SYMMETRIES AND CURRENTS IN LATTICE QUANTUM CHROMODYNAMICS

Thesis

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DECLARATION

Some of the work described in chapters 2 and 3 was performed in collaboration with C.Smith. Some of the work in chapter 4 was done in collaboration with D.Verstegen. All other work in this thesis is entirely my own, except otherwise indicated.

Para dona Marlyse e seu João

I said, 'Isn't it?'

John Fowles: The Magus

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ABSTRACT

After a presentation of the non-Abelian gauge theory description of strong interactions, the necessity for non-perturbative methods is discussed and the implementation of space-time discretization in gauge theories reviewed. The arising of fermion doubling is presented, together with the two main models to cure this problema: Wilson and staggered fermions. The importance of weak coupling expansions on the lattice in order to recover the continuum formulation is stressed and the mass renormalization for Wilson quarks is calculated.

Stemming from an analogy with the continuum, global symmetries for lattice quantum chromodynamics are analysed and it is shown how to obtain the different currents for Wilson fermions. The inclusion of local currents, more satisfying from a computer simulation point of view, is suggested in spite of their more **ad hoc** nature. The renormalization at one-loop is obtained for all the above currents; these radiative corrections correct lattice estimates of meson decay constants, helping to reduce the discrepancy with their experimental values.

A similar discussion is done for staggered fermions where it is shown that the symmetries of the reflection rotation hypercubic group may classify the fermionic bilinears into its representations. In the flavour formulation of staggered fermions, renormalization factors are calculated for a selection of bilinears.

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Chapter 1

INTRODUCTION

In physics, it is generally agreed that a theory has been obtained once a set of ideas on a particular subject has been organized in a coherent, mathematically concise, way and achieves two goals: calculations done with it will explain and recover to some accuracy physical phenomena which had been observed so far, furthermore undiscovered features of nature may or should be predicted and then detected experimentally.

Arguably, the main two and most fundamental physical theories of this century have been relativity and quantum mechanics, and the study of the interaction of photons and electrons from a combined quantum relativistic viewpoint gave rise to a theory which agrees with the experimental results to a remarkable accuracy. The mathematical realization of this quantum electrodynamics is done by renormalization wich arises from divergent perturbation series, where the weakness of the coupling between electrons and photons guarantees the right to make perturbation expansions in the coupling constant. A feature of this theory which now might appear to be the key for its success is that it is locally gauge invariant, that is the electromagnetic field seen as a gauge field enables us to make space-time dependent transformations on the electron field without changing the form of Maxwell's

equations. The importance of this kind of behaviour has led physicists of the 1960's to set up a unified electroweak model which undoubtedly has been promoted to a theory after last year's discovery at CERN of the massive vector bosons, and confirms the importance of what seems to be a new and fundamental symmetry of nature: local gauge invariance.

It was then natural to try to implement local gauge invariance in the study of strong interactions, improving the phenomenological models of quarks and current algebra to obtain a renormalizable theory, making sure to preserve all the features which had described the correct physical behaviour of elementary particles to some extent. One such feature is an approximate global symmetry of the fermionic systems which is still under hard investigation and seems to be necessary for a correct physical description, called chiral symmetry. If one takes a massless Dirac field ψ coupled to an Abelian gauge field (but the discussion below is conceptually correct for non-Abelian gauge fields as well), its Lagrangian will be invariant under the transformation:

$$\psi \to e^{i\alpha\gamma_5}\psi \quad , \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5} \qquad (1.1)$$

in addition to the fermionic number conservation associated symmetry:

$$\psi \to e^{i\alpha}\psi \quad , \quad \bar{\psi} \to \bar{\psi} e^{-i\alpha} \qquad (1.2)$$

The symmetry (1.1) reflects the conservation of positive and negative helicity states separately, conservation which ceases to be true if massive fermions are considered, in which case positive and negative helicity states would be coupled in:

$$m\bar{\psi}\psi = m\left(\psi_L^{\dagger}\psi_R + \psi_R^{\dagger}\psi_L\right) \qquad (1.3)$$

where ψ_R and ψ_L are right and left-handed components of ψ which yield positive and negative helicity states respectively. The conservation arising from (1.1) is really formal and Lee and Nauenberg (1964) have shown that massless spinor electrodynamics contains mass singularities and should be treated as a limit of a massive theory. There is however an anomaly (the so-called axial anomaly), which spoils the conservation of massless axial-vector currents, generated from Lagrangian symmetry under (1.1) as will be discussed in chapter 3. This anomaly could really be absorbed by new definitions of currents, but this will lead to non-gauge invariant quantities. We can see now why the study of elementary particles will require a careful analysis of two sometimes conflicting and necessary symmetries: local gauge invariance and chiral symmetry. At a more mathematical level, the necessity to be "careful" was demonstrated when fermion loop integrations gave wrong results because in linearly divergent integrals, the integration variable cannot be obviously translated.

This anomaly which comes from a subordination of chiral symmetry to the exact gauge symmetry describes the correct physics and should then be present in its own right in fermionic theories, in particular for a theory of strong interactions of quarks.

A trial of constructing a theory of quarks analogous to that of electrons would produce free quarks just as there are free electrons , but this would contradict experimental evidence which shows that not only do quarks appear to interact freely at high energies (or short distances), but that they seem to be confined inside hadrons. To obtain such behaviour, it was found necessary to introduce non-Abelian gauge fields instead of the Abelian one in electrodynamics. Thus we gained in qualitative understanding of the strong interactions, but the fact of having fermions strongly coupled at low energy caused a dramatic setback on quantitative predictions in that regime by the impossibility of using perturbation expansions. The necessity of using non-perturbative methods was then created, the main one of these being lattice gauge theories (or perhaps should we say lattice gauge models).

The idea of replacing space-time by a finite discrete lattice should be seen of course as a mathematical tool, for fundamental space-time symmetries like translational , rotational and Lorentz invariance would be destroyed (although , as has been argued by Zee(1982), there is room for a physical existence of the lattice, as experimental tests of Lorentz invariance hold only up to the order of

10^{-15}m). The main advantage of such a formulation, besides the introduction of an ultra-violet cut-off for perturbative calculations, would be to enable us, through the existence of a finite number of degrees of freedom to make calculations in the non-perturbative region (i.e. large distances)by computer simulations.

In this construction quarks live on sites, gauge fields on links joining neighbouring sites and confinement is explicit at large distances once local gauge invariance has been implemented on the lattice. Hadrons, to be gauge invariant, are made up of quarks linked by gluons which can be thought of as massive strings. To increase the separation between quarks would cost the energy of making the string bits that link them, and at sufficiently large distances, the energy would be large enough to produce a quark anti-quark pair in a typical elementary particles reaction.

This confinement at strong coupling is all very well, but nothing prevents us from making similar constructions with an Abelian gauge field and this would lead to confined electrons! For this case however, it was shown (Lautrup and Nauenberg 1980, Guth 1980) that the system undergoes a transition to a deconfining phase, giving hence strong support that lattice formulations of gauge theories might work after all as physical patterns are recovered at the end of the day.

This brings us to the crucial point of lattice gauge theories: the continuum limit, in which the lattice should

be removed and the real physical world has to be recovered. Effectively when the lattice spacing vanishes, so do the gauge variables which live on bonds. The continuum limit will be obtained when the cut-off dependent bare coupling constant goes to zero (asymptotic freedom states that quarks are essentially free when their separation - by a lattice spacing- is very small). Hadronic measurements on the computer will have to be performed at finite but small coupling. In that regime, it might be of some use to do perturbative corrections of operators that need normalization on the lattice, in order to obtain a correspondence between lattice and continuum operators.

Space-time discretization of non-Abelian gauge models appears then to be the missing step between the original formulation of strong interactions and a fully achieved theory with total predictive power, and the computer simulations undertaken so far have given encouraging results for hadron mass calculations and decay constants, also both analytical and numerical work with lattices seems to guarantee the persistence of a confining phase at all coupling constant values. There is then hope but also a long way to go; computers, despite their growth in the past years, are still too small for better tests of the theory, problems still exist in which should be the best formulation on the lattice, mainly as far as how to implement fermions is concerned. At the root of this problem is the difficulty of retaining the necessary chiral symmetry and at the same time identifying these fermions

properly. It is on this analytical side of lattice gauge theories that we will focus our attention in this thesis.

The next chapter sees a review on the main features of quantum chromodynamics stressing the necessity for a construction of lattice models. We show how they may be constructed and give special treatment to Wilson and staggered fermions for which we have concentrated our attention in this work. We discuss the utility of weak coupling in the last section with a specific calculation on mass renormalization with Wilson fermions.

We begin chapter three with a discussion on the symmetries of continuous quantum chromodynamics, setting the scene for the symmetry and current analysis on the lattice with Wilson fermions. This will be developed in section 3.2 where we shall see that lattice currents may require a finite renormalization due to their non-conservation. We present several possible definitions of currents for Wilson fermions and discuss their properties. Section 3.3 explicitly shows how to obtain renormalization factors due to radiative corrections on the lattice and discusses corrections to meson decay constants calculated on the lattice.

Chapter four brings a discussion on the symmetries of staggered fermions, for which the global symmetries differ from the Wilson formulation. We shall see that rotation and reflection invariances in this picture will require the introduction of flavour matrices in rotation and reflection

lattice operators. This makes the discussion of renormalization of bilinears more delicate than in the previous case, but we shall present a classification of these bilinears ,using properties of the hypercubic rotation reflexion group, charge conjugation and axial symmetry, that had been suggested. The bilinears fall in various representations of the group and components in the same representation should renormalize in the same way. But as the same representation may recur, some operators will have same quantum numbers and will be able to mix via one-loop corrections.

Between the possible site and block variable pictures, we choose the latter, where currents may be better identified than in the former, to calculate renormalization factors, for bilinears with and without mixing.

Chapter 2

LATTICE, WEAK COUPLING AND MASS RENORMALIZATION

Section 2.1 Continuum QCD

Quantum chromodynamics (QCD) is the closest model, in today's physics, to a theory of the strong interactions which governs the behaviour of hadrons (baryons and mesons)(for a review see Huang 1982, Lee 1981, Marciano and Pagels 1978 and Yndurain 1983). QCD has its roots in the quark model (Gell-Mann 1964 and Zweig 1964) which postulated that physical hadrons are composite spin-half-objects, the new "elementary" particles being called the **quarks**, with baryons made up of three quarks and mesons of quark-anti-quark pairs. In their model there were different kinds (or **flavours**) of quark: up,down and strange, and the unitary transformations in flavour space of the three quark states formed the group SU(3). The quarks form the irreducible representation 3 and the antiquarks its adjoint $\bar{3}$. Physical multiplets (which are composed of bound states) are obtained by doing product of representations:

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10 \quad \text{for baryons}$$

$$3 \otimes \bar{3} = 1 \oplus 8 \quad \text{for mesons} \qquad (2.1)$$

The charges of quarks are fractions of the proton charge but the requirement of quarks to be restricted to bound states still ensure that the proton charge is the elementary one. The quark charges in proton charge units are:

$$Q_u = 2/3 \qquad Q_d = Q_s = -1/3$$

The decomposition (2.1) presented the correct results from the experiment and predicted new particles which were later discovered. If one takes the meson pseudoscalar octet, the comparison between the isospin properties of the mesons with those of the quarks shows the quark structure of these particles. For instance:

$$\pi^+ \sim u\bar{d}\,, \qquad \pi^- \sim d\bar{u}$$

$$\pi^0 \sim \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right)$$

$$\eta^0 \sim \frac{1}{\sqrt{6}}\left(2s\bar{s} - u\bar{u} - d\bar{d}\right) \qquad (2.2)$$

The experiments in high energy physics making use of the quark model were very successful, but were never accompanied by the discovery of free quarks. Another disturbing feature of the model was an apparent violation of Fermi-Dirac statistics for half-integer particles: the fundamental state of a composite system is one in which

all relative angular momenta vanish. Thus the spin-3/2 baryon Δ^{++} in its state of lowest energy should be interpreted as made up of

$$|u\uparrow, u\uparrow, u\uparrow\rangle \qquad (2.3)$$

where the wave function (2.3) is clearly symmetric. A solution was proposed by Greenberg (1964), assuming that quarks have another degree of freedom, later called **colour**, so that each quark (i.e. each flavour) comes in three varieties u(c),d(c),s(c), c=1,2,3 (the actual, and irrelevant, names given to each colour vary from author to author, as the flavour ones do - strange or sideways quark etc.-, and many other fancy designations in the realm of particle physics). The wave function (2.3) would then assume the form (and similarly for all baryons):

$$\sum_{i,j,k} \varepsilon^{ijk} |u^i\uparrow, u^j\uparrow, u^k\uparrow\rangle \qquad (2.4)$$

where ε^{ijk} is the fully anti-symmetric tensor; now the Δ^{++} fermion has a perfectly anti-symmetric wave function. Furthermore, the introduction of coloured quarks rules out the existence of the so-called "exotic" states (composed of two or four quarks) if one postulates that all physical hadrons are colourless, i.e. singlets under rotations in colour space:

$$U_{colour} : q^i \to \sum_k U^{ik}_{colour}\, q^k \quad , \quad U_c^{\dagger} U = 1 \qquad (2.5)$$

Such a postulate is important if we do not want to obtain $3^2=9$ meson states for each already existing one. We now see that taking the transformations (2.5) of determinant one (eliminating a trivial overall phase), we can build a new group $SU_{colour}(3)$ (or $SU_c(3)$) for which the singlet representation only appears in

$$3_c \otimes 3_c \otimes 3_c \quad \text{for baryons}$$

$$3_c \otimes \bar{3}_c \quad \text{for mesons} \qquad (2.6)$$

The hypothesis that quarks should be coloured has been supported by some experimental measurements. From the annihilation e^+e^- into hadrons, assuming that in this process a $\bar{q}q$ pair is produced through the electromagnetic interactions and that the hadrons are a product of the emerging quarks fragmentation, one can write the ratio of the cross sections:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

R depends on the different flavours of quarks and of their charge. A factor of three, reflecting different quark colours, is found necessary to account for the observed R

value. Similarly, the electromagnetic decay of $\pi^0 \rightarrow 2\gamma$ requires colour counting to reach the correct experimental value.

It is important to note that the flavour SU(3) symmetry is not an exact one (e.g. the masses of π , K and η in the pseudoscalar octet differ by a few hundred Mev) but the $SU_c(3)$ symmetry is assumed to be exact. The fact that both flavour and colour groups were SU(3) was a coincidence. As a matter of fact, physicists, in their search for symmetry, proposed a model in which there would be the same number of quark flavours and leptons. At the time when only the leptons e, ν_e, μ and ν_μ were known, an SU(4) flavour group was proposed with inclusion of a new flavour **charm** (Glashow,Iliopoulos,Maiani 1970), later confirmed by the discovery of the J/ Ψ particle (Aubert **et al.** 1974 and Augustin **et al.** 1974). Now the **bottom** (or **beauty**) quark had to be included in the flavour group, just as the new lepton τ has been discovered, and inasmuch as a τ neutrino is being sought for, the present state of flavour group would be SU(6), with accelerator physicists throughout the world trying to find (and, now it seems, found at CERN) the new **top** quark experimentally (of course all these "quark discoveries" are made through bound states, as a consequence of the physical "only colour singlet states" assumption).

Another feature of the quark model came from the experiment, when deep inelastic scattering of electrons off nucleons showed that these nucleons are composed of parts

(or **partons** as Feynman called them) which behave essentially as if they were free. These partons were identified with quarks. As for asymptotic freedom, it was shown (Politzer 1973, Gross and Wilczek 1973) that non-Abelian gauge field theories possess its property. In non-Abelian gauge theories (Klein 1938, Yang and Mills 1954), or **Yang-Mills** theories, these authors made the non-exact isospin symmetry a local - as opposed to global- one. Local gauge theories have shown to give a good description of the Abelian electrodynamics (QED), they were then to try to describe the strong interactions.

The main idea of QCD was then to make the colour group, SU(3), a local symmetry. The imposition of a local symmetry requires the introduction of a compensating vector field - the gauge field - , and in the case of an SU(3) gauge group, one must introduce an octet of fields, $G_\mu^a(x)$, $a=1,2,...,8$, which transforms as an adjoint representation of $SU_C(3)$. The octet of gauge fields G_μ^a are called **gluons**, which do not carry flavour, and which interact with the quarks in the fundamental representation of $SU_C(3)$. Following the Yang-Mills prespriction we now can write the minimal Lagrangian density with SU(3) as a gauge group:

$$\mathcal{L} = i\,\bar{\psi}^A_\alpha \left(\not{\partial}_{\alpha\beta} - m_A\right)\psi^A_\beta \qquad (2.7)$$

with α , β $=1,2,3$; $a=1,2,...,8$; $A=1,2,...,N$ (for N flavours) and

using the convention that repeated indices are summed over. The covariant derivative D is defined by:

$$\not{D}_{\alpha\beta}\, q^{A}_{\beta} \equiv \left(\delta_{\alpha\beta}\,\partial_{\mu} - ig\,\frac{\tau^{a}_{\alpha\beta}}{2}\,G^{a}_{\mu}\right)\gamma^{\mu}\, q^{A}_{\beta} \qquad (2.8)$$

and

$$F^{a}_{\mu\nu} \equiv \partial_{\mu} G^{a}_{\nu} - \partial_{\nu} G^{a}_{\mu} + g f^{abc} G^{b}_{\mu} G^{c}_{\nu} \qquad (2.9)$$

and the τ^{a} are the Gell-Mann SU(3) matrices with

$$[\tau^{a}, \tau^{b}] = 2i f^{abc} \tau^{c} \qquad (2.10)$$

f^{abc} being the structure constants of $SU_c(3)$. The fermion mass is **m** and we note that the strength of all interactions defined by (2.7) is specified by a single universal coupling g.

The asumption is then made that the Lagrangian (2.7) describes quarks and gluons which are confined, and then that the colour degrees of freedom cannot be observed. But so far, no one has been able to prove that QCD or any other four dimensional field theory produces confinement. (Note that a suggestion for such a proof has been put forward by Tomboulis 1983).

The QCD Lagrangian is expressed in function of a bare coupling constant g and bare quark masses and some ultra-violet cutoff λ which regularizes the divergences found in the perturbative expansion in g (this cutoff may be thought of as an inverse lattice spacing in a discretized space-time). The renormalization of the theory, as in QED, is done by expressing physical quantities in terms of the bare ones in a given regularization scheme. The renormalization group analysis (Stueckelberg and Peterman 1953, Gell-Mann and Low 1954, Callan 1970, Symanzik 1970) governs the behaviour of bare constants so that a change in the cutoff does not modify physical quantities. The renormalization group defines the β -function for a given momentum scale μ :

$$\beta[g(\mu)] = \mu \frac{d}{d\mu} g(\mu)$$

$$\beta = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \cdots$$

$$\beta_0 = -\frac{1}{6\pi}\left[\frac{33}{2} - (\text{Number of flavours})\right]$$

$$\beta_1 = -\frac{1}{8\pi^2}\left[51 - \frac{19}{3}(\text{Number of flavours})\right] \qquad (2.11a)$$

with $\alpha_s = g^2/4\pi$ - which displays asymptotic freedom as $\mu^2 \to \infty$ for a theory with at the most sixteen flavours - the integration of (2.11-a) gives

$$\alpha_s(\mu^2) = \frac{-1}{\beta_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \qquad (2.11b)$$

where Λ is the renormalization group invariant scale whose value, depending on the renormalization scheme, is given between 50 and 300 Mev (Gasser and Leutwyler 1982). Equation (2.11-b) shows that at low energy, the running coupling constant becomes of order one and in that regime one would expect a breakdown of standard perturbation theory. In the low energy (or at strong coupling) regime, the comparison between physical quantities and QCD will have to be done using methods which were not necessary for QED, and using a model which is to display confinement. Typically one would be interested in mass calculations of quarks and gluons bound states. Lattice gauge theories will represent the required non-perturbative tool that we shall study here.

Locally gauge invariant theories are difficult to quantize because of the gauge fields which are gauge dependent and therefore exhibit an extra non-dynamical degree of freedom which must be dealt with. The standard tool for this problem is given by Feynman's path integral formalism (for a review see Feynman and Hibbs 1965). It is not our aim to discuss quantization of Yang-Mills theory here, but we would like to recall a few steps necessary for doing calculations with QCD.

All the physical quantities in field theory may be derived from the generating functional given by:

$$Z[J] = N^{-1} \int [d\phi] \exp\left\{ \frac{i}{\hbar} \int d^4x \left[\mathcal{L}(x) + J(x)\phi(x) \right] \right\} \qquad (2.12)$$

where J is an external source, N an infinite renormalization factor and $\mathcal{L}(x)$ the Lagrangian density. [dϕ] is a measure representing the integration over all field configurations. We shall not get into details here, but it is important to bear in mind that it is now usual in quantum field theory to perform a Wick rotation ($\tau = -ix_0$) working then in a Euclidean space-time with a positive-definite metric and the generating functionals can be analytically continued back into Minkowski space. A problem with (2.12) is that the measure is not well defined and it is necessary to define a new measure that guarantees no redundant sum on gauge configurations. On the lattice, (2.12) is well defined, with the gauge integration being a product of the integrals over field values defined on the lattice.

Working in Euclidean space turns field theory in a form well known to statistical mechanists, and physical quantities are defined for any operator $\mathcal{O}$ as:

$$\langle \mathcal{O} \rangle = \frac{\int [D\phi]\, \mathcal{O} \exp\left[-S(\phi, \partial_\mu \phi)\right]}{\int [D\phi] \exp\left[-S(\phi, \partial_\mu \phi)\right]} \qquad (2.13)$$

for an action S, and using - from now on - the natural units with $\hbar=c=1$. The denominator is then a partition function in a clear statistical mechanics analogue.

We shall discuss more aspects of continuum QCD in the next chapter when we will focus our attention on flavour symmetries of the action, currents and current algebra. The reason for this splitting in the description of continuum QCD is that we are not trying to give a course in this subject, but rather to illustrate the points which are necessary to an understanding of the lattice gauge construction and that we thought the analogies might appear more clearly by looking at particular aspects rather than at the whole theory at once.

Anyway, we are now ready for a first discussion of lattice gauge theories.

Section 2.2 Lattice QCD: the gauge fields

As we have seen, the formulation of a theory of strong interactions is not yet fully operational, for although QCD describes many experimental evidences in an elegant fashion, it is not possible to prove confinement which seems to be a feature of the model. (Some experimental results showing evidence of particles with fractional electronic charges have been presented but it has not been possible to repeat them to this day. LaRue, Phillips and Fairbank 1981). From a calculation point of

view, we have seen that the coupling constant becomes large at low energy, and when this is the case, it becomes hard to believe in any perturbation expansion which had proved to be so useful and accurate for QED.

The method which was proposed to take care of these problems and that one can consider to be the most successful - by its popularity and even perhaps by the quality of its results - is lattice gauge theory. The idea is of discretizing space and time (although for some time many groups worked with a continuous time variable in the Hamiltonian formalism) having then to work with a field theory in a Euclidean discrete space-time. The lattice provides us with a cutoff which makes any field theory free of ultra-violet divergences, but as there is no evidence of a discretized space-time, physical results will only be obtained when the regulator is removed by taking the lattice spacing to zero, that is taking the **continuum limit**. The fact of having a field theory mathematically well-defined enables one not only to do perturbation theory in the weak coupling regime with a lattice as a regularizing scheme, but also the lattice permits strong coupling expansion calculations where the experience of statistical phyiscs for spin models with high-temperature series expansions may be used. These non-perturbative calculations give then hope of a better understanding of QCD, mainly because confinement is built into the model in this limit.

The problem was initially to find a way of retaining local gauge symmetry on the lattice, which was first solved by Wilson(1974) and Polyakov(1975) generalizing a spin model considered by Wegner (1971) (for a review on lattice gauge theories, see Kogut 1983, Creutz 1983).

The implementation of gauge fields on the lattice goes as follows. For a general gauge group G, one assigns group elements to the links of a hypercubical lattice which represents the discretized space-time, associating an independent element of G for each nearest neighbour pair of lattice sites. If one now requires local gauge invariance and locality of the interactions, it will be possible to write down actions whose continuum limits reproduce the standard Yang-Mills theory for a non-Abelian gauge group. For an SU(3) gauge group, one places an SU(3) matrix on each link; the link variables U will be defined by a site x and a direction μ (μ =1,2,...,d, where d is the dimension of space-time) and can be parametrized as:

$$U_\mu(x) = exp[i a g \tau^i G^i_\mu(x)] \equiv exp[i B_\mu(x)] \qquad (2.14)$$

where a is the lattice spacing, for a gluon field G, the τ's being the SU(3) generators and g the coupling constant. If we take a link in the backward direction, we associate with it $U^+_\mu(x)$:

$$U_{-\mu}(x+\hat{\mu}) \equiv U^\dagger_\mu(x) \qquad (2.15)$$

Bearing in mind that we wish to recover in the continuum limit the classical Yang-Mills theory, and observing that the field strength defined in (2.9) is a generalized curl of the potential, it follows that the gauge part of the lattice action should consist of traces of products of U matrices taken around closed paths. Such closed paths would then be locally gauge invariant because the colour inidices are all contracted into local colour singlets. Taking a minimal contour around a square **plaquette** one obtains the gauge lattice action:

$$S(U) = -\frac{1}{2g^2} \sum_{x,\mu,\nu} \{ \mathrm{Tr}[U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_{-\mu}(x+\hat{\mu}+\hat{\nu})\, U_{-\nu}(x+\hat{\nu})] + \text{Hermitian conjugate} \} \quad (2.16)$$

Before using the action above which enables a study of QCD for all g in a finite and gauge invariant way, it is necessary to check that it has the expected continuum limit. Taking a Taylor expansion of the slowly varying fields $B_\mu(x)$ defined in (2.14) for small a:

$$B_\nu(x+\mu) = B_\nu(x) + a\,\partial_\mu B_\nu(x) + O(a^2)$$

$$B_{-\mu}(x+\mu+\nu) = -B_\mu(x+\nu) = -[B_\mu(x) + a\,\partial_\nu B_\mu(x)] + O(a^2)$$

$$B_{-\nu}(x+\nu) = -B_\nu(x) \quad (2.17)$$

Then

$$U_{\mu\nu} \simeq e^{iB_\mu} e^{i(B_\nu + a\partial_\mu B_\nu)} e^{-i(B_\mu + a\partial_\nu B_\mu)} e^{iB_\nu}$$

$$\simeq \exp\left\{ia(\partial_\mu B_\nu - \partial_\nu B_\mu) - [B_\mu, B_\nu]\right\} \quad (2.18)$$

making use of the identity:

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$$

By taking the explicit form of B_μ in (2.14) we obtain for (2.18) in terms of the field strength:

$$U_{\mu\nu} \simeq e^{ia^2 g F_{\mu\nu}} \quad (2.19)$$

For smooth classical fields with excitations of long wavelengths compared to the lattice spacing we take $a^2 g F_{\mu\nu} << 1$ and one gets for the trace of (2.19), remembering that $\mathrm{Tr} F_{\mu\nu} = 0$:

$$\mathrm{Tr}\,\mathbb{1} + \frac{1}{2} a^4 g^2 \mathrm{Tr} F_{\mu\nu}^2 + \cdots \quad (2.20)$$

Now, after some τ algebra, replacing:

$$\sum_{x,\mu,\nu} \longrightarrow \int \frac{d^4x}{a^4} \sum_{\mu\nu} \qquad (2.21)$$

we get for the gauge action:

$$S = \frac{1}{4} \int d^4x \, (F^i_{\mu\nu})^2 \qquad (2.22)$$

which is the continuum Yang-Mills action. It is important to note that the action (2.16) is by no means unique, other ones may and have been proposed (Wilson 1980, Symanzik 1983) which add to it irrelevant terms which vanish in the continuum limit (and maybe vanish faster), which is the only limitation for a correct theory.

Section 2.3 Lattice QCD: the fermions

In the task of writing down a lattice version of QCD, it would have seemed that the main difficulty would have been to implement local gauge invariance, discussed in the previous section. As a matter of fact, we will now show that somehow any fermion lattice formulation leads to at least one feature which disagrees with the continuum properties of the theory. Different models have been presented which came after the problems linked with the now called **naive** fermions became clear. The two main formulations now used are the Wilson fermions and the staggered fermions, and we are now going to present some

of their characteristics.

Section 2.3.1 Naïve fermions

The gauge fields being associated with links of the lattice, it is natural to assign the fermionic variables with sites. The gauge fields will enable us to transport these matter fields between neighbouring sites and to construct hadrons which will be gauge invariant quantities. Working on a four-dimensional Euclidean hypercubic lattice with lattice spacing a, the sites are given by:

$$x_\mu = \eta_\mu a \quad , \quad \eta_\mu = 0, \pm 1, \pm 2, \cdots \quad , \quad \mu = 1,2,3,4$$

and we can construct the Brillouin zone of now discrete momenta symmetrically around k=0:

$$-\frac{\pi}{a} < k_\mu < \frac{\pi}{a}$$

The most natural way of getting the discrete version of the kinetic term in the free fermionic action:

$$S = \int d^4x \left[-\sum_\mu \bar{\psi}(x) \frac{1}{i} \gamma_\mu \frac{\overleftrightarrow{\partial_\mu}}{2} \psi(x) - m \bar{\psi}(x) \psi(x) \right]$$

is by replacing differentials by differences giving the naïve lattice fermion action (Wilson 1974 and Karsten and Smit 1981 for a review):

$$S = \sum_x \Big\{ \sum_\mu \frac{1}{2a} \Big[\bar{\psi}(x)\gamma_\mu \psi(x+a_\mu) - \bar{\psi}(x+a_\mu)\gamma_\mu \psi(x) \Big] - m\,\bar{\psi}(x)\psi(x) \Big\} \qquad (2.23)$$

where we use the convention of γ-matrices given in Euclidean space such that $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The sum on **x** is over lattice sites and a_μ is a vector of length a in the μ-direction. The action (2.23) is then made gauge invariant by introducing the gauge fields obtaining:

$$S = \sum_x \Big\{ \sum_\mu \frac{1}{2a} \Big[\bar{\psi}(x)\gamma_\mu U_\mu(x)\, \psi(x+a_\mu) - \bar{\psi}(x+a_\mu)\gamma_\mu U_\mu^+(x)\, \psi(x) \Big] - m\,\bar{\psi}(x)\psi(x) \Big\} + S(U) \qquad (2.24)$$

where S(U) is given by (2.16).

If we now make use of the parametrization (2.14) we can take the discrete Fourier transform of the fermion fields in (2.23) and obtain the quark propagator:

$$S(k) = \Big[\sum_\mu -i\gamma_\mu \frac{1}{a} \sin(k_\mu a) + m \Big]^{-1} \qquad (2.25)$$

Shifting the range of momenta to $-\pi/2a < k_\mu < 3\pi/2a$ (using

periodicity over $2\pi/a$) we see that there are sixteen regions in momentum space where the propagator does not vanish in the continuum limit: the regions about $k_\mu = 0$ or π/a. Thus in addition to the expected pole $k^2=-m^2$, there are fifteen other unwanted fermionic states which did not exist in the continuum theory. In these sixteen regions (2.25) is the propagator of a Dirac particle of mass **m**. Furthermore, Karsten and Smit have shown that the vector charge for these sixteen fermions is the same: **+g**, and that their axial-vector charges are +1 for eight of them, -1 for the others, giving thus a theory free of the triangle anomaly (Adler 1969, Bell and Jackiw 1969). This is the doubling problem which more generally gives 2^d fermions degenerate in mass for a naïve discretization in d-dimensions.

It is clear that this fermion doubling is a serious drawback for any practical use of lattice gauge theories in computer simulation, where one would like to reproduce a theory with five, maybe six, but certainly not sixteen flavours. Furthermore, experimental evidence is here to show that masses of different flavours are not necessarily the same. Several methods have been put forward to solve this problem, but each of these will present a price to be paid, by giving away one feature of the continuum theory. The so-called SLAC fermions (Drell, Weinstein and Yankielowicz 1976) present an action which is chiral symmetric and a spectrum without doubling but which by its non-local couplings turns impossible any perturbative

calculations to reproduce the continuum limit. Another method is that of Wilson fermions.

Section 2.3.2 Wilson fermions

To cure the diseases arising from his original formulation, Wilson(1977) proposed to give a mass of the order of the cut-off to the fifteen superfluous particles of the spectrum so that they will diverge when the continuum limit is taken. To do so, he added to the action (2.24) a term which corresponds to a lattice second derivative:

$$\frac{r}{2a}\sum_{x,\mu}\left[\bar{\psi}(x)\,U_\mu(x)\,\psi(x+a_\mu)+\bar{\psi}(x+a_\mu)\,U_\mu^{\dagger}(x)\,\psi(x)-2\bar{\psi}(x)\,\psi(x)\right] \qquad (2.26)$$

where r had been set to one but can be left as an arbitrary parameter. The fermion propagator becomes:

$$S(k)=\left\{\sum_\mu -i\gamma_\mu\frac{1}{a}\sin(k_\mu a)+m+\frac{r}{a}\left[1-\cos(k_\mu a)\right]\right\}^{-1} \qquad (2.27)$$

where the term in r can be seen as a momentum dependent mass term which gives a mass (m + 2nr/a) (n = 1,2,3,4) to fifteen fermions, the propagator being $O(a^0)$ only in the region $p_\mu \approx 0$. We have then recovered a propagator whose continuous limit describes just one fermion. The main disadvantage of this formulation is that chiral symmetry,

whose partial realization seems to be important in continuum QCD, is now strongly broken, as we shall discuss in the next chapter. The staggered fermion method tries to preserve chiral symmetry, even if by keeping a certain amount of fermion doubling.

Section 2.3.3 Staggered fermions

An alternative to Wilson fermions which would reduce the naïve fermion degeneracy without totally eliminating chiral symmetry on the lattice was then proposed and developed by several authors, gaining the name of Susskind, Kogut-Susskind or more generally, staggered fermions (Banks, Kogut and Susskind 1976, Banks **et al.** 1977, Susskind 1977, Sharatchandra, Thun and Weisz 1981, Kawamoto and Smit 1981, Gliozzi 1982, Becher and Joos 1982, Kluberg-Stern **et al.** 1983). The remnant chiral symmetry leaves space for a Goldstone boson on the lattice, and the doubling, now reduced, may be interpreted as representing the different quark flavours.

The idea is to put different components of the fermion spinor on different sites as opposed to the naïve and Wilson prescriptions where the whole spinor lives at one site. The model can easily be visualized in two dimensions, allowing the upper component of the spinors to be only on, say, even sites, and the lower on odd ones. Each component will then live on a new lattice of lattice

spacing twice bigger than the original lattice, and the full action may be written as the sum of two identical and decoupled lattice fermionic actions.

Starting from the naïve action (2.23) in the massless case for an even number of space-time dimensions d, we perform a spin diagonalization (Kawamoto and Smit 1981) operating a change of the fermionic variables, from $\psi(x)$ ($\bar{\psi}(x)$) to $\chi(x)$ ($\bar{\chi}(x)$) (the χ's being Grassmann variables):

$$\psi_\alpha(x) = \gamma_1^{|x_1|}\,\gamma_2^{|x_2|}\cdots\gamma_d^{|x_d|}\,\chi_\alpha(x)$$

$$\bar{\psi}_\alpha(x) = \bar{\chi}_\alpha(x)\,\gamma_1^{|x_1|}\,\gamma_2^{|x_2|}\cdots\gamma_d^{|x_d|} \qquad (2.28)$$

where a lattice site x has components $x_1, x_2, \ldots, x_d$ and $\alpha = 1, 2, \ldots, 2^{d/2}$ labels the spinor components. The fermionic action now reads:

$$S_F = \frac{1}{2a}\sum_{x,\mu}\bar{\chi}_\alpha(x)\,\eta_\mu(x)\,[\chi_\alpha(x+a_\mu) - \chi_\alpha(x-a_\mu)] \qquad (2.29)$$

with

$$\eta_1(x) = 1 \;;\; \eta_2(x) = (-1)^{x_1} \;;\; \eta_3(x) = (-1)^{x_1+x_2} \;;\; \eta_4(x) = (-1)^{x_1+x_2+x_3} \qquad (2.30)$$

We observe that after diagonalization, there are no more

γ-matrices in the action and that the spin degree of freedom α no longer depends on the space-time dimension d, and is free to vary arbitrarily from one to any value f. Also observe that the action then decouples into f identical fermionic species, and that the natural lattice doubling is reduced from 2^d to $2^{d/2}$ (the number of spinor components in ψ). Having f decoupled copies, one is now free to consider the case f=1. The introduction of gauge fields coupled to χ's would not alter this discussion. Including a mass term, we obtain in this case for (2.29):

$$S_F + S_m = \frac{1}{2a} \sum_{x,\mu} \eta_\mu(x) \bar{\chi}(x) [\chi(x+a_\mu) - \chi(x-a_\mu)]$$

$$+ m \sum_x \bar{\chi}(x) \chi(x) \quad - \qquad (2.31)$$

From (2.31) one wants to obtain Dirac fields with $2^{d/2}$ components for which the momentum-space propagator has only one pole, the continuum limit yielding an action of free quarks with $2^{d/2}$ flavours with a flavour-symmetric mass m. The spinors will be defined on d-dimensional hypercubes of the lattice, H(y), with origin at the corners 2y where the original lattice sites are rewritten as:

$$\begin{cases} x_\mu = 2y_\mu + \eta \\ \eta_\mu = 0 \text{ or } 1 \; ; \quad \mu = 1, \cdots, d \end{cases} \tag{2.32}$$

η are the coordinates of the hypercube, there being 2^d d-dimensional vectors η pointing on its corners. Each H(y) contains 2^d χ's and 2^d $\bar{\chi}$'s. Relabelling the χ's as:

$$\chi(2y+\eta) \equiv \chi_\eta(y)$$

$$\bar{\chi}(2y+\eta) \equiv \bar{\chi}_\eta(y) \tag{2.33}$$

it is possible, following Kluberg-Stern et al.(1983) to define quark fields by:

$$\psi^{\alpha a}(y) = \frac{1}{C\sqrt{C}} \sum_\eta \Gamma_\eta^{\alpha a} \chi_\eta(y)$$

$$\bar{\psi}^{\alpha a}(y) = \frac{1}{C\sqrt{C}} \sum_\eta \bar{\chi}_\eta(y) \Gamma_\eta^{*\alpha a} \tag{2.34a}$$

with:

$$C = 2^{d/2} \; ; \quad \Gamma_\eta = \gamma_1^{\eta_1} \gamma_2^{\eta_2} \cdots \gamma_d^{\eta_d} \; ; \quad \alpha, a = 1, \cdots, C \tag{2.34b}$$

and its inverse:

$$\chi_\eta(y) = \sqrt{c} \sum_{\alpha,a} \Gamma_\eta^{*\alpha a} \psi^{\alpha a}(y)$$

$$\bar{\chi}_\eta(y) = \sqrt{c} \sum_{\alpha,a} \bar{\psi}^{\alpha a}(y) \Gamma_\eta^{\alpha a} \qquad (2.35)$$

Defining on the lattice with spacing 2a formed by the origins of the hypercubes the first and second derivatives (respectively):

$$\Delta_\mu f(y) \equiv \frac{1}{4a} \left[f(y+\hat{\mu}) - f(y-\hat{\mu}) \right]$$

$$\delta_\mu f(y) \equiv \frac{1}{4a^2} \left[f(y+\hat{\mu}) + f(y-\hat{\mu}) - 2f(y) \right] \qquad (2.36)$$

the action (2.31) can be expressed in terms of block variables as:

$$S_F + S_m = \sum_{y,\mu} \{ \bar{\psi}(y) (\gamma_\mu \otimes \mathbb{1}) \Delta_\mu \psi(y)$$

$$+ a \bar{\psi}(y) (\gamma_5 \otimes (\gamma_5 \gamma_\mu)^T) \delta_\mu \psi(y)$$

$$+ m \sum_y \bar{\psi}(y) (\mathbb{1} \otimes \mathbb{1}) \psi(y) \qquad (2.37)$$

where the first matrix acts on Greek indices interpreted as Dirac indices, and the second acts on Latin indices in flavour space. We recognize the first term in (2.37) to be the naive action for C Dirac spinors on lattice with spacing 2a and it has got the chiral symmetry U(C) $\circledX$ U(C). The second derivative term, which is order a with respect to the previous one and thus vanishes in the continuum limit, is responsible for lifting the fermion degeneracy. This is clearly seen in the momentum space propagator which is obtained taking the Fourier transform of (2.37):

$$S(k) = \Big[-i \sum_\mu \gamma_\mu \otimes \mathbb{1} \, \frac{1}{a} \sin k_\mu + m \, \mathbb{1} \otimes \mathbb{1} - \frac{1}{a} \sum_\mu (1 - \cos k_\mu) \, \gamma_5 \otimes (\gamma_5 \gamma_\mu)^T \Big]^{-1} \qquad (2.38)$$

which has only one pole at $p_\mu = 0$ in the first Brillouin zone defined on the big lattice (lattice spacing 2a):

$$-\frac{\pi}{2a} \leq p_\mu \leq \frac{\pi}{2a} \qquad (2.39)$$

Note the similarity of (2.27) and (2.39), but here the continuum limit gives C decoupled Dirac spinors of degenerate masses.

So far we have seen that free staggered fermions can be examined via site variables (χ and $\bar{\chi}$) or block variables (ψ and $\bar{\psi}$). The introduction of gauge fields in the fermionic part of the action is now more subtle than in the Wilson case, for there is the alternative of setting the gauge

field on bonds of the small lattice, linking neighbouring site variables, or on the big lattice, linking block variables. The first method might be more practical from a numerical simulations point of view, but the latter presents physical quantities in a form more analogous to the continuum ones (as was the case for Wilson's formulation). The action with gauge fields between blocks is also called Dirac-Kähler, following the work of Becher and Joos (1982) who have shown the correspondence of the staggered fermion degeneracy and a degeneracy already existing in the continuum Kähler-Dirac equation (Kähler 1962).

Section 2.4 Weak coupling and mass renormalization

One of the major requirements for lattice gauge theories to become a faithful model of strong interactions is to make correct quantitative predictions in the low-lying energies of the hadronic world, where perturbative calculations cease to be meaningful. Computer simulations of lattices of dimensions 16^4 - and probably bigger in the future - are being made for that purpose with encouraging results. Expectation values of observables are calculated by Monte Carlo methods where the gauge field integration in (2.13) is replaced by a weighted sum on a sufficiently large number of gauge configurations. The sum has to be weighted for although a sum on all possible configurations would be finite, it hardly would be feasible

by its size, even for small lattices. In analogy with statistical systems we can select a number of gauge field configurations with a statistical weight e^{-S}, making the statistical average over the more relevant ones. There are various algorithms with the same underlying principle: a stochastic procedure produces from a given - generally random - configuration another one which replaces it with a probability weight, so that after many such steps, configurations reach an equilibrium. Because of limitations in the size of computers, SU(3) gauge configurations in four dimensions are still mainly generated in the **quenched** approximation, where one neglects the interaction of the quarks on the gauge fields, ignoring dynamical fermion loops. Unquenched studies are also performed, but with simpler gauge fields or in less than four dimensions.

In this generating of gauge field configurations a value has to be given for the coupling constant g. If we now see the lattice as a cut-off to regularize our theory, it is important when removing the cut-off (lattice spacing goes to zero) that physical quantities should be independent of it, as understood from any renormalization analysis. In particular then, the bare coupling constant will be a-dependent, and from the renormalization group equation it may be obtained that the continuum limit is reached when g tends to zero. In principle, it would appear that numerical simulations should be done at small values of the coupling constant if to reproduce the continuum hadronic spectrum, but in practice, too small a a will produce

hadrons occupying too many lattice sites (not forgetting that masses will be given in lattice units) and not enough "space" for particles to propagate. It will then be a crucial problem for Monte Carlo simulations to move away from a large coupling constant which could hardly reproduce continuum physics, without getting particles too much spread out in the box available in computation.

When g is small, perturbation expansion can be done, and although the lattice regularization was not put forward for that purpose, the fact that non-perturbative analysis is done at small but not vanishing coupling constant indicates that if some operators need any finite renormalizations on the lattice, then these corrections could be included at order g^2 and turn out to be welcome corrections to matrix elements obtained by numerical simulations.

From a more general viewpoint, we know that at strong coupling lattice QCD exhibits confinement, it is then necessary to examine the theory at short distances (weak coupling) and to make sure that there is no transition from the confining phase. It is also important to note that it is in the weak coupling regime that the continuum formulation should be recovered from the lattice one, and hence where results known to be true in the continuum should be obtained.

Having understood the motivations of weak coupling perturbation theory on the lattice, we can now calculate

the mass renormalization, in the case of Wilson fermions, which, albeit with loss of chiral symmetry, presents quarks with unambiguous flavour interpretation. The fermionic Wilson action reads (making use of (2.24) and (2.26):

$$S = \sum_x \Big\{ \sum_\mu \frac{1}{2a} \Big[\bar{\psi}(x)(\gamma_\mu + r) U_\mu(x) \psi(x + a_\mu) - \bar{\psi}(x + a_\mu)(\gamma_\mu - r) U_\mu^\dagger(x) \psi(x) \Big] - M \bar{\psi}(x) \psi(x) \Big\} \qquad (2.40)$$

where, denoting **m** as the renormalized quark mass which should be **a**-independent, we have:

$$M = m + \frac{4r}{a} \qquad (2.41)$$

and we notice that indeed the **r**-term in (2.41) (whose factor of four comes from the sum in μ in (2.26) for four space-time dimensions) cancels out at tree level the **r**-term in the rest of the action when the continuum limit is taken. In this limit the dimensionless parameter **aM** will reach the critical value **4r** at which **1/am** diverges. This is an indication that the physics which will emerge from (2.40) will depend on **m** and **g** but not on the parameter **r**, as it should be.

In the continuum, evaluating the fermion self-energy would amount to calculate the one-loop diagram of figure 1(a). The Feynman rules on the lattice though, will allow also the existence of the diagram 1(b).

To obtain these Feynman rules, we use the exponential parametrization of the gauge field (2.14) and in order to make weak coupling expansions it will be necessary to fix the gauge (we shall pick the Feynman gauge). Taking the action (2.40) into the discrete Fourier space, we can write down momentum space integrals over the Brillouin zone which yield the gluon propagator and the bare vertex functions, which together with the fermion propagator are given in figure 2. We have stopped our weak coupling expansion at order g^2, but it is clear that higher orders would lead to more gluon lines coming out of the vertex of fig.2(d). Note also that this last vertex, non-existent in the continuum, is higher order in a, and thus the correct continuum behaviour is guaranteed, including momentum-independent vertex functions.

Having defined aM_c as the value of aM where the quark mass is zero, we can impose for renormalization condition that the two point vertex should be zero when $M=M_c$:

$$0 = \Gamma^{(2)}(k=0; g, M_c, a) = M_c - \frac{4r}{a} + O(g^2) \qquad (2.42)$$

and therefore in one loop we get for (2.41):

$$M = m + M_c = m + \frac{4r}{a} + \Sigma(0) \qquad (2.43)$$

where $\Sigma(0)$ is the self-energy obtained from the diagrams of fig.1 and given by:

$$\Sigma(P_\mu=0) = \frac{-\bar{g}^2}{(2\pi)^4}\int_{-\frac{\pi}{a}}^{\pi/a} d^4k\, a^3 \sum_{\mu\sigma}\left(-i\gamma_\mu \cos k_\mu\frac{a}{2} + r\sin k_\mu\frac{a}{2}\right)\times$$

$$\times\left(-i\gamma_\sigma \sin k_\sigma a - 2r\sin^2\frac{k_\sigma a}{2}\right)\left(-i\gamma_\mu\cos k_\mu\frac{a}{2} + r\sin k_\mu\frac{a}{2}\right)\times$$

$$\times\left\{\left[\sum_\nu \sin^2 k_\nu a + 4r^2\left(\sum_\nu \sin^2\frac{k_\nu a}{2}\right)^2\right]\left[4\sum_\tau \sin^2\frac{k_\tau a}{2}\right]\right\}^{-1}$$

$$-\frac{\bar{g}^2}{2(2\pi)^4}\int_{-\pi/a}^{\pi/a}\frac{d^4k\, a^3\sum_\mu\left(i\gamma_\mu\sin P_\mu a + r\cos P_\mu a\right)}{\left(4\sum_\tau \sin^2\frac{k_\tau a}{2}\right)} \qquad (2.44)$$

with

$$\bar{g}^2 = \frac{1}{2}g^2\frac{(N^2-1)}{N} \qquad (2.45)$$

and we note the absence of **m** in (2.44) for we are looking at the two point vertex in the critical regime of zero quark mass. These integrals may be evaluated numerically (we used the NAG Fortran subroutine D01FAF which is a Monte Carlo subroutine for numerical multiple integration, giving an approximation to the Riemann integral of irregular functions defined over a hyper-cubical region). For the second integral of (2.44), it is possible to make the so-called exponential trick: we introduce a regulator λ in the denominator, which may be expressed by a Laplace transform:

$$\int_0^\infty ds\, e^{-sy} = \frac{1}{y} \quad \text{with} \quad y = 4\left(\sum_\tau \sin^2\frac{k_\tau}{2} + \lambda\right) \qquad (2.46)$$

We then need to evaluate the integral (where the loop variable has been re-scaled $\mathbf{k} \rightarrow \mathbf{ka}$):

$$\lim_{\lambda \to 0} \frac{1}{4} \int_0^\infty ds \; e^{-\lambda} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} dk \; e^{-s(\sin^2 \frac{k}{2})} \right]^4$$

$$= \left[\int_0^\infty ds \, I_0(2s) e^{-2s} \right]^4 \qquad (2.47)$$

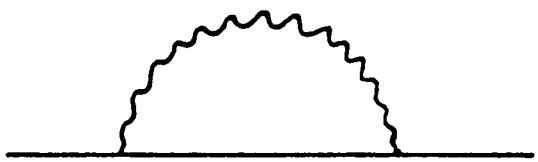
where we have identified the modified Bessel function I_0. For this particular integral, a comparison with the direct numerical evaluation with the above mentioned routine showed a discrepancy of less than 0.1%, and thus provides a test for the accuracy of the NAG routine.

The zero-momentum self-energy contributing to the quark mass renormalization is given as a functions of r in fig.3.

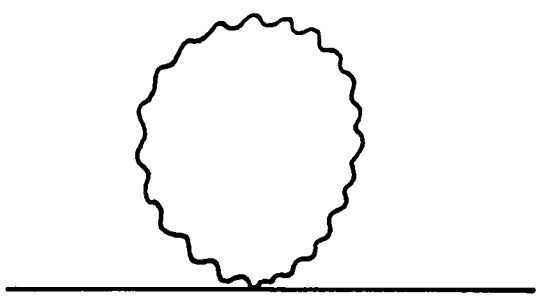
We shall now move to currents and their renormalization on the lattice for Wilson fermions with a special interest in pion operators.

Figure 1

Lattice QCD self-energy



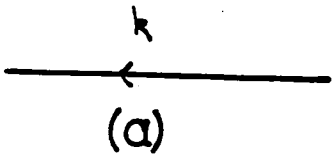
(a)



(b)

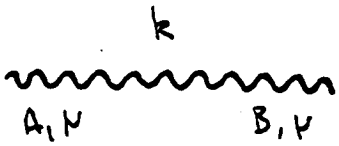
Figure 2

Lattice QCD Feynman rules for Wilson fermions



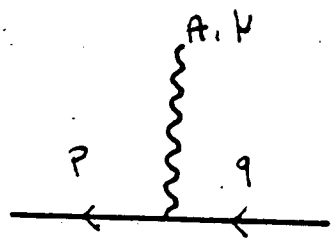
(a)

$$\frac{a}{\sum_\mu -i\gamma_\mu \sin k_\mu a + ma + 2r\sin^2\left(\frac{k_\mu a}{2}\right)}$$



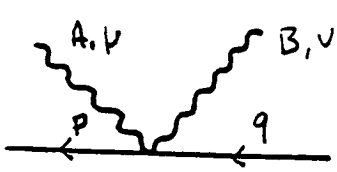
(b)

$$\frac{a^2 \delta_{AB}\,\delta_{\mu\nu}}{4\sum_\mu \sin^2\left(\frac{k_\mu a}{2}\right)}$$



(c)

$$-gT^A\left\{-i\gamma_\mu \cos\left[\tfrac{1}{2}(p_\mu a + q_\mu a)\right] + r\sin\left[\tfrac{1}{2}(p_\mu a + q_\mu a)\right]\right\}$$

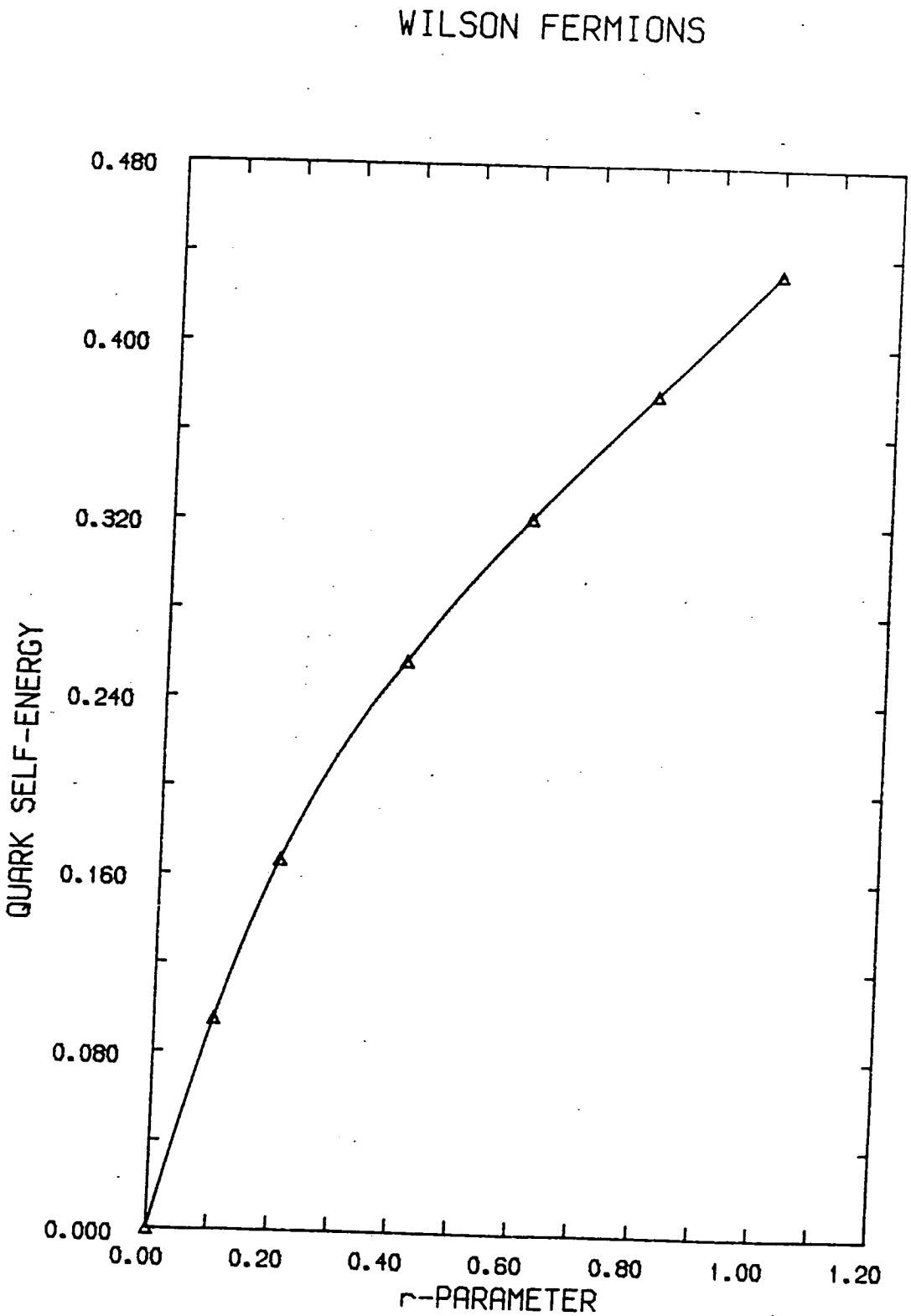


(d)

$$-ag^2\tfrac{1}{2}\left\{T^A, T^B\right\}\delta_{\mu\nu}\left\{i\gamma_\mu \sin\left(\frac{p_\mu a + q_\mu a}{2}\right) + r\cos\left(\frac{p_\mu a + q_\mu a}{2}\right)\right\}$$

Figure 3

Quark self-energy at zero external momentum



Chapter 3

CURRENTS, MESON DECAY CONSTANTS AND THEIR RENORMALIZATION

A requirement for lattice QCD to become a successful theory is that it should correctly describe the phenomenological results confirmed by experiments. In this chapter, we describe the global symmetries of continuum QCD and how currents come about and why they are conserved (or partially conserved). Throughout we underline the differences that will arise on a discretized version of the theory. Section 3.2 presents the lattice QCD symmetries when Wilson fermions are used, and we discuss currents that are originated by these symmetries, as well as some more ad hoc ones. This is done for we shall see that symmetr y currents lo se at finite lattice spacing the nice properties of continuum conserved currents, and will require a renormalization. Hence currents input by hand, as long as they yield the correct continuum structure, may be used instead of the symmetry currents if they are more easily dealt with from a computer simulation point of view.

In section 3.3 we show how to perform the one-loop calculations that give the first order corrections to the non-conserved currents. We also discuss how order g^2 renormalizations correct the values of meson decay constants and help to reduce the discrepancy between lattice estimates and experimental values.

Section 3.1 Symmetries and currents of QCD

As we have seen, the weak coupling regime of lattice QCD should return the continuous behaviour of the theory, introducing corrections caused by the interaction of the fermions with the gauge fields. Such corrections will prove to be of practical importance for a comparison between lattice and continuous quantities, this appearing typically in the study of currents , whose conservation in the continuum will be damaged by the existence of a discretized space-time. At the root of this problem, we find the study of the global symmetries of QCD, and how they get modified on a lattice version, depending too on the way that fermions are introduced. It is important to note that we shall now be discussing global symmetries, of which chiral symmetry is a particular case.

The link between global symmetries and currents is a consequence of Noether's theorem, which guarantees that to each invariance of the Lagrangian under a given continuous transformation, corresponds a four-vector, called current, which will be conserved. Also conserved will be the associated charge, given by the space integral of the zero component of this current.

If we now examine the global symmetries of the continuum Lagrangian (2.7) with N massless flavours, we see an invariance coming from $U(N) \otimes U(N)$, which can be decomposed in the following global symmetry groups:

$$U_V(1):\ \psi(x) \to e^{i\theta \mathbb{1}\otimes\mathbb{1}}\psi(x),\quad \bar\psi(x) \to \bar\psi(x) e^{-i\theta \mathbb{1}\otimes\mathbb{1}} \qquad (3.1a)$$

$$SU_V(1):\ \psi(x) \to e^{i\mathbb{1}\otimes\theta_i\frac{\tau_i}{2}}\psi(x),\quad \bar\psi(x) \to \bar\psi(x) e^{-i\mathbb{1}\otimes\theta_i\frac{\tau_i}{2}} \qquad (3.1b)$$

$$U_A(1):\ \psi(x) \to e^{i\theta\gamma_5\otimes\mathbb{1}}\psi(x),\quad \bar\psi(x) \to \bar\psi(x) e^{i\theta\gamma_5\otimes\mathbb{1}} \qquad (3.1c)$$

$$SU_A(N):\ \psi(x) \to e^{i\gamma_5\otimes\theta_i\frac{\tau_i}{2}}\psi(x),\quad \bar\psi(x) \to \bar\psi(x) e^{i\gamma_5\otimes\theta_i\frac{\tau_i}{2}} \qquad (3.1d)$$

giving the N^2 generators for each U(N), with θ being arbitrary real constants, the $(N^2 - 1)$ τ_i's being the matrices of the adjoint representation of SU(N). The first matrices in (3.1) act in spinor space while the second in flavour space, and we note that $U_V(1)$ and $U_A(1)$ are flavour singlets. The corresponding N^2 currents are respectively:

$$V_\mu(x) = \bar\psi(x)\gamma_\mu\otimes\mathbb{1}\,\psi(x) \qquad (3.2a)$$

$$V_\mu^i(x) = \bar\psi(x)\gamma_\mu\otimes\frac{\tau^i}{2}\psi(x) \quad i = 1,2,\dots,N^2-1 \qquad (3.2b)$$

$$A_\mu(x) = \bar\psi(x)\gamma_\mu\gamma_5\otimes\mathbb{1}\,\psi(x) \qquad (3.2c)$$

$$A_\mu^i(x) = \bar\psi(x)\gamma_\mu\gamma_5\otimes\frac{\tau^i}{2}\psi(x) \quad i = 1,\dots,N^2-1 \qquad (3.2d)$$

In the present massless case all these currents are

conserved, except the axial-vector flavour singlet one (3.2c) which suffers form the so-called triangle anomaly which has proved to be necessary for the first time in the cross section calculation of the pion decay into two photons. The breaking of its conservation is expressed as (Adler 1970):

$$\partial_\mu A_\mu(x) = \frac{\alpha_0}{4\pi} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} F_{\mu\nu} \qquad (3.3)$$

The fact that the anomaly is the divergence of a vector may suggest that a new definition for an axial-vector current could be chosen in order to recover its conservation, but it can be shown that the new current is not gauge invariant (Adler 1970). We now see how it is the (physical) choice that physical fields should be gauge invariant which costs the inclusion of the anomaly as an explicit breaking of $U_A(1)$. Of course, from a phenomenological point of view the anomaly is welcome for it provides a correct decay of the pion, and because the conservation of the axial baryon number is not observed.

The inclusion of mass terms in the Lagrangian alters the picture further and on the assumption of non-degenerate quark masses, only the vector flavour singlet current (3.2a) remains exactly conserved, together with its associated charge, which expresses the conservation of baryonic number (we know of the arguments for a violation of this quantity put forward by the grand

unified theories, but no clear sign for such has so far been found, and this discussion is beyond our scope).

Making use of the Dirac equation for a given flavour f:

$$(i\not{\partial} - m_f)\,\psi_f(x) = 0$$

$$\bar{\psi}_f(x)\,(i\not{\partial} + m_f) = 0 \qquad (3.4)$$

we obtain the divergences for the flavour non-sinlet currents considering the fermion fields as free fields (as the transformation (3.1) commute with the interaction Lagrangian; the gauge fields carry no flavour):

$$\partial_\mu V_\mu^{ff'}(x) = i(m_f - m_{f'})\,\bar{\psi}_f(x)\,\psi_{f'}(x) \qquad (3.5a)$$

$$\partial_\mu A_\mu^{ff'}(x) = i(m_f + m_{f'})\,\bar{\psi}_f(x)\gamma_5\,\psi_{f'}(x) + \text{Anomaly} \qquad (3.5b)$$

This shows explicitly why the isospin symmetry $SU_V(N)$ discussed in section 1.1 will be exact only in the case of degenerate quark masses, and experimental evidence shows that, say, $m_u \ll m_c$. $SU_A(N)$ is softly broken by the mass terms, even in the flavour symmetric case. Such a symmetry would imply the existence of an isospin multiplet with opposite parity and equal mass associated with each existing isospin multiplet. It is the lack of such an even

approximate parity doubling in the spectrum that led to the assumption of a spontaneous breaking of the $SU_V(N) \otimes SU_A(N)$ to $SU_V(N)$ in the ground state, giving a vacuum state not invariant under chiral transformations. We know from Goldstone's theorem (Goldstone 1961) that to each of the generators of the broken SU(N) should correspond a massless pseudoscalar particle. According to this picture then, one would expect the pseudoscalar bosons to be the lightest of all hadrons. This seems to be the case; considering an SU(2) flavour symmetry, we can recognize π^0, π^+ and π^- as the Goldstone bosons, SU(3) yields the eight (π, K,η) and so on. It is true that we broke a $SU_V(N) \otimes SU_A(N)$ symmetry which is valid only for massless quarks. Because the Lagrangian of QCD is not exactly chiral invariant, its Goldstone bosons have got a finite (but small) mass. However, as the departure from chiral symmetry is small, so will be the masses of these nearly Goldstone bosons. This fact is expressed, and proved to lead to correct predictions, in the pre-QCD phenomenological approach of current algebra (conversely, we can visualize the success of this analysis by observing that there exists a vector SU(N) symmetry in the spectrum and hence one should not expect the existence of Goldstone scalar particles. Indeed the lightest scalar mesons lie high up in mass; m_δ =980 Mev against 938 Mev for the proton).

The partial conservation (up to the anomaly) of the axial-vector current (PCAC) even in the flavour symmetric limit (equation 3.5b) is a testable evidence of the hadronic

world and should be obtained in the same right as quark confinement in a theory for strong interactions. This explains the importance of a search for a lattice version of QCD which has the correct chiral behaviour.

To show the link between chiral symmetry and pseudoscalar particles further, observe that states with pion quantum numbers of isospin j can be annihilated by the current operator (3.2d) as:

$$\langle 0 | A_\mu^i(x) | \pi^j(p) \rangle = i \delta_{ij} f_\pi p_\mu e^{-ip.x} \qquad (3.6)$$

f_π being a proportionality constant. The above relation implies (for pions on their mass shell):

$$\langle 0 | \partial_\mu A_\mu^i(x) | \pi^j(p) \rangle = \delta_{ij} f_\pi m_\pi^2 e^{-ip.x} \qquad (3.7)$$

This equation displays clearly how a conserved axial-vector current implies a massless pion (as long as we choose a non-vanishing f_π). Comparison of (3.7) with (3.5b) shows how when the chiral limit of zero quark mass is taken, the up and down quarks should vanish as the square of the pion mass (remember the quark content of the pion in (2.2)). A pion field can now be expressed directly from the divergence of the axial-vector current as:

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$$\pi^i(x) = \frac{1}{m_\pi^2 f_\pi} \partial_\mu A_\mu^i(x) \qquad (3.8)$$

The pion decay constant f_π may be determined from experiment; its value as calculated from the weak decay $\pi^\pm \longrightarrow \mu^\pm \nu$ gives a value around 93 Mev. This dimensional quantity defines so to speak a scale in QCD. It is of importance as it indicates the exact measure of PCAC, which itself should come with all its phenomenological results as the outcome of a theory of strong interactions. This explains the effort that has been put into its calculation in the framework of lattice gauge theories.

The same analysis of quantum numbers that led to eq.(3.6) may be used to obtain the relation between the ϱ-mesons - which are vector particles with a quark wave function similar to that of pions (eq.(2.2)) - and the vector currents (3.2a and 3.2b), relating a ϱ-state with the vacuum according to:

$$\langle 0 | V_\mu^i(x) | \rho^j(P) \rangle = i \delta_{ij} m_\rho^2 f_\rho^{-1} \varepsilon_\mu e^{-iP\cdot x} \qquad (3.9)$$

m_ϱ being the mass of the ϱ-meson and f_ϱ its decay constant. ε_μ is the polarization vector of the spin-one particle.

In a synthesis of the flavour symmetry, Gell-Mann postulated the so-called current algebra establishing the following commutation relations between currents (Lee 1972):

$$\delta(x_0-y_0)[V_0^i(x), J_0^j(y)] = i f^{ijk} J_0^k(x)\,\delta(x-y)$$

$$\delta(x_0-y_0)[V_0^i(x), \vec{J}^j(y)] = i f^{ijk} \vec{J}^k(x)\,\delta(x-y) + \text{Schwinger term}$$

$$\delta(x_0-y_0)[A_0^i(x), A_0^j(y)] = i f^{ijk} V_0^k(x)\,\delta(x-y)$$

$$\delta(x_0-y_0)[A_0^i(x), \vec{A}^j(y)] = i f^{ijk} \vec{V}^k(x)\,\delta(x-y) + \text{Schwinger term} \qquad (3.10)$$

where j_μ^i is either V_μ^i or A_μ^i and the Schwinger term (Schwinger 1959) vanishes in the integration over space, and is introduced to avoid problems that arise when we look at product of currents at the same space-time points. The f^{ijk} are the flavour group structure constants defined as in (2.10).

The above relations are a consequence of PCAC and of the conservation of the vector current. They have fixed group structure factors; this fixes the scale of the currents and so does not allow conserved currents (or partially conserved) to be renormalized.

Because chiral symmetry is strongly broken, in the lattice version this will not necessarily be the case, and a renormalization will be required for each non-conserved current. This renormalization will be obtained by relating the one quark matrix element of the current with the vertex function.

The one-quark matrix element may be written, for a current with momentum transfer **q**, as (Bernstein 1968):

$$\langle P' | V_\mu^j (0) | P \rangle = i \bar{u}(P') \frac{\tau^j}{2} [\gamma_\mu f_v(q^2) + \sigma_{\mu\nu} q_\nu f_M(q^2) + i q_\mu f_s(q^2)] u(P)$$

and

$$\langle P' | A_\mu^j (0) | P \rangle = i \bar{u}(P') \frac{\tau^j}{2} \gamma_5 [\gamma_\mu g_A(q^2) + i q_\mu g_P(q^2) + i (P + P') g_E(q^2)] u(P)$$

We shall keep zero momentum transfer for currents as well as zero external momenta.

We also know that on-shell, the quark propagator has a pole at the quark mass, with residue Z_2.

We know that Z_2 is related to the renormalization of the gluon-quark vertex as there is the Ward identity relating the quark propagator and the vertex Γ_μ^V and Γ_μ^A (up to the anomaly) at all orders in perturbation theory:

$$i\frac{\tau}{2}S^{-1}(p') - iS^{-1}(p)\frac{\tau}{2} = -i(p-p')\Gamma_\mu^V(p',p) \qquad (3.11a)$$

and

$$i\frac{\tau}{2}S^{-1}(p')\gamma_5 + i\gamma_5 S^{-1}(p)\frac{\tau}{2} + 2im\bar{\psi}(x)\gamma_5\psi(x)$$

$$= -i(p-p')\Gamma_\mu^A(p',p) \qquad (3.11b)$$

and the vertex function may be defined in terms of the free, on mass shell spinors $u(p)$, introducing the renormalization factor Z_1 at all orders:

$$\bar{u}(p)\Gamma_\mu^V(p,p)u(p) = \frac{i}{Z_1}\bar{u}(p)\gamma_\mu\frac{\tau}{2}u(p) \qquad (3.12a)$$

and

$$\bar{u}(p)\Gamma_\mu^A(p,p)u(p) = \frac{i}{Z_1}\bar{u}(p)\gamma_\mu\gamma_5\frac{\tau}{2}u(p) \qquad (3.12b)$$

which yields the identity: $Z_2/Z_1 = 1$ (3.13)

The relation (3.13) is directly obtained in the vector case. For the axial-vector one, it is not as immediate, but for the non-anomalous case, it can be shown in a simple way that, up to arbitrary finite factors, it is true (Adler 1970).

The above identity confirms the absence of a need for renormalization for the conserved vector current and for the axial-vector one (excluding here the discussion of axial-vector flavour singlet currents for which the Ward identity as it stands is no longer valid), for we can write:

$$\langle p | V_\mu^j(0) | p \rangle = i \bar{u}(p) \gamma_\mu \frac{\tau^j}{2} u(p) \qquad (3.14)$$

$$\langle p | A_\mu^j(0) | p \rangle = i \bar{u}(p) \gamma_\mu \gamma_5 \frac{\tau^j}{2} u(p) \qquad (3.15)$$

The fact that the above relations are valid at all levels of perturbation theory is a sign that the introduction of gauge fields will not lead to any renormalization. What will change on the lattice is that non-conservation of currents will mean a departure from the tree-level relations and a renormalization of the strength of the currents due to the interaction of the fermions with the colour fields, in a similar way to what happened in the last chapter when the tree-level relation (2.41) had to be corrected into (2.43) when doing a diagramatic expansion.

Strictly speaking, the renormalization of a non-conserved current will only be finite if the symmetry breaking is such that the renormalization will be cut-off independent. The renormalization, if cut-off dependent, will be infinite. This is the case of the anomaly which needs a renormalization proportional to the logarithm of the cut-off; this then rules out a renormalization for flavour singlet axial-vector currents (Adler 1970).

Such a Feynman diagram expansion on the lattice will have as an underlying purpose, the aim of comparing, and hence correcting, values on the lattice with their corresponding continuum partners.

In order to understand, later on, how the pion mass and decay constant may be evaluated numerically on the lattice (Bowler, Pawley, Wallace, Marinari and Rapuano 1983), we shall sketch in analogy with their lattice analysis, how these quantities can be obtained in the continuum.

Define the time slice propagator, for the pion operator $\partial_\mu A^i_\mu$, as:

$$D(x_4) = \int d^3x \, \langle 0 | \partial_\mu A^i_\mu(x) \, \partial_\mu A^i_\mu(0) | 0 \rangle \qquad (3.16)$$

Inserting two complete sets of states, $D(x_4)$ becomes:

$$D(x_4) = \sum_{\pi} \sum_{\pi'} \int d^3x \, \langle 0 | \partial_\mu A^i_\mu(x) | \pi'(x) \rangle \times$$

$$\times \langle \pi'(x) | \pi(0) \rangle \langle \pi(0) | \partial_\mu A^i_\mu(0) | 0 \rangle \qquad (3.17)$$

and writing these one particle states in terms of their propagators:

$$\langle \pi'(x) | \pi(0) \rangle = G^{(2)}_\pi(x,0)\delta_{\pi\pi'} \qquad (3.18)$$

we can obtain for the time slice propagator, after transformation of the fields into Fourier space:

$$D(x_4) \sim \int d^4k\, d^4p\, d^4q\, d^3x\, e^{-ikx} \times$$
$$\times G^{(2)}(k) \langle 0 | \partial_\mu A^i_\mu(x) | \pi(p) \rangle \times$$
$$\times \langle \pi(q) | \partial_\mu A^i_\mu(0) | 0 \rangle \qquad (3.19)$$

The utility of the time slice propagator is that at large "distances" x_4, only the particle with the lowest mass will still contribute, the faster decay of the others enabling us to discard them, and assuming that the pion is the particle with lowest mass (backed by all our previous analysis), we can assume a pion dominance in (3.19) to read:

$$D(x_4) = \delta_{ij} \int dk_4 \, e^{-ik_4 x_4} \, G^{(2)}(k_4, m_\pi) \times$$

$$\times \left| \langle 0 | \partial_\mu A_\mu^i(x) | \pi^j(x) \rangle \right|^2 \qquad (3.20)$$

and the integration finally yields:

$$D(x_4) = \frac{\left| \langle 0 | \partial_\mu A_\mu^i(x) | \pi^i(x) \rangle \right|^2}{2 m_\pi} \, e^{-m_\pi x_4} \qquad (3.21)$$

On the lattice a similar kind of expression will enable to estimate the pion mass. The pion decay constant may be estimated here from the knowledge of PCAC. On the lattice, we shall see that corrections to f_π might not necessarily be strictly estimated via the divergence of an axial-vector current.

Having outlined the behaviour of currents in QCD and shown how their conservation, as a consequence of the global symmetries of the action, produces a correct phenomenological description of nature, we can now move on to an explicit discussion of the points raised above on how lattices may change this picture if symmetries are explicitly broken.

Section 3.2 Currents with Wilson fermions

One of the main guidelines when constructing a lattice version of QCD, besides the implementation of local gauge invariance and the search for a unique confining phase, is to reproduce as close as possible the global symmetries of the action. If on the lattice, more symmetries are present than in the continuum, this will be just as undesirable as if invariances were to be lost in the process of space-time discretization.

When examining the naive lattice action (2.24), we have seen that its main problem is the fermion doubling which makes it of little use for Monte Carlo simulations. Looking from a different viewpoint, the fact that these double fermions cancel out the anomaly, enlarges the symmetry group by picking up a $U_A(1)$ invariance (3.1c), yielding a massless action fully invariant under $U(N) \otimes U(N)$, which is more than the continuum theory, and hence giving very few hopes of a satisfying description of hadrons.

The Wilson action (2.40) which presents a more satisfying discretization, by getting rid of the doubling introduces the loss of a symmetry, and this is not surprising in the light of our discussion in the previous section, where we found that continuous chiral symmetry had to be sacrified to gauge invariance in quantum field theory, and after all, the lattice can always be thought of as a regulator.

The symmetry that is lost is $SU_A(N)$ in (3.1d), which is broken even in the case of zero mass. (2.40) retains of its continuum analogue only the invariance $U_V(N)$. All this can

be visualized if we examine the lattice version of currents and analyse their divergence equation when possible.

In order to preserve the gauge character of lattice theories, we want currents there to be gauge invariant, and also to be local, in the sense that the currents at x should be composed of fields that lie within a finite radius from x.

To obtain currents in analogy with Noether's theorem construction in the continuum, we can start exploiting the symmetry properties of the action, making a point-dependent transformation on the quark fields in flavour space, inside the partition function. We have then a kind of gauge transformation, and the imposition of an invariance of the partition function under these arbitrary changes of variables, by differentiation with respect to its parameters, leads to the divergence equations and from them to the identification of the currents. The study of these symmetry currents has been done by Karsten and Smit (1981).

Specifically, the changes of variables are:

$$\psi(x) \rightarrow e^{i\theta^j(x)\frac{\tau^j}{2}}\psi(x)\ ;\quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{-i\theta^j\frac{\tau^j}{2}}, \qquad (3.22a)$$

and to obtain the axial-vector current:

$$\psi(x) \to e^{i\varphi^j(x)\gamma_5 \frac{\tau^j}{2}} \psi(x)$$

$$\bar{\psi}(x) \to \bar{\psi}(x)\, e^{i\varphi^j(x)\gamma_5 \frac{\tau^j}{2}} \qquad (3.22b)$$

As a first step, the analysis may be done only on the part of the action which is independent of r, leading, after doing the transformation (3.22a), to:

$$\Delta_\mu V_\mu^j(x) = \sum_\mu \left\{ \left[\frac{ir}{2a} \bar{\psi}(x) \frac{\tau^j}{2} U_\mu(x) \psi(x+a_\mu) + h.c. \right] \right.$$

$$\left. - [x \to x - a_\mu] \right\} + \bar{\psi}(x) i \left[\frac{\tau^j}{2}, M \right] \psi(x) \qquad (3.23)$$

where, here,

$$\Delta_\mu f(x) \equiv \frac{1}{a} [f(x) - f(x - a_\mu)] \qquad (3.24)$$

and the vector current of isospin j has been identified as:

$$V_\mu^j(x) = \frac{i}{2} \left[\bar{\psi}(x) \gamma_\mu \frac{\tau^j}{2} U_\mu(x) \psi(x+a_\mu) + \right.$$

$$\left. + \bar{\psi}(x+a_\mu) \gamma_\mu \frac{\tau^j}{2} U_\mu^+(x) \psi(x) \right] \qquad (3.25)$$

Furter, making use of (3.22b) we find the divergence equation:

$$\Delta_\mu A_\mu^j(x) = -\sum_\mu \left\{ \left[\frac{ir}{2a} \bar{\psi}(x) \gamma_5 \frac{\tau^j}{2} U_\mu(x) \psi(x+a_\mu) + h.c. \right] \right.$$

$$\left. + [x \to x - a_\mu] \right\} + \bar{\psi}(x) i \gamma_5 \left\{ \frac{\tau^j}{2}, M \right\} \psi(x) \qquad (3.26)$$

and its corresponding axial-vector current:

$$A_\mu^j(x) = \frac{i}{2} \left[\bar{\psi}(x) \gamma_\mu \gamma_5 \frac{\tau^j}{2} U_\mu(x) \psi(x+a_\mu) + \right.$$

$$\left. + \bar{\psi}(x+a_\mu) \gamma_\mu \gamma_5 \frac{\tau^j}{2} U_\mu^\dagger(x) \psi(x) \right] \qquad (3.27)$$

If we now make the transformation (3.22) on the r-term as well, we obtain different currents. The vector one, that we shall denote by $\hat{V}_\mu^j(x)$, will yield:

$$\Delta_\mu \hat{V}_\mu^j(x) = \bar{\psi}(x) i \left[\frac{\tau^j}{2}, (M - M_c) \right] \psi(x) \qquad (3.28)$$

with the associated current:

$$\hat{V}_\mu^j(x) = \frac{i}{2} \left[\bar{\psi}(x)(\gamma_\mu - r) \frac{\tau^j}{2} U_\mu(x) \psi(x+a_\mu) \right.$$

$$\left. + \bar{\psi}(x+a_\mu)(\gamma_\mu + r) \frac{\tau^j}{2} U_\mu^\dagger(x) \psi(x) \right] \qquad (3.29)$$

Finally, the transformations (3.22b) applied on the whole action produce a current $\hat{A}_\mu^j(x)$:

$$\hat{A}_\mu^j(x) = \frac{i}{2}\left[\bar{\psi}(x)(\gamma_\mu - r)\gamma_5 \frac{\tau^j}{2} U_\mu(x)\psi(x+a_\mu) + h.c.\right] + s\,\bar{\psi}(x+a_\mu)\,i\gamma_5\frac{\tau^j}{2}\psi(x+a_\mu) \qquad (3.30)$$

with a divergence equation:

$$\Delta_\mu \hat{A}_\mu^j(x) = 2M\bar{\psi}(x)\,i\gamma_5\frac{\tau^j}{2}\psi(x) - \sum_\mu\left[\frac{ir}{2a}\bar{\psi}(x-a_\mu)\,i\gamma_5\frac{\tau^j}{2}U_\mu(x)\psi(x) + h.c.\right] + s\sum_\mu\left[\bar{\psi}(x+a_\mu)\,i\gamma_5\frac{\tau^j}{2}\psi(x+a_\mu) - \bar{\psi}(x)\,i\gamma_5\frac{\tau^j}{2}\psi(x)\right] \qquad (3.31)$$

We can now make a few remarks on the above relations. First of all, the divergence equations should really be seen as partition function averages, with zero external sources, but the way they are written is sufficient to display the conservation content of these currents. The existence of a $U_V(N)$ flavour symmetry for Wilson lattice fermions in the case of equal quark masses, as in the continuum, is clearly expressed by (3.28), the right-hand side of which vanishes in the case of flavour

symmetry. We have thus $\hat{V}_\mu^j$ as conserved currents on the lattice. As expected, the behaviour of the axial-vector current has been damaged, and we see that A_μ^j is not conserved, even in the case of zero quark mass, as a result of the generalized mass term introduced in the action. On the other hand, we should expect the axial-vector current to be conserved when the chiral limit $M=M_c$ is taken, when a cancellation of the terms in the right-hand side of (3.26) should take place. It is interesting to remark that there are matrix elements for which different results are obtained depending if we consider flavour singlets or non-singlets. Depending on which matrix element we decide to look at, $< |\Delta_\mu A^j_\mu| >$ might vanish for flavour non-singlets but not for flavour singlets. We know that only the flavour singlets can couple to quark loops, and hence should we calculate $\langle gluons|\Delta_\mu A^j_\mu|0\rangle$, we would find it zero for $j\neq 0$ but non-zero for the singlet $j=0$ (in a correct "anomalous" way). But, if we were to calculate the renormalization factors in weak coupling expansions of non-conserved currents taking one-quark matrix elements $\langle p'|\Delta_\mu A^j_\mu|p\rangle$, it would be zero in the chiral limit for both flavour singlet and non-singlet currents. With such matrix elements at one loop, not only will the renormalization be flavour independent for $j=1,\ldots,N$, but it will be the same for the singlet state $j=0$.

For the current V_μ^j , which as opposed to $\hat{V}_\mu^j$ was not obtained making full use of the action, we also find a non-conserved current, and hence it will suffer also a

finite renormalization, which as in the previous case, will manifest itself by a flavour-independent multiplicative factor that should go to one when the continuum limit (where current conservation is reached) is taken.

The current $\hat{A}_\mu^j$ does not bring more information than A_μ^j, and of course we do not reach a stage of conservation as in the vector case. The s-term had to be introduced by hand in order to have a current with the correct properties in the continuum limit. At the tree-level, s=r, and more generally, s can be adjusted in order that $\hat{A}_\mu^j = A_\mu^j$ in the continuum limit.

We see that these four currents satisfy our assumptions of gauge invariance and locality (in the broad sense, for the currents involve neighbouring sites), and if the continuum limit is taken, all yield the usual Noether currents structure. But as far as the lattice goes, only $\hat{V}_\mu^j$ has a Noether behaviour and should be taken into account. The others are of interest because as the former one, they satisfy known divergence equations and hence Ward identities, but for instance there is no reason why $\hat{A}_\mu^j$ should be considered rather than A_μ^j. In fact, because of the somewhat simpler form of the latter, it is the one that we shall study here.

At this stage we need to keep contact with our initial aim, that of obtaining factors that would give a correction to quantities evaluated directly by computer simulations. And for these, the currents presented above are troublesome for they involve non-local objects. For

that reason , and because the non-conserved point-split currents, despite their symmetry origin, are not particularly good Noether currents, we have decided to examine as well different bilinears, which will lead to the correct continuum currents, but which will be fully local. (Note that we have been abusing the word local - and shall continue doing so! All the currents considered here will be local in the field theory language, but some on the lattice will be point-split, some will not. For these we shall reserve the word local). We then introduce the local vector and axial-vector currents respectively:

$$\tilde{V}_\mu^j(x) = \bar{\psi}(x)\, i\gamma_\mu \frac{\tau^j}{2} \psi(x) \qquad (3.32a)$$

$$\tilde{A}_\mu^j(x) = \bar{\psi}(x)\, i\gamma_\mu\gamma_5 \frac{\tau^j}{2} \psi(x) \qquad (3.32b)$$

The local currents are artificially imposed on the lattice, and the fact that they ignore the nearest neighbour coupling of quark fields in the action explains why it is not an easy task to obtain their divergence equation, and there is even less argument to say that they should be conserved on the lattice, since they are not Noether currents in any sense. Still, and because indeed such local bilinears are used in Monte Carlo simulations, it might be of interest to obtain their renormalization

factors, as well as for V_μ^j and A_μ^j.

Section 3.3 Renormalization of currents and decay constants with Wilson fermions

Knowing that there is a breaking of the current conservation on the lattice, we can move on to the one loop calculations that will produce the renormalization factors. We have already seen why the one quark matrix elements are the more appropriate to examine. Furthermore, having explained in section 3.1 the procedure that may be followed, we can undertake the calculation for generic vector and axial-vector currents, J_μ^{1j}, J_μ^{5j} respectively, in momentum space and at zero momentum transfer.

This matrix element may be written in terms of the vertex function and the residue of the pole in the quark propagator:

$$\langle p | J_\mu^{1,5}(0) | p \rangle = Z_2 \bar{u}(p) \Gamma_\mu^{1,5}(p,p) u(p) \qquad (3.33)$$

and we can define Z_1 factors just as in (3.12):

$$\bar{u}(p) \Gamma_\mu^{J^1}(p,p) u(p) = \frac{i}{Z_1^{J^1}} \bar{u}(p) \gamma_\mu \frac{\tau^j}{2} u(p) \qquad (3.34a)$$

and

$$\bar{u}(p)\,\Gamma_\mu^{J^5}(p,p)\,u(p) = \frac{i}{Z_1^{J^5}}\,\bar{u}(p)\,\gamma_\mu\gamma_5\,\frac{\tau^j}{2}\,u(p) \qquad (3.34b)$$

where form factors have been renormalized to unity at zero momentum transfer and in the flavour symmetry limit.

From (3.33)and (3.34) we obtain :

$$\langle p|\,J_\mu^{1j}(0)\,|p\rangle = K_{J^1}\,\bar{u}(p)\,i\gamma_\mu\,\frac{\tau^j}{2}\,u(p) \qquad (3.35a)$$

and

$$\langle p|\,J_\mu^{5j}(0)\,|p\rangle = K_{J^5}\,\bar{u}(p)\,i\gamma_\mu\gamma_5\,\frac{\tau^j}{2}\,u(p) \qquad (3.35b)$$

where the κ's are the renormalization factors, different for each current but flavour independent for each of them; they are a consequence of the introduction of gauge fields and will be given as a power series in g by:

$$K^{J^{1,5}} = \frac{Z_2}{Z_1^{J^{1,5}}} \qquad (3.36)$$

We know from (3.14) that $\kappa^{\hat{V}}=1$ ($\hat{V}$ being conserved on the lattice) and we shall make use of this fact to calculate the other κ's. Indeed, Z_2 which is given by the

self-energy graph (fig.1) evaluated at non-zero external momentum, needs not be calculated for we can now write (3.36) as:

$$\kappa^{J^{1,5}} = \frac{Z_1^{\hat{V}}}{Z_1^{J^{1,5}}} \qquad (3.37)$$

This substitution is useful to reduce the risks of error as the structure of the vertex for $\hat{V}$ might have some similarities with the other vertices. In particular we shall see that the leading divergence will come with the same coefficient in all cases, and hence its cancellation by the ratio (3.37) will secure finite κ's.

The Feynman rules are obtained by expanding in Fourier space the gauge field as in (2.14) for the different vertices. For the local currents which do not contain gauge fields, we will find the same gauge field independent rules of the continuum. For the non-local currents, there is an infinite number of terms in the expansion, each one picking up a new gluon line, as for the vertices in fig.2, and we shall stop at one loop in our expansion. The Feynman rules are listed in fig.4.

The actual diagrams contributing to the one-quark matrix elements for the currents are listed in fig.5. In view of the Feynman rules, only the diagrams 5.a and 5.b will contribute to the local current expansion, as for the non-local currents, all four diagrams will contribute.

In order to look at a typical weak coupling integral on the lattice, we look first at the diagram 5.b for the conserved $\hat{V}$, as it will be needed to get the overall renormalization factor for all currents, and we should be able to isolate a possible divergence which must be present with the same coefficient for the same diagram with other currents, in order to guarantee a finite renormalization. Using the Feynman rules we obtain (leaving aside the pre-factors):

$$\frac{1}{(2\pi)^4}\int_{-\pi/a}^{\pi/a} d^4k\, a^4 \sum_{\mu\nu\sigma} \left[i\gamma_\rho \cos(k_\rho a + p_\rho a) + r \sin(k_\rho a + p_\rho a)\right] \times$$

$$\times \left[i\gamma_\nu \sin(k_\nu a + p_\nu a) - 2r \sin^2 \frac{1}{2}(k_\nu a + p_\nu a) - ma\right] \times$$

$$\times \left[i\gamma_\sigma \cos\left(\frac{k_\sigma a}{2} + p_\sigma a\right) + r \sin\left(\frac{k_\sigma a}{2} + p_\sigma a\right)\right] \times$$

$$\times \left[i\gamma_\mu \sin(k_\mu a + p_\mu a) - 2r \sin^2 \frac{1}{2}(k_\mu a + p_\mu a) - ma\right] \times$$

$$\times \left[i\gamma_\sigma \cos\left(\frac{k_\sigma a}{2} + p_\sigma a\right) + r \sin\left(\frac{k_\sigma a}{2} + p_\sigma a\right)\right] \times$$

$$\times \left\{ \sum_\alpha \sin^2(k_\alpha a + p_\alpha a) + \left[ma + 2r \sum_\alpha \sin^2 \frac{1}{2}(k_\alpha a + p_\alpha a)\right]^2 \right\}^{-2} \times$$

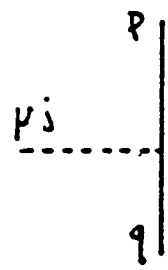
$$\times \left[4 \sum_\alpha \sin^2 \frac{k_\alpha a}{2} \right]^{-1} \qquad (3.38)$$

After rescaling the momenta $k \longrightarrow ka$, we see that we pick up a divergent term which goes as $\ln(m^2a^2)$ and will lead to problems to reach the continuum limit (in other words,

after rescaling (3.38) we find ourselves with a dimensionless integral which could in principle be evaluated numerically, but the integration interval, now from $-\pi$ to π, will cover the origin where the logarithmic divergence would make the integral blow up). Fortunately, but no too surprisingly from out knowledge of the symmetries, the evaluation of the diagram 5.b for all other local and non-local currents will bring a divergent term with the same coefficient as the one arising from (3.38). The ma terms in the numerator may be dropped with respect to leading terms as we are interested in the small quark mass limit, and the renormalization factors being independent of the quark momenta, we can choose the momenta to be zero.

Figure 4

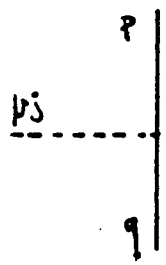
Feynman rules for currents with Wilson fermions



$\tilde{V}, \tilde{A}$

$$-\frac{\tau^j}{2}$$

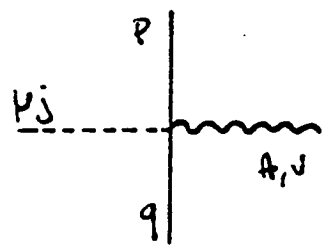
(a)



$V, A, \hat{V}$ ($r=0$ for V, A)

$$-\frac{\tau^j}{2}\left[i\gamma_\mu \cos\left(\frac{p_\mu a + q_\mu a}{2}\right) + r \sin\left(\frac{p_\mu a + q_\mu a}{2}\right)\right](\gamma_5)$$

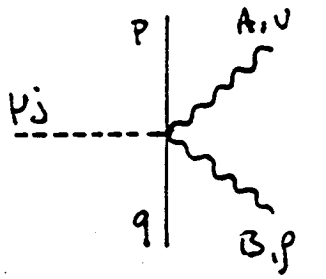
(b)



$V, A, \hat{V}$ ($r=0$ for V, A)

$$-ag\frac{\tau^j}{2}T^A\delta_{\mu\nu}\left[-i\gamma_\mu \sin\left(\frac{p_\mu a + q_\mu a}{2}\right) + r\cos\left(\frac{p_\mu a + q_\mu a}{2}\right)\right](\gamma_5$$

(c)



$V, A, \hat{V}$ ($r=0$ for V, A)

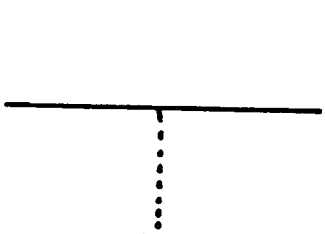
$$a^2 g^2 \frac{\tau^j}{2}\frac{1}{2}\{T^A, T^B\}\delta_{\mu\rho}\delta_{AB} \times$$

$$\times\left[i\gamma_\mu \cos\left(\frac{p_\mu a + q_\mu a}{2}\right) + r\sin\left(\frac{p_\mu a + q_\mu a}{2}\right)\right](\gamma_5)$$

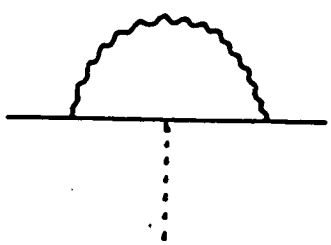
(d)

Figure 5

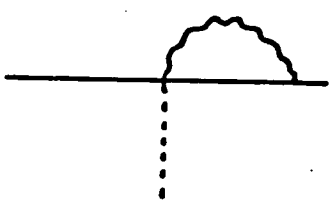
Diagrams which contribute to the matrix elements for currents with Wilson fermions



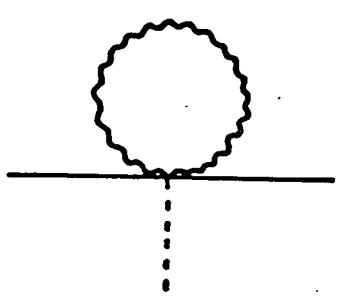
(a)



(b)



(c)



(d)

The integral coming from the diagram 5.d is of similar form as the second integral in (2.44). From fig.5.c we obtain for the non-local, say axial-vector current:

$$\frac{1}{(2\pi)^4}\int_{-\pi/a}^{\pi/a} d^4k\, a^3 \sum_\alpha \Big[-i\gamma_\rho \sin\Big(\frac{k_\rho a}{2}+P_\rho a\Big)+r\cos\Big(\frac{k_\rho a}{2}+P_\rho a\Big)\Big]\gamma_5 \times$$

$$\times\Big[i\gamma_\alpha \sin(k_\alpha a+P_\alpha a-2r\sin^2\tfrac{1}{2}(k_\alpha a+P_\alpha a-ma\Big]\times$$

$$\times\Big[i\gamma_\rho\cos\Big(\frac{k_\rho a}{2}+P_\rho a\Big)+r\sin\Big(\frac{k_\rho a}{2}+P_\rho a\Big)\Big]\times$$

$$\times\Big\{\sum_\beta \sin^2(k_\beta a+P_\beta a)+\Big[ma+2r\sum_\beta \sin^2\Big(\frac{k_\beta a+P_\beta a}{2}\Big)\Big]^2\Big\}^{-1}\times$$

$$\times\Big[4\sum_\beta \sin^2\frac{k_\beta a}{2}\Big]^{-1} \qquad (3.39)$$

We can make the same assumptions about momenta and masses as before; now there are no divergences to worry about and numerical integration can be carried out.

For all these currents, we obtain integrals of the types presented above, and after the delicate γ-algebra, we can obtain integrals that can be evaluated numerically, as the ones of the last chapter. We can note before we start that the diagram of fig.5.d will be present with the same factor for the conserved current and the non-local ones, and hence will be cancelled by the relation (3.37). Its contribution will be explicitly present though for the evaluation of the local current renormalizations.

The results for a gauge group SU(3) with Wilson's parameter r fixed to one (more widely used in computer simulations) were published initially (Meyer and Smith 1983).

We now present the renormalization factors as a function of r in figure 6, where we have plotted the contribution to order g^2 $G_J(r)$ for each current J.

$$K_J = 1 + G_J(r) g^2 + O(g^4) \qquad (3.40)$$

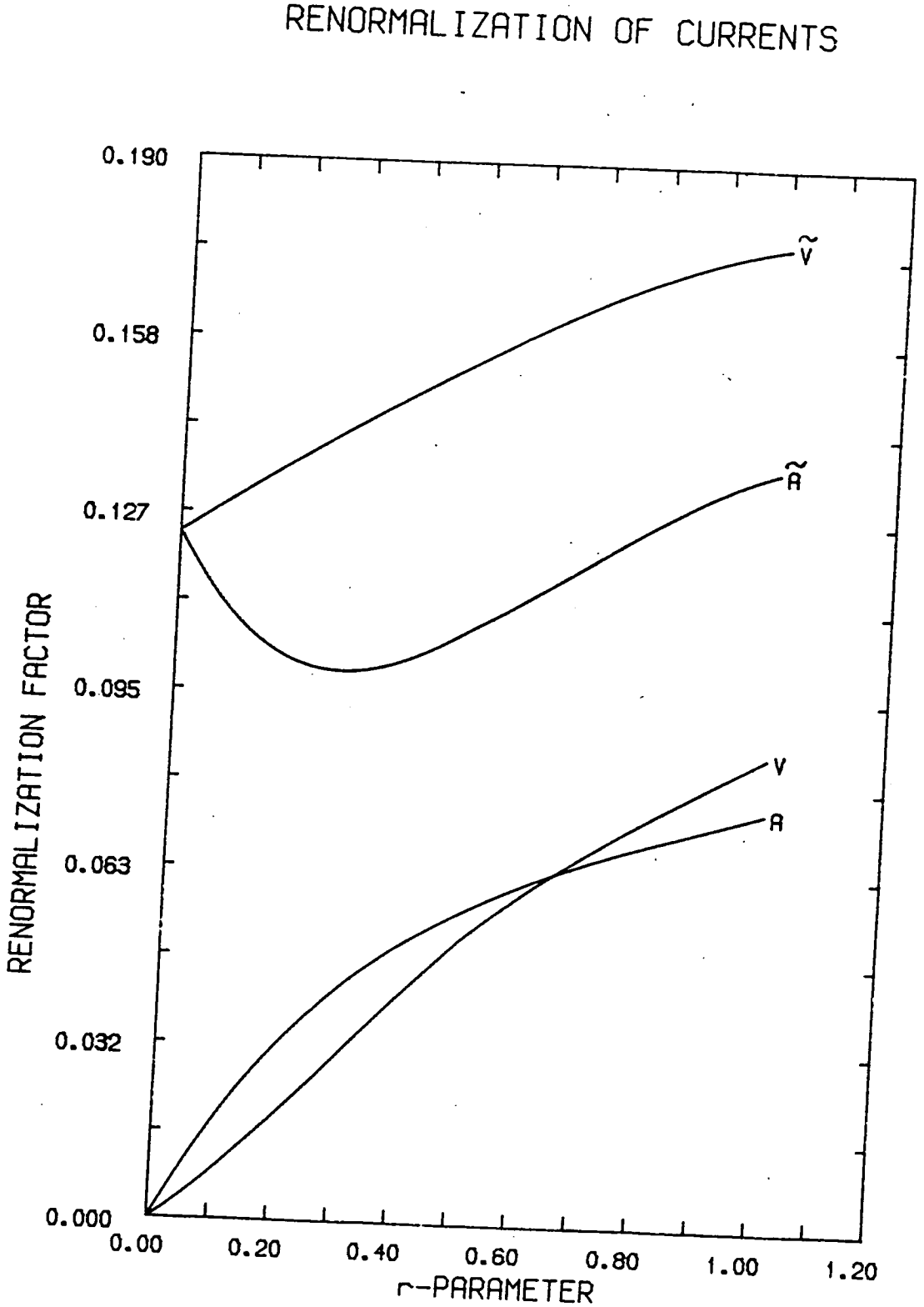
Work by other groups has been performed to calculate these factors (Martinelli and Zhang 1983; Groot, Hoek and Smit 1984). The table below displays the comparison between our results (first column), slightly improved from their published values, the results Martinelli and Zhang in the second column, the third giving the values obtained by Groot **et al.**, all at r=1 for an SU(3) gauge group:

$\tilde{G_V}$	0.176	0.174	0.174081(2)
$\tilde{G_A}$	0.136	0.134	0.133373(2)
G_V	0.085	0.082	0.082629(2)
G_A	0.075	0.073	0.073143(2)

A measure of the error in our numerical evaluation of multiple integrals is given by increasing the number of points sampled by the Monte Carlo routine. These error bars are not large enough to reconcile our results with the ones by the two other groups. However we know from fig.6 that we get the same behaviour as Hoek et al.. The fact that we do not know which kind of routine was used by these other groups makes it difficult to trace the origin of this small discrepancy.

Figure 6

r-dependence of the order g^2 corrections for currents with Wilson fermions and SU(3) gauge group



From fig.6 we see, as expected, that the non-local currents suffer no renormalization as r tends to zero, as they are conserved in the chiral limit. As for the local ones, we could not tell how they behaved, for we do not know their divergence equations, and as they were not constructed from the symmetries of the lattice action, we find, as initially suspected, that they need to be renormalized even at r=0.

We also see confirmed that the correction to local currents is sharply different than that of the non-local ones, and indeed larger, because of the important contribution of the diagram 5.d to them. It can be noted that at this order in the coupling constant, the diagramatic expansion contains no fermion loops, and hence the corrections obtained may be used for quenched as well as unquenched Monte Carlo simulations. Of course corrections to order g^4 would be a different matter, and might actually be a useful thing to do in order to check that the one loop correction is indeed the most important one (we have no guarantee for that), but it is also known that going from one to two loop requires much more than just a couple of diagrams more to evaluate.

Having discussed the renormalization of currents, we shall now move to meson decay constant renormalization, and in a first stage, on how they may be obtained from the above corrections.

We can take the relations (3.6) and (3.9) and assume that they are valid on the lattice, where the currents would now be the lattice ones (local or not) and the decay constants would be lattice decay constants, dependent on the particular category of current that is chosen.

The κ's enabling us to relate lattice currents to the continuum ones, we write the lattice versions of (3.6) and (3.9) making use of the original (3.6) and (3.9) to obtain:

$$\kappa_{\tilde{A}} \langle 0 | A_\mu^{j\,\text{cont.}}(x) | \pi^j(p) \rangle = i \delta_{ij} f_\pi^{\tilde{A}} p_\mu e^{-ip.x} \qquad (3.41a)$$

$$\kappa_{\tilde{V}} \langle 0 | V_\mu^{j\,\text{cont.}}(x) | \rho^j(p) \rangle = i \delta_{ij} m_\rho^2 f_\rho^{\tilde{V}^{-1}} \varepsilon_\mu e^{-ip.x} \qquad (3.41b)$$

and similarly for the non-local currents, which leads directly to a correction to the decay constants from their real continuum values:

$$f_\pi = \kappa_{\tilde{A},A}^{-1} f_\pi^{\tilde{A},A} = \left[1 - G_{\tilde{A},A}(r) g^2 - O(g^4)\right] f_\pi^{\tilde{A},A} \qquad (3.42a)$$

$$f_\rho^{-1} = \kappa_{\tilde{V},V}^{-1} f_\rho^{\tilde{V},V^{-1}} = \left[1 - G_{\tilde{V},V}(r) g^2 - O(g^4)\right] f_\rho^{\tilde{V},V} \qquad (3.42b)$$

A typical Monte Carlo estimate with local vector currents of f_ρ^{-1} was found by Hamber and Parisi (1983) to be 0.5 ± 0.1 at $r=1$ for $a^{-1}=1.5$ Gev. This at $g^2=1$ implies

that:

$$f_\rho^{-1} = 0.83\, \tilde{f}_\rho^{-1} \qquad (3.43)$$

which is not quite enough to obtain the experimental value of 0.19, but at least the correction is going in the right direction. Also the discrepancy between Monte Carlo and experimental values is large and it would have been even worrying that the one loop contribution to a weak coupling expansion might have been enough to take care of it.

As for f_π, the Monte Carlo simulations have not evaluated it by current, but rather by a lattice version of (3.7), where an identification like (3.5b) between divergence of an axial-vector current and a bilinear with pion quantum numbers is assumed to be true on the lattice as well. Strictly speaking we know that the divergence equation on the lattice (3.26) can be visualized as a local term proportional to the renormalized quark mass, like (3.5b), but with the addition of another local term proportional to M_c and one r-dependent non local term. Looking at the local term proportional to the quark mass, we could write a different correction factor for f_π than (3.42a) of the form:

$$\kappa_\pi\, m \langle 0|\bar{\psi}(x)\gamma_5 \frac{\tau^j}{2}\psi(x)|\pi^j(p)\rangle\Big|_{cont.} =$$

$$= m\langle 0|\bar{\psi}(x)\gamma_5\frac{\tau^j}{2}\psi(x)|\pi^j(p)\rangle\Big|_{Latt.} =$$

$$= m_\pi^2 f_\pi^{Latt.} e^{-ip\cdot x} \qquad (3.44)$$

where κ_π would be a renormalization factor for the pion operator $\bar{\psi}(x)\gamma_5 1/2\tau^j\psi(x)$. We would get now instead of (3.42a):

$$f_\pi = \kappa_\pi^{-1} f_\pi^{Latt.} \qquad (3.45)$$

where f_π^{Latt} stands for the pion decay constant on the lattice when it is defined by the local operator in (3.44). We now are left to evaluate the renormalization κ_π.

The analysis done for the other κ's can be repeated, but we shall have to take into account **m** in (3.44) which is renormalized too.

We can take (3.33) as a starting point, with now Γ_μ being a Γ_5 (we are not looking at currents anymore) and define Z^5 by:

$$\bar{u}(p)\Gamma_5(p,p)u(p) = \frac{1}{Z^5}\,\bar{u}(p)\gamma_5\frac{\tau^j}{2}u(p) \qquad (3.46)$$

To renormalize the quark mass, we remember that the idea of the introduction of the bare mass counterterm was that it was to absorb the leading divergence present in the quark self-energy. So that we must look at the quark self-energy, remove the linear divergence, having set the external momentum to zero, to obtain:

$$\frac{m}{(2\pi)^4}\int_{-\pi/a}^{\pi/a} d^4k \sum_\alpha \left[i\gamma_\alpha \cos\frac{k_\alpha a}{2} + r\sin\left(\frac{k_\alpha a}{2}\right)\right] \times$$

$$\times \left[i\gamma_\alpha \cos\frac{k_\alpha a}{2} + r \sin\frac{k_\alpha a}{2} \right] \left[4 \sum_\beta \sin^2 \frac{k_\beta a}{2} \right]^{-1}$$

$$\times \left\{ \sum_\beta \sin^2 k_\beta a + \left[ma + 2r \sum_\beta \sin^2 \frac{k_\beta a}{2} \right]^2 \right\}^{-1} \qquad (3.47)$$

Calling the above contribution $Z_{m'}$ we get for the renormalized mass, considering the two-point vertex function, we obtain:

$$\kappa_m = \frac{1}{Z_2 Z_m} \qquad (3.48)$$

and hence obtain that:

$$m \langle 0 | \bar{\psi}(x) \gamma_5 \frac{\tau^j}{2} \psi(x) | \pi^j(p) \rangle \Big|_{Latt} =$$

$$= \frac{1}{Z_2 Z_m} \, m \langle 0 | \frac{Z_2}{Z_5} \bar{\psi}(x) \gamma_5 \frac{\tau^j}{2} \psi(x) | \pi^j(p) \rangle \Big|_{cont.} \qquad (3.49)$$

which using (3.45) implies that:

$$f_\pi = (Z_m Z_5) f_\pi^{Latt}$$

$$= \left[1 - G_\pi(r) g^2 - O(g^4) \right] f_\pi^{Latt.} \qquad (3.50)$$

We obtain value for G_π that go to zero as r goes to zero. These, together with our renormalization factors for the currents are quoted at the end of this chapter.

f_π on the lattice has been obtained in Monte Carlo simulations using local operators given by (3.44). At r=1, g^2=1, Hamber and Parisi(1983) quote f^{Latt}_π = 150 ± 50 Mev with a^{-1}=1.5 Gev and Bowler et al. (1983) 130 ± 45Mev with a^{-1}=1.4 Gev. As we have f_π = 0.94f_π^{Latt}, we see that our correction, as in the case of f_Q, is not large enough to reach the expermimental value.

We have thus seen examples of how symmetries of the action may be used, in connection with weak coupling expansions, to correct quantities on the lattice that were evaluated by simulations. We have seen how the lattice lead to various possibilities to define continuum equivalents, be it different currents, or ways of approaching the pion decay constant. Many of the operators analysed (the non-local currents and the local pion operator) actually do not need to be renormalized in the continuum limit, but as the computer simulations describe a world not far from the continuum but still at finite g^2, it seems that the renormalization factors are of importance. Their values, assuming that we can trust the conjecture that higher order corrections will contribute less (at least for the flavour non-singlets, which do not couple to the strong chiral symmetry breaker anomaly), will continue being used for new Monte Carlo work, and hopefully, the necessary corrections will help to bring agreement between the lattice estimates and the experimental values.

Table 1

RENORMALIZATION FACTORS OF BILINEARS FOR VARIOUS VALUES OF THE WILSON PARAMETER r

r	G_V	G_A	$G_V^{\sim}$	$G_A^{\sim}$	G_π
1.0	0.085	0.075	0.176	0.136	0.057
0.75	0.070	0.067	0.0168	0.123	0.047
0.5	0.051	0.056	0.155	0.107	0.034
0.25	0.024	0.037	0.140	0.099	0.015
0.2	0.017	0.031	0.136	0.101	0.011
0.1	0.003	0.014	0.127	0.111	0.004
0	0	0	0.123	0.123	0

Chapter 4

SYMMETRIES OF STAGGERED FERMIONS AND RENORMALIZATION FACTORS IN THE FLAVOUR PICTURE

In the last chapter, we have seen the effects of a discretization of QCD on the behaviour of its currents, and how these, when non-conserved, had to be renormalized due to the strength of the colour interaction. All the discussion was done for Wilson fermions for which only $U_V(N)$ is a symmetry of the action.

In this chapter, we shall turn to the same problem but with staggered fermions, for which a remnant of $SU_A(N)$ is still present on the lattice. On the other hand, we shall see that here, because of a breaking in the flavour symmetry, it turns out to be necessary to introduce flavour matrices in the rotation and reflection lattice operators if lattice rotation and reflection invariance are to remain true. In view of these symmetries, together with charge conjugation, a way of classifying bilinears has been given and we shall describe its prescription. A consequence of such will be to limit possible mixings of operators with same quantum numbers via radiative corrections. We shall show how this develops, and in the particular mainframe of block variables ("flavour" formulation), we shall calculate one-loop renormalziations to some operators, with and without mixing.

Section 4.1 Symmetries of staggered fermions

Following Kluberg-Stern et al. (1983), we can start by examining the global symmetries of the massless "staggered" action (2.31) with site variables. The fact that χ's at even (odd) sites can only couple to $\bar{\chi}$'s at odd (even) sites (a site is even (odd) if Σx_μ is even (odd)) gives a global symmetry group:

$$G = U(1) \otimes U(1) \qquad (4.1)$$

with the following transformations for T and V belonging to U(1):

$$\chi \text{ even}: \begin{cases} \chi(x) \rightarrow V \chi(x) \\ \bar{\chi}(x) \rightarrow \bar{\chi}(x) T^\dagger \end{cases}$$

$$\chi \text{ odd}: \begin{cases} \chi(x) \rightarrow T \chi(x) \\ \bar{\chi}(x) \rightarrow \bar{\chi}(x) V^\dagger \end{cases} \qquad (4.2)$$

The mass term which couples χ and $\bar{\chi}$ at the same site

breaks the symmetry (4.1) into U(1) , as we have (4.2) as symmetry transformations for T=V.

As we have discussed in section 2.3.3, although numerical work is more easily performed with site variables, the identification of physical particles and currents as defined in the last chapter becomes clearer using block variables (which from now on we shall call "flavour" formulation). Taking the action (2.37) - introducing the notation of γ's acting in spinor space and T's in flavour space by replacing $(\gamma_5\gamma_\mu)^T$ by iT_μ and T_5 being the only hermitian matrix which anticommutes with all T_μ's (namely $T_1T_2T_3T_4$) - we see that of the invariance $U(N)\otimes U(N)$ of (3.1) we are only left with the following invariances:

$$\psi(x) \to e^{i\theta \mathbb{1}\otimes\mathbb{1}}\psi(x)\ ;\quad \bar{\psi}(x)\to\bar{\psi}(x)e^{-i\theta\mathbb{1}\otimes\mathbb{1}} \qquad (4.3a)$$

$$\psi(x) \to e^{i\theta\gamma_5\otimes T_5}\psi(x)\ ;\quad \bar{\psi}(x)\to\bar{\psi}(x)e^{i\theta\gamma_5\otimes T_5} \qquad (4.3b)$$

The transformation (4.3a) is the vector singlet corresponding to (3.1a), but (4.3b) is not a flavour singlet (T_5 is traceless), and we have then the symmetry:

$$U_V(1)\otimes U_A(1) \qquad (4.4)$$

$U_A(1)$ belongs to $SU_A(N)$ of (3.1d) and is not subject to the flavour singlet anomaly. In fact, just as in (4.1), the mass term is invariant under (4.3a) but not under (4.3b) and (4.4) is broken to $U_V(1)$. We note that this breaking also appears spontaneously, as was shown by dynamical strong coupling calculations (Kluberg-Stern et al. 1981 and 1982, Kawamoto and Smit 1981) and observed in Monte Carlo simulations (Marinari, Parisi and Rebbi 1981, Hamber and Parisi, 1981 and 1983) in the site variable formulation. The axial symmetry which is spontaneously broken being flavour non-singlet, the associated Goldstone boson will be flavour non-singlet like the pion. However, the introduction of gauge fields is not equivalent on the lattice for "staggered" and "flavour" formulations, and there is no pion-like exact Goldstone boson for the latter, as was shown at strong coupling by Napoly (1983).

Except for the soft breaking of the mass term,the transformations (4.3a) and (4.3b) yield a conserved vector and an axial-vector current which are obtained according to the prescription defined in the last chapter. Denoting them by $\hat{V}_\mu$ and $\hat{A}_\mu$ respectively, they are identified as:

$$\hat{V}_\mu(x) = \frac{1}{2}\left[\bar{\psi}(x)(\gamma_\mu \otimes \mathbb{1} - i\gamma_5 \otimes \tau_\mu) U_\mu(x) \psi(x+\hat{\mu}) + h.c.\right] \quad (4.5a)$$

$$\hat{A}_\mu(x) = \frac{1}{2}\left[\bar{\psi}(x)(\gamma_\mu\gamma_5 \otimes \mathbb{1} + i\mathbb{1} \otimes \tau_\mu) U_\mu(x) \psi(x+\hat{\mu}) + h.c.\right] \quad (4.5b)$$

Comparison with the Wilson currents shows that here we have gained a conserved, non-local, axial-vector current. We still have a conserved vector current, but now only in the flavour singlet case (for Wilson fermions we had a $U_V(N)$ symmetry). On top of these conserved currents, we can construct other, non-local, non-conserved currents from the other generators of $U(C) \otimes U(C)$ of the continuum action for C flavours. But we have no particular reason to examine these currents which cumulate the disadvantages of being non-local and not conserved. Instead we shall define, just as in (3.32a) and (3.32b), non-conserved but local operators.

However, before we move on to define these operators, we shall examine a little closer symmetries of the action (2.31) and (2.37).

As was observed by Golterman and Smit (1984), there is a shift invariance of the site variables on the lattice with the small spacing:

$$\chi(x) \rightarrow \varphi_\rho(x)\, \chi(x + a_\rho)$$

$$\bar{\chi}(x) \rightarrow \varphi_\rho(x)\, \bar{\chi}(x + a_\rho) \qquad (4.6)$$

with:

$$\varphi_1(x) = (-1)^{x_2 + x_3 + x_4} \qquad \varphi_2(x) = (-1)^{x_3 + x_4}$$

$$\varphi_3(x) = (-1)^{x_4} \qquad \varphi_4(x) = 1 \qquad (4.7)$$

Sharatchandra, Thun and Weisz (1981) have shown that the transformations like (4.6) imply the absence of mass counterterms. On the bigger lattice of the "flavour" picture, the shift invariance is not obviously defined, moreover, the necessity of mass counterterms in this formulation (Mitra and Weisz 1983) is an indication that shift invariance is broken when gauge fields are introduced.

A 90° rotation R is described by:

$$x \to R^{-1} x \quad ; \quad (R^{-1}x)_\mu = x_\nu$$

$$(R^{-1}x)_\nu = -x_\mu \quad ; \quad (R^{-1}x)_\sigma = x_\sigma \text{ for } \sigma \neq \mu,\nu \qquad (4.8)$$

Rotational invariance at the site level is expressed as (and we shall from now on only consider the fermion fields, the transformations on gauge matrices do not change our following discussion):

$$\chi(x) \to S_R(R^{-1}x)\,\chi(R^{-1}x)$$

$$\bar{\chi}(x) \to S_R(R^{-1}x)\,\bar{\chi}(R^{-1}x) \qquad (4.9)$$

where $S_R(x)$ the phase factor which appears when relating naïve to staggered fermions (Kawamoto and Smit 1981) is defined by:

$$S_R(x) = \frac{1}{2}\left[1 + \eta_\mu(x)\eta_\nu(x) - \xi_\mu(x)\xi_\nu(x) + \right.$$

$$\left. + \eta_\mu(x)\eta_\nu(x)\xi_\mu(x)\xi_\nu(x) \right] \qquad (4.10)$$

and the definition of η was given in eq.(2.30).

If we now want in the flavour formulation rotation transformations in straight analogy with the transformations of continuum spinors, we should have:

$$\psi(y) \rightarrow \left(\frac{\mathbb{1} - \gamma_\mu\gamma_\nu}{\sqrt{2}}\right)\psi(R^{-1}y)$$

$$\bar{\psi}(y) \rightarrow \bar{\psi}(R^{-1}y)\left(\frac{\mathbb{1} + \gamma_\mu\gamma_\nu}{\sqrt{2}}\right) \qquad (4.11)$$

Applying (4.11) to the "flavour" action (2.37), we observe that if its first term as well as the mass term are left invariant, the second term is not. The only way of modifying (4.11) in order to get a full action rotational invariant is to introduce flavour matrices in the rotation

transformations (Mitra and Weisz 1983).

The transformations read:

$$\psi(y) \to \left(\frac{\mathbb{1} - \gamma_\mu \gamma_\nu}{\sqrt{2}}\right)\left(\frac{T_\mu - T_\nu}{\sqrt{2}}\right) T_5 \, \psi(R^{-1}y)$$

$$\bar{\psi}(y) \to \bar{\psi}(R^{-1}y) \, T_5 \left(\frac{T_\mu - T_\nu}{\sqrt{2}}\right)\left(\frac{\mathbb{1} + \gamma_\mu \gamma_\nu}{\sqrt{2}}\right) \qquad (4.12)$$

This gives a striking difference with either the continuum or the lattice Wilson formulation where space-time rotations, to preserve an invariant theory, imply flavour independent transformations. It was also observed by Mitra and Weisz that the most general mass term that stays invariant under (4.12) is:

$$\bar{\psi}(y)\left[m \, \mathbb{1} \otimes \mathbb{1} + \frac{m'}{2} \, \mathbb{1} \otimes \left(\sum_\mu T_\mu\right)\right] \psi(y) \qquad (4.13)$$

Axis reversal I_μ is defined by:

$$x_\mu \to -x_\mu \quad ; \quad x_\sigma \to x_\sigma \text{ for } \sigma \neq \mu \qquad (4.14)$$

and the action (2.31) is invariant at the site level under the reflection operation:

$$\chi(x) \rightarrow (-1)^{x_\mu}\, \chi(I_\mu x)$$
$$\bar{\chi}(x) \rightarrow (-1)^{x_\mu}\, \bar{\chi}(I_\mu x) \qquad (4.15)$$

Axis reversal symmetry in the flavour picture also requires the introduction of flavour matrices through T_5:

$$\psi(y) \rightarrow \gamma_\mu \gamma_5 T_5 \psi(I_\mu y)$$
$$\bar{\psi}(y) \rightarrow \bar{\psi}(I_\mu y) T_5 \gamma_5 \gamma_\mu \qquad (4.16)$$

The massless version of (2.31) is invariant under the interchange:

$$\chi(x) \rightarrow \bar{\chi}(x) \quad , \quad \bar{\chi}(x) \rightarrow \chi(x) \qquad (4.17)$$

Golterman and Smit (1984) show the modifications that should be brought into (4.17) to have an invariance with different types of mass terms in the action. For (2.31) we see that we need to introduce a phase:

$$\chi(x) \rightarrow \exp\left[i \frac{\pi}{2} (-1)^{x_1+x_2+x_3+x_4} \right] \bar{\chi}(x)$$
$$\bar{\chi}(x) \rightarrow \exp\left[i \frac{\pi}{2} (-1)^{x_1+x_2+x_3+x_4} \right] \chi(x) \qquad (4.18)$$

The same authors point out the correspondence of the interchange (4.18) with charge conjugation. We shall see the importance of this symmetry in the flavour formulation in the next section, in the context of bilinears classification.

Section 4.2 Classification of bilinears

By examining the symmetries of staggered fermions we have seen that we are in presence of problems non-existent in the Wilson case, in particular the necessity of a flavour dependence of rotation and reflection invariant transformations.

We have seen how to define bilinears from the symmetries of the action, and were led to the conserved non-local currents (4.5a) and (4.5b). We are working in a four-dimensional space with four flavours (we know from the discussion of chapter 2 that in four dimensions staggered fermions bring four, or a multiple of four flavours), and hence we could construct the other non-local non-conserved currents from the generators of the continuum symmetry $U(4) \otimes U(4)$. Instead, we shall concentrate - backed by our discussion in the last chapter on the freedom of choice of currents once the currents treated are not Noether ones - on the local bilinears constructed from $U(4) \otimes U(4)$.

A possible choice for the generators of U(4) is:

$$\mathbb{1};\ \gamma_\mu;\ \gamma_5;\ \gamma_\mu\gamma_5;\ \frac{1}{2}[\gamma_\mu,\gamma_\nu] \quad \text{for the spinor group} \qquad (4.19a)$$

$$\mathbb{1};\ T_\mu;\ T_5;\ T_\mu T_5;\ \frac{1}{2}[T_\mu,T_\nu] \quad \text{for the flavour group} \qquad (4.19b)$$

We are then left with 256 ($=16^2$) bilinears of the form:

$$\bar{\psi}(x)\ \gamma T\ \psi(x) \qquad (4.20)$$

γ and T being a short hand notation for any of the generators (4.19). None of them is expected to be conserved of course, U(4) ⊗ U(4) not being a symmetry of the lattice action (we have seen that only two non-local currents are conserved from the remnant U(1) ⊗ U(1) on the lattice of the larger continuum group). We are then interested in obtaining their renormalization factors. However, before rushing into the Feynman diagrams, we were interested in taking advantage of the symmetries discussed above in order to simplify the the amount of calculation required to renormalize this legion of bilinears.

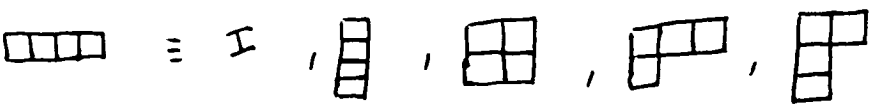
This was not found necessary in our discussion of Wilson currents as there was no mixing of flavour matrices in the lattice rotation group; we had then the same renormalization factors for all the components of each current, the lattice rotation group and the flavour group

transforming one into another. In practice, as long as we knew of the "dangers" of the axial-vector singlet, we could leave the flavour matrices "out" of the actual evaluation of integrals. Now, we face a more subtle situation, having flavour and spinor spaces interwoven.

In this classifying task, the knowledge of the structure and representations of the rotation reflection group of the four-dimensional cubic lattice would be a very useful tool to simplify matters. Such description has actually been recently performed (Mandula, Zweig and Govaerts 1983). Knowing of the existence of this work, Verstegen (1984) was led eventually to an elegant classification of the fermionic bilinears for staggered fermions. We shall sketch the prescription of his work, which showed that bilinears fall in different representations of the hypercubic rotation reflection group, and also that the same representation could recur, implying the mixing of operators with same quantum numbers through radiative corrections.

As described by Mandula et al, the hypercubic group is generated by 90° rotations in each of the six lattice planes. It has 192 elements comprising 13 conjugacy classes. Each element of the group may be described in three different ways: as an element of the four-dimensional rotation group, as an element of $SU(2) \otimes SU(2)/Z_2$ or as a product of a permutation of lattice axes with reflections along lattice axes.

We can briefly describe the 13 irreducible representations (as there are 13 conjugacy classes). The four element permutation group is a subgroup of the hypercubic one and provides five (non-faithful) representations, denoted in terms of Young tableaux by:



(4.21)

of dimensions 1,1,2,3 and 3 respectively.

Four non-trivial representations of O(4) remain irreducible under the hypercubical subgroup:

$$(1,0)\,;\ (0,1)\,;\ \left(\tfrac{1}{2},\tfrac{1}{2}\right);\ \left(\tfrac{1}{2},\tfrac{3}{2}\right) \qquad (4.22)$$

with dimensions 3,3,4 and 8 respectively, the last one being also denoted by $\underset{\sim}{8}$ after its dimension. (In 4.22 the notation of the SU(2) ⊗ SU(2) realization of O(4) has been used). The direct product of the first three (4.22) with gives:

$$\overline{(1,0)}\,;\quad \overline{(0,1)}\,;\quad \overline{\left(\tfrac{1}{2},\tfrac{1}{2}\right)} \qquad (4.23)$$

with dimensions 3,3 and 4 as before. There is a final six-dimensional representation denoted by $\underset{\sim}{6}$.

Under a 90° rotation, using (4.12), a bilinear $\overline{\psi}(0)\gamma T\psi(0)$ goes to:

$$\overline{\mathcal{T}}(0)\left(\frac{\mathbb{1}+\gamma_\mu\gamma_\nu}{\sqrt{2}}\right)\gamma\left(\frac{\mathbb{1}-\gamma_\mu\gamma_\nu}{\sqrt{2}}\right)\times$$

$$\times\ T_5\left(\frac{T_\mu - T_\nu}{\sqrt{2}}\right) T\left(\frac{T_\mu - T_\nu}{\sqrt{2}}\right) T_5\, \psi(0) \qquad (4.24)$$

This can be seen as a product of two representations of the hypercubic group, one acting in spinor space, the other in flavour space. Applying (2.34a) we can write immediately the bilinears in terms of site variables. Nevertheless, because of the clearer form of the flavour picture, the classification of the bilinears will be carried out in term of the ψ's.

We shall not here give the detailed procedure to identify all the bilinears and their representations, this can be found in the work by Verstegen. Just to illustrate the prescription, let us only discuss a few typical cases.

Just using the anticommutation of the matrices involved in the structure (4.24) we see that $\mathbb{1}$ and γ_5 belong to the identity representation of the hypercubic group. However, T_5 changes sign under each 90° rotation, i.e. under each interchange of two axes, and transforms according to the alternate representation. Now take γ_α, under $R_{\mu\nu}(\pi/2)$ it transforms as:

$$\gamma_\mu \to -\gamma_\nu \qquad \gamma_\nu \to \gamma_\mu$$

$$\gamma_\alpha \to \gamma_\alpha \quad (\alpha \neq \mu, \nu) \qquad (4.25)$$

So the matrix asssociated to $R_{12}(\pi/2)$ in this representation is:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is of trace 2. The use of the tables of conjugacy classes and characters of the group given by Mandula et al. enables one to identify the corresponding representation, which in this particular case is $(\frac{1}{2},\frac{1}{2})$. Some cases may lead to reducible representations, which may be reduced using the character (or orthogonality) formula which gives the number of times a given irreducible representation r occurs in a given representation with character χ:

$$a_r = \frac{1}{N} \sum_{\text{all classes } C} n_c \chi(c) \chi^r(C) \qquad (4.26)$$

where N is the number of elements in the group, n_c the number of elements in the class c. Table 2 presents the identification for all the γT.

Further simplification can be obtained for the picture obtained so far bringing in the other symmetries of the action. The extension of the hypercubic group to the group of lattice rotations and reflections was also studied by Mandula et al., and from that it is possible to classify the representations of the rotation reflection group in terms

of the representations of the hypercubic group. Consequently the γ's in (4.19a) may be attributed to parity plus or minus representations, and the T's in (4.19b) to "positive" and "negative" representations, by studying their behaviour under inversion of one axis.

Another sign may be attributed to the representations using, as was done by Verstegen, the charge conjugation property (4.18) transformed in the flavour representation by (2.35). Using the properties of the matrices and remembering the Grassmannian nature of the χ's, he showed that:

$$\bar{\psi}(y)\,\gamma T\,\psi(z) \longrightarrow \bar{\psi}(z)\,\gamma' T'\,\psi(y) \qquad (4.27)$$

where

$$\begin{cases} \gamma' = +\gamma & \text{if} \quad \gamma = \mathbb{1},\ \gamma_5,\ \gamma_\mu\gamma_5 \qquad (4.28a) \\ \gamma' = -\gamma & \text{if} \quad \gamma = \gamma_\mu,\ \gamma_\mu\gamma_\nu \qquad (4.28b) \end{cases}$$

$$\begin{cases} T' = +T & \text{if} \quad T = \mathbb{1},\ T_\mu,\ T_5 \qquad (4.28c) \\ T' = -T & \text{if} \quad T = T_\mu T_5,\ T_\mu T_\nu \qquad (4.28d) \end{cases}$$

Finally, a further distinction between quantities belonging to identical representations was done by taking into account the axial flavour non-singlet transformation (4.3b) which is a symmetry of the theory in the massless limit. The U(1) symmetry can rightly be taken into account for quantities that renormalize finitely, as the **ma** dependence should disappear when **ma** goes to zero. Under (4.3b) bilinears with γT anticommuting with $\gamma_5 \otimes T_5$ are left

unchanged, as those which commute with $\gamma_5 \otimes T_5$ become:

$$\bar{\psi}\,\gamma T \psi \;\longrightarrow\; \bar{\psi}\, e^{2i\theta\gamma_5 T_5}\,\gamma T \psi \qquad (4.29)$$

Thus, no mixing is allowed between components of bilinears with same "J^{PC}" quantum numbers if they are not also related by $U_A(1)$. On the other hand, there will be an interchange between $\gamma_\alpha T_\mu$ and $\gamma_\alpha\gamma_5 T_\mu T_5$ etc. (applying (4.29)) for which the same renormalization constants should be expected. The classification obtained by Verstegen is given in Table 3 without $\gamma_\alpha\gamma_\beta T$ of less physical interest.

Note that the axial-vector flavour singlet current $\gamma_\alpha\gamma_5 \otimes \mathbb{1}$ is the only $\left(\frac{1}{2},\frac{1}{2}\right)^{-+}$ and hence there is no risk of unwanted mixing between quantities that renormalize finetely and others that require infinite corrections.

Finally, note that in the cases of $\gamma_\alpha T_\mu$ and $\gamma_\alpha\gamma_5 T_\mu$ the same representation recurs twice (and only twice), leaving ground for a possible mixing when renormalization factors are calculated, as we shall see explicitly in the next section (the same being of course true for their $U_A(1)$ partners). In the next section we show how to obtain these renormalization factors.

Table 2

CLASSIFICATION OF BILINEARS WITH STAGGERED FERMIONS, FLAVOUR FORMULATION IN REPRESENTATIONS OF THE HYPERCUBIC ROTATION REFLECTION GROUP (VERSTEGEN (1984))

TABLE 2

Reduction of γT products

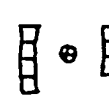
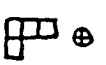
	1 or γ_5	γ_α or $\gamma_\alpha\gamma_5$	$\gamma_\alpha\gamma_\beta$
1	I	$(\frac{1}{2}\,\frac{1}{2})$	$\underline{6}$
T_5		$\overline{(\frac{1}{2}\,\frac{1}{2})}$	$\underline{6}$
T_μ	I $\oplus$	$(\frac{1}{2}\,\frac{1}{2})\oplus(\frac{1}{2}\,\frac{1}{2})\oplus\underline{8}$	$(1,0)\oplus(0,1)\oplus\overline{(1,0)}\oplus\overline{(0,1)}\oplus\underline{6}\oplus\underline{6}$
$T_\mu T_5$		$\overline{(\frac{1}{2}\,\frac{1}{2})}\oplus\overline{(\frac{1}{2}\,\frac{1}{2})}\oplus\underline{8}$	$(1,0)\oplus(0,1)\oplus\overline{(1,0)}\oplus\overline{(0,1)}\oplus\underline{6}\oplus\underline{6}$
$T_\mu T_\nu$		$(\frac{1}{2}\,\frac{1}{2})\oplus\overline{(\frac{1}{2}\,\frac{1}{2})}\oplus\underline{8}\oplus\underline{8}$	$(1,0)\oplus(0,1)\oplus\overline{(1,0)}\oplus\overline{(0,1)}\oplus\underline{6}$ $\oplus(1,0)\oplus(0,1)\oplus\overline{(1,0)}\oplus\overline{(0,1)}\oplus\underline{6}$

Table 3

REFINEMENT OF TABLE 2 WITH PARITY, CHARGE CONJUGATION AND $U_A(1)$ SYMMETRY (VERSTEGEN (1984))

TABLE 3a

Parity and Charge Conjugation Properties

	γ_α	$\gamma_\alpha\gamma_5$
$\mathbb{1}$	$\left(\frac{1}{2},\frac{1}{2}\right)^{+-}$	$\left(\frac{1}{2},\frac{1}{2}\right)^{-+}$
T_5	$\overline{\left(\frac{1}{2}\,\frac{1}{2}\right)}^{+-}$	$\overline{\left(\frac{1}{2}\,\frac{1}{2}\right)}^{-+}$
$T_\mu T_\nu$	$\left(\frac{1}{2},\frac{1}{2}\right)^{++} \oplus \overline{\left(\frac{1}{2},\frac{1}{2}\right)}^{++} \oplus \underset{\sim}{8}^{++} \oplus \underset{\sim}{8}^{++}$	$\left(\frac{1}{2},\frac{1}{2}\right)^{--} \oplus \overline{\left(\frac{1}{2}\,\frac{1}{2}\right)}^{--} \oplus \underset{\sim}{8}^{--} \oplus \underset{\sim}{8}^{--}$

TABLE 3b

Parity and Charge Conjugation Properties

	$\mathbb{1}$	γ_5
T_μ	$I^{-+} \oplus$	$I^{++} \oplus$
$T_\mu T_5$		

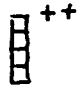
TABLE 3c

Parity and Charge Conjugation Properties of $U(1)_A$ pairs

$\gamma_\alpha\, T_\mu : \left(\frac{1}{2},\frac{1}{2}\right)^{--} \oplus \left(\frac{1}{2},\frac{1}{2}\right)^{--} \oplus \underset{\sim}{8}^{--} \leftrightarrow \gamma_\alpha\gamma_5\, T_\mu T_5 : \overline{\left(\frac{1}{2},\frac{1}{2}\right)}^{+-} \oplus \overline{\left(\frac{1}{2},\frac{1}{2}\right)}^{+-} \oplus \underset{\sim}{8}^{+-}$

$\gamma_\alpha\gamma_5\, T_\mu : \left(\frac{1}{2},\frac{1}{2}\right)^{++} \oplus \left(\frac{1}{2},\frac{1}{2}\right)^{++} \oplus \underset{\sim}{8}^{++} \leftrightarrow \gamma_\alpha\, T_\mu T_5\ \overline{\left(\frac{1}{2},\frac{1}{2}\right)}^{-+} \oplus \overline{\left(\frac{1}{2},\frac{1}{2}\right)}^{-+} \oplus \underset{\sim}{8}^{-+}$

$\mathbb{1}.\mathbb{1} : I^{++} \leftrightarrow \gamma_5\, T_5 :$

$\mathbb{1}.\, T_5 :$  $\leftrightarrow \gamma_5.\mathbb{1} : I^{-+}$

$\mathbb{1}.\, T_\mu T_\nu :$ $\leftrightarrow \gamma_5\, T_\mu T_\nu :$

Section 4.3 Renormalization of fermionic bilinears in the flavour formulation

In the previous chapter, we had defined six currents (although we considered only five in renormalization purposes), the renormalization of each of them being the same irrespect of the flavour (and at one-loop, the flavour singlet-non-singlet splitting is not present).

The analysis of bilinears for staggered fermions in the flavour picture, where the analogy with continuum ones is better established, has proved to produce operators that will **a priori** renormalize differently for different flavours due to the explicit flavour matrix presence in the action (2.37). The discussion of the previous section shows that it is possible to reduce the amount of required computation, as the bilinears are classified in groups for which all their components have the same renormalization, and possible mixings can be constrained, from the knowledge of the symmetries of the theory.

It should be noted from table 3 that the presence of a γ_5 is not enough to ensure that we are dealing with axial quantities. Rather, observe that $\mathbb{1} \otimes T_\mu T_5$ has negative parity and $\gamma_5 \otimes T_\mu T_5$ positive parity.

As for the second sign attributed by Verstegen to the representations in table 3, related to charge conjugation, we may conclude by inspection of table 3 that it is really **minus** charge conjugation. Indeed, taking the negative parity states, we may identify for spin-0 mesons $\cdot$ 6 positive and

10 negative states under charge conjugation, when nature presents instead 10 positive ones ($(\pi^+ + \pi^-)$, $(K^+ + K^-)$, $(F^+ + F^-)$, $(D^+ + D^-)$, π^o, η, η', η_c, D^o, K^o) and six negative ones ($(\pi^+ - \pi^-)$, $(K^+ - K^-)$, $(F^+ - F^-)$, $(D^+ - D^-)$, $\bar{D}^o$, $\bar{K}^o$). This discussion shows that we are keeping contact with the real world, even though we can only do a total counting of states; we can say nothing on which bilinear corresponds to which precise physical state.

The actual calculation of the renormalization factors for the bilinears is done in a similar fashion to the Wilson treatment of the last chapter. For the currents, the conservation of the non-local $\hat{V}_\mu$ comes to our rescue, just as in the Wilson case (remember that there too a non-local conserved current helped to renormalize local ones).

As in (3.35) we may write:

$$\langle p|\bar{\psi}(x)\gamma T \psi(x)|p\rangle = K_{J_\mu}\, \bar{u}(p);\gamma_\mu(\gamma_5)Tu(p) \qquad (4.29)$$

and identifying the residue of the pole in the quark propagator with $Z_1^{\hat{V}}$, we have:

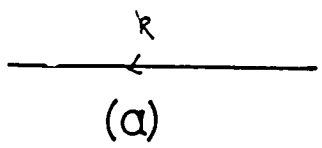
$$K_{J_\mu} = \frac{Z_1^{\hat{V}}}{Z_1^{J_\mu}} \qquad (4.30)$$

The Feynman rules for staggered fermions and the associated currents are given in figure 7 and 8 respectively. All the diagrams of fig.5 contribute to $\hat{V}$, as only fig.5a and 5.b contribute to the local operators.

Figure 7

Feynman rules for staggered fermions

Flavour picture

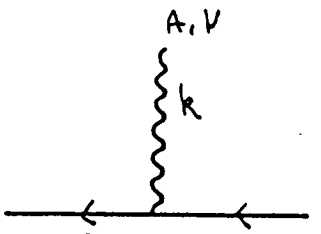


(a)

$$\frac{a}{\sum_\mu -i\gamma_\mu \otimes \mathbb{1}\, \sin k_\mu a + ma\, \mathbb{1}\otimes\mathbb{1} + i\gamma_5 \otimes T_\mu (1-\cos k_\mu)}$$

(b)

$$\frac{a^2 \delta_{AB}\delta_{\mu\nu}}{4\sum_\mu \sin^2\left(\frac{k_\mu a}{2}\right)}$$



(c)

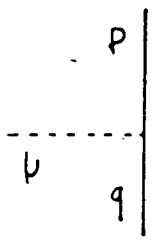
$$-gT^a\left[\gamma_\mu \otimes \mathbb{1}\cos\left(q_\mu a + \frac{k_\mu a}{2}\right) + \right.$$

$$\left. + \gamma_5 \otimes T_\mu \sin\left(q_\mu a + \frac{k_\mu a}{2}\right)\right]$$

Figure 8

Feynman rules for staggered fermions

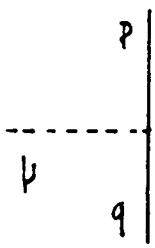
Local γT, non-local conserved V



γT

γT

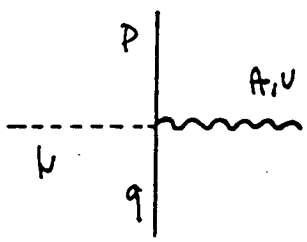
(a)



$$\hat{V}$$

$$\gamma_\mu \otimes \mathbb{1} \cos\left(\frac{p_\mu a + q_\mu a}{2}\right) + \gamma_5 \otimes T_\mu \sin\left(\frac{p_\mu a + q_\mu a}{2}\right)$$

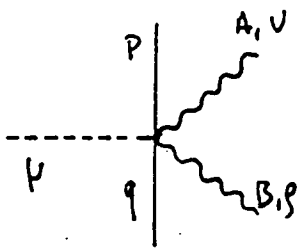
(b)



$$\hat{V}$$

$$a g T^A \left[-\gamma_\mu \otimes \mathbb{1} \sin\left(\frac{p_\mu a + q_\mu a}{2}\right) + \gamma_5 \otimes T_\mu \cos\left(\frac{p_\mu a + q_\mu a}{2}\right) \right]$$

(c)



$$\hat{V}$$

$$a^2 g^2 \frac{1}{2} \{T^A, T^B\} \times$$

$$\times \left[\gamma_\mu \otimes \mathbb{1} \cos\left(\frac{p_\mu a + q_\mu a}{2}\right) + \gamma_5 \otimes T_\mu \sin\left(\frac{p_\mu a + q_\mu a}{2}\right) \right]$$

(d)

Once again a divergence arises from fig.5.b, and eq.4.30 guarantees its cancellation and hence finite corrections to the currents (except the axial flavour non-singlet). These correction are of the form:

$$K_{J_\mu} = 1 + G_{J_\mu} g^2 + O(g^4) \qquad (4.31)$$

For the "pion-like" bilinears (by that we mean operators which look like the $\bar{\psi}(x)\gamma_5\tau^j/2\psi(x)$ of the last chapter, but not necessarily being pion operators), we shall use the fact that the axial-current (4.5b) is partially conserved. Its divergence reads:

$$\Delta_\mu \hat{A}_\mu(x) = 2am\, \bar{\Psi}(x)\, \gamma_5 \otimes T_5\, \Psi(x) \qquad (4.32)$$

and as $\hat{A}_\mu$ suffers no renormalization, the same applies for the right-hand of (4.32). Writing:

$$\langle P|\, \bar{\Psi}(x)\gamma_\mu\gamma_5 \otimes T_5\, \Psi(x)\, |P\rangle =$$

$$= \frac{Z_2}{Z_c}\, \bar{u}(p)\gamma_\mu\gamma_5 \otimes T_5\, u(p)\, e^{-ip\cdot x} \qquad (4.33)$$

we know that $Z_2/Z_c=1$ and using (4.32) we obtain that:

$$\Delta_\mu \langle P|\, \bar{\Psi}(x)\gamma_\mu\gamma_5 \otimes T_5\, \Psi(x)\, |P\rangle =$$

$$= \langle P|\, 2am\, \bar{\Psi}(x)\gamma_5 \otimes T_5 \Psi(x)\, |P\rangle$$

This leads to:

$$-i\bar{u}(p)\not{p}\,\gamma_5\otimes T_5\,u(p)e^{-ip.x} = \langle p|2am\,\bar{\Psi}(0)\gamma_5\otimes T_5\,\Psi(0)|p\rangle e^{-ip.x}$$

$$= 2am\frac{Z_2}{Z_5}\,\bar{u}(p)\gamma_5 T_5\,u(p)\,e^{-ip.x}$$

Introducing the mass renormalization Z_m we write:

$$-i\frac{Z_2}{Z_c}\,\bar{u}(p)\gamma_5 T_5\,u(p) = 2a\frac{1}{2m}m\frac{Z_2}{Z_5}\,\bar{u}(p)\gamma_5 T_5\,u(p)$$

which for on mass-shell quarks yields:

$$\frac{1}{Z_c} = \frac{1}{2mZ_5}$$

Let us call K_J the renormalization to a "pion-like" operator. If we call Z_5' the factor that arises from fig.5.a and 5.b for an operator $\gamma_5 T$ (for any "T"), Z_5 being the contribution of $\gamma_5 T_5$ for the same graphs, we see using the above relations that:

$$m\langle p|\bar{\Psi}(x)\gamma_5\otimes T\,\Psi(x)|p\rangle =$$

$$= \frac{Z_5}{Z_5'}\,m\,\bar{u}(p)\gamma_5 T\,u(p) \qquad (4.34)$$

We have then $K_J=Z_5/Z_5'$, and we have again the familiar cancellation of the triangle of fig.5.b in $\gamma_5 T_5$ and any other

$\gamma_5 T$. We obtain thus one-loop corrections of the form:

$$K_J = 1 - G_J g^2 - O(g^4) \qquad (4.35)$$

First let us examine the corrections to some of the above operators. Making for the integrals similar assumptions and rescaling to the ones of chapter 3, we get for the one-loop contribution to $\gamma_5 T_5$:

$$\sum_{\mu,\alpha,\beta} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left[\gamma_\mu \cos\frac{k_\mu}{2} + \gamma_5 T_\mu \sin\frac{k_\mu}{2} \right] \times$$

$$\times \left[-i\gamma_\alpha \sin k_\alpha - i\gamma_5 T_\alpha (1-\cos k_\alpha) \right] \times$$

$$\times \gamma_5 T_5 \times \left[-i\gamma_\beta \sin k_\beta - i\gamma_5 T_\beta (1-\cos k_\beta) \right] \times$$

$$\times \left[\gamma_\mu \cos\frac{k_\mu}{2} + \gamma_5 T_\mu \sin\frac{k_\mu}{2} \right] \times$$

$$\times \left[4\sum_\sigma \sin^2\frac{k_\sigma}{2} \right]^{-1} \times \left[4\sum_\rho \sin^2\frac{k_\rho}{2} \right]^{-2} \qquad (4.36)$$

which can be easily shown to be :

$$-4\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ \left[4\sum_\alpha \sin^2\frac{k_\alpha}{2} \right] \times \right.$$

$$\left. \times \left[4\sum_\beta \sin^2\frac{k_\beta}{2} \right] \right\}^{-1} \times \gamma_5 T_5 \qquad (4.37)$$

Note that as expected (4.37) is divergent.

Let us take the operator in the representation I^{++} of $\gamma_5 T_\mu$. Before doing explicit calculations, we need the precise form of the fermionic bilinears in each invariant subspace, i.e. bases for these spaces. Bases may be found in Verstegen (1984) for all the representations that we may need, we shall use these and refer the reader to his work for other bases, if more renormalization factors are required.

For the above case, we have $\gamma_5(T_1+T_2+T_3+T_4)\equiv\gamma_5 T$ which replaces $\gamma_5 T_5$ in (4.36). The evaluation of the integral is now more complicated, and careful analysis of the γ and T algebra must be done before arriving at the result:

$$-4\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\left\{\frac{1}{\left[4\sum_\alpha \sin^2\frac{k_\alpha}{2}\right]\left[4\sum_\beta \sin^2\frac{k_\beta}{2}\right]}\right.$$

$$\left.-\frac{1}{64\sum_\mu \sin^2 k_\mu}\right\}\gamma_5 T \qquad (4.38)$$

The first term is divergent with same coefficient as (4.37). The second gives the correction and we get our fist result, for $\gamma_5 T$:

$$K_3 = 1 - 0.013\, g^2 - O(g^4) \qquad (4.39)$$

The procedure is repeated for the bilinear $\gamma_5 T_\mu T_5$ in $\begin{array}{|c|}\hline\\\hline\\\hline\\\hline\end{array}^{+-}$,

for which a base is $\gamma_5(T_1+T_2+T_3+T_4)T_5 \equiv \gamma_5 T T_5$, which replaces $\gamma_5 T_5$ in (4.36) to yield:

$$\sum_{\mu,\nu,\sigma} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ \left[4\sum_{\alpha} \sin^2 \frac{k_\alpha}{2} \right] \left[4\sum_{\beta} \sin^2 \frac{k_\beta}{2} \right]^2 \right\}^{-1} \times$$

$$\times \left\{ -4\left[4\sum_{\alpha} \sin^2 \frac{k_\alpha}{2} \right] + 32 \sin^4 \frac{k_\mu}{2} \right.$$

$$- \frac{1}{2} \sin^2 k_\mu \sin^2 \frac{k_\nu}{2} - 6 \sin^2 \frac{k_\mu}{2} \cdot \sin^2 \frac{k_\nu}{2}$$

$$\left. + 2 \cos k_\sigma \sin^2 \frac{k_\mu}{2} \sin^2 \frac{k_\nu}{2} \right\} \gamma_5 T T_5 \qquad (4.40)$$

We get then for the correction factor of $\gamma_5 T T_5$:

$$K_3 = 1 - 0.086\, g^2 - O(g^4) \qquad (4.41)$$

For negative parity operators it is the same, and from table 3 we choose $\mathbb{1} \otimes T_\mu$ in $\square\!\square^{-+}$ for which $\{T_1-T_2;\ T_2-T_3;\ T_3-T_4\}$ is a suitable base. The replacement of $\gamma_5 T_5$ by $\mathbb{1} \otimes T_1$ in (4.36) leads to terms in $\mathbb{1} \otimes T_1$ as well as some spurious terms (in $\gamma_\mu \gamma_5 T_\mu$ and T_μ) but when the difference (T_1-T_2) is taken, we recover the expected behaviour and if fact the integral:

$$\sum_{\mu,\nu} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ \left[4 \sum_{\alpha} \sin^2 \frac{k_\alpha}{2} \right] \left[4 \sum_{\beta} \sin^2 \frac{k_\beta}{2} \right]^2 \right\}^{-1} \times$$

$$\times \left\{ -4 \left[4 \sum_{\alpha} \sin^2 \frac{k_\alpha}{2} \right] + 8 (1 - \cos k_\mu)^2 + \right.$$

$$+ \left[-\sin^2 k_\mu + (1 - \cos k_\mu)^2 \right] \times$$

$$\left. \times \left[-2 \sin^2 \frac{k_\nu}{2} + 2 \sin^2 \frac{k_1}{2} \right] \right\} (T_1 - T_2) \qquad (4.42)$$

The numerical integration gives:

$$K_J = 1 - 0.126\, g^2 - O(g^4) \qquad (4.43)$$

As the last of these operators, we have calculated $\mathbb{1} \otimes T_\mu T_5$ in $\square^{--}$ with base TT_5. The integral gives:

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ -4 \left[4 \sum_{\alpha} \sin^2 \frac{k_\alpha}{2} \right]^{-1} \times \left[4 \sum_{\beta} \sin^2 \frac{k_\beta}{2} \right]^{-2} + \right.$$

$$\left. + \frac{9}{8} \left[4 \sum_{\mu} \sin^2 \frac{k_\mu}{2} \right]^{-1} \right\} T T_5 \qquad (4.44)$$

The finite correction at order g^2 is:

$$K_S = 1 - 0.232\, g^2 - O(g^4) \qquad (4.45)$$

In order to obtain the correction for currents, we must calculate the finite contribution coming from the conserved current.

The contribution from fig.5.d is numerically the same as in the last chapter, as the contribution from fig.5.c is:

$$\sum_\mu \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ \left[4 \sum_\alpha \sin^2 \frac{k_\alpha}{2} \right] \left[4 \sum_\beta \sin^2 \frac{k_\beta}{2} \right] \right\}^{-1} \times$$

$$\times \left[-\gamma_\rho \sin k_\rho + \gamma_5 T_\rho \cos \frac{k_\rho}{2} \right] \left[-i\gamma_\mu \sin k_\mu - i\gamma_5 T_\mu (1 - \cos k_\mu) \right] \times$$

$$\times \left[\gamma_\rho \cos \frac{k_\rho}{2} + \gamma_5 T_\rho \sin \frac{k_\rho}{2} \right]$$

$$= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left\{ \left[4 \sum_\alpha \sin^2 \frac{k_\alpha}{2} \right] \left[4 \sum_\beta \sin^2 \frac{k_\beta}{2} \right] \right\}^{-1} \times$$

$$\times \left[\sin^2 k_\rho \left(\cos k_\rho - \frac{3}{2} \right) \right] \gamma_\rho \qquad (4.46)$$

Diagram 5.b is again similar to (4.36) in its integral structure, replacing $\gamma_5 T_5$ by $\left[\gamma \cos(k_\rho) + \gamma_5 T_\rho \sin(k_\rho)\right]$.

To obtain the correction to the negative parity current $\gamma_\alpha \gamma_5 T_5$ which falls in $\overline{(\tfrac{1}{2}, \tfrac{1}{2})}^{-+}$, the replacement of

$\gamma_5 T_5$ by $\gamma_\alpha \gamma_5 T_5$ is made in (4.36). Gathering all these factors together we get the correction:

$$K_{J_\mu} = 1 + 0.331 g^2 + O(g^4) \qquad (4.47)$$

Finally, we shall examine a current where mixing occurs. Taking $\gamma_\alpha T_\mu$, we see from table 3 that the representation $(\frac{1}{2},\frac{1}{2})^{--}$ recurs. A base for the first one is $\gamma_\alpha T$, while the second has the invariant vector $\gamma_4(T_1+T_2+T_3-3T_4) \equiv \gamma(T-4T4)$.

This current requires more care than all the previous ones, but the full detailed presentation of its integrals and the simplifications therein, though very long, is not very illuminating. Instead, let us try to obtain a current multiplicatively renormalized which is a combination of the two members.

Denoting $\gamma_4 T$ by θ_1 and $\gamma_4(T-4T_4)$ by θ_2, we get in a one-loop expansion:

$$\begin{matrix} (1+c_1)\theta_1 & + c_2 \theta_2 \\ d_1 \theta_1 & + (1+d_2)\theta_2 \end{matrix} \qquad (4.48)$$

The divergent part of the triangle 5.b is present in c_1 and c_2 with equal coefficients. We are looking for a combination of θ_1 and θ_2 that would give:

$$(\alpha\theta_1 + \beta\theta_2) \to [\alpha(1+c_1) + \beta d_1]\,\theta_1 + [\beta(1+d_2) + \alpha c_2]\,\theta_2$$

$$= M[\alpha\theta_1 + \beta\theta_2] \qquad (4.49)$$

This implies:

$$\frac{\alpha(1+c_1) + \beta d_1}{\alpha} = \frac{\beta(1+d_2) + \alpha c_2}{\beta} \qquad (4.50)$$

Solving a simple quadratic equation we obtain:

$$\frac{\alpha^{\pm}}{\beta} = \frac{c_1 - d_2 \pm \sqrt{(c_1 - d_2)^2 + 4 c_2 d_1}}{2c_2} \qquad (4.51)$$

Hence:

$$M^{\pm} = 1 + c_1 + \frac{\beta}{\alpha} d_1 \qquad (4.52)$$

If then we want to consider a combination of the type (4.49), which has a better physical sense than the isolated θ_1 or θ_2, we need to compare (4.52) with the conserved current. The numerical evaluation of this gives:

$$K^{\pm}_{J_\nu} = 1 \begin{matrix} +0.366\,g^2 \\ +0.285\,g^2 \end{matrix} + O(g^4) \qquad (4.53)$$

We are not giving here any more renormalization factors, but the procedure is now clear and could easily be

extended to any other operators. The results of renormalization factors for staggered fermions will be gathered in a joint publication (Meyer and Verstegen 1984).

Section 4.4 Concluding remarks

In this work we have been able to perform successfully a renormalization procedure doing expansions in the coupling constant. We have seen the importance of such for Wilson fermions, as corrections could be given to quantities evaluated on the lattice. The fact that they are necessary comes about because the Monte Carlo simulations seem to give a behaviour of elementary particles close to the continuum one, even at a finite value of g^2. For the flavour formulation, where a similar treatment has been done, we can say little, as so far no direct simulation work with the block variables has been done. Trying to understand how to perform these renormalizations in the site variable picture would be an interesting and important extension of our work. To our knowledge, no one has yet done this.

Another natural, but quite hard, continuation of what we have done would be to examine the next order in perturbation theory, to refine the corrections, but also to check that the validity of this expansion still holds; indeed, in the power series framework, we should expect smaller contributions from higher orders.

A final point in this discussion is that on the lattice, the breaking of a symmetry may be exploited. We have seen that if currents constructed from the Noether prescription are not conserved, there is no reason why we should limit ourselves to them. We may introduce other currents by hand, as long as they produce the correct continuum limit. We have analysed the local currents in a strict analogy with the continuum, but other objects could equally well be thought of, and also lead to a satisfying continuum behaviour.

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