be settled:

As we have seen the meson is always emitted by the fast moving nucleon. One might wonder whether the nucleon initially at rest is not also capable of emitting a meson. This question has been considered by Williams in the electromagnetic case. His considerations can be immediately extended to our case to show that meson emission by the nucleon initially at rest is negligible.

Our theory is only valid for: $E >> M$, $1 << \varepsilon << E$. We briefly consider qualitatively how our results will change if these conditions are not strictly satisfied. The conditions for ϵ are not of very great importance, the case $\epsilon \sim 1$ being of no interest and if ϵ approaches the value E the cross-section becomes rather small. For $\epsilon = \frac{1}{2}E$ our formulae should still hold approximately and it is very unlikely that they will undergo a change by an order of magnitude if $\epsilon \rightarrow E$. Our formulae will certainly give the right order of magnitude also for this case. Our theory cannot, however, be used if $E < M$. In this case we can obtain a guidance as to the result to be expected from older calculations[11] in which the rate of meson production has been calculated for $E << M$ neglecting the damping. The cross-section is found to be quite small. Thus we have to expect that all our *cross-sections decrease rapidly as* E *becomes comparable with* M *or smaller than* M. Our theory itself indicates that since for $E < M$ the minimum impact parameter Part I (27) is $1/Mv$ instead of $1/M$ and that means larger values for z. All our D's and C decrease exponentially for large z. Thus we expect that meson production practically ceases if $E \sim M$.

In addition to the mode of meson production considered in this paper there is another mode in which mesons can be produced in a collision between two nucleons. A fast moving proton has also a Lorentz-transformed Coulomb-field which is equivalent to a stream of light quanta. A light quantum can be transformed into a meson by the process $\hbar\nu + P \rightarrow N + Y^{+}$ etc.[12] We give a brief estimate of the order of magnitude. The equivalent light quanta spectrum is[2]

$$\frac{1}{137}\,\frac{2}{\pi}\,\frac{d\nu}{\nu}\,\log\frac{E}{\nu}$$

[10] Hamilton, Heitler and Peng, Phys. Rev. in the press.

[11] Cf. for instance Massey and Corben, Proc. Camb. Phil. Soc., 35, 84, 1939.

[12] This kind of meson production has been considered neglecting the damping by Kobayasi and Sato, Scient. Pap. Inst. Phys. Chem. Res. Tokyo, 38, 51, 1940.

The cross-section for transformation into a meson is largest if the meson is longitudinal and has been derived in the following paper by Hamilton and Peng.[9] For $\nu << M$ (the only case for which the problem is solved) the cross-section is $\pi^2 e f / 2\sqrt{2}$. Thus the rate of meson production is

$$\Phi_{long}\, d\epsilon = 5{\cdot}10^{-4} \frac{d\epsilon}{\epsilon} \log \frac{E}{\epsilon} \qquad \text{(N.R.)} \quad (15)$$

(15) should be compared with (8) and (8′). We first notice that the energy dependence is different, (15) is proportional to $d\epsilon / \epsilon \,.\, \log E / \epsilon$ whilst (8′) and (8) are proportional to $d\epsilon / \epsilon^2$ and $d\epsilon / \epsilon^3$. On the other hand (15) has a very small numerical factor. That means that this mode of meson production is negligible for all energies E except the extremely high ones. Unfortunately (15) is only valid for $\epsilon < M$ and we can therefore not give an exact estimate at which energy E (15) would be greater than (8). If we tentatively apply (15) also for $\epsilon > M$, the energy loss due to this kind of meson production would be comparable with the one considered in this paper if $E \simeq 10^4$ which is about 10^{12} e.v. This limit will be lowered if a proton collides with a heavy nucleus because it can be shown that then the effect increases with Z^2. It will be noticed that the contribution of the λ-terms of the cross-section also sets in at very high energies ($E \sim 10^{11}$ e.v.). In addition there are also other processes, for instance the emission of light quanta by collisions between 2 nucleons which become important if the energy exceeds these limits. The formulae given in this paper, especially for the energy loss, do not therefore hold for the extremely high energies.

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XI.

ON THE PRODUCTION OF MESONS BY LIGHT QUANTA AND RELATED PROCESSES.

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1. Introduction.

Until recently it was impossible to apply the meson theory of nuclear forces to collision processes of any kind involving mesons. The reason is that in such collision processes radiation damping is of paramount importance — and a theory of radiation damping is part of the very problem of quantum electrodynamics. To take an example: The process studied below, namely the transformation of a light quantum into a meson during a collision with a proton, say,

$$\hbar\nu + P \longrightarrow N + Y^{+}$$

has, if damping is neglected, a cross-section which would increase with $(\hbar\nu)^2$ (cf. reference (5)). The same would be true for the reverse process. Both are quite incompatible with the actual facts, as has in fact long been known. Recently Heitler and Peng (4) have developed a general theory of damping which may be regarded as an attempt at guessing the correct formulae of a part of quantum electrodynamics, namely, that part which is dealing with transition probabilities. Mathematically, the calculation of transition probabilities requires the solution of a set of simultaneous inhomogeneous integral equations. It is the object of the present paper to show that the above difficulties disappear in this theory of damping. Whilst applications to the actual production of mesons in cosmic radiation will be made in a paper by Heitler and Peng,[1] the contents of this paper are mainly of theoretical interest. In the first place we show that the cross-sections for the process in question and its reverse process are small enough to be at least qualitatively compatible with the experiments and that both processes do not play a very important rôle in cosmic radiation. The method developed here for solving the integral equations can, however, also be used for other, more important, processes, and are actually used in the paper quoted above. In the second place we shall discuss a problem (as it seems of purely theoretical interest), namely, the so-called "principle of

[1] Heitler and Peng, Proc. Roy. Irish Acad., 49, A, 7, 101, 1943.

detailed balance." We shall see that this principle no longer holds in its most general form in the present theory of damping. (If damping is neglected, it is a trivial consequence of the fact that the Hamiltonian of the interactions between all fields and particles is Hermitian.) As far as our present investigation goes, the violation of the principle of detailed balance is of a very special and limited kind; it is only violated if the spin is included in the description of a state, it holds again whenever the average over all spin directions is taken. This, however, we have so far only been able to prove for the processes dealt with in this and the paper ref. 1, but we suppose it to be true more generally. We suppose that it may very well be that the non-validity of the principle of detailed balance is an essential part of future quantum-electrodynamics.

Throughout the paper we shall assume that the proton mass is large compared with the energy of the light quanta or mesons concerned, thus neglecting recoil energies, etc. The form of the meson theory used is a combination of vector and pseudoscalar charged mesons, but, for simplicity, we shall not include any neutral mesons. The inclusion of the latter would make the calculations much more complicated, but for all we know would not alter the order of magnitude of the results.

2. The Interaction of Light and Mesons with Protons and Neutrons.

Light and mesons are treated quantum mechanically according to the usual field quantisation methods. Protons and neutrons are considered as two states of the one particle, which we shall throughout call the *nucleon*.

According to the meson theory in the form suggested by Møller and Rosenfeld,[2] the Hamiltonian for the interaction of light, mesons and nucleons is given by

$$H_{\text{interaction}} = H^{\text{i}} + H^{\text{ii}} + H^{\text{iii}} + H^{\text{iv}}$$

with

$$H^{\text{i}} = \frac{ie}{4\pi}\int \{(\mathrm{A}, \psi^* + \dot{\phi}^*) \operatorname{div} \psi + ([\mathrm{A}, \phi^* - \dot{\psi}^*] \operatorname{curl} \phi) + \Psi^* (\mathrm{A} \operatorname{grad} \Psi)\}\, dV$$
$$+ \text{(adjoint terms)} + \text{(terms quadratic in A)},$$

$$H^{\text{ii}} = \{g\Pi^* \operatorname{div} \psi^* + f\Pi^* (\sigma \operatorname{curl} \phi^*) + f'\Pi^* (\sigma \operatorname{grad} \Psi^*)\} + \text{(adjoint terms)}$$

$$H^{\text{iii}} = ie\{g\Pi^* (\mathrm{A}, \psi^* + \dot{\phi}^*) + f\Pi^* (\sigma[\mathrm{A}, \phi^* - \dot{\psi}^*]) + f'\Pi^* (\sigma\mathrm{A})\Psi^*\}$$
$$+ \text{(adjoint terms}$$

$$H^{\text{iv}} = \frac{e^2}{2M}\Pi\,\Pi^*\mathrm{A}^2. \qquad (1)$$

[2] Møller and Rosenfeld, Kgl. Dansk. Vid. Sels., 17, 8, 1940.

Here the velocity of the nucleon has been assumed small compared with the velocity of light. A, ψ, ϕ, Ψ describe the electromagnetic, longitudinal meson, transverse meson and pseudoscalar meson fields. Π is the operator denoting the transition of the nucleon from the neutron to the proton state. The asterisk denotes, in all cases, the adjoint. σ is the spin vector of the nucleon. e is the elementary electric charge, while g, f, f' are the coupling constants of the nucleon with the longitudinal, transverse, pseudoscalar meson fields respectively. The units are "meson units," chosen so that $\hbar = c = \mu$ (meson mass) $= 1$.

We now give a complete list of the matrix elements of H for all those transitions which will occur below. To explain the notation we take a typical non-vanishing matrix element of H^{i} corresponding to the virtual transition

$$\hbar\nu \longrightarrow Y^{+}_{\text{long}} + Y^{-}_{\text{trans}}.$$

This element we denote by $H^{\mathrm{i}}_{k_3,\, k'_1 \mid \nu_1}$, where $\nu_1, \mathbf{k}'_1, \mathbf{k}_3$ denote respectively the polarization vectors of the photon absorbed, and of the transverse and longitudinal mesons emitted. Further, by ν_3, $\mathbf{k}'_3$, $\mathbf{k}_3$ we denote unit vectors in the directions of propagation of the photon and the mesons. Also $\mathbf{k}_2 = [\mathbf{k}_3\mathbf{k}_1]$. In the case of a meson being pseudoscalar, we use the suffix k_4 in the matrix element, but its unit vector of propagation is always called $\mathbf{k}_3$. Primes are used for the second of two-mesons involved. For example, the element for the virtual process

$$Y^{+}_{\text{trans}} \longrightarrow Y^{+}_{\text{ps}} + \hbar\nu$$

is denoted by

$$H^{\mathrm{i}}_{k'_4\nu_1 \mid k_1}.$$

The H^{i} elements are then as follows:—

$$-H^{\mathrm{i}}_{k_3 \mid \nu_1 k'_3} = H^{\mathrm{i}}_{k_3 k'_3 \mid \nu_1} = \frac{\sqrt{\pi}\, e}{\sqrt{2V\epsilon\epsilon'\nu}}\{k'(\nu_1\mathbf{k}_3) + k(\nu_1\mathbf{k}'_3)\}$$

$$-H^{\mathrm{i}}_{k_1 \mid \nu_1 k'_1} = H^{\mathrm{i}}_{k_1 k'_1 \mid \nu_1} = \frac{\sqrt{\pi}\, e}{\sqrt{2V\epsilon\epsilon'\nu}}\{k'([\nu_1\mathbf{k}_1]\mathbf{k}'_2) + k([\nu_1\mathbf{k}'_1]\mathbf{k}_2)\}$$

$$-H^{\mathrm{i}}_{k_4 \mid \nu_1 k'_4} = H^{\mathrm{i}}_{k_4 k'_4 \mid \nu_1} = \frac{\sqrt{\pi}\, e}{\sqrt{2V\epsilon\epsilon'\nu}}\{k'(\nu_1\mathbf{k}'_3) + k(\nu_1\mathbf{k}_3)\}$$

$$-H^{\mathrm{i}}_{k_3 \mid \nu_1 k'_1} = \frac{\sqrt{\pi}\, e}{\sqrt{2V\epsilon\epsilon'\nu}}\{-i\epsilon k'([\nu_1\mathbf{k}_3]\mathbf{k}'_2) - i\epsilon' k(\nu_1\mathbf{k}'_1)\}$$

$$+H_{k_3 k'_1 \mid \nu} = \frac{\sqrt{\pi}\, e}{\sqrt{2V\epsilon\epsilon'\nu}}\{-i\epsilon k'([\nu_1\mathbf{k}_3]\mathbf{k}'_2) + i\epsilon' k(\nu_1\mathbf{k}'_1)\}. \qquad (2)$$

V is the normalizing volume for the fields. ϵ, ϵ', ν are the energies of the mesons and the photon. $(\epsilon^2 = 1 + k^2)$.

The conservation of momentum necessitates

$$\mathbf{k}' = \mathbf{k} \mp \boldsymbol{\nu} \tag{2'}$$

according as $\mathbf{k}$ refers to a meson of charge $\pm$, where $\mathbf{k}$, $\mathbf{k}'$, $\boldsymbol{\nu}$ are the propagation vectors of the mesons and photon. $(\boldsymbol{\nu} = \nu\, \boldsymbol{\nu}_3,\ \mathbf{k} = k\, \mathbf{k}_3)$.

Using the suffixes P, N to denote proton and neutron states, the other elements can be written:—

$$\left.\begin{aligned} -H^{\text{ii}}_{Nk_3\,|\,P} &= H^{\text{ii}}_{Pk_3\,|\,N} = \sqrt{\frac{2\pi}{\epsilon V}}\, gk \\ -H^{\text{ii}}_{Nk_1\,|\,P} &= H^{\text{ii}}_{Pk_1\,|\,N} = \sqrt{\frac{2\pi}{\epsilon V}}\, fk(\sigma\,\mathbf{k}_2) \\ -H^{\text{ii}}_{Nk_4\,|\,P} &= H^{\text{ii}}_{Pk_4\,|\,N} = \sqrt{\frac{2\pi}{\epsilon V}}\, f'k(\sigma\,\mathbf{k}_3) \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} H^{\text{iii}}_{Nk_3\,|\,P\nu_1} &= -H^{\text{iii}}_{Pk_3\,|\,N\nu_1} = \frac{2\pi e}{V\sqrt{\epsilon\nu}}\{g(\nu_1\,\mathbf{k}_3) - if\epsilon(\sigma[\nu_1\,\mathbf{k}_3])\} \\ H^{\text{iii}}_{Nk\,|\,P\nu_1} &= -H^{\text{iii}}_{Pk_1\,|\,N\nu_1} = \frac{2\pi e}{V\sqrt{\epsilon\nu}}\{ig\epsilon(\nu_1\mathbf{k}_1) + f(\sigma[\nu_1\mathbf{k}_1])\} \\ H^{\text{iii}}_{Nk_4\,|\,P\nu_1} &= -H^{\text{iii}}_{Pk_4\,|\,N\nu_1} = \frac{2\pi e}{V\sqrt{\epsilon\nu}}\, f'(\sigma\,\nu) \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} H^{\text{iv}}_{P\nu'_1\,|\,P\nu_1} &= \frac{\pi e^2}{VM\sqrt{\nu\nu'}}(\nu_1\,\nu_1') \\ H^{\text{iv}}_{N\nu'_1\,|\,N\nu_1} &= 0 \end{aligned}\right\} \tag{5}$$

In (3), (4) and (5) the nucleon is assumed to be initially at rest. Each of the equations in sets (3), (4) and (5) is a two-row-two-column matrix equation, arranged according to the two spin directions of the final and initial states of the nucleon. (Terms free from σ contain the unit 2×2 matrix.)

3. Production of Mesons by Light.

According to the theory of radiation damping[3, 4] the probability for the transition from state B to state A is given by

$$\Gamma_{AB} = 2\pi\rho_A\,|\,U_{AB}\,|^2 \tag{6}$$

---

[3] Heitler, Proc. Camb. Phil. Soc., 37, 291 (1941); Wilson, ibid., 301.
[4] Heitler and Peng, ibid., 38, 296 (1942).

where ρ_A is the density of the states A per unit energy interval. U_{AB} is to be determined from the equation

$$U_{AB} = H_{AB} + i\pi \sum_C H_{AC}\, \rho_C\, U_{CB} \tag{7}$$

where C denotes a state of the same energy as A and B. H_{AB} is the compound matrix element of the lowest non-vanishing order for the transition from state B to state A.

For the production of a meson of energy ϵ by the collision of a photon of the same energy with a nucleon (neglecting the recoil of the nucleon) we consider all final states with one photon or one meson of energy ϵ present. We neglect final states with two or more quanta, since the effect of multiple scattering is small.[4]

The compound matrix elements of the lowest order are as follows:—

$$H_{Nk_j \mid Nk'_l} = \frac{H^{\text{ii}}_{Nk_j \mid P}\, H^{\text{ii}}_{P \mid Nk'_l}}{E_{Nk'_l} - E_P}, \qquad H_{P\nu'_1 \mid P\nu_1} = H^{\text{iv}}_{P\nu'_1 \mid P\nu_1}, \tag{8}$$

$$H_{Nk_j \mid P\nu_1} = H^{\text{iii}}_{Nk_j \mid P\nu_1} + \sum_l \frac{H^{\text{ii}}_{N \mid Pk'_l}\, H^{\text{i}}_{k_j k'_l \mid \nu_1}}{E_{\nu_1} - E_{k_j k'_l}} + \sum_l \frac{H^{\text{i}}_{k_j \mid \nu_1 k'_l}\, H^{\text{ii}}_{Nk'_l \mid P}}{E_P - E_{Nk'_l}}$$

For the calculation of $H_{Nk_j \mid Nk'_l}$ we have to sum over the spin of the nucleon in the intermediate state P. This is automatically taken into account by considering $H^{\text{ii}}_{Nk_j \mid P}\, H^{\text{ii}}_{P \mid Nk'_l}$ as the matrix product of its two factors. The summation (l) is to be made over the four polarizations (2 transverse, 1 longitudinal, 1 pseudoscalar) of the meson k'_l in the intermediate state. The propagation vector $\mathbf{k}'$ is uniquely determined by the law of conservation of momentum $(2')$. The denominators are the energy differences between the initial and intermediate states. The set of equations (8) refers to the case where the total charge of the system is always $+\ e$. A similar set of compound matrix elements corresponding to the case where the total charge of the system is 0, are obtained by interchanging N and P throughout the equations (8).

Using (2), (3), (4) and (5), we obtain for the set (8) the following expressions (dropping the N, P and $\mid$ in the suffix):—

$$H_{k_j k'_l} = \frac{2\pi k^2}{V\varepsilon^2} K_{k_j} K_{k'_l} \tag{9}$$

where

$$K_{k_1} = f(\sigma \mathbf{k}_2), \quad K_{k_2} = -f(\sigma \mathbf{k}_1), \quad K_{k_3} = g, \quad K_{k_4} = f'(\sigma \mathbf{k}_3);$$

$$H_{\nu'_1 \nu_1} = \frac{\pi e^2}{VM\varepsilon} (\nu'_1\, \nu_1) \quad \text{or} \quad 0 \tag{10}$$

according as the total charge is $+ e$ or 0;

$$H_{k_3 \nu_1} = \pm \frac{\pi e}{V \epsilon (\nu, \nu \mp \mathbf{k})} \Big\{ g (1 - k^2) (\nu_1 \mathbf{k}_3) - $$
$$- i f \epsilon (\sigma [\nu_1 \mathbf{k}_3]) - i f \epsilon (\sigma, \mathbf{k} \mp \nu) (\mathbf{k} \mp \nu, [\nu_1 \mathbf{k}_3]) \Big\}$$

$$H_{k_1 \nu_1} = \pm \frac{i \pi e}{V \epsilon (\nu, \nu \mp \mathbf{k})} \Big\{ g \epsilon (\nu_1 \mathbf{k}_1) - i f (\sigma [\nu_1 \mathbf{k}_1]) - \tag{11}$$
$$- i f (\sigma, \mathbf{k} \mp \nu) (\mathbf{k} \mp \nu, [\nu_1 \mathbf{k}_1]) - i f k ([\sigma, \mathbf{k} \mp \nu] [\nu_1 \mathbf{k}_2]) \Big\}$$

$$H_{k_4 \nu_1} = \pm \frac{2 \pi e f'}{V \epsilon} \Big\{ (\sigma \nu_1) - \frac{(\sigma, \mathbf{k} \mp \nu)}{(\nu, \nu \mp \mathbf{k})} (\mathbf{k} \nu_1) \Big\}$$

where the upper *or* lower signs are to be taken throughout according as the system has total charge $+ e$ or 0. (9), (10) and (11) are 2×2 matrix equations, like (3), (4) and (5). (9) gives the matrix elements for the scattering of mesons by a nucleon, (10) the Thompson scattering of light by a proton, while (11) gives the matrix elements for the production of mesons by light. According to the ordinary radiation theory which neglects damping the calculation of the cross-section involves the elements (11) alone. This has been done in an earlier paper by Heitler.[5] When radiation damping is taken into account all the final states are coupled together, and therefore (9) and (10) are also required. In particular, for the production of mesons by light following (7), we have to solve the following simultaneous equations for $U_{k\nu}$

$$U_{k\nu} = H_{k\nu} + i \pi H_{kk'} \rho_{k'} U_{k'\nu} + i \pi H_{k\nu''} \rho_{\nu''} U_{\nu''\nu} \tag{12a}$$

$$U_{\nu'\nu} = H_{\nu'\nu} + i \pi H_{\nu'k'} \rho_{k'} U_{k'\nu} + i \pi H_{\nu'\nu''} \rho_{\nu''} U_{\nu''\nu} \,. \tag{12b}$$

Here the polarization indices have been omitted. $\rho_k = V \dfrac{\epsilon k}{8 \pi^3} d \Omega_k$ and $\rho_\nu = V \dfrac{\epsilon^2}{8 \pi^3} d \Omega_\nu$ give the number of k and ν states with propagation vectors lying in the solid angles $d \Omega_k$ and $d \Omega_\nu$ respectively. The product terms containing the density functions imply summation over all the polarizations of k' and ν'', and integration over all directions of these propagation vectors. $U_{k\nu}$ are 2×2 matrices with rows and columns corresponding to the spin direction of the nucleon in its final and initial states, similar to $H_{k\nu}$. Also terms with products of H and U are matrix products on account of the summation over the spin of the nucleon in the states k' and ν''.

[5] Heitler, Proc. Roy. Soc., 166, 529, 1938.

For the reverse process (the production of a photon by a meson) $U_{\nu k}$ has to be solved from the equations obtained from (12) by interchanging k and ν throughout.

For both processes the cross-sections are then obtained from the transition probabilities, given by (6).

4. General Solution of the Integral Equations.

We now explain the general procedure adopted in solving the simultaneous integral equations (12). One of the variables U is eliminated giving an integral equation for the other. This elimination is carried out by making use of the reciprocal kernel method of solving an integral equation.[6]

If $K(x, y)$ is the kernel of the integral equation

$$f(x) = g(x) + \int K(x, y) f(y) \, dy, \tag{13}$$

and if the reciprocal kernel $k(x, y)$ is defined by the integral equation

$$k(x, y) = K(x, y) + \int K(x, z) k(z, y) \, dz, \tag{14}$$

then the solution of equation (13) is

$$f(x) = g(x) + \int k(x, y) g(y) \, dy. \tag{15}$$

From (14) one can easily derive the important relation between K and its reciprocal k

$$\int K(x, z) k(z, y) \, dz = \int k(x, z) K(z, y) \, dz. \tag{14$'$}$$

With the help of this theorem the problem of the solution of (13) is reduced to that of solving (14), which is often much easier. Clearly this theorem also holds when x and y stand for several variables. To apply it to equations (12) we have to understand by the integration over y in (13) integrations over two continuous variables describing the direction of k' (or ν''), as well as a summation over discrete variables describing the polarization of the meson (or the photon) and the spin of the nucleon. The latter summation over the spin variable is taken into account by the matrix multiplication of H and U. Due to this the reciprocal kernel is also a 2×2 matrix, and the order of the factors in the matrix products in the above equations must not be altered.

We solve (12a) for $U_{k\nu}$ in terms of $U_{\nu'\nu}$. The kernel reciprocal to H_{kk} we denote by $V_{kk'}$, defined by the equation

$$V_{kk'} = H_{kk'} + i\pi H_{kk''} \rho_{k''} V_{k''k'} \tag{16}$$

[6] E.g. see Whittaker and Watson, Modern Analysis (4th ed.), p. 218.

Using the abbreviation

$$V_{k\nu} = H_{k\nu} + i\pi V_{kk'} \rho_{k'} H_{k'\nu} \tag{17}$$

then the solution of (12a) is

$$U_{k\nu} = (H_{k\nu} + i\pi H_{k\nu''} \rho_{\nu''} U_{\nu''\nu}) + i\pi V_{kk'} \rho_{k'} (H_{k'\nu} + i\pi H_{k'\nu''} \rho_{\nu''} U_{\nu''\nu})$$
$$= V_{k\nu} + i\pi V_{k\nu''} \rho_{\nu''} U_{\nu''\nu} \tag{18}$$

Now substituting for $U_{k\nu}$ in (12b) we obtain

$$U_{\nu'\nu} = H^{\nu'\nu} + i\pi H_{\nu'\nu''} \rho_{\nu''} U_{\nu''\nu} + i\pi H_{\nu'k'} \rho_{k'} (V_{k'\nu} + i\pi V_{k'\nu''} \rho_{\nu''} U_{\nu''\nu}).$$

Making the abbreviation

$$V_{\nu'\nu} = H_{\nu'\nu} + i\pi H_{\nu'k'} \rho_{k'} V_{k'\nu} \tag{19}$$

we obtain

$$U_{\nu'\nu} = V_{\nu'\nu} + i\pi V_{\nu'\nu''} \rho_{\nu''} U_{\nu''\nu}. \tag{20}$$

Thus we have transformed the simultaneous equations (12) into two independent integral equations (16) and (20). Equation (18) then gives the solution for $U_{k\nu}$.

For the reverse process, i.e. the production of a photon by a meson, the integral equations can be solved in a similar manner. Defining $V_{\nu k}$ by

$$V_{\nu k} = H_{\nu k} + i\pi H_{\nu k'} \rho_{k'} V_{k'k} \tag{21}$$

we get the solution

$$U_{\nu k} = V_{\nu k} + i\pi U_{\nu\nu'} \rho_{\nu'} V_{\nu' k} \tag{22}$$

where $V_{kk'}$ and $U_{\nu\nu'}$ are again the solutions of equations (16) and (20). Here the relation (14') has repeatedly been used.

5. Low Energy Region, Principle of Detailed Balance.

The first integral equation (16) can be solved easily because the kernel $H_{kk'}$ given by (9) is composed of a factor depending on k followed by one depending on k'. The solution is simply

$$V_{kk'} = \frac{1}{1 - i\kappa} H_{kk'} = \frac{1}{1 - i\kappa} \frac{2\pi k^2}{V\varepsilon^2} K_k K_{k'} \tag{23}$$

where

$$\kappa = (g^2 + 2f^2 + f'^2) k^3 / \varepsilon . \tag{24}$$

Therefore using (17) and (21)

$$V_{k\nu} = H_{k\nu} + \frac{i\kappa}{1 - i\kappa} K_k K_\nu \tag{25}$$

$$V_{\nu k} = H_{\nu k} + \frac{i\kappa}{1 - i\kappa} K_\nu^* K_k \tag{26}$$

where

$$K_{\nu_1} = \frac{1}{g^2 + 2f^2 + f'^2} \cdot \frac{1}{4\pi} \int d\Omega_k \sum_l K_{k_l} H_{k_l \nu_1} \,. \tag{27}$$

K_ν^* is the adjoint of K_ν. Using (9) and (11) after some calculation we obtain from (27)

$$K_{\nu_1} = \frac{i\pi e}{V} \eta\,(\sigma \nu_2) \tag{28}$$

where

$$\eta = \frac{1}{g^2 + 2f^2 + f'^2}\left\{\frac{fg\epsilon}{2k} - \frac{2f^2 + f'^2}{k} - \frac{2fg}{\epsilon k} + \left(\frac{3fg}{2k^2} + \frac{2f^2 + f'^2}{\epsilon k^2}\right)\log(\epsilon + k)\right\}. \tag{29}$$

Using this result it is seen from (19) that $V_{\nu\nu''}\rho_{\nu''}$ is of the order $e^2 f^2 \epsilon^4$. Using (20) it is clear that if

$$e^2 f^2 \epsilon^4 << 1 \tag{30}$$

then $V_{\nu\nu'}$ is a good approximation to $U_{\nu\nu'}$, and hence by (18), under the same condition (30), $V_{k\nu}$ is a good approximation to $U_{k\nu}$. Similarly $V_{\nu k}$ is, in this energy region, a good approximation to $U_{\nu k}$. Using (6) the transition probabilities for the production of mesons by photons and for the reverse process can be calculated from (25) and (26). As (26) is *not* the adjoint of (25) it remains to be seen whether or not these two probabilities are equal (apart from the density functions).

For statistical considerations it is usually postulated or proved that the "principle of detailed balance" holds, more precisely that the following theorem is true: If A and B are two states completely specified, i.e. with given directions, polarizations, spin directions, etc. of all particles involved, the transition probabilities $A \rightarrow B$ and $B \rightarrow A$ are equal if referred to a unit volume of phase space. (I.e. the transition probabilities are equal apart from the density functions).

To discuss this in more detail we restore the polarization indices, and add two indices, α, β describing the initial and final spin directions of the nucleon. Then the principle of detailed balance would require that

$$\left| V^{\beta\alpha}_{k_j \nu_1} \right|^2 = \left| V^{\alpha\beta}_{\nu_1 k_j} \right|^2. \tag{31}$$

Using (25) and (26), however, we find that

$$\left| V^{\beta\alpha}_{k_j \nu_1} \right|^2 - \left| V^{\alpha\beta}_{\nu_1 k_i} \right|^2 =$$

$$= \left(\frac{i\kappa}{1 - i\kappa} + \frac{i\kappa}{1 + i\kappa} \right) \left\{ H^{\alpha\beta}_{\nu_1 k_j} \left(K_{k_j} K_{\nu_1} \right)^{\beta\alpha} - H^{\beta\alpha}_{k_j \nu_1} \left(K^*_{\nu_1} K_{k_j} \right)^{\alpha\beta} \right\} \tag{32}$$

On evaluation it is found that, in general (32) is not zero. However, if we average over the initial and final spin directions of the nucleon then the right-hand side of (32) vanishes. This is essentially due to the fact that

$$\text{spur } \left(H_{\nu_1 k_j} \; K_{k_j} \; K_{\nu_1} \right)$$

is real (because the above matrix is a real function of $i\,\sigma$). Hence the principle of detailed balance is not satisfied if the spin direction of the nucleon is included in the description of a state. It *is* satisfied if an average over the spin directions is taken.

Principle of Detailed Balance.—The above failure of the principle of detailed balance is a feature of the new radiation theory. In the ordinary quantum theory of radiation where damping is neglected, this principle follows from the fact that the perturbation Hamiltonian is Hermitian. The difference is that, whereas in the ordinary theory the probabilities are given by $| H_{AB} |^2$, when damping is taken account of $| U_{AB} |^2$ has to be used.

From equation (7) there is no indication that, in general

$$| U_{AB} |^2 = | U_{BA} |^2 .$$

However, in all the processes so far considered[7], the case just discussed is the only example for which

$$| U_{AB} |^2 \neq | U_{BA} |^2 .$$

Although, due to the fact that on averaging over the spins of the nucleon, this inequality disappears and therefore the failure of the principle is of no practical importance; yet it is of considerable theoretical interest. It may be connected with the fact that in the classical treatment of radiation damping a term $\dddot{x}$ arises, which upsets the time reversibility of the equation of motion.

Cross-sections for the Production of Mesons by Photons.—These cross-sections are actually of the same order of magnitude as those given by

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[7] Cf. especially the paper by Heitler and Peng in these Proceedings.

the ordinary theory neglecting damping in this energy region. We give the values for $\epsilon >> 1$, $e^2 f^2 \epsilon^4 << 1$ of the average total cross-sections for the productions of mesons.

$$\Phi_{\text{long}} = \pi e^2 f^2 \varepsilon^2 + \pi e^2 \varepsilon^2 \frac{\kappa^2}{1 + \kappa^2} \eta g (\eta g - f)$$

$$\Phi_{\text{trans}} = \pi e^2 (2 f^2 \log 2 \varepsilon + g^2 - f^2) + 2 \pi e^2 \varepsilon^2 \frac{\kappa^2}{1 + \kappa^2} \eta^2 f^2 \tag{33}$$

$$\Phi_{\text{ps}} = 4 \pi e^2 f'^2 + \pi e^2 \varepsilon^2 \frac{\kappa^2}{1 + \kappa^2} \eta^2 f'^2 .$$

6. HIGH ENERGY REGION.

We now discuss the energy region where $e^2 f^2 \varepsilon^4 > 1$. $U_{k\nu}$ is no longer closely approximated by $V_{k\nu}$. Thus equation (20) must be solved for $U_{\nu\nu'}$. As now $\varepsilon >> 1$ we only retain the highest powers of ε (or k) in all expressions. From (11) it is seen that $H_{k_3\nu}$ is of the order ef, while $H_{k_1\nu}$ and $H_{k_4\nu}$ are of the order ef/ε. (25) gives

$$V_{k\nu} = H_{k\nu} - K_k K_\nu \tag{34}$$

where we use the approximations, e.g.

$$H_{k_3\nu_1} = \frac{i e f \pi}{V} \frac{(\sigma, \mathbf{k}_3 \mp \nu_3)}{(\nu_3, \nu_3 \mp \mathbf{k}_3)} (\nu_2 \mathbf{k}_3) \tag{35}$$

$$K_{\nu_1} = \frac{i e \pi}{V} \frac{f g}{2 (g^2 + 2 f^2 + f'^2)} (\sigma \nu_2) . \tag{36}$$

Similarly from (26) it follows that

$$V_{\nu k} = H_{\nu k} - K_\nu^* K_k = V_{k\nu}^* . \tag{37}$$

To the same order of approximation the kernel of equation (20) is (by (19))

$$\begin{aligned} V_{\nu'\nu} &= i \pi H_{\nu' k_3} \rho_{k_3} V_{k_3 \nu} = \\ &= i \pi H_{\nu' k_3} \rho_{k_3} H_{k_3 \nu} - i \pi H_{\nu' k_3} \rho_{k_3} K_{k_3} K_\nu . \end{aligned} \tag{38}$$

The integral

$$\begin{aligned} & H_{\nu'_1 k_3} \rho_{k_3} H_{k_3 \nu_1} = \\ & = \frac{e^2 f^2 \varepsilon^2}{V\, 8 \pi} \int d\,\Omega_k \frac{(\nu_2' \mathbf{k}_3) (\sigma, \mathbf{k}_3 \mp \nu_3') (\sigma, \mathbf{k}_3 \mp \nu_3) (\mathbf{k}_3 \nu_2)}{(\nu_3', \nu'_3 \mp \mathbf{k}_3) (\nu_3, \nu_3 \mp \mathbf{k}_3)} \end{aligned} \tag{39}$$

is evaluated in Appendix I. For the second term in (38) we have

$$H_{\nu_1' k_3}\, \rho_{k_3}\, K_{k_3}\, K_{\nu_1} = \frac{V\,\varepsilon^2}{8\,\pi^3}\int d\,\Omega_{k}\, H_{\nu_1' k_3}\, K_{k_3}\, K_{\nu_1} =$$

$$= \frac{V\,\varepsilon^2}{2\,\pi^2}\,(g^2 + 2f^2 + f'^2)\, K^*_{\nu_1'}\, K_{\nu_1} = \frac{c^2 f^2 g^2 \varepsilon^2}{8\,V(g^2 + 2f^2 + f'^2)}\,(\sigma\,\nu_2')(\sigma\,\nu_2) \tag{40}$$

using (27) and (36).

From this and (I. 6) of Appendix I we obtain

$$\begin{aligned} V_{\nu_1'\nu_1} = i\,\pi\,\frac{c^2 f^2 \varepsilon^2}{8\,V}\,[\{1 + (\sigma\,\nu_3')\}\,(\nu_1' - i\,\nu_2', \nu_1 + i\,\nu_2)\,\{1 + (\sigma\,\nu_3)\}\,A((\nu_3'\,\nu_3)) + \\ + \{1 - (\sigma\,\nu_3')\}\,(\nu_1' + i\,\nu_2', \nu_1 - i\,\nu_2)\,\{1 - (\sigma\,\nu_3)\}\,A((\nu_3'\,\nu_3)) + \\ + B_1\,(\sigma\,\nu_1')\,(\sigma\,\nu_1) + B_2\,(\sigma\,\nu_2')\,(\sigma\,\nu_2)] \end{aligned} \tag{41}$$

where

$$\left.\begin{aligned} A((\nu_3'\,\nu_3)) &= \frac{2}{\{1 + (\nu_3'\,\nu_3)\}^2}\left[\frac{3 - (\nu_3'\,\nu_3)}{4} + \frac{1 - (\nu_3'\,\nu_3)}{1 + (\nu_3'\,\nu_3)}\log\frac{1 - (\nu_3'\,\nu_3)}{2}\right] \\ B_1 &= 1 \\ B_2 &= 1 - \frac{g^2}{g^2 + 2f^2 + f'^2}\,. \end{aligned}\right\} \tag{42}$$

(41) shows directly that $V_{\nu'\nu}$ is skew-Hermetian,

$$V_{\nu'\nu} = -\,V^*_{\nu\nu'}\,. \tag{43}$$

Hence from (20) it follows that $U_{\nu'\nu}$ is skew-Hermetian. Comparing (22) and (18) and using the fact that $V_{k\nu}$ is Hermetian (37) we see that $U_{k\nu}$ is Hermetian, i.e.

$$U_{\nu k} = U^*_{k\nu}\,. \tag{44}$$

Hence the cross-section for the production of mesons by light is the same as that for the reverse process in this region of high energies, apart from such differences that arise from the density functions.

Solution of the Second Integral Equation.—The kernel $V_{\nu'\nu}$ of the equation (20) has, in the form (41), been split up into four terms of different types which have the orthogonal property appropriate to this type of integral equation. This has the important consequence that the solution $U_{\nu'\nu}$ has the same dependence on spin and polarization as $V_{\nu'\nu}$. Therefore $U_{\nu_1'\nu_1}$ is given by an expression obtained from (41) by replacing

$A\,((\nu_3'\,\nu_3))$ by a function $X\,((\nu_3'\,\nu_3))$ and replacing B_1 and B_2 by constants Y_1 and Y_2 respectively. On substituting these expressions for $V_{\nu'\nu}$ and $U_{\nu'\nu}$ in (20) it is found that no cross products arising from terms of different types occur in the integral $V_{\nu'\nu''}\,\rho_{\nu''}\,U_{\nu''\nu}$. This is easily seen from the following orthogonality relations

$$\{1 + (\sigma\,\nu_3'')\}\,\{1 - (\sigma\,\nu_3'')\} = 0 \tag{45}$$

$$\int d\,\Omega_{\nu''}\,(\sigma\,\nu_1'')(\sigma\,\nu_2'') = i \int d\,\Omega_{\nu''}\,(\sigma\,\nu_3'') = 0 \tag{46}$$

$$\begin{aligned} &\Sigma\,\{\nu_1'' + i\,\nu_2''\}\,\{1 + (\sigma\,\nu_3'')\}\,(\sigma\,\nu_1'') = \\ &= \Sigma\,\{\nu_1'' + i\,\nu_2''\}\,(\nu_1'' + i\,\nu_2'',\ \sigma) \\ &= \{\nu_1'' + i\,\nu_2''\}\,(\nu_1'' + i\,\nu_2'',\ \sigma) + \{\nu_2'' - i\,\nu_1''\}\,(\nu_2'' - i\,\nu_1'',\ \sigma) \\ &= 0 \end{aligned} \tag{47}$$

as the summation Σ is to extend over the two polarizations of ν''. (The second term is obtained from the first by replacing ν_1'' by ν_2'' and ν_2'' by $-\nu_1''$). Further two terms of the same type yield, in this integral, a product which is again of the same type. Thus (20) breaks up into four equations (arising from the four types of terms) which determine the X, Y_1 and Y_2 separately. (The two equations containing X are identical). These equations are (putting $\lambda = e^2\,f^2\,\epsilon^4\,/\,2$) where

$$\left.\begin{aligned} Y_1 &= B_1 - \frac{\lambda}{4}\,B_1\,Y_1 \\ Y_2 &= B_2 - \frac{\lambda}{4}\,B_2\,Y_2 \end{aligned}\right\} \tag{48}$$

$$X\,((\nu_3'\,\nu_3)) = A\,((\nu_3'\,\nu_3)) - \frac{\lambda}{4\pi}\int d\,\Omega_{\nu''}\,A\,((\nu_3'\,\nu_3''))\,[\ldots]\,X\,((\nu_3''\,\nu_3)) \tag{49}$$

where

$$[\ldots] = \left\{\frac{1 + (\nu_3'\,\nu_3) + (\nu_3'\,\nu_3'') + (\nu_3''\,\nu_3)}{1 + (\nu_3'\,\nu_3)}\right\}^3 - $$

$$- \frac{3}{2}\,\frac{\{1 + (\nu_3'\,\nu_3) + (\nu_3'\,\nu_3'') + (\nu_3''\,\nu_3)\}\,\{1 + (\nu_3'\,\nu_3'')\}\,\{1 + (\nu_3''\,\nu_3)\}}{\{1 + (\nu_3'\,\nu_3)\}^2} \tag{50}$$

arising from the summation of the spins and the photon polarizations ν_1'', ν_2''; as shown in Appendix II.

The solutions of (48) are

$$Y_1 = \frac{B_1}{1 + \frac{\lambda}{4}\,B_1}\,, \qquad Y_2 = \frac{B_2}{1 + \frac{\lambda}{4}\,B_2} \tag{51}$$

To solve (49) we expand A and X in terms of the Legendre functions, thus

$$A((\nu_3'\nu_3)) = \sum_n a_n (2n+1) P_n((\nu_3'\nu_3))$$
$$X((\nu_3'\nu_3)) = \sum_n x_n (2n+1) P_n((\nu_3'\nu_3)) \qquad (52)$$

The integral equation (49) then on substitution becomes equivalent to an infinite set of algebraic equations determining x_n. As shown in Appendix III this infinite set of equations can be solved exactly, the result being given in the form of a recurrence formula for the x_n's. Defining $x_{n+\frac{3}{2}}$, by

$$x_{n+\frac{3}{2}} = \left(1 - \frac{3}{2}\cdot\frac{n+1}{2n+3}\right)x_n + \frac{3}{2}\cdot\frac{n+3}{2n+5}x_{n+1} + \frac{3}{2}\cdot\frac{n+1}{2n+3}x_{n+2} + \left(1 - \frac{3}{2}\cdot\frac{n+3}{2n+5}\right)x_{n+3} \qquad (n = 0, 1, 2, ..) \qquad (53)$$

and $a_{n+\frac{3}{2}}$ similarly in terms of $a_n, a_{n+1}, .. a_{n+3}$ we obtain for the solution of (19) the independent equations

$$x_{n+\frac{3}{2}} = a_{n+\frac{3}{2}} - \lambda a_{n+\frac{3}{2}} x_{n+\frac{3}{2}} \qquad (54)$$

or

$$x_{n+\frac{3}{2}} = \frac{a_{n+\frac{3}{2}}}{1+\lambda a_{n+\frac{3}{2}}} \qquad (n = 0, 1, 2, ...) \qquad (55)$$

Here the $a_{n+\frac{3}{2}}$ are to be evaluated from (42), using the definitions (52) and (53), giving

$$a_{n+\frac{3}{2}} = \frac{1}{(n+1)(n+2)^2(n+3)}. \qquad (56)$$

However, (56) is obtained more easily by expanding

$$\{1 + (\nu_3'\nu_3)\}^3 A((\nu_3'\nu_3))$$

and using the methods of Appendix III.

The above gives the solution of the integral equation (20) for $U_{\nu'\nu}$. $U_{k\nu}$ has then to be calculated using (18). For this purpose we use relation (34) giving

$$U_{k\nu} = H_{k\nu} + i\pi H_{k\nu'}\rho_{\nu'} U_{\nu'\nu} - K_k\{K_\nu + i\pi K_{\nu'}\rho_{\nu'} U_{\nu'\nu}\}. \qquad (57)$$

Since, from (36), $K_{\nu_1'}$ contains only a term of the type $(\sigma\nu_2')$ it only combines with the same type of term in $U_{\nu'\nu}$. Thus we have for the combination occurring in (57)

$$K_\nu + i\pi K_{\nu'}\rho_{\nu'} U_{\nu'\nu} = \frac{1}{1+\frac{\lambda}{4}B_2} K_\nu. \qquad (58)$$

Total Cross-section for the Production of Longitudinal Mesons.

This is to be calculated from the formula

$$\Phi_{\text{long}} = 2\pi V\, U^{*}_{k_3 \nu}\, \rho_{k_3}\, U_{k_3 \nu}\,. \tag{59}$$

Using the fact that $U_{k\nu}$ is Hermitian we have with (22)

$$U^{*}_{k\nu} = U_{\nu k} = V_{\nu k} + i\pi U_{\nu\nu'}\, \rho_{\nu'}\, V_{\nu' k} \tag{60}$$

We substitute the expressions (57) and (60) in (59) and obtain

$$\begin{aligned}\Phi_{\text{long}} = 2\pi V \Big\{ V_{\nu k_3} + i\pi U_{\nu\nu'}\, \rho_{\nu'}\, V_{\nu' k_3} \Big\} \rho_{k_3} \Big\{ H_{k_3\nu} + \\ + i\pi H_{k_3 \nu''}\, \rho_{\nu''}\, U_{\nu''\nu} - \frac{1}{1+\frac{\lambda}{4}B_2} K_{k_3} K_\nu \Big\}\end{aligned} \tag{61}$$

We first evaluate the integrals over k_3. There are just two types of these integrals. The first is given by the adjoint of (38) namely

$$i\pi V_{\nu' k_3}\, \rho_{k_3}\, H_{k_3 \nu} = V_{\nu'\nu}\,. \tag{62}$$

To obtain this we note that $V_{\nu k}$ is Hermitian while $V_{\nu'\nu}$ is skew-Hermitian. ((37) and (43)). The second integral is

$$\begin{aligned} V_{\nu' k_3}\, \rho_{k_3}\, K_{k_3} &= H_{\nu' k_3}\, \rho_{k_3}\, K_{k_3} - K^{*}_{\nu'}\, K_{k_3}\, \rho_{k_3}\, K_{k_3} \\ &= \frac{V\varepsilon^2}{2\pi^2}(2f^2 + f'^2)\, K^{*}_{\nu'} \end{aligned} \tag{63}$$

using the adjoint of (34) and the definition K^{*}_{ν}, the adjoint of (27).

Making use of (62) and (63), and manipulating the resultant expression by means of the integral equation (20) and the formula (58) we obtain

$$\begin{aligned}\Phi_{\text{long}} = 2\pi V \Big\{ \frac{U_{\nu\nu}}{i\pi} + U_{\nu\nu''}\, \rho_{\nu''}\, U_{\nu''\nu} \Big\} - \\ - \frac{V^2 \varepsilon^2}{\pi}(2f^2 + f'^2) \frac{K^{*}_{\nu} K_{\nu}}{\left(1+\frac{\lambda}{4}B_2\right)^2}\,.\end{aligned} \tag{64}$$

Here ν represents the incident photon, and the only summation is that denoted by $\rho_{\nu''}$. As is shown in Appendix IV this leads to the result that the total cross-section for the production of a longitudinal meson by a photon is

$$\Phi_{\text{long}} = \frac{\pi e^2 f^2 \varepsilon^2}{4}\left[8 \sum_n (n+2) \frac{a_{n+\frac{3}{2}}}{\left(1 + \lambda a_{n+\frac{3}{2}}\right)^2} + \frac{1}{\left(1 + \frac{\lambda}{4}\right)^2} + \frac{B_2^2}{\left(1 + \frac{\lambda}{4} B_2\right)^2}\right] \tag{65}$$

where

$$\lambda = \frac{e^2 f^2 \varepsilon^4}{2} \qquad B_2 = \frac{2f^2 + f'^2}{g^2 + 2f^2 + f'^2}$$

and

$$a_{n+\frac{3}{2}} = \frac{1}{(n+1)(n+2)^2(n+3)} \tag{66}$$

The Asymptotic Form of the Cross-section for large $\lambda = e^2 f^2 \epsilon^4 / 2$.

The series $\sum_n$ in (65) contributes the term of highest order. Replacing the sum by an integral we obtain

$$\begin{aligned}\Phi_{\text{long}} &= \frac{\pi e^2 f^2 \varepsilon^2}{\sqrt{\lambda}} \int_0^\infty \frac{y^2}{(1+y^2)^2}\, dy \\ &= \frac{\pi^2 e f}{2\sqrt{2}} \qquad \text{for large } \lambda .\end{aligned} \tag{67}$$

This cross-section does not increase with ϵ as the latter becomes large. Its numerical value is roughly $5 \cdot 10^{-27}$ cm^2.

Total Cross-section for the Production of Pseudoscalar Mesons.

As before we have

$$\Phi_{\text{ps}} = 2\pi V U^*_{k_4 \nu}\, \rho_{k_4}\, U_{k_4 \nu}$$

To find the asymptotic behaviour of the total cross-section, as before, we need only consider the series term $\sum_n$. Thus in (57) we only use the terms

$$U_{k\nu} = H_{k\nu} + i\pi H_{k\nu'}\, \rho_{\nu'}\, U_{\nu'\nu} .$$

Further we use the relation

$$H_{\nu' k_4}\, \rho_{k_4}\, H_{k_4 \nu} = \frac{4 f'^2}{f^2 \varepsilon^2} H_{\nu' k_3}\, \rho_{k_3}\, H_{k_3 \nu}$$

which follows directly from (11) and (35). Thus we obtain the asymptotic formula

$$\Phi_{ps} = \frac{4f'^2}{f^2\,\varepsilon^2}\,.\,\Phi_{long} = \frac{\sqrt{2}\,\pi^2\,e\,f'^2}{f\,\varepsilon^2} \tag{68}$$

which decreases with increasing ϵ.

The cross-section for the production of transverse mesons has not been calculated in detail, but it can be seen that it is probably of the order

$$\Phi_{long}\,/\,\varepsilon^2\,.$$

As shown above the evaluation of the total cross-sections e.g. (61) is rendered simple by performing the integration over $\mathbf{k}$ before that over ν' or ν''. Since this cannot be done for the calculation of the differential cross-section, the calculation is then much more complicated.

Scattering of Light by the Nucleon.

The theory leads also to a scattering of light by a nucleon due to the interaction of light with the meson field of the nucleon. The total cross-section for this process is given by the formula

$$\Phi_{scattering} = 2\pi V\,U^*_{\nu'\nu}\,\rho_{\nu'}\,U_{\nu'\nu}$$

$$= e^4 f^4 \varepsilon^6 \left\{\pi\sum_n (n+2)\left(\frac{a_{n+\frac{3}{2}}}{1+\lambda a_{n+\frac{3}{2}}}\right)^2 + \frac{\pi}{32}(Y_1^2+Y_2^2)\right\}.$$

This relation holds exactly if $\epsilon \gg 1$. Using the same method as above we find the asymptotic formula

$$\Phi_{scattering} = \frac{\pi^3\,e\,f}{2\sqrt{2}} \quad \text{for large } \lambda \tag{69}$$

(which equals Φ_{long}).

7. PHYSICAL DISCUSSION.

The results to be discussed are summarized in the following formulae (putting now $f' = f$):

$$\text{cross-section for } \hbar\nu + P \rightarrow N + Y_{long} : \frac{\pi^2 e f}{2\sqrt{2}}$$

$$\text{,, \quad ,, \quad } \hbar\nu + P \rightarrow N + Y_{ps} : \frac{\sqrt{2}\,\pi^2\,e\,f}{\varepsilon^2}$$

(For the reverse processes a factor 2 has to be added because of the 2 polarizations of the light quantum.)

cross-section for the scattering of a light quantum by a nucleon:

$$\frac{\pi^2 c f}{2\sqrt{2}} .$$

We first discuss the transformation of a meson into a light quantum. The process is observable since the light quantum will at once give rise to a cascade shower. The production of showers by mesons is known to be quite small, and considerable difficulties would arise if we would neglect the damping. The cross-section would then be far too large to be compatible with the experiments. The production of showers by mesons has been studied by Lovell.[8] He found that the average cross-section for a meson at sea level to produce a cascade shower in Pb is, per one nucleon, $0 \cdot 7 \times 10^{-29}$ cm². (For this figure one must take into account that only half the number of nucleons is effective because of the conservation of charge.) The mesons at sea level are all pseudoscalar mesons. Their average energy may be taken to be about 25 in our units. Using this value for ϵ the above formula would give a cross-section of $5 \cdot 10^{-29}$ cm². This is still far larger than the experiments permit, especially if we take into account that the experimental showers are certainly mostly knock-on showers. We must remember, however, that our formulae are only valid (i) for $\epsilon < 10$, and (ii) if no neutral mesons exist. Both facts may change the numerical factors in our formulae quite considerably. In addition it may also well be that the binding of the nucleons inside the nucleus or the overlapping of the meson fields in the nucleus diminishes the cross-section. However this may be, the chief difficulty, namely, the increase of the cross-section with energy, is removed in our theory and the results are not in disastrous disagreement with the facts.

The creation of mesons by light quanta leads to the occurrence of mesons in big cascade showers. Those mesons have actually been observed. The effect is discussed more in detail in a paper by Hamilton, Heitler and Peng[9] and is found to be in reasonable agreement with the experiments.

The meson field surrounding a nucleon gives also rise to a scattering of light by a nucleon. The cross-section given above is much larger than the corresponding Thomson cross-section which is

$$\frac{8\pi}{3}\left(\frac{e^2}{Mc^2}\right)^2 .$$

It is even, for the energies in question ($\epsilon \sim 10$ say), larger than the cross-section for the scattering of light by an electron which is of the

[8] Lovell, Proc. Roy. Soc., 172, 568 (1939).
[9] Hamilton, Heitler and Peng, Phys. Rev. 64, 78, 1943.

order of the magnitude $\frac{e^4}{mc^3\,\varepsilon}$ (Klein-Nishina formula). Nevertheless the effect is difficult to observe since light quanta of this energy produce pairs with a cross-section varying with Z^2, whereas our effect varies with Z. Except for hydrogen the cross-section for pair production is very much bigger than that for scattering.

APPENDIX I.

Spin and Polarization Dependence of the Kernel $V_{\nu'\nu}$.

We evaluate the integral (39)

$$H_{\nu_1' k_3}\ \rho_{k_3}\ H_{k_3 \nu_1} = \frac{e^2 f^2 \varepsilon^2}{V\,8\pi} \int d\,\Omega_k \frac{(\nu_2'\,\mathbf{k}_3)\,(\sigma, \mathbf{k}_3 \mp \nu_3')\,(\sigma, \mathbf{k}_3 \mp \nu_3)\,(\mathbf{k}_3\,\nu_2)}{(\nu_3', \nu_3' \mp \mathbf{k}_3)\,(\nu_3, \nu_3 \mp \mathbf{k}_3)}$$

by choosing the following co-ordinate axes. The z-axis lies along the bisector of ν_3 and ν_3' and the y-axis is perpendicular to the plane of ν_3 and ν_3'. This gives the unit co-ordinate vectors

$$\mathbf{i}_z = \frac{\nu_3 + \nu_3'}{2\,C}, \qquad \mathbf{i}_x = \frac{\nu_3' - \nu_3}{2\,S}, \qquad \mathbf{i}_y = \frac{[\nu_3\,\nu_3']}{2\,S\,C} \qquad \text{(I. 1)}$$

where C and S denote the cosine and sine of half the angle between ν_3 and ν_3' respectively. Further we use polar-co-ordinates with $\mathbf{i}_z$ as polar axis. Hence we put

$$\pm\ \mathbf{k}_3 = \mathbf{i}_z \cos\theta + \mathbf{i}_x \sin\theta\cos\phi + \mathbf{i}_y \sin\theta \sin\phi\ .$$

Here we choose the $\pm$ signs so as to give the integral the same form in both cases. It follows that

$$(\sigma, \mathbf{k}_3 \mp \nu_3')\,(\sigma, \mathbf{k}_3 \mp \nu_3) = \left(1 - \frac{\cos\theta}{C}\right)\left\{1 + (\sigma\,\nu_3')\,(\sigma\,\nu_3)\right\} - \\ - i\,\frac{S}{C}\sin\theta\sin\phi\ (\sigma, \nu_3 + \nu_3')\ .$$

The other terms in the numerator can be written (using matrix notation)

$$(\nu_2'\,\mathbf{k}_3)\,(\mathbf{k}_3\,\nu_2) = \left[\frac{(\nu_2'\,\nu_3)}{2\,C}, -\frac{(\nu_2'\,\nu_3)}{2\,S}, -\frac{(\nu_1'\,\nu_3)}{2\,S\,C}\right]$$

$$\begin{bmatrix} \cos^2\theta, & \cos\theta\sin\theta\cos\phi, & \cos\theta\sin\theta\sin\phi \\ \cos\theta\sin\theta\cos\phi, & \sin^2\theta\cos^2\phi, & \sin^2\theta\cos\phi\sin\phi \\ \cos\theta\sin\theta\sin\phi, & \sin^2\theta\cos\phi\sin\phi, & \sin^2\theta\sin^2\phi \end{bmatrix} \begin{bmatrix} \frac{(\nu_3'\,\nu_2)}{2\,C} \\ \frac{(\nu_3'\,\nu_2)}{2\,S} \\ \frac{(\nu_3'\,\nu_1)}{2\,S\,C} \end{bmatrix}.$$

The denominator of the integrand is an even function of ϕ

$$(\nu_3', \nu_3' \mp \mathbf{k}_3)(\nu_3', \nu_3 \mp \mathbf{k}_3) = (1 - C\cos\theta)^2 - S^2\sin^2\theta\sin^2\phi .$$

Thus it is only necessary to pick out the terms in the numerator which are even in ϕ. After elementary integrations, we obtain

$$H_{\nu_1' k_3}\,\rho_{k_3}\,H_{k_3\nu_1} = -\frac{e^2 f^2 \varepsilon^2}{2V} \times$$

$$\left[\frac{(\nu_1'\nu_3)(\nu_3'\nu_1) + (\nu_2'\nu_3)(\nu_3'\nu_2)}{4\,S^2 C^2} \cdot \frac{1 + (\sigma\nu_3')(\sigma\nu_3)}{2C^2}\left\{1 + \frac{S^2}{C^2}\log S^2\right\} + \right.$$

$$\left. + i\,\frac{(\nu_1'\nu_3)(\nu_3'\nu_1) - (\nu_2'\nu_3)(\nu_3'\nu_2)}{4\,S^2 C^2} \cdot \frac{(\sigma,\ \nu_3' + \nu_3)}{2C^2}\left\{S^2 + \frac{S^2}{C^2}\log S^2\right\}\right]. \tag{I. 2}$$

This can be further simplified, as follows. Expressing the scalar products $(\nu_i'\nu_j)$ $(i, j = 1 \text{ and } 2)$ in terms of their components in the system (I. 1) we obtain the relations

$$\left.\begin{aligned}(\nu_1'\nu_1) + (\nu_2'\nu_2) &= -\frac{(\nu_1'\nu_3)(\nu_3'\nu_1) + (\nu_2'\nu_3)(\nu_3'\nu_2)}{2\,S^2}\\ (\nu_1'\nu_2) - (\nu_2'\nu_1) &= -\frac{(\nu_1'\nu_3)(\nu_3'\nu_2) - (\nu_2'\nu_3)(\nu_3'\nu_1)}{2\,S^2}\end{aligned}\right\} \tag{I. 3}$$

Similarly we obtain

$$(\sigma\nu_1')(\sigma\nu_1) + (\sigma\nu_2')(\sigma\nu_2) = \{1 + (\sigma\nu_3')(\sigma\nu_3)\}\,\frac{(\nu_1'\nu_1) + (\nu_2'\nu_2)}{2\,C^2} +$$

$$+ i\,(\sigma,\ \nu_3' + \nu_3)\,\frac{(\nu_2'\nu_1) - (\nu_1'\nu_2)}{2\,C^2}. \tag{I. 4}$$

Further by direct manipulation

$$\begin{aligned}&\tfrac{1}{2}\{1 + (\sigma\nu_3')\}(\nu_1' - i\nu_2',\ \ \nu_1 + i\nu_2)\{1 + (\sigma\nu_3)\} +\\ &+ \tfrac{1}{2}\{1 - (\sigma\nu_3')\}(\nu_1' + i\nu_2',\ \ \nu_1 - i\nu_2)\{1 - (\sigma\nu_3)\} =\\ &= \{(\nu_1'\nu_1) + (\nu_2'\nu_2)\}\{1 + (\sigma\nu_3')(\sigma\nu_3)\} +\\ &+ i\{(\nu_1'\nu_2) - (\nu_2'\nu_1)\}(\sigma,\ \nu_3' + \nu_3).\end{aligned} \tag{I. 5}$$

Introducing (I. 3), (I. 4) and (I. 5) into (I. 2) we can now write (39) in the orthogonal form

$$\begin{aligned}H_{\nu_1' k_3}\,\rho_{k_3}\,H_{k_3\nu_1} = \frac{e^2 f^2 \varepsilon^2}{8V}\,&[\{1 + (\sigma\nu_3')\}(\nu_1' - i\nu_2',\ \nu_1 + i\nu_2)\{1 + (\sigma\nu_3)\}A((\nu_3'\nu_3)) +\\ &+ \{1 - (\sigma\nu_3')\}(\nu_1' + i\nu_2',\ \nu_1 - i\nu_2)\{1 - (\sigma\nu_3)\}A((\nu_3'\nu_3)) +\\ &+ (\sigma\nu_1')(\sigma\nu_1) + (\sigma\nu_2')(\sigma\nu_2)]\end{aligned} \tag{I. 6}$$

where

$$A((\nu_3'\nu_3)) = \frac{1}{2C^4}\left\{\frac{1 + S^2}{2} + \frac{S}{C^2}\log S^2\right\}$$

giving the form (42).

APPENDIX II.

The Kernel of the Integral Equation for $X((\nu_3'\nu_3))$.

We have to show, how using terms of the first type for $V_{\nu'\nu''}$ and $U_{\nu''\nu}$ as given in (41) we arrive at equation (49).

For this purpose we show that

$$\int d\Omega_{\nu''}\sum\{1+(\sigma\nu_3')\}(\nu_1'-i\nu_2',\ \nu_1''+i\nu_2'')\{1+(\sigma\nu_3'')\}A((\nu_3'\nu_3''))\{1+(\sigma\nu_3'')\}$$
$$(\nu_1''-i\nu_2'',\ \nu_1+i\nu_2)\{1+(\sigma\nu_3)\}X((\nu_3''\nu_3)) =$$
$$= 8\{1+(\sigma\nu_3')\}(\nu_1'-i\nu_2',\ \nu_1+i\nu_2)\{1+(\sigma\nu_3)\}\ \times$$
$$\int d\Omega_{\nu''}A((\nu_3'\nu_3''))\left[\left\{\frac{1+(\nu_3'\nu_3)+(\nu_3'\nu_3'')+(\nu_3''\nu_3)}{1+(\nu_3'\nu_3)}\right\}^3-\right.$$
$$\left.-\frac{3}{2}\cdot\frac{1+(\nu_3'\nu_3)+(\nu_3'\nu_3'')+(\nu_3''\nu_3)}{1+(\nu'_3\nu_3)}\cdot\frac{\{1+(\nu_3'\nu_3'')\}\{1+(\nu_3'\nu_3)\}}{1+(\nu_3'\nu_3)}\right]X((\nu_3''\nu_3)). \tag{II. 1}$$

Σ means summation over the polarizations of ν''. We use the same axes and co-ordinates as in Appendix I, where θ and ϕ now refer to ν_3''.

First we deal with the spin terms:

$$\{1+(\sigma\nu_3')\}\{1+(\sigma\nu_3'')\}^2\{1+(\sigma\nu_3)\} = 2\{1+(\sigma\nu_3')\}\{1+(\sigma\nu_3'')\}\{1+(\sigma\nu_3)\}.$$

Using the co-ordinate system we can write

$$\{1+(\sigma\nu_3'')\} = 1+\frac{\cos\theta}{2C}(\sigma,\ \nu_3'+\nu_3)+\frac{\sin\theta\cos\phi}{2S}(\sigma,\ \nu_3'-\nu_3)+$$
$$+\frac{\sin\theta\sin\phi}{2SC}(\sigma[\nu_3\nu_3']).$$

On substitution we obtain

$$\{1+(\sigma\nu_3')\}\{1+(\sigma\nu_3'')\}^2\{1+(\sigma\nu_3)\} =$$
$$= 2\{1+(\sigma\nu_3')\}\{1+(\sigma\nu_3)\}\left\{1+\frac{\cos\theta}{C}+i\frac{S}{C}\sin\theta\sin\phi\right\} \tag{II. 2}$$

giving again the correct spin dependence for terms of the first type.

Next we consider the polarization terms

$$\Sigma(\nu_1'-i\nu_2',\ \nu_1''+i\nu_2'')(\nu_1''-i\nu_2'',\ \nu_1+i\nu_2) =$$
$$= 2\{(\nu_1'-i\nu_2',\ \nu_1+i\nu_2)-(\nu_1'-i\nu_2',\ \nu_3'')(\nu_3'',\ \nu_1+i\nu_2)+$$
$$+(\nu_1'-i\nu_2',\ [i\nu_3'',\ \nu_1+i\nu_2])\}.$$

Calculating the various terms on the right-hand side by means of the same co-ordinate system we obtain

$$\Sigma(\nu_1' - i\nu_2',\ \nu_1'' + i\nu_2'')(\nu_1'' - i\nu_2'',\ \nu_1 + i\nu_2) =$$

$$= 2(\nu_1' - i\nu_2',\ \nu_1 + i\nu_2)\left[\left(1 + \frac{\cos\theta}{C}\right) + \right.$$

$$\left. + 2S^2\left\{\frac{\cos^2\theta}{4C^2} - \frac{\sin^2\theta\sin^2\phi}{4S^2C^2} - \frac{\sin^2\theta\cos^2\phi}{4S^2} + i\frac{\sin\theta\sin\phi}{2SC}\left(1 + \frac{\cos\theta}{C}\right)\right\}\right] \tag{II. 3}$$

Thus the polarization terms also lead to an expression of the same type as before.

On substitution in (II. 1) we need only retain the terms which are even functions of ϕ in the product of the expressions (II. 2) and (II. 3), as $A((\nu_3'\nu_3''))$ and $X((\nu_3''\nu_3))$ are even functions of ϕ. Thus the term in (II. 1) giving the angular dependence is, after some reduction,

$$8\left(1 + \frac{\cos\theta}{C}\right)\left[\left(1 + \frac{\cos\theta}{C}\right)^2 - \frac{3}{4C^2}\left\{(1 + C\cos\theta)^2 - S^2\sin^2\theta\cos^2\phi\right\}\right]. \tag{II. 4}$$

The form used in (I. 1) is obtained from this by returning to the vector notation.

By changing the sign of σ and i in the above we conclude that the same factor (II. 4) arises from the product of the terms of second type of $V_{\nu'\nu''}$ and $U_{\nu''\nu}$.

APPENDIX III.

Solution of the Integral Equation for $X((\nu_3'\nu_3))$.

We have to solve the integral equation (49) by using the series expansions (52).

The equation (49) is complicated due to the factor (50) which arises from the spin and polarization summations. If this factor did not occur the solution of (49) would be very simple, due to the relation

$$\frac{1}{4\pi}\int d\Omega_{\nu''} P_n((\nu_3'\nu_3''))P_m((\nu_3''\nu_3)) = \frac{1}{2n+1}\delta_{nm}P_n((\nu_3'\nu_3)). \tag{III. 1}$$

X is then given by the independent equations

$$x_n = a_n - \lambda a_n x_n \qquad (n = 0, 1, 2, \ldots) \tag{III. 2}$$

The method of solution can be best seen by considering the next simplest case, where *only* the factor introduced by the spin terms is taken into account.

In this case the factor (50) is replaced by the factor

$$\frac{1 + (\nu_3' \nu_3) + (\nu_3' \nu_3'') + (\nu_3'' \nu_3)}{1 + (\nu_3' \nu_3)} \tag{III. 3}$$

arising from (II. 2) (the sin ϕ term gives no contribution). Then the equation (49) becomes, on multiplying by the factor $\{1 + (\nu_3' \nu_3)\}$ throughout

$$\{1 + (\nu' \nu)\} \sum_n x_n (2n + 1) P_n((\nu' \nu)) =$$

$$= \{1 + (\nu' \nu)\} \sum_n a_n (2n + 1) P_n((\nu' \nu)) -$$

$$- \lambda \sum_n \sum_m a_n (2n + 1) x_m (2m + 1) \frac{1}{4\pi} \int d\Omega_{\nu''} P_n((\nu' \nu'')) \times$$

$$\left\{1 + ((\nu' \nu)) + (\nu' \nu'') + (\nu'' \nu)\right\} P_m((\nu'' \nu)) \tag{III. 4}$$

(where the index 3 is dropped throughout).[10] We now use the recurrence relations for the Legendre functions in the form

$$z P_n(z) = \underline{n} P_{n-1}(z) + \bar{n} P_{n+1}(z) \tag{III. 5}$$

where

$$\underline{n} = \frac{n}{2n + 1}, \qquad \bar{n} = \frac{n + 1}{2n + 1}.$$

To facilitate the application of this relation to the equation (III. 4) we use the following artifice. Consider the expression

$$z \sum_n a_n (2n + 1) P_n(z). \tag{III. 6}$$

Using (III. 5) this becomes

$$\sum_n \left\{ \underline{n}\, a_{n-1} + \bar{n}_{n+1} \right\} (2n + 1) P_n(z).$$

---

[10] All summations are for $n = 0, 1, 2, \ldots$

If we put

$$P_{-1} = P_{-2} = \ldots = 0, \quad a_{-1} = a_{-2} = \ldots = 0, \quad x_{-1} = x_{-2} = \ldots = 0$$

all the formulae in this appendix hold without limitation of n.

This can be written as

$$a \sum_n a_n (2n + 1) P_n(z) \tag{III. 7}$$

where a operates *only* on the a_n, as follows

$$a\, a_n = \underline{n}\, a_{n-1} + \bar{n}\, a_{n+1} \tag{III. 8}$$

a relation analogous to (III. 5).

Similarly,

$$z \sum_n x_n (2n + 1) P_n(z) = \xi \sum_n x_n (2n + 1) P_n(z)$$

where ξ operates on the x_n only in the same way as a operates on the a_n.

Using this the factor $\{1 + (\nu' \nu) + (\nu' \nu'') + (\nu'' \nu)\}$ in the integrand of (III. 4) can be replaced by the operator

$$\{1 + (\nu' \nu) + a + \xi\}$$

which is now removed outside the integral sign. Using (III. 1) to evaluate the integral we obtain

$$(1 + \xi) \sum_n x_n (2n + 1) P_n((\nu' \nu)) = (1 + a) \sum_n a_n (2n + 1) P_n((\nu' \nu)) -$$

$$- \lambda \{1 + (\nu' \nu) + a + \xi\} \sum_n a_n x_n (2n + 1) P_n((\nu' \nu)) . \tag{III. 9}$$

Using a similar trick $(\nu'\nu)$ multiplying the last series can be replaced by an operator γ which operates on $c_n (= a_n x_n)$, the coefficient of $(2n + 1) P_n((\nu'\nu))$ in this series, according to the relation

$$\gamma c_n = \underline{n}\, c_{n-1} + \bar{n}\, c_{n+1} .$$

Thus the equation (III. 9) becomes, on picking out the coefficient of $(2n + 1) P_n((\nu'\nu))$ throughout,

$$(1 + \xi) x_n = (1 + a) a_n - \lambda (1 + \gamma + a + \xi) a_n x_n . \tag{III. 10}$$

$(1 + \gamma + a + \xi) a_n x_n$ consists of 7 terms with coefficients indicated by the following scheme:—

$$\begin{array}{c|ccc} & x_{n-1} & x_n & x_{n+1} \\ \hline a_{n-1} & \underline{n} & \underline{n} & \\ a_n & \underline{n} & 1 & \bar{n} \\ a_{n+1} & & \bar{n} & \bar{n} \end{array} \tag{III. 11}$$

Using the relation $\underline{n} + \bar{n} = 1$ these coefficients can also be written as the sum of two sets

$$\begin{array}{c|cc} & x_{n-1} & x_n \\ \hline a_{n-1} & \underline{n} & \underline{n} \\ a_n & \underline{n} & \underline{n} \end{array} \quad + \quad \begin{array}{c|cc} & x_n & x_{n+1} \\ \hline a_n & \bar{n} & \bar{n} \\ a_{n+1} & \bar{n} & \bar{n} \end{array} \tag{III. 12}$$

Defining

$$\begin{aligned} a_{n+\frac{1}{2}} &= a_n + a_{n+1} \\ x_{n+\frac{1}{2}} &= x_n + x_{n+1} \end{aligned} \tag{III. 13}$$

we see from (III. 12) that $(1 + \gamma + a + \xi)\, a_n\, x_n$ can be written in the form

$$(1 + \gamma + a + \xi)\, a_n\, x_n = \underline{n}\, a_{n-\frac{1}{2}}\, x_{n-\frac{1}{2}} + \bar{n}\, a_{n+\frac{1}{2}}\, x_{n+\frac{1}{2}} \tag{III. 14}$$

Also we have

$$\begin{aligned} (1 + \xi)\, x_n &= x_n + \underline{n}\, x_{n-1} + \bar{n}\, x_{n+1} \\ &= \underline{n}\, x_{n-\frac{1}{2}} + \bar{n}\, x_{n+\frac{1}{2}} \end{aligned} \tag{III. 15}$$

and similarly for $(1 + a)\, a_n$.

Using (III. 14) and (III. 15) we can write (III. 10) in the form

$$\underline{n}\, R_{n-\frac{1}{2}} + \bar{n}\, R_{n+\frac{1}{2}} = 0 \tag{III. 16}$$

where

$$R_{n+\frac{1}{2}} \equiv x_{n+\frac{1}{2}} - a_{n+\frac{1}{2}} + \lambda\, a_{n+\frac{1}{2}}\, x_{n+\frac{1}{2}}. \tag{III. 17}$$

Choosing $n = 0$ we see that we must have

$$R_{\frac{1}{2}} = 0\,,$$

and hence by induction, the solution of (III. 16) is

$$R_{n+\frac{1}{2}} = 0\,, \qquad (n = 0,\ 1,\ 2,\ \ldots). \tag{III. 18}$$

Thus the solution of the integral equation (49) using (III. 3) instead of (50) would be:

$$x_{n+\frac{1}{2}} = a_{n+\frac{1}{2}} - \lambda\, a_{n+\frac{1}{2}}\, x_{n+\frac{1}{2}}\,, \tag{III. 19}$$

a result which is formally the same as (III. 2).
Actually we have then:

$$x_n + x_{n+1} = \frac{a_n + a_{n+1}}{1 + \lambda(a_n + a_{n+1})} \tag{III. 20}$$

from which relations the x_n can be found.

Now returning to the actual, and more complicated, equation (49) where the factor in the integrand is given by (50), we obtain, analogous to (III. 10), the relation

$$\begin{aligned}(1+\xi)^3 x_n &= (1+\alpha)^3 a_n - \\ &- \lambda\left[(1+\gamma+\alpha+\xi)^3 - \frac{3}{2}(1+\gamma+\alpha+\xi)(1+\gamma)(1+\alpha)(1+\xi)\right] a_n x_n .\end{aligned} \tag{III. 21}$$

The expression corresponding to (III. 11) contains the principal diagonal and three other diagonals on each side of it. This can then, similarly to (III. 12), be split up into four squares of four rows and four columns each. Each square is a multiple of the product $a_{n+\frac{3}{2}}\, x_{n+\frac{3}{2}}$, where $a_{n+\frac{3}{2}}$, $x_{n+\frac{3}{2}}$ are given by (53). Corresponding to (III 14) we now have

$$\begin{aligned}&\left[(1+\gamma+\alpha+\xi)^3 - \frac{3}{2}(1+\gamma+\alpha+\xi)(1+\gamma)(1+\alpha)(1+\xi)\right] a_n x_n = \\ &= \frac{\underline{n-2}\,\underline{n-1}\,\underline{n}}{1-\frac{3}{2}\underline{n-2}}\, a_{n-\frac{3}{2}}\, x_{n-\frac{3}{2}} + \\ &\qquad + \frac{3\,\underline{n-1}\,\underline{n}}{1-\frac{3}{2}\underline{n-1}}\left(1 - \frac{\frac{1}{2}\,\underline{n-2}\,\overline{n-1}}{1-\frac{3}{2}\underline{n-2}}\right) a_{n-\frac{1}{2}}\, x_{n-\frac{1}{2}} +\end{aligned}$$

$$+ \frac{3\,\overline{n+1}\;\overline{n}}{1 - \frac{3}{2}\,\overline{n+1}} \left(1 - \frac{\frac{1}{2}\,\overline{n+2}\;\overline{n+1}}{1 - \frac{3}{2}\,\overline{n+2}}\right) a_{n+\frac{1}{2}}\, x_{n+\frac{1}{2}} + \frac{\overline{n+2}\;\overline{n+1}\;\overline{n}}{1 - \frac{3}{2}\,\overline{n+2}}\, a_{n+\frac{3}{2}}\, x_{n+\frac{3}{2}} . \tag{III. 22}$$

$(1 + \xi)^3 x_n$ and $(1 + a)^3 a_n$ by direct manipulation (similar to (III. 15)) give expressions, which are identical with (III. 22) with either the a's or x's omitted.

Thus for $n > 2$ we have four terms in the equation corresponding to (III. 16). By considering the cases $n = 0, 1, 2$ and by induction we obtain the solution:

$$x_{n+\frac{3}{2}} = a_{n+\frac{3}{2}} - \lambda\, a_{n+\frac{3}{2}}\, x_{n+\frac{3}{2}} . \tag{54}$$

$$(n = 0, 1, 2, \ldots)$$

APPENDIX IV.

Evaluation of the Cross-section.

In (64) the term $U_{\nu\nu}$ gives immediately on considering (41)

$$U_{\nu\nu} = i\pi\, \frac{c^2 f^2 \varepsilon^2}{8V} \left\{ 8X((\nu\,\nu)) + Y_1 + Y_2 \right\} \tag{IV. 1}$$

(The index 3 of ν is again dropped throughout this Appendix.) We have from (52)

$$X((\nu\,\nu)) = \sum_n x_n\,(2n + 1) = \sum_n (n + 2)\, x_{n+\frac{3}{2}}$$

after some manipulation. The term $U_{\nu\nu''}\ \rho_{\nu''}\ U_{\nu''\nu}$ according to (II. 1) gives

$$\frac{V\varepsilon^2}{8\pi^3}\left(\frac{i\pi c^2 f^2 \varepsilon^2}{8V}\right)^2 \left\{ 8^2 \int_{(\nu'=\nu)} d\Omega_{\nu''}\, X((\nu'\nu''))\,[\ldots]\, X((\nu''\nu)) + 8\pi Y_1^2 + 8\pi Y_2^2 \right\}. \tag{IV. 2}$$

The expression $[\ldots]$ is given by (50). After integration we put $\nu' = \nu$.

This integral is evaluated by (III. 22) which gives

$$\left\{1 + (\nu'\,\nu)\right\}^3 \frac{1}{4\pi} \int d\Omega_{\nu''}\, X((\nu'\,\nu''))\,[\ldots]\, X((\nu''\,\nu)) =$$

$$= \sum_n \left\{ \frac{\overline{n-2}\;\overline{n-1}\;n}{1-\frac{3}{2}\,\overline{n-2}}\, x^2_{n-\frac{3}{2}} + \frac{3\,\overline{n-1}\;n}{1-\frac{3}{2}\,\overline{n-1}} \left(1 - \frac{\frac{1}{2}\,\overline{n-2}\;\overline{n-1}}{1-\frac{3}{2}\,\overline{n-2}} \right) x^2_{n-\frac{1}{2}} + \right.$$

$$+ \frac{3\,\overline{n+1}\;n}{1-\frac{3}{2}\,\overline{n+1}} \left(1 - \frac{\frac{1}{2}\,\overline{n+2}\;\overline{n+1}}{1-\frac{3}{2}\,\overline{n+2}} \right) x^2_{n+\frac{1}{2}} +$$

$$\left. + \frac{\overline{n+2}\;\overline{n+1}\;n}{1-\frac{3}{2}\,\overline{n+2}}\, x^2_{n+\frac{3}{2}} \right\} (2n+1)\, P_n\left((\nu'\,\nu) \right) . \qquad \text{(IV. 3)}$$

Putting $\nu' = \nu$, and collecting terms we obtain

$$\frac{1}{4\pi} \int_{(\nu'=\nu)} d\,\Omega_{\nu''}\, X\left((\nu'\,\nu'') \right) [\,.\,.\,.\,]\, X\left((\nu''\,\nu) \right) = \sum_n (n+2)\, x^2_{n+\frac{3}{2}} .$$

Combining the various terms in (64) now leads directly to (65).

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XIV.

ON THE CASCADE PRODUCTION OF MESONS.

By H. W. PENG.

(From the Dublin Institute for Advanced Studies.)

[Read 30 November, 1943. Published 25 February, 1944.]

Introduction.

In a previous paper by Heitler and myself[1] it has been found that a nucleon (i.e. a proton or a neutron) in collisions with other nucleons can produce mesons. In traversing matter a fast nucleon will thus produce a number of mesons and gradually lose its energy. If the energy lost by the primary nucleon in a single collision is large, say larger than its rest energy, a *recoil nucleon* is also produced accompanying the production of the meson. The recoil nucleon on further traversing matter produces further mesons and recoil nucleons in the same way as the primary nucleon does. The successive generations of nucleons together with the primary nucleon form what may be called for convenience a "nucleon cascade."

In the subsequent applications[2] of the results obtained in I to the production of mesons in the atmosphere by the protons which form the primary cosmic radiation the complication due to this cascade process has been evaded by the following, very crude, assumption: It has been assumed that in each collision the *whole* energy lost by the primary nucleon is taken up by the meson and no energy given to the recoil nucleon. In this way the energy loss of the primary nucleon is represented correctly, and also the total energy given to mesons by the primary nucleon until it is stopped (because eventually nearly all its energy is, via recoil nucleons, turned into mesons). On the other hand the energy distribution and the total number of mesons is not given correctly by this crude assumption. The agreement between the theory and the various cosmic ray experiments found in II is very satisfactory. Nevertheless it seems desirable to study the described cascade process more in detail and to find out how good the above simplifying assumption really is.

In the present paper the production of mesons by a fast nucleon traversing matter will be treated in detail by taking a proper account of

[1] These Proceedings, 49 A, 101, 1943, quoted below as I.

[2] Hamilton, Heitler and Peng, Phys. Rev., 67, 78, 1943, quoted below as II.

the nucleon cascade. The results will be compared with the crude results obtained by making the above simplifying assumption. It will be seen that the difference is not great. We shall confine ourselves to the case of a fast nucleon traversing dense matter as is commonly used in laboratory experiments. This enables us to neglect the β-decay of the mesons while traversing the matter. From our results we may conclude that also for the case of meson production in air the results will not be very different from those obtained in II by the simplifying assumption.

1. *The nucleon cascade.*

We shall measure the thickness of the matter in terms of equivalent cm. H_2O. We shall consider the simple case that but a single nucleon of energy E_0 falls on the top of the matter. We shall then determine, at any given depth x from the top of the matter, the number of nucleons (including the primary one and recoil ones of all generations) having energy between E and $E + dE$, which we shall denote by $F(E, x)dE$.

It has been found in I that a nucleon with energy smaller than a threshold energy T, say, ceases to produce mesons on traversing matter. Hence we shall confine ourselves to nucleons of energy larger than T. T is of the order of magnitude of the rest energy M of the nucleon. Throughout the following we shall measure energies in units of the rest energy of the meson so that we have approximately $T \doteq M \doteq 10$.

The energy loss for a nucleon of energy E is given by the formula (12) of Part III of reference 1, namely,[3]

$$-dE\,/\,dx \doteq 0{\cdot}97 \log 0{\cdot}30E = b \text{ (say)}, \quad (E > M). \tag{1}$$

The number of nucleons of energy between E and $E + dE$ produced within the thickness dx cm. H_2O by a nucleon of energy E', say, is easily obtained by multiplying the cross-section for the production of a recoil nucleon (equ. (6) of Part III of reference 1) by the number of nucleons encountered in the thickness dx. The number of nucleons produced in dx and in the energy interval dE by one nucleon with energy larger than E is then

$$cE^{-2}\,dE dx, \quad c \doteq 0{\cdot}49, \quad (M < E < E'). \tag{2}$$

The number of nucleons with energy larger than E at a depth x $F(E, x)$ satisfies the equation

$$\frac{\partial F}{\partial x} - b\frac{\partial F}{\partial E} = \frac{c}{E^2}\int_E^\infty F(E', x)\,dE' \tag{3}$$

---

[3] We adopt here, as in II, the "symmetrical form of the meson theory."

with the initial condition at the top of the matter $x = 0$:

$$F(E, 0) = \delta(E - E_0). \tag{3_0}$$

δ being Dirac's delta function.

If we put $F = F_0 + F_1 + F_2 + \ldots$ where F_n is the number of recoil nucleons produced in the n-th generation and consider the primary nucleon by itself, we obtain

$$\frac{\partial F_n}{\partial x} - b\frac{\partial F_n}{\partial E} = \frac{c}{E^2}\int_E^\infty F_{n-1}(E', x)\, dE' \qquad (n = 1, 2, 3, \ldots) \tag{4}$$

$$\frac{\partial F_0}{\partial x} - b\frac{\partial F_0}{\partial E} = 0. \tag{4_0}$$

The equations (4), taken all together, are of course equivalent to (3).

Without committing a serious error we shall neglect the slight variation of b and c with E. The solution of (4) and (4_0) with the initial condition (3_0) is then

$$F_n(E, x) = c\int_0^x \frac{dx'}{(E + bx - bx')^2}\int_{E+bx-bx'}^\infty F_{n-1}(E', x')\, dE', \qquad (n = 1, 2, \ldots) \tag{5}$$

$$F_0(E, x) = \delta(E_0 - E - bx). \tag{5_0}$$

F_1, F_2, etc., are thus obtained in succession.

It is convenient to introduce Heaviside's function

$$H(x) = \int_{-x}^\infty \delta(x')\, dx' = \begin{cases} 1 & (x > 0) \\ 0 & (x < 0). \end{cases} \tag{6}$$

By mathematical induction it follows from (5) and (5_0) that

$$F_n(E, x) = H(E_0 - E - bx) \cdot G_n(E, x), \quad (n = 1, 2, \ldots) \tag{7}$$

where G_n and G_{n-1} are related to each other by a relation obtainable from (5) by changing the F's into G's and the infinite upper limit of integration by $E_0 - bx'$. (7) implies that no recoil nucleons can be found at x with energy larger than $E_0 - bx$, which is an evident consequence of the energy loss suffered by all nucleons.

Let $\Phi(E, x) = \Phi_0 + \Phi_1 + \Phi_2 + \ldots$ be the total number of nucleons at x having energy above E,

$$\Phi_n(E, x) = \int_E^\infty F_n(E', x)\, dE' \tag{8}$$

where the suffix n may be 0, 1, 2, etc., or may be omitted. From (5_0), (6), (7) and (8) it follows that

$$\Phi_n(E, x) = H(E_0 - E - bx) \cdot \Psi_n(E, x) \tag{9}$$

where

$$\Psi_0(E, x) = 1, \ \Psi_n(E, x) = \int_E^{E_0 - bx} G_n(E', x)\, dE' \quad (n = 1, 2, \ldots). \tag{10}$$

Between Ψ_n and Ψ_{n-1} we obtain from (5) the relation

$$\Psi_n(E, x) = c \int_E^{E - bx} dE' \int_0^x \frac{dx'}{(E' + bx - bx')^2} \Psi_{n-1}(E' + bx - bx', x')$$
$$(n = 1, 2, \ldots) \quad (11)$$

It will be shown that $\Psi_n(E, x)$, $n = 0, 1, 2$, etc., are functions of a single variable W:

$$W = bx(E_0 - E - bx) / E_0 E. \tag{12}$$

For this purpose let us suppose that $\Psi_{n-1}(E, x)$ is a function of W only: $\Psi_{n-1}(W)$, say. Then the integrand of (11) contains a function $\Psi_{n-1}(U)$ of the variable U which is obtained from the right-hand-side of (12) by replacing E and x by $E' + bx - bx'$ and x' respectively,

$$U = bx'(E_0 - E' - bx) / E_0(E' + bx - bx'). \tag{13}$$

By using this as a new variable of integration in place of x', say, (11) becomes

$$\Psi_n(E, x) = \frac{c}{b} \int_E^{E_0 - bx} dE' \frac{E_0}{(E' + bx)(E_0 - E' - bx)} \int_0^V \Psi_{n-1}(U)\, dU$$
$$(n = 1, 2, \ldots) \quad (14)$$

where the two limits of integration of U are obtained from the right-hand-side of (13) by putting $x' = 0$ and x respectively and are therefore 0 and V,

$$V = bx(E_0 - E' - bx) / E_0 E'. \tag{15}$$

Now use V as the new variable of integration in place of E'. (14) becomes

$$\Psi_n(E, x) = \frac{c}{b} \int_0^W \frac{dV}{V(1 + V)} \int_0^V \Psi_{n-1}(U)\, dU = \Psi_n(W)$$
$$(n = 1, 2, \ldots) \quad (16)$$

where the two limits of integration of V, as obtained from the right-hand-side of (15) by putting $E' = E_0 - bx$ and E respectively, are 0 and W, remembering (12). (16) shows that $\Psi_n(E, x)$ is a function of W if $\Psi_{n-1}(E, x)$ is so. Since in the case of $n = 1$ $\Psi_0(E, x)$ is, by 10, a trivial function of W, we see by mathematical induction that $\Psi_n(E, x)$, $n = 1, 2$, etc., and thus also $\Psi = \Psi_0 + \Psi_1 + \ldots$ are functions of the single variable W.

The above consideration of the successive generations of the nucleon cascade was only needed to show that $\Psi(E, x)$ is a function of W only. There is no need any more to calculate the $\Psi_n(W)$, etc., individually.

For, when we add up (16) for all generations $n = 1, 2$, etc., and invert the integrations on the right-hand-side into differentiations on the left-hand-side we obtain simply, taking into account that $\Psi_0(W) = 1$, a differential equation for $\Psi(E, x) = \Psi(W)$, viz.:

$$\frac{d}{dW} W(1+W)\frac{d\Psi}{dW} = \frac{c}{b}\Psi . \qquad (17)$$

The initial conditions can easily be derived from (10), and (14)–(16) and work out to be

$$\Psi = 1, \quad d\Psi/dW = c/b \quad \text{at } W = 0. \qquad (18)$$

We can bring (17) into the standard form of Legendre's differential equation by using $1 + 2W$ as the independent variable and putting

$$\alpha(\alpha + 1) = c/b \doteq 0{\cdot}5/\log 0{\cdot}3E , \qquad (19)$$

α being then the order of Legendre's polynomials. The solution conforming to the initial condition (18) is then simply Legendre's function of the first kind

$$\Psi = P_\alpha(1 + 2W). \qquad (20)$$

α is, of course, not a whole number.
Numerical values for such Legendre's functions can be obtained from the hypergeometric series

$$P_\alpha(1 + 2W) = F(\alpha + 1, -\alpha; 1; -W) \quad (W < 1) \qquad (21)$$

or from the descending series

$$P_\alpha(z) = \frac{2^\alpha\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)\,\Gamma(\frac{1}{2})} z^\alpha F\left(\frac{1-\alpha}{2}, -\frac{\alpha}{2}; \frac{1}{2} - \alpha; \frac{1}{z^2}\right) + \frac{2^{-\alpha-1}\,\Gamma(-\alpha-\frac{1}{2})}{\Gamma(-\alpha)\,\Gamma(\frac{1}{2})} z^{-\alpha-1} F\left(\frac{\alpha}{2}+1, \frac{\alpha+1}{2}; \alpha+\frac{3}{2}; \frac{1}{z^2}\right),$$
$$(z + 1 + 2W > 1) \qquad (22)$$

cf. e.g. Whittaker and Watson's "Modern Analysis" (4th ed.). p. 312 and p. 334.

For definiteness, α will always denote the bigger root of (19). It is positive and smaller than one, namely,

$$\alpha = 0{\cdot}34 \text{ for } E = 10; \quad \alpha = 0{\cdot}23 \text{ for } E = 20, \text{ etc.} \qquad (23)$$

Ψ exceeds $\Psi_0 = 1$ (which describes the primary nucleon) appreciably only for $W > 1$, when the leading term of (22) gives a good approximation to Ψ,

$$\Psi \doteq \frac{2^\alpha\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)\,\Gamma(\frac{1}{2})}(1+2W)^\alpha, \quad (W > 1) . \qquad (24)$$

$W < 1$ means that the recoil nucleons are not yet produced appreciably or are largely absorbed already.

It is to be noted that W is a homogeneous function of degree zero in the variables E_0, E and x. Hence the total number of nucleons of energy above E at the depth x produced by a primary nucleon of energy originally E_0 is the same as the total number of nucleons of energy above mE at the depth mx associated with a primary nucleon of energy originally mE_0, m being an arbitrary number. Though this exact correspondence is slightly impaired by the fact that b and a vary slightly with E, it is still fair to say that the nucleon cascades produced by primary nucleons of different energies are similar in their general features.

On re-examining the calculation made above from the beginning of this section one sees that the error committed by treating the energy loss b as constant can be diminished considerably by using in the final result (20) for W the following expression:

$$W = x(y_0 - y - x) / y_0 y, \quad y_0 = E_0 / b_0, \quad y = E / b. \qquad (25)$$

Here the energy loss b per cm. H_2O is considered as a function of the energy E of the nucleon, so that b_0 is obtained from b of (1) by replacing E by E_0. y and y_0 are thus measured in cm. H_2O units. A few values obtained from (24) with the improvement (25) are given in the table below in order to show the main features of the nucleon cascade.

TABLE I.—*Total number of nucleons (including the primary nucleon) with energy above E meson units at x cm. H_2O depth produced by a primary nucleon of energy E_0 = 5,000 meson units falling on the top of the matter (i.e. y_0 = 710 cm. H_2O).*

(*1 meson unit of energy* = $0{\cdot}94 \times 10^8$ *e.v.*)

x \ E	10	20	50	100	200
150	2·7	1·8	1·4	1·25	1·13
300	3·1	2·0	1·5	1·3	1·16
450	3·0	2·0	1·5	1·3	1·15
600	2·4	1·7	1·3	1·2	1·06

The total number of nucleons of energy above E reaches its maximum at the depth $x = (y_0 - y) / 2$, that is, half way between the top and the depth where the energy of the primary nucleon has decreased to E. The maximum number is, by (24) and (25),

$$\Psi_{\max} \doteq \frac{\Gamma(a + \frac{1}{2})}{\Gamma(a + 1)\,\Gamma(\frac{1}{2})} \left\{\frac{y_0}{y} + \frac{y}{y_0}\right\}^a, \quad (y_0 > 6y). \qquad (26)$$

Owing to the smallness of the exponent a the maximum size of the nucleon cascade does not increase very much as the energy of the primary nucleon increases, as illustrated in the following table:—

TABLE II.—*Increase of the size of the nucleon cascade with the energy of the primary nucleon. (All values of energy are given in meson units. 1 meson unit = $0{\cdot}94 \times 10^8$ e.v.)*

Energy of primary nucleon ...	200	500	1,000	2,000	5,000	10,000
Max. no. of nucleons of energy $E > 10$ ...	1·3	1·6	1·9	2·3	3·1	3·8
Max. no. of nucleons of energy $E > 20$ ...	1·1	1·3	1·5	1·7	2·0	2·3

The above values include the primary nucleon itself.

By partial differentiation of (8) and (9) we get for the energy distribution of nucleons at x

$$F(E, x) = -\frac{\partial}{\partial E}\Phi(E, x)$$

$$= \delta(E_0 - E - bx) + H(E_0 - E - bx)\,.\left(-\frac{\partial W}{\partial E}\right)\frac{d\Psi}{dW}\,. \qquad (27)$$

It is seen that the energy of the primary nucleon remains always the highest at all depths. The majority of the recoil nucleons have low energy which is more readily seen from the numerical values given in Table I. This is, of course, a consequence of the fact that the cross-section for the production of recoil nucleons favours low energies because of the factor E^{-2} of (2).

2. *The mesons associated with the cascade.*

Let the density of the matter traversed by the nucleons and mesons be as high as that of a liquid or solid. The β-decay of vector mesons as well as pseudoscalar mesons can then be neglected. We consider vector mesons[4] and pseudoscalar mesons together. If $f(\epsilon, x)d\epsilon$ is the total number of mesons of energy between ϵ and $\epsilon + d\epsilon$ at the depth x, a the energy loss by ionization of a meson of energy ϵ per cm. H_2O, and $s(\epsilon, x)\,d\epsilon\,dx$ the number of mesons of energy between ϵ and $\epsilon + d\epsilon$ produced between the depths x and $x + dx$, then $f(\epsilon, x)$ satisfies the equation

$$\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial \epsilon} = s\,. \qquad (28)$$

By adding together the cross-sections given in the formulae (2) and (4) of Part III of reference 1 for the production of vector and pseudoscalar

[4] The vector mesons concerned with are essentially transverse mesons, because longitudinal mesons are rarely produced during the collision of two nucleons, according to I.

mesons and multiplying the result by the number of nucleons encountered by a nucleon per cm. H_2O path we obtain for the number of mesons produced per cm. H_2O by a nucleon of energy E

$$N(\varepsilon, E)\, d\varepsilon = 0{\cdot}014\, (D_{ps} + D_{tr})\, \varepsilon^{-3}\, d\varepsilon \qquad (2{\cdot}8 < \varepsilon < M) \qquad (29a)$$

$$N(\varepsilon, E)\, d\varepsilon = 0{\cdot}0056\, d\varepsilon \int_{\epsilon} (D_{ps} + \tfrac{1}{2} D_{tr})\, \varepsilon'^{-3}\, d\varepsilon' \qquad (M < \varepsilon < E)\,. \qquad (29b)$$

D_{ps} and D_{tr} are negligibly small if E is less than a certain threshold energy T which is of the order of M. For $E > T$ the two combinations of the D's occurring in (29a) and (29b) are approximately constants, viz., 200 and 115. So we have, after carrying out the integration of (29b),

$$N(\varepsilon, E) = 2{\cdot}8\, \varepsilon^{-3}\, H(E - T) \qquad (2{\cdot}8 < \varepsilon < T) \qquad (30a)$$

$$N(\varepsilon, E) = 0{\cdot}32\, \varepsilon^{-2} \qquad (T < \varepsilon < E)\,. \qquad (30b)$$

Taking account of all the nucleons having energy $E > \epsilon$ at the depth x we get for $s(\epsilon, x)$

$$s(\varepsilon, x) = \int_{\epsilon}^{\infty} N(\varepsilon, E)\, F(E, x)\, dE\,; \qquad (31)$$

or, making use of the results obtained for the nucleon cascade in the preceding section,

$$s(\varepsilon, x) = 2{\cdot}8\, \varepsilon^{-3}\, H(E_0 - T - bx)\,.\, P_a(1 + 2t) \qquad (2{\cdot}8 < \varepsilon < M) \qquad (31a)$$

$$s(\varepsilon, x) = 0{\cdot}32\, \varepsilon^{-2}\, H(E_0 - \varepsilon - bx)\,.\, P_a(1 + 2w) \qquad (M < \varepsilon) \qquad (31b)$$

where t and w are obtained from the right-hand-side of (12) by putting $E = T$ and ϵ respectively,

$$t = bx(E_0 - T - bx) \,/\, E_0 T\,, \qquad (32a)$$

$$w = bx(E_0 - \epsilon - bx) \,/\, E_0 \epsilon\,. \qquad (32b)$$

The ionization loss a varies but little with the energy ϵ of the meson. In meson units for the energy and the cm. H_2O units of length of path we have

$$a = 0{\cdot}021, \quad 0{\cdot}022, \quad 0{\cdot}037 \quad \text{for } \epsilon = 1, \; 10, \; 100. \qquad (33)$$

Hence regarding a as constant we can solve (28) by an integral:

$$f(\varepsilon, x) = \int_0^x s(\varepsilon + ax - ax', x')\, dx'\,, \qquad (34)$$

the initial condition being clearly $f = 0$ at $x = 0$. With the function s given by (31a) and (31b) it is difficult to perform this integration analytically.

Let $\phi(\epsilon, x)$ be the total number of mesons of energy above ϵ at the depth x, that is

$$\phi(\epsilon, x) = \int_\epsilon^\infty f(\epsilon', x)\, d\epsilon' . \tag{35}$$

Insert here (34) for f and change the order of integration, regarding a as a constant. We get then

$$\varphi(\epsilon, x) = \int_0^x \sigma(\epsilon + ax - ax', x')\, dx' \tag{36}$$

with

$$\sigma(\epsilon, x) = \int_\epsilon^\infty s(\epsilon', x)\, d\epsilon' . \tag{37}$$

It is easy to perform the integration (37) analytically, but not the integration (36).

For $\epsilon > M$ we replace the ϵ of (31b) by ϵ' and insert the result into (37). So we obtain

$$\sigma(\epsilon, x) = 0{\cdot}32\, H(E_0 - \epsilon - bx) \int_\epsilon^{E_0 - bx} P_\alpha(1 + 2w')\, \epsilon'^{-2}\, d\epsilon' \qquad (\epsilon > M) \tag{38b}$$

where, by (32b),

$$w' = bx(E_0 - \epsilon' - bx) / E_0 \epsilon' . \tag{39b}$$

Transform the variable of integration of (38b) from ϵ' to w' and apply the well-known relation of Legendre's functions

$$\frac{dP_{\alpha+1}(z)}{dz} - \frac{dP_{\alpha-1}(z)}{dz} = (2\alpha + 1) P_\alpha(z) . \tag{40}$$

We obtain then for $\sigma(\epsilon, x)$

$$\sigma(\epsilon, x) = 0{\cdot}32\, H(E^0 - \epsilon - bx) . \frac{E_0\{P_{\alpha+1}(1 + 2w) - P_{\alpha+1}(1 + 2w)\}}{(4\alpha + 2)\, bx\, (E_0 - bx)} \qquad (\epsilon > M) \tag{41b}$$

w being again (32b).

For $\epsilon < M$ we have to divide the integral of (37) into two parts, and use both (31a) and (31b). The part arising from (31b) is simply $\sigma(M, x)$ obtained from (41b) by putting $\epsilon = M$. To this we add the part arising from (31a) and obtain for $\sigma(\epsilon, x)$

$$\sigma(\epsilon, x) = 2{\cdot}8\, H(E_0 - T - bx)\, P_\alpha(1 + 2t) \frac{1}{2}\left(\frac{1}{\epsilon^2} - \frac{1}{M^2}\right) + \sigma(M, x) \qquad (\epsilon < M) . \tag{41a}$$

The slight variation of the energy loss b with the energy E of the nucleon cannot be ignored completely because the latter diminishes rapidly as the nucleon travels. On re-examining the calculation made in this section from the beginning one sees easily how this effect can be roughly taken into account by simply measuring the energy of a nucleon in cm. H_2O units, as introduced towards the end of section 1. Let y_0 (or η) be the measure of the energy E_0 (or ϵ) of a nucleon in cm. H_2O units, that is to say,

$$y_0 = E_0 / b_0, \qquad \eta = \epsilon / \beta \tag{42}$$

where b_0 (or β) is the energy loss per cm. H_2O by a nucleon of energy E_0 (or ϵ), viz., by (1),

$$b_0 = 0{\cdot}97 \log 0{\cdot}30 E_0, \qquad \beta = 0{\cdot}97 \log 0{\cdot}30\epsilon. \tag{43}$$

Let similarly τ be the measure of the threshold energy T of a nucleon in cm. H_2O units,

$$\tau = T / (0{\cdot}97 \log 0{\cdot}30 T) = 9{\cdot}4 \tag{44}$$

for $T = M = 10$, say. The improved expressions instead of (41a) and (41b) are then

$$\sigma(\epsilon, x) = 0{\cdot}32\, H(y_0 - \eta - x) \,.\, \frac{y_0 \{P_{a+1}(1 + 2w) - P_{a-1}(1 + 2w)\}}{(4a + 2)\, \beta x\, (y_0 - x)} \qquad (\epsilon > M) \tag{45b}$$

where now

$$w = x(y_0 - \eta - x) / y_0 \eta, \quad a(a + 1) = 0{\cdot}5 / \log 0{\cdot}3\epsilon, \tag{46b}$$

and

$$\sigma(\epsilon, x) = 2{\cdot}8\, H(y_0 - \tau - x)\, P_a(1 + 2t)\, \frac{1}{2}\left(\frac{1}{\epsilon^2} - \frac{1}{M^2}\right) + \sigma(M, x) \qquad (\epsilon < M) \tag{45a}$$

where now

$$\begin{aligned} t &= x(y_0 - \tau - x) / y_0 \tau = x(y_0 - 9{\cdot}4 - x) / 9{\cdot}4 y_0, \\ a(a + 1) &= 0{\cdot}5 / \log 0{\cdot}3T = 0{\cdot}45. \end{aligned} \tag{46a}$$

Since we have to use different expressions for $\sigma(\epsilon, x)$ according to $\epsilon \gtrless M$ it is convenient to write (36) in the form

$$\phi(\epsilon, x) = \int_{\epsilon}^{\epsilon + ax} \sigma\left(\epsilon', x - \frac{\epsilon' - \epsilon}{a}\right) \frac{d\epsilon'}{a} \tag{47}$$

which is obtained from (36) by the transformation of variables

$$\epsilon' = \epsilon + ax - ax'.$$

The integration (47) has been carried out by graphical methods for some selected values of ϵ, x and E_0. Most of the numerical values of the Legendre functions that occur in the integrand are calculated from the descending series (22).

3. *Results and discussion.*

Before giving the results derived from the formulae of the preceding section we treat the same problem by making the simplifying assumption mentioned in the introduction, i.e. we assume that in each individual collision the whole energy lost by the fast nucleon is given to the meson produced and no energy given to the recoil nucleon. This simplification has been used throughout in II, and it is our aim to show that it is a reasonably good approximation.

We use the same notation as in the previous section, but apply an asterisk to all quantities derived from this simplification.

The number of mesons produced in the energy range $d\epsilon$ by a nucleon of energy E per cm. H_2O then becomes, instead of (29b) and (30b),

$$N^*(\varepsilon, E)\, d\varepsilon = 0{\cdot}0056\,(D_{ps} + \tfrac{1}{2} D_{tr})\,\varepsilon^{-2}\, d\varepsilon = 0{\cdot}64\,\varepsilon^{-2}\, d\varepsilon \qquad (M < \varepsilon < E)\,. \qquad (30b^*)$$

At every depth now only the primary nucleon is effective in producing mesons. (31) is therefore to be replaced by

$$s^*(\varepsilon, x) = \int_{\epsilon}^{\infty} N^*(\varepsilon, E)\, F_0(E, x)\, dE\,. \qquad (31^*)$$

Using the results of section 1 for $F_0(E, x)$ we find in place of (31a) and (31b)

$$s^*(\epsilon, x) = 2{\cdot}8\,\epsilon^{-3} H(E_0 - T - bx)\,, \quad (2{\cdot}8 < \epsilon < M)\,, \qquad (31a^*)$$

$$s^*(\epsilon, x) = 0{\cdot}64\,\epsilon^{-2} H(E_0 - \epsilon - bx), \quad (M < \epsilon)\,, \qquad (31b^*)$$

These differ from the correct expressions (31a) and (31b) in that $P_\alpha(1 + 2t)$ has been replaced by unity and $P_\alpha(1 + 2w)$ by 2. Thus we see that we underestimate the production of mesons of energy $\epsilon < M$ and, by consulting Table I, overestimate at almost all depths the production of mesons of energy $\epsilon > 50$ if the primary nucleon has originally as high an energy as 5,000.

After an elementary integration and by taking into account the slight variation of the energy loss with the energy of a nucleon in the same manner as above, we obtain for $\sigma^*(\epsilon, x)$ in place of (45a) and (45b)

$$\sigma^*(\varepsilon, x) = 0{\cdot}64\, H(y_0 - \eta - x)\left\{\frac{1}{\varepsilon} - \frac{1}{\beta(y_0 - x)}\right\}. \qquad (\varepsilon > M) \qquad (45b^*)$$

$$\sigma^*(\varepsilon, x) = 1{\cdot}4\, H(y_0 - \tau - x)\,(\varepsilon^{-2} - M^{-2}) + \sigma^*(M, x) \qquad (\varepsilon < M) \qquad (45a^*)$$

We recall that $\eta = \epsilon / \beta$, β being the energy loss per cm. H_2O for a nucleon of energy ϵ.

It remains to carry out the integration (47) which is now elementary. Owing to the small ionization loss of the meson the two limits of integration of (47) are sufficiently close to each other so that the variation of β in the integrand can be neglected. We have to distinguish between three cases according to whether $(aa)\ \epsilon + ax < M$ or $(bb)\ \epsilon > M$ or $(ab)\ \epsilon + ax > M > \epsilon$ when (45a*) or (45b*) or both are to be used for the integrand. We give the results:

$$\phi^*(\varepsilon, x) = 0{\cdot}64\, H\left(y_0 - \frac{\varepsilon + ax}{\beta}\right)\left\{\frac{1}{a}\log\frac{\varepsilon + ax}{\varepsilon_b} - \frac{1}{\beta}\log\frac{y_0}{y_0 - x_b}\right\} \qquad (\varepsilon > M) \qquad \text{(47bb*)}$$

$$\phi^*(\varepsilon, x) = \frac{1.4}{a} H(y_0 - \tau)\left\{\frac{1}{\varepsilon_a} - \frac{1}{\varepsilon + ax} - \frac{\varepsilon + ax - \varepsilon_a}{M^2}\right\} + 0{\cdot}64\, H(y_0 - \tau)\left\{\frac{x_a}{M} - \frac{\tau}{M}\log\frac{y_0}{y_0 - x_a}\right\} \qquad (\varepsilon + ax < M) \qquad \text{(47aa*)}$$

$$\phi^*(\varepsilon, x) = \frac{1.4}{a} H(y_0 - \tau - x_M)\left\{\frac{1}{\varepsilon_a} - \frac{1}{M} - \frac{M - \varepsilon_a}{M^2}\right\} + 0{\cdot}64\, H(y_0 - \tau - x_M)\left\{\frac{x_a - x_M}{M} - \frac{\tau}{M}\log\frac{y_0 - x_M}{y_0 - x_a}\right\} + 0{\cdot}64\, H\left(y_0 - \tau\frac{\varepsilon + ax}{M}\right)\left\{\frac{1}{a}\log\frac{\varepsilon + ax}{\varepsilon_{bM}} - \frac{\tau}{M}\log\frac{y_0}{y_0 - x_{bM}}\right\} \qquad (\varepsilon + ax > M > \varepsilon) \qquad \text{(47ab*)}$$

where $x_M = x - (M - \varepsilon)/a$ and $\varepsilon_b = \varepsilon,\ x_b = x$ if $y_0 > x + \eta$,

$$\varepsilon_b = \frac{\varepsilon + ax - ay_0}{1 - a/\beta}, \quad x_b = \left(y_0 = \frac{\varepsilon + ax}{\beta}\right)\Big/\left(1 - \frac{a}{\beta}\right) \quad \text{if} \quad y_0 < x + \eta,$$

$$\varepsilon_a = \epsilon, \quad x_a = x \quad \text{if} \quad y_0 > x + \tau,$$

$$\varepsilon_a = \varepsilon + ax + a\tau - ay_0, \quad x_a \doteq y_0 - \tau \quad \text{if} \quad y_0 < x + \tau,$$

$$\varepsilon_{bM} = M, \quad x_{bM} = x_M \quad \text{if} \quad y_0 > x_M + \tau,$$

$$\varepsilon_{bM} = \frac{\varepsilon + ax - ay_0}{1 - \tau a/M}, \quad x_{bM} = \left(y_0 - \tau\frac{\varepsilon + ax}{M}\right)\Big/\left(1 - \frac{\tau a}{M}\right) \quad \text{if} \quad y_0 < x_M + \tau.$$

The values ϕ^* calculated from these formulae for the total number of mesons produced at the depth x of energy above ϵ by a nucleon of energy originally E_0 are compared in the tables below with the accurate values ϕ calculated by the method described in section 2, where the nucleon cascade is taken into account properly.

TABLE III.

ϕ and ϕ^* for $E_0 = 5000$

$x \backslash \epsilon$	2·8	10
343	44 34*	12·6 16*
700	50 49*	18 25*

TABLE IV.

ϕ and ϕ^* for $E_0 = 200$

$x \backslash \epsilon$	2·8	10
20	2·8 3·8*	0·48 0·95*
40·6	6·0 6·6*	0·85 1·47*

It is clear that the production of mesons of energy above M has been overestimated and that of small energies underestimated, as was to be expected. For each value of E_0 the two depths considered in the tables above are roughly half the distance and the whole distance the primary nucleon travels until the production of mesons ceases.

On the whole, for most values of ϵ and x the difference between ϕ and ϕ^* is not very great. Especially the total number of mesons, i.e. the number of mesons with energy above $\epsilon = 2{\cdot}8$ never deviates from the exact number by more than 30 per cent. if ϕ^* is used instead of ϕ. We can safely conclude that also for the case of a nucleon traversing air (when the β-decay of mesons cannot be neglected) the results will not be very different and that the use of ϕ^* instead of ϕ will not cause very large errors. The nucleon cascade never gives rise to a marked shower phenomenon, because in the production of recoil nucleons low energies are favoured too strongly.

In II the energy spectrum of mesons at sea level has been compared with the theory using the ϕ^* function instead of ϕ. It was found that the experimental spectrum falls off somewhat more steeply than the theoretical one at high energies. Looking at tables III and IV we see immediately that the agreement would be improved if the exact results of this paper were used by taking into account the nucleon cascade.

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Theory of Cosmic-Ray Mesons

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The quantum theory of damping developed by two of us (Heitler and Peng) is applied to the production of mesons by proton-proton collisions. For this purpose the modification of the meson theory proposed by Møller and Rosenfeld is used. A primary radiation consisting of protons with a suitable energy spectrum is assumed, and it is shown that the rate of meson production is so high that nearly all mesons are produced in a top layer of the atmosphere of thickness 15–30 cm H_2O. The variation of the meson intensity with energy, height, and geomagnetic latitude is found to be in good agreement with the experiments. The transverse mesons, which have a very short lifetime, are seen to give, by decay, a satisfactory account of the soft component in the high atmosphere. A number of other effects (meson showers, transformation into neutretto's) are discussed in Sections VI and VII.

I. INTRODUCTION

UNTIL recently it has not been possible to apply Yukawa's meson theory of the nuclear forces to cosmic-ray mesons and thus to establish the identity of the particles predicted by Yukawa with the cosmic-ray mesons. The reason for this deep-rooted difficulty is the following: The interaction between a meson and a nuclear particle is, in contrast to the electron-light interaction, a strong one, and becomes increasingly stronger at high energies. This makes a proper treatment of the reaction forces exerted by the meson field on the nuclear particles imperative. However, as is well known, a treatment of the radiation reaction is intimately connected with the divergence difficulties occurring in every quantized field theory. To remove this difficulty two distinctly different sets of ideas have been put forward recently. Their difference can best be understood by remembering Lorentz's expansion of the reaction force which a light wave emitted by an electron exerts on the electron. This reaction force can be expanded according to powers of the electronic radius r: The first term is proportional to the acceleration and to r^{-1} thus diverging for a point particle. This term is usually thought to be included in the inertia of the particle. The second term, the usual damping term, is independent of r and proportional to the time derivative of the acceleration. Higher terms are proportional to positive powers of r and are, as a rule, neglected. In the meson case it is the charge-and-spin degrees of freedom of the nuclear particle which are coupled strongly with the meson field. We expect, therefore, that the reaction force will produce a twofold effect: (i) a large inertia to be attributed to these degrees of freedom, (ii) a large damping. In the first set of theories[1] mentioned above attention is concentrated on the first effect. It is clear that in order to make the inertia term finite, a finite particle radius has to be introduced which makes a relativistic treatment of the nuclear particle so far impossible. In the second kind of theory no physical reality is attributed to the first term of the reaction force at all. The particle is strictly considered as a point particle. By suitable subtraction the diverging inertia term of the reaction force is made to vanish (and so are the other diverging integrals occurring in the theory). The only finite part of the reaction force is then the damping term. Along this line Dirac's[2] new quantum-electrodynamics (confined so far to the electromagnetic field) is based. Independently of Dirac but in the same spirit, though less general, two of us (Heitler and Peng)[3] have made an attempt at "guessing" the correct equations of

[1] The so-called "strong coupling theory" put forward by Wentzel [Helv. Phys. Acta. **13**, 269 (1940)], Oppenheimer and Schwinger [Phys. Rev. **60**, 150 (1941)], and recently Pauli and Dancoff [Phys. Rev. **62**, 85 (1942)]. This theory is closely connected with former suggestions made by Bhabha, Ma, and Heitler, in which it was assumed that a proton and neutron can exist in excited charge and spin states.

[2] Dirac, Proc. Roy. Soc. **180**, 1 (1942) and lectures given at the Dublin Institute for Advanced Studies, to appear shortly in the Communication of the Dublin Institute for Advance Studies.

[3] Heitler and Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942).

quantum electrodynamics by omitting consistently and systematically all the diverging features from the quantized field theories. A simple new set of equations has been obtained which, however, can so far only be used for non-static problems (transition probabilities) whilst the static field appears to be a problem of a higher order of difficulty. This paper[3] will, in the following, be referred to as *I*. In contrast to the older expansion method used previously for the calculation of transition probabilities the new theory includes a damping force. In cases where a classical analogue exists it can be shown that the theory is equivalent to a treatment of the second classical damping term. The transition probabilities are to be calculated from a set of simultaneous inhomogeneous integral equations. The application to multiple processes in *I* shows that the results are at least "reasonable." We hope to be able to show that our theory can, at any rate as a good approximation, be derived from Dirac's quantum-electrodynamics.[4]

It should be possible to decide between the two kinds of theories experimentally. The cross section for the scattering of mesons by a proton depends upon the energy in a different way in the two theories. More precise measurements of the scattering cross section especially as a function of the energy are therefore very desirable. An experimental decision would virtually decide whether the elementary particles have a finite radius or are point particles (*cum grano salis*).

Further applications of our theory to the creation of mesons by light quanta and through proton-proton collisions are made in two papers,[5] the results of which are the basis of the present work (referred to as *II* and *III*, respectively). They are summarized as far as they are needed in Section II. The aim of this paper is to show that the theory gives a satisfactory account—as far as we can see—of all the chief cosmic-ray phenomena connected with mesons, including their creation, their diffusion through the atmosphere, meson showers, and the transformation into neutrettos.

We shall assume that mesons are created by a primary radiation consisting of protons, in accordance with the geomagnetic evidence.

It will be seen that the rate of production of mesons is so great that practically all the mesons are produced in a very thin top layer of the atmosphere. This is indeed what recent experiments[6] have shown to be the case.

A certain ambiguity arises from the fact that at present it is not quite known which of the various modifications of the meson theory is to be used. As we endeavor to give a *connected account of cosmic-ray and nuclear phenomena* we choose that form of the meson theory which gives the best account of the nuclear forces: This is undoubtedly the form of the theory proposed by Møller and Rosenfeld.[7] Thus we assume that vector and pseudoscalar mesons exist (with equal coupling constants for transverse and pseudoscalar mesons) and that neutrettos (neutral mesons) also exist, whose coupling constants are half of those of charged mesons.

In accordance with this theory and with the fact that only pseudoscalar mesons are found at sea level we assume that vector mesons have a much smaller lifetime than pseudoscalar mesons. A lifetime of 10^{-8} sec. at rest suffices to explain that all vector mesons decay practically at the point where they are created. The meson producing layer of the atmosphere will be seen to have a thickness of less than 15–30 cm H_2O. Most of the electrons arising from the decay of the transverse mesons are, therefore, also produced in a very thin top layer of the atmosphere. We shall see in Section V that the number of these decay electrons and their energy spectrum is just of the right magnitude and type to produce, by cascade multiplication, the soft component in the high atmosphere. The observed intensity curve is in good agreement with the calculated one. Thus the *soft component can be explained as tertiary products of the incoming primary protons.*

II. PRODUCTION OF MESONS BY PROTON-PROTON COLLISIONS. RANGE OF FAST PROTONS

If a fast proton collides with another nuclear particle it may emit a meson in analogy to the

[4] This does not mean that our results should be regarded as a proof for Dirac's theory, even if our equations should prove to be in quantitative agreement with the experiments. Indeed many of the special features of Dirac's theory only become apparent when static problems are treated.

[5] Hamilton and Peng; Heitler and Peng, to appear shortly in the Proceedings of the Royal Irish Academy.

[6] Schein, Jesse, and Wollan, Phys. Rev. **59**, 615 (1941).

[7] Møller and Rosenfeld, Kgl. Danske Vid. Sels. Math.-Fys. Medd. **17** (1940).

"bremsstrahlung" emitted when a fast charged particle is deflected in a Coulomb field. The rate of production has been calculated in *III* by applying the method of Weizsäcker-Williams which has proved useful in the electromagnetic case: The field of the fast proton is expanded into a Fourier series and is, at sufficiently high energies, equivalent (i) to a beam of light quanta arising from the expansion of the Lorentz-transformed Coulomb field and (ii) to a beam of mesons arising from the Lorentz-transformed nuclear field. Both types of quanta interact with the nuclear particle at rest and can produce a meson, (i) by the process,

$$h\nu + P \rightarrow N + Y^{+} \tag{1}$$

and (ii) by scattering. It has been shown in *III* that the contribution of (i) is entirely negligible compared with that of (ii) except if the proton has an energy $>10^{12}$ ev which is not of great interest in the present work.

Thus the cross section for meson production is obtained by multiplying the number of mesons of a given energy occurring in the spectrum of the transformed nuclear field by the cross section of scattering of mesons by the nuclear particle at rest.

Throughout this paper we shall use "natural meson" units, putting $\hbar = c = \mu$ (meson mass) $=1$. Energies are then measured in units of $\mu c^2 \sim 90$ Mev and cross sections in units of $(\hbar/\mu c)^2 = 4.3\times 10^{-26}$ cm^2. Furthermore, we introduce a unit for the thickness of matter traversed, namely, that thickness which a fast particle with unit charge has to traverse in order to lose an energy by ionization equal to $\mu c^2 = 1$. This unit thickness is about 45 cm H_2O. Thicknesses measured in these units are denoted by x. The coupling constants of longitudinal, transverse, and pseudoscalar mesons with a nuclear particle are denoted by g^2, f^2, f'^2 (dimensionless), respectively, and their numerical values taken from the theory of nuclear forces (cf. reference 7):

$$g^2 = 0.054, \quad f^2 = f'^2 = 0.13. \tag{2}$$

The coupling constants for neutrettos are halves of these values ($f_0^2 = f_0'^2 = 0.065$, $g_0^2 = 0.027$). For the meson mass we assume the value $\frac{1}{10}$ of the proton mass.

In the "equivalent spectrum" of the field of the fast proton with energy E, charged and neutral mesons of all polarizations occur. The number of longitudinal mesons, however, is negligible. The number of transverse and pseudoscalar mesons with energy between ϵ and $\epsilon + d\epsilon$ was found in *III* to be

$$q_{tr}^{(\epsilon)d\epsilon} = \frac{d\epsilon}{\pi\epsilon} f^2 D_{tr}, \quad q_{ps}^{(\epsilon)d\epsilon} = \frac{d\epsilon}{\pi\epsilon} f^2 D_{ps}. \tag{3}$$

Here the D's are rather complicated functions of ϵ/E, involving Hankel functions, but when plotted against ϵ/E it turns out that they can quite well be represented by the following simple functions (with errors less than 30 percent)

$$\begin{aligned} D_{tr} &= 165, & D_{ps} &= 50(\epsilon/E)^{\frac{1}{2}}, \\ D_{tr} + D_{ps} &= 200, & \tfrac{1}{2}D_{tr} + D_{ps} &= 115. \end{aligned} \tag{4}$$

Only these combinations of the D's will occur. D_{tr} comprises both transverse polarizations. The number of neutrettos occurring in the spectrum is half of (3).

The above method and therefore the expressions (3) are only accurate if: (i) $\epsilon \gg 1$, (ii) $E \gg M$ (proton mass). For smaller proton energies the Weizsäcker-Williams method fails completely. (For a more detailed discussion of the validity of the method cf. *III.*) In the following we shall only be interested in meson energies $\epsilon > 1/f \simeq 3$ for which (i) is fairly well satisfied. We shall use (3) for energies E down to values $\sim M$. This is certainly crude, but cannot involve very large errors because the slower protons are not very effective in producing mesons, on account of their small energy. If $E < M$ a proton may still produce mesons. A guidance as to the order of magnitude of this effect can be obtained in the following way: If the damping is neglected altogether and if $E \ll M$ the rate of meson production can quite easily be calculated directly by the old methods.[8] The result can be compared with that obtained by using the Weizsäcker-Williams method, also, of course, neglecting the damping, which means $\epsilon < 1/f \simeq 3$. The result is that the actual rate of meson production is about 10 times smaller than that obtained by using (3). We therefore expect that the *rate of meson production drops rapidly for*

[8] Cf. for example: Nordheim and Nordheim, Phys. Rev. 54, 254 (1938).

$E<M$, and we shall neglect the contribution from protons with $E<M$.

The cross section for scattering of a meson by a nuclear particle has also been derived in *III*.[9] Since the exact expressions would be very complicated, we use only their asymptotic forms for $\epsilon\gg 1/f$. ϵ may then still be either $<M$ ("nonrelativistic case") or $>M$ ("extreme relativistic case"). Since a meson with given charge and polarization may be "scattered" into a meson with different charge (a charged meson may be transformed into a neutretto and vice versa) and different polarization, we write the result in the form of a matrix attributing rows to the primary and columns to the secondary particles. Attributing rows and columns to the various polarizations as indicated in the formulae we found for the scattering cross sections:

$$\Phi=\frac{4\pi}{\epsilon^2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{3} & \frac{1}{3} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{3} & \frac{1}{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{3} & \frac{1}{3} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{3} & \frac{1}{3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l}\text{long.}\\ \text{transv.}\\ \text{pseud.}\end{array}\right\}\text{charged} \\ \left.\begin{array}{l}\text{pseud.}\\ \text{transv.}\\ \text{long.}\end{array}\right\}\text{neutral} \end{array} \qquad (\epsilon\ll M) \qquad (5a)$$

$$\Phi=\frac{16\pi}{\epsilon M}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{4} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l}\text{long.}\\ \text{transv.}\\ \text{pseud.}\end{array}\right\}\text{charged} \\ \left.\begin{array}{l}\text{pseud.}\\ \text{transv.}\\ \text{long.}\end{array}\right\}\text{neutral} \end{array} \qquad (\epsilon\gg M) \qquad (5b)$$

The rows and columns marked transv. are to be understood as giving the transition probabilities into *any one* of the two transverse polarizations of the secondary meson and thus have actually to be understood as a submatrix: for instance $\begin{pmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{4}\end{pmatrix}$ instead of $\frac{1}{4}$ in (5b). At the places marked by dots in (5) expressions occur which are of a smaller order of magnitude containing high negative powers of ϵ. We shall use (5a) and (5b) for energies $\epsilon\lesssim M$ and $\epsilon\gtrsim M$, respectively.

In the rows and columns marked transv. of (5b) an additional diagonal term has been omitted which increases with ϵ but is multiplied by such a small factor that it becomes only appreciable for $\epsilon\gtrsim 1000$. There are also other processes, not considered in this paper at all, which become important for such extremely high energies. Throughout this paper it must be kept in mind that our formulae are no longer valid whenever such high energies are involved.

The most remarkable feature of (5) is the occurrence of *selection rules*. A meson has the tendency to conserve its charge as well as—to some extent—its polarization, and this becomes a strict law in the limit of high energies. The occurrence of these selection rules is a special feature of our theory and entirely due to the damping. No such selection rules occur if damping is neglected (which is permissible only for $\epsilon\lesssim 1/f$) and our matrices (5) would be filled everywhere with expressions of the same order of magnitude. For the interpretation of cosmic rays it will be of particular importance that charged mesons cannot be transformed into neutrettos in a collision with a nuclear particle, except at very small energies ($\epsilon\sim 1/f$) (cf. Section VII).

Another remarkable fact is that the energy dependence changes from a $1/\epsilon^2$ to a $1/\epsilon$ law for $\epsilon>M$. This fact will be fundamental for our understanding of cosmic rays.

Multiplying (5) by (3) we obtain the cross section for the production of a meson of given energy and polarization. If $\epsilon<M$ it is seen from (5a) that the pseudoscalar and the transverse parts of the equivalent spectrum combine to produce either transverse or pseudoscalar mesons. Thus the cross sections for the production of a meson with energy ϵ become

$$\Phi_{tr}(\epsilon)d\epsilon=\frac{8}{3}f^2\frac{d\epsilon}{\epsilon^3}(D_{tr}+D_{ps}), \qquad (6a)$$

$$(\epsilon<M)$$

$$\Phi_{ps}(\epsilon)d\epsilon=\frac{4}{3}f^2\frac{d\epsilon}{\epsilon^3}(D_{tr}+D_{ps}). \qquad (6b)$$

Equation (6a) is already summed over both transverse polarizations. If $\epsilon>M$ the pseudoscalar part of the equivalent spectrum produces

[9] Earlier work assuming charged mesons only: Heitler, Proc. Camb. Phil. Soc. **37**, 291 (1941), and Wilson, *ibid.*, 301. (In these two papers only the non-relativistic case is treated.) Exact relativistic expression (for charged vector mesons only) by S. T. Ma, *ibid.*, in the press. Cf. also Fierz, Helv. Phys. Acta. **14**, 105 (1941); Landau, J. Phys. U. S. S. R. **2**, 483 (1940).

only pseudoscalar mesons and the transverse part only transverse mesons. Thus

$$\Phi_{tr}(\epsilon)d\epsilon=(8/M)f^2\frac{d\epsilon}{\epsilon^2}D_{tr}, \quad (6c)$$

$$(\epsilon>M)$$

$$\Phi_{ps}(\epsilon)d\epsilon=(16/M)f^2\frac{d\epsilon}{\epsilon^2}D_{ps}. \quad (6d)$$

In (6c, d) ϵ is not the energy of the meson produced but the energy lost by the moving proton, i.e., the energy of the meson produced plus the recoil energy of the nuclear particle originally at rest. The probability for the meson to take up an energy ϵ' leaving the energy $\epsilon-\epsilon'$ to the recoil particle is simply $d\epsilon'/\epsilon$. This is true if, as in fact is the case, the angular distribution of scattering is uniform in a Lorentz system where the meson and nuclear particle are colliding with opposite and equal momenta; thus the cross section for producing a meson of energy ϵ' is in the extreme relativistic case:

$$d\epsilon'\int_{\epsilon'}^{E}\Phi_{tr}(\epsilon)\frac{d\epsilon}{\epsilon}\backsimeq\tfrac{1}{2}\Phi_{tr}(\epsilon')d\epsilon', \quad (6'c)$$

$$d\epsilon'\int_{\epsilon'}^{E}\Phi_{ps}(\epsilon)\frac{d\epsilon}{\epsilon}\backsimeq\tfrac{2}{3}\Phi_{ps}(\epsilon')d\epsilon'. \quad (\text{for } E\ll\epsilon') \quad (6'd)$$

The energy distribution remains the same but the number of mesons is only $\frac{1}{2}$ or $\frac{2}{3}$ of what it would be if all the energy would be taken up by the meson. The rest of the energy is taken up by the recoil particles. Their energy distribution is also given by (6c, d) and their number is (6c, d) multiplied by $\frac{1}{2}$ and $\frac{1}{3}$, respectively. These recoil particles are further capable of producing mesons and recoil particles. The process repeats itself until the energy has degenerated to a sufficiently low value to make further meson production impossible. The process very much resembles the *cascade multiplication* of electrons but is not so pronounced because the energy distribution (6) favors low energies more strongly than in the case of the bremsstrahlung emitted by a fast electron. Thus the energy degenerates more quickly than in the electromagnetic case. A detailed treatment of this cascade process lies outside the scope of this paper which is only intended to give a first orientation. We can take account of it in a crude way by using (6c, d) instead of (6′c, d) also for the number of mesons produced. By doing so we represent the energy loss of the primary proton correctly. Also the total energy content of the charged mesons is correct because the energy given by the recoil particles to neutrettos is compensated by the energy of those charged mesons which are produced by the recoil particles from neutrettos. Using (6c, d), we overrate the number of fast mesons by a factor 2 or $\frac{3}{2}$, respectively, but we also underrate the number of slower mesons. Thus the error committed by using (6c, d) is a slight distortion of the energy spectrum in favor of high energies, whilst the total number of mesons will be somewhat bigger than what we obtain in this way.

We multiply (6) by the number of nuclear particles contained in a cylinder of unit length, measured in our x units, and with cross section equal to our unit cross section. For water or air this figure is 1.18, but is not very different for other materials (for Pb it is about 1.3).

We then obtain the number of mesons with energy ϵ produced by a proton in traveling the distance dx (using (2) and (4)):

$$\left.\begin{aligned}\Phi_{tr}(\epsilon)d\epsilon dx&=82\frac{d\epsilon}{\epsilon^3}dx \quad (7a)\\ \Phi_{ps}(\epsilon)d\epsilon dx&=41\frac{d\epsilon}{\epsilon^3}dx \quad (7b)\end{aligned}\right\}(\epsilon<M),$$

$$\left.\begin{aligned}\Phi_{tr}(\epsilon)d\epsilon dx&=21\frac{d\epsilon}{\epsilon^2}dx \quad (7c)\\ \Phi_{ps}(\epsilon)d\epsilon dx&=12.3\left(\frac{\epsilon}{E}\right)^{\frac{1}{3}}\frac{d\epsilon}{\epsilon^2}dx \quad (7d)\end{aligned}\right\}(\epsilon>M).$$

For neutrettos the above figures are to be halved. All these formulae are only valid for $\epsilon>1/f\cong3$.

From (7) it is seen immediately that a very fast proton produces more mesons with energy between $1/f$ and M than with $\epsilon>M$, but far the greater part of the *energy* is contained in the fast mesons.

The mesons with $\epsilon>M$ are all emitted in the forward direction, within a very small angle. This is not so for the mesons with $\epsilon<M$. The latter play a not very important role, except in

TABLE I. Range of fast proton.

E_0	20	50	100	300	1000	10,000
$x_{E_0,M}$	0.18	0.45	0.83	2.0	4.1	28

the top part of the atmosphere. For the calculation of intensities, the error will not be *very* large if we disregard any angular dependence and assume that *all particles* are always *emitted in the forward direction*. This we shall do throughout this paper. For this reason, and because we use in many places asymptotic laws instead of exact ones, and finally because we have only taken a crude account of the cascade process mentioned above, we must not expect too high an accuracy for our calculations. On the average, errors will be of the order of magnitude of, say, a factor 2.

From (7) the energy loss of a proton can be obtained immediately. Multiplying by $\frac{3}{2}$ to account for the energy lost by producing also neutral particles and summing over-all polarizations we find (for $E>M$, of course)

$$-\frac{dE}{dx}=\sum_{\text{polar.}}\int_{1/f}^{E}\epsilon\Phi(\epsilon)d\epsilon=43\log 0.3E. \qquad (8)$$

The energy loss is very high. It depends in a similar way on E to the ordinary energy loss by ionization but is roughly a hundred times larger. Per cm Pb (about $\frac{1}{5}$th of our x units) a proton with an energy of 3×10^9 ev loses an energy of 2×10^9 ev.

Accordingly the *range of a fast proton* is very small. Of course, (8) is only valid for $E>M$; therefore, we can calculate only the distance a proton travels until it is slowed down to an energy $\sim M$. We find

$$x_{E_0,M}=\int_M^{E_0}dE\bigg/\left(-\frac{dE}{dx}\right)$$
$$=\frac{1}{13}\{\mathrm{li}(0.3E)-\mathrm{li}(0.3M)\} \qquad (9)$$

where "*li*" is the integral-logarithm. For practical purposes *lix* can well be replaced by $x/\log x$ (this is exact for large x). For $x_{E_0,M}$ we thus find for the range the values given in Table I. In our units the thickness of the atmosphere is 22. Thus a proton needs an energy of more than 7000, i.e., 7×10^{11} ev in order to penetrate through the whole atmosphere and still retain an energy M.

The majority of the protons entering the atmosphere at a latitude of 50° ($E\sim22-50$) lose the effective part of their energy in distances of 0.2–0.5 (9–23 cm H_2O).

The theory does not tell us how quickly a proton loses its energy *after* having been slowed down to an energy M. Although meson production is then negligible compared with its rate at higher energies, the energy loss may still be much greater than that due to ionization. If our above estimate is correct (rate of meson production $\frac{1}{10}$ of that at higher energies), the energy loss would be 10 times greater than that due to ionization and the range of a proton with $E\sim M$ about 2 of our units (1 m H_2O).

Below we shall need the distance a proton travels in order to lose energy from E_0 to E:

$$x_{E_0,E}=\frac{E_0}{43\log 0.3E_0}-\frac{E}{43\log 0.3E} \qquad (10)$$

in which we have replaced the $li(x)$ function by $x/\log x$.

III. PRODUCTION AND DIFFUSION OF PSEUDOSCALAR MESONS IN THE ATMOSPHERE

Since the primary protons have an extremely short range the majority of the mesons is produced in a thin top layer of the atmosphere. The transverse mesons are expected to decay almost at once, and only the pseudoscalar mesons will travel through the atmosphere. Their absorption is due to two factors: (i) energy loss by ionization (ii) β-decay. The latter depends upon the distance the meson travels but not on the amount of matter traversed. We assume for simplicity that the density of the atmosphere at a depth x below the top is proportional to x. The probability of a meson at a depth x and with energy ϵ decaying while traveling the distance dx is therefore $dxb/\epsilon x$, where b is inversely proportional to the lifetime τ of a meson at rest. We choose τ from the results of Rossi and Hall,[10] who measured the ratio τ/μ. For $\mu=185$, τ becomes $2\times7\times10^{-6}$ sec. b is then

$$b=\frac{1}{c\tau}\times(\text{distance in cm corresponding to one } x \text{ unit at sea level})\times(\text{height of atmosphere in } x \text{ units})$$

or $b=13$.

[10] Rossi and Hall, Phys. Rev. 59, 223 (1941).

Let $f(\epsilon, x)d\epsilon$ be the number of mesons with energy ϵ at depth x and $S(\epsilon, x)dx$ the number of mesons produced at depth x in the distance dx. Then f satisfies the diffusion equation

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial \epsilon}-\frac{b}{\epsilon x}f+S. \tag{11}$$

This differs from the diffusion equation considered and solved previously by Euler and Heisenberg[11] by the inclusion of the source function S (which is determined by our theory). The term $\partial f/\partial\epsilon$ accounts for the energy loss by ionization. Equation (11) can be solved by introducing new variables

$$\eta=\epsilon+x,\ x$$

instead of ϵ and x. The solution with the boundary condition $f=0$ for $x=0$ is

$$f(\eta, x)=\left(\frac{\eta-x}{x}\right)^{b/\eta}\int_0^x S(\eta, \xi)\left(\frac{\eta-\xi}{\xi}\right)^{-b/\eta}d\xi,$$

$$\eta=\epsilon+x. \tag{12}$$

S is determined by our theory if we know the number of protons of each energy and at each depth x. The latter depends, of course, on the energy spectrum of the protons falling onto the top of the atmosphere. From the measurements at great depths it will be seen below that the primary spectrum is within certain comparatively wide energy regions a simple power law. For our purpose it is convenient to include also the logarithmic term occurring in (8) in the expression for the primary spectrum; thus we assume that the number of primary protons with energy larger than E is of the form $A\{E/43\log 0.3E\}^{-\alpha}$. There is, of course, not the slightest reason why α should be exactly a constant. The experiments only show that α does not vary much if E changes by a factor 10 or so. Indeed, the measurements at extreme depths indicate an increase of α with increasing E. We shall determine α from underground measurements and shall find $\alpha=2.2$ for E between 100 and 1000. For larger E, α is bigger. Consequently we expect α to be smaller for $E<100$. The latter energy region will be of importance for the upper regions of the atmosphere and for the intensity curve of the soft component. α can be determined then from the shape of the Regener-Pfotzer-curve. We found $\alpha=1.3$ satisfactory for $E<100$. Thus the chief phenomena of cosmic radiation will be explained by the following crude but simple primary differential spectrum:

$$\begin{aligned} dF(E)&=\frac{\partial}{\partial E}\frac{A}{(E/43\log 0.3E)^{2.2}}\quad(\text{for } E>100),\\ dF(E)&=\frac{\partial}{\partial E}\frac{B}{(E/43\log 0.3E)^{1.3}}\quad(\text{for } E<100),\end{aligned} \tag{13}$$

$$B=\frac{2.2}{1.3}\left(\frac{43\log 30}{100}\right)^{0.9}A=2.4A,$$

the connection between B and A being determined by the continuity of the primary spectrum. In reality α will change gradually from smaller to larger values as E increases. For any depth x larger than 2 only the high energy part of (13) will be important. The number of protons with energy larger than E at depth x is, according to (10) and (13):

$$F(E, x)=A\left(\frac{E}{43\log 0.3E}+x\right)^{-2.2},$$

$$F(E, x)=B\left(\frac{E}{43\log 0.3E}+x\right)^{-1.3}-B\left(\frac{100}{43\log 30}+x\right)^{-1.3}+A\left(\frac{100}{43\log 30}+x\right)^{-2.2}, \tag{14}$$

according to whether

$$\frac{E}{43\log 0.3E}+x\gtrless\frac{100}{43\log 30}.$$

The source function $S(\epsilon, x)$ for pseudoscalar mesons is then, by (7b, d),

$$S(\epsilon, x)=41\frac{1}{\epsilon^3}F(M, x)\quad(\epsilon<M), \tag{15a}$$

$$S(\epsilon, x)=12.3\frac{1}{\epsilon^2}\int_\epsilon^\infty\left(\frac{\epsilon}{E}\right)^{\frac{1}{2}}\frac{\partial F(E, x)}{\partial E}dE\quad(\epsilon>M). \tag{15b}$$

[11] Euler and Heisenberg, Ergeb. d. exakt. Naturwiss. **17** (1938).

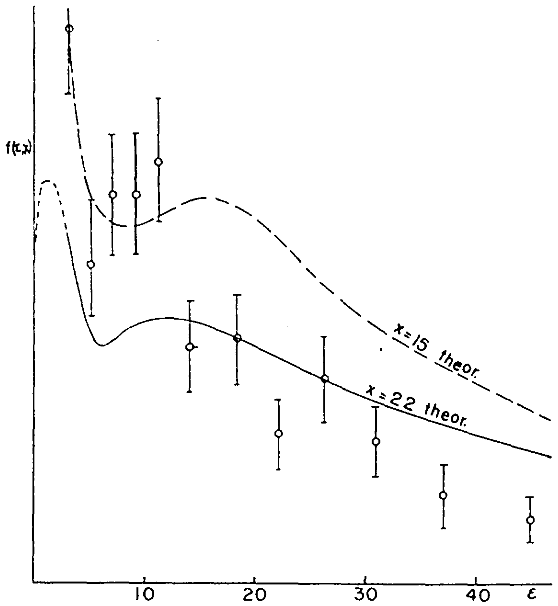

FIG. 1. Energy spectrum of pseudoscalar mesons at sea level, theoretical and experimental. Dotted curve: theoretical spectrum at a height of 4000 m (50 cm Hg).

Thus $f(\epsilon, x)$ becomes:

$$f(\epsilon, x)=41\left(\frac{\eta-x}{x}\right)^{b/\eta}\int_0^x \frac{\xi^{b/\eta}}{(\eta-\xi)^{3+b/\eta}}F(M, \xi)d\xi \qquad (\eta-\xi<M), \quad (16a)$$

$$f(\epsilon, x)=12.3\left(\frac{\eta-x}{x}\right)^{b/\eta}\int_0^x d\xi\int_{\eta-\xi}^{\infty} dE\left[\frac{(\eta-\xi)}{E}\right]^{\frac{1}{2}} \times\frac{\xi^{b/\eta}}{(\eta-\xi)^{2+b/\eta}}\frac{\partial F(E, \xi)}{\partial E} \qquad (\eta-\xi>M), \quad (16b)$$

$$\eta=\epsilon+x.$$

In (16) the limits 0, x have to be replaced: in (16a) by $\eta-M\cdots x$ and in (16b) by $0\cdots\eta-M$ if $\eta-M$ lies between 0 and x. After the integration η is again to be replaced by $\epsilon+x$. In (16) the expression (14) for $F(E, X)$ is to be inserted. Actually each integral (16) consists of two parts with two different values of α. It turns out, however, that for most values of ϵ, x, only one part of (14) is predominant, the other leading only to minor corrections; in particular for any depth $x\geqslant 2$ only that part of F with $\alpha=2.2$ is important.

The integrals (16) cannot be worked out exactly in closed form, but for most values of ϵ, x good approximations can be found. Equation (16a) is needed only for a few values of x and ϵ ($\epsilon<M$) and has been worked out numerically. In (16b) the logarithm occurring in F may be replaced by an average value. The chief contribution arises from small ξ (corresponding to the fact that the protons have extremely small range). For any value of $x>2$, and even with reasonably good approximation for $x=1$, ξ can be neglected against $\eta=\epsilon+x$ and the integration be extended to ∞ instead of x. The exact condition for this to be true is

$$x\gg\frac{\eta}{43\log 0.3\eta} \quad (17)$$

which is satisfied for all but extremely small x or extremely large ϵ. Equation (16b) then becomes, if only the high energy part of F, ($\alpha=2.2$) is used

$$f(\epsilon, x)=12.3A\frac{\Gamma(1+b/\eta)\Gamma(\alpha-b/\eta)}{\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2}-b/\eta)}\left(\frac{\epsilon}{x}\right)^{b/\eta}\eta^{-1-\alpha} \times(43\log 0.3\eta)^{\alpha-1-b/\eta} \quad \eta=\epsilon+x. \quad (18)$$

For $x<1$ the contribution of (16b) has only been estimated but here (16a) is much more important than (16b).

From (18) the asymptotic laws can immediately be read off:

(i) *Tail end of the energy spectrum.* Consider $\epsilon\gg x$ which for the lower part of the atmosphere also implies $\epsilon\gg b=13$.

$$f(\epsilon, x)d\epsilon\propto d\epsilon\cdot\epsilon^{-1-\alpha}(\log 0.3\epsilon)^{\alpha-1}. \quad (19)$$

Since $\alpha=2.2$ the energy spectrum falls off a little less rapidly (because of the logarithmic term) than $\epsilon^{-3.2}$.

(ii) *Great depths.* For underground measurements in dense materials the decay constant b should be put equal to zero. $f(\epsilon, x)$ is then a function of $\epsilon+x$ only. Integrating over ϵ we find for the total number of mesons:

$$I(x)=\int_{1/l}^{\infty} f(\epsilon, x)d\epsilon\propto x^{-\alpha}(\log 0.3x)^{\alpha-1}. \quad (20)$$

Thus the total intensity decreases like $x^{-\alpha}$, again apart from a logarithmic modification, and it is in this way that $\alpha=2.2$ was determined.

In Fig. 1 the theoretical energy spectrum is compared with the measurements by Blackett[12] for small and medium energies. It is seen that the shape of the theoretical curve is nearly the same as the experimental one. On the whole the theoretical spectrum falls off less rapidly than the experimental one. This was to be expected because of the crude way in which we have accounted for the cascade production of mesons as explained in Section II. The experimental minimum at $\epsilon=20$ does not appear in the theoretical

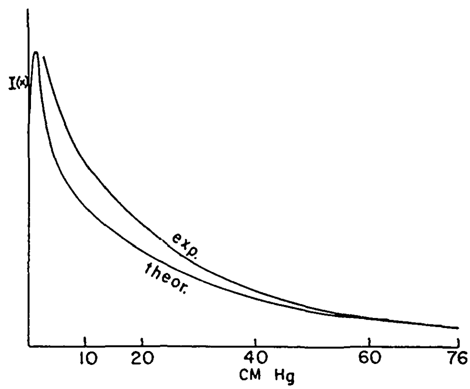

FIG. 2. Theoretical number of pseudoscalar mesons and experimental number of penetrating particles as function of height.

curve and is probably due to a special absorption process as was supposed by Blackett (production of proton pairs?). The maximum at $\epsilon=10$ is clearly indicated in the experiments as well as in the theoretical curve. It is difficult to trace the origin of this maximum as many factors combine to produce it. This maximum did not appear, however, in the calculations of Heisenberg and Euler, in which the exact source function was not used. For the low energy end ($\epsilon\sim3$) of $f(\epsilon, x)$ also the contribution from those very few very energetic protons which penetrate to the lower parts of the atmosphere and produce a large number of slow mesons is important. Hence the rise of the spectrum for small ϵ. Actually the curve will bend down to zero for $\epsilon<3$, as indicated in the figure, but in order to reproduce this part of the curve correctly it would be necessary to use the exact cross section for scattering instead of the asymptotic one (5).

There is another reason why we must expect to find few slow mesons for $\epsilon<3$, say. It will be seen in Section VII that a meson, once having reached an energy $\epsilon<4$, has a probability of 79 percent to be transformed into a neutretto before reaching the end of its range. This would remove a large fraction of mesons from the lower end of the spectrum.

The high energy tail end of the spectrum, i.e., the decrease of $f(\epsilon, x)\propto\epsilon^{-3.2}$ (modified by the logarithmic term) has already been adequately discussed by Euler and Heisenberg[11] and was found to agree with the measurements. In order to see how the energy spectrum varies with height we have also plotted the theoretical spectrum for $x=15$ (50 cm Hg). Slow mesons become relatively more predominant.

In Fig. 2 the total number of mesons

$$I(x)=\int_{1/f}^{\infty} f(\epsilon, x)d\epsilon$$

is plotted against height and compared with the recent measurements by Schein, Jesse, and Wollan.[13] Here also the agreement is as good as can be expected. The maximum occurs at an extreme height of $x=0.5$ corresponding to only 22 cm H_2O or 1.8 cm Hg. In the measurements at extreme heights presumably some of the primary protons also are included as "penetrating particles," but it is difficult to say how many protons have to be classed as "penetrating" because we do not know exactly the range of a proton with energy less than M. We estimate the range to be about 2 x units (=9 cm Pb) for $E=M$ and accordingly smaller for $E<M$. The theoretical curve of Fig. 2 refers to pseudoscalar mesons only. The two curves are normalized so as to agree at sea level.

IV. LATITUDE EFFECT

In the calculations of Section III no account has been taken yet of the fact that the primary energy spectrum is cut off for energies smaller than E_ϑ depending on the geomagnetic latitude ϑ. $E_\vartheta=22$ for $\vartheta=50°$ and $E_\vartheta=150$ for $\vartheta=0°$ (equator). The correction can easily be made. From (16) the contribution from those protons which, at the top of the atmosphere had an

[12] Blackett, Proc. Roy. Soc. **159**, 1 (1937), **165**, 11 (1938). Compare also: Jones, Rev. Mod. Phys. **11**, 235 (1939); Hughes, Phys. Rev. **57**, 592 (1940).

[13] Schein, Jesse, and Wollan, Phys. Rev. **59**, 615 (1941).

energy $<E_\vartheta$ has to be subtracted, i.e., for which

$$\xi+E/43\log 0.3E<E_\vartheta/43\log 0.3E_\vartheta.$$

It has turned out that for "normal latitudes" 50°, say, the correction is negligible throughout the atmosphere. This means that *no increase of intensity is to be expected for more northern latitudes,* not only at sea level but also at any height above sea level, except the very top of the atmosphere, and even there the increase is small. This explains the well known *knee of the latitude effect* at 50°.

The reason is easily to be understood. Protons stop producing mesons for $E<M=10$.

Even the contribution from protons with energy between 10 and 22 is very small on account of their smaller energy (in fact even smaller than assumed in this paper where the asymptotic law for meson production for $E\gg 10$ has been extended to $E=10$). Thus the reason for the knee of the latitude effect is simply the fact that protons of energy less than $E_{50°}$ cease to produce, or produce very few mesons. The same applies to the soft component. We shall see in Section V, that the soft component is probably due to the decay of transverse mesons in the high atmosphere, the former being also produced by the same primary protons. Thus the same knee of the latitude effect is to be expected for electrons, which is also found experimentally.[14] The fact that the soft component shows the same latitude knee as the hard component supports the assumption that both are produced by the same primary radiation, i.e., by protons, as explained by our theory.

TABLE II. Latitude effect. $(I_{50°}-I_{\mathrm{equ}})/I_{50°}$.

x	mesons 22 (sea level)	15	2	1	soft component $x=2$
theor.	0.13	0.13	0.18	0.33	0.62
exp.	0.1	—	—	—	0.75

For latitudes $<50°$ an appreciable latitude effect is to be expected. We calculate the *latitude defect* $\Delta_\vartheta f(\epsilon,x)$, i.e., the difference of $f(\epsilon,x)$ between the North Pole and the latitude ϑ. As mentioned above $\Delta_\vartheta f(\epsilon,x)$ is negligible for $\vartheta>50°$. From (16) those contributions have to be subtracted for which $\xi+E/43\log 0.3E<E_\vartheta/43\log 0.3E_\vartheta$. Thus $\Delta_\vartheta f(\epsilon,x)$ is given by (for $\eta-\xi>M$):

$$\Delta_\vartheta f(\epsilon,x)=12.3\left(\frac{\eta-x}{x}\right)^{b/\eta}\int_0^{\xi_\vartheta}d\xi\int_{\eta-\xi}^{E_{\vartheta\xi}}dE \times\left[\frac{(\eta-\xi)}{E}\right]^{\frac{1}{2}}\frac{\xi^{b/\eta}}{(\eta-\xi)^{2+b/\eta}}\frac{\partial F(E,\xi)}{\partial E},\qquad(21)$$

with

$$\frac{E_{\vartheta\xi}}{43\log 0.3E_{\vartheta\xi}}+\xi=\frac{E_\vartheta}{43\log 0.3E_\vartheta},$$

$$\xi_\vartheta=\frac{E_\vartheta}{43\log 0.3E_\vartheta}-\frac{\eta}{43\log 0.3\eta}$$

$$\eta=\epsilon+x.$$

(In the condition for ξ_ϑ, ξ has been neglected compared with η). $\Delta_\vartheta f$ is, of course, zero whenever in (21) an upper limit is smaller than the lower limit. A similar integral holds for $\eta-\xi<M$.

We have worked out the integrals numerically. Here the change of α at $E=100$ is important. Integrating over ϵ we find the latitude effect for the total number of mesons:

$$I_{50°}-I_\vartheta=\int_{1/f}^{\infty}\Delta_\vartheta f(\epsilon,x)d\epsilon.$$

The results are given in Table II. The latitude effect is fairly constant for low levels and only increases for $x<2$. The figure for sea level is in good agreement with the experiments, whilst at greater elevations no measurements seem to have been made yet for the hard component. For the figure concerning the soft component see Section V.

V. THE TRANSVERSE MESONS AND THE SOFT COMPONENT

In addition to pseudoscalar mesons the primary protons produce a large number of transverse mesons which we assume to decay at once. If the mesons have energy $\epsilon\gg 1$—which is nearly always the case—the decay electrons are emitted in the forward direction and take up with equal probability, any amount of energy between 0 and ϵ. Thus a large number of slow and fast electrons are produced. The fast ones multiply by cascade multiplication and it will be seen that a satisfactory account of the soft component can be

[14] Carmichael and Dymond, Proc. Roy. Soc. 171, 321 (1939).

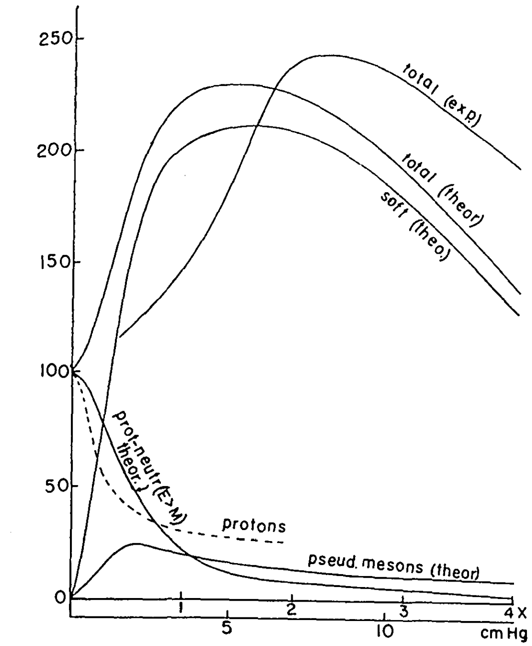

FIG. 3. Cosmic-ray intensities in the top part of the atmosphere for 100 primary protons. Soft component (theor.), pseudoscalar mesons (theor.), primary protons and neutrons (theor.), and total intensity (theor. and exp.).

given in this way. Let $F(\epsilon, x)$ be the number of protons at depth x with energy larger than ϵ. [F is given by (14).] Then the number of electrons produced at x in the energy interval $d\backepsilon$ is

$$d\backepsilon \int_{\backepsilon}^{\infty} F(\epsilon, x)\Phi_{tr}(\epsilon)\frac{d\epsilon}{\epsilon}.$$

Let now $C(\backepsilon, \xi)$ be the cascade multiplication function (i.e., the number of electrons produced in a depth ξ by one primary electron of energy $\backepsilon$), the total number of electrons at depth x is

$$Z_{tr}(x) = \int^{x} d\xi \int_0^{\infty} d\backepsilon \int_{\backepsilon}^{\infty} \frac{d\epsilon}{\epsilon}\Phi_{tr}(\epsilon) \times F(\epsilon, \xi)C(\backepsilon, x-\xi). \quad (22)$$

Equation (22) can be worked out partly by making suitable approximations and partly numerically. For $\Phi_{tr}(\epsilon)$ the expressions (7a, c) and for F (14) are to be used. For the high atmosphere ($x<4$, say) practically only the low energy part of $F(\alpha=1.3)$ is important. For C we have used the figures given by Arley.[15]

To obtain an idea of the effect to be expected we consider large values of x. ($x>5$, say.) We can then make the following approximations: $F(\epsilon, \xi)$ decreases rapidly with ξ for any ϵ of importance. $C(\backepsilon, x-\xi)$ can be replaced by $C(\backepsilon, x)$ and the integral over ξ extended to infinity instead of to x. This amounts to assuming that all the mesons produced in the atmosphere are practically produced at the very top and, decaying at once, produce a "primary" electron spectrum $N(\backepsilon)d\backepsilon$ say, which later produces cascade effects. $N(\backepsilon)$ is given by

$$N(\backepsilon) = \int_{\backepsilon}^{\infty} \Phi_{tr}(\epsilon)\frac{d\epsilon}{\epsilon}\int_0^{\infty} F(\epsilon, \xi)d\xi. \quad (23)$$

We divide the range of energy $\backepsilon$ into 4 regions, and find for $N(\backepsilon)$ (considering logarithmic terms as constants, and using the low energy part of F only):

$$\begin{aligned} N(\backepsilon)d\backepsilon &= 33B\frac{d\backepsilon}{\backepsilon^2}\left(\frac{43 \log 0.3\backepsilon}{\backepsilon}\right)^{0.3} && (\backepsilon > E_\vartheta = 22) \\ &= Bd\backepsilon\left\{\frac{61}{\backepsilon^2} - \frac{99}{\backepsilon 43 \log 0.3\backepsilon} + 0.1\right\} && (E_\vartheta > \backepsilon > M = 10) \\ &= Bd\backepsilon\left\{\frac{142}{\backepsilon^3} + 0.34\right\} && (10 > \backepsilon > 1/f = 3) \\ &= 5.6Bd\backepsilon && (\backepsilon < 3). \end{aligned} \quad (24)$$

Equation (24) plays the role of the primary electronic energy spectrum responsible for the soft component. $N(\backepsilon)$ decreases approximately like $1/\backepsilon^2$, at any rate in the high energy region. But there are also a large number of soft electrons present. Apart from the latter $N(\backepsilon)$ is indeed very similar to the primary electron spectrum deduced first by Nordheim and Heitler[16] from the intensity curve of the soft component on grounds of the cascade theory and later found by Bowen, Millikan, and Neher[17] by latitude measurements. We therefore expect that (24) should give rise to an intensity curve for the soft component very similar to the observed one. The effect of the low energy electrons with small range also present in

[15] Arley, Proc. Roy. Soc. **A168**, 519 (1938). In the cascade theory different units for the thickness of matter traversed are usually used. For air or water one cascade unit is about $\frac{2}{3}$ of our x units.

[16] Nordheim, Phys. Rev. **51**, 1110 (1937); Heitler, Proc. Roy. Soc. **161**, 261 (1937).

[17] Bowen, Millikan, and Neher, Phys. Rev. **52**, 80 (1937) and **53**, 855 (1938).

(24) will be that the intensity curve does not fall down very steeply near the top but becomes in fact rather flat.

For the most interesting part of the atmosphere ($x=2, 3$, maximum of the Regener-Hotzer-curve) the use of (24) is too crude an approximation. In fact quite a number of low energy mesons are still produced at levels $x>2$. We have therefore used (22) directly and evaluated the integrals. We find that on the average one primary proton produces about 1.5 electrons at $x=2$ or 3 in this way.

In addition to the electrons produced by the decay of transverse mesons there are also electrons produced by the decaying pseudoscalar mesons. Their number is smaller but not negligible compared with the number of electrons due to transverse mesons. We easily find:

$$Z_{ps}=\int_0^x d\xi\int_0^\infty d\exists\int_\exists^\infty \frac{d\epsilon}{\epsilon}\frac{b}{\xi}f(\epsilon,\,\xi)C(\exists,\,x-\xi),\quad(25)$$

where $f(\epsilon,\ \xi)$ is the energy spectrum of pseudoscalar mesons as calculated in Section III. The result is for instance the following for $x=2$ and for one primary proton: $Z_{tr}=1.5$, $Z_{ps}=0.6$, $Z_{tr}+Z_{ps}=2.1$. Thus one primary proton produces in the average altogether about 2 electrons at the Regener-maximum, roughly $\frac{1}{3}$ of which are due to pseudoscalar and $\frac{2}{3}$ to transverse mesons. This is in very good agreement with the experiments. In Fig. 3 we have plotted the intensities of all the cosmic-ray components for the top part of the atmosphere (normalized for 100 incident protons). The agreement with the Regener-Hotzer-curve for the total intensity is as good as can be expected. The position of the maximum is, however, slightly shifted towards greater heights, than is found experimentally. The reason is probably that we have been overrating the production of slow mesons (and therefore slow electrons) by protons of energy just above M.

The latitude effect of the soft component is very large. We have calculated it for $x=2$ and included the result in Table III. The agreement with the measurements by Bowen, Millikan, and Neher[17] is good. The latter measurements refer to the total intensities, of which, however, the soft component is far the strongest.

There is one experiment which is apparently in contradiction to our assumption that the soft component originates from primary protons. Whilst the mesons in the lower atmosphere show an east-west asymmetry as big as the latitude effect proving that they originate from positive primaries no such east-west asymmetry has been found for the soft component in the high atmosphere although the latter is also—and very strongly—latitude sensitive.[18] If this result is taken at its face value it would mean that the soft component originates from equal numbers of positive and negative primary particles. We must remember, however, that the soft component is, if our picture is correct, created by the primary protons in a rather indirect way: The protons first produce mesons which decay into electrons, which then multiply by means of the cascade process. An east-west asymmetry can only be expected if in each of these three processes the particles are always strictly emitted in the forward direction with no appreciable angular straggling. This is surely the case if the secondary mesons have energies large compared with M, and if these mesons then decay while still having an energy large compared with one.

In the high atmosphere, however, low energy particles are predominant. Indeed, as it appears from the calculations of this section, most of the electrons in the high atmosphere originate from mesons with energy less or not much greater than M. These mesons are bound to be emitted in all directions with only a slight favoring of forward directions. Even for the more energetic particles the predominance of forward directions will be impaired to some extent by the many processes which have to take place to produce the soft component. It might, therefore, well be that the east-west effect is completely masked by the large angular straggling of the soft component.

TABLE III. Cosmic-ray intensities (100 primary particles).*

	soft comp. $x=2$	soft+hard $x=2$	mesons sea level	energetic protons and neutrons sea level
theor.	208	220	1.7	1/500
exp.	—	225	4	1/2000

* The column soft+hard ($x=2$) does not include the primary protons at this depth. Their number should be added to the theoretical figure 220 but is probably less than 20 (cf. Section VI).

[18] Johnson, Phys. Rev. 56, 219 (1939).

For a judgment of this theory it must be remarked that, once the number of primary protons is normalized, and their energy spectrum given, the theory allows one to calculate the *absolute intensities* of all cosmic-ray components at each depth without further adjusting our intensities to those of the experiments. It is clear that the values of the intensities depend entirely on the constants occurring in the theory, especially on the nuclear coupling constants g, f. All these constants are either determined from nuclear facts (g, f) or by direct experiments (meson mass, decay constants). It is interesting to compare the intensities obtained by our theory with the experimental ones. Table III gives a selection of data. The absolute number of mesons at sea level is about twice of that calculated. This is quite what we had to expect as was remarked in Section II. (The last column refers to measurements by Jannossy, discussed in Section VI.)

VI. PROTONS, NEUTRONS, MESON-SHOWERS

The primary protons entering the atmosphere are very quickly slowed down to energy ~ 10. Since their energy loss is mainly due to producing charged mesons we must expect that half of these "protons" have in fact become neutrons after having travelled only a very small distance. In Fig. 3 we have plotted the number of these protons or neutrons with energy $>M$ as a function of depth for the primary energy spectrum (13), the number simply being given by the relation (14). It is seen that their intensity decreases very rapidly. For a discussion of the experiments, however, we would rather like to know the number of protons (not neutrons) with all energies, including $E<M$. This number can only be guessed at present, since we do not know at what rate protons lose their energy once E is less than M. All we can say is that the number of protons drops very quickly to half the initial value (the other half have become neutrons). We might estimate that the energy loss of a proton becomes say 10 times smaller as E becomes less than M (cf. Section II)—a figure which can only be a very crude guess. The intensity curve thus obtained is dotted in Fig. 3. The recoil protons and neutrons are not included in these curves. The curve marked "total" does not include these protons except those at the very top of the atmosphere.

After having reached an energy $E<M$ the protons and neutrons have a much larger range than the distance that a proton or neutron can travel while having an energy $>M$. It is therefore not surprising that a large number of slow protons and neutrons are found in the upper atmosphere. Fluctuations of the range are large, since energy is lost in large portions, thus a small but appreciable fraction of them may even manage to travel through a considerable part of the atmosphere. In addition there are numerous fast and slow recoil protons. The experiments show a rapid increase of slow protons and neutrons with height, though their number is still very small at a height of 4000 m compared for instance with the number of mesons. A quantitative discussion is not yet possible. A certain fraction of these slow protons and neutrons are recoil products or are produced by photoelectric nuclear disintegrations, etc.

In addition to these slow particles we must expect, in any part of the lower atmosphere, a very small number of very energetic protons and neutrons, namely, the primaries themselves which have sufficient energy to travel such large distances. Let us estimate their number, at sea level. The energy necessary to penetrate through the atmosphere is at least 7×10^{11} ev. The primary energy spectrum (13) is valid for E up to 1000 (10^{11} ev). For higher energies we know from the measurements at great depths that α increases with increasing energy. Thus for the energies in question we may assume $\alpha=3$, for $E>1000$, say. The number of primary protons at sea level is then given by $A'x^{-3}$, $x=22$ where A' is determined by A [Eq. (13)] and the condition that the spectrum is continuous at $E=1000$, thus $A'=23$ if A, B, A' are normalized for 100 incident protons. Thus we find 1/500 energetic protons or neutrons at sea level for 100 primary protons at the top of the atmosphere.

This figure may in fact be still too large, for the following reason: As was mentioned in Section II, our formula for the energy loss of a fast proton is no longer valid for energies $>5\times10^{11}$ ev, because then other modes of energy

TABLE IV. Number of mesons (with $\epsilon>10$) produced by energetic proton or neutron.

E_0	100	300	1000
$x_{E_0,M}$	0.9	2.1	4.3
$N_{ps}+N_{tr}$	3.3	6.3	11.0

loss become appreciable.[19] Thus still higher energies may be required for a proton to penetrate to sea level making their number still smaller. From measurements discussed below Janossy found that the number of protons or neutrons at sea level is about 1/12,000 of the total number of cosmic-ray particles at sea level. Putting the latter value to be about 6 (for 100 incident protons) the experiments would give the number of these energetic protons and neutrons as 1/2000 which may be considered to be in reasonable agreement with our theoretical estimate 1/500.

When these energetic particles travel through matter they will produce a number of mesons in succession thus producing a *meson-shower*. Showers consisting of penetrating particles have been observed by several authors.[20] The most extensive measurements are due to Janossy: A radiation whose intensity was found to be 1/12,000th of the total cosmic-ray intensity at sea level was found to produce showers consisting of penetrating particles which are certainly mesons with energy $\epsilon>10$. The transition curve of these showers in lead reaches saturation at about 5 cm Pb (about 1 x unit). About a third of the primary radiation producing these showers consists of neutral particles. Janossy found, however, that for showers produced by the neutral primaries saturation is only reached after 10 cm of Pb. Each shower consists of an average number of 2–6 *recorded* penetrating particles. The actual number of particles in each shower may be larger.

We can only give a very preliminary discussion of these experiments. The most obvious explanation is that the very energetic protons and neutrons expected from our theory are responsible for these showers. There should be about equal numbers of protons and neutrons. Janossy found about a third to be neutral particles. Let $E\gg M$ be the average energy of the protons and neutrons and $x_{E,M}$ their range as given in Table I (i.e., the distance travelled until the energy is M).

The number of mesons with energy >10 produced is then according to (7):

$$N_{tr}=21\int_M^{E_0}\frac{dE}{-dE/dx}\int_{10}^{E}\frac{d\epsilon}{\epsilon^2}\sim 2.1x_{E_0,M},$$

$$N_{ps}\sim 15.5E_0^{-\frac{1}{2}}x_{E_0,M}.$$

Thus the total number of mesons produced is for various energies E_0 (from Table I) given by Table IV.

These values cannot, however, be compared directly with the experiments. It has kindly been pointed out to us by Janossy, that the saturation point of the transition curve is not at all identical with the range of the primary radiation but marks the point where enough mesons are produced to be recorded as a shower. This figure is about 2–3. A further increase of the lead thickness will only increase the size of the showers but not the number of the showers. Experimentally, we find, therefore, that 2 or 3 mesons are produced in the first 5 cm Pb or in the first x unit. This is in very good agreement with Eq. (26), the theoretical number of mesons produced in $x=1$ being $2.1+15.5E_0^{-\frac{1}{2}}$. We do not know E_0 exactly, but it certainly is rather large (>100). For greater thickness of Pb, Table IV shows that the size of the showers increases very much and even large showers are quite possible. On the other hand it is difficult to understand why neutrons should give rise to showers with a larger saturation thickness, than that for showers produced by protons. The processes considered in this paper lead to no explanation of this asymmetry.

There are, however, a number of points, omitted so far in the present theory, which will have to be considered before a final discussion of the penetrating showers can be given:

(i) As was mentioned in Section I, there are further modes of energy loss by a fast proton, expected from the present theory, including also a higher rate of production of transverse mesons,

[19] The limit of validity for the energy loss formula is higher than that for the scattering cross section because the former is an integral of the latter.

[20] Wataghin, de Souza Santos, and Pompeia, Phys. Rev. **57**, 61 and 339 (1940). Janossy, Proc. Roy. Soc. **179**, 361 (1942); Janossy, McCusker, and Rochester, Nature **148**, 660 (1941).

which become important if the energy exceeds the value 5000, say. The primary protons would then hardly have a chance at all to reach sea level. The protons and neutrons responsible for the penetrating showers must be the fast recoil particles mentioned above which naturally have also smaller energies than the primaries. (If this argument holds the last column in Table III has no meaning.) Some of the processes taking place at extremely high energies depend on the Coulomb field of the proton which might perhaps indirectly account for the difference of the behavior of the protons and neutrons.

(ii) Throughout this paper we have neglected the occurrence of multiple processes. It is indeed possible that a fast proton creates mesons, not only one by one in succession, but several mesons in one elementary act. We obtain the rate of occurrence of this event by multiplying the equivalent meson spectrum of a fast proton by the cross section for the splitting up of a meson into several mesons instead of by the scattering cross section (5). The splitting up processes have been calculated in *I* for the case when all energies concerned are smaller than M. In this case it has been found, indeed, that the splitting-up cross section for such multiple processes is always at least 10 times smaller than that for ordinary scattering (which is the reason why multiple processes were neglected in this paper). It may well be—but this could only be decided by further, rather difficult, calculations—that the ratio is more in favor of multiple processes if the energy of the proton is $>M$. If this should turn out to be true, the range of the fast protons would be diminished and the number of mesons increased.

(iii) We have always assumed that the particles in the nuclei of the matter traversed act independently. It might very well be that the nuclear fields of the nuclear particles overlap and interfere (as their Coulomb field does) producing an effect which increases with a different power of the atomic weight than the first power. This might lead to some changes of our theoretical results for heavy materials but hardly for air.

(iv) So far we have considered only protons and neutrons as primary agents. It is to be expected that mesons also could create penetrating showers themselves, by the very splitting up process mentioned in (ii). Again no quantitative discussion is possible unless the calculations are performed for energies $>M$. A crude estimate on grounds of the non-relativistic calculations shows indeed that the rate of occurrence of this process is of the observed order of magnitude.

Mesons have also been found to occur in very large cascade showers. This is easily understood. According to our theory, a large cascade shower owes its origin to a very energetic primary proton emitting an energetic transverse meson which, by decaying, produces the very energetic electron responsible for the cascade. Naturally, the primary proton will also produce numerous other mesons, including pseudoscalar ones, during its path through the atmosphere. These mesons will, of course, occur together with the cascade shower. In addition, energetic light quanta, have a small chance of creating mesons themselves by process (1) instead of producing an electron pair. The cross section for (1) was found in *II* to be $\sqrt{2}\pi^2 ef/\epsilon^2$ for pseudoscalar mesons, if $\epsilon<M$. Comparing this with the cross section for pair production $\sim 15Z^2/137$ we find that the chance for a light quantum producing a meson with $\epsilon=10$, say, is in air 1/200. Since in a big cascade shower at the point of its maximum several thousand light quanta may occur, quite a number mesons are to be expected. The mesons accompany the cascade shower but are much more slowly absorbed; the ratio of the number of mesons to that of soft particles may therefore be quite appreciable at sea level.

VII. NEUTRETTOS

According to our theory neutrettos should be very frequent particles in cosmic radiation. The total number produced is half the total number of charged mesons produced. We know nothing, however, about their absorption and possible β-decay nor does it seem that they could produce any noticeable secondary effects. It is true, though, that in colliding with a nuclear particle a neutretto can be transformed into a charged meson. But, as mentioned in Section II, the cross section is very small at high energies decreasing like ϵ^{-6} and it is unlikely that this effect will ever be observed. Slow neutrettos must be very rare because there is nothing that would slow down a once fast neutretto.

There is, however, one observable effect, by which the existence of neutrettos could be established experimentally, namely, the transformation of a charged meson into a neutretto. At high energies $\epsilon \gg 1/f \sim 3$ again the cross section Φ^0 for this transformation is negligible ($\Phi^0 \propto \epsilon^{-6}$) compared with the cross section for scattering ($\propto \epsilon^{-2}$). This is, indeed, an important result of our theory because it explains why all attempts at detecting this transformation[21] have failed whereas the anomalous scattering of mesons has been observed. This is no longer true for energies of the order of magnitude $\epsilon \sim 3$. The cross section becomes then appreciable and, as we shall see, big enough to lead to observable effects.

The method for calculating Φ^0 as a function of ϵ is described in detail in *III* and the calculations are carried out up to a point where numerical evaluation is possible. The exact formula would be a rather lengthy and complicated expression because no approximation can be made in this case except that we can assume ϵ to be small compared with M. We shall not give the formula here. Φ^0 is, for very small energies, proportional to $p^2=\epsilon^2-1$, rises to a maximum at $\epsilon=2$ and decreases rapidly for higher values. The value of Φ^0 at the maximum ($\epsilon=2$) is in our units $\Phi^0_{\max}=0.56$. This refers to the transformation of a pseudoscalar charged meson into neutrettos of all polarizations (longitudinal, transverse, or pseudoscalar).

We shall calculate the total probability for a meson to be transformed into a neutretto while traveling through matter, starting with a high initial energy $\epsilon \gg 1/f$ until it is stopped. If $-\partial\epsilon/\partial x$ is the energy loss by ionization this probability is

$$w=\frac{NA}{2}\int_1^\infty \frac{\Phi^0(\epsilon)}{-\partial\epsilon/\partial x}d\epsilon, \qquad (27)$$

where N is the number of nuclei per cm^3, A the atomic weight. The factor $\frac{1}{2}$ is due to the fact that a positive meson can only be transformed into a neutretto if it collides with a neutron and a negative meson only if it collides with a proton. The average number of "active" nuclear particles is therefore $A/2$. We have worked out (27) by numerical integration for Pb and found[22]

$$w=0.79. \qquad (28)$$

For any other materials w will not be very different because $\partial\epsilon/\partial x$ is nearly proportional to NZ which differs only slightly from $NA/2$. Practically the whole contribution to the integral (27) arises from energies between $\epsilon=\frac{3}{2}$ and $\epsilon=4$.

We see that, once a meson has reached an energy as small as 4, the chance is so great that it will be transformed into a neutretto. Only 21 percent of the mesons reach the end of their range as charged mesons. It may very well be that this is one of the reasons why so very few slow mesons with $\epsilon<2$ occur in cosmic radiation.

The length of path a meson has to travel in order to lose energy from $\epsilon=4$ to $\epsilon=\frac{3}{2}$ is about 1 km of normal air or 18 cm of Pb. While traveling through the gas of a cloud chamber the chance is so negligible that the effect could be observed. But, if mesons are stopped in a thick block of lead an appreciable fraction of the mesons should not reach the end of their range but should be transformed *earlier* into neutrettos. This is important for the type of experiments carried out by Rassetti.[23] Rassetti found that if mesons are stopped in 10 cm of aluminum (equivalent to 2.8 cm of Pb) only a certain fraction (42 ± 15 percent) give rise to decay-electrons. The effect was interpreted by Rassetti in the following way: Once a meson is stopped entirely and retains only an energy of a few electron volts it may either decay or else be captured by a nucleus giving its rest energy to the nucleus. Only negative mesons, however, can be captured in this way, since a positive meson with such a small energy could not come in contact with the nucleus. Calculations referred to by Rassetti indicate indeed, that for such slow negative mesons the capture-probability is larger than the decay probability. It may very well be, that transformation into neutrettos as well as nuclear capture is responsible for the fact that only a fraction of the mesons are really

[21] Cf., for instance, Nishina and Birus, Sci. Pap. Inst. Phys. Chem. Res. Tokyo **38**, 360 (1941).

[22] For this result the numerical values (2) for the coupling constants have been used.

[23] Rassetti, Phys. Rev. **60**, 198 (1941).

decaying. In the above experiment 12 percent of the mesons should be transformed into neutrettos before being stopped. Of the remaining 88 percent half should be captured leaving 44 percent decaying into electrons.

The experiments are not accurate enough to decide whether the number of decay electrons is actually appreciably less than 50 percent. The actual percentage should depend on the thickness of the absorber.

VIII. CONCLUSIONS

On the whole the theory was seen to give a satisfactory account of all the chief cosmic-ray phenomena. Although a number of points, especially the cascade production of mesons (cf. Section II), the east-west effect of the soft component and the penetrating showers, must await further investigation, it is probable that this theory is fundamentally correct. The conclusions that can be drawn from our results are twofold: (i) The theory of damping used as a basis for all our calculations will be a correct part of future quantum electrodynamics or at least a good approximation. (ii) Cosmic-ray mesons are in fact identical with the quanta predicted by Yukawa which are responsible for the nuclear fields. It may perhaps be too early to say that the special form of the meson theory used in this paper (the one suggested by Møller and Rosenfeld) is the correct form of the meson theory. We have carried out all the calculations also with a different form of the theory (assuming that only charged mesons exist) and found the agreement with cosmic-ray experiments less good, though this form of the theory cannot be wholly excluded. (The energy loss of fast protons would be three times smaller.) It is satisfactory that the form of the meson theory which gives the best account of the nuclear forces also agrees best with cosmic-ray facts. Especially, the assumption of both pseudoscalar and transverse mesons is strongly supported by the fact that all components of cosmic radiation can be explained as owing their origin to one kind of primary particles, i.e., protons.

We are very much indebted to Dr. L. Janossy for a most useful comment on this paper and also for communicating some of his experimental results to us before publication.

VII.—Quantum Mechanics of Fields. I. Pure Fields.

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Introduction

The difficulties met in the usual treatment of quantised field theories seem to us somewhat similar to those which occurred in Bohr's semi-classical quantum mechanics of particles. In this theory the orbits were described by Fourier series in the time; there was no exact correspondence between the periodic terms of this series and quantum transitions, but only an approximate one for terms of high order. Matrix mechanics considers not the Fourier series, but the single terms which are generalised into matrix elements having not one but two indices. This generalisation is founded on Ritz's combination principle.

In the existing field theories each field component is considered as a matrix for each point of space, and, as a function of the latter, can be expanded into an ordinary Fourier series, the coefficients of which are matrices, each element having two indices, which refer to the excitation of the corresponding vibration. But the three indices (k_x, k_y, k_z forming the wave vector $\mathbf{k}$), which enumerate the Fourier terms, are not included in the matrix indices. This situation with regard to space is therefore very similar to the pre-quantum mechanical treatment of particles by Bohr with regard to time. We suggest a similar remedy. According to relativity, space and time, or the conjugate quantities momentum and energy (wave-vector and frequency), have to be treated on the same footing, and Ritz's combination principle for frequencies has to be supplemented by one for wave vectors. Just as in the case of the transition from Bohr's theory to quantum mechanics we deny that the Fourier series for the field components have any exact significance; and we consider only single terms, but replace these by matrix elements. Each field component is thus as a whole represented by a matrix, just as a co-ordinate of a particle is represented by a matrix in ordinary quantum mechanics.

This programme seems to imply marked deviation from the usual procedure where the total energy and total momentum are obtained as integrals over space. However, in the application to all specific fields the boundary conditions are always chosen in such a simple way (very large box) that this integration means nothing more than multiplication by the total volume. We therefore suggest to use the densities of momentum and energy (multiplied by the volume) in the same way as energy is used in particle mechanics. These densities taken from classical theory are always bilinear in the field components, and as multiplication now means matrix multiplication the densities really depend on the field as a whole just as the total momentum-energy components in the older theory. Thus the discrepancy between the new procedure and the old is not so marked as it first appears, and we shall show below that it leads to satisfactory results. The advantages of the new theory are these: the formalism is much simpler and of the same type as ordinary quantum mechanics. Further, the new theory can be applied to non-linear field equations, *i.e.* to energy-densities which are very general functions of the field components, without any mathematical complication. This is not so in the usual theory; if here energy-density is a relatively simple function in the space representation, it is in general extremely involved in the momentum representation (as the Fourier coefficient of a function of functions is a complicated expression of the Fourier coefficients of the latter). But it is a well-known theorem that a matrix equation is unchanged by transforming to another representation.

The fields considered below are classified according to their transformation properties; scalar field, vector field and spinor field are treated for the general (non-linear) case. The scalar-meson field, the vector-meson field, the linear and non-linear electromagnetic field, and Dirac's electron field are contained as special cases. In the classical treatment the field variables are in general taken to be complex, so that what is usually referred to as a real field is obtained as a special case by a supplementary condition. In the quantum treatment the supplementary condition restricts the class of wave functions admitted.

1. Scalar Field: Classical Treatment

We shall explain our method with the help of a simple example, the scalar field.

We use first the notation of tensor calculus, considering all quantities as dimensionless. Let x_α be the independent variables, which are considered as real, $x_\alpha^* = x_\alpha$ (although x_4 is proportional to the time multiplied by the relativity factor $\iota = \sqrt{-1}$. This is to be distinguished from the imaginary unit i, which changes sign on passing to the conjugate, $i^* = -i$). Let ϕ be the potential, ϕ^* its conjugate, and

$$\phi_\alpha = \frac{\partial \phi}{\partial x_\alpha}, \qquad \phi_\alpha^* = \frac{\partial \phi^*}{\partial x_\alpha}. \tag{1.1}$$

Let $\Lambda(\phi, \phi^*; \phi_1, \phi_1^*; \phi_2, \phi_2^*; \ldots)$ be the Lagrangian density, which is assumed to be real, $\Lambda = \Lambda^*$; and

$$\Phi^* = \frac{\partial \Lambda}{\partial \phi}, \quad \Phi = \frac{\partial \Lambda}{\partial \phi^*}; \qquad \Phi_\alpha^* = \frac{\partial \Lambda}{\partial \phi_\alpha}, \quad \Phi_\alpha = \frac{\partial \Lambda}{\partial \phi_\alpha^*}. \tag{1.2}$$

Then the field equations are

$$\sum_\alpha \frac{\partial \Phi_\alpha}{\partial x_\alpha} = \Phi, \qquad \sum_\alpha \frac{\partial \Phi_\alpha^*}{\partial x_\alpha} = \Phi^*. \tag{1.3}$$

From these a set of conservation laws can be deduced:

$$\sum_\beta \frac{\partial T_{\alpha\beta}}{\partial x_\beta} = 0, \qquad T_{\alpha\beta} = \phi_\alpha \Phi_\beta^* + \Phi_\beta \phi_\alpha^* - \Lambda \delta_{\alpha\beta}, \tag{1.4}$$

where $T_{\alpha\beta}$ are the components of the energy-momentum tensor;† these are symmetric if Λ is relativistically invariant. To show this let us apply an infinitesimal Lorentz-transformation, the components ϕ_α being transformed like a four-vector, *i.e.*

$$\phi_\alpha' = \phi_\alpha + \epsilon \sum_\beta f_{\alpha\beta} \phi_\beta \tag{1.5}$$

with real constants $f_{\alpha\beta} = -f_{\beta\alpha}$ and ϵ small. Regarding Λ as a function of ϕ_α, ϕ_α^* besides the scalars ϕ, ϕ^*, we obtain for the transformed Λ:

$$\Lambda' = \Lambda + \epsilon \sum_{\alpha,\beta} \left(f_{\alpha\beta} \phi_\beta \frac{\partial \Lambda}{\partial \phi_\alpha} + \frac{\partial \Lambda}{\partial \phi_\alpha^*} f_{\alpha\beta} \phi_\beta^* \right),$$

i.e.

$$\Lambda' - \Lambda = \epsilon \sum_{\alpha,\beta} f_{\alpha\beta} (\phi_\beta \Phi_\alpha^* + \Phi_\alpha \phi_\beta^*). \tag{1.6}$$

If now Λ is an invariant the right-hand side of (1.6) must vanish. Since $f_{\alpha\beta}$ is skew-symmetric, $\phi_\beta \Phi_\alpha^* + \Phi_\alpha \phi_\beta^*$ must be symmetric and hence also $T_{\alpha\beta}$.

By specialising the transformation (1.5) to be a rotation in space, (1.6) shows that if Λ is invariant in space then $T_{ik} = T_{ki}$ ($i, k = 1, 2, 3$). Conversely, by (1.6), the invariance of the Lagrangian follows from the symmetry of the energy-momentum tensor.

If Λ is also gauge-invariant (with respect to the transformation $\phi \to \phi e^{i\gamma}$, γ an arbitrary real constant), it satisfies the further identity

$$i\left(\phi \frac{\partial \Lambda}{\partial \phi} + \sum_\alpha \phi_\alpha \frac{\partial \Lambda}{\partial \phi_\alpha^*} - \frac{\partial \Lambda}{\partial \phi^*} \phi^* - \sum_\alpha \frac{\partial \Lambda}{\partial \phi_\alpha^*} \phi_\alpha^* \right) = 0,$$

† The order of the factors in (1.4) and later equations, although so far arbitrary, is chosen with regard to later application to non-commuting quantities.

i.e.

$$i\left(\phi\Phi^* + \sum_\alpha \phi_\alpha \Phi^*_\alpha - \Phi\phi^* - \sum_\alpha \Phi^*_\alpha \phi^*_\alpha\right) = 0. \tag{1.7}$$

One has then a further conservation law which is usually called conservation of charge. We prefer, however, to use another expression, load, for the more natural quantity which differs from the charge by the factor e (the elementary charge).

Making use of the field equation (1.3), one has

$$\sum_\alpha \frac{\partial s_\alpha}{\partial x_\alpha} = 0, \qquad s_\alpha = i(\phi\Phi^*_\alpha - \Phi_\alpha\phi^*). \tag{1.8}$$

The allocation of the asterisks of the definition (1.2) is made so that Φ and Φ_α transform like ϕ under a gauge transformation, if Λ is invariant.

We shall make use especially of the fourth column of the tensor $T_{\alpha\beta}$ and the fourth component of the vector s_α:

$$\left.\begin{aligned} T_{k4} &= \phi_k\Phi^*_4 + \Phi_4\phi^*_k, \qquad (k = 1, 2, 3) \\ T_{44} &= \phi_4\Phi^*_4 + \Phi_4\phi^*_4 - \Lambda, \\ s_4 &= i(\phi\Phi^*_4 - \Phi_4\phi^*). \end{aligned}\right\} \tag{1.9}$$

As explained in the Introduction we may consider these densities as constant in an arbitrary (dimensionless) volume ω; then the total momentum, energy and load contained in this volume are obtained by multiplying the densities by the volume ω.

We now go over to quantities with ordinary dimensions and introduce a unit length a. With the help of $\hbar$ and c we can express all quantities in certain units: volume in a^3, momentum in $\hbar/a$, energy in $\hbar c/a$, load in the dimensionless unit 1, Lagrangian in $\hbar c/a^4$, potential in $\sqrt{\hbar c}/a$, field in $\sqrt{\hbar c}/a^2$, anti-potential (*i.e.* the derivative of the Lagrangian with respect to the potential) in $\sqrt{\hbar c}/a^3$. We define:

$$\left.\begin{aligned} &x = ax_1, \qquad y = ax_2, \qquad z = ax_3, \qquad ict = ax_4; \\ &L = \frac{\hbar c}{a^4}\Lambda, \qquad v = \frac{\sqrt{\hbar c}}{a}\phi, \qquad V = \frac{\sqrt{\hbar c}}{a^3}\Phi; \\ &\mathbf{f} = (f_x, f_y, f_z) = \frac{\sqrt{\hbar c}}{a^2}(\phi_1, \phi_2, \phi_3), \qquad ig = \frac{\sqrt{\hbar c}}{a^2}\phi_4; \\ &\mathbf{F} = (F_x, F_y, F_z) = \frac{\sqrt{\hbar c}}{a^2}(\Phi_1, \Phi_2, \Phi_3), \qquad iG = \frac{\sqrt{\hbar c}}{a^2}\Phi_4. \end{aligned}\right\} \tag{1.10}$$

Further let Ω be the ordinary volume, $\mathbf{p}$, E and q its momentum, energy and load content:

$$\left.\begin{aligned} &\Omega = a^3\omega, \\ &\mathbf{p} = (p_x, p_y, p_z) = -i\frac{\hbar\Omega}{a^4}(T_{14}, T_{24}, T_{34}), \\ &E = -\frac{\hbar c\Omega}{a^4}T_{44}, \\ &q = -i\frac{\Omega}{a^3}s_4. \end{aligned}\right\} \tag{1.11}$$

Then we have from (1.9):

$$\left.\begin{aligned} &\mathbf{p} = \frac{\Omega}{c}(\mathbf{f}G^* + G\mathbf{f}^*), \\ &E = \Omega(gG^* + Gg^* + L), \\ &q = \frac{i\Omega}{\hbar c}(vG^* - Gv^*). \end{aligned}\right\} \tag{1.12}$$

These equations are relativistically covariant if Ω is transformed as a volume.

The field equations (1.1) and (1.3) now become:

$$\mathbf{f}=\operatorname{grad} v, \qquad g=-\frac{1}{c}\frac{\partial v}{\partial t}, \qquad \operatorname{div}\mathbf{F}+\frac{1}{c}\frac{\partial G}{\partial t}=V. \tag{1.13}$$

The dimension of the Lagrangian L was chosen so that the definition (1.2) now becomes:

$$V^*=\frac{\partial L}{\partial v}, \qquad G^*=-\frac{\partial L}{\partial g}, \qquad F_x^*=\frac{\partial L}{\partial f_x}, \ldots \tag{1.14}$$

As in particle mechanics the quantum treatment of fields which we are going to develop corresponds most closely to the Hamiltonian form of the classical treatment. Instead of working with a given Lagrangian depending on v, g, $\mathbf{f}$ and their conjugates we make use of the energy E given by (1.12), and regard E as a function of v, G, $\mathbf{f}$ and their conjugates. Then in virtue of (1.14) we have

$$V^*=\frac{1}{\Omega}\frac{\partial E}{\partial v}, \qquad g^*=\frac{1}{\Omega}\frac{\partial E}{\partial G}, \qquad F_x^*=\frac{1}{\Omega}\frac{\partial E}{\partial f_x}, \ldots \tag{1.15}$$

E is an arbitrary function of v, v^*; G, G*; $\mathbf{f}$, $\mathbf{f}^*$ apart from the condition that the relativistic invariance is ensured or, as shown above, that the energy-momentum tensor is symmetric:

$$\mathbf{f}G^*+G\mathbf{f}^*=g\mathbf{F}^*+\mathbf{F}g^*. \tag{1.16}$$

For the special case with the simple Lagrangian

$$\Lambda=\phi\phi^*+\sum_{\alpha}\phi_\alpha\phi_\alpha^* \tag{1.17}$$

Φ coincides with ϕ and Φ_α with ϕ_α. We have then

$$E=\Omega(GG^*+\mathbf{f}\,.\,\mathbf{f}^*+\eta^2 vv^*), \qquad \eta\equiv 1/a. \tag{1.18}$$

The field equations for this case, being obtained from (1.13) by identifying g, $\mathbf{F}$, V respectively with G, $\mathbf{f}$, $\eta^2 v$, are linear. This is the customary description of the scalar-meson field; the rest-mass μ of the meson is related to the unit length chosen above by

$$\frac{\mu c}{\hbar}=\frac{1}{a}=\eta. \tag{1.19}$$

We shall refer to this case as the linear case.

2. Scalar Field: Quantum Treatment for the Linear Case

In this section we shall quantise the scalar field treated above, confining ourselves to the linear case as defined at the end of the preceding section. We now consider all field components as q-numbers obeying the laws of non-cummutative algebra and interpret G*, v^*, etc. as the adjoint of G, v, etc. Hence for q-numbers the asterisk signifies the operation of passing to the adjoint. For abbreviation we use for the commutator of two q-numbers the symbol

$$[A, B]=AB-BA, \quad \text{so} \quad [A, B]^*=B^*A^*-A^*B^*=[B^*, A^*]. \tag{2.1}$$

The passage from classical to quantum mechanics consists in the transcription:

$$\frac{\partial f}{\partial x}\to\frac{i}{\hbar}[p_x, f], \qquad \frac{\partial f}{\partial t}\to-\frac{i}{\hbar}[E, f], \tag{2.2}$$

which is to be valid for all field components. We shall postulate certain relations among the field quantities so that the field equations (1.13) quantised by (2.2) will then all follow with

$\mathbf{p}$ given by (1.12) and E given by (1.18). It is easily verified that we may assume for this purpose the following relations:—

$$[v, G^*] = i\hbar c/\Omega, \tag{2.3a}$$

$$[\mathbf{f}, G^*] = [G, \mathbf{f}^*], \tag{2.3b}$$

while all other commutators (except of course the conjugate ones of those given above) vanish. For then we have

$$[\mathbf{p}, v] = \frac{\Omega}{c}[\mathbf{f}G^* + G\mathbf{f}^*, v] = -i\hbar\mathbf{f}, \tag{2.4a}$$

$$\frac{1}{c}[E, v] = \frac{\Omega}{c}[GG^* + \mathbf{f} \cdot \mathbf{f}^* + \eta^2 vv^*, v] = -i\hbar G, \tag{2.4b}$$

$$\sum_x [p_x, f_x] - \frac{1}{c}[E, G] = \frac{\Omega}{c}\left\{\sum_x f_x[G^*, f_x] - \sum_x f_x[f_x^*, G] - \eta^2 v[v^*, G]\right\} = -i\hbar\eta^2 v, \tag{2.4c}$$

which are the equations (1.13) adapted to the linear case and quantised according to (2.2).

In view of (2.3*b*) it is convenient to introduce the abbreviation

$$\mathbf{k} \equiv \frac{\Omega}{\hbar c}[G^*, \mathbf{f}] = \frac{\Omega}{\hbar c}[\mathbf{f}^*, G], \tag{2.5}$$

which is, by (2.1), self-adjoint. By using the identity $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$ one sees that in virtue of the first form $\mathbf{k}$ commutes with v^*, G, $\mathbf{f}^*$, and in virtue of the second form with v, G^*, $\mathbf{f}$. Thus all the three components of $\mathbf{k}$ commute with all field quantities and hence, by (2.5) again, also among themselves.

In order that the transcription (2.2) may be self-consistent, the commutators among the energy E and the components p_x, p_y, p_z of the momentum must commute with all field components. From (1.12) and (1.18) we obtain:

$$[p_x, p_y] = \frac{\hbar\Omega}{c}\{(f_x k_y - f_y k_x)G^* - G(k_y f_x^* - k_x f_y^*)\}, \tag{2.6}$$

$$[E, p_x] = \hbar\Omega\{k_y(f_y f_x^* - f_x f_y^*) + k_z(f_z f_x^* - f_x f_z^*) + i(f_x v^* + v f_x^*)\}, \tag{2.7}$$

and similar expressions for the remaining commutators. Since (2.6) has to commute with v we are led, with (2.3), to the condition

$$f_x k_y - f_y k_x = 0. \tag{2.8}$$

Similarly, commuting (2.7) with G we obtain with (2.5) the condition

$$k_y(f_y k_x - f_x k_y) + k_z(f_z k_x - f_x k_z) - f_x + ivk_x = 0. \tag{2.9}$$

Combining these and taking account of the similar conditions obtained from them by cyclic permutation of the suffices x, y, z, we see that

$$\mathbf{f} = iv\mathbf{k}, \quad \text{so} \quad \mathbf{f}^* = -iv^*\mathbf{k}. \tag{2.10}$$

This condition is also sufficient for the transcription (2.2) to be self-consistent; for all the commutators among p_x, p_y, p_z, E vanish when (2.10) is introduced into (2.6) and (2.7). Since $\mathbf{k}$ commutes with all field components, (2.10) is also consistent with the commutation relations (2.3*a*) and (2.3*b*), remembering the definition of $\mathbf{k}$ by (2.5). Hence it is possible to eliminate $\mathbf{f}$ according to (2.10) by introducing $\mathbf{k}$, all components of which commute with all the field quantities, among themselves, as well as with the energy and momentum.

Hence it is convenient to work in a representation in which the matrices representing the self-adjoint k_x, k_y, k_z are all diagonal with the eigenvalues of each ordered according to their magnitude. The field quantities commute with all of them; their matrices therefore consist of submatrices placed along the diagonal which correspond to the submatrices of k_x, k_y, k_z, which are multiples of the unit matrix. This representation is thus completely reduced into the submatrices corresponding to various eigenvalues (k), and we shall need to consider, in what follows, only a particular value of (k), which is then used as superscript.

From (1.12), (1.18) and (2.10) we obtain the submatrices for the momentum, energy and load:

$$\left.\begin{aligned} \mathbf{p}^{(k)} &= \frac{i\Omega}{c}\mathbf{k}\{v^{(k)}G^{*(k)} - G^{(k)}v^{*(k)}\}, \\ E^{(k)} &= \Omega\{G^{(k)}G^{*(k)} + (\eta^2 + \mathbf{k}^2)v^{(k)}v^{*(k)}\}, \\ q^{(k)} &= \frac{i\Omega}{\hbar c}\{v^{(k)}G^{*(k)} - G^{(k)}v^{*(k)}\}. \end{aligned}\right\} \tag{2.11}$$

The only non-vanishing commutator is, by (2.3*a*),

$$[v^{(k)}, G^{*(k)}] = i\hbar c/\Omega. \tag{2.12}$$

The above relations (2.11) and (2.12) are the same as those for the Fourier coefficients in the Heisenberg-Pauli quantisation method. The remaining problem is to make $\mathbf{p}^{(k)}$, $E^{(k)}$ simultaneously diagonal. The transformation for this purpose is the same as that given by Pauli and Weisskopf,† viz.

$$\left.\begin{aligned} v^{(k)} &= 2^{-\frac{1}{2}}(\eta^2 + \mathbf{k}^2)^{-\frac{1}{4}}\{a^{(k)} + b^{*(k)}\}, \\ G^{*(k)} &= 2^{-\frac{1}{2}}(\eta^2 + \mathbf{k}^2)^{\frac{1}{4}}i\{b^{(k)} - a^{*(k)}\}, \end{aligned}\right\} \tag{2.13}$$

which we need only outline here.

The only non-vanishing commutators among the new variables are, by (2.12),

$$[a^{*(k)}, a^{(k)}] = [b^{*(k)}, b^{(k)}] = \hbar c/\Omega. \tag{2.14}$$

By the transformation (2.13), (2.11) becomes

$$\left.\begin{aligned} \mathbf{p}^{(k)} &= \hbar\mathbf{k}\{N_+^{(k)} - N_-^{(k)} - 1\}, \\ E^{(k)} &= \hbar c(\eta^2 + \mathbf{k}^2)^{\frac{1}{2}}\{N_+^{(k)} + N_-^{(k)} + 1\}, \\ q^{(k)} &= N_+^{(k)} - N_-^{(k)} - 1, \end{aligned}\right\} \tag{2.15}$$

where

$$N_+^{(k)} = \frac{\Omega}{\hbar c}a^{(k)}a^{*(k)}, \qquad N_-^{(k)} = \frac{\Omega}{\hbar c}b^{(k)}b^{*(k)} \tag{2.16}$$

have, in virtue of (2.14), the eigenvalues 0, 1, 2, 3, . . . From (2.15) it follows that $N_+^{(k)}$ are the number of positively loaded mesons of momentum $\hbar\mathbf{k}$ and energy $\hbar c\sqrt{\eta^2 + \mathbf{k}^2}$, and $N_-^{(k)}$ are the number of negatively loaded mesons of momentum $-\hbar\mathbf{k}$ and energy $\hbar c\sqrt{\eta^2 + \mathbf{k}^2}$. The zero-point energy, momentum and load of (2.15) correspond to one negatively loaded quantum, but this is due to the particular order in which the factors of (1.12) are arranged. One can instead allocate half a quantum to both the positively and negatively loaded particles by making the classical expressions (1.12) and (1.18) for the momentum and energy symmetric with respect to the order of the factors. Then in (2.15) the -1 in $\mathbf{p}^{(k)}$ and $q^{(k)}$ disappears but the $+1$ in $E^{(k)}$ remains.

In physical application to electrically charged particles, *e.g.* the charged mesons, the load may be expected to be electric charge. The positively loaded particle carries a positive charge $+e$ and the negatively loaded particle a negative charge $-e$, e being the numerical value of the elementary charge. This result, however, can only be obtained when the interaction between the scalar field and the electromagnetic field is introduced. We shall not treat the interaction of fields in this paper.

3. Scalar Field: Quantum Treatment for the General Case

Although the new quantum treatment leads to results identical with those obtained by the Heisenberg-Pauli quantisation for the linear case considered above, we note that there is this fundamental difference: we have decomposed the matrices representing the field components into a set of submatrices—while in the usual treatment the field is additively decomposed, as a Fourier series. Now the sequence of Fourier coefficients obeys entirely

† *Helv. Phys. Acta*, VII, 1934, 709.

different transformation laws from those of the array of matrix elements. This difference will exhibit itself more clearly when we no longer confine ourselves to the case (1.18) with the energy a quadratic expression in the field quantities. It will be seen that for the general case the present formalism of quantisation is by far the simpler.

Consider now the general case. We note that the expressions for the momentum and load remain the same as given by (1.12). We shall not specialise the energy E except by demanding that the corresponding Lagrangian satisfies the requirements of relativity and gauge invariance. According to (1.14) and (1.15) the energy is to be regarded as a function of v, G, **f** and their conjugates.

Since the energy and momentum depend on the same set of field variables as before, we naturally take over the whole set of commutation relations obtained in § 2. The quantity **k** defined by (2.5) commutes with all field quantities, *in virtue of the commutation relations alone*. It follows further from the commutation relations alone that $\mathbf{f}-i\mathbf{k}v$ commutes with all field quantities. We therefore take over (2.10) also, anticipating that this would again ensure the self-consistency of the transcription (2.2) for quantisation.

It remains to verify this and also that the field equations (1.13) are satisfied in their quantised form.

We form the commutator of the momentum with the field components v, **f**, G in turn. With (2.3*a*) and (2.10) we obtain

$$[\mathbf{p}, v]=\frac{\Omega}{c}[\mathbf{f}G^*+G\mathbf{f}^*, v]=-i\hbar\mathbf{f}=\hbar\mathbf{k}v, \tag{3.1}$$

and with (2.5) and (2.10) we obtain

$$[\mathbf{p}, f_x]=\frac{\Omega}{c}[\mathbf{f}G^*+G\mathbf{f}^*, f_x]=\mathbf{f}\hbar k_x=\hbar\mathbf{k}f_x, \tag{3.2}$$

$$[\mathbf{p}, G]=\frac{\Omega}{c}[\mathbf{f}G^*+G\mathbf{f}^*, G]=\hbar\mathbf{k}G. \tag{3.3}$$

Taking into account the equations adjoint to these we obtain the rule that the commutator of **p** with the field component is simply $\pm\hbar\mathbf{k}$ times the field component, with the sign minus or plus according as the component carries an asterisk or not. Hence, in general, the commutor of **p** with a product of m field components with the asterisk and n components without is $(n-m)\hbar\mathbf{k}$ times the original product. We note that this product transforms under a gauge transformation like v^{n-m}. A gauge-invariant quantity is therefore necessarily † a series of terms each of which contains an equal number of the field components with and without the asterisk. It therefore commutes with the momentum.

For a gauge-invariant Lagrangian the energy E is, by (1.12), gauge-invariant. Therefore E commutes with all components of the momentum. Further, the latter commute among themselves as before. Thus by means of (2.10) the self-consistency of the transcription (2.2) is ensured.

The quantities V, g, **F** defined by (1.15) transform under gauge transformation like v. Hence we have for their commutators with the momentum (putting $n-m=1$)

$$[\mathbf{p}, F_x]=\hbar\mathbf{k}F_x, \text{ etc.} \tag{3.4}$$

The commutator of the field components with the energy E, given as an arbitrary function of the field components, can be obtained as follows:—

Let f be an arbitrary function of the independent variables $q_1, q_2, \ldots q_n$, and let the commutators $[q_r, q_s]$ among the independent variables commute with any of the variables $q_1, q_2, \ldots q_n$. Then it is easily seen that

$$[f, q_s]=\sum_{r=1}^{n}\frac{\partial f}{\partial q_r}[q_r, q_s], \tag{3.4a}$$

† The reciprocals of the field components can be admitted also. Then in counting the number of factors with and without the asterisk the reciprocals are to be subtracted,

which can be proved by induction with respect to the two basic operations of building up new functions, (i) $f=f_1+f_2$, (ii) $f=f_1 f_2$, the formula (3.4a) being assumed to hold for f_1 and f_2.

In the present case the field variables v, v^*, G, G*, f_x, f_x^*, . . . satisfy the condition that their commutators commute with all the field variables. Hence applying the formula (3.4a) with E for f we obtain with the definitions (1.15),

$$[\mathrm{E}, v]=\frac{\partial \mathrm{E}}{\partial \mathrm{G}^*}[\mathrm{G}^*, v]=-i\hbar cg, \tag{3.5}$$

and similarly

$$[\mathrm{E}, \mathrm{G}]=\frac{\partial \mathrm{E}}{\partial v^*}[v^*, \mathrm{G}]+\sum_x \frac{\partial \mathrm{E}}{\partial f_x^*}[f_x^*, \mathrm{G}]=i\hbar c\mathrm{V}+\hbar c\mathbf{k}\ .\ \mathbf{F}. \tag{3.6}$$

By (3.4) the last equation can be written as

$$\frac{1}{c}[\mathrm{E}, \mathrm{G}]-\sum_x [p_x, \mathrm{F}_x]=i\hbar\mathrm{V}. \tag{3.7}$$

In (3.1), (3.5) and (3.7) we have verified the quantised field equation, the classical equations being (1.13).

Having justified our choice of the commutation laws we can proceed further: we introduce **k** and eliminate **f** by (2.10); we represent k_x, k_y, k_z by diagonal matrices and reduce all matrices representing either the field or its energy-momentum-load into their submatrices. The submatrices of momentum and load are the same as those given above, (2.11). The submatrices of the energy of course depend on the particular expression given for the energy and are obtained from the latter by replacing v, G, **f** by $v^{(k)}$, $\mathrm{G}^{(k)}$, $i\mathbf{k}v^{(k)}$ respectively. Thus $\mathrm{E}^{(k)}$ is a function of $v^{(k)}$, $\mathrm{G}^{(k)}$ and their conjugates only:

$$\mathrm{E}^{(k)}=\mathrm{E}^{(k)}(v^{(k)}, v^{*(k)}, \mathrm{G}^{(k)}, \mathrm{G}^{*(k)}). \tag{3.8}$$

In view of the commutation law (2.12) the problem of making $\mathrm{E}^{(k)}$ diagonal can be dealt with as an eigenvalue problem of partial differential equations

$$\mathrm{E}_{\mathrm{op}}^{(k)}\psi^{(k)}=\lambda^{(k)}\psi^{(k)}, \tag{3.9}$$

where the operator $\mathrm{E}_{\mathrm{op}}^{(k)}$ is obtained from $\mathrm{E}^{(k)}$, say, by the transcription

$$\mathrm{G}^{*(k)}\rightarrow -\frac{i\hbar c}{\Omega}\frac{\partial}{\partial v^{(k)}}, \qquad v^{*(k)}\rightarrow -\frac{i\hbar c}{\Omega}\frac{\partial}{\partial \mathrm{G}^{(k)}},$$

if $\psi^{(k)}$ is taken as a function of $v^{(k)}$ and $\mathrm{G}^{(k)}$, the latter being regarded as ordinary complex numbers.

The Pauli-Weisskopf transformation of variables can be applied to the general case also; the factor $(\eta^2+\mathbf{k}^2)^{\frac{1}{4}}$ in (2.13) can be omitted without modifying the commutation laws (2.14) of the new variables $a^{(k)}$, $b^{(k)}$, $a^{*(k)}$, $b^{*(k)}$. The differential equation (3.9) holds with $\psi^{(k)}$ depending on $a^{(k)}$, $b^{(k)}$ (now regarded as ordinary complex numbers), and with $\mathrm{E}_{\mathrm{op}}^{(k)}$ depending on $a^{(k)}$, $b^{(k)}$,

$$a^{*(k)}=\frac{\hbar c}{\Omega}\frac{\partial}{\partial a^{(k)}} \quad \text{and} \quad b^{*(k)}=\frac{\hbar c}{\Omega}\frac{\partial}{\partial b^{(k)}}. \tag{3.10}$$

The expressions for the momentum and the load are now given by (2.15). The operators $\mathrm{N}_+^{(k)}$ and $\mathrm{N}_-^{(k)}$, (2.16), are represented by

$$\mathrm{N}_+^{(k)}=a^{(k)}\frac{\partial}{\partial a^{(k)}}, \qquad \mathrm{N}_-^{(k)}=b^{(k)}\frac{\partial}{\partial b^{(k)}}. \tag{3.11}$$

The use of the complex numbers a, b for the argument of the wave function was initiated by Fock † and is particularly convenient for a system where the number of quanta is not precisely known.

† *Zeits. f. Phys.*, XLIX, 1928, 339; *Phys. Zeits. Sowjet.*, VI, 1934, 425.

The reduction of the energy E, given as an arbitrary (invariant) function of the field components, into its submatrices $E^{(k)}$ represents the central feature of this new treatment of quantised fields. For it allows us to handle "non-linear" field equations, which were practically intractable by the ordinary methods, with the same ease as linear equations. In fact this distinction loses its proper meaning; for the new theory is a linear wave theory (see equation (3.9)) for any arbitrary energy function E, and the superposition principle of states holds in any case. The non-linearity refers to the dependence of the derivatives of E with respect to the field components: they may be non-linear functions of the field components; but as the latter are linear operators, these functions are also linear operators. The so-called non-linear field theories, so far violently rejected by all experts in the quantum theory of fields, are no more non-linear than for instance the theory of the relativistic electron with the Hamiltonian $H=\sqrt{m^2c^4+c^2p^2}$ in ordinary quantum mechanics. This change of the viewpoint also means a very great simplification of the mathematical method. Each field variable is represented by a matrix, and the indices of each matrix element refer (according to general principles of quantum theory) to pure states, described by a characteristic or pure wave function. Any arbitrary state is represented by a superposition of such pure wave functions.

In particular, the variable $\mathbf{k}$ appears in our treatment as a "constant of motion" defining a particularly simple set of pure states, for which all field quantities have matrix elements of the simple form

$$\phi_{kk'}=\phi^{(k)}\delta_{kk'}. \tag{3.12}$$

The same holds for the energy and the components of momentum. If a state of the field is a superposition of such pure states of different $\mathbf{k}$, the expectation value for the energy of the system is, according to the rules of quantum mechanics,

$$(\psi\ .\ \mathrm{E}\psi)=\sum_k(\psi^{(k)}\ .\ \mathrm{E}^{(k)}\psi^{(k)}).$$

There is no interaction among the various proper modes of vibration (k) of one and the same mechanical system.

In the Heisenberg and Pauli quantisation method each field is decomposed classically as a Fourier sum, viz.

$$\phi(\mathbf{r})=\text{const.}\sum_k\phi_k e^{i\mathbf{k}.\mathbf{r}}, \tag{3.12a}$$

and a "model" mechanical system, usually an oscillator, is assigned to each individual Fourier coefficient. These "model" systems are then treated by quantum mechanics. Except for the linear case instanced in § 2 there is always interaction among the various "model" mechanical systems which makes the quantisation impracticable.

It is easy to establish the correspondence of our submatrices with Heisenberg-Pauli's individual Fourier terms. For this purpose we shall work in the co-ordinate representation, the matrix element of the co-ordinate x and the momentum operator $\mathbf{p}=\frac{\hbar}{i}\text{grad}$ being the well-known delta-functions and their derivatives:

$$\left.\begin{aligned}x_{r'r''}&=x'\delta(x'-x'')\delta(y'-y'')\delta(z'-z''),\\ \mathbf{p}_{r'r''}&=\frac{\hbar}{i}\,\text{grad}'\,\delta(x'-x'')\delta(y'-y'')\delta(z'-z'').\end{aligned}\right\} \tag{3.13}$$

Now the submatrices $\phi^{(k)}$ of all the field components without the asterisk satisfy, by (3.1)–(3.4), the matrix equation

$$[\mathbf{p},\phi^{(k)}]=\hbar\mathbf{k}\phi^{(k)}, \tag{3.14}$$

where $\mathbf{k}$ on the right-hand side is a multiple of the unit submatrix. Since by (3.13) the matrix elements of $[\mathbf{p},\phi^{(k)}]$ are the same as those of the operator $\frac{\hbar}{i}$ grad $\phi^{(k)}$, the solution of (3.14) in operator form is

$$\phi^{(k)}=\phi_k e^{i\mathbf{k}.\mathbf{r}}, \tag{3.15}$$

with matrix elements

$$\phi^{(k)}_{r'r''} = \phi_k e^{i\mathbf{k}\cdot\mathbf{r}'}\delta(x'-x'')\delta(y'-y'')\delta(z'-z''), \tag{3.16}$$

the operator or matrix ϕ_k being independent of the co-ordinates. Hence our submatrix $\phi^{(k)}$ corresponds to the kth Fourier coefficient.

4. Scalar Field: Quantum Treatment of a Real Field

In this section we shall explain the quantum treatment of a real field (*i.e.* a field described classically by real field quantities), using again the scalar field as an example. In physical application real fields are used to describe electrically neutral particles.

It is evident that a real field is something like half a complex field and can therefore be obtained from the complex field treated above by specialisation. In the semi-classical treatment all field quantities (which are functions of one point in space-time) are restricted by

$$\phi^*(\mathbf{r}) = \phi(\mathbf{r}). \tag{4.1}$$

In terms of the Fourier coefficients ϕ_k of (3.12*a*) this is

$$\phi_{-k} = \phi_k^*. \tag{4.2}$$

According to the correspondence (3.15) just established the case of real field in our quantum treatment should be described by the restriction between pairs of submatrices:

$$\phi^{(-k)} = \phi^{*(k)}. \tag{4.3}$$

This does not mean that the matrix ϕ is self-adjoint; for this would imply

$$\phi^{*(k)} = \phi^{(k)}. \tag{4.3a}$$

It is gratifying that the restriction (4.3) is compatible with all the commutation relations established above. The restriction (4.3*a*), however, contradicts these commutation laws. This we shall now show. (For the scalar field we found it impossible to obtain a new set of commutation laws compatible with the restriction (4.3*a*).)

It is convenient to work in the k-representation, and we may confine our attention to the two sets of submatrices with the superscripts (k) and ($-k$). The commutation laws (2.3*a*) and (2.5) obtained for the whole matrices contain the following for the submatrices:—

$$[v^{(k)}, G^{*(k)}] = i\hbar c/\Omega, \qquad [G^{*(k)}, \mathbf{f}^{(k)}] = \hbar c\mathbf{k}/\Omega, \tag{4.4}$$

$$[v^{(-k)}, G^{*(-k)}] = i\hbar c/\Omega, \qquad [G^{*(-k)}, \mathbf{f}^{(-k)}] = -\hbar c\mathbf{k}/\Omega. \tag{4.5}$$

We take the adjoint of (4.5) and reverse the sign throughout. If now (4.3) is used, by putting $G^{*(k)} = G^{(-k)}$, . . ., the result of the transformation of (4.5) is just identical with (4.4); hence (4.3) is compatible with the commutation laws. The incompatibility of (4.3*a*) with these laws follows from (4.4) together with the trivial commutation laws:

$$[v^{(k)}, G^{(k)}] = 0, \qquad [G^{*(k)}, \mathbf{f}^{*(k)}] = 0. \tag{4.6}$$

Suppose the energy-momentum tensor has been made symmetric with respect to the order of factors. Under the restriction (4.3) the two sets of submatrices of the energy and momentum distinguished by (k) and ($-k$) respectively can be shown to be the same. For example, the submatrices $\mathbf{p}^{(k)}$, $\mathbf{p}^{(-k)}$ are obtained from the general expression (given by (1.12) symmetrised) for $\mathbf{p}$ simply by adding corresponding superscripts to all the field components. According to (4.3), instead of adding the superscript ($-k$) to a field component one can add the superscript (k) together with an asterisk. Since the interchange of factors is supposed not to affect $\mathbf{p}$, $\mathbf{p}^{(-k)}$ is the same as the adjoint of $\mathbf{p}^{(k)}$, which equals $\mathbf{p}^{(k)}$. The same consideration holds for the submatrices of the energy:

$$\mathbf{p}^{(k)} = \mathbf{p}^{(-k)}, \qquad E^{(k)} = E^{(-k)}. \tag{4.7}$$

For the load q the same consideration leads to the result that $q^{(k)}/i$ is the adjoint of $q^{(-k)}/i$. Hence

$$q^{(k)} = -q^{(-k)}. \tag{4.8}$$

The restriction (4.3) can also be stated in terms of the wave function ψ of the whole system, namely that the parts $\psi^{(k)}$ and $\psi^{(-k)}$ are related by

$$\psi^{(k)}(a^{(k)}, b^{(k)}) = \psi^{(-k)}(b^{(-k)}, a^{(-k)}). \tag{4.9}$$

That is to say, as functions of the two Pauli-Weisskopf arguments a and b, $\psi^{(-k)}$ differs from $\psi^{(k)}$ only by an interchange of the two arguments. Operating on wave functions thus restricted we have always, using (3.10) for the operators $a^{*(k)}$, $b^{*(k)}$,

$$a^{(-k)} = b^{(k)}, \qquad a^{(k)} = b^{(-k)}, \qquad a^{*(-k)} = b^{*(k)}, \qquad a^{*(k)} = b^{*(-k)}. \tag{4.10}$$

This differs from (4.3) only by a transformation (2.13) of the field variables. The submatrix elements of the energy, momentum and load with respect to such wave functions are thus connected by (4.7) and (4.8). In virtue of the latter, the matrix elements of the load with respect to the total wave functions thus restricted all vanish.

5. Vector Field

After having explained our method in all details with the help of the scalar field as an example, we shall now proceed to treat other fields by the same method, discussing only such new points as may arise.

We consider here the vector field described (in the space-vector notation) by a scalar potential v, a vector potential $\mathbf{u}$, a six-vector $\mathbf{f}$, $\mathbf{g}$ which is the four-dimensional curl of the potentials

$$\mathbf{f} = -\operatorname{grad} v - \frac{1}{c}\frac{\partial \mathbf{u}}{\partial t}, \qquad \mathbf{g} = \operatorname{curl} \mathbf{u}, \tag{5.1}$$

and their conjugates. The Lagrangian L (dimensions $\hbar c/a^4$) depends on these quantities and is invariant for relativity and gauge transformation. We put

$$F_x^* = -\frac{\partial L}{\partial f_x}, \ldots, \qquad G_x^* = \frac{\partial L}{\partial g_x}, \ldots, \qquad U_x^* = \frac{\partial L}{\partial u_x}, \ldots, \qquad V^* = -\frac{\partial L}{\partial v}. \tag{5.2}$$

By the same method as used in § 1 we get the field equations

$$\operatorname{div} \mathbf{F} + V = 0, \qquad \operatorname{curl} \mathbf{G} - \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} + \mathbf{U} = 0. \tag{5.3}$$

Further, we have for the momentum, energy and load:

$$\left.\begin{aligned} p_x &= \frac{\Omega}{c}\{g_z F_y^* - g_y F_z^* + F_y g_z^* - F_z g_y^* + u_x V^* + V u_x^*\}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ E &= \Omega\{\mathbf{f} \cdot \mathbf{F}^* + \mathbf{F} \cdot \mathbf{f}^* + vV^* + Vv^* + L\}, \\ q &= \frac{i\Omega}{\hbar c}\{\mathbf{u} \cdot \mathbf{F}^* - \mathbf{F} \cdot \mathbf{u}^*\}. \end{aligned}\right\} \tag{5.4}$$

Regarding E as a function of $\mathbf{F}$, $\mathbf{g}$, $\mathbf{u}$, V and their conjugates, we obtain $\mathbf{f}$, $\mathbf{G}$, $\mathbf{U}$, v by partial differentiation:

$$f_x^* = \frac{1}{\Omega}\frac{\partial E}{\partial F_x}, \ldots, \qquad G_x^* = \frac{1}{\Omega}\frac{\partial E}{\partial g_x}, \ldots, \qquad U_x^* = \frac{1}{\Omega}\frac{\partial E}{\partial u_x}, \ldots, \qquad v^* = \frac{1}{\Omega}\frac{\partial E}{\partial V}. \tag{5.5}$$

For relativity invariance we must have

$$\left.\begin{aligned} &g_z F_y^* - g_y F_z^* + F_y g_z^* - F_z g_y^* + u_x V^* + V u_x^* \\ &= G_z f_y^* - G_y f_z^* + f_y G_z^* - f_z G_y^* + U_x v^* + v U_x^*, \end{aligned}\right\} \tag{5.6}$$

and similar equations obtained by permuting the indices x, y, z.

In order to quantise these equations we have to distinguish between two cases: (i) L does involve the potentials $\mathbf{u}$, v as well as their curl, $\mathbf{f}$, $\mathbf{g}$; (ii) L involves only $\mathbf{f}$, $\mathbf{g}$ and not the potentials $\mathbf{u}$, v.

For the case (i) we postulate that all commutators vanish except the following ones and those adjoint to these or obtainable from these by a cyclic permutation of the indices x, y, z:

$$[u_x^*, F_x] = i\hbar c/\Omega, \tag{5.7}$$

$$\left.\begin{aligned} [V^*, u_x] &= [g_z^*, F_y] = -[g_y^*, F_z] \\ = [u_x^*, V] &= [F_y^*, g_z] = -[F_z^*, g_y] \end{aligned}\right\} \equiv \frac{\hbar c}{\Omega} k_x. \tag{5.8}$$

Here we have introduced the symbol k_x for the common value of all the commutators of (5.8). Using some form or other of (5.8) for defining k_x, we see that k_x actually commutes with all field variables. k_x, k_y, k_z are thus all "constants of motion."

We further assume

$$\left.\begin{aligned} g_x &= i(k_y u_z - k_z u_y), \ldots \\ V &= -i\mathbf{k} \cdot \mathbf{F}, \end{aligned}\right\} \tag{5.9}$$

which are compatible with (5.7) and (5.8). The justification of these assumptions runs exactly parallel to what we have done in § 3 for the scalar field and hence will be omitted here.

(5.9) can be used to eliminate $\mathbf{g}$, V in terms of $\mathbf{k}$, and the k-representation can be introduced in order to reduce all matrices representing the field variables, etc. into submatrices. Then the commutation laws are, by (5.7),

$$[u_x^{*(k)}, F_y^{(k)}] = \frac{i\hbar c}{\Omega}\delta_{xy} \tag{5.10}$$

and their adjoints.

We have applied this quantisation to the linear case of the customary vector-meson field and obtained the usual results.

For the case (ii) where the Lagrangian does not involve the four potentials the energy and momentum are functions of $\mathbf{F}$, $\mathbf{g}$ and their conjugates. The four potentials admit the gauge transformation of the first kind (*i.e.* the addition of an arbitrary four-dimensional gradient) which does not affect the energy, momentum or load (*cf.* p. 52). In fact one can do away with the potentials altogether, and replace (5.1) by

$$\operatorname{div} \mathbf{g} = 0, \qquad \operatorname{curl} \mathbf{f} + \frac{1}{c}\frac{\partial \mathbf{g}}{\partial t} = 0. \tag{5.11}$$

(5.3) now reduces to similar equations:

$$\operatorname{div} \mathbf{F} = 0, \qquad \operatorname{curl} \mathbf{G} - \frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} = 0. \tag{5.12}$$

For quantisation we postulate the non-vanishing commutator relations

$$\left.\begin{aligned} [g_z^*, F_y] &= -[g_y^*, F_z] \\ = [F_y^*, g_z] &= -[F_z^*, g_y] \end{aligned}\right\} \equiv \frac{\hbar c}{\Omega} k_x. \tag{5.13}$$

k_x, thus defined, commutes, in virtue of the commutation laws assumed, with all the field variables $\mathbf{F}$, $\mathbf{g}$, $\mathbf{F}^*$, $\mathbf{g}^*$.

We further restrict g_x, g_y, g_z and F_x, F_y, F_z by

$$\mathbf{k} \cdot \mathbf{g} = 0, \qquad \mathbf{k} \cdot \mathbf{F} = 0, \tag{5.14}$$

which are also compatible with (5.13).

Then the field equations (5.11), (5.12) are reproduced in the quantised form. The commutator of the momentum with one of the vectors $\mathbf{F}$, $\mathbf{g}$, $\mathbf{F}^*$, $\mathbf{g}^*$ is $\pm\hbar\mathbf{k}$ times the same vector. Further, E, p_x, p_y, p_z all commute.

The solution of the quantised equations of (5.11) is therefore given by the quantised form of (5.1); the potentials thus introduced are determined only to an arbitrary additive term, since one can add

$$-\frac{i}{\hbar c}[\mathrm{E},f] \text{ to } v, \qquad \frac{i}{\hbar}[\mathbf{p},f] \text{ to } \mathbf{u}, \tag{5.15}$$

f being an arbitrary function of $\mathbf{F}$, $\mathbf{g}$, $\mathbf{F}^*$, $\mathbf{g}^*$. The commutators of $\mathbf{p}$ with any function of $\mathbf{F}$, $\mathbf{g}$, $\mathbf{F}^*$, $\mathbf{g}^*$ is proportional to $\mathbf{k}$. Hence the indeterminate part of $\mathbf{u}$ is proportional to $\mathbf{k}$, and since $\mathbf{k}\,.\,\mathbf{F}=0$ this has no effect on the total load q given by (5.4).

If one prefers to take (5.1) and (5.12) as the field equations and thus include the potentials as field variables, one is left with some arbitrariness in choosing the commutation relations involving the potentials. To avoid this arbitrariness one can make the potentials determinate by adding the restriction

$$\sum_x [p_x, u_x]=0, \qquad i.e. \quad \mathbf{k}\,.\,\mathbf{u}=0, \tag{5.16}$$

or classically

$$\operatorname{div}\mathbf{u}=0. \tag{5.16a}$$

Still regarding the field components $\mathbf{F}$, $\mathbf{g}$, $\mathbf{F}^*$, $\mathbf{g}^*$ as the independent variables, the potentials can be defined in terms of them by

$$v=\frac{i\mathbf{k}\,.\,\mathbf{f}}{\mathbf{k}^2}, \qquad u=\frac{i\mathbf{k}\wedge\mathbf{g}}{\mathbf{k}^2}. \tag{5.17}$$

Here $\mathbf{k}$ and $\mathbf{f}$ are abbreviations standing for functions of the field components $\mathbf{F}$, $\mathbf{g}$, $\mathbf{F}^*$, $\mathbf{g}^*$, as given by (5.13) and (5.5). From (5.17) and (5.13) we have the commutation relations

$$[u_x^{*(k)}, \mathrm{F}_y^{(k)}]=\frac{i\hbar c}{\Omega}\left(\delta_{xy}-\frac{k_x k_y}{\mathbf{k}^2}\right). \tag{5.18}$$

This is different from the corresponding relation for the case (i), and depends † on the arbitrary restriction for the determination of the potentials. The quantised equations of (5.1) and (5.12) then follow from (5.13), (5.14) and (5.17).

The case of a real vector field (*e.g.* the electromagnetic field) can be obtained from the above by the restriction (4.3) for the field components. It is easily verified that the restriction is compatible with all the commutation laws obtained here for the vector field.

6. Spinor Field

We now consider a spinor field which is described by a spinor v, with four components v_s, $s=1, 2, 3, 4$, its adjoint v^* and the space and time derivatives

$$\mathbf{f}_s=\operatorname{grad} v_s, \qquad g_s=-\frac{1}{c}\frac{\partial v_s}{\partial t}. \tag{6.1}$$

For a Lorentz transformation the v_s are subject to a linear transformation in such a way that

$$\left.\begin{aligned}(vav^*)&=\sum_{r,s=1}^{4} a_{rs}v_r v_s^*,\\ (vv^*)&=\sum_{r=1}^{4} v_r v_s^*\end{aligned}\right\} \tag{6.2}$$

transform as a four vector. The first of these formulæ shows that v has to be considered as a 1-row matrix, v^* as a 1-column matrix in spinor space. (In Dirac's theory this is generally done the other way, the 4 spinor components of the wave function are considered as a 1-column matrix.)

Let the Lagrangian L be an invariant real function of v_s, $\mathbf{f}_s$, g_s and their conjugates. Let us introduce

$$\mathrm{V}_s^*=\frac{\partial \mathrm{L}}{\partial v_s}, \qquad \mathrm{G}_s^*=-\frac{\partial \mathrm{L}}{\partial g_s}, \qquad \mathrm{F}_{xs}^*=\frac{\partial \mathrm{L}}{\partial f_{xs}}, \ldots \tag{6.3}$$

† The commutation laws in this form for Fourier coefficients have been already obtained in the case of Maxwell's electromagnetic field by Novobatzki, *Zeits. f. Physik*, CXI, 1938, 293.

and their conjugate quantities. The field equations are then

$$\operatorname{div} \mathbf{F}_s + \frac{1}{c}\frac{\partial G_s}{\partial t} = V_s. \tag{6.4}$$

The canonical energy-momentum tensor

$$\left.\begin{aligned} T_{00} &= (gG^*) + (Gg^*) + L, \\ T_{x0} &= (f_xG^*) + (Gf_x^*), \\ T_{0y} &= (gF_y^*) + (F_yg^*), \\ T_{xy} &= (f_xF_y^*) + (F_yf_x^*) - \delta_{xy}L \end{aligned}\right\} \tag{6.5}$$

satisfies the conservation laws

$$\frac{\partial T_{ax}}{\partial x} + \frac{\partial T_{ay}}{\partial y} + \frac{\partial T_{az}}{\partial z} + \frac{1}{c}\frac{\partial T_{a0}}{\partial t} = 0, \qquad (a = 0, x, y, z). \tag{6.6}$$

For a relativistically invariant Lagrangian it can be shown † that a symmetrical energy-momentum tensor exists, which also satisfies the conservation laws.

Concerning the total energy and momentum in a volume Ω it does not matter whether it is calculated from the last row (or column) of this symmetrical tensor by integration over space, or from the components T_{00} and T_{x0} (but not T_{0y}) of the canonical (unsymmetrical) tensor (6.5).

According to our procedure of treating the densities as constants, we take for the energy and momentum the expressions obtained from the canonical tensor:

$$E = \Omega\{(gG^*) + (Gg^*) + L\}, \tag{6.7}$$

$$\mathbf{p} = \frac{\Omega}{c}\{(\mathbf{f}G^*) + (G\mathbf{f}^*)\}. \tag{6.8}$$

The dimension of the Lagrangian is assumed to be that of energy-density.

If the Lagrangian is also gauge invariant (*i.e.* with respect to the transformation $v_s \rightarrow v_s e^{i\gamma}$), we have further the conservation law:

$$\operatorname{div} \mathbf{s} + \frac{1}{c}\frac{\partial s_0}{\partial t} = 0,$$

where

$$\left.\begin{aligned} \mathbf{s} &= i\{(v\mathbf{F}^*) - (\mathbf{F}v^*)\}, \\ s_0 &= i\{(vG^*) - (Gv^*)\}. \end{aligned}\right\} \tag{6.9}$$

We thus obtain the load q (dimensionless):

$$q = \frac{i\Omega}{\hbar c}\{(vG^*) - (Gv^*)\}. \tag{6.10}$$

When the energy E is regarded as a function of v_s, $\mathbf{f}_s$, G_s and their conjugates, V_s^*, $\mathbf{F}_s^*$, g_s^*, etc. can be obtained by differentiation:

$$V_s^* = \frac{1}{\Omega}\frac{\partial E}{\partial v_s}, \qquad g_s^* = \frac{1}{\Omega}\frac{\delta E}{\delta G_s}, \qquad F_{xs}^* = \frac{1}{\Omega}\frac{\partial E}{\partial f_{xs}}, \ldots \tag{6.11}$$

For gauge invariance E is necessarily given by a series of terms each containing an equal number of factors with the asterisk and without, the total number of factors of each term being thus always even. E/Ω has also to satisfy the condition which expresses that the Lagrangian is relativistically invariant.

The procedure for quantising the field is the same as above, (6.1) and (6.4) being replaced by commutator equations according to (2.2). But this time we assume "anti-commutation" laws, the non-vanishing ones being

† *Cf.* W. Pauli, *Rev. Mod. Phys.*, XIII, 1941, 204.

$$[G_r^*, v_s]_+ = -\frac{ihc}{\Omega}\delta_{rs}, \tag{6.12a}$$

$$[G_r^*, \mathbf{f}_s]_+ = [f_r^*, G_s]_+ = \frac{hc}{\Omega}\mathbf{k}\delta_{rs}. \tag{6.12b}$$

The last equation defines **k**. The anti-commutator of A and B is defined by

$$[A, B]_+ = AB + BA; \tag{6.13}$$

whenever desirable the ordinary commutator will be denoted by $[A, B]_-$. We note the following elementary algebraic identities containing anti-commutators:—

$$[AB, C]_- \equiv A[B, C]_+ - [A, C]_+ B, \tag{6.14}$$

$$[[A, B]_+, C]_- \equiv [A, [B, C]_+]_- + [B, [A, C]_+]_-. \tag{6.15}$$

By applying (6.15) it follows from the anti-commutation laws alone that **k** commutes with all field components. Further, it follows from the anti-commutation laws that

$$\mathbf{f}_s = i\mathbf{k}v_s \tag{6.16}$$

because the difference $\mathbf{f}_s - i\mathbf{k}v_s$ anti-commutes with all field components.

We proceed to verify the field equations. Commute the momentum (6.8) with v_s:

$$[\mathbf{p}, v_s]_- = \frac{\Omega}{c}\sum_{r=1}[\mathbf{f}_r G_r^* + G_r \mathbf{f}_r^*, v_s]_-. \tag{6.17}$$

Apply the formula (6.14) and note that most anti-commutators vanish. Hence

$$[\mathbf{p}, v_s]_- = \frac{\Omega}{c}\mathbf{f}_s[G_s^*, v_s]_+ = -ih\mathbf{f}_s, \tag{6.18}$$

which is the quantised form of the first equation of (6.1). Combining (6.18) with (6.16) we have

$$[\mathbf{p}, v_s]_- = h\mathbf{k}v_s. \tag{6.19}$$

Evaluating the commutator of the momentum with $\mathbf{f}_s$ or G_s by (6.14) and the anti-commutation laws, we have further

$$[\mathbf{p}, f_{xs}]_- = h\mathbf{k}f_{xs}, \qquad [\mathbf{p}, G_s]_- = h\mathbf{k}G_s. \tag{6.20}$$

By the same consideration about the gauge invariance of the energy as given above (§ 3, p. 46), it follows from (6.19), (6.20) and their adjoint equations that the components of the momentum and energy all commute, and further

$$[\mathbf{p}, F_{xs}]_- = h\mathbf{k}F_{xs}, \text{ etc.} \tag{6.21}$$

We also need the commutators of the energy with the field components. In virtue of the anti-commutation laws we may assume, without any loss of generality, that the energy E (supposed to be gauge invariant) is given as a series of terms, each term being the product of a number of "doublets"—a doublet being the product of two field components, the right-hand factor carrying an asterisk and the left-hand factor not. The commutator of a doublet with a third field component can be evaluated by the formula (6.14) and contains one anti-commutator only, the other vanishes. Since the anti-commutators of the field components commute with all field components, we have, by induction with respect to the addition and multiplication of doublets,

$$\left.\begin{aligned} [f, v_s]_- &= \frac{\partial f}{\partial G_s^*}[G_s^*, v_s]_+, \\ [f, G_s]_- &= \frac{\partial f}{\partial v_s^*}[v_s^*, G_s]_+ + \sum_x \frac{\partial f}{\partial f_{xs}^*}[f_{xs}^*, G_s]_+, \\ [f, \mathbf{f}_s]_- &= \frac{\partial f}{\partial G_s^*}[G_s^*, \mathbf{f}_s]_+; \end{aligned}\right\} \tag{6.22}$$

$$\left.\begin{aligned}[f, v_s^*]_- &= -[G_s, v_s^*]_+\frac{\partial f}{\partial G_s},\\ [f, G_s^*]_- &= -[v_s, G_s^*]_+\frac{\partial f}{\partial v_s} - \sum_x [f_{xs}, G_s^*]_+\frac{\partial f}{\partial f_{xs}},\\ [f, \mathbf{f}_s^*]_- &= -[G_s, \mathbf{f}_s^*]_+\frac{\partial f}{\partial G_s},\end{aligned}\right\} \tag{6.23}$$

where f denotes an arbitrary series of terms, each being the product of a number of doublets. The negative sign of (6.23) originates from that of the second term of (6.14).

Taking the energy E for f, and using the definitions of (6.11) and the values for the anti-commutators given by (6.12), we get from (6.22)

$$\left.\begin{aligned}\frac{1}{c}[E, v_s]_- &= \frac{\Omega}{c} g_s [G_s^*, v_s]_+ = -i\hbar g_s,\\ \frac{1}{c}[E, G_s]_- &= i\hbar V_s + \hbar \mathbf{k} \cdot \mathbf{F}_s = i\hbar V_s + \sum_x [p_x, F_{xs}]_-,\end{aligned}\right\} \tag{6.24}$$

which are the remaining quantised field equations to be verified. The adjoint equations of (6.24) follow from (6.23).

The special case of Dirac's electron field is described by the Lagrangian

$$L = \frac{\hbar c}{2i}\{(vg^*) - (gv^*) + (\mathbf{f} \cdot \alpha v^*) - (v\alpha \cdot \mathbf{f}^*)\} + mc^2(v\beta v^*), \tag{6.25}$$

where mc^2 is the rest energy of the electron and v_s of dimensions $(\text{length})^{-3/2}$ as usual. α (components α_x, α_y, α_z) and β are Dirac's numerical matrices, their columns and rows being here transposed. From (6.25) we get, according to the last equation of (6.3),

$$G_s^* = -\frac{\partial L}{\partial g_s} = \frac{\hbar c}{2i} v_s^*, \tag{6.26}$$

which is independent of g_s. Here we have a singular case, the above transformation from L to E based on the change of variables from g_s to G_s being now impossible. Hence we consider this case separately.

From the Lagrangian (6.25) the following expressions for the energy, momentum and load are derived:—

$$\left.\begin{aligned}E &= \frac{\hbar c\Omega}{2i}\{(\mathbf{f} \cdot \alpha v^*) - (v\alpha \cdot \mathbf{f}^*)\} + mc^2\Omega(v\beta v^*),\\ \mathbf{p} &= \frac{\hbar\Omega}{2i}\{(\mathbf{f}v^*) - (v\mathbf{f}^*)\},\\ q &= \Omega(vv^*).\end{aligned}\right\} \tag{6.27}$$

Here $\mathbf{p}$ and q are obtained from the general expressions (6.8) and (6.10) by the specialisation (6.26). E is obtained from L by the canonical transformation (6.7), the terms of L linear in g and g^* disappear in E. The energy and momentum are thus expressed in terms of v_s, $\mathbf{f}_s$ and their conjugates. No other variable need be introduced.

It can easily be verified that for this case the anti-commutation laws are the following:—

$$\left.\begin{aligned}[v_r^*, v_s]_+ &= \delta_{rs}/\Omega,\\ [v_r^*, f_s]_+ &= -[v_r, f_s^*]_+ \equiv i\mathbf{k}\delta_{rs}/\Omega,\end{aligned}\right\} \tag{6.28}$$

while all the other anti-commutators vanish.

If in the expressions (6.27) for the energy and momentum we had taken the factors in the reverse order we should have obtained formally the same anti-commutation laws as (6.28) but with a negative sign on the right-hand side. Such an equation as

$$[v_r^*, v_s]_+ = -\delta_{rs}/\Omega \tag{6.29}$$

is self-contradictory for $r=s$ as then the left-hand side is always positive. This negative sign originates from the second term of (6.14).

It is interesting to compare the quantisation of tensor fields on the one hand and spinor fields on the other. The momentum and for linear fields also the energy are bilinear expressions of the field components, and it is always the commutators of the energy and momentum with the field components that correspond to the time and space derivatives of the latter. The branching into commutation and anti-commutation laws has its root in the following elementary algebraic identities:—

$$[AB, C]_- \equiv +A[B, C]_- + [A, C]_- B, \tag{6.30}$$

$$[AB, C]_- \equiv +A[B, C]_+ - [A, C]_+ B. \tag{6.14}$$

The two plus signs on the right-hand side of (6.30) have the result that for tensor fields the quantum expression for a classical product AB^* can be either $+AB^*$ or $+B^*A$. On the other hand the plus and minus signs on the right-hand side of (6.14) have the result that for spinor fields the quantum expression for a classical product AB^* has to be $+AB^*$ or $-B^*A$. Undoubtedly the most symmetrical choice would be the mean $\frac{1}{2}AB^* + \frac{1}{2}B^*A$ for tensor fields and $\frac{1}{2}AB^* - \frac{1}{2}B^*A$ for spinor fields. The differences of the various quantum expressions for the same classical product are in most cases constants or zero, expressible in terms of commutators for tensor fields and anti-commutators for spinor fields.

Returning to Dirac's electron field, we can proceed further by eliminating $\mathbf{f}_s$ and again considering the submatrices:

$$\left.\begin{aligned} E^{(k)} &= \hbar c\Omega\mathbf{k}\,.\,(v^{(k)}\alpha v^{*(k)}) + mc^2\Omega(v^{(k)}\beta v^{*(k)}), \\ \mathbf{p}^{(k)} &= \hbar\mathbf{k}\Omega(v^{(k)}v^{*(k)}), \\ q^{(k)} &= \Omega(v^{(k)}v^{*(k)}). \end{aligned}\right\} \tag{6.31}$$

The transformation to principal axes for the energy is achieved by

$$v_s^{(k)} = \sum_{\tau=1}^{4} v^{(k)\tau} S_{\tau s}^{(k)}, \tag{6.32}$$

where all the transformation coefficients $S_{\tau s}^{(k)}$ are c-numbers. The index τ is used in order to distinguish the new variables from the old ones and to emphasise the fact that $v^{(k)\tau}$ does not constitute a spinor. The $S_{\tau s}^{(k)}$ are the usual normalised solutions of Dirac's equation for a free particle of momentum $\hbar\mathbf{k}$; the index τ distinguishes the two orientations of the spin and the two signs of the energy-value, while the index s enumerates the four components of the spinor. Thus $S_{\tau s}^{(k)}$ forms a unitary matrix satisfying

$$S^{(k)}\{\hbar c\mathbf{k}\,.\,\alpha + mc^2\beta\}S^{*(k)} = \epsilon^{(k)}, \tag{6.33}$$

which is a diagonal matrix. In the usual representation of Dirac we have

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \epsilon^{(k)} = \epsilon_k\beta, \qquad \epsilon_k = +c\sqrt{\hbar^2\mathbf{k}^2 + m^2c^2}. \tag{6.34}$$

The expressions for the momentum and load are invariants of this transformation of variables. We thus have

$$\left.\begin{aligned} E^{(k)} &= \epsilon_k\Omega\{v^{(k)1}v^{*(k)1} + v^{(k)2}v^{*(k)2} - v^{(k)3}v^{*(k)3} - v^{(k)4}v^{*(k)4}\}, \\ \mathbf{p}^{(k)} &= \hbar\mathbf{k}\Omega\{v^{(k)1}v^{*(k)1} + v^{(k)2}v^{*(k)2} + v^{(k)3}v^{*(k)3} + v^{(k)4}v^{*(k)4}\}, \\ q^{(k)} &= \Omega\{v^{(k)1}v^{*(k)1} + v^{(k)2}v^{*(k)2} + v^{(k)3}v^{*(k)3} + v^{(k)4}v^{*(k)4}\}. \end{aligned}\right\} \tag{6.35}$$

The anti-commutation law is transformed into

$$[v^{*(k)\tau}, v^{(k)\sigma}]_+ = \delta_{\tau\sigma}/\Omega. \tag{6.36}$$

If we define $\Omega v^{(k)\tau} v^{*(k)\tau}$, which by the anti-commutation law has the eigenvalues 0 and 1, to be the number of quanta of the kind distinguished by the superscripts, we have then to assign negative energy-values to the quanta of the third and fourth kind. But since $\Omega v^{*(k)\tau} v^{(k)\tau}$ has also the eigenvalues 0 and 1, we can define the number of quanta in a different way:

$$\left.\begin{aligned} N^{(k)\tau} &= \Omega v^{(k)\tau} v^{*(k)\tau}, \qquad (\tau = 1, 2), \\ N^{(k)\tau} &= \Omega v^{*(k)\tau} v^{(k)\tau}, \qquad (\tau = 3, 4). \end{aligned}\right\} \tag{6.37}$$

Then we can assign negative load but positive energy to the quanta of the third and fourth kinds:

$$\left.\begin{aligned} E^{(k)} &= \epsilon_k \Omega \{N^{(k)1} + N^{(k)2} + N^{(k)3} + N^{(k)4} - 2\}, \\ \mathbf{p}^{(k)} &= \hbar \mathbf{k} \{N^{(k)1} + N^{(k)2} - N^{(k)3} - N^{(k)4} + 2\}, \\ q^{(k)} &= N^{(k)1} + N^{(k)2} - N^{(k)3} - N^{(k)4} + 2. \end{aligned}\right\} \tag{6.38}$$

Here the zero-point energy, momentum and load correspond to -1 quantum for the quanta of the third and fourth kinds. It could be just as well distributed as $-\frac{1}{2}$ quantum for quanta of all kinds if the mean value $\frac{1}{2}vv^* - \frac{1}{2}v^*v$ were adopted for the transcription of the classical expression vv^*. Then in $\mathbf{p}^{(k)}$ and $q^{(k)}$ the $+2$ disappears while in $E^{(k)}$ the -2 is preserved.

Conclusion

A main feature of the new theory is the fact that the superposition principle holds for each state of a pure field. Interaction between states of different $\mathbf{k}$ is excluded. It can exist only between different types of fields if a proper interaction energy is introduced. This situation determines clearly the programme for further work. One has to introduce a coupling between different fields in an invariant way. It seems to be possible in this way to account for the production of new particles from the primary ones. It remains to be seen whether this idea leads to an understanding of the ultimate particles.

(*Issued separately March* 7, 1944)

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Statistical Mechanics of Fields and the 'Apeiron'

IN view of the difficulties encountered in the quantum theory of fields, we have developed a new approach to this subject which bears a much closer resemblance to ordinary quantum mechanics of particles than the existing theories (Heisenberg and Pauli). We consider the field in a finite volume Ω as a mechanical system described by its total energy and momentum. The latter are obtained from the classical expressions for the energy-momentum densities simply by multiplication with Ω. Each field component is considered as an operator (for the whole volume Ω) and is not regarded as a function of the co-ordinates at all. Between the field components we have established very simple commutation laws, analogous to the ordinary law $pq - qp = \hbar/i$, which depend on the transformation character of the field considered. For example, for the scalar meson field (potential v, its gradient f, its time derivative g; $*$ means Hermitean adjoint; $[a, b]$ means $ab - ba$):

$$[v, g^*] = [v^*, g] = i\frac{\hbar c}{\Omega},$$

$$[g^*, f] = [f^*, g] = k\frac{\hbar c}{\Omega}.$$

Here the vector k is just an abbreviation for the commutators; it is a dynamical variable, but in virtue of the commutation laws a constant of motion, that is, it commutes with all field quantities, with the energy and the momentum.

We have verified[1] that in this case, and also for all other known fields, these commutation laws (or in the case of electrons similar anti-commutation laws) lead, with the transcription

$$[p_x, f] = \frac{\hbar}{i}\frac{\partial f}{\partial x}, \; [E, f] = -\frac{\hbar}{i}\frac{\partial f}{\partial t},$$

to the complete set of field equations and yield the same results as the usual quantization method (integral number of quanta with half-quantum zero energy).

But there appears a *completely new feature* of fundamental importance in the theory, connected with the vector k introduced above. It turns out that k corresponds to the geometrical variable 'wave vector' introduced by Fourier analysis in the usual theory. But it is here a real dynamical variable; its eigen values together with other quantum numbers define the pure states of the field. The distribution of these eigen values depends on the shape of the volume Ω, but can be assumed for a large volume to be uniform in the k_x, k_y, k_z-space (forming a cubic lattice).

In quantum mechanics it is necessary to introduce, apart from pure states, linear combinations of such, called mixtures; but it is not necessary that every possible state should appear in a mixture. (That is the main difference between our new theory and the usual one, because there each Fourier coefficient, considered as a q-number, necessarily appears, having at least its zero energy). A mixture contains in general a selection of $\boldsymbol{k}$-points, each of which may still be occupied by any one of the quanta of the kind considered.

It is necessary to have a name for this sub-group of pure states belonging to the same $\boldsymbol{k}$-value, which is something intermediate between the ordinary notion of a quantum (or particle) and a mechanical system. We suggest to use the word 'apeiron', introduced by the Greek philosopher Anaximander (about 550 B.C.) for the boundless and structureless primordial matter.

For a definite system (pure field) no mechanical knowledge is available to decide which apeiron may appear (which $\boldsymbol{k}$-points are occupied by apeirons). Therefore we have to apply statistical methods. This is a new type of statistics, to be distinguished sharply from the ordinary distribution of quanta over the possible quantum states. For example, in the case of the scalar meson field referred to above, one has not only to assign the number of positive and negative mesons N_k^+, N_k^- (equal to 0, or 1, or 2, . . .) of a specified apeiron $\boldsymbol{k}$, but also the number Δ_k (equal to 0 or 1) which indicates whether a place in the k-space is occupied by an apeiron or not. Both distributions must be made simultaneously; the N_k satisfy in this case the Bose-Einstein statistics (in the case of electrons the Fermi-Dirac statistics), while the n_k evidently always belong to the Fermi-Dirac type of statistics.

If, therefore, the total number of apeirons is a given number n, the distribution of $\bar{\Delta}_k$, the mean value of Δ_k, as a function in the k-space has the features well known from the electron theory of metals. For low temperatures all k-points are occupied up to a finite limit $|\,\boldsymbol{k}\,| = k_m$, while all places with higher $|\,\boldsymbol{k}\,|$ are empty. For higher temperatures this rectangular distribution curve becomes rounded off at the end, but in any event the function $\bar{\Delta}_k$ falls exponentially to zero for $|k| \to \infty$. If $\varepsilon_k \left(N_k^+, N_k^-\right)$ is the energy for the apeiron k in the occupation state N_k^+, N_k^- (for example, for the scalar meson field: $\varepsilon_k = \varepsilon_k^0 \left(N_k^+ + N_k^- + 1\right)$) the total energy is $E = \sum_k \Delta_k \varepsilon_k$; it behaves as if the field were a Fermi-Dirac gas of molecules each of which was an apeiron and therefore had an infinity of states corresponding to the numbers N_k^+, N_k^- of quanta. If now Δ_k is replaced by its average $\bar{\Delta}_k$, the sum E will converge in spite of the zero energy ε_k^0 which s proportional to $|k|$.

It is to be expected that all divergent sums and integrals of the usual photon and meson theory are made convergent by the proper application of the apeiron statistics. In the case of collision phenomena, one has no thermal equilibrium; but the transition probabilities depend not only on the initial and final state but also on all possible 'intermediate' states. The cross-section integrals are extended over the statistical apeiron distribution.

The magnitude of the total number n of apeirons in thermal equilibrium has to be chosen in such a way that the self-energies of the elementary particles are correctly represented; that means that the upper limit k_m of the filled part of the k-space is connected with the radius r_0 of the electron by a relation of the kind $k_m r_0 \sim \hbar$.

The case of the electronic field can be treated in such a way that the contribution to the energy of each electron and positron is positive. But then there is a negative zero energy, and therefore no thermal equilibrium of the apeiron is possible. This does not seem to us to be a real difficulty, because a set of electrons left to themselves is not in equilibrium. Electric clouds like those in metals exist only on account of the fact that in this case the interaction energy with other particles (which is omitted in our considerations) is large compared with the thermal energy.

Investigations on the application of this theory to the elementary particles, their masses and their mutual collision cross-sections are in progress.

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XII.—Quantum Mechanics of Fields. II. Statistics of Pure Fields.

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Introduction

THE quantum mechanics of fields * recently developed by us leads to a modification of statistical mechanics of elementary particles which seems to overcome some of the difficulties (divergence of integrals) occurring in the usual quantum theory of fields. The main difference between the new theory and the usual one is as follows.

In the usual theory the wave-vector **k** is introduced classically and, so to speak, kinematically by the Fourier analysis of the field. The Fourier coefficients of the field components are then treated according to quantum mechanics as non-commuting quantities; those belonging to the wave-vector **k** describe the corresponding "model" mechanical system, namely the **k**th radiation oscillator. But the statement that the Fourier coefficients belonging to a certain **k** all vanish, which statement classically is significant, is now meaningless because there is a lowest state with zero-point energy for each radiation oscillator. The field is thus made to be equivalent to the assembly of radiation oscillators of all possible wave-vectors which, being necessarily infinite in number, contribute an infinite zero-point energy for the pure field and lead to other divergent integrals for the interaction between different fields.

In the new theory, on the other hand, the whole field occupying a certain volume is treated as one mechanical system. The field components, each as a whole, are non-commuting quantities, while the energy and momentum are simply algebraic functions of these. Besides the energy and the momentum there occur three other constants of motion k_x, k_y, k_z forming a vector **k**. The latter are primarily defined as certain commutators of the field components and are, as such, self-adjoint dynamical variables referring to the whole volume of field. It turns out that they commute with all field components, and hence with the energy and the momentum, and also among themselves. Hence if matrices are used to represent the non-commuting quantities, and if the representation is chosen in such a way that the matrices for k_x, k_y, k_z are all diagonal with the eigenvalues (*i.e.* the diagonal elements) of each matrix grouped together when they are equal, then all the matrices representing the field components and also their functions will appear as step-matrices with each step belonging to a certain set of eigenvalues of k_x, k_y, k_z (or, for shortness, belonging to a certain (k)). In view of the way the eigenvalues of k_x, k_y, k_z are grouped together, the different steps always belong to different sets of eigenvalues (*i.e.* different (k)).

This decomposition into submatrices belonging to different (k) in the new theory corresponds to, and replaces, the expansion into Fourier terms of different wave-vectors in the usual theory. As shown in I, the submatrices belonging to (k) for all the field components, the momentum and, when the field equations are linear, the energy, coincide respectively with the matrices representing the corresponding Fourier coefficients of the field components, the momentum and the energy of the **k**th radiation oscillator of the usual theory. When the boundary conditions are taken into account the possible values for (k) are exactly those values which the wave-vector when introduced kinematically can take. (This will be shown in § 1 below.) These possible values for (k) may be taken in the usual way as discrete, forming a three-dimensional **k**-lattice, if the volume of the field is large but finite.

But in the decomposition of the matrices, including those representing the field components, there is no reason why all submatrices belonging to all possible (k) should occur. Exactly which of these actually occur depends on the dynamical state of the system (*i.e.* the volume

* We shall refer to Part I of this series of papers by simply quoting I. See these *Proceedings*, LXII, 1944, 40–57.

of field as a whole) and is described expressly by the constants of motion k_x, k_y, k_z for this dynamical state. This elevation from kinematics to dynamics of the wave-vector $\mathbf{k}$ of the field as a whole is then the main difference between the new theory and the usual one.

It may be remarked in passing that in the usual theory the use of the Fourier expansion, though convenient, is of course not necessary. Indeed the general idea of the usual theory has to be formulated primarily in the co-ordinate representation, namely that the field at the *infinitesimal* neighbourhood of each point $\mathbf{r}$ forms a "model" mechanical system. In this representation the main difference between the new theory and the usual one exhibits itself no less fundamentally in that in the new theory no kinematical analysis of the volume of field into *infinitesimal* neighbourhoods of points is allowed. The only kinematical notion still retained is the size of the volume. In applications this volume refers always to that concerning a measurement. If one analyses this volume into smaller but finite volumes one has to compare the results with those of finer measurements, which is a problem the details of which we need not go into for the present. For the whole volume there exists only *one dynamical* co-ordinate vector, which is the quantum-mechanical conjugate variable of the total momentum contained in this volume of field and undoubtedly serves its only purpose when the motion of the volume of field as a whole is being investigated.

Having thus emphasised the deep-lying character of the difference between the new theory and the usual one, we return to the $\mathbf{k}$-representation and seek for a physical interpretation of the dynamical variables k_x, k_y, k_z of the new theory.

To begin with, we may recall the fact that in ordinary quantum mechanics the properties of a system of *known constitution* (by this term we mean, *e.g.*, the masses of the particles which are assumed to be known, or the frequencies of the oscillators) are described by a set of matrices which are *essentially irreducible*; *i.e.* they cannot all be transformed simultaneously into the form (A here denoting any matrix of the set)

$$\begin{pmatrix} A' & O & \cdots \\ O & A'' & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

without the step-matrices being simply tautological reproduction of the first step,* $A' = A'' = \ldots$ In such cases one can derive all information about the system from the set of the first-step step-matrices, which set is also the only irreducible representation of the non-commuting set of dynamical variables describing the system.

But in the new quantum mechanics of fields the properties of a volume of field turn out to be described by a set of matrices which are really reducible, because the step-matrices belonging to different (k) are distinct. The transition from the type of commutation laws which lead to the irreducible matrices of ordinary quantum mechanics, and the new type of commutation laws which admit non-trivial reducible solutions, corresponds in classical mechanics to the transition from the mechanics of point masses, represented by ordinary differential equations (one independent variable, time t), to the mechanics of continuous matter, represented by partial differential equations of the hyperbolic type (4 independent variables x, y, z, t with the signature $+ + + -$). This is a new aspect of the relation of differential equations to the algebra of non-commuting quantities. It also constitutes a generalisation of the representation theory of the ordinary quantum mechanics dealing with a system of known constitution. Some physical hypothesis has to be introduced for interpreting the use of such reducible representations, and thus for enabling us to decide how the pure states † of the volume of field are to be described in this kind of representation and what the corresponding energy-values are.

* For the demonstration of this theorem see Born and Jordan, *Elementare Quantenmechanik*, Springer, 1930, § 23, for a single particle, and § 17 for a system of particles.

† The notion of "pure state" was used in I, section 3, in a way which did not take account of the fundamental difference between ordinary quantum mechanics and the present field theory concerning the irreducibility of the matrices. We have to correct these statements according to our present better understanding of the mathematical structure of the new theory.

We shall now show that such reducible representation could have been used also in the ordinary quantum mechanics to describe an assembly of systems of known constitution; yet the advantage of using such reducible representation only appears when one wishes to describe an assembly or a grand assembly of systems of variable constitution. To illustrate this we consider for simplicity an assembly of two harmonic oscillators of known frequencies ν_1, ν_2 which are unequal. Instead of studying the Hamiltonian

$$H=\tfrac{1}{2}(p_1^2+\nu_1^2q_1^2)+\tfrac{1}{2}(p_2^2+\nu_2^2q_2^2)$$

with the commutation law $[p_j, q_l]=-i\hbar\delta_{jl}$ which is the usual method, working with essentially irreducible representations, one can also study that reducible representation of the energy-operator $E=\frac{1}{2}(p^2+\nu^2q^2)$ with the commutation law $[p, q]=-i\hbar$ in which the frequency-operator is represented by $\nu=\begin{pmatrix}\nu_1 & 0\\ 0 & \nu_2\end{pmatrix}$, ν_1, ν_2 being the step-matrices. By comparing the two ways of describing the same assembly we obtain the rule for the enumeration of the pure states.

If an assembly is described by a reducible representation, the pure states of the assembly are obtained by combining the pure states of the various step-matrices in all possible ways. The energy-value for a pure state of the assembly is obtained by summing the corresponding eigenvalues of the various step-matrices of the energy-operator.

However, if the above assembly of two harmonic oscillators were of variable frequencies the values of which depend on the knowledge derived from measurements, the method working with the reducible representation is then by far the simpler to apply. This method can also be used for a grand assembly of varying number of systems which then corresponds to various dimensions of reducible representations. The totality of the pure states of the various reducible representations gives the pure states of the assembly or grand assembly of systems of variable constitution.

Adopting this use of reducible representations, we may conversely consider a volume of field as an assembly of systems of variable (k). We suggest the term *apeiron* * for the system described by the step-matrices; an apeiron of wave-vector (k) is the **k**th radiation oscillator,† considered not as a virtual assembly of states, but as a kind of material particle (like a photon). It has as many degrees of freedom as the number of independent polarisations for a real field and twice this number for a complex field. For each degree of freedom there exists a quantum number N = 0, 1, 2, . . ., which in the usual way may be interpreted as the number of quanta of a certain kind occupying the apeiron.

The reducible representation with k_x, k_y, k_z represented by

$$k_x=\begin{pmatrix} k_x' & & & & \\ & k_x'' & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & k_x^{(n)}\end{pmatrix},\quad k_y=\begin{pmatrix} k_y' & & & & \\ & k_y'' & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & k_y^{(n)}\end{pmatrix},\quad k_z=\begin{pmatrix} k_z' & & & & \\ & k_z'' & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & k_z^{(n)}\end{pmatrix}$$

where the step-matrices k_x', k_y'', etc. are multiples of unit matrices, then describes an assembly of n apeirons of wave-vectors $\mathbf{k}'$, $\mathbf{k}''$, . . . $\mathbf{k}^{(n)}$. We thus reach the following physical interpretation for the constants of motion k_x, k_y, k_z of the field; namely that, by their being reducible into various distinct steps, they describe the entire configuration of the wave-vectors. (They may therefore be called the configuration wave-vector of the assembly of apeirons.) Since the step-matrices belonging to a certain (k) are essentially irreducible, no two apeirons can have the same value for their wave-vector. This exclusion principle follows from the way we are using reducible representations.

* The term ἄπειρον was introduced by Anaximander of Miletos (about 550 B.C.) for the boundless and shapeless primordial matter which is the first product (arché, ἀρχή) of the creation and develops into the specific types of ordinary matter.

† The term radiation oscillator used throughout this paper differs from its current use in the minor point that we do not analyse it into simple harmonic oscillators, one for each polarisation.

The pure states of the volume of field are to be enumerated by indicating those wave-vectors for which apeirons appear, together with the quantum numbers indicating how each apeiron is occupied by quanta. The volume of field is thus equivalent to a grand assembly of apeirons, and can be compared to a volume of gaseous molecules.

We have therefore to treat a volume of field by statistical methods. In the following sections we shall consider in detail the canonical distribution of the apeirons, particularly for the electromagnetic field.* We shall see that Planck's law is essentially valid. The apeiron distribution is like that of a Fermi gas with a high degeneration temperature, if we assume that the number of apeirons in a finite volume is finite and take the zero-energy of the apeirons seriously for statistical consideration. Since this zero-energy cannot be measured as heat, the "temperature" parameter characterising the apeiron distribution may be unrelated to the temperature measured with thermometers. The distribution of the apeirons affects, however, many observable quantities, such as the cross-sections for collision processes which will be treated in subsequent papers. But it is clear that with such a distribution of the apeirons all sums over the assembly of apeirons are convergent.

1. Probability of Distribution

The mechanical treatment of a field, as explained in I, led to the following description of a pure field. Using the reducible **k**-representation (see I and the Introduction of this paper), the Schroedinger wave-equations for the volume of field are reduced to the following for the apeirons:—

$$\mathbf{p}^{(k)}\Psi^{(k)}=\frac{\hbar}{i}\,\mathrm{grad}\,\Psi^{(k)};\qquad \mathrm{E}^{(k)}\Psi^{(k)}=-\frac{\hbar}{i}\frac{\partial\Psi^{(k)}}{\partial t}. \tag{1.1}$$

The step-matrices $\mathbf{p}^{(k)}$, $\mathrm{E}^{(k)}$ are here represented by operators; these are functions of the operators $a_\tau^{(k)}$, $b_\tau^{(k)}$ and their adjoints which represent the step-matrices of the field components (*cf.* I, § 2). Here the index τ indicates the polarisation; the total number of independent polarisations is 1 for a scalar field, 2 for the electromagnetic field, and 3 for the vector-meson field (I, § 5). We shall confine ourselves to such fields. Owing to the fact that the field components are complex quantities there are two operators, denoted here by a and b, for each wave-vector † **k** and polarisation τ. For a real field the a's and the b's of opposite wave-vectors are identical, thus reducing these field variables by one-half.

The wave-functions involve x, y, z, t as parameters; as their arguments one may take the eigenvalues $a_{k\tau}$, $b_{k\tau}$ of the operators $a_\tau^{(k)}$, $b_\tau^{(k)}$; *i.e.*

$$\Psi^{(k)}=\Psi_k(a_{k\tau},b_{k\tau};\ x,y,z,t). \tag{1.2}$$

The operators $a_\tau^{(k)}$, $b_\tau^{(k)}$, applied to (1.2), simply multiply the wave-function by $a_{k\tau}$, $b_{k\tau}$ respectively; while their adjoint operators, applied to (1.2), produce the partial derivatives with respect to $a_{k\tau}$, $b_{k\tau}$ respectively (see I, § 3, equation (3.10)).

Since the operators $\mathbf{p}^{(k)}$ and $\mathrm{E}^{(k)}$ do not involve x, y, z, t, the solution of (1.1) is, by the method of separation of variables, a linear combination of the eigenfunctions

$$\psi_{k,\mathrm{N}}(a_{k\tau},b_{k\tau})e^{i(\mathbf{p}_k\cdot\mathbf{r}-\mathrm{E}_k t)/\hbar}. \tag{1.3}$$

The eigenvalues $\mathbf{p}_k$ and E_k are (after the symmetrisation of factors, see I)

$$\mathbf{p}_k=\hbar\mathbf{k}\sum_\tau(\mathrm{N}_{k\tau+}-\mathrm{N}_{k\tau-}), \tag{1.4}$$

$$\mathrm{E}_k=\epsilon_k\sum_\tau(\mathrm{N}_{k\tau+}+\mathrm{N}_{k\tau-}+1),\qquad \epsilon_k=\hbar c\sqrt{\eta^2+k^2}, \tag{1.5}$$

---

* It can be seen that the application of statistical mechanics to the apeirons of a pure electron (Dirac) field leads to no canonical equilibrium.

† Here and in what follows **k** denotes the wave-vector of one apeiron. We regret that the same symbol has been used in I both for this and for the configuration wave-vector of the assembly of apeirons, but the context and the other symbols which occur in an equation will determine the meaning of **k**.

where the last equation is restricted to the case of linear field equations, and $\hbar\eta/c$ is the rest mass of the quanta (zero for photons). For a complex field the eigenvalues $N_{k\tau\pm}$ represent the number of quanta of positive and negative load (*i.e.* charge in the elementary unit), the eigenvalue of the load being

$$q_k = \sum_{\tau} (N_{k\tau+} - N_{k\tau-}). \tag{1.6}$$

For a real field the eigenvalues $N_{k\tau\pm}$ satisfy the restriction

$$N_{\bar{k}\tau+} = N_{k\tau-} \equiv N_{\bar{k}\tau}, \qquad N_{k\tau+} = N_{\bar{k}\tau-} \equiv N_{k\tau}, \tag{1.7}$$

where $\bar{k}$ and k refer to any two apeirons of equal but opposite wave-vectors.

In thermodynamical considerations the assembly of apeirons has to be considered as enclosed in a box with walls. If the volume Ω of the box is large and if the shape of the box is rectangular, one can replace the (unknown) boundary conditions at the walls by the postulate of periodicity of the wave-functions of the apeirons. Since $\sum_{\tau}(N_{k\tau+} - N_{k\tau-})$ can assume all integral values, one has from (1.3), for a cube of volume Ω,

$$k_x = \frac{2\pi}{\sqrt[3]{\Omega}} h_x, \qquad k_y = \frac{2\pi}{\sqrt[3]{\Omega}} h_y, \qquad k_z = \frac{2\pi}{\sqrt[3]{\Omega}} h_z, \tag{1.8}$$

with integral values h_x, h_y, h_z forming a **k**-lattice. By the usual consideration there are thus

$$g_s = \frac{\Omega}{(2\pi)^3} dk_x dk_y dk_z \tag{1.9}$$

possible values for the wave-vector of an apeiron to lie in the sth interval, $\mathbf{k}$ to $\mathbf{k} + d\mathbf{k}$.

Let $\Delta_k = 1$ or 0 if the possible value $\mathbf{k}$ for the wave-vector (called the $\mathbf{k}$th radiation oscillator in the Introduction) is taken up by an apeiron or not. If $\Delta_k = 1$ the apeiron can be occupied by a certain number $N_{k\tau\pm}$ of quanta of the kind $\tau\pm$. Hence the total number of apeirons and the total number of quanta of the kind $\tau\pm$ in the sth interval are respectively:

$$n_s = \sum_{\mathbf{k}}^{\mathbf{k}+d\mathbf{k}} \Delta_k, \qquad \nu_{s\tau\pm} = \sum_{\mathbf{k}}^{\mathbf{k}+d\mathbf{k}} \Delta_k N_{k\tau\pm}. \tag{1.10}$$

We have confined ourselves to such fields as the electromagnetic field where the number of quanta can assume the values 0, 1, 2, 3, . . . The occupation of the n_s apeirons by the $\nu_{s\tau\pm}$ quanta thus obeys Bose's statistics. Since the occupation by quanta of different kinds is uncorrelated for a complex field, we have for the probability for the distribution where there are $\nu_{s\tau\pm}$ quanta of the kind $\tau\pm$ occupying some or all of the n_s apeirons of the sth interval, and so on for all other intervals,

$$\log W_B = \sum_s \sum_\tau \sum_\pm B_{s\tau\pm} \tag{1.11}$$

$$B_{s\tau\pm} = (n_s + \nu_{s\tau\pm}) \log (n_s + \nu_{s\tau\pm}) - n_s \log n_s - \nu_{s\tau\pm} \log \nu_{s\tau\pm}. \tag{1.12}$$

On the other hand, since Δ_k can only be 0 or 1, we apply Fermi's statistics and get for the probability for the distribution for which n_s apeirons occupy some or all of the g_s possible $\mathbf{k}$-values of the sth interval, and so on for all other intervals,

$$\log W_F = \sum_s F_s \tag{1.13}$$

$$F_s = g_s \log g_s - n_s \log n_s - (g_s - n_s) \log (g_s - n_s). \tag{1.14}$$

The compound probability W for the distribution where n_s apeirons and $\nu_{s\tau\pm}$ quanta of the kind $\tau\pm$ are in the sth interval, and so on for all other intervals, is therefore given by

$$\log W = \log (W_F W_B) = \sum_s F_s + \sum_s \sum_\tau \sum_\pm B_{s\tau\pm}. \tag{1.15}$$

We note that for this distribution the total momentum, the total energy, the total load and the total number of apeirons of the assembly are respectively, by (1.10) and (1.4), (1.5), (1.6),

$$\mathbf{p}=\sum_k \Delta_k \mathbf{p}_k=\sum_s \hbar\mathbf{k}_s \sum_\tau (\nu_{s\tau+}-\nu_{s\tau-}); \tag{1.16}$$

$$\mathrm{E}=\sum_k \Delta_k \mathrm{E}_k=\sum_s \epsilon_s \sum_\tau (\nu_{s\tau+}+\nu_{s\tau-}+1), \qquad \epsilon_s=\hbar c\sqrt{\eta^2+k_s^2}; \tag{1.17}$$

$$q=\sum_k \Delta_k q_k=\sum_s \sum_\tau (\nu_{s\tau+}-\nu_{s\tau-}); \tag{1.18}$$

$$n=\sum_k \Delta_k =\sum_s n_s. \tag{1.19}$$

For a real field we have, from (1.7) and (1.10),

$$n_{\bar{s}}=n_s; \qquad \nu_{\bar{s}\tau+}=\nu_{s\tau-}\equiv\nu_{\bar{s}\tau}, \qquad \nu_{s\tau+}=\nu_{\bar{s}\tau-}\equiv\nu_{s\tau}; \tag{1.20}$$

so that, by (1.18), $q=0$; the load-operator becomes insignificant. The remaining equations given above hold, provided we add a factor one-half everywhere in connection with the summation over s. This factor comes in in (1.11) and (1.13) because the distribution is now correlated by (1.20), and in (1.16), (1.17), (1.19) because the step-matrices are now identical in pairs.

It is convenient to adopt throughout this paper the convention that a factor one-half is always hidden in the summation symbol over s for real fields.

2. Canonical Distribution of the Quanta

We assume in this section that the apeiron distribution is arbitrarily given. The canonical distribution of the quanta is then obtained by making (1.15) a maximum under suitable subsidiary conditions.

Since we have enclosed the assembly of apeirons and quanta in a box with walls, the total momentum (1.16) of the assembly in equilibrium is zero. The total energy of the quanta is constant

$$\mathrm{U}=\sum_k \Delta_k \epsilon_k \sum_\tau (\mathrm{N}_{k\tau+}+\mathrm{N}_{k\tau-})=\sum_s \epsilon_s \sum_\tau (\nu_{s\tau+}+\nu_{s\tau-}). \tag{2.1}$$

(This differs from the energy-value (1.17) of the assembly merely by the zero-energy of the apeirons, which is known from the given apeiron distribution.) The total load (1.18) is also constant; but it vanishes identically for a real field.

Taking into account these subsidiary conditions we get, for the determination of the canonical distribution of the quanta,

$$\frac{\partial \log \mathrm{W}}{\partial \nu_{s\tau\pm}}=\boldsymbol{\alpha}\cdot\frac{\partial \mathbf{p}}{\partial \nu_{s\tau\pm}}+\beta\frac{\partial \mathrm{U}}{\partial \nu_{s\tau\pm}}+\gamma\frac{\partial q}{\partial \nu_{s\tau\pm}}, \tag{2.2}$$

where $\boldsymbol{\alpha}$, β, γ are Lagrange multipliers. Substituting (1.15), (1.12), (1.16), (2.1) and (1.18) into (2.2), we get

$$\log\frac{n_s+\nu_{s\tau\pm}}{\nu_{s\tau\pm}}=\pm\boldsymbol{\alpha}\cdot\hbar\mathbf{k}_s+\beta\epsilon_s\pm\gamma \tag{2.3}$$

or

$$\nu_{s\tau\pm}=\frac{n_s}{e^{\pm\boldsymbol{\alpha}\cdot\hbar\mathbf{k}_s+\beta\epsilon_s\pm\gamma}-1}. \tag{2.4}$$

This indicates that the distribution is independent of the polarisation. It is easily seen that $\boldsymbol{\alpha}$ vanishes identically because the total momentum vanishes. γ is to be determined from the total load and vanishes if the latter vanishes. Indeed for a real field γ should have never been introduced because the total load vanishes *identically*. We have therefore for the canonical distribution of photons,

$$\nu_{s\tau}=\frac{n_s}{e^{\beta\epsilon_s}-1}=\nu_{\bar{s}\tau}. \tag{2.5}$$

We shall prove that the multiplier β is inversely proportional to the temperature T. Consider a volume of electromagnetic radiation and change the volume Ω slowly by a reversible process, allowing transfer of heat to take place which means a variation of the number of quanta. We have, for the total variation of log W, **p** and U,

$$\left.\begin{aligned}\delta \log W &= \sum_s \sum_\tau \sum_\pm \frac{\partial \log W}{\partial \nu_{s\tau\pm}} \delta\nu_{s\tau\pm} \\ &= \sum_s \sum_\tau \sum_\pm (\pm \boldsymbol{\alpha} \cdot \hbar\mathbf{k}_s + \beta\epsilon_s)\delta\nu_{s\tau\pm},\end{aligned}\right\} \tag{2.6}$$

$$\delta\mathbf{p} = \sum_s \sum_\tau \hbar\mathbf{k}_s(\delta\nu_{s\tau+} - \delta\nu_{s\tau-}) + \sum_s \sum_\tau (\nu_{s\tau+} - \nu_{s\tau-})\frac{\partial\hbar\mathbf{k}_s}{\partial\Omega}\delta\Omega, \tag{2.7}$$

$$\delta U = \sum_s \sum_\tau \epsilon_s(\delta\nu_{s\tau+} + \delta\nu_{s\tau-}) + \sum_s \sum_\tau (\nu_{s\tau+} + \nu_{s\tau-})\frac{\partial\epsilon_s}{\partial\Omega}\delta\Omega. \tag{2.8}$$

The second equality of (2.6) follows from the assumption that the radiation was originally a canonical distribution of quanta. The terms of (2.7), (2.8) containing the variation of $\mathbf{k}_s$ with respect to the volume Ω are to be calculated from (1.8) without * changing the quantum numbers h_x, h_y, h_z.

On combining (2.6), (2.7), and (2.8) we get, on dividing both sides by β,

$$\frac{1}{\beta}\delta \log W = \delta U + \frac{1}{\beta}\boldsymbol{\alpha} \cdot \delta\mathbf{p} + P\delta\Omega, \tag{2.9}$$

where P stands for the expression

$$P = -\sum_s \sum_\tau (\nu_{s\tau+} + \nu_{s\tau-})\frac{\partial\epsilon_s}{\partial\Omega} - \frac{1}{\beta}\boldsymbol{\alpha} \cdot \sum_s \sum_\tau (\nu_{s\tau+} - \nu_{s\tau-})\frac{\partial\hbar\mathbf{k}_s}{\partial\Omega}. \tag{2.10}$$

Since we can treat the motion of the total volume of radiation by classical mechanics, we have $\delta\mathbf{p} = \mathbf{f}\delta t$, where $\mathbf{f}$ is the total external force acting on the volume of radiation during the interval of time δt. Thus the last two terms of (2.9) represent the total work done by the system

$$-\delta w = P\delta\Omega - \mathbf{v} \cdot \mathbf{f}\delta t, \tag{2.11}$$

and hence P signifies the pressure and

$$\mathbf{v} = -\boldsymbol{\alpha}/\beta \tag{2.12}$$

is the molar velocity of the total system.

Comparing (2.9) with the first law of thermodynamics

$$\delta Q = \delta U - \delta w, \tag{2.13}$$

we see that β is an integrating factor of the heat transfer δQ and hence, according to the second law of thermodynamics, proportional to $1/T$. As usual we put (here k denotes the Boltzmann constant)

$$\beta = 1/kT, \qquad S = k \log W, \tag{2.14}$$

then S is the entropy.

(2.9) shows at once that $(-\boldsymbol{\alpha}, \beta) = (\mathbf{v}/kT, 1/kT)$ form a four-vector. Hence temperature is not a scalar but its reciprocal is the fourth component of a four-vector. This has been already considered in a formal way by Bergmann.†

In our present case, however, we have, by our use of the **k**-representation which is the analogue of the usual Fourier analysis, committed ourselves to the system of reference which is at rest with the walls of the volume Ω. Hence, by (2.12), we see again that $\boldsymbol{\alpha} = 0$.

In equilibrium the distribution of the quanta must be isotropic. This imposes, by (2.5), a condition on the distribution of the apeirons. For an isotropic distribution of quanta it

* This follows from the adiabatic principle of quantum mechanics.

† Bergmann, *Phys. Rev.*, LIX, 1941, 928.

can be easily verified that the pressure (putting $\alpha=0$ in (2.10))

$$P = -\sum_s \sum_\tau \sum_\pm \nu_{s\tau\pm} \frac{\partial \epsilon_s}{\partial \Omega} \tag{2.15}$$

can be considered as produced by the rebounding of the quanta on the walls. For this purpose consider the group velocity of the waves which, according to de Broglie, represents the particle velocity and is defined by the derivate of the energy with respect to the momentum. In our case the momentum is

$$\hbar \mathbf{k}_{s\pm} = \pm \hbar \mathbf{k}_s \tag{2.16}$$

for the positive or negative quanta; the energy is ϵ_s; and therefore the group velocity is

$$\mathbf{u}_{s\pm} = \frac{\partial \epsilon_s}{\partial \hbar \mathbf{k}_{s\pm}}. \tag{2.17}$$

Introducing this into (2.15) and using (1.8) we get for the pressure

$$P = -\sum_s \sum_\tau \sum_\pm \nu_{s\tau\pm} \frac{\partial \epsilon_s}{\partial \hbar \mathbf{k}_{s\pm}} \cdot \frac{\partial \hbar \mathbf{k}_{s\pm}}{\partial \Omega} = \frac{1}{3\Omega} \sum_s \sum_\tau \sum_\pm \nu_{s\tau\pm} (\mathbf{u}_{s\pm} \cdot \hbar \mathbf{k}_{s\pm}). \tag{2.18}$$

Consider, on the other hand, the momentum imparted to the wall by the quanta, the momenta of which are $\hbar \mathbf{k}_{s\pm}$; their velocities $\mathbf{u}_{s\pm}$ and their number for unit volume $\nu_{s\tau\pm}/\Omega$. The pressure produced by the impact and rebounding on the wall is, by the considerati familiar from the kinetic theory of gases,

$$P_x = \frac{1}{\Omega} \sum_s \sum_\tau \sum_\pm \nu_{s\tau\pm}\, u_{xs\pm}\, \hbar k_{xs\pm}, \tag{2.19}$$

where the inward normal to the wall is chosen to be the x-axis. The summation over s extends over all directions of $\mathbf{k}$, including the quanta hitting the wall ($\hbar k_{xs\pm} < 0$) as well as the quanta rebounding from the wall ($\hbar k_{xs\pm} > 0$). For an isotropic distribution of quanta comparison of (2.18) and (2.19) shows immediately

$$P_x = P_y = P_z = P. \tag{2.20}$$

For a real field the same result holds. The hidden factor one-half in the summation symbol compensates the superfluous summation over the positive and negative quanta which are physically indistinguishable.

For photons, since the rest mass vanishes, (2.15) gives the well-known relation between the radiation pressure and density:

$$P = \frac{1}{3\Omega} \sum_s \sum_\tau \sum_\pm \nu_{s\tau\pm} \epsilon_s = \frac{1}{3} \frac{U}{\Omega}. \tag{2.21}$$

We have thus established the physical meaning of U and β. Comparing (2.5) with Planck's law of black-body radiation, we conclude that for all frequencies up to the ultra-violet where Planck's law has been experimentally verified n_s cannot differ appreciably from g_s. That is to say, all the harmonic oscillators of low frequencies must be occupied by apeirons for the electromagnetic field.

3. Canonical Distribution of the Apeirons

We consider now the distribution of the quanta as arbitrarily given and determine the canonical distribution of the apeirons by making (1.15) a maximum under suitable subsidiary conditions.

For the subsidiary conditions we assume the total number of apeirons (1.19) and the total zero-energy U′ of the apeirons are constant,

$$U' = \sum_k \Delta_k \sum_\tau \epsilon_k = \varpi \sum_s n_s \epsilon_s. \tag{3.1}$$

Here ϖ denotes the total number of independent polarisations of the field. U′ differs from the energy-value (1.17) by the known energy of the quanta.

Taking (1.19) and (3.1) into account, we get for the determination of the canonical distribution of the apeirons

$$\frac{\partial \log W}{\partial n_s} = \beta' \frac{\partial U'}{\partial n_s} - \beta' \mu \frac{\partial n}{\partial n_s}. \tag{3.2}$$

The two Lagrange multipliers are here denoted by β' and $-\beta'\mu$. Substituting (1.15), (1.19), and (3.1) into (3.2), we get

$$\log \frac{g_s - n_s}{n_s} + \sum_{\tau} \sum_{\pm} \log \frac{n_s + \nu_{s\tau\pm}}{n_s} = \beta'(\varpi \epsilon_s - \mu) \tag{3.3}$$

or

$$n_s = \frac{g_s}{e^{\beta'(\varpi\epsilon_s - \mu)} \prod_{\tau} \prod_{\pm} \frac{n_s}{n_s + \nu_{s\tau\pm}} + 1}. \tag{3.4}$$

This is essentially a Fermi distribution with a degeneration temperature proportional to μ. The factor involving $\nu_{s\tau\pm}$ indicates how the apeiron distribution is influenced by the quanta occupying the apeirons. But this influence is only appreciable where the first term of the denominator of (3.4) is comparable to unity, *i.e.* where the occupation of the harmonic oscillators by the apeirons is falling off. In this region the distribution of the apeirons depends very much on how they are being occupied by quanta.

If the given distribution of the quanta is canonical we introduce (2.4) (putting $\alpha = 0$) into (3.4) and get

$$n_s = \frac{g_s}{e^{\beta'(\varpi\epsilon_s - \mu)}(1 - 2e^{-\beta\epsilon_s}\cosh\gamma + e^{-2\beta\epsilon_s})^{\varpi} + 1}. \tag{3.5}$$

In the case of the electromagnetic field we must suppose $\beta'\mu >> 1$ so that the apeiron distribution is degenerate in order to meet the demand $n_s = g_s$ for all frequencies up to the ultra-violet where Planck's law has been verified. The falling off of the apeiron distribution then occurs at such high energy ϵ_s that the factor of (3.5) containing $e^{-\beta\epsilon_s}$ can be replaced by unity for all ordinary temperatures. Substituting (1.9) for g_s and making use of the fact that the rest mass for the photons vanishes, we get from (3.5) for the number of electromagnetic apeirons of an energy between ϵ and $\epsilon + d\epsilon$:

$$n_s = \frac{\Omega}{2\pi^2\hbar^3c^3} \cdot \frac{\epsilon^2 d\epsilon}{e^{\beta'(2\epsilon - \mu)} + 1}. \tag{3.6}$$

The number of polarisations is $\varpi = 2$. Remembering the hidden factor one-half in the summation symbol over s, we get from (1.19) and (3.1)

$$n = \frac{\Omega}{4\pi^2\hbar^3c^3} \int_0^{\infty} \frac{\epsilon^2 d\epsilon}{e^{\beta'(2\epsilon - \mu)} + 1}, \tag{3.7}$$

$$U' = \frac{\Omega}{2\pi^2\hbar^3c^3} \int_0^{\infty} \frac{\epsilon^3 d\epsilon}{e^{\beta'(2\epsilon - \mu)} + 1}. \tag{3.8}$$

These integrals can be treated by the usual method dealing with a degenerate electron gas, but the results in our case are much simpler and more exact because the numerators of the integrand are rational. The expansion into powers series of $1/\beta'\mu$ now terminates; the maximum percentage error in n or U′ is due to the neglect of $e^{-\beta'\mu}(\beta'\mu)^2$ or $e^{-\beta'\mu}(\beta'\mu)^4$ in comparison to unity. We get thus for n and U′

$$n = \frac{\Omega\mu^3}{96\pi^2\hbar^3c^3}\left\{1 + \left(\frac{\pi}{\beta'\mu}\right)^2\right\}, \tag{3.9}$$

$$U' = \frac{\Omega\mu^4}{128\pi^2\hbar^3c^3}\left\{1 + 2\left(\frac{\pi}{\beta'\mu}\right)^2 + \frac{7}{15}\left(\frac{\pi}{\beta'\mu}\right)^4\right\}. \tag{3.10}$$

Since μ has to be much larger than the energy of an ultra-violet photon, (3.9) shows that the number of apeirons per unit volume must be larger than one per ultra-violet wave-length cube. The energy-density due to the zero-energy of the apeirons is, by (3.10), larger than that of one ultra-violet photon per ultra-violet wave-length cube, and is therefore larger than the average density of matter (times c^2) of the universe.

One might speculate that the number of apeirons per unit volume is even larger so that there will be one apeiron per volume of an elementary particle, *i.e.* per cube of the "electron radius."

Owing to the fact that the physical rôle played by the zero-energy is as yet unknown, we mention a few possibilities of the canonical distribution.

(*a*) One might treat the zero-energy on the same footing as that associated with the emission and absorption of quanta. For the canonical distribution of the quanta and apeirons one has then to introduce one parameter β for the subsidiary condition $E = U + U' =$ constant (see (1.17), (2.1), (3.1)). This makes β' identical with β which, as shown in section 2, is inversely proportional to the temperature measured with thermometers. Since the zero-energy does not contribute to the specific heat of a volume of black-body radiation, one has

$$U' = U_0' = \frac{\Omega\mu_0^4}{128\pi^2\hbar^3c^3}, \tag{3.11}$$

where μ_0 is the value of μ at zero temperature. Then (3.9) gives the variation of the total number of apeirons per unit volume with respect to temperature:

$$\left.\begin{aligned} n &= n_0\sqrt{1+\tfrac{8}{15}\xi^2}\sqrt{\sqrt{1+\tfrac{8}{15}\xi^2}-\xi} \\ &= n_0(1-\tfrac{1}{2}\xi+\tfrac{11}{40}\xi^2-\;.\;.\;.), \\ n_0 &= \frac{\Omega\mu_0^3}{96\pi^2\hbar^3c^3}, \qquad \xi \equiv \left(\frac{\pi kT}{\mu_0}\right)^2 << 1 \end{aligned}\right\} \tag{3.12}$$

(*b*) One might consider the hypothesis that the zero-energy has a separate physical significance unrelated to heat. Then β' is unrelated to the temperature.

If the material walls enclosing the radiation interact with the radiation only in so far as to emit and absorb quanta, the walls may be regarded as permeable for those apeirons, which are unoccupied by quanta. β' is then a cosmic parameter which essentially determines the distribution of the apeirons throughout the universe.

(*c*) In both cases considered above the enormous total zero-energy creates a difficulty for the existing theory of gravitation. Hence one might reject the existence of the zero-energy of the apeiron. This can be done in the new quantum mechanics of fields, in the same way as in the usual theory, by simply omitting the zero-energy because the latter amounts only to a constant and does not affect the equations of motion. One is then at liberty to assume any apeiron distribution guided by known facts, such as the experiments verifying Planck's law and the finite cross-section for collision processes.

We shall leave the decision among these possibilities to future investigation.

We remark that the canonical distribution of the quanta and the apeirons can very simply be obtained by means of the grand partition function *

$$\Xi = \sum_{(q.\,no.)} e^{-\beta\sum\limits_{k}\sum\limits_{\tau}\Delta_k(N_{k\tau+}+N_{k\tau-})\epsilon_k-\gamma\sum\limits_{k}\sum\limits_{\tau}\Delta_k(N_{k\tau+}-N_{k\tau-})-\beta'\sum\limits_{k}\Delta_k\tilde{\omega}\epsilon_k+\zeta\sum\limits_{k}\Delta_k} \tag{3.13}$$

Here the summation (*q. no.*) is over all the quantum numbers Δ_k, $N_{k\tau\pm}$ that are needed to specify a pure state of the volume of field. The parameters β, γ, β', ζ define the grand canonical distribution and are related to the average values of U, q, U$'$, n by

$$U = -\frac{\partial\log\Xi}{\partial\beta}, \qquad q = -\frac{\partial\log\Xi}{\partial\gamma}, \qquad U' = -\frac{\partial\log\Xi}{\partial\beta'}, \qquad n = \frac{\partial\log\Xi}{\partial\zeta}. \tag{3.14}$$

* For the use of grand partition functions see, *e.g.*, Fowler and Guggenheim, *Statistical Thermodynamics*, Cambridge University Press, 1939, Chap. VI.

The evaluation of Ξ is elementary and simple; it yields

$$\log \Xi = \sum_k \log \{1 + e^{-\beta' \tilde{\omega} \epsilon_k + \zeta} (1 - 2e^{-\beta \epsilon_k} \cosh \gamma + e^{-2\beta \epsilon_k})^{\varpi}\}. \tag{3.15}$$

Substituting this in (3.14) and comparing the resultant expressions with those given above expressing U, q, U′, n as sums over the distribution of the apeirons and quanta, we obtain the same results (2.4) (with $\alpha = 0$) and (3.5) (with $\zeta = \beta' \mu$) for the canonical distribution. This method is particularly satisfactory because it deals with the total system (*i.e.* the volume of field or, what is the same, the assembly of apeirons and quanta) as a whole, expressing all its statistical properties with the help of one partition function which automatically leads to the Bose statistics for the quanta and the Fermi statistics for the apeirons (treated separately in §§ 1, 2 and 3). It also opens the way to applying the powerful methods of Darwin and Fowler.*

* See, for instance, Fowler, *Statistical Mechanics*, Cambridge University Press, 1936.

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XV.—Quantum Mechanics of Fields. III. Electromagnetic Field and Electron Field in Interaction.

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Introduction

STUDYING the interaction of different pure fields, we have been led to some essential modifications of the ideas on which our quantum mechanics of fields is based. We shall explain these here for the example of the interaction of the Maxwell and the Dirac field.

In Part I * we showed that a pure field in a given volume Ω can be described by considering the potentials and field components as matrices, not attached to single points in Ω (as the theory of Heisenberg and Pauli), but to the whole volume. Further, we assumed the total energy and momentum to be the product of Ω and the corresponding densities. In Part II † we showed that this conception has to be modified; the eigenvalues of the energy and momentum as defined in Part I represent neither the states of single particles nor of a system of particles, but of something intermediate which corresponds to the simple oscillators of Heisenberg-Pauli and which we have called *apeirons*. The total energy and momentum of the system is a sum over the contributions of an assembly of apeirons. Mathematically the differences of the quantum mechanics of a field from that of a set of mass points (as treated in ordinary quantum mechanics) is the fact that the matrices representing a field are reducible (while those representing co-ordinates of mass points are irreducible); each irreducible submatrix corresponds to an apeiron.

The considerations of Part II make it obvious that the correct theory of quantised fields must be a much closer union of mechanics and statistics than we had anticipated in Part I.

A second indication of the need for more precise definitions and modifications is now obtained from the consideration of the nature of the interaction terms in the Lagrangian. We have assumed in Part I that the Lagrangian characterising a system is given by the usual function of the field quantities taken from the classical theory. So long as one has to do with pure fields this leads to no ambiguity. In the classical theories of the photon, meson, or electron field the Lagrangian is of the second degree in the field quantities. If one introduces a Fourier transformation (transition to the momentum space) the new expressions are still of the second degree and essentially the same as the original ones. It is true, we have also considered the case of arbitrary functions (non-linear equations); but we cannot ascertain whether the general function assumed to be the Lagrangian has as arguments the field quantities in the ordinary space or in the Fourier space. Hence the general theory is formally correct but has very little concrete content.

If we now consider the coupling of two pure fields we cannot avoid a decision about the function by which we represent the interaction. In the case (Maxwell plus Dirac)-field this function is of the 3rd degree in the space representation; its Fourier transformed, also of the 3rd degree, is a quite different and much more complicated function. Which of them is the correct one? If this question has been decided, the function chosen will then be considered as a matrix function and remains as such unchanged for every canonical (matrix) transformation. Hence if we have to choose the Fourier transformed function we would maintain that this rather complicated function is also the correct expression for the quantum theory of the fields in the space representation; the simpler classical expressions would then be only approximately true, in the sense of the correspondence principle.

Now we have a guide for this decision in the fact that the Heisenberg-Pauli theory, apart from the well-known awkward divergent terms, satisfactorily represents the facts of absorption,

* These *Proceedings*, LXII, 1944, 40. † *Ibid.*, 92.

emission, scattering, etc. This theory is based on the Fourier representation (momentum space). Hence we have to consider this representation as fundamental, and we have to choose the Lagrangian correspondingly, replacing Fourier coefficients by matrices in the same way as in the first approach to quantum mechanics (Heisenberg-Born-Jordan).

The whole argument which leads to a new and satisfactory theory for interacting fields is based on the correspondence postulate, and in order to make this clear it seemed to us advisable to go back to first principles. Therefore we begin with the classical form of the theory and introduce then the quantisation method of Heisenberg and Pauli. It is now an easy step to replace this semi-classical procedure by our new method, which is a generalised quantum mechanics using only non-commuting quantities.

A main feature of this theory is the definition of the total energy and momentum by the traces of the matrices representing the densities; this is an obvious generalisation of the apeiron sums used in Part II. It is further necessary to generalise the commutation laws for the field quantities given in Part I in such a way that they include the commutation of any quantity with that which is produced from it by permuting the apeirons; in this manner the full correspondence with the Heisenberg-Pauli commutation laws is obtained.

That the new theory is not less satisfactory than that of Heisenberg and Pauli is obvious from its derivation. The difference can be expressed by saying that the new theory admits an arbitrary apeiron distribution, while that of Heisenberg and Pauli assumes a uniform apeiron distribution (in momentum space). Hence all results not involving this distribution will be the same, while the divergent integrals produced by the uniform distribution become now convergent and may lead to new results.

We think, however, that the new theory may have more far-reaching consequences concerning the connection of the ultimate particles. These difficult problems, which have to be considered in relation to the most general principles of quantum theory, will be postponed for another article.

1. The Introduction of the Interaction between the Electron Field and the Electromagnetic Field in general

Without interaction, let the electromagnetic field be described in general by a Lagrangian $L'(u_{gh})$ of the field strengths

$$u_{gh}=\frac{\partial u_h}{\partial x_g}-\frac{\partial u_g}{\partial x_h}, \tag{1.1}$$

and the electron field be described by a Lagrangian L'' of the spinors v, v^* and their derivatives v_g, v_g^*. The combined field, together with interaction, is then † described by the Lagrangian

$$L=L'(u_{gh})+L''(v, v^*;\ v_g, v_g^*), \tag{1.2}$$

where only the definition of v_g and v_g^* is now altered so as to take account of the interaction:

$$v_g=D_g v, \qquad v_g^*=D_g^* v^*; \qquad D_g=\frac{\partial}{\partial x_g}+\frac{ie}{\hbar c}u_g, \qquad D_g^*=\frac{\partial}{\partial x_g}-\frac{ie}{\hbar c}u_g. \tag{1.3}$$

For abbreviation let

$$U_{gh}=\frac{\partial L}{\partial u_{gh}}; \qquad V^*=\frac{\partial L}{\partial v}, \qquad V_g^*=\frac{\partial L}{\partial v_g} \tag{1.4}$$

and similarly for the conjugates. The variational equations with respect to u_h, v^* and v are

$$-\frac{\partial U_{gh}}{\partial x_g}+\frac{ie}{\hbar c}\{(vV_h^*)-(V_h v^*)\}=0. \tag{1.5}$$

$$-D_g V_g+V=0, \qquad -D_g^* V_g^*+V^*=0. \tag{1.6}$$

† This section represents a generalisation of the introduction of the interaction with Maxwell's field given by Pauli, *Rev. Mod. Phys.*, XIII, 1941, 207.

In order that (1.5) may be integrable the charge-current vector

$$s_h = -\frac{ie}{\hbar c}\{(v\mathrm{V}_h^*) - (\mathrm{V}_h v^*)\} \tag{1.7}$$

must satisfy the equation of continuity, which is by (1.3) and (1.6)

$$\frac{\partial s_h}{\partial x_h} = -\frac{ie}{\hbar c}\{(v_h\mathrm{V}_h^*) + (v\mathrm{V}^*) - (\mathrm{V}_h v_h^*) - (\mathrm{V}v^*)\} = 0. \tag{1.8}$$

This is the case if the Lagrangian is invariant under the general gauge transformation

$$v \rightarrow ve^{i\gamma}, \qquad u_g \rightarrow u_g - \frac{\hbar c}{e}\frac{\partial\gamma}{\partial x_g} \quad \text{and hence} \quad v_g \rightarrow v_g e^{i\gamma}, \tag{1.9}$$

where γ is an arbitrary real function of the co-ordinates. Then (1.8) is verified by varying v, v_g and their conjugates according to (1.9) and demanding the total variation of L with respect to γ to vanish.

The energy-momentum tensor T_{gh} for the combined field including the interaction can also be partitioned into two parts in a gauge-invariant way:

$$\mathrm{T}_{gh} = \mathrm{T}'_{gh} + \mathrm{T}''_{gh}, \tag{1.10}$$

$$\mathrm{T}'_{gh} = u_{gj}\mathrm{U}_{hj} - \mathrm{L}'\delta_{gh}, \tag{1.11}$$

$$\mathrm{T}''_{gh} = (v_g\mathrm{V}_h^*) + (\mathrm{V}_h v_g^*) - \mathrm{L}''\delta_{gh}. \tag{1.12}$$

The divergence of the part T'_{gh} which may be attributed to the electromagnetic field is, by using the cyclic divergence equations resulting from (1.1) and in virtue of the source term of (1.5),

$$\frac{\partial \mathrm{T}'_{gh}}{\partial x_h} = u_{gj}\frac{\partial \mathrm{U}_{hj}}{\partial x_h} = -u_{gj}s_j. \tag{1.13}$$

The divergence of the part T''_{gh} which may be attributed to the electrons in the presence of the electromagnetic field is, by using the identity

$$\frac{\partial}{\partial x_j}v_g\mathrm{V}_h^* = \mathrm{D}_j v_g \,.\, \mathrm{V}_h^* + v_g \,.\, \mathrm{D}_j^*\mathrm{V}_h^* \tag{1.14}$$

and (1.6) and (1.4),

$$\begin{aligned}\frac{\partial \mathrm{T}''_{gh}}{\partial x_h} &= (\mathrm{D}_h v_g \,.\, \mathrm{V}_h^*) + (\mathrm{V}_h \,.\, \mathrm{D}_h^* v_g) + (v_g\mathrm{V}^*) + (\mathrm{V}v_g^*) \\ &\qquad - \left(\frac{\partial v}{\partial x_g}\mathrm{V}^*\right) - \left(\frac{\partial v_h}{\partial x_g}\mathrm{V}_h^*\right) - \left(\mathrm{V}\frac{\partial v^*}{\partial x_g}\right) - \left(\mathrm{V}_h\frac{\partial v_h^*}{\partial x_g}\right).\end{aligned} \tag{1.15}$$

By using (1.8) the $\partial/\partial x_g$ here can be replaced by D_g where it operates on v and v_h, and by D_g^* where it operates on v^* and v_h^*. By using (1.13) and cancelling terms (1.15) becomes

$$\frac{\partial \mathrm{T}''_{gh}}{\partial x_h} = (\mathrm{D}_h v_g - \mathrm{D}_g v_h,\ \mathrm{V}_h^*) + (\mathrm{V}_h,\ \mathrm{D}_h^* v_g^* - \mathrm{D}_g^* v_h^*). \tag{1.16}$$

By using (1.3) again and noting that, by (1.1),

$$\mathrm{D}_h\mathrm{D}_g - \mathrm{D}_g\mathrm{D}_h = \frac{ie}{\hbar c}u_{hg}, \qquad \mathrm{D}_h^*\mathrm{D}_g^* - \mathrm{D}_g^*\mathrm{D}_h^* = -\frac{ie}{\hbar c}u_{hg}, \tag{1.17}$$

(1.16) becomes, by using also (1.7),

$$\frac{\partial \mathrm{T}''_{gh}}{\partial x_h} = \frac{ie}{\hbar c}u_{hg}\{(v\mathrm{V}_h^*) - (\mathrm{V}_h v^*)\} = -u_{hg}s_h = u_{gh}s_h. \tag{1.18}$$

This verifies the Lorentz law of force in general. The sum of (1.13) and (1.18) yields the conservation laws for the energy and momentum

$$\frac{\partial T_{gh}}{\partial x_h} = 0. \tag{1.19}$$

2. Change of Notation for the Passage from the Lagrangian Formalism to the Hamiltonian Formalism. Maxwell's Field and Dirac's Field

In order to prepare the classical theory of § 1 for quantisation by the method of Heisenberg and Pauli or by the new method, it is necessary to pass from the Lagrangian formalism to the Hamiltonian formalism by treating the time differently from the spatial co-ordinates. It is then appropriate to use the space-vector notation.

We specialise to Maxwell's field and Dirac's field in interaction described by the Lagrangian

$$L = \tfrac{1}{4} u_{gh} u_{gh} + \frac{\hbar c}{2i}\{(v_g \alpha_g v^*) - (v\alpha_g v_g^*)\} + mc^2(v\beta v^*), \tag{2.1}$$

where α_1, α_2, α_3 and β are Dirac's matrices, but $\alpha_4 = \iota = \sqrt{-1}$ in consequence of our use of pseudo-Euclidian metric. In space-vector notation let

$$\left.\begin{array}{ll}
(x_1, x_2, x_3) = \mathbf{r}, & x_4 = \iota ct; \\
(u_1, u_2, u_3) = \mathbf{A}, & u_4 = \iota\phi; \\
(u_{23}, u_{31}, u_{12}) = \mathbf{H}, & (u_{41}, u_{42}, u_{43}) = \iota\mathbf{E}; \\
(\alpha_1, \alpha_2, \alpha_3) = \boldsymbol{\alpha}, & \alpha_4 = \iota; \\
(s_1, s_2, s_3) = \mathbf{j}, & s_4 = \iota\rho; \\
(v_1, v_2, v_3) = \mathbf{f} + \dfrac{ie}{\hbar c}\mathbf{A}v, & v_4 = \iota\left(-\dfrac{1}{c}\dfrac{\partial v}{\partial t} + \dfrac{ie}{\hbar c}\phi v\right).
\end{array}\right\} \tag{2.2}$$

Among the components of the energy-momentum tensor we shall need only T_{g4}, which by integration over a volume Ω give the total momentum $\mathbf{p}$ and energy E contained in Ω, as follows:—

$$\int_\Omega (T_{14}, T_{24}, T_{34}) d\tau = \iota c\mathbf{p}, \qquad \int_\Omega T_{44} d\tau = -E. \tag{2.3}$$

The equations of § 1 will now be specialised for the Lagrangian (2.1) and written in the notation (2.2) and (2.3), as far as they will be needed in the following sections. They can be ordered in four groups:

(*a*) Equations which are merely definitions of the densities of current and charge, and the total momentum and energy:

$$\mathbf{j} = -e(v\boldsymbol{\alpha}v^*), \qquad \rho = -e(vv^*), \tag{2.4), (2.5}$$

$$\mathbf{p} = \int_\Omega \left(\frac{1}{c}\mathbf{E}\wedge\mathbf{H} - \frac{1}{c}\mathbf{A}\rho + \frac{\hbar}{2i}\{(\mathbf{f}v^*) - (v\mathbf{f}^*)\}\right) d\tau, \tag{2.6}$$

$$E = \int_\Omega \left(\frac{\mathbf{E}^2 + \mathbf{H}^2}{2} - \mathbf{A}.\mathbf{j} + \frac{\hbar c}{2i}\{(\mathbf{f}.\boldsymbol{\alpha}v^*) - (v\boldsymbol{\alpha}.\mathbf{f}^*)\} + mc^2(v\beta v^*)\right) d\tau. \tag{2.7}$$

(*b*) The field equations in space:

$$\operatorname{grad} v = \mathbf{f}, \qquad \operatorname{curl} \mathbf{A} = \mathbf{H}, \qquad \operatorname{div} \mathbf{E} = \rho, \tag{2.8), (2.9), (2.10}$$

(*c*) The field equations in time:

$$-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{E} + \operatorname{grad}\phi, \qquad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl}\mathbf{H} - \mathbf{j}, \tag{2.11), (2.12}$$

$$-\frac{\hbar}{i}\frac{\partial v}{\partial t} = -ev\phi + \frac{\hbar c}{i}\mathbf{f}.\boldsymbol{\alpha} + ev\boldsymbol{\alpha}.\mathbf{A} + mc^2 v\beta. \tag{2.13}$$

(*d*) The equation of continuity which is a consequence of (2.4), (2.5), 2.8), and (2.13):

$$\operatorname{div} \mathbf{j} + \frac{1}{c}\frac{\partial \rho}{\partial t} = 0. \tag{2.14}$$

In addition, in order to avoid the arbitrary gradient which can be added to the vector potential **A** according to the gauge transformation (1.9), it is practical to adopt the restriction

$$\operatorname{div} \mathbf{A} = 0 \tag{2.15}$$

for any time t. This restriction fixes the electromagnetic potentials and leaves only a constant phase factor undetermined for the electron field. The scalar potential is then determined by the density of charge according to Poisson's equation

$$\operatorname{div} \operatorname{grad} \phi = -\rho, \tag{2.16}$$

which follows easily from (2.15), 2.11), and (2.10).

3. Separation of the Electromagnetic Field into a Transversal and a Longitudinal Part in the Wave-Vector Representation

It is a well-known theorem that any vector field can be decomposed into a divergence-free part plus a curl-free part. For the electromagnetic field **H** is, by (2.9), divergence-free and **A** is, by (2.15), chosen to be so. **E** is not divergence-free, but its divergence is determined, by (2.10) and (2.5), by the fundamental variables of the electron field.

Hence as the fundamental variables of the total field we shall take those of the electron field plus those of the divergence-free part of the electromagnetic field, because the curl-free part of the electromagnetic field can be expressed in terms of the variables of the electron field. This can be done explicitly by resolving all field variables into their Fourier coefficients.

We enclose the field in a rectangular box and expand all the field components into three-dimensional Fourier series, assuming the usual periodic boundary conditions. For the electron field the field components are complex quantities. Let †

$$v(\mathbf{r}) = \sum_l v_l e^{i\mathbf{l}.\mathbf{r}}, \qquad \mathbf{f}^*(\mathbf{r}) = \sum_l \mathbf{f}_l^* e^{-i\mathbf{l}.\mathbf{r}}, \qquad \text{etc.} \tag{3.1}$$

The wave-vector **l** covers the whole reciprocal space. For the electromagnetic field the field components are real. Therefore † we write

$$\mathbf{A}(\mathbf{r}) = \sum_k (\mathbf{A}_k e^{i\mathbf{k}.\mathbf{r}} + \mathbf{A}_k^* e^{-i\mathbf{k}.\mathbf{r}}), \quad \text{etc.} \tag{3.2}$$

Here the wave-vector **k** covers only half the reciprocal space in order that all the Fourier coefficients $\mathbf{A}_k$'s and $\mathbf{A}_k^*$'s may be independent. Then the Fourier coefficients of the divergence-free part of any vector field are perpendicular to the wave-vectors and will be called transversal, while those of the curl-free part are parallel to the wave-vectors and will be called longitudinal.

By inserting the Fourier series for **E** and ρ in (2.10) and equating the coefficients of $\exp(i\mathbf{k}.\mathbf{r})$ on both sides we get

$$i\mathbf{k}.\mathbf{E}_k = \rho_k. \tag{3.3}$$

Thus the longitudinal part of $\mathbf{E}_k$ is, by projection along the direction parallel to **k**,

$$\mathbf{E}_{k,\,\text{long.}} = \mathbf{k}\frac{\mathbf{k}.\mathbf{E}_k}{k^2} = -i\mathbf{k}\frac{\rho_k}{k^2}. \tag{3.4}$$

The transversal part of $\mathbf{E}_k$ is therefore

$$\mathbf{E}_{k,\,\text{tr.}} = \mathbf{E}_k - \mathbf{E}_{k,\,\text{long.}} = \mathbf{E}_k + i\mathbf{k}\rho_k/k^2. \tag{3.5}$$

† Unlike the usual practice, no normalisation factor is introduced in the Fourier analysis. The Fourier coefficients of a quantity are then of the same physical dimension as that of the quantity itself.

We shall, however, regard this equation as the definition of $\mathbf{E}_k$ in terms of $\mathbf{E}_{k,\,\mathrm{tr.}}$ and take the latter as a fundamental variable. In fact, as will be shown immediately, the field equations for the electromagnetic field can be expressed in terms of the variables of the transversal part alone.

The Fourier coefficients of the scalar potential are determined by virtue of (2.16), by the electron field

$$\phi_k = \rho_k/k^2. \tag{3.6}$$

By this and (3.5), (2.11) yields simply

$$-\frac{1}{c}\frac{\partial \mathbf{A}_k}{\partial t} = \mathbf{E}_k + i\mathbf{k}\phi_k = \mathbf{E}_k + i\mathbf{k}\frac{\rho_k}{k^2} = \mathbf{E}_{k,\,\mathrm{tr.}} \tag{3.7}$$

From (2.12) follows the vector equation

$$\frac{1}{c}\frac{\partial \mathbf{E}_k}{\partial t} = i\mathbf{k}\wedge\mathbf{H}_k - \mathbf{j}_k, \tag{3.8}$$

but the longitudinal part of this is, by (3.3), nothing but the equation of continuity

$$\frac{1}{c}\frac{\partial \rho_k}{\partial t} = -i\mathbf{k}.\mathbf{j}_k, \tag{3.9}$$

which involves only the variables of the electron field, and is a consequence of the field equations for the latter. Hence (3.8) is, by (3.9), equivalent to

$$\frac{1}{c}\frac{\partial \mathbf{E}_{k,\,\mathrm{tr.}}}{\partial t} = i\mathbf{k}\wedge\mathbf{H}_k - \mathbf{j}_k + \mathbf{k}\frac{\mathbf{k}.\mathbf{j}_k}{k^2}. \tag{3.10}$$

The other variables, $\mathbf{H}_k$ and $\mathbf{A}_k$, are entirely transversal. Either may be taken as a fundamental variable, the other can then be expressed in terms of it. By (2.9), $\mathbf{H}_k$ can be expressed in terms of $\mathbf{A}_k$ thus

$$\mathbf{H}_k = i\mathbf{k}\wedge\mathbf{A}_k. \tag{3.11}$$

Conversely, this can be solved for $\mathbf{A}_k$ (because the latter is transversal) and yields

$$\mathbf{A}_k = i\mathbf{k}\wedge\mathbf{H}_k/k^2. \tag{3.12}$$

After inserting in (2.6) the Fourier series for the field quantities the integration gives for the total momentum

$$\mathbf{p} = \Omega\Big(\sum_k \frac{1}{c}\{\mathbf{E}_k\wedge\mathbf{H}_k^* + \mathbf{E}_k^*\wedge\mathbf{H}_k - \mathbf{A}_k\rho_k^* - \mathbf{A}_k^*\rho_k\} + \frac{\hbar}{2i}\sum_l\{(\mathbf{f}_l v_l^*) - (v_l\mathbf{f}_l^*)\}\Big). \tag{3.13}$$

By (3.12) and (3.5) this becomes the sum of contributions by the transversal electromagnetic field and the electron field without any interaction term,

$$\mathbf{p} = \Omega\Big(\sum_k \frac{1}{c}\{\mathbf{E}_{k,\,\mathrm{tr.}}\wedge\mathbf{H}_k^* + \mathbf{E}_{k,\,\mathrm{tr.}}^*\wedge\mathbf{H}_k\} + \frac{\hbar}{2i}\sum_l\{(\mathbf{f}_l v_l^*) - (v_l\mathbf{f}_l^*)\}\Big). \tag{3.14}$$

However, the total energy E still contains interaction terms:

$$\begin{aligned} \mathrm{E} = \Omega\Big(&\sum_k\{\mathbf{E}_{k,\,\mathrm{tr.}}.\mathbf{E}_{k,\,\mathrm{tr.}}^* + \mathbf{H}_k.\mathbf{H}_k^* + \rho_k\rho_k^*/k^2 - \mathbf{A}_k.\mathbf{j}_k^* - \mathbf{A}_k^*.\mathbf{j}_k\} \\ &+ \frac{\hbar c}{2i}\sum_l\{(\mathbf{f}_l.\boldsymbol{\alpha}v_l^*) - (v_l\boldsymbol{\alpha}.\mathbf{f}_l^*)\} + mc^2\sum_l(v_l\beta v_l^*)\Big). \end{aligned} \tag{3.15}$$

The equations (2.4), (2.5), (2.8), and (2.13), which are mainly concerned with the electron field, become, in terms of the Fourier coefficients,

$$\mathbf{j}_k = -e\sum_l(v_{l+k}\boldsymbol{\alpha}v_l^*), \qquad \rho_k = -e\sum_l(v_{l+k}v_l^*), \tag{3.16), (3.17}$$

$$i\mathbf{l}v_l = \mathbf{f}_l, \tag{3.18}$$

$$-\frac{\hbar}{i}\frac{\partial v_l}{\partial t} = -e\sum_k\{\phi_k v_{l-k} + v_{l+k}\phi_k^*\} + \frac{\hbar c}{i}\mathbf{f}_l.\boldsymbol{\alpha} + e\sum_k\{v_{l-k}\mathbf{A}_k.\boldsymbol{\alpha} + v_{l+k}\mathbf{A}_k^*.\boldsymbol{\alpha}\} + mc^2 v_l\beta. \tag{3.19}$$

The total charge q contained in the volume Ω is, by integrating (2.5),

$$q = -e\Omega \sum_l (v_l v_l^*). \tag{3.20}$$

4. Quantisation of Maxwell's Field and Dirac's Field in Interaction by the Method of Heisenberg and Pauli

(a) *Simple Treatment*

A simple and practical way of applying the method of Heisenberg and Pauli for the quantisation of a field—in our case the combined field of Maxwell and Dirac—is the following: after the field quantities have been resolved into their Fourier coefficients (what we have done in § 3), the field is treated as an assembly of oscillators characterised by the wave-vectors. Then the field equations in time (§ 2 (c)) give rise to the equations of motion of the oscillators, viz. in our case (3.7), (3.10), (3.19) and their adjoint equations. The canonical variables, which can be read off from the time-derivative terms of these equations, are the following conjugate pairs:—

$$v_l,\ v_l^*; \qquad \mathbf{A}_k,\ \mathbf{E}^*_{k,\,\mathrm{tr.}}; \qquad \mathbf{E}_{k,\,\mathrm{tr.}},\ \mathbf{A}^*_k.$$

The equations which arise from space-derivatives are, according to this view, to be regarded partly as definitions of some auxiliary variables (namely, (3.11) as the definition of $\mathbf{H}_k$ and (3.18) of $\mathbf{f}_l$) and partly as constraints among the above canonical variables (namely,

$$i\mathbf{k}.\mathbf{A}_k = 0, \qquad i\mathbf{k}.\mathbf{E}_{k,\,\mathrm{tr.}} = 0 \tag{4.1}$$

and their adjoint equations). Quantisation consists in considering all the canonically conjugate pairs of variables as q-numbers which satisfy simple commutation or anti-commutation laws, as follows.

For the spinor components (s denoting the spinor index) anti-commutation laws hold †

$$[v_{ls},\ v^*_{ls}]_+ \equiv v_{ls}v^*_{ls} + v^*_{ls}v_{ls} = 1/\Omega. \tag{4.2}$$

All other anti-commutators vanish.

For the vector components let the axes be chosen so that $\mathbf{k}$ lies on the z-axis. The x- and y-components of $\mathbf{A}_k$ and $\mathbf{E}^*_{k,\,\mathrm{tr.}}$ satisfy the commutation laws

$$[\mathrm{A}_{kx},\ \mathrm{E}^*_{kx,\,\mathrm{tr.}}] \equiv \mathrm{E}^*_{kx,\,\mathrm{tr.}}\mathrm{A}_{kx} - \mathrm{A}_{kx}\mathrm{E}^*_{kx,\,\mathrm{tr.}} = i\hbar c/\Omega, \qquad [\mathrm{A}_{ky},\ \mathrm{E}^*_{ky,\,\mathrm{tr.}}] = i\hbar c/\Omega, \qquad (k_x = k_y = 0). \tag{4.3}$$

All other commutators vanish.

By an arbitrary rotation of the co-ordinate axes (4.3) becomes in general ‡

$$[\mathrm{A}_{kx},\ \mathrm{E}^*_{ky,\,\mathrm{tr.}}] = \frac{i\hbar c}{\Omega}\left(\delta_{xy} - \frac{k_x k_y}{k^2}\right). \tag{4.4}$$

All the vector components of $\mathbf{A}_k$, $\mathbf{E}^*_{k,\,\mathrm{tr.}}$, $\mathbf{E}_{k,\,\mathrm{tr.}}$ and $\mathbf{A}^*_k$ commute with all the spinor components of v_l and v_l^*.

The quantised equations of motion are

$$\frac{\partial \mathrm{F}}{\partial t} = -\frac{i}{\hbar}[\mathrm{E},\ \mathrm{F}], \qquad \mathrm{F} = \mathbf{A}_k,\ \mathbf{E}^*_{k,\,\mathrm{tr.}};\ \mathbf{E}_{k,\,\mathrm{tr.}},\ \mathbf{A}^*_k;\ v_l,\ v_l^*; \tag{4.5}$$

where the total energy E of the assembly of oscillators is given by the q-number expression § of (3.15). The close formal analogy between the classical and quantum equations of motion can be demonstrated by working out the commutators of (4.5). If F is a component of $\mathbf{A}_k$, in (3.15) only the term $\mathbf{E}_{k,\,\mathrm{tr.}}.\mathbf{E}^*_{k,\,\mathrm{tr.}}$ does not commute with $\mathbf{A}_k$. By (4.4) and (4.1) the

† Jordan and Wigner, *Zeits. f. Physik*, XLVII, 1928, 631.

‡ Novobatzky, *loc. cit.*

§ Since the order of factors can be arbitrarily changed in a c-number expression but not so in a q-number expression the q-number expression of (3.15) (as well as that of (3.14), etc.) is slightly ambiguous. This has no effect on the anti-commutation laws or commutation laws. The zero-point momentum and charge, however, can be avoided by taking the mean of the two possible q-number expressions as discussed in Part I.

commutator has the value

$$[\mathrm{E}, \mathbf{A}_k] = -i\hbar c \mathbf{E}_{k,\,\mathrm{tr.}} \tag{4.6}$$

Hence (4.5) with $\mathrm{F}=\mathbf{A}_k$ is formally identical with (3.7). If $\mathrm{F}=\mathbf{E}_{k,\,\mathrm{tr.}}$ the relevant term of (3.15) can be written, using the adjoint of (3.11), in the form

$$\mathbf{H}_k . \mathbf{H}_k^* - \mathbf{A}_k^* . \mathbf{j}_k = (i\mathbf{k}\wedge\mathbf{H}_k) . \mathbf{A}_k^* - \mathbf{A}_k^* . \mathbf{j}_k. \tag{4.7}$$

With the help of the adjoint of (4.4), (4.5) with $\mathrm{F}=\mathbf{E}_{k,\,\mathrm{tr.}}$ is seen to be formally identical with (3.10). Making use of (3.6), (3.16), (3.17), (3.18), their adjoint equations and (4.2) the quantised equation of motion (4.5) for v_l is formally identical with (3.19), if the order of the factors ϕ and v in the latter equation is properly adjusted.

Instead of working with the Heisenberg representation (4.5) one can consider the variables of (4.5) as operators satisfying (4.2) and (4.4) and describe the field (which is here treated as an assembly of oscillators) by the Schroedinger wave function Ψ. Ψ contains, besides the independent variable t, the variables on which the operators of (4.5) act. One then replaces (4.5) by the wave equation

$$\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\mathrm{E}\Psi, \tag{4.8}$$

where the Hamiltonian E is the operator expression of (3.15).

(*b*) *Complete Treatment*

In the complete and rigorous application of Heisenberg and Pauli's method for the quantisation of a field the equations (2.8), (2.9), (2.10), and (2.15), where only space derivatives appear, are to be treated on the same footing as the equations containing time-derivatives. The quantised equations of motion in space supplement those in time (4.5), namely

$$\operatorname{grad} \mathrm{F} = \frac{i}{\hbar}[\mathbf{p}, \mathrm{F}]; \qquad \mathrm{F}=\mathrm{F}(\mathbf{r}), \text{ the field variables at the point } \mathbf{r}. \tag{4.9}$$

$\mathbf{p}$ denotes the total momentum of the field contained in the volume Ω. If the Fourier series (3.1) and (3.2) for the field variables are used and their Fourier coefficients considered as q-numbers (the $\mathbf{r}$ in field theories is a c-number which may be compared to the time t in the quantum mechanics of particles) (4.9) becomes

$$[\mathbf{p}, \mathrm{F}_l] = \hbar\mathbf{l}\mathrm{F}_l, \quad \mathrm{F}_l = v_l, \mathbf{f}_l; \qquad [\mathbf{p}, \mathrm{F}_k] = \hbar\mathbf{k}\mathrm{F}_k, \quad \mathrm{F}_k = \mathbf{A}_k, \mathbf{E}_{k,\,\mathrm{tr.}}, \mathbf{H}_k, \mathbf{j}_k, \rho_k. \tag{4.10}$$

One has now to add to the canonical variables considered in the simple treatment the quantities $\mathbf{f}_l$, $\mathbf{H}_k$ and their adjoints as fundamental field variables. The following additional commutation and anti-commutation laws containing the additional field variables have to hold and to be considered as fundamental as those given above, (4.2) and (4.4):

$$[\mathbf{f}_{ls}, v_{ls}^*]_+ = -[\mathbf{f}_{ls}^*, v_{ls}]_+ = i\mathbf{l}/\Omega, \qquad [f_{lsx} f_{lsy}^*]_+ = l_x l_y/\Omega; \tag{4.11}$$

$$[\mathrm{H}_{kx}, \mathrm{E}_{ky,\,\mathrm{tr.}}^*] = -[\mathrm{H}_{ky}, \mathrm{E}_{kx,\,\mathrm{tr.}}^*] = [\mathrm{E}_{ky,\,\mathrm{tr.}}, \mathrm{H}_{kx}^*] = -[\mathrm{E}_{kx,\,\mathrm{tr.}}, \mathrm{H}_{ky}^*] = \hbar c k_z/\Omega. \tag{4.12}$$

It is understood that the vector indices x, y, z in (4.12) may be cyclically permuted. With $\mathbf{p}$ given by the q-number expression of (3.14), (4.10) follows directly from the totality of commutation and anti-commutation laws if one chooses $\mathrm{F}_l=v_l$, $\mathbf{f}_l$ and $\mathrm{F}_k=\mathbf{A}_k$, $\mathbf{E}_{k,\,\mathrm{tr.}}$, $\mathbf{H}_k$. By using the definition (3.16) and (3.17), and also (4.10) with $\mathrm{F}_l=v_l$ together with the adjoint equation, one sees that (4.9) holds for $\mathrm{F}_k=\mathbf{j}_k$ and ρ_k; for instance

$$\begin{aligned}[\mathbf{p}, \rho_k] &= -e\sum_l \{([\mathbf{p}, v_{l+k}]v_l^*) + (v_{l+k}[\mathbf{p}, v_l^*])\} \\ &= -e\sum_l \{\hbar(\mathbf{l}+\mathbf{k})(v_{l+k}v_l^*) - (v_{l+k}v_l^*)\hbar\mathbf{l}\} = \hbar\mathbf{k}\rho_k. \end{aligned} \tag{4.13}$$

By (4.2) and (4.11) the combination $\mathbf{f}_{ls} - i\mathbf{l}v_{ls}$ anti-commutes with all the field variables of the electron field and commutes with those of the electromagnetic field. Hence it vanishes. By (4.4) and (4.10) the combinations $\mathbf{H}_k - i\mathbf{k}\wedge\mathbf{A}_k$, $i\mathbf{k}.\mathbf{E}_{k,\,\mathrm{tr.}}$ and $i\mathbf{k}.\mathbf{A}_k$ commute with all field variables. Hence these expressions commute with the total momentum and the total energy of the field; therefore they vanish in virtue of (4.9) and properly chosen boundary conditions, or in virtue of (4.5) and proper initial conditions.

Thus in the complete treatment (3.18), (3.11), and (4.1) as q-number equations result from the commutation and anti-commutation laws. In virtue of (4.10) and (4.9) these equations can be written in a way formally identical with the classical equations (2.8), (2.9), (2.10), and (2.15).

The demonstration of the formal identity between the classical and quantum equation of motion in time can now be carried out in the same manner as in the simple treatment.

It follows from (4.10) and its adjoint that the components of the momentum and the energy all commute with each other.

$$[p_x, p_y] = 0, \quad \text{etc.,} \qquad [\mathbf{p}, \mathrm{E}] = 0. \tag{4.14}$$

In the Schroedinger representation (4.5) and (4.9) are to be replaced by the equations

$$\frac{\hbar}{i}\frac{\partial \Psi}{\partial t} = -\mathrm{E}\Psi, \qquad \frac{\hbar}{i}\operatorname{grad}\Psi = \mathbf{p}\Psi, \tag{4.15}$$

where E and $\mathbf{p}$ are the operator expressions of (3.15) and (3.14). Because of (4.14) the wave equations (4.15) are compatible.

5. New Method of Quantisation

The Heisenberg-Pauli method developed in the last section can be considered as semi-classical; while it uses the classical representation of the field variables by Fourier series (3.1) and (3.2) which are ordinary functions of the position vector $\mathbf{r}$, it considers the Fourier coefficients not as functions of time, but as q-numbers. Now we proceed to a method of complete quantisation in which space and time are treated on the same footing. We discard the Fourier series as a sum of terms and replace for each field variable the set of its Fourier coefficients by an array of elements which form a more complicated matrix † than that needed to represent the individual q-number Fourier coefficients by matrices in the Heisenberg and Pauli quantisation. The total matrix belongs to the volume Ω as a whole and contains, as will soon appear, the totality of information about the field variable it represents throughout this volume. The only non-vanishing anti-commutation and commutation laws for the total matrices are those given in Part I, § 5 and § 6 (equations (5.13), (5.18), and (6.28)); they are the following (s denoting the spinor index):—

$$[v_s, v_s^*]_+ = 1/\Omega, \tag{5.1}$$

$$[\mathbf{f}_s, v_s^*]_+ = -[\mathbf{f}_s^*, v_s]_+ \equiv i\mathbf{l}/\Omega, \qquad [f_{sx}, f_{y}^*]_+ = l_x l_y \Omega, \tag{5.2}$$

$$[\mathrm{H}_x, \mathrm{E}^*_{y,\,\mathrm{tr.}}] = -[\mathrm{H}_y, \mathrm{E}^*_{x,\,\mathrm{tr.}}] = [\mathrm{E}_{y,\,\mathrm{tr.}}, \mathrm{H}_x^*] = -[\mathrm{E}_{x,\,\mathrm{tr.}}, \mathrm{H}_y^*] \equiv \hbar c k_z/\Omega, \tag{5.3}$$

where x, y, z may be cyclically permuted;

$$[\mathrm{A}_x, \mathrm{E}^*_{y,\,\mathrm{tr.}}] = [\mathrm{A}_x^*, \mathrm{E}_{y,\,\mathrm{tr.}}] = \frac{i\hbar c}{\Omega}\{\delta_{xy} - (k_x^2 + k_y^2 + k_z^2)^{-1}k_x k_y\}. \tag{5.4}$$

(5.2) and (5.3) contain also the definitions of the new self-adjoint variables $\mathbf{l}$ and $\mathbf{k}$ thus introduced. These variables, like all the field variables, are represented by total matrices and are not to be confused with the c-number wave-vectors used in § 3 or § 4 for Fourier analysis. As shown in Part I, § 6, the three components of $\mathbf{l}$ commute with all the field variables of the electron field by virtue of the anti-commutation laws (5.1) and (5.2) only. By (5.2) again they commute among themselves. In the present case, because the field variables of the electron field all commute with those of the transversal electromagnetic field, l_x, l_y, and l_z commute also with all variables of the transversal electromagnetic field. Similarly, as a consequence of (5.3) and (5.4), k_x, k_y, and k_z commute with all field variables, among themselves and also with l_x, l_y, l_z.

† Such a matrix will be referred to in what follows as a "total matrix."

Since all the field variables commute with k_x, k_y, k_z, l_x, l_y, and l_z the set of matrices which represent them is reducible. In the representation where k_x, k_y, k_z, l_x, l_y, and l_z are simultaneously diagonal the total matrices for all the field variables will appear as being composed of submatrices placed along the diagonal. The representation is the *direct product* of the two representations for the two pure fields considered in Part I, § 5 and § 6, separately. In this representation we have

$$\mathbf{k}_{k'l',k''l''} = \mathbf{k}'\delta_{k'k''}\delta_{l'l''}, \qquad \mathbf{l}_{k'l',k''l''} = \delta_{k'k''}\mathbf{l}'\delta_{l'l''}; \tag{5.5}$$

$$\mathrm{F}_{k'l',k''l''} = \mathrm{F}_{k'}\delta_{k'k''}\delta_{l'l''} \quad \text{for} \quad \mathrm{F} = \mathbf{H}, \mathbf{E}_{\mathrm{tr.}}, \mathbf{A}, \mathbf{j}, \rho \quad \text{and their adjoints}; \tag{5.6}$$

$$\mathrm{F}_{k'l',k''l''} = \delta_{k'k''}\mathrm{F}_{l'}\delta_{l'l''} \quad \text{for} \quad \mathrm{F} = v, \mathbf{f} \quad \text{and their adjoints}; \tag{5.7}$$

where the submatrices k'_x, k'_y, k'_z, l'_x, l'_y, and l'_z are scalar (*i.e.* a number multiplied by a unit matrix) and the submatrices $\mathrm{F}_{k'}$ or $\mathrm{F}_{l'}$ will be shown to be the same as those matrices used in the Heisenberg and Pauli quantisation to represent the corresponding q-number Fourier coefficients.

As in the Heisenberg and Pauli quantisation where it is convenient to treat the field as an assembly of oscillators each of which is described by the Fourier coefficients belonging to a wave-vector, so in the new method of quantisation it is convenient to treat the field as an assembly of *apeirons* each of which is described by the submatrices belonging to an eigenvalue of the total matrices $\mathbf{k}$ or $\mathbf{l}$ introduced by (5.3) and (5.2). Yet in the new method of quantisation the eigenvalue $\mathbf{k}'$ or $\mathbf{l}'$ needs only to assume *a selection of* the possible values of the wave-vector, while all of them are automatically included in the Heisenberg and Pauli quantisation where the wave-vector is introduced by means of Fourier analysis.

The non-vanishing anti-commutation laws and commutation laws obtained by taking the diagonal submatrices $k'l'$, $k'l'$ of (5.1), (5.2), (5.3), and (5.4) coincide exactly with those of Heisenberg and Pauli's Fourier coefficients, namely (4.2), (4.10), (4.11), and (4.4), except that now k' and l' are written in place of the former k and l. The vanishing anti-commutation and commutation brackets obtained by taking the non-diagonal submatrices $k'l'$, $k''l''$ (either $\mathbf{k}' \neq \mathbf{k}''$ or $\mathbf{l}' \neq \mathbf{l}''$ or both) of (5.1), (5.2), (5.3), and (5.4) are trivial identities because no field quantity, by virtue of its being reducible, contains any non-vanishing non-diagonal submatrices.

In order to obtain the full correspondence of the commutation laws and anti-commutation laws for the submatrices with those for the Fourier coefficients we have to supplement (5.1) by

$$[\mathrm{P}v_s\mathrm{P}^*, v_s^*]_+ = 0, \tag{5.8}$$

where P denotes any permutation matrix permuting all the l'-apeirons. It is sufficient, however, to take P to be the set of cyclic permutations. (5.2), (5.3), and (5.4) are to be supplemented in a similar way.

Let I denote the unit matrix $\delta_{k'k''}$ and $\mathrm{trace}_{(k)}$ A denote the sum $\Sigma_{k'}\mathrm{A}_{k'k'}$. Then the operation I $\mathrm{trace}_{(k)}$ produces a scalar matrix of the same number of rows and columns as the matrix on which it operates. Let J $\mathrm{trace}_{(l)}$ have a similar significance where J denotes the unit matrix $\delta_{l'l''}$. Corresponding to (3.14) and (3.15), which express the total momentum and energy as summations over the Fourier coefficients, we use now the summations over the submatrices, namely

$$\mathbf{p} = \Omega\left(\frac{1}{c}\mathrm{I}\ \mathrm{trace}_{(k)}\{\mathbf{E}_{\mathrm{tr.}}\wedge\mathbf{H}^* + \mathbf{E}^*_{\mathrm{tr.}}\wedge\mathbf{H}\} + \frac{\hbar}{2i}\mathrm{J}\ \mathrm{trace}_{(l)}\{(\mathbf{f}v^*) - (v\mathbf{f}^*)\}\right). \tag{5.9}$$

$$\begin{aligned}\mathrm{E} = \Omega\Big(&\mathrm{I}\ \mathrm{trace}_{(k)}\{\mathbf{E}_{\mathrm{tr.}}\cdot\mathbf{E}^*_{\mathrm{tr.}} + \mathbf{H}\cdot\mathbf{H}^* + \rho(k_x^2+k_y^2+k_z^2)^{-1}\rho^* - \mathbf{A}\cdot\mathbf{j}^* - \mathbf{A}^*\cdot\mathbf{j}\}\\ &+ \frac{\hbar c}{2i}\mathrm{J}\ \mathrm{trace}_{(l)}\{(\mathbf{f}\cdot\boldsymbol{\alpha}v^*) - (v\boldsymbol{\alpha}\cdot\mathbf{f}^*)\} + mc^2\mathrm{J}\ \mathrm{trace}_{(l)}(v\beta v^*)\Big),\end{aligned} \tag{5.10}$$

The density ρ and current $\mathbf{j}$ are now defined by

$$\mathbf{j} = -e\mathrm{J}\ \mathrm{trace}_{(l)}(\mathrm{C}v\mathrm{C}^*\boldsymbol{\alpha}v^*), \qquad \rho = -e\mathrm{J}\ \mathrm{trace}_{(l)}(\mathrm{C}v\mathrm{C}^*v^*), \qquad (5.11), (5.12)$$

where the matrix C satisfies the following conditions:—

$$\mathbf{k}C = C\mathbf{k}, \qquad C\mathbf{l} - \mathbf{l}C = \mathbf{k}C, \qquad CC^* = 1. \tag{5.13}$$

It follows from (5.13) that $C_{k'l', k''l''}$ vanishes except if $\mathbf{k}' = \mathbf{k}''$ and $\mathbf{l}'' - \mathbf{l}' = \mathbf{k}'$ when it is of modulus unity. Hence we have

$$(CvC^*)_{k'l', k''l''} = \delta_{k'k''} v_{l'+k'} \delta_{l'l''} \tag{5.14}$$

and the correspondence of (5.11) and (5.12) with (3.16) and (3.17) is apparent.

The total charge is

$$q = -e\Omega J \,\text{trace}_{(l)}(vv^*). \tag{5.15}$$

The correspondence of the submatrices of the new method of quantisation with the Fourier coefficients of the Heisenberg and Pauli quantisation is so close that the demonstration of the field equations given in § 4 for the Fourier coefficients may be now taken over for the submatrices with corresponding formal change, which is too obvious to be repeated here.

In the Schroedinger representation (4.15) holds with E and **p** now considered as the operator expression of (5.10) and (5.9).

Since the total matrices representing **k** and **l** commute with those representing the energy E and the momentum **p**, **k** and **l** are constants of motion in time and space. That is, the distribution of both the electromagnetic and the electronic apeirons (which is described by the values **k**′, **k**″, etc., and **l**′, **l**″, etc., that actually occur as submatrices of **k** and **l**, *cf.* Part II) remains the same throughout space and time in spite of the interaction between Maxwell's field and Dirac's field. The interaction affects only the number of quanta occupying the apeirons, the quanta being electrons and photons respectively. Hence our attempt made in Part II to determine the apeiron distribution by statistical considerations cannot be based on the theory in its present form, which is very likely only provisional. Then there may be another way, also mentioned in Part II, to determine this distribution, namely by studying its effect on the self-energies of the quanta and the transition probabilities of collision processes.

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The Divergence Difficulty of Quantized Field Theories

In the theory of interaction of particles like photons and electrons or mesons and nucleons, etc., the experiments are always made in such a way that the different kinds of particles are observed in different parts of space with different instruments. In these parts of space one can speak of pure kinds of particles and describe them by pure quantized fields—Maxwell's field, Dirac's field, etc. However, in the part of space where the collisions really take place, the pure fields are not simply additive but have to be supplemented by an interaction field. All effects due to the interaction can be obtained by considering the stationary states of the whole system.

Because of the complication of the problem, one has to use a method of successive approximation called the perturbation method in which the interaction is regarded as small. It is well known that then special care must be taken to remove the degeneracy of the states of the unperturbed system in a way so as to anticipate the eigenstates of the perturbed system. This preliminary step has, however, been ignored in the usual practice for the interaction of fields; and consequently divergent expressions appear as soon as the next higher order of the perturbation method is attempted.

The results obtained by the usual practice were in good agreement with observations in the case of the photon-electron interaction, although the divergence of the higher approximation indicates that some mistake must have been made, which might be physical or mathematical. No such agreement with observation has been found in the case of the meson field. Some time ago an improved method[1] based on physical reasoning was developed which takes account, to the first approximation only, of what is classically known as the 'radiation reaction'. This method is well confirmed by its applications, especially to the meson field, but hitherto its theoretical basis was not satisfactorily established.

It has now been found that this provisional method can be rigorously established by a systematic application of the ordinary perturbation theory for degenerate systems adapted to the case of the continuous spectrum. I have found that the treatment of the radiation reaction referred to above constitutes exactly the preliminary step of the removal of degeneracy. The continuation of the perturbation method to higher approximations is then possible without any difficulty.

With this mathematical improvement, it might be that the present field theories (without any change—such as that proposed by Dirac[2], or that by Born and Peng[3]) are sufficient to explain most of the known facts. For example, the anomalous value of the magnetic moment of the proton or the neutron can now be rigorously dealt with. It seems possible that the infinite self-energy of the point electron, which has always been a difficulty in the classical theory, will also become finite in the quantum theory.

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[1] Heitler and Peng, *Proc. Camb. Phil. Soc.*, **38**, 296 (1942).
[2] Dirac, *Proc. Roy. Soc.*, A, **180**, 1 (1942).
[3] Born and Peng, *Proc. Roy. Soc. Edinburgh*, A, **62**, 40 (1944).

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