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**Quantum Groups and
Cohomological
Donaldson-Thomas Theory**

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Shivang Jindal)

To my Parents

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Abstract

We calculate the deformed and non-deformed cohomological Hall algebra (CoHA) of the preprojective algebra for the case of cyclic quivers by studying the Kontsevich-Soibelman CoHA and using tools from cohomological Donaldson-Thomas theory. We show that for the cyclic quiver of length K , this algebra is the universal enveloping algebra of the positive half of a certain extension of matrix differential operators on \mathbb{C}^* , while its deformation gives a positive half of an explicit integral form of Guay's Affine Yangian $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K))$. By the main theorems of [BD23] and [SV23], we also determine the Maulik-Okounkov Yangian for the case of cyclic quivers. Furthermore, we provide evidence for the strong rationality conjecture, calculate the spherical subalgebra of the non-deformed CoHA for any quiver without loops, the associated graded algebra of non deformed CoHA with respect to perverse filtration for any quiver, recover results about the CoHA of compactly supported semistable sheaves on the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_K$, identify a commutative algebra inside the additive shuffle algebra associated to the cyclic quiver and following the ideas of Davison-Hennecart-Kinjo-Schiffmann-Vasserot, we prove existence of cocommutative coproduct on the CoHA. We end by conjecturally relating the obtained integral form with the algebra defined by Gaiotto-Rapčák-Zhou, in the context of twisted M-theory. This thesis is based on [Jin24].

Lay Summary

This thesis concerns connections between two seemingly unrelated fields of study, Enumerative geometry and Quantum Groups. One of the biggest challenges in fundamental physics is finding a unified theory that successfully merges Einstein's General Relativity, which governs the physics of large-scale phenomena like galaxies, with Quantum Theory, which governs very small-scale phenomena such as atoms. String Theory is a theoretical framework that attempts to achieve this unification, predicting that the dimension of space-time is not four but ten. The additional six dimensions arise from a space called Calabi-Yau 3-fold, which is curled up with an extremely small radius. My thesis focuses on invariants associated to the Calabi-Yau 3-fold. These numbers have a remarkable property: the number remains unchanged under small changes to the underlying geometry for deep and sometimes mysterious reasons. In String Theory, these invariants are called the number of BPS states. Due to the work of several people, these numbers can be obtained by computing the dimension of certain mysterious vector spaces. Since joining two particles gives a new particle, one expects that these vector spaces have a multiplication structure.

In 2008, Kontsevich and Soibelman proved that there is indeed such a multiplication rule on these vector spaces, called Cohomological Hall algebra for Quivers with potential, which provides a local model of Calabi-Yau 3-folds. By considerations coming from M-theory, it was conjectured by Kevin Costello for a special case of cyclic quivers that these algebras are a certain explicit algebra, called the Affine Yangian. Affine Yangians are a type of quantum group, which are rich algebraic structures whose origins are deeply rooted in physics but a source of a lot of mathematical advancements. My thesis concerns the relationship between these two fields of study and, in particular, I prove this expectation of Kevin Costello.

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Chapter 1

Introduction

1.1 Motivations

Given a quiver Q and potential $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, where $\mathbb{C}Q$ denotes the path algebra, in [KS11] Kontsevich and Soibelman defined a cohomological Hall algebra structure on the critical cohomology

$$\mathcal{A}_{Q,W} := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} H(\mathfrak{M}_{\mathbf{d}}(Q), {}^p\Phi_{\mathrm{Tr}(W)} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}(Q)}^{\mathrm{vir}})$$

as a mathematical definition of the algebra of BPS states. Here ${}^p\Phi_{\mathrm{Tr}(W)}$ is the vanishing cycle functor associated to the function $\mathrm{Tr}(W)_{\mathbf{d}}$ on the moduli space of \mathbf{d} dimensional representations $\mathfrak{M}_{\mathbf{d}}(Q)$ of the quiver Q . The main result of this thesis calculates the algebra and its deformations $\mathcal{A}_{Q,W}^T$ for the quiver $\widetilde{Q^K}$ and potential $\widetilde{W^K}$ as defined in Example 2. This choice of quiver with potential, often called the tripled quiver with canonical potential, provides a presentation of the moduli space of representations of the preprojective algebra of the cyclic quiver Q^K , equipped with an endomorphism, as a global critical locus. This gives rise to new integral forms of affine Yangians and Lie algebras, which are of independent interest. We start by explaining our main motivations, coming from enumerative geometry and the local geometry of the moduli space of objects in 2-Calabi-Yau categories.

1.1.1 Noncommutative Donaldson-Thomas theory

Given any projective 3-Calabi-Yau (CY) variety X , indivisible Chern class $\alpha \in H^{\mathrm{even}}(X, \mathbb{Z})$ and a generic stability condition ζ , the Donaldson-Thomas invariants $\mathrm{DT}_{\alpha}^{\zeta}$ defined in [Tho00] are virtual counts of semistable sheaves on X of given Chern class α . They are defined by taking the degree of the virtual fundamental class of the moduli space of sheaves $\mathcal{M}_{\alpha}^{\zeta}(X)$. Later, given any scheme Y , Behrend defined a constructible function $\nu_Y : Y \rightarrow \mathbb{C}$ and showed that $\mathrm{DT}_{\alpha}^{\zeta}$ is precisely the weighted Euler characteristic $\chi(\mathcal{M}_{\alpha}^{\zeta}(X), \nu_{\mathcal{M}_{\alpha}^{\zeta}(X)})$ [Beh09][Theorem 4.18]. When Y can be written as the critical locus of a function $f : \widetilde{Y} \rightarrow \mathbb{C}$ where \widetilde{Y} is smooth, then there exists the vanishing cycle sheaf ${}^p\Phi_f \mathbb{Q}^{\mathrm{vir}}$ which satisfies (see [KL16][2.1])

for details)

$$\sum (-1)^i \dim H^i(\tilde{Y}, {}^p\Phi_f \mathbb{Q}^{\text{vir}}) = \chi(Y, v_Y).$$

Thus, the cohomology

$$H(\tilde{Y}, {}^p\Phi_f \mathbb{Q}^{\text{vir}})$$

could be understood as a categorification of the DT invariant.

For any 3CY variety X , the moduli space $\mathfrak{M}_\alpha^\zeta(X)$ can rarely be written as a global critical locus. However, by the result of [Tod18] analytically locally these moduli spaces can be written as the moduli space of representations of a Jacobi algebra $\text{Jac}(Q, W)$ (Definition 2.0.7)¹. Furthermore, the moduli stack of representations of a Jacobi algebra can be written as the critical locus of a smooth function $\text{Tr}(W)$ on a smooth moduli space $\mathfrak{M}_d(Q)$. The study of the DT invariants of a non-compact Calabi–Yau threefold via virtual counts of moduli of $\text{Jac}(Q, W)$ modules, where $\text{Jac}(Q, W)$ is the Jacobi algebra derived equivalent to X , was initiated by Szendrői in the case of the conifold [Sze08]. For any quiver Q , we can define the tripled quiver \tilde{Q} with the canonical cubic potential \tilde{W} (Example 2). When $X = \mathbb{C}^3$ the moduli stack $\text{Coh}_n(\mathbb{C}^3)$ of torsion sheaves of length n can be described as a moduli stack of n dimensional representations of the Jacobi algebra of the tripled Jordan quiver \tilde{Q}_{Jor} with canonical potential $\tilde{W}_{\text{Jor}} := x[y, z]$.

The example concerning the main result of this paper captures the geometry of Kleinian surfaces. Let $S_K \rightarrow \mathbb{C}^2/\mathbb{Z}_{K+1}$ be the minimal resolution of the quotient singularity where the action of \mathbb{Z}_{K+1} is given by $(x, y) \rightarrow (\omega x, \omega^{-1}y)$ and $\omega^{K+1} = 1$ is a primitive root of unity. We then have an equivalence of derived categories

$$D^b(\text{Coh}(S_K \times \mathbb{C})) \simeq D^b(\text{Jac}(\tilde{Q}^K, \tilde{W}^K)) \quad (1.1)$$

where Q^K is the cyclic quiver with $K+1$ vertices. The Jacobi algebra $\text{Jac}(\tilde{Q}^K, \tilde{W}^K)$ is the 3CY completion of the preprojective algebra Π_{Q^K} [KW21].

1.1.2 Cohomological Hall algebras of preprojective algebras

There is a deep relationship between cohomological Donaldson-Thomas Theory of objects in the 3CY completion $\tilde{\mathcal{C}}$ of a 2CY category \mathcal{C} and the Borel-Moore homology of objects in \mathcal{C} which goes under the name of dimensional reduction ([Kin22], [DP22], [Dav17][Appendix]). In particular, the dimensional reduction theorem implies that for any quiver Q , there is an isomorphism²

$$H(\mathfrak{M}_d(\tilde{Q}), {}^p\Phi_{\text{Tr}(\tilde{W})} \mathbb{Q}_{\mathfrak{M}_d(\tilde{Q})}^{\text{vir}}) \simeq H^{\text{BM}}(\mathfrak{M}_d(\Pi_Q), \mathbb{Q}^{\text{vir}}) \quad (1.2)$$

The mixed Hodge structure on the right, i.e. on the BM homology of the stack of representations of the preprojective algebra Π_Q , plays an important role in

¹To be more precise, Toda’s theorem gives an analytic potential and so we need to consider an analytic Jacobi algebra $\text{Jac}\{Q, W\}$ as defined in [Dav23b][Definition 2.7].

²Note that the quotient stack $\mathfrak{M}_d(\Pi_Q)$ is not smooth, so here by \mathbb{Q}^{vir} we mean the shifted complex $\mathbb{Q}[-2(\mathbf{d}, \mathbf{d})]$, where $(-, -)$ denotes the Euler form, as defined in Definition 2.0.4.

multiple branches of mathematics. In particular, by the Ext quiver construction, the stack of representations of the preprojective algebra étale locally models the geometry of the stack of objects in arbitrary 2CY categories possessing good moduli spaces [Dav23a]. Examples of such moduli spaces include Higgs bundles on smooth projective curves, moduli of local systems on Riemann surfaces, and coherent sheaves on K3 surfaces. The first two examples also make studying this space a valuable tool in the study of non-abelian Hodge theory, while on the other hand, there is an algebra structure on

$$\mathcal{A}_{\Pi_Q} := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathbf{d}}(\Pi_Q), \mathbb{Q}^{\mathrm{vir}})$$

which has been defined in the context of equivariant cohomology for the case of Jordan quiver [SV13] in the context of AGT conjecture and later defined for arbitrary quivers in [YZ18b]. This algebra acts on the cohomology of Nakajima quiver varieties by raising operators and, in some sense, is the biggest possible algebra acting on the cohomology of Nakajima quiver varieties, making it an important object in geometric representation theory. It is shown in [RS17][Appendix A] and [YZ20a] that up to a sign twist, the isomorphism in (1.2) in fact preserves the algebra structure and thus is an isomorphism of algebras. Thus studying $\mathcal{A}_{Q,W}$ completely determines \mathcal{A}_{Π_Q} .

1.2 Results

1.2.1 Integral matrix $\mathcal{W}_{1+\infty}$ Lie algebras

Let \hbar be a formal variable and let $D_{\hbar}(\mathbb{C}^*)$ be the algebra of \hbar -differential operators on \mathbb{C}^* . It is defined as a unital associative $\mathbb{C}[\hbar]$ linear algebra generated by $z^{\pm 1}, D$ subject to the relations:

$$Dz = z(D + \hbar), \quad z^{+1}z^{-1} = z^{-1}z^{+1} = 1. \quad (1.3)$$

We will view $D_{\hbar}(\mathbb{C}^*)$ as a $\mathbb{C}[\hbar]$ linear Lie algebra with the commutator Lie bracket $[\cdot, \cdot]$, coming from the associative algebra structure on $D_{\hbar}(\mathbb{C}^*)$. A central extension of the Lie algebra $D_{\hbar}(\mathbb{C}^*)$, after specialization at $\hbar = 1$, is called $\mathcal{W}_{1+\infty}$ Lie algebra. The $\mathcal{W}_{1+\infty}$ Lie algebra was first introduced in [KP81]. Its connection with the vertex algebras $\mathcal{W}(\mathfrak{gl}_N)$ for central charge N , where N is any natural number, has been developed in [Fre+95] [KR96]. Their matrix generalization was considered in [Awa+95] (see also [Str21], [EP19] and [CH19] for a more physical perspective).

Let $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$ be the $\mathbb{C}[\hbar]$ linear associative algebra of $K \times K$ matrices with values in $D_{\hbar}(\mathbb{C}^*)$, i.e. for any $f(z, D) \otimes X$ and $g(z, D) \otimes Y$ in $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$ where $f(z, D), g(z, D) \in D_{\hbar}(\mathbb{C}^*)$ and $X, Y \in \mathfrak{gl}_K$, the multiplication is given by

$$(f(z, D) \otimes X)(g(z, D) \otimes Y) = f(z, D)g(z, D) \otimes XY.$$

We view $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$ as a $\mathbb{C}[\hbar]$ linear Lie algebra with the commutator Lie

bracket coming from the associative algebra structure on $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$, i.e. Lie bracket is given by

$$[f(z, D) \otimes X, g(z, D) \otimes Y] = (f(z, D)g(z, D)) \otimes XY - (g(z, D)f(z, D)) \otimes YX$$

for any $f(z, D) \otimes X, g(z, D) \otimes Y$ in $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$. The elements

$$T_{k,a}(X) = z^k D^a \otimes X \text{ where } k \in \mathbb{Z}, a \geq 0, X \in \mathfrak{gl}_K$$

form a spanning subset of $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$. We slightly extend this Lie algebra by defining an integral form.

Definition 1.2.1. Let

$$\mathcal{W}_K \subset (D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm 1}]$$

be the $\mathbb{C}[\hbar]$ linear subspace spanned by $T_{k,a}(X), X \in \mathfrak{gl}_K$ and

$$t_{k,a} := T_{k,a}(1)/\hbar$$

where $k \in \mathbb{Z}, a \geq 0, X \in \mathfrak{gl}_K$ and by 1, we mean the identity matrix $\text{Id} \in \mathfrak{gl}_K$.

The subspace $\mathcal{W}_K \subset (D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm 1}]$ in fact forms a Lie subalgebra (Proposition 8.1.1). We remark that the Lie algebra \mathcal{W}_K is also considered in [GRZ23]. We consider the classical limit as $\hbar \rightarrow 0$ of \mathcal{W}_K .

Let $\mathcal{O}(\mathbb{T}^*\mathbb{C}^*)$ be the ring of functions on $\mathbb{T}^*\mathbb{C}^*$. We denote by $\mathfrak{po}(\mathbb{T}^*\mathbb{C}^*)$, the Lie algebra on $\mathcal{O}(\mathbb{T}^*\mathbb{C}^*)$, where the Lie bracket is given by the standard Poisson bracket $\{-, -\}$. Let $\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K$ be the Lie algebra with the Lie bracket given by

$$[f \otimes X, g \otimes Y] = fg \otimes [X, Y]$$

for any $X, Y \in \mathfrak{sl}_K$ and $f, g \in \mathcal{O}(\mathbb{T}^*\mathbb{C}^*)$.

The Lie algebra $\mathfrak{po}(\mathbb{T}^*\mathbb{C}^*)$ acts on $\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K$ via the Poisson bracket with the first tensor product, i.e.

$$f \cdot (g \otimes X) := \{f, g\} \otimes X,$$

for any $f \in \mathfrak{po}(\mathbb{T}^*\mathbb{C}^*)$ and $g \otimes X \in \mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K$. Let

$$\mathfrak{po}(\mathbb{T}^*\mathbb{C}^*) \ltimes (\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K)$$

be the Lie algebra given by the semidirect product of $\mathfrak{po}(\mathbb{T}^*\mathbb{C}^*)$ with $\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K$. The classical limit $\hbar \rightarrow 0$ of \mathcal{W}_K is exactly the Lie algebra $\mathfrak{po}(\mathbb{T}^*\mathbb{C}^*) \ltimes (\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K)$, i.e. There is an isomorphism of Lie algebras

$$(\mathcal{W}_K)/(\hbar = 0) \simeq \mathfrak{po}(\mathbb{T}^*\mathbb{C}^*) \ltimes (\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K).$$

In Proposition 8.1.3, we present this Lie algebra via generators and relations. Our first main result describes the cohomological Hall algebra $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}$ as a universal enveloping algebra of the positive half of the Lie algebra $\mathfrak{po}(\mathbb{T}^*\mathbb{C}^*) \ltimes (\mathcal{O}(\mathbb{T}^*\mathbb{C}^*) \otimes \mathfrak{sl}_K)$.

Theorem A (Theorem 8.2.6). *Let $K \geq 1$. There is an isomorphism of Lie algebras*

$$\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}} \simeq (\mathfrak{po}(T^* \mathbb{C}^*) \ltimes (\mathcal{O}(T^* \mathbb{C}^*) \otimes \mathfrak{sl}_{K+1}))^+$$

between the the affinized BPS Lie algebra $\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}}$ for the cyclic quiver (Section 6.1.17) and the positive half $(\mathfrak{po}(T^* \mathbb{C}^*) \ltimes (\mathcal{O}(T^* \mathbb{C}^*) \otimes \mathfrak{sl}_K))^+ \subset \mathfrak{po}(T^* \mathbb{C}^*) \ltimes (\mathcal{O}(T^* \mathbb{C}^*) \otimes \mathfrak{sl}_K)$, defined in Definition 8.1.4. This gives an isomorphism of algebras

$$\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^\chi \simeq \mathbf{U}((\mathfrak{po}(T^* \mathbb{C}^*) \ltimes (\mathcal{O}(T^* \mathbb{C}^*) \otimes \mathfrak{sl}_{K+1}))^+),$$

where χ is a sign twist, defined in Section 4.1.4.

It is also possible to recover the positive half of \mathcal{W}_K itself, without taking the classical limit. To do so, we consider the deformed cohomological Hall algebra. For any torus T acting on the quiver $\widetilde{Q^K}$, leaving the potential $\widetilde{W^K}$ invariant, there is a cohomological Hall algebra (Section 4.1) structure on the equivariant vanishing cycle cohomology

$$\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^T := \bigoplus_{\mathbf{d} \in \mathbb{N}^{K+1}} \mathrm{H}_T(\mathfrak{M}_{\mathbf{d}}(\widetilde{Q^K}), {}^p\Phi_{\mathrm{Tr}(\widetilde{W^K})} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}(\widetilde{Q^K})}^{\mathrm{vir}}).$$

Our next main result, calculates this algebra for $T = \mathbb{C}^*$, whose action on $\mathfrak{M}_{\mathbf{d}}(\widetilde{Q^K})$, leaves the added loops ω_i invariant for all $i \in Q_0$. We set up a bit more notation. For any commutative ring R and a R linear Lie algebra \mathfrak{g} , its tensor algebra is defined to be

$$T_R(\mathfrak{g}) := \bigoplus_{m \geq 0} \underbrace{\mathfrak{g} \otimes_R \cdots \otimes_R \mathfrak{g}}_{m \text{ times}}$$

and correspondingly, the universal enveloping algebra over R is defined to be the quotient

$$\mathbf{U}_R(\mathfrak{g}) := T_R(\mathfrak{g}) / \langle ab - ba - [a, b] \mid a, b \in \mathfrak{g} \rangle.$$

Theorem B (Theorem 11.0.3). *For $K \geq 1$, let \mathbb{C}^* act on representations of the tripled quiver $\widetilde{Q^K}$ by acting on all the original arrows a with weight 1, all opposite arrows a^* with weight -1 and with weight 0 on added loops $\omega_i, i \in [0, K]$ (Section 5.0.2). Then there is an isomorphism of $\mathbb{C}[\hbar]$ linear Lie algebras*

$$\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}, \mathbb{C}^*} \simeq (\mathcal{W}_{K+1})^+$$

between the deformed affinized BPS Lie algebra, defined in the Section 6.1.17 and the positive half of an integral form of differential operators on \mathbb{C}^* valued in matrices, defined in Defintion 8.1.2. This gives an isomorphism of $\mathbb{C}[\hbar]$ linear algebras

$$\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{C}^*, \chi} \simeq \mathbf{U}_{\mathbb{C}[\hbar]}((\mathcal{W}_{K+1})^+)$$

where χ is a sign twist, defined in Section 4.1.4.

Next, we consider the action of a larger torus $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$, which acts on the added loops ω_i for all $i \in Q_0$ non-trivially. This gives rise to quantum groups.

1.2.2 Integral form of Yangians

Given a semisimple Lie algebra \mathfrak{g} , in [Dri85] Drinfeld defined the Yangian $\mathcal{Y}_h(\mathfrak{g})$ as a deformation of the universal enveloping algebra $\mathbf{U}(\mathfrak{g}[x])$ of the polynomial current Lie algebra $\mathfrak{g}[x]$. Yangians form a family of quantum groups related to rational solutions of classical Yang-Baxter equations. For any quiver Q , in [MO19], the authors defined the Yangian \mathbf{Y}_Q^{MO} by geometrically defining R matrices using stable envelopes on the singular cohomology of Nakajima quiver varieties and applying the FRT formalism to construct quantum groups. One of Maulik and Okounkov's motivations was to realise the Quantum cohomology ring as a maximal commutative subalgebra of the Yangian. From the calculations in [MO09] for the quiver variety for cyclic quivers, it was clear that one needs slightly larger algebra than the usual affine Yangians for the case of cyclic quivers. They conjectured a strong relation between the quantum cohomology of the Nakajima quiver varieties and the Bethe subalgebras of \mathbf{Y}_Q^{MO} . However, the algebras \mathbf{Y}_Q^{MO} are not known for cases beyond ADE and Jordan quivers. The computation for the Jordan quiver occupies much of the latter half of [MO19], where it is shown to be isomorphic to the affine Yangian $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\widehat{\mathfrak{gl}(1)})$ of $\widehat{\mathfrak{gl}(1)}$ and used in their proof of AGT conjecture. For the ADE quivers Q , a central quotient of \mathbf{Y}_Q^{MO} is shown to be isomorphic to the Yangian $\mathcal{Y}_h(\mathfrak{g}_Q)$, as defined by Drinfeld [McB13].

For any quiver with potential Q, W , the cohomological Hall algebras are also known to satisfy a similar property, i.e. there is a Lie algebra $\mathfrak{g}_{Q,W}^{\text{BPS},T}$ such that $\mathcal{A}_{Q,W}^T \simeq \text{Sym}_{\text{H}_T(\text{pt})}(\mathfrak{g}_{Q,W}^{\text{BPS},T}[u])$ as graded vector spaces [DM20], allowing us to think of $\mathcal{A}_{Q,W}^T$ as generalized Yangians. For particular choices of torus, it is shown that for ADE quivers, there is an isomorphism of algebras $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{\mathbb{T}} \simeq Y_h^+(\mathfrak{g}_Q)$ ([YZ18b]+[SV17]) and for Jordan quiver, there is an isomorphism of algebras $\mathcal{A}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}}^{\mathbb{T}} \simeq \check{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(1))$ by ([Dav22a]+[Rap+23]) and [SV13] where $+$ denotes the positive half. Furthermore, for all tripled quivers with canonical potential $\widetilde{Q}, \widetilde{W}$, there is an isomorphism of algebras

$$\mathbf{Y}_Q^{\text{MO},+} \simeq \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^T$$

as proved in [BD23] and isomorphism

$$\mathbf{Y}_Q^{\text{MO},-} \simeq \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \widetilde{\mathcal{N}}}$$

is proved in [SV23], where $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \widetilde{\mathcal{N}}}$ is a nilpotent cohomological Hall algebra. Here, T is an appropriately chosen torus. Strictly speaking, Maulik-Okounkov Yangians are only defined with respect to T -equivariant cohomology, where T is some torus that scales the natural symplectic form on the Nakajima quiver varieties associated to the quiver Q , non-trivially.

We now consider the cyclic quiver Q^K and choose $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ to be a 2 torus. The spherical subalgebra of the localized K -theoretic version of the cohomological Hall algebra [Păd23] of the preprojective algebra, defined by considering K theory instead of cohomology, was considered in [Neg15] for the case of cyclic quivers. It is shown to be isomorphic to the positive half of the quantum toroidal algebra $\check{\mathbf{U}}_{q,t}(\mathfrak{gl}(K))$. Its rational counterpart, the affine Yangian, has been defined in

[Gua07] to be a $\mathbb{C}[\hbar_1, \hbar_2]$ algebra $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K))$ for $K > 2$ as a deformation of a central extension of $\mathbf{U}(\mathfrak{sl}_K[u^\pm, v])$. The case when $K = 2$ was first defined by Kodera in [Kod19] and also considered by Bershtein-Tsybaliuk in [BT19]. A specialization of this algebra at $\hbar_1 = \hbar_2$ has been shown to be isomorphic to a spherical subalgebra of $\mathcal{A}_{\widehat{Q}^K, \widehat{W}^K}^{\mathbb{C}^*}$ where $\mathbb{C}^* \subset \mathbb{T}$ acts by scaling the arrows with the same weight in [YZ20b]. However $\mathcal{A}_{\widehat{Q}, \widehat{W}}^{\mathbb{T}}$ is not spherically generated and this makes computing $\mathcal{A}_{\widehat{Q}, \widehat{W}}^{\mathbb{T}}$ essentially a new problem. In Proposition 10.2.6 implies that $\mathcal{A}_{\widehat{Q}^K, \widehat{W}^K}^{\mathbb{T}} \otimes_{\text{H}_{\mathbb{T}}(\text{pt})} \text{Frac}(\text{H}_{\mathbb{T}}(\text{pt})) \simeq \check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K)) \otimes_{\text{H}_{\mathbb{T}}(\text{pt})} \text{Frac}(\text{H}_{\mathbb{T}}(\text{pt}))$. Motivated by this, we slightly enlarge the algebra by including the non-spherically generated part. Note that our only addition is to add elements $\mathfrak{R}_{\pm}^{(r)}$ to the existing definition of Affine Yangian. We use the conventions in [BT19]. We define

Definition 1.2.2 (CoHA Affine Yangian). Assume $n \geq 2$. Let $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), \text{CoHA}}$ be the $\mathbb{C}[\hbar_1, \hbar_2]$ algebra generated by $X_{i,r}^{\pm}, H_{i,r}$ for any $i \in \mathbb{Z}/n\mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$ and $\mathfrak{R}_{+}^{(r)}, \mathfrak{R}_{-}^{(r)}$ for $r \in \mathbb{Z}_{\geq 0}$, with the relations

$$[X_{i,r}^{+}, X_{j,s}^{-}] = \delta_{i,j} H_{i,r+s} \quad (\text{R0})$$

$$[H_{i,r}, H_{j,s}] = 0 \quad (\text{R1})$$

for any $n \geq 2$. When $n > 2$,

$$[X_{i,r+1}^{\pm}, X_{j,s}^{\pm}] - [X_{i,r}^{\pm}, X_{j,s+1}^{\pm}] = -m_{ij}(\hbar_1 + \hbar_2/2)[X_{i,r}^{\pm}, X_{j,s}^{\pm}] \pm a_{ij}\hbar_2/2\{X_{i,r}^{\pm}, X_{j,s}^{\pm}\} \quad (\text{R2})$$

$$[H_{i,r+1}, X_{j,s}^{\pm}] - [H_{i,r}, X_{j,s+1}^{\pm}] = -m_{ij}(\hbar_1 + \hbar_2/2)[H_{i,r}, X_{j,s}^{\pm}] \pm a_{ij}\hbar_2/2\{H_{i,r}, X_{j,s}^{\pm}\} \quad (\text{R3})$$

$$[X_{i,r}^{\pm}, X_{j,s}^{\pm}] = 0 \text{ for all } |i - j| > 1 \quad (\text{R4})$$

$$\sum_{\sigma \in S_2} [X_{i,r_{\sigma(1)}}^{\pm}, [X_{i,r_{\sigma(2)}}^{\pm}, X_{i\pm 1,s}^{\pm}]] = 0 ; [H_{i,0}, X_{j,s}^{\pm}] = \pm a_{ij} X_{j,s}^{\pm} \quad (\text{R5})$$

$$\begin{aligned} (\hbar_1)(\hbar_1 + \hbar_2)\mathfrak{R}_{\pm}^{(r)} &= \text{T}^r \left(\sum_{i \in \mathbb{Z}/n\mathbb{Z}} [X_{i,0}^{\pm}, [X_{i+1,1}^{\pm}, [X_{i+2,0}^{\pm}, [\cdots, X_{i+n-1,0}^{\pm}]]]] \right) \quad (\text{Int1}) \\ &\quad - \hbar_2 \text{T}^r \left(\sum_{i \in \mathbb{Z}/n\mathbb{Z}} X_{i,0}^{\pm} ([X_{i+1,0}^{\pm}, [X_{i+2,0}^{\pm}, [\cdots, X_{i+n-1,0}^{\pm}]]]) \right) \end{aligned}$$

for any $i, j \in \mathbb{Z}/n\mathbb{Z}$ and $r, s \in \mathbb{Z}_{\geq 0}$ where $m_{ij} = -\delta_{i+1,j} + \delta_{i,j+1}$, $a_{ij} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$. Here T is an operator defined so that $\text{T}(X_{i,r}^{\pm}) := \pm(X_{i,r+1}^{\pm})$, $\text{T}(H_{i,r}) = 0$ and T is a derivation, i.e. $\text{T}(ab) := \text{T}(a)b + a\text{T}(b)$ for any a, b . By the curly brackets, we mean $\{a, b\} := ab + ba$ and finally, S_2 is the symmetric group in 2 elements.

When $n = 2$, we have the following relations:

$$[X_{i,r+1}^{\pm}, X_{i,s}^{\pm}] - [X_{i,r}^{\pm}, X_{i,s+1}^{\pm}] = \pm \hbar_2 \{X_{i,r}^{\pm}, X_{i,s}^{\pm}\}, \quad (\text{R2.1})$$

$$[X_{i,r+2}^\pm, X_{j,s}^\pm] - 2[X_{i,r+1}^\pm, X_{j,s+1}^\pm] + [X_{i,r}^\pm, X_{j,s+2}^\pm] = \hbar_1(\hbar_1 + \hbar_2)[X_{i,r}^\pm, X_{j,s}^\pm] \mp \hbar_2(\{X_{i,r+1}^\pm, X_{j,s}^\pm\} - \{X_{i,r}^\pm, X_{j,s+1}^\pm\}) \text{ for } j \neq i, \quad (\text{R2.2})$$

$$[H_{i,r+1}, X_{i,s}^\pm] - [H_{i,r}^\pm, X_{i,s+1}^\pm] = \pm \hbar_2 \{H_{i,r}, X_{i,s}^\pm\}, \quad (\text{R3.1})$$

$$[H_{i,r+2}, X_{j,s}^\pm] - 2[H_{i,r+1}, X_{j,s+1}^\pm] + [H_{i,r}, X_{j,s+2}^\pm] = \hbar_1(\hbar_1 + \hbar_2)[H_{i,r}, X_{j,s}^\pm] \mp \hbar_2(\{H_{i,r+1}, X_{j,s}^\pm\} - \{X_{i,r}, X_{j,s+1}^\pm\}) \text{ for } j \neq i, \quad (\text{R3.2})$$

$$[H_{i,0}, X_{j,s}^\pm] = \pm a_{i,j} X_{j,s}^\pm, \quad [H_{i,1}, X_{i+1,s}^\pm] = \mp (2H_{i+1,s+1}^\pm + \hbar_2 \{H_{i,0}, X_{i+1,s}^\pm\}), \quad (\text{R4.1})$$

$$\text{Sym}_{r_1, r_2, r_3} [X_{i,r_1}^\pm, [X_{i,r_2}^\pm, [X_{i,r_3}^\pm, X_{i+1,s}^\pm]]] = 0. \quad (\text{5.1})$$

$$(\hbar_1)(\hbar_1 + \hbar_2) \mathfrak{K}_\pm^{(r)} = \text{Tr} ([X_{0,0}^\pm, X_{1,1}^\pm] + [X_{1,0}^\pm, X_{0,1}^\pm]) - \hbar_2 \text{Tr} (X_{0,0}^\pm X_{1,0}^+ + X_{1,0}^+ X_{0,0}^+) \quad (\text{Int2})$$

Clearly $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), \text{CoHA}} \otimes_{\mathbb{C}[\hbar_1, \hbar_2]} \mathbb{C}[\hbar_1^{\pm 1}, (\hbar_1 + \hbar_2)^{\pm 1}] \simeq \check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(n)) \otimes_{\mathbb{C}[\hbar_1, \hbar_2]} \mathbb{C}[\hbar_1^{\pm 1}, (\hbar_1 + \hbar_2)^{\pm 1}]$, since we have only added elements $\mathfrak{K}_\pm^{(r)}$ to the original definition of $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K))$ which by relations (Int1) and (Int2), after multiplication by $(\hbar_1)(\hbar_1 + \hbar_2)$ belong to $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(n))$. We then have

Theorem C (Theorem 10.2.10). *For $K \geq 1$, let $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ be the 2-torus which acts on representations of the tripled quiver \widetilde{Q}^K by scaling all the original arrows a with weight $(1, 0)$, all opposite arrows a^* with weight $(0, 1)$ and added loops $\omega_i, i \in [0, K]$ with weight $(-1, -1)$ (Section 5.0.2). Then we have an isomorphism of $\mathbb{C}[\hbar_1, \hbar_2]$ algebras*

$$\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \chi} \simeq \mathcal{Y}_{-\hbar_2, \hbar_1 + \hbar_2}^{(K+1), +, \text{CoHA}}$$

where $\mathcal{Y}_{-\hbar_2, \hbar_1 + \hbar_2}^{(K+1), +, \text{CoHA}} \subset \mathcal{Y}_{-\hbar_2, \hbar_1 + \hbar_2}^{(K+1), \text{CoHA}}$ is the subalgebra generated by $X_{i,r}^+$ and $\mathfrak{K}_+^{(r)}$ for $i \in [0, K], r \geq 0$ and χ is a sign twist, defined in Section 4.1.4.

Let $\overline{\mathcal{Y}}_{\hbar_1, \hbar_2}^{n, \text{CoHA}}$ be the subalgebra of $\mathcal{Y}_{\hbar_1, \hbar_2}^{n, \text{CoHA}}$ which is generated by generators $\langle \mathfrak{K}_+^{(r)}, X_{i,r}^\pm, H_{i,r} \mid i \in \mathbb{Z}/n\mathbb{Z}, r \geq 0 \rangle$ (So we only exclude the generators $\mathfrak{K}_-^{(r)}$ for $r \geq 0$). Let $\overline{\mathcal{Y}}_{\hbar_1, \hbar_2}^{n, \text{CoHA}, \mathfrak{e}} = \overline{\mathcal{Y}}_{\hbar_1, \hbar_2}^{n, \text{CoHA}}[\langle k_{i,r} \rangle]$ be the extended algebra, where we have added central elements $\langle k_{i,r} \mid i \in \mathbb{Z}/n\mathbb{Z}, r \geq 0 \rangle$ to the list of generators. Then we have

Corollary A (Theorem 12). *For $K \geq 1$, the Maulik-Okounkov Yangian $\mathbf{Y}_{Q^K}^{\text{MO}, \mathbb{T}}$ is isomorphic to an extended integral form of Guay's affine Yangian. i.e. There is an isomorphism of $\mathbb{C}[\hbar_1, \hbar_2]$ linear algebras*

$$\mathbf{Y}_{Q^K}^{\text{MO}, \mathbb{T}} \simeq \overline{\mathcal{Y}}_{-\hbar_2, \hbar_1 + \hbar_2}^{(K+1), \text{CoHA}, \mathfrak{e}}.$$

Proof Strategy for Theorems A, B, C

We remark that although by taking the classical limit, Theorem C \implies Theorem B \implies Theorem A, to prove them, we first prove Theorem A. There exists a cocommutative coproduct on $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\chi}$ which is compatible with the cohomological

Hall algebra structure on $\mathcal{A}_{Q^k, \widetilde{W}^k}^x$ (Section 6.1). By the Milnor-Moore theorem, $\mathcal{A}_{Q^k, \widetilde{W}^k}^x \simeq \mathbf{U}(\mathcal{P})$ where \mathcal{P} is the Lie algebra of the primitive elements with respect to this coproduct. We refer to this Lie algebra as the affinized BPS Lie algebra $\widehat{\mathfrak{g}}_{Q^k, \widetilde{W}^k}^{\text{BPS}}$. There exists a filtration, referred to as the perverse filtration \mathfrak{F} on $\mathcal{A}_{Q^k, \widetilde{W}^k}^x$, given by a decomposition type theorem. The associated graded algebra with respect to this filtration is isomorphic to $\text{Sym}(\mathfrak{g}_{Q^k, \widetilde{W}^k}^{\text{BPS}}[u])$, where $\mathfrak{g}_{Q^k, \widetilde{W}^k}^{\text{BPS}}$ is the non-affinized BPS Lie algebra (Section 4.1.5). This implies that, as a vector space $\widehat{\mathfrak{g}}_{Q^k, \widetilde{W}^k}^{\text{BPS}} \simeq \mathfrak{g}_{Q^k, \widetilde{W}^k}^{\text{BPS}}[u]$, but not as a Lie algebra (Section 5.0.2). With the help of an action of the three-dimensional Heisenberg Lie algebra (Section 6.3), we show that a much smaller subspace of $\widehat{\mathfrak{g}}_{Q^k, \widetilde{W}^k}^{\text{BPS}}$, and the relations among them, determine the affinized BPS Lie algebra. Finally, to compute the relations, we construct a geometric action of $\mathcal{A}_{Q^k, \widetilde{W}^k}^x$ on the cohomology of the equivariant Hilbert scheme $\text{Hilb}^{\mathbb{Z}^{k+1}}(\mathbb{C}^2)$ and the cohomology of Hilbert scheme of points on the minimal resolution $\text{Hilb}(S_k)$ (Chapter 7). By identifying the induced action of $\mathfrak{g}_{Q^k, \widetilde{W}^k}^{\text{BPS}}$ with the action of the affine Lie Algebra and the infinite-dimensional Heisenberg Lie algebra due to Nakajima, we prove Theorem A in Section 8.2.

To calculate $\mathcal{A}_{Q^k, \widetilde{W}^k}^{\mathbb{T}, x}$, we use the embedding into the cohomological Hall algebra $\mathcal{A}_{Q^k}^{\mathbb{T}, x}$ of the same quiver but without potential. The algebra structure on $\mathcal{A}_{Q^k}^{\mathbb{T}, x}$ can be explicitly described in terms of a shuffle product (Section 10.1). The calculation of the undeformed version gives a minimal set of generators of $\mathcal{A}_{Q^k, \widetilde{W}^k}^{\mathbb{T}, x}$ (Proposition 10.2.1). We find the image of these generators in $\mathcal{A}_{Q^k, \widetilde{W}^k}^{\mathbb{T}, x}$ (Proposition 10.2.2). This allows us to construct a homomorphism of algebras from the positive half of Guay's affine Yangian to the algebra $\mathcal{A}_{Q^k, \widetilde{W}^k}^{\mathbb{T}, x}$ (Section 10.2.1). Finally, with the help of the PBW theorem for $\mathcal{A}_{Q^k, \widetilde{W}^k}^{\mathbb{T}, x}$, we prove that this morphism is an isomorphism, proving Theorem C (Proposition 10.2.6 and Theorem 10.2.10). Finally, we take the classical limit $\hbar_1 + \hbar_2 \rightarrow 0$ to calculate the \mathbb{C}^* deformed algebra $\mathcal{A}_{Q^k, \widetilde{W}^k}^{\mathbb{C}^*, x}$, proving Theorem B.

1.2.3 Spherical subalgebra

The spherical subalgebra, i.e, the subalgebra generated by the smallest possible dimension vectors of the deformed version of the cohomological Hall algebra, has been studied extensively in [YZ18a]. An important tool used in [YZ18a] is embedding into the cohomological Hall algebra of quiver without potential. However, no such injection exists in the non-deformed case (Section 10.1). As an application of general tools from [Dav22a], developed in Section 6.3, we can precisely calculate the spherical part for any quivers without loops.

Theorem D (Theorem 6.4.2). *Let Q be any quiver without loops. Then there is an isomorphism of algebras*

$$\mathbf{U}(\mathfrak{n}_Q^+[D]) \simeq \mathcal{S}\mathcal{A}_{Q, \widetilde{W}}^x$$

where \mathfrak{n}_Q^+ is the positive half of the Kac-Moody Lie algebra (Definition 4.2.2) associated with Q , $\mathfrak{n}_Q^+[D]$ is the current Lie algebra (Definition 6.3.8), $\mathcal{SA}_{\widetilde{Q}, \widetilde{W}}^X \subset \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^X$ is the subalgebra generated by $\mathcal{A}_{\widetilde{Q}, \widetilde{W}, \delta_i}^X$ for $i \in Q_0$ and χ is a sign twist, defined in Section 4.1.4.

We remark that for quivers without loops, $\mathcal{SA}_{\widetilde{Q}, \widetilde{W}}^X = \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^X$ if and only if the quiver is an orientation of a finite type ADE Dynkin quiver [Dav22a][Proposition 5.7].

1.2.4 Less Perverse Associated Graded Algebra

For any quiver Q , let \overline{Q} be double of the quiver Q , obtained by adding opposite arrow a^* to Q for any arrow $a \in Q$. Let $\Pi_Q = \mathbb{C}\overline{Q}/(\sum_{a \in Q_1} [a, a^*])$ be the preprojective algebra associated to Q . Let $\mathcal{A}_{\Pi_Q}^X$ be the preprojective algebra associated to Q , where χ is a sign twist defined in Section 4.1.4.

Let $\text{JH} : \mathfrak{M}(\Pi_Q) \rightarrow \mathcal{M}(\Pi_Q)$ be the affinization morphism. Then in [Dav22a], Davison proved that even though the stack $\mathfrak{M}(\Pi_Q)$ is not smooth and the morphism JH is not proper, one still have splitting

$$\text{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)}^{\text{vir}} \simeq \bigoplus_{i \in 2\mathbb{Z}_{\geq 0}} {}^p\mathcal{H}(\text{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)}^{\text{vir}})[-i].$$

As a result of this splitting, [Dav22a] defined an increasing filtration $\mathfrak{L}^\bullet \mathcal{A}_{\Pi_Q}$ on $\mathcal{A}_{\Pi_Q}^X$, which they call the *less perverse filtration* (Section 4.2.2). This filtration is defined by setting

$$\mathfrak{L}_i \mathcal{A}_{\Pi_Q} := \text{H}(\mathcal{M}(\Pi_Q), {}^p\tau^{\leq i} \text{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)}^{\text{vir}}).$$

Perverse filtrations defined in this way have played an important role in non-abelian Hodge theory and representation theory and are important objects of study.

Davison further showed that this filtration respects the algebra structure on $\mathcal{A}_{\Pi_Q}^X$ and in fact that there is an isomorphism of algebras $\mathfrak{L}_0 \mathcal{A}_{\Pi_Q}^X \simeq \mathbf{U}(\mathfrak{g}_{\widetilde{Q}, \widetilde{W}})$. Our next theorem proves a refinement of this result.

Theorem E (Theorem 6.5.1). *Let Q be any quiver. Then there is an isomorphism of algebras*

$$\mathbf{U}(\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}[D]) \simeq \text{Gr}_{\mathfrak{L}^\bullet}(\mathcal{A}_{\Pi_Q}^X)$$

where $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is the BPS Lie algebra defined in Section 5 and $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}[D]$ is the current Lie algebra (Definition 6.3.8).

1.2.5 Strong rationality conjecture

Given $f: X \rightarrow Y$ a three-fold flopping contraction with exactly one exceptional fibre, which is isomorphic to a rational curve C , one can associate numerous invariants. One is the Gopakumar-Vafa invariant $n_{C,r}$. In [Dav23b], the author gives a cohomological categorification $\text{BPS}_{r,d}$ of these numbers in terms of BPS

cohomology of the contraction algebra associated with the flopping curve C , where d is the Euler characteristic of the coherent sheaves on X . The invariance of BPS invariants under the choice of Euler characteristic for 3CY varieties has been conjectured in the work of Pandharipande and Thomas [PT09]. By analogy, the conjecture [Dav23b][Conjecture 5.9] is the statement that this invariance lifts to an isomorphism of BPS cohomology

$$\mathrm{BPS}_{r,d} \simeq \mathrm{BPS}_{r,d+1}.$$

Here, the isomorphism is constructed by defining a Hecke correspondence from the cohomological Hall algebra. In particular, it implies the strong rationality conjectures in [Tod23][PT09] for flopping curves. We can consider a similar Hecke correspondence for the case when $X = S_K \times \mathbb{C}$ where $S_K \rightarrow \mathbb{C}^2/\mathbb{Z}_{K+1}$ is the minimal resolution of a Kleinian singularity. Then the proof of Theorem C implies that the Hecke correspondence indeed gives an isomorphism between BPS cohomology, as we vary d .

Corollary B (Corollary 8.2.3.1). *The cohomological lift of the strong rationality conjecture is true for $S_K \times \mathbb{C}$.*

1.2.6 Cohomological Hall algebra of Kleinian surfaces

Given any smooth quasi-projective surface S , in [KV23], the authors defined a cohomological Hall algebra structure on the Borel-Moore homology of the stack of compactly supported sheaves on S . Furthermore, this structure was upgraded to the categorical level in [PS23]. Given any finite subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$, let $S_G \rightarrow \mathbb{C}^2/G$ be the minimal resolution. Then in [DPS23], the authors study the cohomological Hall algebra $\mathcal{A}_\mu^\omega(S_G)$ of ω -semistable properly supported sheaves on S_G with fixed slope μ . They show that there is an isomorphism of algebras

$$\mathcal{A}_\mu^\omega(S_G) \simeq \mathcal{A}_0^\zeta(\Pi_{Q_G})$$

where Q_G is the affine ADE quiver associated to the group G by the McKay correspondence. Here $\mathcal{A}_0^\zeta(\Pi_Q)$ is the cohomological Hall algebra of slope 0, ζ -semistable representations of the preprojective algebra Q , where the stability condition ζ is explicitly determined in terms of ω and μ . Now assume $G = \mathbb{Z}_{K+1}$. The main result and the properties of the affinized BPS Lie algebra allow us to explicitly determine $\mathcal{A}_{\mu'}^{\zeta'}(\Pi_{Q_K})$ as a subalgebra of $\mathcal{A}_{\Pi_{Q_K}}$, for any choice of ζ' and μ' . In particular, as a corollary, we recover the calculation of the cohomological Hall algebra of 0-dimensional sheaves $\mathcal{A}(S_K)$ as in [Mel+23] and of $\mathcal{A}_\mu^\omega(S_K)$ as in [DPS23].

Corollary C. 1. *Let $W(S_K)$ be the Lie algebra defined in the Corollary (9.0.2.1). We then have an isomorphism of algebras (Corollary (9.0.2.1))*

$$\mathcal{A}(S_K) \simeq \mathbf{U}(W(S_K)).$$

2. *Let $W_\mu^\omega(S_K)$ be the Lie algebra as defined in the Definition 9.0.3. We then*

have an isomorphism of algebras (Corollary 9.0.3.1)

$$\mathcal{A}_\mu^\omega(S_K) \simeq \mathbf{U}(W_\mu^\omega(S_K)).$$

1.2.7 Shuffle algebra realization and commutative subalgebras

It is an interesting problem to construct the realization of quantum groups in terms of a shuffle algebra, which is an algebra structure on the symmetric polynomial ring with a shuffle product. This idea dates back to the work of Enriques in [Enr01]. In [Neg23b], the localized K-theoretic Hall algebras are realized in terms of polynomials in the shuffle algebra satisfying Feigin-Odesskii type wheel conditions³. However, determining the explicit integral form is far from being known. In [Neg22], the author described the seminilpotent K-theoretic Hall algebra of the preprojective algebra of the Jordan quiver as a shuffle algebra with polynomials satisfying certain divisibility conditions, giving a shuffle realization of an integral form of $\check{\mathbf{U}}_{q,t}(\mathfrak{gl}(1))$. In [Zha19], the author gives some necessary but not sufficient divisibility conditions in the shuffle algebra associated to the surface to be in the image of the K-theoretic Hall algebra of any smooth projective surface. In [Tsy17a], the author describes an integral form of the Yangian of \mathfrak{sl}_{n+1} by divisibility conditions.

In Section 10.2.2, we describe the image of $\mathcal{A}_{Q^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$ in the shuffle algebra $\mathcal{A}_{Q^K}^{\mathbb{T}, \chi}$ as a subalgebra generated by certain specific polynomials. We remark that the additive shuffle algebra (without wheel conditions) considered in [BT19][Section 5.2] is isomorphic to the algebra $\mathcal{A}_{Q^K}^{\mathbb{T}, \chi}$ after localization (Remark 10.1.2).

We now state an interesting corollary of this embedding. By computing the image of a natural trivial Lie subalgebra of $\mathfrak{g}_{Q^K, \widetilde{W}^K}^{\text{BPS}, \mathbb{T}}$, we define elements $L_n, n \geq 1$ in the shuffle algebra (Corollary 10.2.2.1) such that

Corollary D (Corollary 10.2.2.1). *For all $n \geq 1$, let $A_K^{\mathbb{T}} \subset \mathcal{A}_{Q^K}^{\mathbb{T}, \chi}$ be the subalgebra generated by L_n . Then $A_K^{\mathbb{T}} \simeq \mathbb{C}[L_1, L_2, \dots]$ is a polynomial subalgebra.*

For the quantum toroidal algebra $\check{\mathbf{U}}_{q,t}(\mathfrak{gl}(1))$, a commutative polynomial subalgebra is defined in [FT11][Theorem 2.4] and its action on the K theory of $\text{Hilb}^n(\mathbb{C}^2)$ has been studied in relation to the theory of Macdonald polynomials. Its rational counterpart, has been defined in [Tsy17b][Theorem 7.7], and it can be seen as the image of the commutative Lie algebra $\mathfrak{g}_{Q_{\text{Jor}}, \widetilde{W}_{\text{Jor}}}^{\text{BPS}, \mathbb{T}}$ in the shuffle algebra from [Dav22a][Section 3.13].

1.2.8 Algebras in physics literature

In [GRZ23], for every $K \neq 2$, the authors construct certain double deformed current algebras A^K as the algebra of gauge-invariant local observables on M2 branes, in the twisted M-theory background. They further construct a $\mathbb{C}[\hbar_1, \hbar_2]$

³In some references, the shuffle algebra is defined to be the subalgebra of symmetric Laurent polynomials satisfying the wheel conditions. See [Neg23b][Definition 2.9].

linear algebra Y^K by gluing two copies of A^K . They show that Y^K is also an integral form of the affine Yangian $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K))$ and after quotienting by a central element, this algebra deforms \mathcal{W}_K . We expect that (see Section 13)

Conjecture 1. *For any $K > 2$, there is an isomorphism of algebras*

$$Y^K \simeq \mathcal{Y}_{\hbar_1, \hbar_2}^{(K), \text{CoHA}}.$$

This would also explain the expectation in [Cos17][Section 1.8] concerning the embedding of the cohomological Hall algebra in Costello's deformed double current algebra $\mathcal{O}_{\hbar_1}(\mathfrak{M}_{\hbar_2}(\bullet, K))$, since it is shown in [GRZ23][Theorem 2] that Costello's algebra is isomorphic to a particular integral form of the algebra A^K which contains the positive half of Y^K [GRZ23][Definition 7.01].

1.2.9 Notations

- All schemes/stacks are considered over \mathbb{C} .
- For any algebraic group G , $\text{BG} := \text{pt}/G$ is the stack theoretic quotient, while $\text{H}_G := \text{H}(\text{BG}, \mathbb{Q})$.

Chapter 2

Quivers and Moduli Spaces

In this chapter, we will introduce our main object of study. We first set up some notations.

A quiver is a directed graph $Q = (Q_0, Q_1, s, t)$, where Q_0, Q_1 are finite sets and $s, t : Q_1 \rightarrow Q_0$. The set Q_0 is the set of vertices, and Q_1 is the set of edges. For every edge $\alpha \in Q_1$, $s(\alpha)$ is the source of the edge, and $t(\alpha)$ is the target of the edge.

Definition 2.0.1 (Path algebra). Let $\mathbb{C}Q$ be the path algebra of Q , i.e., the algebra over \mathbb{C} having the paths in Q as a basis, with multiplication given by concatenation of paths. For each vertex $i \in Q_0$, there is a lazy path of length 0 starting and ending at i , and we denote by e_i the resulting element of $\mathbb{C}Q$.

Given a quiver Q , the doubled quiver \overline{Q} is the quiver obtained by adding a new arrow a^* for each arrow $a \in Q_1$, with $s(a^*) = t(a)$ and $t(a^*) = s(a)$. We then define

Definition 2.0.2 (Preprojective algebra). The preprojective algebra is defined as the quotient

$$\Pi_Q := \mathbb{C}\overline{Q} / \langle \sum [a, a^*] \rangle.$$

From now on, by a representation of a quiver, we mean a left $\mathbb{C}Q$ module.

Definition 2.0.3 (Dimension). A left $\mathbb{C}Q$ module ρ is said to be of dimension $\mathbf{d} \in \mathbb{N}^{Q_0}$ defined by

$$\mathbf{d}_i := \dim(e_i \cdot \rho).$$

Convention 1. For quiver Q , and for any $0 \leq i \leq |Q_0| - 1$ we set $\delta_i = (\underbrace{0, \dots, 0}_i, 1, 0, \dots, 0) \in \mathbb{N}^{Q_0}$ and for any $m \leq |Q_0| - 1$, we set $[i, m) = \sum_{j=i}^{j=i+m \bmod (|Q_0|)} \delta_j \in \mathbb{N}^{Q_0}$ is a cyclic string of 0s and 1s with $m+1$ consecutive 1s starting from position i and remaining terms 0. Note that $[i, 0) = \delta_i$ and $[i+1, |Q_0| - 2) = \delta - \delta_i$ where

$\delta = (\underbrace{1, \dots, 1}_{|Q_0| \text{ times}})$.

Given two representations ρ, ρ' of dimension \mathbf{d}, \mathbf{e} respectively, we consider the Euler form

$$(\rho, \rho') := \dim(\text{Hom}(\rho, \rho')) - \dim(\text{Ext}^1(\rho, \rho')),$$

which in this case, only depends on the dimensions \mathbf{d}, \mathbf{e} .

Definition 2.0.4 (Euler form). Given a pair of dimension vectors $\mathbf{d}, \mathbf{e} \in \mathbb{N}^{Q_0}$, we define the Euler form

$$\chi_Q(\mathbf{d}, \mathbf{e}) := \sum_{i \in Q_0} \mathbf{d}_i \mathbf{e}_i - \sum_{\alpha \in Q_1} \mathbf{d}_{s(\alpha)} \mathbf{e}_{t(\alpha)}.$$

Definition 2.0.5 (Serre Subcategory). A Serre subcategory $\mathcal{S} \subset \mathbb{C}Q\text{-mod}$ such that for every short exact sequence of representations

$$0 \rightarrow \rho_1 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow 0,$$

ρ_2 is in \mathcal{S} if and only if ρ_1, ρ_3 are in \mathcal{S} .

Definition 2.0.6 (Potential). A potential¹ on a quiver Q is an element $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$. A potential is given by a linear combination of cyclic words in Q , where two cyclic words are considered to be the same if one can be cyclically permuted to be the other. If W is a single cyclic word and $a \in Q_1$, then we define

$$\partial W / \partial a = \sum_{W=cac'} c'c$$

And we extend this definition linearly to general W .

Definition 2.0.7 (Jacobi algebra). The Jacobi algebra associated to (Q, W) is defined to be

$$\text{Jac}(Q, W) := \mathbb{C}Q / \langle \partial W / \partial a \mid a \in Q_1 \rangle.$$

Although the definition might seem artificial, these algebras provide a way to study important examples of non-compact Calabi-Yau 3-folds.

Example 1. Let \tilde{Q}_{Jor} be a 3 loop quiver with one vertex and three loops.



and let $\tilde{W} = X[Y, Z]$ be the potential. Then

$$\text{Jac}(\tilde{Q}_{\text{Jor}}, \tilde{W}) \simeq \mathbb{C}[X, Y, Z].$$

The above example has a natural generalization.

Example 2 (Canonical Cubic Potential). Given a quiver Q , we consider the quiver \tilde{Q} , given by adding loops ω_i to each vertex $i \in \overline{Q}_0$. Then the tripled quiver $\tilde{\tilde{Q}}$ carries the canonical cubic potential

$$\tilde{\tilde{W}} = \left(\sum_{i \in \overline{Q}_0} \omega_i \right) \left(\sum_{a \in \overline{Q}_1} [a, a^*] \right).$$

¹Also called superpotential in physics terminology

We have an isomorphism of algebras

$$\Pi_Q[\omega] \simeq \text{Jac}(\widetilde{Q}, \widetilde{W})$$

for the polynomial ring $\Pi_Q[\omega]$ where $\omega \mapsto \sum \omega_i$.

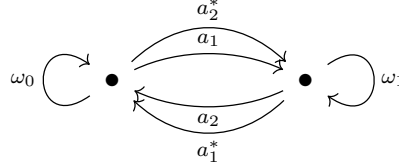
Example 3. Let $G \subset \text{SL}(\mathbb{C}^2)$ be any finite subgroup of $\text{SL}(\mathbb{C}^2)$. Let X_G be the minimal resolution of the affine quotient \mathbb{C}^2/G . The McKay correspondence associates an affine ADE quiver Q_G to the group G . Let $Y = X_G \times \mathbb{A}^1$. Then it follows from the Derived McKay correspondence [KV00] that there is an equivalence of derived categories

$$\text{D}_c^b(\text{Coh}(X_G \times \mathbb{A}^1)) \simeq \text{D}^b(\text{Jac}(\widetilde{Q}_G, \widetilde{W}_{Q_G}) - \text{mod}).$$

In particular, let $G = \mathbb{Z}_2$. Then $\mathbb{C}^2/\mathbb{Z}_2 = \{(x, y, z) \mid x^2 = yz\}$ and the minimal resolution is given by $\mathbb{T}^*\mathbb{P}^1$. We can identify $Y = \text{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ and we have derived equivalence

$$\text{D}_c^b(\text{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))) \simeq \text{D}^b(\text{Jac}(\widetilde{Q}^2, \widetilde{W}^2) - \text{mod}),$$

where quiver \widetilde{Q}^2 is given by



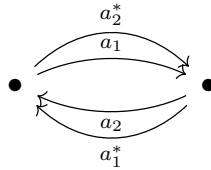
with the potential $\widetilde{W}^2 = (\omega_0 + \omega_1)([a_1, a_1^*] + [a_2, a_2^*])$.

For any toric 3CY manifold X , there is a procedure developed in mathematical physics, known as dimer models, through which one can construct a quiver with potential. These quiver encodes D-branes on X . Mathematically, this procedure construct a tilting bundle $\mathcal{T} := \oplus \mathcal{T}_i$ on X such that

$$\text{RHom}(\mathcal{T}, -) : \text{D}_c^b(\text{Coh}(X)) \simeq \text{D}^b(\text{Jac}(Q, W) - \text{mod})$$

There is a derived equivalence between the bounded derived category of compactly supported coherent sheaves on X and the bounded derived category of finite-dimensional modules over the Jacobi algebra $\text{Jac}(Q, W) = \text{End}_X(\mathcal{T})$. See [IU15] for details.

Example 4. Let $X = \text{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ be the resolved conifold. Let $\mathcal{T}_1 = \pi^*\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{T}_2 = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ where $\pi : \text{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow \mathbb{P}^1$ is the projection map. Then it can be checked that $\text{End}_X(\mathcal{T}) = \text{Jac}(Q_{\text{con}}, W_{\text{KW}})$ where Q_{con} is the quiver



and $W_{KW} = a_1 a_2 a_2^* a_1^* - a_1 a_1^* a_2^* a_2$ is the Klebanov–Witten potential.

Definition 2.0.8 (Nilpotent Representation). We define the notion of nilpotency as considered in [BSV20], which is based on the work of Lusztig [Lus91]. Let $\mathcal{N}_{\overline{Q}} \subset \mathbb{C}\overline{Q}\text{-mod}$ be the subcategory of semi-nilpotent representations of the quiver \overline{Q} , i.e representations ρ for which there exist some l such that $\mathbb{C}\overline{Q}_{\geq l} \cdot \rho = 0$. Furthermore,

- We say that a representation of Π_Q is nilpotent if underlining $\mathbb{C}\overline{Q}$ module is nilpotent. We shall denote that subcategory of representations by \mathcal{N} .
- Let \tilde{Q} be the tripled quiver introduced in Example 2. We define $\tilde{\mathcal{N}} \subset \mathbb{C}\tilde{Q}\text{-mod}$ to be the subcategory for which underlining \overline{Q} module is in $\mathcal{N}_{\overline{Q}}$.

We will also consider representations under stability conditions.

Definition 2.0.9 (Semistability). Given a quiver Q , a stability condition is a tuple $\zeta \in \mathbb{Q}^{Q_0}$. For a non-zero representation ρ of Q , we define the slope

$$\mu_\zeta(\rho) := \frac{\chi_\zeta(\dim(\rho))}{|\dim(\rho)|} := \frac{\zeta \cdot \dim(\rho)}{|\dim(\rho)|}.$$

Note that it only depends on $\dim(\rho)$. A representation ρ is semistable if for all non-zero, proper $\rho' \subset \rho$, we have $\mu_\zeta(\rho') \leq \mu_\zeta(\rho)$ and stable if the inequalities are strict. We say that ζ is generic for dimension \mathbf{d} if for all $\mathbf{d}' < \mathbf{d}$, we have $\mu_\zeta(\mathbf{d}') \neq \mu_\zeta(\mathbf{d})$.

For $\mu \in (-\infty, \infty)$ a slope, we denote by

$$\Lambda_\mu^\zeta \subset \mathbb{N}^{Q_0}$$

the submonoid of dimension vectors \mathbf{d} such that $\mathbf{d} = 0$ or $\mu_\zeta(\rho) = \mu$.

2.0.1 Torus Actions

Given any quiver Q and a lattice $N = \mathbb{Z}^r$, let the map $\mathbf{w} : Q_1 \rightarrow N$ be a weighting function on the edges. Fix $T = \text{Hom}_{\text{Group}}(N, \mathbb{C}^*)$. For every $a \in Q_1$, we can define a homomorphism $T \rightarrow \mathbb{C}^*$ given by $t \rightarrow t(\mathbf{w}(a))$. This gives a morphism of stacks $\lambda_a : \text{BT} \rightarrow \text{BC}^*$. Taking pullback, defines a cohomology class $\mathbf{t}(a) \in H^2(\text{BT}, \mathbb{Q})$ given by $\mathbf{t}(a) = \lambda_a^*(u)$ where u is the generator of $H^2(\text{BC}^*, \mathbb{Q}) \simeq \mathbb{Q}[u]$. Given a path $p = a_1 a_2 \cdots a_n$ on the quiver Q , we define its weight to be $\mathbf{w} = \sum_{i=1}^n \mathbf{w}(a_i)$. A linear combination of paths is said to be homogeneous if the weights of each path in the linear combination are the same.

2.1 Moduli Spaces of quiver representations

Given a quiver Q and dimension \mathbf{d} , define

$$\begin{aligned}\mathrm{Rep}_{\mathbf{d}}(Q) &:= \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}(s(a))}, \mathbb{C}^{\mathbf{d}(t(a))}) \\ \mathrm{GL}_{\mathbf{d}} &:= \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}(i)}\end{aligned}$$

where the group $\mathrm{GL}_{\mathbf{d}}$ acts on $\mathrm{Rep}_{\mathbf{d}}(Q)$ by conjugation. Let

$$\mathfrak{M}_{\mathbf{d}}(Q) = \mathrm{Rep}_{\mathbf{d}}(Q) / \mathrm{GL}_{\mathbf{d}}(Q)$$

be the moduli stack of \mathbf{d} dimensional representations of Q . Let $\mathcal{M}_{\mathbf{d}}(Q) = \mathrm{Spec}(\mathbb{C}[\mathrm{Rep}_{\mathbf{d}}(Q)]^{\mathrm{GL}_{\mathbf{d}}})$ be the coarse moduli space parametrizing semisimple points. We then have the affinization map given by taking the direct sum of successive quotients in the Jordan-Holder filtration of the representation

$$\mathrm{JH}_{\mathbf{d}}: \mathfrak{M}_{\mathbf{d}}(Q) \rightarrow \mathcal{M}_{\mathbf{d}}(Q).$$

There is an action of T on $\mathrm{Rep}_{\mathbf{d}}(Q)$ by

$$(t, (\rho_a)_{a \in Q_1}) \rightarrow (t(\mathbf{w}(a)) \cdot \rho_a)_{a \in Q_1}$$

and then we define

$$\mathfrak{M}_{\mathbf{d}}^T(Q) := \mathrm{Rep}_{\mathbf{d}}(Q) / (\mathrm{GL}_{\mathbf{d}}(Q) \times T).$$

Similarly, we define the stack

$$\mathcal{M}_{\mathbf{d}}^T(Q) := \mathcal{M}_{\mathbf{d}}(Q) / T$$

where T acts on $\mathcal{M}_{\mathbf{d}}(Q)$ since the action of $\mathrm{Rep}_{\mathbf{d}}(Q)$ commutes with the action of the gauge group $\mathrm{GL}_{\mathbf{d}}$. We have an induced morphism

$$\mathrm{JH}_{\mathbf{d}}^T: \mathfrak{M}_{\mathbf{d}}^T(Q) \rightarrow \mathcal{M}_{\mathbf{d}}^T(Q).$$

We can also consider the versions of these objects with stability conditions. Given a King's stability condition ζ , we define $\mathrm{Rep}_{\mathbf{d}}^{\zeta\text{-ss}}(Q) \subset \mathrm{Rep}_{\mathbf{d}}(Q)$ to be the open subvariety of ζ semistable representations. We refer [Rei08] for an excellent introduction to quiver moduli spaces.

We define the quotient stack $\mathfrak{M}_{\mathbf{d}}^{T, \zeta\text{-ss}}(Q) := \mathrm{Rep}_{\mathbf{d}}^{\zeta\text{-ss}}(Q) / (\mathrm{GL}_{\mathbf{d}}(Q) \times T)$. When there is no T , we have the coarse moduli space $\mathcal{M}_{\mathbf{d}}^{\zeta\text{-ss}}(Q)$ given by GIT Quotient. Since the points of this variety are in bijection with polystable representations, it is easy to see that T acts on $\mathcal{M}_{\mathbf{d}}^{\zeta\text{-ss}}(Q)$. We define $\mathcal{M}_{\mathbf{d}}^{T, \zeta\text{-ss}}(Q)$ to be the quotient stack $\mathcal{M}_{\mathbf{d}}^{\zeta\text{-ss}}(Q) / T$. We then have the induced morphism

$$\mathrm{JH}_{\mathbf{d}}^{T, \zeta}: \mathfrak{M}_{\mathbf{d}}^{T, \zeta\text{-ss}}(Q) \rightarrow \mathcal{M}_{\mathbf{d}}^{T, \zeta\text{-ss}}(Q).$$

We denote by $\text{JH}_\mu^{T,\zeta} = \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta} \text{JH}_\mathbf{d}^{T,\zeta} : \mathfrak{M}_\mu^{T,\zeta\text{-ss}}(Q) \rightarrow \mathcal{M}_\mu^{T,\zeta\text{-ss}}(Q)$, where $\mathfrak{M}_\mu^{T,\zeta\text{-ss}} := \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta} \mathfrak{M}_\mathbf{d}^{T,\zeta\text{-ss}}(Q)$ and $\mathcal{M}_\mu^{T,\zeta\text{-ss}} := \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta} \mathcal{M}_\mathbf{d}^{T,\zeta\text{-ss}}(Q)$. Finally for two dimension vectors \mathbf{d}' and \mathbf{d}'' , we define $\text{Rep}_{\mathbf{d}',\mathbf{d}''}^{T,\zeta\text{-ss}} \subset \text{Rep}_{\mathbf{d}'+\mathbf{d}''}^{T,\zeta\text{-ss}}$ to be the subvariety of representations such that the flag $\mathbb{C}^{\mathbf{d}'_i} \subset \mathbb{C}^{\mathbf{d}'_i+\mathbf{d}''_i}$ for all $i \in Q_0$ is preserved. Let $\text{GL}_{\mathbf{d}',\mathbf{d}''} \times T \subset \text{GL}_{\mathbf{d}'+\mathbf{d}''} \times T$ be the subgroup preserving this flag. We then define the quotient stack of extensions

$$\text{Ext}_{\mathbf{d}',\mathbf{d}''}^{T,\zeta\text{-ss}}(Q) := \text{Rep}_{\mathbf{d}',\mathbf{d}''}^{T,\zeta\text{-ss}}(Q) / (\text{GL}_{\mathbf{d}',\mathbf{d}''} \times T).$$

For any algebra A given by the quotient of $\mathbb{C}Q$ by a two sided ideal which is preserved by T , we define quotient stack $\mathfrak{M}_\mathbf{d}^{T,\zeta\text{-ss}}(A)$ as the $\text{GL}_\mathbf{d}$ quotient of ζ semistable \mathbf{d} dimensional representations of A . We similarly have stacks $\text{Ext}_{\mathbf{d}',\mathbf{d}''}^{T,\zeta\text{-ss}}(Q)$ and coarse moduli spaces $\mathcal{M}_\mathbf{d}^{T,\zeta\text{-ss}}(A)$. We will be focusing especially on the case when $A = \Pi_Q$, which is the quotient of the path algebra of the doubled quiver \overline{Q} . This stack can be more conceptually understood in the following way.

2.1.1 Representations of preprojective algebra

Let \overline{Q} be the doubled quiver. Then we can identify

$$\text{Rep}_\mathbf{d}(\overline{Q}) \simeq T^* \text{Rep}_\mathbf{d}(Q)$$

by the pairing between $\text{Hom}(\mathbb{C}^{\mathbf{d}(s(a))}, \mathbb{C}^{\mathbf{d}(t(a))})$ and $\text{Hom}(\mathbb{C}^{\mathbf{d}(t(a))}, \mathbb{C}^{\mathbf{d}(s(a))})$ for any arrow $a \in Q_1$. The symplectic manifold $T^* \text{Rep}_\mathbf{d}(Q)$ carries the action of Gauge group $\text{GL}_\mathbf{d}$ with moment map

$$\mu_{Q,\mathbf{d}} : \text{Rep}_\mathbf{d}(\overline{Q}) \rightarrow \mathfrak{gl}_\mathbf{d} := \prod_{i \in Q_0} \mathfrak{gl}_{\mathbf{d}_i}$$

given by

$$(\rho(a), \rho(a^*))_{a \in Q_1} \mapsto \sum_{a \in Q_1} [\rho(a), \rho(a^*)].$$

Here we have identified $\mathfrak{gl}_\mathbf{d}$ with $\mathfrak{gl}_\mathbf{d}^\vee$ by the trace pairing. Then a substack of $\mathfrak{M}_\mathbf{d}(\overline{Q})$, the quotient stack $\mu_{\overline{Q},\mathbf{d}}^{-1}(0)/\text{GL}_\mathbf{d}$ can be identified with the stack of representations of the preprojective algebra $\mathfrak{M}_\mathbf{d}(\Pi_Q)$.

Let $\mathfrak{M}_\mathbf{d}^{\zeta\text{-ss}}(\Pi_Q) = \mu^{-1,\zeta\text{-ss}}(0)/\text{GL}_\mathbf{d}$ be the open substack of ζ semisstable stable quiver representations. We may also consider GIT quotients $\mathcal{M}_\mathbf{d}^{\zeta\text{-ss}}(\Pi_Q)$. Similar to quiver representations, we have semisimplification morphism

$$\text{JH} : \mathfrak{M}_\mathbf{d}(\Pi_Q) \rightarrow \mathcal{M}_\mathbf{d}(\Pi_Q).$$

When the torus T preserves the moment map relations, we can construct moduli spaces $\mathfrak{M}_\mathbf{d}^{T,\zeta\text{-ss}}(\Pi_Q)$ and $\mathcal{M}_\mathbf{d}^{T,\zeta\text{-ss}}(\Pi_Q)$ in exactly similar way as before.

These stacks are often highly singular, as seen in the following example.

Example 5. Let Q be the Jordan quiver. Then the $\mu_{Q_{\text{Jor},\mathbf{d}}}^{-1}(0)$ is space of com-

muting matrices and thus the stack of preprojective algebra representations is

$$\mathfrak{M}_d(\Pi_{Q_{\text{Jor}}}) = \{A, B \in \text{End}(\mathbb{C}^n) \mid [A, B] = 0\} / \text{GL}_d.$$

This can be equivalently described as the stack $\text{Coh}_d(\mathbb{A}^2)$ of length d torsion sheaves on \mathbb{A}^2 and the coarse moduli space $\mathcal{M}_d(\Pi_{Q_{\text{Jor}}})$ can be identified with symmetric space of d points on plane $\text{Sym}^n(\mathbb{C}^2)$. Finally, the Jordan holder map is identified with the support morphism $\text{Supp} : \text{Coh}_d(\mathbb{A}^2) \rightarrow \text{Sym}_n(\mathbb{C}^2)$.

Earlier, we defined the notion of nilpotency \mathcal{N} for preprojective algebra representation. Following Lusztig, we can define Lusztig nilpotent stack $\mathfrak{M}_d^{\mathcal{N}}(\Pi_Q)$ as the closed substack $\mu_d^{-1, \mathcal{N}}(0) / \text{GL}_d$ of nilpotent representations of Π_Q where $\mu_d^{-1, \mathcal{N}}(0) := (\mu_d^{-1}(0) \cap \text{Rep}_d^{\mathcal{N}}(\overline{Q}))$.

Example 6. Let Q be the Jordan quiver. Then the Lusztig nilpotent stack $\mathfrak{M}_d^{\mathcal{N}}(\Pi_{Q_{\text{Jor}}})$ parametrizes commuting nilpotent matrices of dimension d .

Chapter 3

Perverse Sheaves and Vanishing Cycles

In this chapter, we will briefly introduce the theory of Perverse sheaves and introduce Vanishing cycles, a central object in the cohomological Donaldson-Thomas theory, and explain the dimension reduction theorems, which provide a way to compute vanishing cycles.

3.1 Perverse Sheaves

Our reference for this section is [Ach21]. Let X be any algebraic variety and let X^{an} denote the analytification of X . Let $\text{Sh}(X)$ denote the abelian category of sheaves over \mathbb{Q} on X . For a \mathbb{Q} vector space V , $V_X \in \text{Sh}(X)$ denote the constant sheaf on X associated to V . We say that a sheaf is constructible if there exists a stratification $X = \coprod X_i$ such that the restriction of the sheaf \mathcal{F} to each X_i is a locally constant sheaf.

Let $D^b(X)$ denote the derived category of sheaves on X . An object \mathcal{F} is constructible if the cohomologies $\mathcal{H}^i(\mathcal{F})$ are constructible. Let $D_c^b(X)$ denote the derived category of constructible complexes. This category comes with six functors $f_!$, f_* , $f^!$, f^* , $\mathcal{H}\text{om}(\mathcal{F}, -)$ and $\mathcal{F} \otimes -$.

Let $p : X \rightarrow \text{pt}$ be a constant map. Then the dualizing complex $\mathbb{D}\mathbb{Q}_X$ is defined to be $p^!\mathbb{Q}_{\text{pt}}$. This defines the verdier duality functor $\mathbb{D} : D_c^b(X)^{\text{op}} \rightarrow D_c^b(X)$ given by $\mathbb{D}(\mathcal{F}) = \mathcal{H}\text{om}(-, \mathbb{D}\mathbb{Q}_X)$.

We then define *perverse t-structure* on $D_c^b(X)$ by

$$\begin{aligned} {}^pD_c^{\leq 0}(X) &:= \{ \mathcal{F} \in D_c^b(X) \mid \dim(\text{Supp}(\mathcal{H}^i(\mathcal{F}))) \leq -i, \forall i \in \mathbb{Z} \} \\ {}^pD_c^{\geq 0}(X) &:= \{ \mathcal{F} \in D_c^b(X) \mid \mathbb{D}(\mathcal{F}) \in {}^pD_c^{\leq 0}(X) \} \end{aligned}$$

Then, finally, we define

Definition 3.1.1. The category of Perverse sheaves on X is defined to be the heart of the perverse t structure, i.e.

$$\mathbf{Perv}(X) := {}^pD_c^{\leq 0}(X) \cap {}^pD_c^{\geq 0}(X).$$

We will often denote the i the cohomology functor associated to Perverse- t -structure by ${}^p\mathcal{H}^i(-)$.

Example 7. If X is any smooth irreducible variety, then $\mathcal{L}[\dim(X)]$ is a Perverse sheaf on X . In the cases when X is smooth, we will denote by $\mathbb{Q}_X^{\text{vir}} := \mathbb{Q}_X[\dim(X)]$ as the constant perverse sheaf on X .

3.1.1 Perverse Sheaves on Stacks

In earlier sections, we only defined the category of perverse sheaves when the underlying space is an algebraic variety. There is a way to define these categories for smooth Artin stacks \mathfrak{X} .

The first step is to define the constructible category $D_c^b(\mathfrak{X})$ with six-functor formalism. This is done in [LO08a] and [LO08b]. In the setting when \mathfrak{X} is the quotient stack X/G , which is the case we will be dealing with in this thesis, the constructible derived category $D_c^b(X/G)$ is defined to be the equivariant derived category $D_G^b(X)$.

Then the second step is to define a perverse t structure, which is done in [LO09], which defines the category of Perverse sheaves. For any stack \mathfrak{X} , let $\mathbf{Perv}(\mathfrak{X})$ denote the category of Perverse sheaves on \mathfrak{X} . In the case when \mathfrak{X} is smooth, we will denote by $\mathbb{Q}_{\mathfrak{X}}^{\text{vir}}$ the perverse sheaf $\mathbb{Q}_{\mathfrak{X}}[\dim(\mathfrak{X})]$.

Example 8. We can interpret standard cohomology theories via the six functor formalism. For any stack \mathfrak{X} any $\mathcal{F} \in D_c^b(\mathfrak{X})$ and the morphism to point $p : \mathfrak{X} \rightarrow \text{pt}$, we write $H(\mathfrak{X}, \mathcal{F}) := p_*(\mathcal{F})$ and $H^i(\mathfrak{X}, \mathcal{F}) := H^i(p_*(\mathcal{F}))$. We can define

$$\begin{aligned} H^{\text{BM}}(X, \mathcal{F}) &:= p_*(\mathbb{D}\mathcal{F}) \\ H_c(X, \mathcal{F}) &:= p_!(\mathcal{F}) \end{aligned}$$

where \mathbb{D} is the Verdier duality functor.

Convention 2. Given any space X , let $D^b(\mathbf{Perv}(X))$ denote the unbounded derived category of constructible complexes on X such that for each connected component $X' \subset X$, the cohomology sheaves ${}^p\mathcal{H}^i(\mathcal{F})$ vanish for $i \ll 0$.

3.1.2 Decomposition Theorem and Intersection Cohomology Sheaves

Despite the abstract definition, the category of Perverse sheaves behaves quite nicely. The category of perverse sheaves is known to be a finite length category (both Artinian and Noetherian). Furthermore, there is a complete description of the simple Perverse sheaves.

Let $i : Z \hookrightarrow X$ be any locally closed, irreducible, smooth subvariety. Then in general, it is not true that either of the functors $i_!$ or i_* preserves perverse heart. However, there exist intermediate extension functor $i_{!*}$ defined by $i_{!*}\mathcal{F} = \text{Im}({}^p\mathcal{H}^0(i_!\mathcal{F}) \rightarrow {}^p\mathcal{H}^0(i_*\mathcal{F}))$ which preserves perverse t structure i.e. We have a functor $i_{!*} : \mathbf{Perv}(X) \rightarrow \mathbf{Perv}(Y)$.

Originally defined in work of Goresky-Macpherson [GM77], given a local system \mathcal{L} on Z , $\mathcal{L}[\dim(Z)] \in \mathbf{Perv}(Z)$. We define the IC sheaves as the image of this perverse sheaf via the intermediate extension functor, i.e.

$$\mathcal{IC}_Z(\mathcal{L}) := i_{!*}(\mathcal{L}[\dim(Z)]).$$

For any variety X , for which each connected component is irreducible, we define the intersection complex

$$\mathcal{IC}_X := \bigoplus_{Z \in \pi_0(X)} \mathcal{IC}_Z(\mathbb{Q}_{Z^{\text{sm}}}[\dim(Z)]),$$

i.e. it is the intermediate extension of the constant perverse sheaf on the smooth locus. We then have the following fundamental theorem.

Theorem 3.1.2. *Let \mathcal{L} be a irreducible local system on Z , then $\mathcal{IC}_Z(\mathcal{L})$ is a simple object in $\mathbf{Perv}(X)$ and all simple objects in $\mathbf{Perv}(X)$ are of this form. Furthermore, every perverse sheaf admits a finite filtration whose subquotients are simple IC sheaves $\mathcal{IC}_Z(\mathcal{L})$.*

Suppose one is in a setting where one wants to study the cohomology of some space X and assume further that there is a morphism $f : X \rightarrow Y$ such that we understand the cohomology of the fibers, but maybe some of the fibers are ill-behaved. It might still be possible that these fibers don't contribute much to the overall cohomology. Thus, studying the pushforward $f_*\mathbb{Q}_X$ is worthwhile, which might tell us something about the support. This technique is quite heavily used in some of the main theorems of this thesis.

The following theorem of Beilinson–Bernstein–Deligne–Gabber [Bei+18], called the BBDG decomposition theorem, is precisely of this nature. We have

Theorem 3.1.3. *If $f : X \rightarrow Y$ is a proper morphism from a smooth algebraic variety. Then we have splitting*

$$f_*\mathbb{Q}_X^{\text{vir}} \simeq \bigoplus_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(f_*\mathbb{Q}_X^{\text{vir}})[-i]$$

and each of ${}^p\mathcal{H}^i(f_*\mathbb{Q}_X^{\text{vir}})$ are semisimple perverse sheaves.

Note that $H(Y, {}^p\tau^{\leq i}(f_*\mathbb{Q}_X^{\text{vir}})) \subset H(X, \mathbb{Q}_X^{\text{vir}})$ defines a canonical filtration on the cohomology, often called Perverse filtration. In this thesis, we will see several instances of such filtrations. The Splitting 4.3 for the semisimplification morphism JH can be seen as an instance where the first part of this theorem is true, even when the underlying morphism JH is not proper. The decomposition in equation 4.15 is even more interesting as neither the underlying space $\mathfrak{M}_d(\Pi_Q)$ is smooth, nor the morphism JH is proper. Yet, we have both parts of the BBDG decomposition theorem.

3.2 Vanishing Cycles

Vanishing cycles provide a coefficient system of counting points in the moduli space of objects in 3CY categories, which are locally a critical locus. In this section, we will introduce vanishing cycles and the properties we need in later sections. For more details, refer to [KS90] and [DM20]. Let $f : X \rightarrow \mathbb{C}$ be a function on a smooth complex variety X . Then we consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & i & & & \\
 & & & \curvearrowright & & & \\
 X_0 & \xrightarrow{i_0} & X_{\leq 0} & \xrightarrow{i_{\leq 0}} & X & \xleftarrow{r} & X_{>0} \\
 \downarrow & & \downarrow & & \downarrow f & & \downarrow \\
 \{z = 0\} & \longleftrightarrow & \{z \in \mathbb{C} \mid \Re(z) \leq 0\} & \longleftrightarrow & \mathbb{A}^1 & \longleftrightarrow & \{z \in \mathbb{C} \mid \Re(z) > 0\}
 \end{array}$$

Definition 3.2.1 (Nearby cycle functor). The perverse¹ nearby cycle functor

$${}^p\Psi_f : D^+(X) \rightarrow D^+(X)$$

is defined to be

$$\Psi_f := i_* i^* r_* r^* [-1]$$

By adjunction, there is a natural transformation $\text{id} \rightarrow r_* r^*$, precomposing with $i_* i^*$ yields a natural transformation

$$i_* i^* [-1] \rightarrow {}^p\Psi_f. \quad (3.1)$$

Definition 3.2.2 (Vanishing cycle functor). The vanishing cycle functor

$${}^p\Phi_f : D^+(X) \rightarrow D^+(X)$$

is defined to be the cone of above morphism

$${}^p\Phi_f := \text{Cone}(i_* i^* [-1] \rightarrow {}^p\Psi_f)$$

These functors behave nicely with the perverse structure. As we have

Proposition 3.2.3 ([Ach21], Theorem 4.2.8). *Let $f : X \rightarrow \mathbb{A}^1$ be a regular function. The functors ${}^p\Phi_f$ and ${}^p\Psi_f$ are t -exact for the perverse- t structure.*

Thus in particular, the **vanishing cycle sheaf** ${}^p\Phi_f(\mathbb{Q}_X^{\text{vir}})$ is a perverse sheaf on X . The vanishing cycle sheaf captures the singularity of the function. Let's consider the smoothest function of all first.

Example 9. Consider the identity function $\text{id} : \mathbb{C} \rightarrow \mathbb{C}$, $\text{id}(z) = z$. Then by definition, we have ${}^p\Psi_{\text{id}} \mathbb{Q}_{\mathbb{A}^1}[1] = i_* i^* r_* \mathbb{Q}_{\mathbb{A}^1_{>0}}$. So we have to compute the talk $(r_* \mathbb{Q}_{\mathbb{A}^1_{>0}})_0$. By [Ach21][Lemma 1.2.10], we have

$$H^k(i_* r_* \mathbb{Q}_{\mathbb{A}^1_{>0}}) = \lim_{0 \in U} H^k(U, (r_* \mathbb{Q}_{\mathbb{A}^1_{>0}})|_U) = \lim_{0 \in U} H^k(\mathbb{A}^1_{>0} \cap U, \mathbb{Q}_{\mathbb{A}^1_{>0} \cap U}) = H^k(\mathbb{A}^1_{>0}, \mathbb{Q}_{\mathbb{A}^1_{>0}})$$

¹We have shifted the usual functor by $[-1]$.

where last equality is true since for any open disc U containing 0, the inclusion $\mathbb{A}_{>0}^1 \cap U \hookrightarrow \mathbb{A}_{>0}^1$ is a homotopy equivalence. Since this vanishes unless $k = 0$, this is just \mathbb{Q} . Thus ${}^p\Psi_{\text{id}} = i_*\mathbb{Q}_0$. And so ${}^p\Phi_{\text{id}}(\mathbb{Q}_{\mathbb{A}^1}[1]) = 0$.

Proposition 3.2.4. *The vanishing cycle sheaf ${}^p\Phi_f \mathbb{Q}_X^{\text{vir}}$ is supported on the critical locus $\text{Crit}(f) = \{df = 0\}$.*

Proof. Let $x \in X$ which is not in the critical locus. Thus, for a neighbourhood $x \in U$, we may assume that $f : U \rightarrow \mathbb{A}^1$ is smooth. Let $\text{id} : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be the identity function. Then by smooth pullback (Section 3.2.2), we have $0 = f^* {}^p\Phi_{\text{id}} \mathbb{Q}_{\mathbb{A}^1} \simeq {}^p\Phi_f f^* \mathbb{Q}_{\mathbb{A}^1} = {}^p\Phi_f \mathbb{Q}_X$. \square

Let's consider another example where f is not a smooth function.

Example 10. Consider the function $f : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ given by $f(z) = z^n$ for some $n \geq 2$. Let $(\mathbb{A}_{>0}^1)^{1/n} = f^{-1}(\mathbb{A}_{>0}^1)$. By definition ${}^p\Psi_f(\mathbb{Q}_{\mathbb{A}^1}[1]) = i_* i^* r_* \mathbb{Q}_{(\mathbb{A}_{>0}^1)^{1/n}}$. Now just like Example 9, we have

$$H^k(i^* r_* \mathbb{Q}_{(\mathbb{A}_{>0}^1)^{1/n}}) = H((\mathbb{A}_{>0}^1)^{1/n}, \mathbb{Q}_{(\mathbb{A}_{>0}^1)^{1/n}}) = \mathbb{Q}^n$$

since there are n connected components. So ${}^p\Psi_{z^n} \mathbb{Q}_{\mathbb{A}^1}^{\text{vir}} = i_* \mathbb{Q}_0^n$ and thus

$${}^p\Phi_{z^n} \mathbb{Q}_{\mathbb{A}^1}^{\text{vir}} = i_* \mathbb{Q}_0^{n-1}$$

Example 11. Let Q be any quiver, \mathbf{d} be any dimension vector, and W be any potential. Consider the morphism $\text{Tr}(W) : \mathfrak{M}_{\mathbf{d}}(Q) \rightarrow \mathbb{C}$. Then

$$\text{crit}(\text{Tr}(W)) = \mathfrak{M}_{\mathbf{d}}(\text{Jac}(Q, W)).$$

So in particular,

$$H(\mathfrak{M}_{\mathbf{d}}(Q), {}^p\Phi \mathbb{Q}_{\text{Tr}(W)}) = H(\mathfrak{M}_{\mathbf{d}}(\text{Jac}(Q, W)), {}^p\Phi \mathbb{Q}_{\text{Tr}(W)}).$$

Like perverse sheaves, vanishing cycles can be generalized to Artin stacks. In this thesis, we will mainly be dealing with the case of smooth quotient stacks X/G , where one can understand its cohomology as the equivariant cohomology $H_G(X, {}^p\Phi_f)$.

Definition 3.2.5 (Critical cohomology). Given a function $f : \mathfrak{X} \rightarrow \mathbb{A}^1$, we define critical cohomology with slight abuse of notation:

$$H(\mathfrak{X}, {}^p\Phi_f) := H(\mathfrak{X}, {}^p\Phi_f \mathbb{Q}_{\mathfrak{X}}^{\text{vir}}).$$

The vanishing cycle functor enjoys many nice properties, which we shall explain now. We refer [DM20] for more details. We assume that all the stacks considered are quotient stacks.

3.2.1 Thom-Sebastiani

There is an analogue of the Künneth isomorphism for the critical cohomology. Let f and g be two functions on smooth stacks \mathfrak{X} and \mathfrak{Y} respectively. The Thom-

Sebastiani theorem states that there is a natural equivalence

$$\mathrm{TS} : {}^p\Phi_{f\boxplus g}(\mathcal{F} \boxtimes \mathcal{G})|_{f^{-1}(0) \times g^{-1}(0)} \simeq {}^p\Phi_f \mathcal{F} \boxtimes {}^p\Phi_g \mathcal{G}$$

and when $\mathrm{Supp}({}^p\Phi_{f\boxplus g}(\mathcal{F} \boxtimes \mathcal{G})) \subset f^{-1}(0) \times g^{-1}(0)$, we get an isomorphism on cohomology

$$\mathrm{TS} : \mathrm{H}(\mathfrak{X}, {}^p\Phi_f \mathcal{F}) \otimes \mathrm{H}(\mathfrak{Y}, {}^p\Phi_g \mathcal{G}) \simeq \mathrm{H}(\mathfrak{X} \times \mathfrak{Y}, {}^p\Phi_{f\boxplus g}(\mathcal{F} \boxtimes \mathcal{G})).$$

3.2.2 Pullback

Given a morphism $p: \mathfrak{X} \rightarrow \mathfrak{Y}$ between smooth stacks, there is a natural morphism

$$p^* {}^p\Phi_f \rightarrow {}^p\Phi_{fp} p^*$$

which is an isomorphism when the map p is smooth. Applying this to $\mathbb{Q}_{\mathfrak{Y}}$ and pushing forward to the point, we get a pullback map

$$p^* : \mathrm{H}(\mathfrak{Y}, {}^p\Phi_f \mathbb{Q}_{\mathfrak{Y}}) \rightarrow \mathrm{H}(\mathfrak{X}, {}^p\Phi_{fp} \mathbb{Q}_{\mathfrak{X}}).$$

3.2.3 Verdier Duality

There is a natural isomorphism $\mathbb{D} {}^p\Phi_f \simeq {}^p\Phi_f \mathbb{D}$ where \mathbb{D} is the Verdier duality functor.

3.2.4 Pushforward

Given any morphism $p: \mathfrak{X} \rightarrow \mathfrak{Y}$ and a function $f: \mathfrak{Y} \rightarrow \mathbb{C}$ there is a natural map

$${}^p\Phi_f p_* \rightarrow p_* {}^p\Phi_{fp}$$

which is an isomorphism when p is proper. Applying this to $\mathcal{F} = \mathbb{Q}_{\mathfrak{X}}$ and composing gives a morphism ${}^p\Phi_f \mathbb{Q}_{\mathfrak{Y}} \rightarrow p_*({}^p\Phi_{fp} \mathbb{Q}_{\mathfrak{X}})$. Applying \mathbb{D} and pushing forward to the point yields a map for p proper:

$$p_* : \mathrm{H}(\mathfrak{X}, {}^p\Phi_{fp} \mathbb{Q}_{\mathfrak{X}}) \rightarrow \mathrm{H}(\mathfrak{Y}, {}^p\Phi_f \mathbb{Q}_{\mathfrak{Y}})[2(\dim(\mathfrak{Y}) - \dim(\mathfrak{X}))].$$

3.3 Dimension Reduction

Let $X = X' \times \mathbb{A}^n$ be a smooth variety, let f be a function on X of weight one for the scaling action on \mathbb{A}^n and trivial action on X' and let $\pi: X \rightarrow X'$ be the projection map so that we can write

$$f = \sum_{1 \leq i \leq n} x_i \pi^*(f_i)$$

where f_i are functions on X' and x_i are coordinates on \mathbb{A}^n . Let

$$Z' := \{x \in X' \mid f_i(x) = 0 \forall x \in X'\}$$

and let $Z := \pi^{-1}(Z')$. Let $i_{Z'} : Z' \hookrightarrow X'$, and $i : Z \rightarrow X$ be the inclusions. Then the natural map

$$\pi_! \mathbb{P}\Phi_f \pi^* \rightarrow \pi_! \pi^*(i_{Z'})_*(i_{Z'})^*$$

is an isomorphism [Dav17][Appendix A].

Example 12. We apply dimension reduction isomorphism to the sheaf $\mathcal{F} = \mathbb{Q}_{X'}$. This gives an isomorphism of complexes

$$\pi_! \mathbb{P}\Phi_f \mathbb{Q}_X \simeq \pi_! \pi^*(i_{Z'})_* \mathbb{Q}_{Z'} \simeq \pi_! i_* \mathbb{Q}_Z = \mathbb{Q}_{Z'}[-2n].$$

Taking global sections gives an isomorphism of cohomology

$$H_c(X, \mathbb{P}\Phi_f \mathbb{Q}_X) \simeq H_c(Z, \mathbb{Q}_Z) \simeq H_c(Z', \mathbb{Q}_{Z'}[-2n]). \quad (3.2)$$

Applying Verdier duality gives an isomorphism of cohomology:

$$H^{\text{BM}}(Z', \mathbb{Q}_{Z'}[-2n]) \simeq H(X, \mathbb{P}\Phi_f(\mathbb{Q}_X^{\text{vir}})[\dim(X)]).$$

This isomorphism also extends to quotient stacks. Suppose that an algebraic group G acts on X' and \mathbb{A}^n such that the function f is also G invariant. Then we get an isomorphism of sheaves

$$\pi_! \mathbb{P}\Phi_f \mathbb{Q}_{X/G} \simeq \mathbb{Q}_{Z'/G}[-2n].$$

Taking Verdier dual gives an isomorphism of sheaves

$$\mathbb{D}\mathbb{Q}_{Z'/G}[2n - (\dim(X/G))] \simeq \pi_* \mathbb{P}\Phi_f \mathbb{Q}_{X/G}^{\text{vir}}. \quad (3.3)$$

We shall be applying isomorphism 3.3 to dimensionally reduce from Kontsevich-Soibelman Cohomological Hall algebras to the Schiffmann-Vasserot Cohomological Hall algebra of preprojective algebra in Section 4.2.1.

Example 13. Let $f : \mathbb{A}^2 \rightarrow \mathbb{C}$ given by $f = xy$. Then we can write f as $x f_y$ where $f_y : \mathbb{A}^1 \rightarrow \mathbb{C}$, $f_y(y) = y$. Then applying dimension reduction gives

$$H(\mathbb{A}^2, \mathbb{P}\Phi_f \mathbb{Q}_{\mathbb{A}^2}^{\text{vir}}) \simeq H^{\text{BM}}(\text{pt}, \mathbb{Q}) \simeq \mathbb{Q}.$$

Notice that we could have rather written $f = y f_x$, and we would have gotten another natural isomorphism

$$H(\mathbb{A}^2, \mathbb{P}\Phi_f \mathbb{Q}_{\mathbb{A}^2}^{\text{vir}}) \simeq H^{\text{BM}}(\text{pt}, \mathbb{Q}) \simeq \mathbb{Q}.$$

These two morphisms differ by a sign. They are one of the major reasons why we sometimes twist multiplication rules in the cohomological Hall algebra by specific choices of signs. See Proposition 6.1.12 to see this manifestation in the coproduct.

Notice that there is an underlying assumption that there is a weight 1 scaling action. This condition can be slightly relaxed. We have the following theorem of Davison-Padurariu [DP22], which we write in the generality we need.

Theorem 3.3.1 (Deformed Dimension Reduction). *Let G be any algebraic group acting on X' and the affine space \mathbb{A}^n . Assume that \mathbb{A}^n is given \mathbb{C}^* action by non-negative weights, which commutes with G action. Let \mathbb{C}^* acts on $X = X' \times \mathbb{A}^n$ by trivial action on X' and given action on \mathbb{A}^n . Now assume that f is G invariant and \mathbb{C}^* semi-invariant, with strictly positive weight. Assume that one can write $\mathbb{A}^n = \mathbb{A}^m \times \mathbb{A}^{n-m}$, $G \times \mathbb{C}^*$ equivariantly and one can write*

$$f = f_0 + \sum_{1 \leq i \leq m} f_i x_i$$

where f_0, f_1, \dots, f_m are pulled back from $X \times \mathbb{A}^{n-m}$ and x_i are the coordinates of \mathbb{A}^m . Then there is a natural isomorphism of sheaves

$$\pi_! {}^p\Phi_f \mathbb{Q}_{X/G} \simeq q_! {}^p\Phi_{f_0} \mathbb{Q}_{Z/G}[-2m],$$

where $Z \subset X \times \mathbb{A}^{n-m}$ is the vanishing locus of functions $f_i, 1 \leq i \leq m$ and $q : X \times \mathbb{A}^{n-m} \rightarrow X$ is the projection morphism.

Notice the case when $n = m$ is the usual dimension reduction theorem, particularly isomorphism 3.3.

Example 14. Let f be a function $f : \mathbb{A}^2 \rightarrow \mathbb{C}$ given by $f(x, y) = xy + x^n, n \geq 2$. Then we define a \mathbb{C}^* action so that it scales y by weight $n - 1$ and x by weight 1. We can write $\mathbb{A}^2 = \mathbb{A}_x^1 \times \mathbb{A}_y^1$ where we see x^n, x as pullback of functions on \mathbb{A}_x^1 and y as the coordinate. Then deformed dimension reduction theorem with $\mathbb{A}^m = \mathbb{A}_y^1, \mathbb{A}^{n-m} = \mathbb{A}_x^1$ and $X' = \text{pt}$ gives isomorphism

$$H(\mathbb{A}^2, {}^p\Phi_f \mathbb{Q}_{\mathbb{A}^2}^{\text{vir}}) \simeq \mathbb{Q}$$

3.3.1 Saito's Mixed Hodge Modules

Given X any algebraic variety X , we saw that the constructible category $D_c^b(X)$, and hence the category $\mathbf{Perv}(X)$, provides a way to study the cohomology of the space X . When X is a smooth and projective variety, it is well known that the cohomology $H(X, \mathbb{Q})$ has a Hodge decomposition. This is, however, not true for an arbitrary algebraic variety. In [Del71], Deligne proved that the cohomology of any complex algebraic variety admits a canonical Mixed Hodge structure.

In [Sai86a] and [Sai86b], Saito introduced the theory of Mixed Hodge modules. This theory is a way to unify the Mixed Hodge structures and Perverse sheaves formalism. For any algebraic variety X , let $\mathbf{MHM}(X)$ denote the category of polarizable mixed Hodge modules. When X is a point, $\mathbf{MHM}(\text{pt})$ is just the category of polarizable mixed Hodge structures. The derived category of mixed Hodge complexes $D^b(\mathbf{MHM}(X))$ can be seen as an analog of $D_c^b(X)$ and admits six functor formalism. Finally there exist a conservative functor

$$\text{rat}_X : D^b(\mathbf{MHM}(X)) \rightarrow D_c^b(X)$$

which maps the abelian category $\mathbf{MHM}(X)$ to $\mathbf{Perv}(X)$.

This generalization is particularly useful in cohomological Donaldson-Thomas theory, as now we can study the mixed Hodge structure on the critical cohomology, which provides refinements to DT invariants. In this thesis, we will be working with Perverse sheaves, but all the objects defined can be upgraded at the level of Mixed Hodge modules, so every cohomology considered has a Mixed hodge structure. One of the main theorems (PBW isomorphism [4.6](#)), which we shall be using in this thesis, is proved via going through this upgrade.

Chapter 4

Cohomological Hall algebra

4.1 Kontsevich-Soibelman Cohomological Hall algebra

We recall the construction of the cohomological Hall algebra of a quiver with potential from [KS11]. Let Q be any quiver with potential W and let T be any torus defined by weighting $\mathbf{w} : Q_1 \rightarrow \mathbb{Z}^r$ such that the weight of potential W is 0. Then given any stability condition ζ and slope μ , we consider the graded vector space

$$\mathcal{A}_{Q,W,\mu}^{T,\zeta} := \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta} \mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta}$$

where

$$\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta} := \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q), {}^p\Phi_{\mathrm{Tr}_{\mathbf{d}}^\zeta(W)} \mathbb{Q}^{\mathrm{vir}}),$$

and $\mathrm{Tr}(W)_{\mathbf{d}}^\zeta : \mathfrak{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q) \rightarrow \mathbb{C}$ is the trace function¹, the grading is given by dimension and cohomology and $\mathbb{Q}^{\mathrm{vir}}$ is the constant T equivariant perverse sheaf defined² by

$$\mathbb{Q}^{\mathrm{vir}}|_{\mathfrak{M}_{\mathbf{d}}^T(Q)} = \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(Q)}[-(\mathbf{d}, \mathbf{d})].$$

We then consider the convolution diagram

$$\begin{array}{ccc} & \mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T, \zeta-\mathrm{ss}}(Q) & \\ \swarrow^{\pi_1 \times \pi_3} & & \searrow^{\pi_2} \\ \mathfrak{M}_{\mathbf{d}_1}^{T, \zeta-\mathrm{ss}}(Q) \times_{[\mathrm{pt}/T]} \mathfrak{M}_{\mathbf{d}_2}^{T, \zeta-\mathrm{ss}}(Q) & & \mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}^{T, \zeta-\mathrm{ss}}(Q) \end{array}$$

¹We shall ignore the subscripts from $\mathrm{Tr}(W)_{\mathbf{d}}^\zeta$ when context is clear.

²We choose this twist since we want to think of $\mathcal{A}_{Q,W}^T$ as version of $\mathcal{A}_{Q,W}$ with equivariant parameters. For $\mathcal{A}_{Q,W}$, the sheaf $\mathbb{Q}^{\mathrm{vir}} = \mathbb{Q}[-(\mathbf{d}, \mathbf{d})]$ is perverse, while for $\mathcal{A}_{Q,W}^T$ we choose the sheaf so that restriction to the fibres of projection $\mathfrak{M}_{\mathbf{d}}^T(Q) \rightarrow \mathrm{pt}/T$ are perverse.

By taking the inverse of the Thom-Sebastiani isomorphism (3.2.1), we have the isomorphism of graded vector spaces:

$$\begin{aligned} \text{TS}^{-1}: \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}}) \otimes \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}}) \\ \simeq \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q) \times \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)_{\mathbf{d}_1} \boxplus \mathrm{Tr}(W)_{\mathbf{d}_2}} \mathbb{Q}^{\mathrm{vir}}). \end{aligned}$$

By the natural map $[\mathrm{pt}/T] \rightarrow \mathrm{pt}$, we have a forgetful morphism $j: \mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q) \times_{[\mathrm{pt}/T]} \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q) \rightarrow \mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q) \times \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q)$, so by taking the pullback (3.2.2), we have the morphism of graded vector spaces

$$\begin{aligned} j^*: \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q) \times \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)_{\mathbf{d}_1} \boxplus \mathrm{Tr}(W)_{\mathbf{d}_2}} \mathbb{Q}^{\mathrm{vir}}) \\ \simeq \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q) \times_{[\mathrm{pt}/T]} \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)_{\mathbf{d}_1} \boxplus \mathrm{Tr}(W)_{\mathbf{d}_2}} \mathbb{Q}^{\mathrm{vir}}). \end{aligned}$$

By taking pullback (3.2.2) along the morphism $\pi_1 \times \pi_3$ we have the morphism of graded vector spaces

$$\begin{aligned} (\pi_1 \times \pi_3)^*: \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{T,\zeta\text{-ss}}(Q) \times_{[\mathrm{pt}/T]} \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)_{\mathbf{d}_1} \boxplus \mathrm{Tr}(W)_{\mathbf{d}_2}} \mathbb{Q}^{\mathrm{vir}}) \rightarrow \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)|_{\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q)}} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q)})[-(\mathbf{d}_1, \mathbf{d}_1) - (\mathbf{d}_2, \mathbf{d}_2)]. \end{aligned}$$

since $\dim(\mathfrak{M}_{\mathbf{d}}(Q)) = -(\mathbf{d}, \mathbf{d})$. Then finally doing the pushforward (3.2.4) along π_2 , gives a morphism of graded vector spaces

$$\begin{aligned} (\pi_2)_*: \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)|_{\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q)}} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q)})[-(\mathbf{d}_1, \mathbf{d}_1) - (\mathbf{d}_2, \mathbf{d}_2)] \rightarrow \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q), {}^p\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}}). \end{aligned}$$

Note that above respects the cohomological grading since

$$\dim(\mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}^{T,\zeta\text{-ss}}) - 2 \dim(\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta\text{-ss}}(Q)) = -(\mathbf{d}_1, \mathbf{d}_1) - (\mathbf{d}_2, \mathbf{d}_2).$$

Composing the above morphisms together gives a morphism of cohomologically graded vector spaces

$$m_{\mathbf{d}_1, \mathbf{d}_2}^T := (\pi_2)_*(\pi_1 \times \pi_3)^* j^* \text{TS}^{-1}: \mathcal{A}_{Q, W, \mathbf{d}_1}^{T, \zeta} \otimes \mathcal{A}_{Q, W, \mathbf{d}_2}^{T, \zeta} \rightarrow \mathcal{A}_{Q, W, \mathbf{d}_1 + \mathbf{d}_2}^{T, \zeta}.$$

Taking the direct sum of maps $m_{\mathbf{d}_1, \mathbf{d}_2}$ over all $(\mathbf{d}_1, \mathbf{d}_2) \in \Lambda_\mu^\zeta \times \Lambda_\mu^\zeta$ defines a map of $\Lambda_\mu^\zeta \times \mathbb{Z}$ graded vector spaces

$$m_\mu^\zeta: \mathcal{A}_{Q, W, \mu}^{T, \zeta} \otimes \mathcal{A}_{Q, W, \mu}^{T, \zeta} \rightarrow \mathcal{A}_{Q, W, \mu}^{T, \zeta}$$

where by Λ_μ^ζ we mean the dimension grading while \mathbb{Z} is the cohomological grading. The map m_μ^ζ in fact gives $\mathcal{A}_{Q, W, \mu}^{T, \zeta}$ a structure of $\Lambda_\mu^\zeta \times \mathbb{Z}$ graded associative algebra structure [KS11][Section 2.3]. We will often denote the multiplication by \star . We note that when $\zeta = (0, \dots, 0)$ and $\mu = 0$ then $\mathfrak{M}_{\mathbf{d}}^{T, \zeta\text{-ss}}(Q) = \mathfrak{M}_{\mathbf{d}}^T(Q)$ and the direct sum is over all \mathbf{d} in \mathbb{N}^{Q_0} . We will denote the resulting algebra by $\mathcal{A}_{Q, W}^T$

and $\mathcal{A}_{Q,W,\mu}$ for the case without T . Note that by definition $\mathcal{A}_{Q,W,\mu}^{T,\zeta}$ is a $\mathrm{H}_T(\mathrm{pt})$ linear algebra.

4.1.1 Nilpotent 3d CoHA

Following [DM20], one can define a CoHA structure for any geometrically defined Serre subcategory of the quiver representations. In this thesis, we will only consider a case of this construction which is given by the Serre category $\tilde{\mathcal{N}}$ of nilpotent representations of the tripled quiver \tilde{Q} , defined in Section 2.0.8. Let $\bar{\omega} : \mathfrak{M}_{\mathbf{d}}^{\tilde{\mathcal{N}},\zeta-\mathrm{ss}}(\tilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}}^{\zeta-\mathrm{ss}}(\tilde{Q})$ denote the inclusion of the stacks, where $\mathfrak{M}_{\mathbf{d}}^{\tilde{\mathcal{N}}}(\tilde{Q})$ is the stack of ζ semistable \mathbf{d} dimensional representations of quiver \tilde{Q} . We consider the graded vector space

$$\mathcal{A}_{Q,W,\mu}^{T,\zeta,\tilde{\mathcal{N}}} := \bigoplus_{\mathbf{d} \in \Lambda_{\mu}^{\zeta}} \mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\tilde{\mathcal{N}}}$$

where we define

$$\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\tilde{\mathcal{N}}} := \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q), \bar{\omega}_*(\bar{\omega})^! \mathfrak{p}\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}}).$$

Then, in the same way as the last section, the CoHA convolution diagram constructs an associative algebra structure on $\mathcal{A}_{Q,W,\mu}^{T,\zeta,\tilde{\mathcal{N}}}$.

4.1.2 Relative CoHA

The algebra $\mathcal{A}_{Q,W,\mu}^{T,\zeta}$ admits a lift to an algebra object $\mathcal{R}\mathcal{A}_{Q,W,\mu}^{T,\zeta}$ in the derived category $\mathrm{D}^+(\mathbf{Perv}'(\mathcal{M}_{\mu}^{T,\zeta}(Q)))$ of perverse sheaves on the coarse moduli space.³ This lift is sometimes useful in studying this algebra as many of the products in algebra can then be interpreted as morphisms of semisimple Perverse sheaves, which are relatively easier to understand. Let

$$\mathfrak{M}_{\mu}^{T,\zeta-\mathrm{ss}}(Q) := \coprod_{\mathbf{d} \in \Lambda_{\mu}^{\zeta}} \mathfrak{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q)$$

and similarly let

$$\mathcal{M}_{\mu}^{\zeta-\mathrm{ss}}(Q) := \coprod_{\mathbf{d} \in \Lambda_{\mu}^{\zeta}} \mathcal{M}_{\mathbf{d}}^{\zeta-\mathrm{ss}}(Q).$$

We denote $\mathfrak{M}_{\mu}^{T,\zeta-\mathrm{ss}}(Q)$ and $\mathcal{M}_{\mu}^{T,\zeta-\mathrm{ss}}(Q)$ by $\mathfrak{M}^T(Q)$ and $\mathcal{M}_{\mu}^{T,\zeta-\mathrm{ss}}(Q)$ respectively, in the case when $\zeta = 0, \mu = 0$. Finally let $\mathrm{JH}_{\mu}^T := \coprod_{\mathbf{d} \in \Lambda_{\mu}^{\zeta}} \mathrm{JH}_{\mathbf{d}}^T : \mathfrak{M}_{\mu}^{T,\zeta-\mathrm{ss}}(Q) \rightarrow \mathcal{M}_{\mu}^{T,\zeta-\mathrm{ss}}(Q)$ to be the semisimplification morphism.

We consider the direct sum map

$$\mathcal{M}_{\mu}^{\zeta}(Q) \times \mathcal{M}_{\mu}^{\zeta}(Q) \xrightarrow{\oplus} \mathcal{M}_{\mu}^{\zeta}(Q)$$

³The shifted perverse t structure on $\mathrm{D}_c^b(\mathcal{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q))$ is defined by setting $\mathfrak{p}'_{\tau \leq i} := \mathfrak{p}_{\tau \leq i - \dim(T)}$ and $\mathfrak{p}'_{\tau \geq i} := \mathfrak{p}_{\tau \geq i - \dim(T)}$. Let $\mathbf{Perv}'(\mathfrak{X}/T)$ to be heart with respect to this shifted perverse structure. This is defined so that for any $\mathcal{F} \in \mathbf{Perv}'(\mathfrak{X}/T)$, $(\mathfrak{X} \rightarrow \mathfrak{X}/T)^*(\mathcal{F}) \in \mathbf{Perv}(\mathfrak{X})$.

It is proved in [MR19] that this morphism is a finite and commutative morphism of schemes. This morphism lifts to the T quotients

$$\mathcal{M}_\mu^{T,\zeta}(Q) \times_{[\text{pt}/T]} \mathcal{M}_\mu^{T,\zeta}(Q) \xrightarrow{\oplus^T} \mathcal{M}_\mu^{T,\zeta}(Q)$$

is also finite and commutative. We consider the monoidal product

$$\mathcal{F} \boxtimes_{\oplus^T} \mathcal{G} := \oplus_*^T(\mathcal{F} \boxtimes_{[\text{pt}/T]} \mathcal{G})$$

over $[\text{pt}/T]$ on $\mathbf{D}^+(\mathbf{Perv}'(\mathcal{M}_\mu^{T,\zeta}(Q)))$, allowing us to consider algebra objects in $\mathbf{D}^+(\mathbf{Perv}'(\mathcal{M}_\mu^{T,\zeta}(Q)))$. Since the morphism \oplus^T is finite, it implies that the monoidal product is bi-exact and symmetric.

Let $\Lambda_\mu^\zeta \times [\text{pt}/T]$ be the stack consisting of the copy of $[\text{pt}/T]$ for every dimension vector $\mathbf{d} \in \Lambda_\mu^\zeta$. Then the Λ_μ^ζ graded algebra structure on $\mathcal{A}_{Q,W,\mu}^{T,\zeta-ss}$ can be understood as the algebra structure on the object $p_* {}^p\Phi_{\text{Tr}(W)} \mathbb{Q}_{\mathfrak{M}_\mu^{T,\zeta}(Q)}^{\text{vir}} \in \mathbf{D}^+(\Lambda_\mu^\zeta \times [\text{pt}/T])$ where $p : \mathfrak{M}_\mu^\zeta(Q) \rightarrow \Lambda_\mu^\zeta \times [\text{pt}/T]$ is the canonical morphism and $\Lambda_\mu^\zeta \times [\text{pt}/T]$ is a monoid over $[\text{pt}/T]$ given by the sum of dimension vectors. We now consider something intermediate.

Let

$$\mathcal{R}\mathcal{A}_{Q,W,\mu}^{T,\zeta} := (\text{JH}_{\mu}^{T,\zeta})_* {}^p\Phi_{\text{Tr}(W)}(\mathbb{Q}_{\mathfrak{M}_\mu^{T,\zeta}(Q)}^{\text{vir}}).$$

Since, after doing semisimplification, non-trivial extensions are the same as the direct sum, the CoHA convolution diagram lifts to the commutative diagram

$$\begin{array}{ccc} & \text{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^{T,\zeta-ss}(Q) & \\ \pi_1 \times \pi_3 \swarrow & & \searrow \pi_2 \\ \mathfrak{M}_{\mathbf{d}_1}^{T,\zeta-ss}(Q) \times_{[\text{pt}/T]} \mathfrak{M}_{\mathbf{d}_2}^{T,\zeta-ss}(Q) & & \mathfrak{M}_{\mathbf{d}_1+\mathbf{d}_2}^{T,\zeta-ss}(Q) \\ \text{JH}_{\mathbf{d}_1}^T \times \text{JH}_{\mathbf{d}_2}^T \downarrow & & \downarrow \text{JH}_{\mathbf{d}_1+\mathbf{d}_2}^T \\ \mathcal{M}_{\mathbf{d}_1}^{T,\zeta-ss}(Q) \times_{[\text{pt}/T]} \mathcal{M}_{\mathbf{d}_2}^{T,\zeta-ss}(Q) & \xrightarrow{\oplus^T} & \mathcal{M}_{\mathbf{d}_1+\mathbf{d}_2}^{T,\zeta-ss}(Q) \end{array}$$

Now, we can do the same game as in the definition of Kontsevich-Soibelman CoHA, but instead we push forward underlying sheaves to $\mathcal{M}_\mu^{T,\zeta-ss}(Q)$. It is proved in [DM20] and [Dav22b] that this defines a morphism

$$\mathcal{R}m : \mathcal{R}\mathcal{A}_{Q,W,\mu}^{T,\zeta} \boxtimes_{\oplus^T} \mathcal{R}\mathcal{A}_{Q,W,\mu}^{T,\zeta} \rightarrow \mathcal{R}\mathcal{A}_{Q,W,\mu}^{T,\zeta}$$

which makes $\mathcal{R}\mathcal{A}_{Q,W,\mu}^{T,\zeta}$ an algebra object inside the monoidal category $\mathbf{D}^+(\mathbf{Perv}'(\mathcal{M}_\mu^{T,\zeta}(Q)))$. We will often refer to this as the relative CoHA. Taking cohomology of the morphism $\mathcal{R}m$ above, we obtain the usual CoHA $\mathcal{A}_{Q,W,\mu}^{T,\zeta}$.

4.1.3 Tautological bundles

Given a family \mathcal{F} of \mathbf{d} dimensional ζ semistable representations of $\mathbb{C}Q$ we can obtain a line bundle by taking $\text{Det}(\mathcal{F})$. This gives a map $\text{Det} : \mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q) \rightarrow \text{BC}_u^*$

where $\mathrm{BC}_u^* = \mathrm{pt}/\mathbb{C}_u^*$ and we use variable u to emphasise that $u \in \mathrm{H}^2(\mathrm{BC}_u^*, \mathbb{Q})$ is the tautological generator for this \mathbb{C}^* . Thus, we have a map of stacks

$$(\mathrm{Det} \times \mathrm{id}): \mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q) \rightarrow \mathrm{BC}_u^* \times \mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q) \quad (4.1)$$

which induces a natural map in the constructible category

$${}^{\mathrm{p}}\Phi_{0 \in \mathrm{Tr}(W)_{\mathbf{d}}} \mathbb{Q}_{\mathrm{BC}_u^* \times \mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q)} \rightarrow (\mathrm{Det} \times \mathrm{id})_* {}^{\mathrm{p}}\Phi_{\mathrm{Tr}(W)_{\mathbf{d}}} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q)} \quad (4.2)$$

while by Thom-Sebastiani ${}^{\mathrm{p}}\Phi_{0 \in \mathrm{Tr}(W)_{\mathbf{d}}} \mathbb{Q}_{\mathrm{BC}_u^* \times \mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q)} \simeq \mathbb{Q}_{\mathrm{BC}_u^*} \boxtimes {}^{\mathrm{p}}\Phi_{\mathrm{Tr}(W)_{\mathbf{d}}} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q)}$ and so after shifting by $[(\mathbf{d}, \mathbf{d})]$ and taking global sections, we get

$$(\mathrm{Det} \times \mathrm{id})^*: \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q}) \otimes \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q), {}^{\mathrm{p}}\Phi_{\mathrm{Tr}(W)_{\mathbf{d}}}) \rightarrow \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^{T, \zeta - \mathrm{ss}}(Q), {}^{\mathrm{p}}\Phi_{\mathrm{Tr}(W)_{\mathbf{d}}})$$

This gives an action of $\mathbb{Q}[u] = \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q})$ on $\mathcal{A}_{Q, W, \mu}^{T, \zeta}$ which we denote by \cdot . We will see later that this action acts by derivation.

We may also pushforward the morphism 4.2 by $\mathrm{JH}_{\mathbf{d}}^{T, \zeta}$ to induce similar action of $\mathbb{Q}[u]$ on $\mathcal{R}\mathcal{A}_{Q, W, \mu}^{T, \zeta}$.

4.1.4 Perverse filtration and BPS Lie algebra

Sign Twist

We slightly twist the multiplication on $\mathcal{A}_{Q, W}^{T, \zeta}$ so that the resulting algebra satisfies the PBW theorem. Given dimension vectors $\mathbf{d}', \mathbf{d}''$, we define

$$\tau(\mathbf{d}', \mathbf{d}'') = \chi_Q(\mathbf{d}', \mathbf{d}')\chi_Q(\mathbf{d}'', \mathbf{d}'') + \chi_Q(\mathbf{d}', \mathbf{d}'')$$

Then a bilinear form ψ on $(\mathbb{Z}/2\mathbb{Z})^{Q_0}$ is defined so that we have

$$\psi(\mathbf{d}', \mathbf{d}'') + \psi(\mathbf{d}'', \mathbf{d}') \equiv \tau(\mathbf{d}', \mathbf{d}'') \pmod{2}.$$

Then $\mathcal{A}_{Q, W}^{T, \zeta, \psi}$ is defined as an algebra on the same vector space, but with the multiplication $m_{\mathbf{d}', \mathbf{d}''}^{\psi} = (-1)^{\psi(\mathbf{d}', \mathbf{d}'')} m_{\mathbf{d}', \mathbf{d}''}$. Similarly we define $\mathcal{R}\mathcal{A}_{Q, W, \mu}^{T, \zeta, \psi}$ to be the algebra object with the same underlying complex of perverse sheaves on $\mathcal{R}\mathcal{A}_{Q, W, \mu}^{T, \zeta, \psi}$ but with the modified multiplication $\mathcal{R}m_{\mathbf{d}', \mathbf{d}''}^{\psi} = (-1)^{\psi(\mathbf{d}', \mathbf{d}'')} \mathcal{R}m_{\mathbf{d}', \mathbf{d}''}$.

For a general situation of quiver Q with potential W , we fix a choice of ψ . For the tripled quiver \tilde{Q} and canonical potential \tilde{W} , we have

$$\tau_{\tilde{Q}}(\mathbf{d}', \mathbf{d}'') \equiv \sum_{\alpha: i \rightarrow j} \mathbf{d}'_i \mathbf{d}''_j + \mathbf{d}'_j \mathbf{d}''_i$$

and so we choose

$$\psi(\mathbf{d}', \mathbf{d}'') = \chi_Q(\mathbf{d}', \mathbf{d}'') = \sum_{i \in Q_0} \mathbf{d}'_i \mathbf{d}''_i - \sum_{\alpha: i \rightarrow j} \mathbf{d}'_i \mathbf{d}''_j.$$

when the quiver Q is fixed, we will denote this sign twist by χ .

4.1.5 Perverse filtration and integrality

We assume that quiver Q is symmetric from now on⁴. The morphism $\mathrm{JH}_{\mathbf{d}}^{T,\zeta}$ behaves like a proper map [DM20] by which we mean that it satisfies an analog of the decomposition theorem, i.e. the direct image $(\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)}$ splits, that is we have a non-canonical isomorphism in $\mathrm{D}^b(\mathbf{Perv}'(\mathcal{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q)))$:

$$(\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{P}'\mathcal{H}^i((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)})[-i]. \quad (4.3)$$

Furthermore, it is proved in [DM20] that $\mathbb{P}'\mathcal{H}^i((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)}) = 0$ for all $i < 1$, so the above splitting starts from $i \geq 1$. For any i , applying the natural transformation $\mathbb{P}'(\tau^{\leq i}) \rightarrow \mathrm{id}$, gives a split morphism

$$\mathbb{P}'_{\tau^{\leq i}}((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}}) \rightarrow ((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}}).$$

Thus pushing forward all the way to the point, gives an injection

$$\mathfrak{P}^i \mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi} := \mathrm{H}(\mathcal{M}_{\mathbf{d}}^{T,\zeta-\mathrm{ss}}(Q), \mathbb{P}'_{\tau^{\leq i}}((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W)} \mathbb{Q}^{\mathrm{vir}})) \rightarrow \mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi},$$

giving an increasing filtration

$$\mathfrak{P}^\bullet(\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi}) := 0 \subset \mathfrak{P}^1(\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi}) \subset \mathfrak{P}^2(\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi}) \cdots \subset \mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi}$$

of $\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi}$. This filtration respects the algebra structure on $\mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}$. It is shown in [Dav22a][Theorem 3.3] that in fact the vector space $\bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta} \mathfrak{P}^1(\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi})$ is preserved under the Lie bracket induced by the associative algebra structure on $\mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}$. The BPS Lie algebra is defined to be the $\Lambda_\mu^\zeta \times \mathbb{Z}$ graded vector subspace⁵

$$\mathfrak{g}_{Q,W,\mu}^{\mathrm{BPS},T,\zeta} := \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \mathbf{0}} \mathfrak{P}^1(\mathcal{A}_{Q,W,\mathbf{d}}^{T,\zeta,\psi}) \subset \mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}.$$

Furthermore, by the action of tautological bundle (Section 4.1.3), we have a morphism of graded vector spaces $\mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes \mathfrak{g}_{Q,W,\mu}^{\mathrm{BPS},T,\zeta} \rightarrow \mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}$ which induces a morphism from the tensor algebra

$$\mathrm{T}_{\mathrm{H}_T(\mathrm{pt})}(\mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes \mathfrak{g}_{Q,W,\mu}^{\mathrm{BPS},T,\zeta}) \rightarrow \mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}. \quad (4.4)$$

Let $\mathrm{Sym}_{\mathrm{H}_T(\mathrm{pt})}(\mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes \mathfrak{g}_{Q,W,\mu}^{\mathrm{BPS},T,\zeta}) \subset \mathrm{T}(\mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes \mathfrak{g}_{Q,W,\mu}^{\mathrm{BPS},T,\zeta})$ be the subspace of symmetric tensors, then restricting morphism (4.4) yields a map of $\Lambda_\mu^\zeta \times \mathbb{Z}$ graded $\mathrm{H}_T(\mathrm{pt})$ modules

$$\mathrm{Sym}_{\mathrm{H}_T(\mathrm{pt})}(\mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes \mathfrak{g}_{Q,W,\mu}^{\mathrm{BPS},T,\zeta}) \rightarrow \mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}. \quad (4.5)$$

We then have

⁴This is not essential and there are more general results involving μ generic stability conditions in [DM20]

⁵Since we define the BPS Lie algebra after fixing the twist ψ , we omit it from the notation.

Theorem 4.1.1 (PBW Isomorphism [Dav22a], [DM20]). *The morphism (4.5) is an isomorphism of $\Lambda_\mu^\zeta \times \mathbb{Z}$ graded $\mathbf{H}_T(\text{pt})$ modules. In fact, let $\text{Gr}_{\mathfrak{P}}(\mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi})$ be the associated graded algebra with respect to the Perverse filtration. Then there is an isomorphism of algebras*

$$\text{Gr}_{\mathfrak{P}}(\mathcal{A}_{Q,W,\mu}^{T,\zeta,\psi}) \simeq \mathbf{Sym}_{\mathbf{H}_T(\text{pt})}(\mathbf{H}(\text{BC}_u^*, \mathbb{Q}) \otimes \mathfrak{g}_{Q,W,\mu}^{\text{BPS},T,\zeta}) \quad (4.6)$$

where for any $\mathbf{H}_T(\text{pt})$ module V , $\mathbf{Sym}_{\mathbf{H}_T(\text{pt})}(V) = T_{\mathbf{H}_T(\text{pt})}(V) / \langle x \otimes y - y \otimes x = 0 \mid x, y \in V \rangle$ is the quotient of the tensor algebra.

Relative Integrality Theorem

There is a lift of Proposition 4.1.1 to the Relative CoHA. Following [DM20], we define the BPS sheaf $\mathcal{BPS}_{Q,W,\mu}^{T,\zeta}$ to be

$$\mathcal{BPS}_{Q,W,\mu}^{T,\zeta} := {}^{p'}\tau^{\leq 1}((\text{JH}_\mu^{T,\zeta})_*({}^p\Phi_{\text{Tr}(W)})) [1]. \quad (4.7)$$

So that

$$\mathfrak{g}_{Q,W,\mu}^{\text{BPS},T,\zeta} \simeq \mathbf{H}(\mathfrak{M}_\mu^{T,\zeta-\text{ss}}(Q), \mathcal{BPS}_{Q,W,\mu}^{T,\zeta}[-1]). \quad (4.8)$$

By definition, we have a canonical morphism

$$\mathcal{BPS}_{Q,W,\mu}^{T,\zeta}[-1] \rightarrow \mathcal{RA}_{Q,W,\mu}^{T,\zeta}.$$

Applying the action of $\mathbb{Q}[u]$ introduced in Section 4.1.3, yields a morphism of complexes

$$\mathbf{H}(\text{BC}_u^*, \mathbb{Q}) \otimes \mathcal{BPS}_{Q,W,\mu}^{T,\zeta}[-1] \rightarrow \mathcal{RA}_{Q,W,\mu}^{T,\zeta}.$$

Since the category $\mathbf{D}^+(\mathbf{Perv}'(\mathcal{M}_{Q,W,\mu}^{T,\zeta}(Q)))$ is symmetric monoidal over $[\text{pt}/T]$, for any object $\mathcal{F} \in \mathbf{D}^+(\mathbf{Perv}'(\mathcal{M}_{Q,W,\mu}^{T,\zeta}(Q)))$, we may define

$$\text{Sym}_{\oplus T}(\mathcal{F}) := \bigoplus_{i \geq 0} \text{Sym}_{\oplus T}^i(\mathcal{F})$$

where $\text{Sym}_{\oplus T}^i(\mathcal{F})$ is the S_i invariant part of

$$\underbrace{\mathcal{F} \boxtimes_{\oplus T} \cdots \boxtimes_{\oplus T} \mathcal{F}}_{i \text{ times}}.$$

Thus, in exactly the same way as before, since $\mathcal{RA}_{Q,W,\mu}^{T,\zeta}$ has relative CoHA structure, the above morphism lifts to a morphism

$$\text{Sym}_{\oplus T} \left(\mathbf{H}(\text{BC}_u^*, \mathbb{Q}) \otimes \mathcal{BPS}_{Q,W,\mu}^{T,\zeta}[-1] \right) \rightarrow \mathcal{RA}_{Q,W,\mu}^{T,\zeta} \quad (4.9)$$

of objects in $\mathbf{D}^+(\mathbf{Perv}'(\mathcal{M}_{Q,W,\mu}^{T,\zeta}(Q)))$. We then have

Theorem 4.1.2 ([DM20]). *The morphism 4.9 is an isomorphism and for any $\mathbf{d} \in \Lambda_\mu^\zeta \setminus 0$, there is an isomorphism of perverse sheaves*

$$\mathcal{BPS}_{Q,W,\mathbf{d}}^{T,\zeta} \simeq \begin{cases} {}^p\Phi_{\mathrm{Tr}_d^\zeta(W)} \mathcal{IC}_{\mathcal{M}_d^{T,\zeta-ss}(Q)} & \text{if } \mathcal{M}_d^{\zeta-st}(Q) \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.10)$$

where $\mathrm{Tr}_d^\zeta(W) : \mathcal{M}_d^{T,\zeta-ss}(Q) \rightarrow \mathbb{C}$ is the unique function through which the of morphism $\mathrm{Tr}_d^\zeta(W) : \mathfrak{M}_d^{T,\zeta-ss}(Q) \rightarrow \mathbb{C}$ factors.

Remark 4.1.3. • This theorem builds on the work of Reineke and Meinhardt in [MR19], who proved a version of this theorem for the case when $W = 0$. Then one proves the above statement by showing that the vanishing cycle functor, in some sense, commutes with the semisimplification morphism.

- We have stated this theorem at the level of Perverse sheaves, but these isomorphisms lift to the level of Monodromic Mixed Hodge modules, which is how they are proved.

Example 15. Let Q be the Jordan quiver and $W = x^n$ for some $n > 1$. Then calculating the critical cohomology $H(\mathfrak{M}_d(Q_{\mathrm{Jor}}), {}^p\Phi_{\mathrm{Tr}(x^n)})$ by definitions is already quite difficult. However, by the PBW theorem, we have an isomorphism of graded vector spaces

$$\bigoplus_{d \geq 0} H(\mathfrak{M}_d(Q_{\mathrm{Jor}}), {}^p\Phi_{\mathrm{Tr}(x^n)}) \simeq \mathrm{Sym}(\mathfrak{g}_{Q_{\mathrm{Jor}},x^n}^{\mathrm{BPS}} \otimes H(\mathrm{BC}_u^*, \mathbb{Q})).$$

Furthermore, there aren't any simple modules of dimension $d > 1$ for the Jordan quiver since the path algebra is the polynomial ring $\mathbb{C}[x]$. Thus

$$\mathfrak{g}_{Q_{\mathrm{Jor}},x^n}^{\mathrm{BPS}} = \mathfrak{g}_{Q_{\mathrm{Jor}},x^n,1}^{\mathrm{BPS}} = H(\mathbb{A}^1, {}^p\Phi_{x^n}) = \mathbb{Q}^{n-1}$$

by Theorem 4.1.2 and Example 10. Thus we have

$$\bigoplus_{d \geq 0} H(\mathfrak{M}_d(Q_{\mathrm{Jor}}), {}^p\Phi_{\mathrm{Tr}(x^n)}) \simeq \mathrm{Sym}(\mathbb{Q}^{n-1} \otimes \mathbb{Q}[u])$$

which completely determines the vanishing cycle cohomology $H(\mathfrak{M}_d(Q_{\mathrm{Jor}}), {}^p\Phi_{\mathrm{Tr}(x^n)})$.

Note that $\mathfrak{g}_{Q_{\mathrm{Jor}},x^n}^{\mathrm{BPS}}$ is a graded Lie algebra, but there aren't any components for dimension > 1 . Thus infact $\mathfrak{g}_{Q_{\mathrm{Jor}},x^n}^{\mathrm{BPS}} \simeq \mathbb{Q}^{n-1}$ is a trivial Lie algebra.

Although it may look like that above theorem completely determines \mathcal{BPS} sheaves, in practice it is quite difficult to calculate the vanishing cycle of the intersection cohomology sheaves. We recall the following theorem of Toda, which relates the BPS sheaves with or without stability.

Proposition 4.1.4 ([Tod23]). *For any symmetric quiver Q , potential W , $\zeta \in \mathbb{Q}^{\mathcal{Q}_0}$ a stability condition and $\mathbf{d} \in \mathbb{N}^{\mathcal{Q}_0}$ a dimension vector, there is a natural isomorphism*

$$(q_{\mathbf{d}}^\zeta)_* \mathcal{BPS}_{Q,W,\mathbf{d}}^\zeta \simeq \mathcal{BPS}_{Q,W,\mathbf{d}}$$

where $q_{\mathbf{d}}^\zeta$ is the GIT affinization map

$$q_{\mathbf{d}}^\zeta : \mathcal{M}_d^{\zeta-ss}(Q) \rightarrow \mathcal{M}_d(Q).$$

Note that the Lie bracket on $\mathfrak{g}_{Q,W,\mu}^{\text{BPS},\zeta}$ lifts to a morphism of complexes in $\text{D}^b(\mathbf{Perv}(\mathcal{M}_\mu^{\zeta\text{-ss}}(Q)))$

$$[\cdot, \cdot]: \mathcal{BPS}_{Q,W,\mu}^\zeta[-1] \boxtimes_{\oplus} \mathcal{BPS}_{Q,W,\mu}^\zeta[-1] \rightarrow \mathcal{BPS}_{Q,W,\mu}^\zeta[-1]$$

which after pushforward along the map $q_{\mathbf{d}}$ and applying Toda's theorem, gives

Proposition 4.1.5. *There is an isomorphism of Lie algebras*

$$\mathfrak{g}_{Q,W,\mu}^{\text{BPS},\zeta} \simeq \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \mathbf{0}} \mathfrak{g}_{Q,W,\mathbf{d}}^{\text{BPS}}. \quad (4.11)$$

The structure of these Lie algebras is not known and is an active area of research. However, a lot more can be said in the case of tripled quiver with potential, i.e, when $Q = \tilde{Q}$ and $W = \tilde{W}$. In fact, in that case, it gives rise to the CoHA of preprojective algebras.

4.2 Preprojective Cohomological Hall algebra

Let $\bar{w} : \bar{Q} \rightarrow N$ be a torus weighting on the doubled quiver such that the preprojective algebra relation $\sum_{a \in Q_1} [a, a^*]$ is homogeneous with respect to this weighting. We then consider the $\Lambda_\mu^\zeta \times \mathbb{Z}$ graded $\text{H}_T(\text{pt})$ module ⁶

$$\mathcal{A}_{\Pi_Q}^T := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \text{H}^{\text{BM}}(\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q), \mathbb{Q}^{\text{vir}})$$

where we define ⁷ $\mathbb{Q}^{\text{vir}}|_{\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q)} := \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q)}[-2(\mathbf{d}, \mathbf{d})]$. In [YZ18b] (For any quiver) and [SV13] (For the Jordan quiver), by doing virtual pullback and push forward, the authors define a $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded associative algebra structure $\mathcal{A}_{\Pi_Q}^T$ which we refer to as the preprojective cohomological Hall algebra. We can also twist the multiplication in $\mathcal{A}_{\Pi_Q}^T$ as we did for $\mathcal{A}_{\tilde{Q}, \tilde{W}}$ in Section 4.1.4. Let us denote the resulting algebra by $\mathcal{A}_{\Pi_Q}^{T,\chi}$. In a similar way, in [SV20], authors define a $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded algebra structure on

$$\mathcal{A}_{\Pi_Q}^{T,\mathcal{N}} := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \text{H}^{\text{BM}}(\mathfrak{M}_{\mathbf{d}}^{T,\mathcal{N}}(\Pi_Q), \mathbb{Q}^{\text{vir}}).$$

These algebras can be seen as a particular case of the Kontsevich-Soibelman cohomological Hall algebra.

⁶There also exist versions with a stability condition, however, we don't need to consider that for our applications.

⁷Note that the quotient stack $\mathfrak{M}_{\mathbf{d}}(\Pi_Q)$ is not smooth, so the defined shifted complex is not the same as the intersection complex. Note that $\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q)}^{\text{vir}} = (\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q)})[2(\mathbf{d}, \mathbf{d})]$.

4.2.1 Tripled quiver with canonical cubic potential

In Example 2, we saw that for the tripled quiver \widetilde{Q} with canonical cubic potential \widetilde{W} , we have an isomorphism of algebras $\text{Jac}(\widetilde{Q}, \widetilde{W}) \simeq \Pi_Q[\omega]$; this in fact allows us to see the algebra $\mathcal{A}_{\Pi_Q}^T$ as the dimension reduction of $\mathcal{A}_{\widetilde{Q}, \widetilde{W}, \mu}^T$ where the torus T action on $\mathfrak{M}_{\mathbf{d}}(\widetilde{Q})$ is defined by uniquely extending the action on $\mathfrak{M}_{\mathbf{d}}(\overline{Q})$ such that it leaves the potential \widetilde{W} invariant. Let

$$\pi : \mathfrak{M}_{\mathbf{d}}^T(\widetilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}}^T(\overline{Q})$$

be natural forgetful morphism. We have decomposition

$$\mathfrak{M}_{\mathbf{d}}^T(\widetilde{Q}) = \mathfrak{M}_{\mathbf{d}}(\overline{Q}) \times \mathfrak{M}_{\mathbf{d}}(L) \quad (4.12)$$

where L is quiver with vertices Q_0 and loops ω_i on each vertex i . We may consider the \mathbb{C}^* action, which scales the loops. Then applying dimension reduction (3.3) gives an isomorphism of complexes

$$\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q)}^{\text{vir}} \simeq \pi_* \, {}^p\Phi_{\text{Tr}(\widetilde{W})} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(\overline{Q})}^{\text{vir}}. \quad (4.13)$$

Taking cohomology, gives an isomorphism of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded $H_T(\text{pt})$ modules

$$\text{DR}_{\mathbf{d}} : H(\mathfrak{M}_{\mathbf{d}}^T(\widetilde{Q}), {}^p\Phi_{\text{Tr}(W)}) \simeq H(\mathfrak{M}_{\mathbf{d}}^T(\Pi_Q), \mathbb{Q}^{\text{vir}}). \quad (4.14)$$

In [RS17][Appendix A] and [YZ20a], it shown that taking sum of maps $\widetilde{\text{DR}}_{\mathbf{d}} := \text{DR}_{\mathbf{d}} \cdot \binom{\mathbf{d}}{2}$ over all $\mathbf{d} \in \Lambda_{\mu}^{\zeta}$, where $\binom{\mathbf{d}}{2} := \prod_{i \in Q_0} \binom{d_i}{2}$ gives an isomorphism of $\Lambda_{\mu}^{\zeta} \times \mathbb{Z}$ graded algebras

$$\widetilde{\text{DR}} : \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^T \simeq \mathcal{A}_{\Pi_Q}^{T, \zeta}.$$

A similar analysis for the case of nilpotent representations, as done in [Dav22a], one obtains an isomorphism of algebras

$$\widetilde{\text{DR}} : \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \mathcal{N}} \simeq \mathcal{A}_{\Pi_Q}^{T, \mathcal{N}}.$$

4.2.2 Support lemma and Less Perverse Filtration

In this section, we restrict ourselves to the non-equivariant setting. We remark that these results also hold when any torus T . We saw that morphism 4.13 is an isomorphism of complexes. Combined with the Integrality Theorem 4.1.2, this gives a integrality theorem for the pushforward $\text{JH}_{*}^{\overline{Q}} \mathbb{D}\mathbb{Q}_{\mathfrak{M}^T(\Pi_Q)}^{\text{vir}}$, where we treat $\mathbb{D}\mathbb{Q}_{\mathfrak{M}^T(\Pi_Q)}^{\text{vir}}$ as a complex on $\mathfrak{M}(\overline{Q})$. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}(\widetilde{Q}) & \xrightarrow{\pi} & \mathfrak{M}(\overline{Q}) \\ \text{JH}_{\overline{Q}} \downarrow & & \downarrow \text{JH}_{\overline{Q}} \\ \mathcal{M}(\widetilde{Q}) & \xrightarrow{\pi'} & \mathcal{M}(\overline{Q}) \end{array}$$

Thus, we have an isomorphism of complexes

$$\mathrm{JH}_*^{\overline{Q}} \mathbb{D}\mathcal{Q}_{\mathfrak{M}(\Pi_Q)}^{\mathrm{vir}} \simeq \pi'_* \mathrm{JH}_*^{\overline{Q}} \mathcal{Q}_{\mathfrak{M}_d(\overline{Q})}^{\mathrm{vir}} \simeq \mathrm{Sym}_{\oplus} \left(q_* \mathcal{BPS}_{\overline{Q}, \widetilde{W}}[-1] \otimes \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q}) \right)$$

It turns out that the complex $q_* \mathcal{BPS}_{\overline{Q}, \widetilde{W}}[-1]$ itself becomes a perverse sheaf on $\mathcal{M}(\overline{Q})$. This is explained by the support lemma. Let $l : \mathfrak{M}_d(\overline{Q}) \times \mathbb{A}^1 \rightarrow \mathfrak{M}_d(\widetilde{Q})$ be an embedding of stacks induced by the closed inclusion $\mathrm{Rep}_d(\overline{Q}) \times \mathbb{A}^1 \rightarrow \mathrm{Rep}_d(\widetilde{Q})$ defined by sending a representation ρ of \overline{Q} and a scalar λ to a representation of \widetilde{Q} whose underlying \overline{Q} representation is ρ and the loops ω_i act by multiplication with the scalar λ . Then it is proved in [Dav23c][Lemma 4.1] that

Theorem 4.2.1 (Support Lemma). *There exist a unique complex*

$$\mathcal{BPS}_{\Pi_Q} \in \mathrm{D}^+(\mathbf{Perv}(\mathcal{M}(\overline{Q})))$$

such that

$$\mathcal{BPS}_{\overline{Q}, \widetilde{W}} \simeq l_*(\mathcal{BPS}_{\Pi_Q} \otimes \mathbb{Q}_{\mathbb{A}^1}[1]).$$

Thus, we have an isomorphism of complexes

$$\mathrm{JH}_*^{\overline{Q}} \mathbb{D}\mathcal{Q}_{\mathfrak{M}(\Pi_Q)}^{\mathrm{vir}} \simeq \mathrm{Sym}(\mathcal{BPS}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q})) \quad (4.15)$$

Furthermore, it is shown in [Dav22b] that \mathcal{BPS}_{Π_Q} is in fact a semisimple perverse sheaf. Thus it follows that the complex $(\mathrm{JH}_d^{\overline{Q}})_* \mathbb{D}\mathcal{Q}_{\mathfrak{M}_d(\Pi_Q)}^{\mathrm{vir}}$ splits, infact we have

$$(\mathrm{JH}_d^{\overline{Q}})_* \mathbb{D}\mathcal{Q}_{\mathfrak{M}_d(\Pi_Q)}^{\mathrm{vir}} \simeq \bigoplus_{i \in 2\mathbb{Z}_{\geq 0}} \bigoplus_{j \in \mathcal{S}_i} \mathcal{IC}_{Z_j}(\mathcal{L}_j[\dim(Z_j)])[-i].$$

where $Z_j \subset \mathcal{M}_d(\Pi_Q)$ are locally closed irreducible smooth subvarieties of $\mathcal{M}_d(\Pi_Q)$ and \mathcal{L}_j are simple local systems on Z_j . As we did in Section 4.1.5, we can apply the truncation functors to get another filtration. It follows that the morphism

$$\mathcal{L}^i \mathcal{A}_{\Pi_Q, d}^X := \mathrm{H}(\mathcal{M}_d(\Pi_Q), {}^{\mathrm{p}_T \leq i}(\mathrm{JH}_d^{\overline{Q}})_* \mathbb{D}\mathcal{Q}_{\mathfrak{M}_d(\Pi_Q)}^{\mathrm{vir}}) \rightarrow \mathcal{A}_{\Pi_Q, d}^X$$

is an injection, giving an increasing filtration, called the *Less Perverse Filtration* which starts from degree 0, i.e we have

$$\mathcal{L}^\bullet \mathcal{A}_{\Pi_Q, d} := 0 \subset \mathcal{L}^0 \mathcal{A}_{\Pi_Q, d} \subset \mathcal{L}^1 \mathcal{A}_{\Pi_Q, d} \subset \cdots \mathcal{A}_{\Pi_Q, d}.$$

By exactness of monoidal product, it is shown in [Dav22b][Proposition 5.1] that the algebra structure on $\mathcal{A}_{\Pi_Q}^X$ is compatible with the Less perverse filtration. Since the algebras $\mathcal{A}_{\Pi_Q}^X$ and $\mathcal{A}_{\overline{Q}, \widetilde{W}}^X$ are isomorphic via dimension reduction, this gives a filtration \mathcal{L}' on the algebra $\mathcal{A}_{\overline{Q}, \widetilde{W}}^X$. Note that this filtration, which we call the Less Perverse filtration on $\mathcal{A}_{\overline{Q}, \widetilde{W}}^X$, is different from the Perverse filtration defined in Section 4.1.5. For example by definition,

$$\mathcal{L}'^0(\mathcal{A}_{\overline{Q}, \widetilde{W}}) \simeq \mathrm{Sym}(\mathfrak{g}_{\overline{Q}, \widetilde{W}}^{\mathrm{BPS}})$$

while the $\mathfrak{P}^0(\mathcal{A}_{\tilde{Q}, \tilde{W}}) = 0$. Let us denote $\text{Gr}_{\mathcal{L}'}(\mathcal{A}_{\tilde{Q}, \tilde{W}})$ the associated graded algebra with respect to the Less perverse filtration. In Theorem 6.5.1, we shall determine the associated graded algebra $\text{Gr}_{\mathcal{L}}(\mathcal{A}_{\Pi_Q})$, providing an analog of PBW theorem 4.1.1.

4.2.3 Kac polynomials and Graded Dimension of BPS Lie algebra

Given any quiver Q , for any finite field \mathbb{F}_q and any dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$. A representation is said to be *indecomposable* if it cannot be non-trivially written as a direct sum of two representations of Q . A representation ρ is said to be *absolutely indecomposable* if it remains indecomposable after tensoring with the closure of \mathbb{F}_q , i.e., $\rho \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is indecomposable.

Let $a_{Q, \mathbf{d}}(q)$ be the number of isomorphism classes of absolutely indecomposable \mathbf{d} dimensional representations of Q over \mathbb{F}_q . There is an intricate relationship between the Kac polynomials and the Lie algebra associated with the quiver, which can be considered as the generalization of Gabriel's Theorem, which classifies finite type quivers. We recall

Definition 4.2.2 (Associated Kac-Moody Lie algebra). For any quiver Q , the real subquiver Q' is defined to be the subquiver of Q containing those vertices of Q that do not support any edge loops, along with all arrows between these vertices. Let $\mathfrak{g}_{Q'}$ be the associated Kac-Moody Lie algebra. It is defined as a Lie algebra generated by symbols $e_i, f_i, h_i; i \in Q'_0$ with the relations

$$\begin{aligned} [h_i, e_j] &= a_{ij}e_j \\ [h_i, f_j] &= -a_{ij}f_j \\ [e_i, f_j] &= \delta_{ij}h_i \\ [e_i, -]^{a_{ij}+1}(e_j) &= 0; \\ [f_i, -]^{a_{ij}+1}(f_j) &= 0; \end{aligned}$$

where a_{ij} is the number of edges joining i and j in the graph obtained from Q' . We identify $\mathbb{Z}^{Q'_0}$ along with the symmetrized Euler form $\chi_Q(\mathbf{d}, \mathbf{e}) + \chi_Q(\mathbf{e}, \mathbf{d})$ with the root lattice $\Delta_{\mathfrak{g}_{Q'}}$ of the Lie algebra $\mathfrak{g}_{Q'}$ along with the Cartan pairing via the morphism $\mathbf{d} \mapsto \sum \mathbf{d}_i \alpha_i$ where α_i are the simple roots of $\mathfrak{g}_{Q'}$. Let $\mathfrak{n}_{Q'}^+$ be the Lie subalgebra generated by e_i and $\mathfrak{n}_{Q'}^-$ denote the Lie subalgebra generated by f_i .

Then it is a Theorem of Kac that:

Theorem 4.2.3 ([Kac80]). $a_{Q, \mathbf{d}}(q)$ is a monic polynomial in q of degree $1 - (\mathbf{d}, \mathbf{d})$ with coefficients in \mathbb{Z} . It doesn't depend on the orientation of Q . For a quiver without loops, $a_{Q, \mathbf{d}}(q)$ is non-zero if and only if \mathbf{d} is a positive root of the associated Lie algebra \mathfrak{g}_Q .

Example 16 (Finite type). If Q is finite type ADE quiver then by theorem of

Gabriel,

$$a_{Q^{ADE}, \mathbf{d}}(q) = \begin{cases} 1, & \text{when } (\mathbf{d}, \mathbf{d}) = 1 (\mathbf{d} \text{ is a positive root}) \\ 0, & \text{otherwise} \end{cases} \quad (4.16)$$

Example 17 (Jordan Quiver). If Q is the Jordan quiver, then the path algebra is just the polynomial ring $\mathbb{F}_q[x]$. By classification of modules over PID, absolutely indecomposable representation of dimension n is isomorphic to $\overline{\mathbb{F}}_q[x]/(x-a)^n$ for some n . Thus

$$a_{Q_{\text{Jor}}, d}(q) = q, \forall d \in \mathbb{Z}_{>0}.$$

More generally, for an arbitrary quiver Q , there exists a formula to compute the Kac Polynomial due to Hua [Hua00]. Kac conjectured that the coefficients of $a_{Q, \mathbf{d}}(Q)$ are always positive. For indivisible dimensional vectors, this was proved in 2004 by Crawley-Boevey and Van den Bergh ([CV04]). Then, using DT invariants, Mozgovoy proved this for a quiver without loops in [Moz13]. Finally, this conjecture was settled by Hausel, Letellier, and Rodriguez-Villegas in [HLR13] for arbitrary quivers. There is also a proof of this conjecture using BPS Lie algebras. In fact, in [Dav23a], Davison proved this conjecture by realizing the coefficients of Kac polynomials as the graded dimensions of BPS Lie algebras $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$. We have

Theorem 4.2.4 ([Dav23c]). *For any quiver Q ,*

$$\sum_i (-1)^i \dim(\mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i}) q^{i/2} = a_{Q, \mathbf{d}}(q^{-1}). \quad (4.17)$$

Example 18 (BPS Lie algebra for the Jordan quiver). Let Q_{Jor} be the Jordan quiver. Then by Equation 4.17 and calculation of Kac polynomial for Jordan quiver in Example 17, it follows that for all $n > 0$,

$$\mathfrak{g}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}, n}^{\text{BPS}} = \mathbb{Q}[2].$$

Since all of these spaces are 1 dimensional, for all n , choose any non-zero element α_n in $\mathfrak{g}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}, n}^{\text{BPS}}$. Since the BPS Lie algebra is graded, it follows that the commutator

$$[\alpha_n, \alpha_m] = 0$$

since the commutator must be of cohomological degree -4 , but there isn't any element of cohomological degree -4 in dimension $m+n$. Thus

$$\mathfrak{g}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}}^{\text{BPS}} \simeq \bigoplus_{n \geq 1} \mathbb{Q}[2].$$

Although the Lie bracket is trivial, one should think of this as the positive half of the infinite-dimensional Heisenberg Lie algebra, as we shall see that this Lie algebra acts on the cohomology of the Hilbert scheme of points on a plane and its action lifts to the action of the full infinite-dimensional Heisenberg Lie algebra.

Example 19. We can also calculate the size of the Nilpotent cohomological Hall algebra. Let $i : \mathcal{M}_{\mathbf{d}}^{\mathcal{N}_{\overline{Q}}}(\overline{Q}) \rightarrow \mathcal{M}_{\mathbf{d}}(\overline{Q})$ be the inclusion of the coarse moduli space of

nilpotent \bar{Q} representations, clearly $\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q})$ is projective. We have a Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}) & \xrightarrow{i'} & \mathfrak{M}_d(\bar{Q}) \\ \downarrow \text{JH} & \lrcorner & \downarrow \text{JH} \\ \mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\tilde{Q}) & \xrightarrow{i} & \mathcal{M}_d(\tilde{Q}) \end{array}$$

and so we have

$$\text{JH}_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}^{\mathcal{N}}(\Pi_Q)}^{\text{vir}} \simeq i^! \text{JH}_*^{\bar{Q}} \mathbb{D}\mathbb{Q}_{\mathfrak{M}(\Pi_Q)}^{\text{vir}} \simeq \text{Sym}(i^! \mathcal{BPS}_{\Pi_Q} \otimes \text{H}(\text{BC}_u^*, \mathbb{Q}))$$

and thus there is an isomorphism of graded vector spaces

$$\mathcal{A}_{\tilde{Q}, \tilde{W}}^{\tilde{N}} \simeq \text{Sym}\left(\text{H}(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^! \mathcal{BPS}_{\Pi_Q}) \otimes \text{H}(\text{BC}_u^*, \mathbb{Q})\right)$$

But since Verdier duality functor \mathbb{D} commutes with the vanishing cycles and intersection cohomology sheaf, it follows that

$$\text{H}(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^! \mathcal{BPS}_{\Pi_Q}) \simeq (\text{H}_c(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^* \mathcal{BPS}_{\Pi_Q}))^\vee$$

Note that $(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}))$ is homotopic to $(\mathcal{M}_d(\bar{Q}))$ via torus T action which scales the arrows, for which the BPS sheaf is equivariant. Thus we have

$$\text{H}_c(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^* \mathcal{BPS}_{\Pi_Q}) = \text{H}(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^* \mathcal{BPS}_{\Pi_Q}) \simeq \text{H}(\mathcal{M}_d(\bar{Q}), i^* \mathcal{BPS}_{\Pi_Q}) \simeq \mathfrak{g}_{\bar{Q}, \tilde{W}}^{\text{BPS}}.$$

Thus we have isomorphism of graded vector spaces

$$\text{H}(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^! \mathcal{BPS}_{\Pi_Q}) \simeq (\mathfrak{g}_{\bar{Q}, \tilde{W}}^{\text{BPS}})^\vee$$

and so

$$\sum_i (-1)^i \dim(\text{H}^i(\mathcal{M}_d^{\mathcal{N}_{\bar{Q}}}(\bar{Q}), i^! \mathcal{BPS}_{\Pi_Q})) q^{i/2} = a_{Q, \mathbf{d}}(q). \quad (4.18)$$

Example 20. In Example 6, we saw that $\mathfrak{M}_d^{\mathcal{N}}(\Pi_{Q^{\text{Jor}}})$ parametrizes commuting nilpotent matrices. Now, by the above two examples, and dimension reduction in Section 4.2.1, we have an isomorphism of vector spaces

$$\bigoplus_{d \geq 0} \text{H}^{\text{BM}}(\mathfrak{M}_d^{\mathcal{N}}(\Pi_{Q^{\text{Jor}}}), \mathbb{Q}) \simeq \text{Sym}\left(\bigoplus_{n \geq 1} \mathbb{Q}[u][[-2]]\right)$$

Note that for n , smallest cohomological degree in $\mathbb{Q}[u][[-2]]$ is 2. We conclude that the lowest cohomological degree of $\text{H}^{\text{BM}}(\mathfrak{M}_d^{\mathcal{N}}(\Pi_{Q^{\text{Jor}}}), \mathbb{Q})$ is 2 and we have

$$\dim(\text{H}^{\text{BM}, 2}(\mathfrak{M}_d^{\mathcal{N}}(\Pi_{Q^{\text{Jor}}}), \mathbb{Q})) = 1.$$

Thus, the top dimensional homology degree of the space $\mathcal{N}_2(\mathbb{C}^n)$ of commuting nilpotent matrices is $2n^2 - 2$ and of dimension 1, implying that $\mathcal{N}_2(\mathbb{C}^n)$ is irre-

ducible and of dimension $n^2 - 1$, recovering the theorem of Branovsky [Bar01].

Kac also conjectured that the constant term $a_{Q,\mathbf{d}}(0)$ in the case when quiver Q is the same as the dimension $\dim(\mathfrak{n}_Q^+)$ of the Lie algebra associated to Q . Hausel proved this [Hau10] by counting points over finite fields on Nakajima quiver varieties. Thus we have

$$a_{Q,\mathbf{d}}(0) = \dim(\mathfrak{n}_{Q',\mathbf{d}}^+). \quad (4.19)$$

The above relations allow us to conclude non-trivial statements about the BPS Lie algebra. For any quiver Q , since $\mathrm{Tr}(\widetilde{W}) = 0$ for $\mathbf{d} = \delta_i$, we have an isomorphism of vector spaces $\mathcal{A}_{Q,\widetilde{W},\delta_i}^X \simeq \mathbb{C}[x_{i,1}]$. We define $\alpha_i \in \mathcal{A}_{Q,\widetilde{W}}^X$ to be the inverse of $1 \in \mathbb{C}[x_{i,1}]$. Clearly $\alpha_i \in \mathfrak{g}_{Q,\widetilde{W},\delta_i}^{\mathrm{BPS},0}$ when the vertex i doesn't support any edge loops. We claim that for any i, j , $a_{Q,\delta_j+(1+a_{ij})\delta_i}(q) = 0$, this is because Kac's polynomials are orientation independent. So if we choose the orientation such that all the arrows are from vertex i to j , then there is no indecomposable representation of dimension $\delta_j + (1 + a_{ij})\delta_i$. We thus have a map of Lie algebras

$$\mathfrak{n}_{Q'}^+ \rightarrow \mathfrak{g}_{Q,\widetilde{W}}^{\mathrm{BPS},0} \quad (4.20)$$

given by $e_i \mapsto \alpha_i$ where \mathfrak{n}_Q^+ . The equality mentioned above suffices to show that this morphism is an injection. This is done by comparing the action on the Nakajima quiver varieties (See Chapter 7.3.1) with Nakajima's action, which is known to be faithful. We have

Proposition 4.2.5 ([Dav22b], Theorem 6.6). *The map (4.20) gives an isomorphism of Lie algebras⁸*

$$\mathfrak{n}_{Q'}^+ \simeq \mathfrak{g}_{Q,\widetilde{W}}^{\mathrm{BPS},0}.$$

4.2.4 Flat deformation

In the case when the natural mixed Hodge structure on $\mathcal{A}_{Q,W}$ is pure, the deformed cohomological Hall algebra $\mathcal{A}_{Q,W}^{T,\psi}$ forms a flat deformation of $\mathcal{A}_{Q,W}^\psi$.

Proposition 4.2.6. *Let $\mathbf{w}: Q_1 \rightarrow \mathbb{Z}^s$ be a W invariant weighting function and let $v: \mathbb{Z}^s \rightarrow \mathbb{Z}^{s'}$ be a surjective morphism inducing an inclusion of tori $T' \rightarrow T$ where T' is the torus associated to weighting $v\mathbf{w}$. Choose a splitting of $v: \mathbb{Z}^s \rightarrow \mathbb{Z}^{s'}$ inducing an isomorphism $H_T(\mathrm{pt}) \simeq H_{T'}(\mathrm{pt}) \otimes H_{T''}(\mathrm{pt})$. Assume the mixed Hodge structure on $\mathcal{A}_{Q,W}$ is pure. Then we have an isomorphism of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded $H_T(\mathrm{pt})$ modules*

$$\begin{aligned} \mathcal{A}_{Q,W}^T &\simeq \mathcal{A}_{Q,W}^{T'} \otimes H_{T''}(\mathrm{pt}) \\ \mathfrak{g}_{Q,W}^{\mathrm{BPS},T} &\simeq \mathfrak{g}_{Q,W}^{\mathrm{BPS},T'} \otimes H_{T''}(\mathrm{pt}), \end{aligned}$$

⁸In [Dav22b], one writes the above isomorphism for the negative half. But to our perspective, it is more natural to identify it to the positive half as the cohomological Hall algebra acts by *creation* operators on the cohomology of Nakajima quiver varieties.

while there is an isomorphism of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded $H_{T'}(\text{pt})$ algebras

$$\mathcal{A}_{Q,W}^{T,\psi} \otimes_{H_T(\text{pt})} H_{T'}(\text{pt}) \simeq \mathcal{A}_{Q,W}^{T',\psi}$$

and $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded $H_{T'}(\text{pt})$ Lie algebras

$$\mathfrak{g}_{Q,W}^{\text{BPS},T} \otimes_{H_T(\text{pt})} H_{T'}(\text{pt}) \simeq \mathfrak{g}_{Q,W}^{\text{BPS},T'}.$$

Proof. This statement is proved for $\widetilde{Q}, \widetilde{W}$ in [Dav23c][Theorem 9.6]. The proof works as it is in this case, as it only relies on the purity of $\mathcal{A}_{Q,W}$. The statement $\mathfrak{g}_{Q,W}^{\text{BPS},T} \simeq \mathfrak{g}_{Q,W}^{\text{BPS},T'} \otimes H_{T'}(\text{pt})$ as vector spaces follows from the PBW theorem (Proposition 4.1.1) while the isomorphism of algebras $\mathcal{A}_{Q,W}^{T,\psi} \otimes_{H_T(\text{pt})} H_{T'}(\text{pt}) \simeq \mathcal{A}_{Q,W}^{T',\psi}$ implies in particular that there is map of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded Lie algebras $\mathfrak{g}_{Q,W}^{\text{BPS},T} \otimes_{H_T(\text{pt})} H_{T'}(\text{pt}) \simeq \mathfrak{g}_{Q,W}^{\text{BPS},T'}$, which then becomes an isomorphism by a size argument. \square

We will, in particular, be applying the above proposition in the setting of the tripled quiver \widetilde{Q} with tripled canonical potential \widetilde{W} , which is shown to be of pure mixed Hodge structure in [Dav23c][Theorem A]. In particular, this implies that

Theorem 4.2.7. *The cohomological Hall algebra $\mathcal{A}_{\widetilde{Q},\widetilde{W}}^T$ is free module over $H_T(\text{pt})$.*

A similar theorem holds for the case of nilpotent CoHA. It is proved in [Dav23c] and [SV17] that

Theorem 4.2.8. *There is an isomorphism of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded $H_T(\text{pt})$ modules $\mathcal{A}_{\widetilde{Q},\widetilde{W}}^{T,\widetilde{N}} \simeq \mathcal{A}_{\widetilde{Q},\widetilde{W}}^{\widetilde{N}} \otimes H_T(\text{pt})$ i.e. The nilpotent cohomological Hall algebra $\mathcal{A}_{\widetilde{Q},\widetilde{W}}^{T,\widetilde{N}}$ is free module over $H_T(\text{pt})$.*

Chapter 5

BPS Lie algebra of Cyclic quivers

Let $Q^K = \widetilde{A}_K$ be the cyclic quiver of length $K + 1$, i.e. we have $K + 1$ vertices $0, 1, \dots, K$ with arrows $i \rightarrow i + 1$ for $i = 0, \dots, K$ and an arrow $K \rightarrow 0$. We then consider the tripled cyclic quiver \widetilde{Q}^K with the canonical potential \widetilde{W}^K as in Example 2 (See Figure 5.1). Let T be any torus action which leaves the potential \widetilde{W}_K invariant. We consider two distinct choices for the torus T .

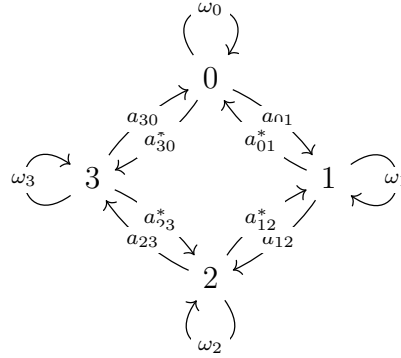


Figure 5.1: $\widetilde{Q}^3, \widetilde{W}^3 = \sum_{i \in \mathbb{Z}/4\mathbb{Z}} \omega_i (a_{i,i+1}^* a_{i,i+1} - a_{i,i-1} a_{i,i-1}^*)$

- Action of \mathbb{C}^* given by weighting function $\mathbf{w} : \widetilde{Q}^K \rightarrow \mathbb{Z}$, such that $\mathbf{w}(a) = 1$, $\mathbf{w}(a^*) = -1$ and $\mathbf{w}(\omega_i) = 0$ for all vertices $i \in Q_0^K$.
- Action of $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ given by weighting function $\mathbf{w} : \widetilde{Q}^K \rightarrow \mathbb{Z}^2$ where $\mathbf{w}(a) = (1, 0)$, $\mathbf{w}(a^*) = (0, 1)$ and $\mathbf{w}(\omega_i) = (-1, -1)$ for all vertices $i \in Q_0^K$.

We denote by $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^\chi$, the cohomological Hall algebra of the tripled cyclic quiver with canonical potential and its \mathbb{C}^* and \mathbb{T} deformed versions by $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathbb{C}^*, \chi}$ and $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathbb{T}^*, \chi}$ respectively.

(Deformed) BPS Lie algebra of Cyclic quiver

In this section, we shall calculate the BPS Lie algebra for tripled cyclic quivers with a canonical cubic potential. Following the work of Davison, Hennecart, and Schelegel Mejia, these Lie algebras are now known for arbitrary tripled quivers with canonical potential (See Section 5.1).

In the case of cyclic quivers, we can calculate these algebras more directly, as we explain next. We first calculate the Kac polynomial for cyclic quivers, which tells us about the graded dimensions of BPS Lie algebras.

Proposition 5.0.1. *The Kac polynomial of the cyclic quiver Q^K is given by*

$$a_{Q,\mathbf{d}}(q) = \begin{cases} q + K & \text{when } \mathbf{d} \in \mathbb{Z}_{>0} \cdot \delta \text{ (positive imaginary root)} \\ 1 & \text{when } \mathbf{d} \in \Delta_{re}^+ \text{ (positive real root)} \end{cases} \quad (5.1)$$

where Δ_{re}^+ and $\delta = (1, \dots, 1)$ are respectively, the real positive roots and the primitive imaginary root of the affine Lie algebra $\widetilde{\mathfrak{sl}}_{K+1}$.

Proof. By Kac's Theorem, $a_{Q,\mathbf{d}}(q)$ is non-zero if and only if \mathbf{d} is a positive root of \widetilde{A}_K . When \mathbf{d} is a positive real root, i.e., when $(\mathbf{d}, \mathbf{d}) = 1$, we have that the highest degree of the Kac polynomial is $1 - (\mathbf{d}, \mathbf{d}) = 0$. To compute the constant, we note that $\dim(\mathfrak{g}_{Q,\mathbf{d}}) = 1$ and thus $a_{Q,\mathbf{d}} = 1$ in this case. When \mathbf{d} is a positive imaginary root, then the degree of the Kac polynomial is given by $1 - (\mathbf{d}, \mathbf{d}) = 1$. Thus it is enough to compute the constant, i.e to compute $\dim(\mathfrak{n}_{A_K, n\delta}^+)$ which is well known to be K , giving $a_{Q,\mathbf{d}}(q) = q + K$ in this case.

The case when $\mathbf{d} = \delta$ is easier to visualize. Either all arrows are non-zero, in that the product of all arrows determines the representation, i.e., $q - 1$ choices, or when one of them is non-zero, which gives $K + 1$ choices (choice of vertex). Thus, in total, we get $q - 1 + K + 1 = q + K$ absolutely indecomposables. \square

Proposition 5.0.2. *There is an isomorphism of Lie algebras*

$$\begin{aligned} \mathfrak{g}_{Q^K, \widetilde{W^K}}^{\text{BPS}} &\simeq \mathfrak{n}_{Q^K}^+ \oplus s\mathbb{Q}[s] \\ \mathfrak{g}_{Q^K, \widetilde{W^K}}^{\text{BPS}, \mathbb{C}^*} &\simeq (\mathfrak{n}_{Q^K}^+ \oplus s\mathbb{Q}[s]) \otimes H_{\mathbb{C}^*}(\text{pt}) \\ \mathfrak{g}_{Q^K, \widetilde{W^K}}^{\text{BPS}, \mathbb{T}} &\simeq (\mathfrak{n}_{Q^K}^+ \oplus s\mathbb{Q}[s]) \otimes H_{\mathbb{T}}(\text{pt}) \end{aligned}$$

where $\mathfrak{n}_{Q^K}^+ \oplus s\mathbb{Q}[s]$ is a trivial central extension of \mathfrak{n}_{Q^K} and $\deg(s^n) = n \cdot \delta$.

Proof. From Proposition 4.2.5, we have an isomorphism of Lie algebras

$$\mathfrak{n}_{Q^K}^+ \simeq H^0(\mathfrak{g}_{Q^K, \widetilde{W^K}}^{\text{BPS}}).$$

In particular this also shows that $H^0(\mathfrak{g}_{Q^K, \widetilde{W^K}}^{\text{BPS}})$ is generated as a Lie algebra by α_i , where we recall that $\alpha_i \in \mathfrak{g}_{Q^K, \widetilde{W^K}, \delta_i}^{\text{BPS}}$. The Kac polynomial of the cyclic quiver (5.0.1) and it's relation with BPS Lie algebra (4.17) also tells us that for every $n \in \mathbb{Z}_{>0}$, there exist $\gamma_{n\delta} \in H^{-2}(\mathfrak{g}_{Q^K, \widetilde{W^K}, n\delta}^{\text{BPS}})$. By equation (5.0.1), there is no

element of cohomological degree -4 in dimension $\mathbb{Z}_{>0}\delta$. Thus it follows that $[\gamma_{n\cdot\delta}, \gamma_{m\cdot\delta}] = 0$. Again, by equation 5.0.1, there is no -2 cohomological degree term in dimension $\delta + \delta_i$ for any i , thus it follows that $[\gamma_{n\cdot\delta}, \alpha_i] = 0$. But then α_i generates $H^0(\mathfrak{g}_{\widetilde{Q^k}, \widetilde{W^k}}^{\text{BPS}})$ and all of $H^{-2}(\mathfrak{g}_{\widetilde{Q^k}, \widetilde{W^k}}^{\text{BPS}})$ is central and so we are done. This argument does not work for deformed versions. However in [DHM24][Corollary 11.9], it is shown that for any quiver Q and for any torus action such that the canonical cubic potential \widetilde{W} remains invariant, $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, T}$ is a trivial deformation of $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$, i.e. there is an isomorphism of Lie algebras

$$\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, T} \simeq \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}} \otimes H_T(\text{pt}).$$

Hence, the conclusion follows. \square

The BPS Lie algebras $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ for tripled quiver \widetilde{Q} and potential \widetilde{W} are now completely known in terms of generators and relations. This also gives a candidate for a double of the BPS Lie algebra.

5.1 Generalized Kac-Moody Lie algebra

In this section, we will recall the construction of the Generalized Kac-Moody Lie algebra for a monoid with a weight function defined in [DHM24], based on the earlier construction of Borchers-Bozec Kac-Moody Lie algebras in [Boz16].

Let $(M, +)$ be a monoid. Let $(-, -) : M \times M \rightarrow \mathbb{Z}$ be the symmetric bilinear form. Then we define

Definition 5.1.1 (Primitive Positive Roots). The set of primitive positive roots $\sum_{M, (-, -)}$ is defined to be the set of $m \in M, m \neq 0$ such that $2 - (m, m) \geq 0$ and $2 - (m, m) > \sum_{j=1}^s (2 - (m_j, m_j))$ for any non-trivial decomposition $m = \sum_{j=1}^s m_j$ with $m_j \in M, m_j \neq 0$.

Definition 5.1.2 (Simple Positive Roots). The set of simple positive roots $\Phi_{M, (-, -)}^+$ is defined to be the union of the set of primitive positive roots $\sum_{M, (-, -)}$ and $lm, l \geq 1$ where $m \in \sum_{M, (-, -)}$ and $(m, m) = 0$.

Then define subsets

$$\begin{aligned} \Phi_{M, (-, -)}^{+, \text{real}} &:= \{m \in \Phi_{M, (-, -)}^+ \mid (m, m) = 2\} \text{ (Real positive roots)} \\ \Phi_{M, (-, -)}^{+, \text{im}} &:= \{m \in \Phi_{M, (-, -)}^+ \mid (m, m) \leq 0\} \text{ (Imaginary positive roots)} \\ \Phi_{M, (-, -)}^{+, \text{iso}} &:= \{m \in \Phi_{M, (-, -)}^+ \mid (m, m) = 0\} \text{ (Isotopic positive roots)} \\ \Phi_{M, (-, -)}^{+, \text{hyp}} &:= \{m \in \Phi_{M, (-, -)}^+ \mid (m, m) < 0\} \text{ (Hyperbolic positive roots)} \end{aligned}$$

Definition 5.1.3 (Cartan Matrix). Let

$$A_{M, (-, -)} = (a_{m, n} = (m, n))_{m, n \in \Phi_{M, (-, -)}^+}$$

to be the Cartan Matrix for $M, (-, -)$.

The final piece of the data is the weight function.

Definition 5.1.4 (Weight Function). Let

$$P: \Phi_{M,(-,-)}^+ \rightarrow \mathbb{N}[t^{\pm 1/2}]$$

given by $P \mapsto P_m(t^{1/2})$ be the weight function. We assume that $P_m(t^{1/2}) = 1$ whenever $(m, m) = 2$.

Let $P_m(t^{1/2}) = \sum_{j \in \mathbb{Z}} p_{m,j} t^{j/2}$ with $p_{m,j} \in \mathbb{N}$.

Definition 5.1.5 (Generalized Kac Moody Lie algebra for Monoid and Weight Function). Given a monoid M with symmetric bilinear form $(-, -)$, such that $(m, n) \leq 0$ if $m \neq n$ and $(m, m) \in 2\mathbb{Z}_{\leq 1}$. The generalized Kac Moody Lie algebra $\mathfrak{g}_{M,(-,-),P}$ is defined to be the Lie algebra generated by $e_{m,j,l}, f_{m,j,l}, h_m$ with $m \in \Phi_{M,(-,-)}^+, j \in \mathbb{Z}$ and $1 \leq l \leq p_{m,j}$, with the relations

$$\begin{aligned} h_{m+n} &= h_m + h_n \\ [h_m, h_n] &= 0 \\ [h_m, e_{n,j,l}] &= a_{m,n} e_{n,j,l} \\ [h_m, f_{n,j,l}] &= -a_{m,n} f_{n,j,l} \\ [e_{m,j,l}, f_{n,j',l'}] &= \delta_{m,n} \delta_{j,j'} \delta_{l,l'} h_n \\ \text{ad}(e_{m,j,l})^{1-a_{m,n}}(e_{n,j',l'}) &= 0 \text{ if } a_{m,n} = 2 \\ \text{ad}(f_{m,j,l})^{1-a_{m,n}}(f_{n,j',l'}) &= 0 \text{ if } a_{m,n} = 2 \\ [e_{m,j,l}, e_{n,j',l'}] &= [f_{m,j,l}, f_{n,j',l'}] = 0 \text{ if } a_{m,n} = 0 \end{aligned}$$

We place $e_{m,j,l}$ in cohomological degree j , each $f_{m,j,l}$ in cohomological degree $-j$ and h_m of cohomological degree 0. We define positive half $\mathfrak{g}_{M,(-,-),P}^+ \subset \mathfrak{g}_{M,(-,-),P}$ by Lie subalgebra generated by $e_{m,j,l}$. Similarly define Cartan $\mathfrak{h}_{M,(-,-),P}$ as the Lie subalgebra generated by h_m . We then have the following theorem.

Theorem 5.1.6 ([DHM24]). *Let Q be any quiver. Let $M = \mathbb{N}^{Q_0}$ and let $(\mathbf{d}, \mathbf{e})_{\Pi_Q} = \chi_Q(\mathbf{d}, \mathbf{e}) + \chi_Q(\mathbf{e}, \mathbf{d})$ be the bilinear form. Let*

$$P(\mathbf{d}) = \begin{cases} \sum_{j \in \mathbb{Z}} \dim H^j(\mathcal{IC}_{\mathcal{M}_{\mathbf{d}}(\Pi_Q)}) t^{j/2} & \text{when } \mathbf{d} \in \Phi_{\mathbb{N}^{Q_0},(-,-)_{\Pi_Q}}^{+, \text{real}} \setminus \Phi_{\mathbb{N}^{Q_0},(-,-)_{\Pi_Q}}^{+, \text{iso}} \\ \sum_{j \in \mathbb{Z}} \dim H^j(\mathcal{IC}_{\mathcal{M}_{\mathbf{d}'}(\Pi_Q)}) t^{j/2} & \text{when } \mathbf{d} = l\mathbf{d}' \in \Phi_{\mathbb{N}^{Q_0},(-,-)_{\Pi_Q}}^{+, \text{iso}}, \mathbf{d}' \in \sum_{\mathbb{N}^{Q_0},(-,-)_{\Pi_Q}}^+ \end{cases}$$

be the weight function. Then there is an isomorphism of Lie algebras:

$$\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}} \simeq \mathfrak{g}_{\mathbb{N}^{Q_0},(-,-)_{\Pi_Q},P}^+$$

We will refer to the Lie algebras $\mathfrak{g}_{\mathbb{N}^{Q_0},(-,-)_{\Pi_Q},P}$ by $\mathfrak{g}_Q^{\text{GKM}}$.

Example 21 (Jordan Quiver). Let Q be the Jordan quiver. Then $(m, n)_{\Pi_Q} = \chi_Q(m, n) + \chi_Q(n, m) = 0$. Thus all $m \in \mathbb{Z}_{>0}$ are isotopic positive roots. Then

$$P(m) = \dim(H^j(\mathcal{IC}_{\mathcal{M}_1(\Pi_Q)})) t^{j/2} = \dim(H^j(\mathcal{IC}_{\mathbb{A}^2})) t^{j/2} = t^{-1}.$$

Thus the associated generalized Kac Moody Lie algebra is generated by $e_m := e_{m,-1,1}, f_m := f_{m,-1,1}, h_m$ for $m \geq 1$ with the relation $h_m = mh_1$, h_i commutes with e_j, f_j and $[e_i, f_j] = \delta_{i,j}h_i$ and thus this is just infinite dimensional Heisenberg Lie algebra $gl(\hat{1})$.

Example 22 (Cyclic Quivers). Let Q be the cyclic quiver Q^K . Then the Symmetrized euler form is given by

$$2 \left(\sum_{i=0}^K (d_i - d_{i+1})^2 \right)$$

So only solutions to $2 - (\mathbf{d}, \mathbf{d})_{\Pi_Q} \geq 0$ is given by strings of type $(n, \dots, n, n+1, n+1, \dots, n+1, n, \dots, n)$, i.e $\mathbf{d} = n \cdot \delta + [i, m]$ or $(n+1) \cdot \delta$ for some $n \geq 0$, $i \in [0, K]$ and $0 \leq m \leq K-1$ (See section 1 for Notation).

Moreover if $\mathbf{d} = n \cdot \delta + [i, m]$, then it cannot be a primitive root since one can write $2 - (\mathbf{d}, \mathbf{d}) = 0$, but $2 - (n \cdot \delta, n \cdot \delta)_{\Pi_{Q^K}} = 2$. Similarly if $\mathbf{d} = [i, m] = \delta_i + \delta_{i+1} + \dots + \delta_{i+m-1}$ then also it can't be a primitive root as then $2 - (\mathbf{d}, \mathbf{d})_{\Pi_{Q^K}} = 0$ but then so does $2 - (\delta_i, \delta_i)_{\Pi_{Q^K}}$ which contradicts the primitiveness of \mathbf{d} . Thus, we have real positive roots δ_i for $i \in [0, K]$ and imaginary positive roots $l\delta, l \geq 1$. Now we consider the weight function.

$$\begin{aligned} P(\delta_i) &= \sum \dim(H^j(\mathbb{Q}_{\text{pt}}))t^{j/2} = 1 \\ P(l\delta) &= \sum \dim(H^j(\mathcal{IC}_{\mathcal{M}_\delta(\Pi_{Q^K})}))t^{j/2} = t^{-1} \end{aligned}$$

So we have generators e_i, f_i, h_i for $i \in [0, K]$ of cohomological degree 0 and $e_{l\delta}, f_{l\delta}, h_{l\delta}$ of cohomological degree $-2, 0, 2$ respectively satisfying the relations above, i.e the affine Lie algebra $\hat{\mathfrak{gl}}(K+1)$.

We can also define the notion of a Verma Module.

Definition 5.1.7 (Verma Module of Generalized Kac Moody Lie algebra). Let $\mathfrak{g}_Q^{\text{GKM}, \leq 0}$ be the negative Borel, i.e., Lie subalgebra generated by $f_{m,j,l}$ and h_m . For $\mathbf{f} \in \text{Hom}(\mathbb{Z}^{Q_0} \rightarrow \mathbb{Z})$, we define the Verma module as

$$V_{\mathfrak{g}_Q^{\text{GKM}}, \mathbf{f}} = \mathbf{U}(\mathfrak{g}_Q^{\text{GKM}}) \otimes_{\mathbf{U}(\mathfrak{g}_Q^{\text{GKM}, \leq 0})} \mathbb{C}$$

where \mathbb{C} becomes a $\mathbf{U}(\mathfrak{g}_Q^{\text{GKM}, \leq 0})$ module via the quotient $\mathbf{U}(\mathfrak{g}_Q^{\text{GKM}, \leq 0}) \rightarrow \mathbf{U}(\mathfrak{h}_M)$ and $h_m \cdot 1 = \mathbf{f}(m)$.

Then, analogous to the usual case of semisimple Lie algebras, one defines

Definition 5.1.8. The Lowest Weight module $L_{\mathfrak{g}_Q^{\text{GKM}}, \mathbf{f}}$ is defined to be quotient of $V_{\mathfrak{g}_Q^{\text{GKM}}, \mathbf{f}}$ by the unique maximal submodule of $V_{\mathfrak{g}_Q^{\text{GKM}}, \mathbf{f}}$ which doesn't contain $1 \otimes 1$.

Example 23 (Jordan Quiver). Let Q be the Jordan quiver. Choose $\mathbf{f}_{-1} \in \text{Hom}(\mathbb{Z} \rightarrow \mathbb{Z})$ given by $\mathbf{f}(1) = -1$. Then

$$V_{gl(\hat{1}), \mathbf{f}_1} = L_{gl(\hat{1}), \mathbf{f}_1} \simeq \mathbb{C}[e_1, e_2, e_3, \dots,].$$

Chapter 6

Affinized BPS Lie Algebra

For any symmetric quiver Q and potential W , we saw in the Section 4.1.5, that there exists a Lie algebra $\mathfrak{g}_{Q,W}^{\text{BPS}}$ such that $\text{Sym}(\mathfrak{g}_{Q,W}^{\text{BPS}}[u]) \simeq \mathcal{A}_{Q,W}^\psi$ as vector spaces. This makes it natural to wonder if the vector subspace $\mathfrak{g}_{Q,W}^{\text{BPS}}[u] \subset \mathcal{A}_{Q,W}^\psi$ is itself closed under the commutator Lie bracket coming from the associative algebra structure on $\mathcal{A}_{Q,W}^\psi$. The answer to this question is no for a general quiver with potential Q, W ¹. However, the answer becomes yes in the case of the tripled quiver with potential \tilde{Q}, \tilde{W} . This is obtained by constructing a coproduct which uses factorization structure on the stack of representations of the Jacobi algebra for the tripled quiver with potential and is outlined by Davison in [Dav22a]. We prove that this construction works here. The construction for more general 3-Calabi–Yau completions of 2-Calabi–Yau categories will appear in the joint work of Davison, Hennecart, Kinjo, Schiffmann and Vasserot.

6.1 Factorization coproduct and the Affinized BPS Lie algebra

In the case of tripled quiver with canonical potential, there is an isomorphism of algebras $\text{Jac}(\tilde{Q}, \tilde{W}) \simeq \Pi_Q[\omega]$ and thus a representation of $\text{Jac}(\tilde{Q}, \tilde{W})$ is the same as a representation of preprojective algebra Π_Q with an endomorphism. This allows us to define

Definition 6.1.1. Let $U \subset \mathbb{C}$ be any open subset. Then we define $\mathfrak{M}_d^U(\tilde{Q}) \subset \mathfrak{M}_d(\tilde{Q})$ to be the open substack of representations, such that the generalized eigenvalues of ω_i for all $i \in Q_0$ are all inside U . We can similarly define coarse moduli space $\mathcal{M}_d^U(\tilde{Q})$. We will denote this moduli space by $\mathfrak{M}^U(\tilde{Q})$ when the dimension vector is clear from the context.

Given a short exact sequence of representations

$$0 \rightarrow \rho_1 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow 0,$$

¹This can be seen when Q, W comes from the derived category of resolved conifold and will be explained in future work.

the eigenvalues of $\rho_1(\omega)$ and $\rho_3(\omega)$ are in U if and only if eigenvalues of $\rho_2(\omega)$ are in U . This means that $\mathfrak{M}^U(\tilde{Q})$ forms a Serre subcategory and hence from [DM20][Section 3], we have

Proposition 6.1.2. *For any open $U \subset \mathbb{C}$, let*

$$\mathcal{A}_{\tilde{Q}, \tilde{W}}^U := \bigoplus_{\mathbf{d} \in \mathbb{N}^{\mathcal{Q}_0}} \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})} |U).$$

Then $\mathcal{A}_{\tilde{Q}, \tilde{W}}^U$ has a structure of cohomological Hall algebra.

For any $U \subset U'$, we have open immersions $\mathfrak{M}_{\mathbf{d}}^U(\tilde{Q}) \hookrightarrow \mathfrak{M}_{\mathbf{d}}^{U'}(\tilde{Q})$ and thus the restriction morphism

$$\mathrm{H}(\mathfrak{M}_{\mathbf{d}}^{U'}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})} |U') \rightarrow \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})} |U)$$

gives a morphism of cohomologically graded vector spaces

$$\mathcal{A}_{\tilde{Q}, \tilde{W}}^{U'} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}^U.$$

Proposition 6.1.3. *For any open $U \subset U' \subset \mathbb{C}$, the restriction morphism $\rho_{U', U}: \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U'} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}^U$ is morphism of associative algebras.*

Proof. Since the proof follows the same strategy, we only prove the case when $U' = \mathbb{C}$. We have a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}(\tilde{Q}) & & \\ & \swarrow^{\pi_1 \times \pi_3} & \uparrow & \searrow^{\pi_2} & \\ \mathfrak{M}_{\mathbf{d}_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}(\tilde{Q}) & & \mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^U(\tilde{Q}) & & \mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}(\tilde{Q}) \\ \uparrow^{i_{U, \mathbf{d}_1} \times i_{U, \mathbf{d}_2}} & \swarrow^{(\pi_1 \times \pi_3)|U} & & \searrow^{(\pi_2)|U} & \uparrow^{i_{U, \mathbf{d}_1 + \mathbf{d}_2}} \\ \mathfrak{M}_{\mathbf{d}_1}^U(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^U(\tilde{Q}) & & & & \mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}^U(\tilde{Q}) \end{array}$$

where $\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^U(\tilde{Q})$ is the stack of short exact sequences where $\rho(\omega)$ for each term in the short exact sequence has generalized eigenvalues inside U . Taking pullback on the cohomology in the first square, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}) \otimes \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}) & \xrightarrow{(\pi_1 \times \pi_3)^*} & \mathrm{H}(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}) \\ \downarrow^{i_{U, \mathbf{d}_1}^* \otimes i_{U, \mathbf{d}_2}^*} & & \downarrow^{i^*} \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)} |U) \otimes \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)} |U) & \xrightarrow{((\pi_1 \times \pi_3)|U)^*} & \mathrm{H}(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)} |U) \end{array}$$

The right square is cartesian and the morphism $\pi_2, (\pi_2)|U$ is proper, so ignoring the cohomological shifts, by proper base change (Lemma 6.1.9(a)), the

following square commutes

$$\begin{array}{ccc}
\mathrm{H}(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}) & \xrightarrow{(\pi_2)^*} & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}) \\
\downarrow i^* & & \downarrow i_{U, \mathbf{d}_1 + \mathbf{d}_2}^* \\
\mathrm{H}(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}|_U) & \xrightarrow{((\pi_2)|_U)^*} & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)}|_U)
\end{array}$$

Thus, the proposition follows by composing the above two commutative squares. \square

We shall write ρ_U when $U' = \mathbb{C}^2$. When U is an open ball, the restriction morphism ρ_U is an isomorphism.

Proposition 6.1.4. *For U an open ball, the restriction morphism $\rho_U: \mathcal{A}_{\tilde{Q}, \tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}^U$ is an isomorphism of algebras. i.e we have*

$$\mathrm{H}(\mathfrak{M}_{\mathbf{d}}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)_{\mathbf{d}}}) \simeq \mathrm{H}(\mathfrak{M}_{\mathbf{d}}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)_{\mathbf{d}}|_U}).$$

Proof. Let $i_U: \mathcal{M}^U(\tilde{Q}) \hookrightarrow \mathcal{M}(\tilde{Q})$ be the inclusion. Then

$$\mathrm{JH}_* {}^p\Phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}^U(\tilde{Q})} = (i_U)^*(\mathrm{JH}_* {}^p\Phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathcal{M}(\tilde{Q})}) \quad (6.1)$$

$$= (i_U)^* \mathrm{Sym} \left(\mathcal{BPS}_{\tilde{Q}, \tilde{W}} \otimes \mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}^{\mathrm{vir}}) \right) \quad (6.2)$$

$$= \mathrm{Sym} \left((\mathcal{BPS}_{\tilde{Q}, \tilde{W}})_{|\mathcal{M}^U(\tilde{Q})} \otimes \mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}^{\mathrm{vir}}) \right) \quad (6.3)$$

Now by the support lemma (Theorem 4.2.1), $\mathcal{BPS}_{\tilde{Q}, \tilde{W}} = l_*(\mathcal{BPS}_{\Pi_Q} \boxtimes \mathbb{Q}_{\mathbb{A}^1}[1])$ where

$$l: \mathcal{M}(\Pi_Q) \times \mathbb{A}^1 \rightarrow \mathcal{M}(\tilde{Q})$$

is morphism which takes a representation ρ of Π_Q and hence of \overline{Q} and a scalar λ and outputs a representation of \tilde{Q} where $\rho(\omega_i)$ is scalar λ and rest is defined by original representation $\rho(a)$ for $a \in \overline{Q}_1$. We have a Cartesian diagram

$$\begin{array}{ccc}
\mathcal{M}(\Pi_Q) \times \mathbb{A}^1 & \xrightarrow{l} & \mathcal{M}(\tilde{Q}) \\
i_U \uparrow & & \uparrow i_U \\
\mathcal{M}(\Pi_Q) \times U & \xrightarrow{l'} & \mathcal{M}^U(\tilde{Q})
\end{array}$$

and so by open base change

$$i_U^* \mathcal{BPS}_{\tilde{Q}, \tilde{W}} = i_U^* l_*(\mathcal{BPS}_{\Pi_Q} \boxtimes \mathbb{Q}_{\mathbb{A}^1}[1]) = l'_*(\mathcal{BPS}_{\Pi_Q} \boxtimes \mathbb{Q}_U[1]).$$

Thus

$$\mathrm{H}(\mathfrak{M}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)|_U}) = \mathrm{Sym} \left(\mathrm{H}(\mathcal{M}^U(\tilde{Q}), (\mathcal{BPS}_{\tilde{Q}, \tilde{W}})_{|\mathcal{M}^U(\tilde{Q})}) \otimes \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q}^{\mathrm{vir}}) \right) \quad (6.4)$$

$$= \mathrm{Sym} \left(\mathrm{H}(\mathcal{M}(\Pi_Q), \mathcal{BPS}_{\Pi_Q}) \otimes H(U, \mathbb{Q}) \otimes \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q}) \right) \quad (6.5)$$

Since U is an open ball, $H(U, \mathbb{Q}) = \mathbb{Q}$ and thus we have isomorphism of cohomologically graded vector spaces

$$\mathrm{H}(\mathfrak{M}^U(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})|_U}) \simeq \mathrm{Sym} \left(\mathrm{H}(\mathcal{M}(\Pi_Q), \mathcal{BPS}_{\Pi_Q}) \otimes \mathrm{H}(\mathrm{BC}_u^*, \mathbb{Q}) \right) \simeq \mathrm{H}(\mathfrak{M}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})}).$$

Since ρ_U respects the cohomological grading and algebra morphism, it is an isomorphism of algebras. \square

Furthermore, for two disjoint open balls U_1 and U_2 , since any point $\rho \in \mathfrak{M}^{U_1 \amalg U_2}(\mathrm{Jac}(\tilde{Q}, \tilde{W}))$ admits a canonical direct sum decomposition $\rho_1 \oplus \rho_2$ such that generalized eigenvalues of $\rho_1(\omega)$ are in U_1 and $\rho_2(\omega)$ are in U_2 . we have an isomorphism of stacks

$$\mathfrak{M}_*^{U_1 \amalg U_2}(\mathrm{Jac}(\tilde{Q}, \tilde{W})) \simeq \mathfrak{M}_*^{U_1}(\mathrm{Jac}(\tilde{Q}, \tilde{W})) \times \mathfrak{M}_*^{U_2}(\mathrm{Jac}(\tilde{Q}, \tilde{W})). \quad (6.6)$$

Let $\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}$ be the stack of $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$ representations of \tilde{Q} where $(\mathbf{d}_1)_i$ of the generalized eigenvalues of $\rho(\omega_i)$ are in U_1 and $(\mathbf{d}_2)_i$ of the generalized eigenvalues of $\rho(\omega_i)$ are in U_2 or equivalently the image of $\mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q})$ in the above isomorphism. We have closed embedding

$$i_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2} : \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q})$$

given by the direct sum map. We will now calculate the pullback $i^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q})}^{\mathrm{vir}}$.

We remark that even though there is an isomorphism of supports (Equation 6.6, we can't apply the Thom-Sebastiani isomorphism theorem directly since the underlying smooth stacks are not isomorphic, i.e $i : \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \not\simeq \mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q})$.

Note that $\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q})$ is the quotient stack $\mathrm{Rep}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}(\tilde{Q}) / (\mathrm{GL}_{\mathbf{d}_1} \times \mathrm{GL}_{\mathbf{d}_2})$ where $\mathrm{Rep}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}(\tilde{Q})$ is the space of representations where $\rho(\omega_i)$ is a block diagonal matrix, where the top block is of dimension $(\mathbf{d}_1)_i$ and has all its eigenvalues in U_1 and bottom block is of dimension $(\mathbf{d}_2)_i$ and all its eigenvalues are in U_2 . When the context is clear, we will ignore writing \tilde{Q} .

Let $\mathrm{Tr}(\tilde{W})_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}$ be the restriction of the function $\mathrm{Tr}(\tilde{W})$ on $\mathrm{Rep}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}$. Since the function $\mathrm{Tr}(\tilde{W})_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}$ is $\mathrm{GL}_{\mathbf{d}_1} \times \mathrm{GL}_{\mathbf{d}_2}$ invariant, we may assume that all of $\rho(\omega_i)$ for $i \in Q_0$ are in some Jordan normal form. So we may assume that the entries $\rho(\omega_i)_{k_i, l_i}$ of the $\mathbf{d}_i \times \mathbf{d}_i$ matrix $\rho(\omega_i)$ are 0 except for the diagonal entries $\rho(\omega_i)_{k, k}$, $1 \leq k \leq \mathbf{d}_i$ and entries right above diagonal entries $\rho_{k, k+1}$, $1 \leq k \leq \mathbf{d}_i - 1$. Finally since $(\mathbf{d}_1)_i$ of the generalized eigenvalues of $\rho(\omega_i)$ are in U_i , we may assume

that $\rho(\omega_i)_{k,k} \in U_1$ for $1 \leq k \leq (\mathbf{d}_1)_i$ and $\rho(\omega_i)_{k,k} \in U_2$ for $(\mathbf{d}_1)_i + 1 \leq k \leq \mathbf{d}_i$.

$$\begin{bmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & & & & \\ & & \lambda_2 & & & \\ & & & \lambda_3 & 1 & \\ & & & & \lambda_3 & \end{bmatrix} \quad (6.7)$$

Example 6.1.1: $d_1 = 3, d_2 = 2, (\lambda_1, \lambda_2, \lambda_3), \lambda_1, \lambda_2 \in U_1; \lambda_3 \in U_2$
Then for any representation ρ , we have

$$\begin{aligned} \mathrm{Tr}(\widetilde{W})_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}(\rho) &= \sum_{i \in Q_0} \mathrm{Tr} \left(\left(\sum_{i \in Q_0} \rho(\omega_i) \right) [\rho(a), \rho(a^*)] \right) \\ &= \sum_{a \in Q_1} \mathrm{Tr} (\rho(\omega_{t(a)})\rho(a)\rho(a^*) - \rho(\omega_{s(a)})\rho(a^*)\rho(a)) \\ &= \sum_{a \in Q_0} \left(\sum_{1 \leq i, j \leq (\mathbf{d})_{t(a)}} \rho(\omega_{t(a)})_{ij} (\rho(a)\rho(a^*))_{ji} - \sum_{1 \leq i, j \leq (\mathbf{d})_{s(a)}} \rho(\omega_{t(a)})_{ij} (\rho(a^*)\rho(a))_{ji} \right) \end{aligned}$$

Since the matrix $\rho(\omega_i)$ is block diagonal, we have

$$\begin{aligned} \mathrm{Tr}(\widetilde{W})_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}(\rho) &= \sum_{a \in Q_0} \left(\sum_{1 \leq i, j \leq (\mathbf{d}_1)_{t(a)}} \rho(\omega_{t(a)})_{ij} (\rho(a)\rho(a^*))_{ji} - \sum_{1 \leq i, j \leq (\mathbf{d}_1)_{s(a)}} \rho(\omega_{t(a)})_{ij} (\rho(a^*)\rho(a))_{ji} \right) \\ &\quad - \sum_{a \in Q_0} \left(\sum_{1 \leq i, j \leq (\mathbf{d}_1)_{t(a)}} \rho(\omega_{t(a)})_{ij} (\rho(a)\rho(a^*))_{ji} - \sum_{1 \leq i, j \leq (\mathbf{d}_1)_{s(a)}} \rho(\omega_{t(a)})_{ij} (\rho(a^*)\rho(a))_{ji} \right) \\ &= \sum_{a \in Q_1} \left(\sum_{1 \leq i, j \leq (\mathbf{d}_1)_{t(a)}} \rho(\omega_{t(a)})_{ij} \left(\sum_{1 \leq k \leq \mathbf{d}_{s(a)}} \rho(a)_{jk} \rho(a^*)_{ki} \right) \right) - \\ &\quad \sum_{a \in Q_1} \left(\sum_{1 \leq i, j \leq (\mathbf{d}_1)_{s(a)}} \rho(\omega_{s(a)})_{ij} \left(\sum_{1 \leq k \leq \mathbf{d}_{t(a)}} \rho(a^*)_{jk} \rho(a)_{ki} \right) \right) + \\ &\quad \sum_{a \in Q_1} \left(\sum_{(\mathbf{d}_1)_{t(a)} + 1 \leq i, j \leq (\mathbf{d})_{t(a)}} \rho(\omega_{t(a)})_{ij} \left(\sum_{1 \leq k \leq \mathbf{d}_{s(a)}} \rho(a)_{jk} \rho(a^*)_{ki} \right) \right) - \\ &\quad \sum_{a \in Q_1} \left(\sum_{(\mathbf{d}_1)_{s(a)} + 1 \leq i, j \leq (\mathbf{d})_{s(a)}} \rho(\omega_{s(a)})_{ij} \left(\sum_{1 \leq k \leq \mathbf{d}_{t(a)}} \rho(a^*)_{jk} \rho(a)_{ki} \right) \right) \\ &= \mathrm{Tr}(\widetilde{W})_{\mathbf{d}_1}^{U_1} + \mathrm{Tr}(\widetilde{W})_{\mathbf{d}_2}^{U_2} + G_{\mathbf{d}_1, \mathbf{d}_2} \end{aligned} \quad (6.8)$$

where $\mathrm{Tr}(\widetilde{W})_{\mathbf{d}_i}^{U_i} : \mathfrak{M}_{\mathbf{d}_i}^{U_i}(\widetilde{Q}) \rightarrow \mathbb{C}$ is the restriction of the function $\mathrm{Tr}(\widetilde{W})_{\mathbf{d}_1}$ and

$$G_{\mathbf{d}_1, \mathbf{d}_2} = \sum_{a \in Q_1} \left(\sum_{\substack{1 \leq j \leq (\mathbf{d}_1)_{t(a)} \\ (\mathbf{d}_1)_{s(a)} + 1 \leq k \leq \mathbf{d}_{s(a)}}} \rho(a)_{jk} g_{jk}(a) + \sum_{\substack{1 \leq j \leq (\mathbf{d}_1)_{s(a)} \\ (\mathbf{d}_1)_{t(a)} + 1 \leq k \leq \mathbf{d}_{t(a)}}} \rho(a^*)_{jk} g_{jk}^*(a^*) \right) \quad (6.9)$$

where

$$g_{jk}(a) = \sum_{1 \leq i \leq (\mathbf{d}_1)_{t(a)}} \rho(\omega_{t(a)})_{ij} \rho(a^*)_{ki} - \sum_{(\mathbf{d}_1)_{s(a)} + 1 \leq i \leq \mathbf{d}_{s(a)}} \rho(\omega_{s(a)})_{ki} \rho(a^*)_{ij}$$

and

$$g_{jk}^*(a^*) = \sum_{(\mathbf{d}_1)_{t(a)} + 1 \leq i \leq \mathbf{d}_{t(a)}} \rho(\omega_{t(a)})_{kj} \rho(a)_{ij} - \sum_{1 \leq i \leq (\mathbf{d}_1)_{s(a)}} \rho(\omega_{s(a)})_{ij} \rho(a)_{ki}.$$

We then have

Proposition 6.1.5. *The coefficients of upper right block of $\rho(a)_{ij}$ for all arrows $a, a^* \in Q_1$ are 0 iff all the entries of lower left block in $\rho(a)_{ij}$ for all $a, a^* \in Q_1$ are 0. i.e*

The functions $g_{jk}(a) = 0$ for all $1 \leq j \leq (\mathbf{d}_1)_{t(a)}$, $(\mathbf{d}_1)_{s(a)} + 1 \leq k \leq \mathbf{d}_{s(a)}$ and $g_{jk}(a^) = 0$ for all $1 \leq j \leq (\mathbf{d}_1)_{s(a)}$, $(\mathbf{d}_1)_{t(a)} + 1 \leq k \leq \mathbf{d}_{t(a)}$ if and only if $\rho(a)_{ki} = 0$ for all $(\mathbf{d}_1)_{t(a)} + 1 \leq k \leq \mathbf{d}_{t(a)}$, $1 \leq i \leq (\mathbf{d}_1)_{s(a)}$ and $\rho(a^*)_{ki} = 0$ for $(\mathbf{d}_1)_{s(a)} + 1 \leq k \leq \mathbf{d}_{s(a)}$, $1 \leq i \leq (\mathbf{d}_1)_{t(a)}$.*

Proof. The proposition relies on the fact that the difference between the generalized eigenvalues of the top diagonal and lower diagonal matrix can never be zero since they are in disjoint sets.

By assumption the matrices $\rho(\omega_i)$ are of Jordan normal type, so the entries $\rho(\omega_i)_{kl} = 0$ for $k \neq l$ and $k + 1 \neq l$. So for admissible choices of j, k , we have

$$g_{jk}(a) = (\rho(\omega_{t(a)})_{j,j} - \rho(\omega_{s(a)})_{k,k}) \rho(a^*)_{k,j} + \rho(\omega_{t(a)})_{j-1,j} \rho(a^*)_{k,j-1} - \rho(\omega_{s(a)})_{k,k+1} \rho(a^*)_{k+1,j} \quad (6.10)$$

- We plug $k = \mathbf{d}_{s(a)}$ and $j = 1$. Then

$$g_{\mathbf{d}_{s(a)}, 1} = 0 = (\rho(\omega_{t(a)})_{1,1} - \rho(\omega_{s(a)})_{\mathbf{d}_{s(a)}, \mathbf{d}_{s(a)}}) \rho(a^*)_{\mathbf{d}_{s(a)}, 1}$$

implies that $\rho(a^*)_{\mathbf{d}_{s(a)}, 1} = 0$ since $\rho(\omega_{t(a)})_{1,1} \in U_1$ and $\rho(\omega_{s(a)})_{\mathbf{d}_{s(a)}, \mathbf{d}_{s(a)}} \in U_2$ so $\rho(\omega_{s(a)})_{\mathbf{d}_{s(a)}, \mathbf{d}_{s(a)}}$ can never be disjoint.

- We now plug $k = \mathbf{d}_{s(a)}$ and $j = 2$. Then since $\rho(a^*)_{\mathbf{d}_{s(a)}, 1} = 0$, equation 6.10 implies that $(\rho(\omega_{t(a)})_{2,2} - \rho(\omega_{s(a)})_{\mathbf{d}_{s(a)}, \mathbf{d}_{s(a)}}) \rho(a^*)_{\mathbf{d}_{s(a)}, 2} = 0$ then again since $\rho(\omega_{t(a)})_{2,2}$ and $\rho(\omega_{s(a)})_{\mathbf{d}_{s(a)}, \mathbf{d}_{s(a)}}$ are in disjoint sets, $\rho(a^*)_{\mathbf{d}_{s(a)}, 2} = 0$. We then inductively conclude that $\rho(a^*)_{\mathbf{d}_{s(a)}, j} = 0$ for all $1 \leq j \leq (\mathbf{d}_1)_{t(a)}$.

- Now we may plug $k = \mathbf{d}_{s(a)} - 1$ and $j = 1$. Then since we already proved $\rho(a^*)_{\mathbf{d}_{s(a)},j} = 0$, by same argument as above we conclude that $\rho(a^*)_{\mathbf{d}_{s(a)}-1,1} = 0$ thus inductively $\rho(a^*)_{\mathbf{d}_{s(a)}-1,j} = 0$ for all $1 \leq j \leq (\mathbf{d}_1)_{t(a)}$. Then by induction on k , we finally conclude that $\rho(a^*)_{k,j} = 0$ for all $(\mathbf{d}_{s(a)} + 1 \leq k \leq \mathbf{d}_{t(a)})$ and $1 \leq j \leq (\mathbf{d}_1)_{t(a)}$.

The proof of vanishing of all admissible $\rho(a)_{ki}$ is exactly the same. □

Let $X = \text{Rep}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}$. Then it admits a $\text{GL}_{\mathbf{d}_1} \times \text{GL}_{\mathbf{d}_2}$ equivariant decomposition

$$\text{Rep}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}(L) \times \text{Rep}_{\mathbf{d}_1}(\overline{Q}) \times \text{Rep}_{\mathbf{d}_2}(\overline{Q}) \times \prod_{a \in \overline{Q}_1} \text{Hom}(\mathbb{C}^{(d_1)_{s(a)}}, \mathbb{C}^{(d_2)_{t(a)}}) \times \prod_{a \in \overline{Q}_1} \text{Hom}(\mathbb{C}^{(d_2)_{s(a)}}, \mathbb{C}^{(d_1)_{t(a)}}).$$

Let

$$\begin{aligned} X' &= \text{Rep}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}(L) \\ Y &= \text{Rep}_{\mathbf{d}_1}(\overline{Q}) \times \text{Rep}_{\mathbf{d}_2}(\overline{Q}) \\ Z_1 &= \prod_{a \in \overline{Q}_1} \text{Hom}(\mathbb{C}^{(d_1)_{s(a)}}, \mathbb{C}^{(d_2)_{t(a)}}) \\ Z_2 &= \prod_{a \in \overline{Q}_1} \text{Hom}(\mathbb{C}^{(d_2)_{s(a)}}, \mathbb{C}^{(d_1)_{t(a)}}) \end{aligned}$$

Let \mathbb{C}^* acts on X' trivially and with weight 1 on each of Y, Z_1, Z_2 by scaling. This action makes the function $\text{Tr}(W)_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}$ semi-invariant and commutes with the action of $\text{GL}_{\mathbf{d}_1} \times \text{GL}_{\mathbf{d}_2}$. Let $\mathbb{A}^n = Y \times Z_1 \times Z_2$, $\mathbb{A}^m = Y_2$, $\mathbb{A}^{n-m} = Y \times Z_1$. Then, by deformed dimension reduction (Theorem 3.3.1), and Proposition 6.1.5, it follows that there is a natural isomorphism of vector spaces

$$\text{H}(\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q}), {}^p\Phi_{\text{Tr}(\tilde{W})_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}}) \simeq \text{H}(\mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}), {}^p\Phi_{\text{Tr}(\tilde{W})_{\mathbf{d}_1}^{U_1}} \boxtimes {}^p\Phi_{\text{Tr}(\tilde{W})_{\mathbf{d}_2}^{U_2}}). \quad (6.11)$$

The isomorphism 6.11 is natural but not canonical, as there is a choice. We could have taken $\mathbb{A}^m = Y_1$ and $\mathbb{A}^{n-m} = Y \times Z_2$ and still get the isomorphism of vector spaces in Equation 6.11. However, as it turns out, they differ by sign. To see this, we prove isomorphism 6.11 in a slightly different way, seeing it as a case of Thoma-Sebastiani isomorphism and a more straightforward application of the usual dimension reduction theorem. This allows us to have the above isomorphism at the level of sheaves and under the above sign issue. We have the following lemma:

Lemma 6.1.6. *Let X be any smooth variety and G be any algebraic group acting on X and on an affine space \mathbb{A}^{2m} . Let*

$$f: \overline{X} = X \times \mathbb{A}^{2m} \rightarrow \mathbb{C}$$

be a G invariant function. Assume that there is G equivariant decomposition

$\mathbb{A}^{2m} = \mathbb{A}_1^m \times \mathbb{A}_2^m$ such that

$$f = f_0 + \sum_{1 \leq i \leq m} \left(\sum_{1 \leq j \leq m} g_{ij} y_j \right) x_i$$

such that g_0 is a lift of a function on X , y_1, \dots, y_m are the coordinates on \mathbb{A}_1^m and x_1, \dots, x_m are the coordinates on \mathbb{A}_2^m and $g_{ij}, 1 \leq i, j \leq m$ are linear functions on X such that matrix $G = \{g_{ij}, 1 \leq i, j \leq m\}$ is invertible. Let $i : X \rightarrow X \times \mathbb{A}^{2n}$ be the zero section. Then we have two different natural isomorphisms

$$D_1 : i^* \mathbb{P}\Phi_f \mathbb{Q}_X^{\text{vir}} \simeq \mathbb{P}\Phi_{f_0} \mathbb{Q}_X^{\text{vir}} \otimes \mathbb{H}(Z(y_1, \dots, y_m)) \quad (6.12)$$

and

$$D_2 : i^* \mathbb{P}\Phi_f \mathbb{Q}_X^{\text{vir}} \simeq \mathbb{P}\Phi_{f_0} \mathbb{Q}_X^{\text{vir}} \otimes \mathbb{H}(Z(x_1, \dots, x_m))$$

such that the following diagram is almost commutative

$$\begin{array}{ccc} i^* \mathbb{P}\Phi_f \mathbb{Q}_X^{\text{vir}} & \xrightarrow{D_1} & \mathbb{P}\Phi_{f_0} \mathbb{Q}_X^{\text{vir}} \otimes \mathbb{H}(Z(y_1, \dots, y_m)) \\ D_2 \downarrow & & \downarrow \pi_2 \\ \mathbb{P}\Phi_{f_0} \mathbb{Q}_X^{\text{vir}} \otimes \mathbb{H}(Z(x_1, \dots, x_m)) & \xrightarrow{\pi_1} & i^* \mathbb{P}\Phi_f \end{array}$$

i.e $\pi_2 D_1 = (-1)^m \pi_1 D_2$

Proof. Since g_{ij} is invertible, we have an isomorphism of complex analytic spaces

$$G : X \times \text{Spec}(\mathbb{C}[x_1, \dots, x_m]) \times \text{Spec}(\mathbb{C}[y_1, \dots, y_m]) \rightarrow X \times \text{Spec}(\mathbb{C}[x_1, \dots, x_m]) \times \text{Spec}(\mathbb{C}[\tilde{y}_1, \dots, \tilde{y}_m])$$

given by $\tilde{y}_i = \sum_{1 \leq j \leq m} g_{ij} y_j$ and rest is identity. With this choice, the following diagram

$$\begin{array}{ccc} X \times \text{Spec}(\mathbb{C}[x_1, \dots, x_m]) \times \text{Spec}(\mathbb{C}[y_1, \dots, y_m]) & \xrightarrow{f} & \mathbb{A}^1 \\ G \downarrow & \nearrow \tilde{f} & \\ X \times \text{Spec}(\mathbb{C}[x_1, \dots, x_m]) \times \text{Spec}(\mathbb{C}[\tilde{y}_1, \dots, \tilde{y}_m]) & & \end{array}$$

commutes, where $\tilde{f} = f_0 + \sum_{1 \leq i \leq m} x_i y_i$. By Thom-Sebastiani isomorphism, we have a natural isomorphism $\mathbb{P}\Phi_{\tilde{f}} \mathbb{Q}_X^{\text{vir}} \simeq \mathbb{P}\Phi_{f_0} \mathbb{Q}_X^{\text{vir}} \boxtimes \mathbb{P}\Phi_{\sum_{1 \leq i \leq m} x_i y_i} \mathbb{Q}_{\mathbb{A}^{2m}}^{\text{vir}}$. We thus have natural isomorphism

$$\pi_* \mathbb{P}\Phi_f \mathbb{Q}_X^{\text{vir}} \simeq \mathbb{P}\Phi_{f_0} \mathbb{Q}_X^{\text{vir}} \otimes \mathbb{H}(\mathbb{A}^{2m}, \mathbb{P}\Phi_{\sum_{1 \leq i \leq m} x_i y_i} \mathbb{Q}_{\mathbb{A}^{2m}}^{\text{vir}}).$$

To compute $\mathbb{H}(\mathbb{A}^{2m}, \mathbb{P}\Phi_{\sum_{1 \leq i \leq m} x_i y_i} \mathbb{Q}_{\mathbb{A}^{2m}}^{\text{vir}})$. We may consider the projection $\pi_1 : \mathbb{A}^{2m} = \mathbb{A}_1^m \times \mathbb{A}_2^m \rightarrow \mathbb{A}_1^m$ and think of $f = \sum_{1 \leq i \leq m} x_i y_i$ as function of weight 1, where we think of y_i as the coordinates of \mathbb{A}_2^m and x_i as lift of function on \mathbb{A}_1^m . Then, via the dimension reduction theorem, we have a natural isomorphism

$$\mathrm{H}(\mathbb{A}^{2m}, {}^p\Phi_{\sum_{1 \leq i \leq m} x_i y_i} \mathbb{Q}_{\mathbb{A}^{2m}}^{\mathrm{vir}}) \simeq \mathrm{H}(Z(y_1, \dots, y_m), \mathbb{Q}).$$

On the other hand, we may also consider the projection $\pi_2 : \mathbb{A}_1^n \times \mathbb{A}_2^n \rightarrow \mathbb{A}_2^n$ and do the dimension reduction where we think of x_i as coordinates. Then we get a natural isomorphism:

$$\mathrm{H}(\mathbb{A}^{2m}, {}^p\Phi_{\sum_{1 \leq i \leq m} x_i y_i} \mathbb{Q}_{\mathbb{A}^{2n}}^{\mathrm{vir}}) \simeq \mathrm{H}(Z(x_1, \dots, x_m), \mathbb{Q}).$$

Then the statement follows from work of Davison in [RS17][Lemma 4.1], where they check they check that the diagram

$$\begin{array}{ccc} \mathrm{H}(\mathbb{A}^{2m}, {}^p\Phi_{\sum_{1 \leq i \leq m} x_i y_i} \mathbb{Q}_{\mathbb{A}^{2m}}^{\mathrm{vir}}) & \xrightarrow{(\pi_1)_*} & \mathrm{H}(Z(y_1, \dots, y_m), \mathbb{Q}) \\ \downarrow (\pi_2)_* & & \downarrow = \\ \mathrm{H}(Z(x_1, \dots, x_m), \mathbb{Q}) & \xrightarrow{=} & \mathbb{Q} \end{array}$$

commutes upto a sign, i.e $(\pi_1)_* = (-1)^m (\pi_2)_*$. Then the statement follows, since ${}^p\Phi_f \mathbb{Q}_{\bar{X}}^{\mathrm{vir}}$ is supported on the image of the zero section i . □

We will now apply the above lemma to the situation when

$$\begin{aligned} X &= \mathrm{Rep}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathrm{Rep}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \\ \mathbb{A}_2^m &= \prod_{a \in \bar{Q}_1} \mathrm{Hom}(\mathbb{C}^{(d_1)_{s(a)}}, \mathbb{C}^{(d_2)_{t(a)}}) \\ \mathbb{A}_1^m &= \prod_{a \in \bar{Q}_1} \mathrm{Hom}(\mathbb{C}^{(d_2)_{s(a)}}, \mathbb{C}^{(d_1)_{t(a)}}) \end{aligned}$$

Then we have

$$D_1 : (i_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2})^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q})}^{\mathrm{vir}}) \simeq {}^p\Phi_{\mathrm{Tr}(\tilde{W})_{\mathbf{d}_1}^{U_1}}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q})}^{\mathrm{vir}}) \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})_{\mathbf{d}_2}^{U_2}}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q})}^{\mathrm{vir}})$$

Taking disjoint union over all decomposition $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$, we have

Proposition 6.1.7. *We have a natural isomorphism*

$$D_1 : i^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}(\mathbb{Q}_{\mathfrak{M}^{U_1 \amalg U_2}(\tilde{Q})}^{\mathrm{vir}}) \simeq {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}}(\mathbb{Q}_{\mathfrak{M}^{U_1}(\tilde{Q})}^{\mathrm{vir}}) \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}}(\mathbb{Q}_{\mathfrak{M}^{U_2}(\tilde{Q})}^{\mathrm{vir}})$$

Thus, we have natural isomorphisms:

$$\begin{array}{ccc} {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}(\mathbb{Q}_{\mathfrak{M}^{U_1 \amalg U_2}(\tilde{Q})}^{\mathrm{vir}}) & \xrightarrow{\eta} & i_* i^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}(\mathbb{Q}_{\mathfrak{M}^{U_1 \amalg U_2}(\tilde{Q})}^{\mathrm{vir}}) \\ & \searrow & \downarrow D_1 \\ & & i_* ({}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}}(\mathbb{Q}_{\mathfrak{M}^{U_1}(\tilde{Q})}^{\mathrm{vir}}) \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}}(\mathbb{Q}_{\mathfrak{M}^{U_2}(\tilde{Q})}^{\mathrm{vir}})) \end{array}$$

The first morphism is an isomorphism since the morphism i is an isomorphism

on the support of the underlying sheaves. Pushing forward to the point yields an isomorphism of vector spaces

$$I^{U_1 \amalg U_2} : H(\mathfrak{M}^{U_1 \amalg U_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)|_{U_1 \amalg U_2}}) \rightarrow H(\mathfrak{M}^{U_1}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)|_{U_1}}) \otimes H(\mathfrak{M}^{U_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(W)|_{U_2}}). \quad (6.13)$$

Proposition 6.1.8. *The morphism $I^{U_1 \amalg U_2} : \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1 \amalg U_2} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2}$ is an isomorphism of algebras.*

Proof. We proceed as in proof of Proposition 6.1.3. We have

$$\begin{array}{ccccc} & & \mathrm{Ext}_{(\mathbf{d}_1, \mathbf{e}_1), (\mathbf{d}_2, \mathbf{e}_2)}^{U_1 \amalg U_2} & & \\ & \swarrow^{\pi_1 \times \pi_3} & \uparrow^E & \searrow^{\pi_2} & \\ \mathfrak{M}_{\mathbf{d}_1, \mathbf{e}_1}^{U_1 \amalg U_2} \times \mathfrak{M}_{\mathbf{d}_2, \mathbf{e}_2}^{U_1 \amalg U_2} & & \mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1} \times \mathrm{Ext}_{\mathbf{e}_1, \mathbf{e}_2}^{U_2} & & \mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2, \mathbf{e}_1 + \mathbf{e}_2}^{U_1 \amalg U_2} \\ \uparrow^{i_{\mathbf{d}_1, \mathbf{e}_1}^{U_1 \amalg U_2} \times i_{\mathbf{d}_2, \mathbf{e}_2}^{U_1 \amalg U_2}} & \swarrow^{\Pi_{1,3} := \pi_1^{U_1} \times \pi_1^{U_2} \times \pi_3^{U_1} \times \pi_3^{U_2}} & & \searrow^{\Pi_2 := \pi_2^{U_1} \times \pi_2^{U_2}} & \uparrow^{i_{\mathbf{d}_1 + \mathbf{d}_2, \mathbf{e}_1 + \mathbf{e}_2}^{U_1 \amalg U_2}} \\ (\mathfrak{M}_{\mathbf{d}_1}^{U_1} \times \mathfrak{M}_{\mathbf{e}_1}^{U_2}) \times (\mathfrak{M}_{\mathbf{d}_2}^{U_1} \times \mathfrak{M}_{\mathbf{e}_2}^{U_2}) & & & & \mathfrak{M}_{\mathbf{d}_1 + \mathbf{d}_2}^{U_1} \times \mathfrak{M}_{\mathbf{e}_1 + \mathbf{e}_2}^{U_2} \end{array} \quad (6.14)$$

Lets denote $\mathfrak{M}_{\mathbf{d}, \mathbf{e}}^{U_1 \amalg U_2} = \overline{X}_{\mathbf{d}, \mathbf{e}}$ and $\mathfrak{M}_{\mathbf{d}}^{U_1} \times \mathfrak{M}_{\mathbf{e}}^{U_2} = X_{\mathbf{d}, \mathbf{e}}$ for any \mathbf{d}, \mathbf{e} , $I_{\mathbf{d}, \mathbf{e}}$ for any $i_{\mathbf{d}, \mathbf{e}}^{U_1 \amalg U_2}$ and when context is clear, I for $i_{\mathbf{d}_1, \mathbf{e}_1}^{U_1 \amalg U_2} \times i_{\mathbf{d}_2, \mathbf{e}_2}^{U_1 \amalg U_2}$.

Here in the first square, we can take the pullback, as in proposition 6.1.3. Since $\pi_1 \times \pi_3$ and $\Pi_{1,3}$ are affine fibrations, the dimension reduction morphism is also compatible, and the following diagram is commutative

$$\begin{array}{ccc} H(\overline{X}_{\mathbf{d}_1, \mathbf{e}_1} \times \overline{X}_{\mathbf{d}_2, \mathbf{e}_2}, \overline{{}^p\Phi} \boxtimes \overline{{}^p\Phi}) & \xrightarrow{(\pi_1 \times \pi_3)^*} & H(\mathrm{Ext}_{(\mathbf{d}_1, \mathbf{e}_1), (\mathbf{d}_2, \mathbf{e}_2)}^{U_1 \amalg U_2}, \overline{{}^p\Phi}) \\ (I_{\mathbf{d}_1, \mathbf{e}_1} \times I_{\mathbf{d}_2, \mathbf{e}_2})^* \downarrow & & E^* \downarrow \\ H(X_{\mathbf{d}_1, \mathbf{e}_1} \times X_{\mathbf{d}_2, \mathbf{e}_2}, I_{\mathbf{d}_1, \mathbf{e}_1}^*(\overline{{}^p\Phi}) \boxtimes I_{\mathbf{d}_2, \mathbf{e}_2}^*(\overline{{}^p\Phi})) & \xrightarrow{\Pi_{1,3}^*} & H(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1} \times \mathrm{Ext}_{\mathbf{e}_1, \mathbf{e}_2}^{U_2}, \Pi_{1,3}^* I^*(\overline{{}^p\Phi} \boxtimes \overline{{}^p\Phi})) \\ D_1 \times D_1 \downarrow & & D_1 \downarrow \\ H(X_{\mathbf{d}_1, \mathbf{e}_1} \times X_{\mathbf{d}_2, \mathbf{e}_2}, {}^p\Phi \boxtimes {}^p\Phi) & \xrightarrow{\Pi_{1,3}^*} & H(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1} \times \mathrm{Ext}_{\mathbf{e}_1, \mathbf{e}_2}^{U_2}, \Pi_{1,3}^* ({}^p\Phi \boxtimes {}^p\Phi)) \end{array}$$

The right-hand square in the commutative diagram 6.14 is not Cartesian. However, it is on the critical locus since given a short exact sequence of $\Pi_Q[\omega]$ modules

$$0 \rightarrow \rho_1 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow 0$$

where ρ_2 splits into $\rho_2^{U_1} \oplus \rho_2^{U_2}$ then ρ_1 and ρ_3 splits into direct sum $\rho_1^{U_1} \oplus \rho_1^{U_2}$ and $\rho_3^{U_1} \oplus \rho_3^{U_2}$ respectively such that we have short exact sequences $0 \rightarrow \rho_1^{U_i} \rightarrow \rho_2^{U_i} \rightarrow \rho_3^{U_i} \rightarrow 0$ for $i = 1, 2$. Thus by Lemma 6.1.9 and the fact that dimension reduction commutes with Kunneth isomorphism [Dav17][Proposition A.8], we have commutative diagram

$$\begin{array}{ccc}
\mathrm{H}(\mathrm{Ext}_{(\mathbf{d}_1, \mathbf{e}_1), (\mathbf{d}_2, \mathbf{e}_2)}^{U_1 \amalg U_2}, {}^p\Phi) & \xrightarrow{(\pi_2)_*} & \mathrm{H}(\overline{X}_{\mathbf{d}_1 + \mathbf{d}_2, \mathbf{e}_1 + \mathbf{e}_2}, \overline{{}^p\Phi}) \\
\downarrow E^* & & \downarrow I_{\mathbf{d}_1 + \mathbf{e}_1, \mathbf{d}_2 + \mathbf{e}_2}^* \\
\mathrm{H}(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1} \times \mathrm{Ext}_{\mathbf{e}_1, \mathbf{e}_2}^{U_2}, \Pi_{1,3}^* I^*(\overline{{}^p\Phi} \boxtimes \overline{{}^p\Phi})) & \xrightarrow{(\Pi_2)_*} & \mathrm{H}(X_{\mathbf{d}_1 + \mathbf{d}_2, \mathbf{e}_1 + \mathbf{e}_2}, I_{\mathbf{d}_1 + \mathbf{e}_1, \mathbf{d}_2 + \mathbf{e}_2}^* \overline{{}^p\Phi}) \\
\downarrow D_1 & & \downarrow D_1 \\
\mathrm{H}(\mathrm{Ext}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1} \times \mathrm{Ext}_{\mathbf{e}_1, \mathbf{e}_2}^{U_2}, \Pi_{1,3}^* ({}^p\Phi \boxtimes {}^p\Phi)) & \xrightarrow{(\Pi_2)_*} & \mathrm{H}(X_{\mathbf{d}_1 + \mathbf{e}_1, \mathbf{d}_2 + \mathbf{e}_2}, {}^p\Phi)
\end{array}$$

Thus we are done by composing the two commutative diagrams, as the top row defines CoHA multiplication for $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{U_1} \otimes \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{U_2}$ and the bottom row is the CoHA multiplication for $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{U_1 \amalg U_2}$ □

Lemma 6.1.9. *Let*

$$\begin{array}{ccc}
X & \xrightarrow{g'} & Y \\
f' \downarrow & & \downarrow f \\
Z & \xrightarrow{g} & Z'
\end{array} \tag{6.15}$$

be a commutative square of complex manifolds where morphism g, g' is proper and $A: Z' \rightarrow \mathbb{C}$ is a holomorphic function. Then the following diagram commutes

$$\begin{array}{ccc}
\mathrm{H}(X, (f')^* {}^p\Phi_{Ag} \mathbb{Q}_Z) & \longrightarrow & \mathrm{H}(Y, f^* {}^p\Phi_A \mathbb{Q}_{Z'}[2(\dim(Z') - \dim(Z))]) \\
\uparrow & & \uparrow \\
\mathrm{H}(Z, {}^p\Phi_{Ag} \mathbb{Q}_Z) & \longrightarrow & \mathrm{H}(Z', {}^p\Phi_A \mathbb{Q}_{Z'}[2(\dim(Z') - \dim(Z))])
\end{array}$$

if any of the following is true:

- (a) Above square is Cartesian and the morphisms f, f' are open inclusions.
- (b) f', f are closed immersions and in the commutative cube

$$\begin{array}{ccccc}
& & X & \xrightarrow{g'} & Y \\
& \nearrow i_X & \downarrow & & \downarrow f \\
\mathrm{crit}(Agf') & \xrightarrow{g'_c} & \mathrm{crit}(Af) & & \\
\downarrow f'_c & & \downarrow f_c & & \\
& \nearrow i_Z & Z & \xrightarrow{g} & Z' \\
\mathrm{crit}(Ag) & \xrightarrow{g_c} & \mathrm{crit}(A) & & \\
& \searrow i_{Z'} & & &
\end{array}$$

all faces except the back are known to be Cartesian and f'_c, f_c are isomor-

phism.

Proof. Part (a) follows from proper base change, since there is natural isomorphism $g'_*(f')^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) \simeq f^*g_* {}^p\Phi_{Ag}(\mathbb{Q}_Z)$. See [Dav17][Proposition 2.14] for how this implies commutativity at the level of cohomology. For part (b), since ${}^p\Phi_{Ag} \mathbb{Q}_Z$ is supported on $\text{crit}(Ag)$, we have isomorphism ${}^p\Phi_{Ag}(\mathbb{Q}_Z) = (i_Z)_*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z)$. Thus we have

$$\begin{aligned}
g'_*(f')^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) &= g'_*(f')^*(i_Z)_*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) \\
&= g'_*(i_X)_*(f'_c)^*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) \\
&= (i_Y)_*(g'_c)_*(f'_c)^*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) \\
&= (i_Y)_*(f_c)^*(g_c)_*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) \\
&= f^*(i_{Z'})_*(g_c)_*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) \\
&= f^*g_*(i_Z)_*(i_Z)^* {}^p\Phi_{Ag}(\mathbb{Q}_Z) = f^*g_* {}^p\Phi_{Ag}(\mathbb{Q}_Z)
\end{aligned}$$

and so the commutativity follows. \square

Thus, for any disjoint balls U_1 and U_2 , we have series of morphism of algebras

$$\mathcal{A}_{\tilde{Q}, \tilde{W}} \xrightarrow{\rho_{U_1} \amalg U_2} \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1 \amalg U_2} \xrightarrow{I^{U_1} \amalg U_2} \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2} \xrightarrow{\rho_{U_1}^{-1} \otimes \rho_{U_2}^{-1}} \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}$$

Composing gives a morphism of algebras

$$\Delta : \mathcal{A}_{\tilde{Q}, \tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}.$$

Proposition 6.1.10. *The morphism Δ doesn't depend on the choice of disjoint sets U_1 and U_2 .*

Proof. Note that if ρ is some \mathbf{d} dimensional representation of quiver \tilde{Q} then for any z , so does ρ' where $\rho'(\omega_i) = \rho(\omega_i) + z \text{Id}_{\mathbf{d}_i \times \mathbf{d}_i}$ for all $a \in \tilde{Q}_1 \neq \omega_i$ and $\rho'(a) = \rho(a)$ otherwise. We note that this transformation leaves $\text{Tr}(\tilde{W})(\rho)$ invariant. So we have an automorphism $T_z : \mathfrak{M}_{\mathbf{d}}(\tilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}}(\tilde{Q})$ which leaves the potential invariant and thus induces an isomorphism on the critical cohomology. Similarly for any $\lambda \neq 0$, we may scale the action i.e. we define a representation ρ' where $\rho'(\omega_i) = \lambda \rho(\omega_i)$, $\rho'(a) = \rho(a)/\lambda$ and $\rho'(a^*) = \rho'(a^*)$ for $a \in \tilde{Q}_1$. This defines an automorphism $S_\lambda : \mathfrak{M}_{\mathbf{d}}(\tilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}}(\tilde{Q})$ which leaves the potential invariant and thus induces an isomorphism on the critical cohomology.

Now suppose we are given two distinct balls U'_1 and U'_2 . By Proposition 6.1.3 and 6.1.4, we may assume that balls U_1, U_2, U'_1, U'_2 have sufficiently small radius ϵ . Then, doing translation S_λ for any appropriate choice of λ , we may assume that U'_1 and U'_2 are apart by the same distance as U_1 and U_2 . Then by the translation, there exist some z such that there is an isomorphism $(T^z)^* : \mathcal{A}_{\tilde{Q}, \tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}$ such

that following diagram commutes

$$\begin{array}{ccccc}
& & \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1 \amalg U_2} & \xrightarrow{\simeq} & \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_2} \\
& \nearrow^{\rho_{U_1 \amalg U_2}} & \uparrow & & \uparrow \\
\mathcal{A}_{\tilde{Q},\tilde{W}} & & (T_z)^*_{|U'_1 \amalg U'_2} & & (T_z)^*_{|U'_1} \otimes (T_z)^*_{|U'_2} \\
& \searrow & \downarrow & & \downarrow \\
& & \mathcal{A}_{\tilde{Q},\tilde{W}}^{U'_1 \amalg U'_2} & \xrightarrow{\simeq} & \mathcal{A}_{\tilde{Q},\tilde{W}}^{U'_1} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}^{U'_2} \\
& & & & \downarrow^{\rho_{U'_1}^{-1} \otimes \rho_{U'_2}^{-1}} \\
& & & & \mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}
\end{array}$$

and so morphism Δ is independent of the choice of disjoint sets U_1 and U_2 . \square

Proposition 6.1.11. *The morphism Δ is coassociative, i.e., the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{A}_{\tilde{Q},\tilde{W}} & \xrightarrow{\Delta} & \mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}} \\
\Delta \downarrow & & \downarrow \Delta \otimes 1 \\
\mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}} & \xrightarrow{1 \otimes \Delta} & \mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}
\end{array}$$

Proof. Let U_1, U_2 , and U_3 be three disjoint open balls. Then we have the restriction morphism

$$\rho_{U_1 \amalg U_2 \amalg U_3} : \mathcal{A}_{\tilde{Q},\tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1 \amalg U_2 \amalg U_3}$$

Furthermore, by taking the direct sum, we have a morphism $i^{U_1 \amalg U_2 \amalg U_3} : \mathfrak{M}^{U_1}(\tilde{Q}) \times \mathfrak{M}^{U_2}(\tilde{Q}) \times \mathfrak{M}^{U_3}(\tilde{Q}) \rightarrow \mathfrak{M}^{U_1 \amalg U_2 \amalg U_3}(\tilde{Q})$ which induces isomorphism

$$\mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1 \amalg U_2 \amalg U_3} \simeq \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_2} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_3}.$$

Finally we again have inverse of restriction maps giving series of morphisms

$$\Delta_2 : \mathcal{A}_{\tilde{Q},\tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1 \amalg U_2 \amalg U_3} \simeq \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_2} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}^{U_3} \xrightarrow{\rho_{U_1}^{-1} \otimes \rho_{U_2}^{-1} \otimes \rho_{U_3}^{-1}} \mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}} \otimes \mathcal{A}_{\tilde{Q},\tilde{W}}.$$

By exactly the same argument as in Proposition 6.1.10, the morphism Δ_2 is independent of the choice of disjoint sets U_1, U_2 , and U_3 . We claim that we have $(\Delta \otimes 1) \Delta = \Delta_2 = (1 \otimes \Delta) \Delta$, which proves the desired statement. Since the proof is the same, we will only show $\Delta_2 = (1 \otimes \Delta) \Delta$.

Let U'_2 and U''_2 be two disjoint balls inside ball U_2 . Then clearly the following

diagram commutes

$$\begin{array}{ccc}
\mathcal{A}_{\tilde{Q}, \tilde{W}} & & \\
\downarrow \rho_{U_1 \amalg U_2} & \searrow \rho_{U_1 \amalg U_2' \amalg U_2''} & \\
\mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1 \amalg U_2} & \xrightarrow{\rho_{U_1 \amalg U_2, U_1 \amalg U_2' \amalg U_2''}} & \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1 \amalg U_2' \amalg U_2''} \\
\downarrow i^{U_1 \amalg U_2} & & \downarrow i^{U_1, U_2' \amalg U_2''} \\
\mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2} & \xrightarrow{\rho_{U_2, U_2' \amalg U_2''}} & \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2' \amalg U_2''} \\
\downarrow \rho_{U_1}^{-1} \otimes \rho_{U_2}^{-1} & & \downarrow \rho_{U_1}^{-1} \otimes \text{id} \\
\mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}} & \xrightarrow{\text{id} \otimes \rho_{U_2' \amalg U_2''}^{-1}} & \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2' \amalg U_2''}
\end{array}$$

where the morphism $i^{U_1, U_2' \amalg U_2''}$ is the pullback of direct sum map $\mathfrak{M}^{U_1}(\tilde{Q}) \times \mathfrak{M}^{U_2' \amalg U_2''}(\tilde{Q}) \rightarrow \mathfrak{M}^{U_1 \amalg U_2' \amalg U_2''}(\tilde{Q})$. Since the morphism $\mathfrak{M}^{U_1}(\tilde{Q}) \times \mathfrak{M}^{U_2'}(\tilde{Q}) \times \mathfrak{M}^{U_2''}(\tilde{Q}) \rightarrow \mathfrak{M}^{U_1 \amalg U_2' \amalg U_2''}(\tilde{Q})$ is same as the composition $\mathfrak{M}^{U_1}(\tilde{Q}) \times \mathfrak{M}^{U_2'}(\tilde{Q}) \times \mathfrak{M}^{U_2''}(\tilde{Q}) \rightarrow \mathfrak{M}^{U_1}(\tilde{Q}) \times \mathfrak{M}^{U_2' \amalg U_2''}(\tilde{Q}) \rightarrow \mathfrak{M}^{U_1 \amalg U_2' \amalg U_2''}(\tilde{Q})$, we have commutative diagram

$$\begin{array}{ccc}
\mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1 \amalg U_2' \amalg U_2''} & & \\
\downarrow i^{U_1, U_2' \amalg U_2''} & \searrow i^{U_1 \amalg U_2' \amalg U_2''} & \\
\mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2' \amalg U_2''} & \xrightarrow{\text{id} \otimes i_{U_2' \amalg U_2''}^{-1}} & \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2'} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2''} \\
\downarrow \rho_{U_1}^{-1} \otimes \text{id} & & \downarrow \rho_{U_1}^{-1} \otimes \text{id} \otimes \text{id} \\
\mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2' \amalg U_2''} & \xrightarrow{\text{id} \otimes i_{U_2' \amalg U_2''}^{-1}} & \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2'} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2''} \xrightarrow{\rho_{U_1}^{-1} \otimes \rho_{U_2'}^{-1} \otimes \rho_{U_2''}^{-1}} \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}
\end{array}$$

Composing the two, we see that the bottom row is $(1 \otimes \Delta)$, the left vertical row is Δ , and the diagonal is Δ_2 . Thus, the statement follows. \square

Intuitively, since U_1 and U_2 can be exchanged via a homotopy, one can expect the coproduct to be cocommutative. We now claim that the morphism Δ is cocommutative to a sign. We have

Proposition 6.1.12. *We have*

$$\Delta_{\mathbf{d}, \mathbf{e}} = (-1)^{\sum_{a \in Q_1} \mathbf{d}_{s(a)} e_{t(a)} + e_{s(a)} \mathbf{d}_{t(a)}} \sigma(\Delta_{\mathbf{e}, \mathbf{d}}).$$

where $\sigma: \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}$ is the morphism defined by $\sigma(a \otimes b) = b \otimes a$.

Proof. Let $\sigma': \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q})$ be an isomorphism of stacks defined by sending $(\rho, \rho') \mapsto (\rho', \rho)$ and let $\sigma'': \mathfrak{M}_{\mathbf{d}_2, \mathbf{d}_1}^{U_2 \amalg U_1}(\tilde{Q}) \rightarrow \mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q})$

be a change of coordinates by exchanging the diagonal blocks and the offdiagonal blocks. In formulas, for any arrow $a \in \tilde{Q}$, it can be defined as

$$\sigma'' : \rho(a)_{ij} \mapsto \begin{cases} \rho(a)_{i+(\mathbf{d}_2)_{t(a),j+(\mathbf{d}_2)_{s(a)}}} & \text{when } (i, j) \in [1, (\mathbf{d}_1)_{t(a)}] \times [1, (\mathbf{d}_1)_{s(a)}] \\ \rho(a)_{i-(\mathbf{d}_1)_{t(a),j-(\mathbf{d}_1)_{s(a)}}} & \text{when } (i, j) \in [(\mathbf{d}_1)_{t(a)} + 1, \mathbf{d}_{t(a)}] \times [(\mathbf{d}_1)_{s(a)} + 1, (\mathbf{d}_1)_{s(a)}] \\ \rho(a)_{i-(\mathbf{d}_1)_{t(a),j+(\mathbf{d}_2)_{s(a)}}} & \text{when } (i, j) \in [(\mathbf{d}_1)_{t(a)} + 1, \mathbf{d}_{t(a)}] \times [1, (\mathbf{d}_1)_{s(a)}] \\ \rho(a)_{i+(\mathbf{d}_2)_{t(a),j-(\mathbf{d}_1)_{s(a)}}} & \text{when } (i, j) \in [1, (\mathbf{d}_1)_{t(a)}] \times [(\mathbf{d}_1)_{s(a)} + 1, (\mathbf{d}_1)_{s(a)}] \end{cases}$$

So we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) & \xrightarrow{i_{\mathbf{d}_1, \mathbf{d}_2}^{U_1, U_2}} & \mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}(\tilde{Q}) \hookrightarrow \mathfrak{M}^{U_1 \amalg U_2}(\tilde{Q}) \\ \sigma' \uparrow & & \uparrow \sigma'' \\ \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) & \xrightarrow{i_{\mathbf{d}_2, \mathbf{d}_1}^{U_2, U_1}} & \mathfrak{M}_{\mathbf{d}_2, \mathbf{d}_1}^{U_2 \amalg U_1}(\tilde{Q}) \end{array}$$

Taking pullback, the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1, \mathbf{d}_2}^{U_1 \amalg U_2}, {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) & \longrightarrow & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}), i_{\mathbf{d}_1, \mathbf{d}_2}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) \\ (\sigma'')^* \downarrow & & (\sigma')^* \downarrow \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2, \mathbf{d}_1}^{U_2 \amalg U_1}, (\sigma')^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) & \longrightarrow & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}), (\sigma')^* i_{\mathbf{d}_1, \mathbf{d}_2}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) \\ = \downarrow & & = \downarrow \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2, \mathbf{d}_1}^{U_2 \amalg U_1}, {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2 \amalg U_1}}) & \longrightarrow & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}), i_{\mathbf{d}_2, \mathbf{d}_1}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2 \amalg U_1}}) \end{array}$$

The morphism $D_1 : i_{\mathbf{d}_1, \mathbf{d}_2}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}} \rightarrow {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}} \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}}$ is given by dimension reduction of top right block of $\rho(a)$ and $\rho(a^*)$, i.e. the space $\prod_{a \in \tilde{Q}_1} \mathrm{Hom}(\mathbb{C}^{(d_1)_{s(a)}}, \mathbb{C}^{(d_2)_{t(a)}})$. So, due to the change of coordinates σ , the morphism $D_1 : (\sigma'')^*(i_{\mathbf{d}_2, \mathbf{d}_1}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) \rightarrow {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}} \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}}$ is then exactly same as dimensionally reducing the space $\prod_{a \in \tilde{Q}_1} \mathrm{Hom}(\mathbb{C}^{(d_2)_{s(a)}}, \mathbb{C}^{(d_1)_{t(a)}})$. i.e., it is the morphism $D_2 : (i_{\mathbf{d}_2, \mathbf{d}_1}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2 \amalg U_1}}) \rightarrow {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}} \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}}$. Thus, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}), i_{\mathbf{d}_1, \mathbf{d}_2}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) & \xrightarrow{D_1} & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}} \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}}) \\ (\sigma')^* \downarrow & & \downarrow (\sigma')^* \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}), (\sigma')^* i_{\mathbf{d}_1, \mathbf{d}_2}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1 \amalg U_2}}) & \xrightarrow{D_2} & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}} \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}}) \\ = \downarrow & & \downarrow (-1)^{\dim(\prod_{a \in \tilde{Q}_1} \mathrm{Hom}(\mathbb{C}^{(d_2)_{s(a)}}, \mathbb{C}^{(d_1)_{t(a)}}))} \\ \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}), i_{\mathbf{d}_2, \mathbf{d}_1}^* {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2 \amalg U_1}}) & \xrightarrow{D_1} & \mathrm{H}(\mathfrak{M}_{\mathbf{d}_2}^{U_2}(\tilde{Q}) \times \mathfrak{M}_{\mathbf{d}_1}^{U_1}(\tilde{Q}), {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_2}} \boxtimes {}^p\Phi_{\mathrm{Tr}(\tilde{W})^{U_1}}) \end{array}$$

Where the bottom square commutes by the Lemma 6.1.6. Composing the above two commutative diagrams gives the desired result. \square

In earlier sections, we twisted m . Consider twisted coproduct Δ^x defined by

$$\Delta_{\mathbf{d},\mathbf{e}}^x = (-1)^{\chi_Q(\mathbf{d},\mathbf{e})} \Delta_{\mathbf{d},\mathbf{e}}.$$

Since Δ is coassociative, clearly Δ^x is also coassociative. By this sign twist, we now have a cocommutative coproduct.

Proposition 6.1.13. *The twisted coproduct Δ^x is cocommutative and compatible with m^x .*

Proof. Co-commutativity is clear by Proposition 6.1.1 since

$$\begin{aligned} \sigma(\Delta_{\mathbf{d},\mathbf{e}}^x) &= (-1)^{\chi_Q(\mathbf{d},\mathbf{e})} \sigma(\Delta_{\mathbf{d},\mathbf{e}}) = (-1)^{\chi_Q(\mathbf{d},\mathbf{e}) + \sum_{a \in Q_1} \mathbf{d}_{s(a)} \mathbf{e}_{t(a)} + \mathbf{e}_{s(a)} \mathbf{d}_{t(a)}} \Delta_{\mathbf{e},\mathbf{d}} \\ &= (-1)^{\chi_Q(\mathbf{e},\mathbf{d})} \Delta_{\mathbf{e},\mathbf{d}} = \Delta_{\mathbf{e},\mathbf{d}}^x. \end{aligned}$$

and for compatibility, let a and b be elements of dimension vector \mathbf{d}_1 and \mathbf{d}_2 respectively. Then

$$\begin{aligned} \Delta_{\mathbf{d},\mathbf{e}}^x(m^x(a \otimes b)) &= (-1)^{\chi_Q(\mathbf{d},\mathbf{e})} \Delta_{\mathbf{d},\mathbf{e}}(m^x(a \otimes b)) \\ &= (-1)^{\chi_Q(\mathbf{d},\mathbf{e}) + \chi_Q(\mathbf{d}_1,\mathbf{d}_2)} \Delta_{\mathbf{d},\mathbf{e}}(m(a \otimes b)) \\ &= (-1)^{\chi_Q(\mathbf{d},\mathbf{e}) + \chi_Q(\mathbf{d}_1,\mathbf{d}_2)} (m(\Delta(a) \otimes \Delta(b))) \end{aligned}$$

which is the same as

$$\begin{aligned} &(-1)^{\chi_Q(\mathbf{d},\mathbf{e}) + \chi_Q(\mathbf{d}_1,\mathbf{d}_2)} \sum_{\substack{x_1+x_2=\mathbf{d} \\ y_1+y_2=\mathbf{e} \\ x_1+y_1=\mathbf{d}_1 \\ x_2+y_2=\mathbf{d}_2}} m(\Delta_{x_1,y_1}(a) \otimes \Delta_{x_2,y_2}(b)) \\ &= (-1)^{\chi_Q(\mathbf{d},\mathbf{e}) + \chi_Q(\mathbf{d}_1,\mathbf{d}_2)} \sum_{\substack{x_1+x_2=\mathbf{d} \\ y_1+y_2=\mathbf{e} \\ x_1+y_1=\mathbf{d}_1 \\ x_2+y_2=\mathbf{d}_2}} (-1)^{\chi_Q(x_1,x_2) + \chi_Q(y_1,y_2)} m^x(\Delta_{x_1,y_1}(a) \otimes \Delta_{x_2,y_2}(b)) \\ &(-1)^{\chi_Q(\mathbf{d},\mathbf{e}) + \chi_Q(\mathbf{d}_1,\mathbf{d}_2)} \sum_{\substack{x_1+x_2=\mathbf{d} \\ y_1+y_2=\mathbf{e} \\ x_1+y_1=\mathbf{d}_1 \\ x_2+y_2=\mathbf{d}_2}} (-1)^{\chi_Q(x_1,x_2) + \chi_Q(y_1,y_2) + \chi_Q(x_1,y_1) + \chi_Q(x_2,y_2)} m^x(\Delta_{x_1,y_1}(a) \otimes \Delta_{x_2,y_2}(b)) \\ &= \sum_{\substack{x_1+x_2=\mathbf{d} \\ y_1+y_2=\mathbf{e} \\ x_1+y_1=\mathbf{d}_1 \\ x_2+y_2=\mathbf{d}_2}} m^x(\Delta_{x_1,y_1}^x(a) \otimes \Delta_{x_2,y_2}^x(b)) \end{aligned}$$

as desired. □

So Δ^x is a cocommutative coproduct on a connected algebra². Any connected graded bialgebra is a Hopf algebra. Thus by Milnor-Milnor-Moore theorem $\mathcal{A}_{\widetilde{Q},\widetilde{W}}^x \simeq \mathbf{U}(\mathcal{P})$ where \mathcal{P} are the primitive elements with respect to the coproduct

²A graded algebra A is connected if A_0 is the ground field and $A_n = 0 \forall n \leq -1$.

Δ^x . We will now compute primitive elements for the coproduct Δ^x . We first observe that

Proposition 6.1.14. *The action of tautological bundle $(u \cdot): \mathcal{A}_{\tilde{Q}, \tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}$ is a coderivation, i.e we have*

$$\Delta^x(u \cdot x) = (u \otimes 1 + 1 \otimes u) \cdot \Delta^x(x).$$

Proof. Clearly the restriction morphism $\mathcal{A}_{\tilde{Q}, \tilde{W}} \rightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}^U$ is u linear since we have commutative diagram

$$\begin{array}{ccc} \mathrm{BC}_u^* \times \mathfrak{M}_d(\tilde{Q}) & \xleftarrow{\mathrm{act}} & \mathfrak{M}_d(\tilde{Q}) \\ \uparrow \mathrm{id} \times i_U & & i_U \uparrow \\ \mathrm{BC}_u^* \times \mathfrak{M}_d^U(\tilde{Q}) & \xleftarrow{\mathrm{act}} & \mathfrak{M}_d^U(\tilde{Q}) \end{array}$$

On the other hand, we have a commutative diagram:

$$\begin{array}{ccc} \mathfrak{M}_d^{U_1}(\tilde{Q}) \times \mathfrak{M}_e^{U_2}(\tilde{Q}) & \xrightarrow{i_{d,e}} & \mathfrak{M}^{U_1 \amalg U_2}(\tilde{Q}) \\ \mathrm{Det} \otimes \mathrm{Det} \downarrow & & \downarrow \mathrm{Det} \\ \mathrm{BC}_u^* \times \mathrm{BC}_u^* & \xrightarrow{\otimes} & \mathrm{BC}_u^* \end{array}$$

and thus the isomorphism $i^*: \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1} \otimes \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_2} \simeq \mathcal{A}_{\tilde{Q}, \tilde{W}}^{U_1 \amalg U_2}$ satisfies $i_{d,e}^*(u \cdot x) = \otimes^*(u) \cdot i_{d,e}^*(x) = (u \otimes 1 + 1 \otimes u) i_{d,e}^*(x)$. Thus

$$\begin{aligned} \Delta(u \cdot x) &= (\rho_{U_1}^{-1} \otimes \rho_{U_2}^{-1})(i_{d,e}^*)(\rho_{U_1 \amalg U_2}(u \cdot x)) = (\rho_{U_1}^{-1} \otimes \rho_{U_2}^{-1})(i_{d,e}^*)(u \cdot \rho_{U_1 \amalg U_2}(x)) \\ &= (u \otimes 1 + 1 \otimes u) \Delta(x) \end{aligned}$$

Then the same holds for the twisted by coproduct Δ^x . □

Proposition 6.1.15. *The elements of $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}} \otimes \mathbb{C}[u]$ are primitive with respect to the coproduct Δ^x .*

Proof. By the support lemma [Dav23c][Lemma 4.1], $\mathcal{BPS}_{\tilde{Q}, \tilde{W}}$ is supported on the space with only one eigenvalue. Thus $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}}$ is primitive for the Δ coproduct. Then the claim follows from the previous proposition. □

Remark 6.1.16. We can now, in fact, give a shorter proof of cocommutativity. We know from integrality theorem that $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}} \otimes \mathbb{C}[u]$ generates the algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}}^x$. Since $\mathcal{A}_{\tilde{Q}, \tilde{W}}^x$ is compatible with Δ^x , it suffices to check the cocommutativity of the generators. But by the previous proposition, the elements inside the vector subspace $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}} \otimes \mathbb{C}[u]$ are primitive with respect to Δ^x and hence cocommutative.

By the PBW theorem (Proposition 4.1.1), $\mathcal{A}_{\tilde{Q}, \tilde{W}}^x \simeq \mathrm{Sym}(\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}} \otimes \mathbb{C}[u])$ and so by considering the size of \mathcal{P} , it follows that $\mathcal{P} = \mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}} \otimes \mathbb{C}[u]$. This shows that the sub vector space $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\mathrm{BPS}} \otimes \mathbb{C}[u] \hookrightarrow \mathcal{A}_{\tilde{Q}, \tilde{W}}^x$ is closed under the Lie bracket.

Definition 6.1.17 (The affinized BPS Lie algebra). For any quiver Q , the affinized BPS Lie algebra $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is defined to be the Lie algebra structure on the $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded vector space

$$\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}} := \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}} \otimes \mathbb{C}[u] \hookrightarrow \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{\chi}$$

where the Lie bracket is defined by taking the commutator Lie bracket coming from the associative algebra structure on $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{\chi}$.

Let $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i}$ denote the dimension \mathbf{d} , cohomological degree i piece of $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$. By Proposition 4.2.3, for a fixed dimension vector \mathbf{d} , the Poincare series of $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is given by

$$\sum_{i \in \mathbb{Z}} \dim(\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i}) q^{i/2} = (1 - q)^{-1} a_{Q, \mathbf{d}}(q^{-1}) \quad (6.16)$$

where $a_{Q, \mathbf{d}}(q)$ is the Kac Polynomial. Note that $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is not necessarily isomorphic to the trivial extension of $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ as *Lie algebras*, even though they are isomorphic as vector spaces.

Convention 3. Given an element $\alpha \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$, we denote by

$$\alpha^{(n)} := u^n \cdot \alpha \in \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$$

6.2 Deformed Affinized BPS Lie Algebra

Let $\mathbf{w} : \widetilde{Q} \rightarrow N$ be a torus weighting such that $\mathbf{w}(\omega_i) = 0$ for all $i \in Q_0$ ³. Let $T' = \text{Hom}(N, \mathbb{C}^*)$. Then since $\mathbf{w}(\omega) = 0$, where $\omega = \sum_{i \in Q_0} \omega_i$, the morphism

$$\lambda : \mathfrak{M}_{\mathbf{d}}(\widetilde{Q}) \rightarrow \text{Sym}(\mathbb{A}^1)$$

which records the generalized eigenvalue, is T' equivariant. This induces a morphism

$$\lambda^{T'} : \mathfrak{M}_{\mathbf{d}}^{T'}(\widetilde{Q}) \rightarrow \text{Sym}(\mathbb{A}^1).$$

We can apply the construction of factorization coproduct to $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T', \chi}$, where we consider $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T', \chi}$ as an object in the category of $H_{T'}(\text{pt})$ modules, i.e. we have a $H_{T'}(\text{pt})$ module morphism

$$\Delta^{T'} : \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T', \chi} \rightarrow \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T', \chi} \otimes_{H_{T'}(\text{pt})} \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T', \chi}.$$

Considering the primitive elements of the cocommutative coproduct $\Delta^{T'}$, shows that the $H_{T'}(\text{pt})$ module

$$\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, T'} := \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, T'} \otimes \mathbb{C}[u] \subset \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T', \chi}$$

³This condition is crucial. This construction doesn't work, for example, in the case of tripled ADE quiver with canonical potential with torus weighting $\mathbf{w}(a) = 1, \mathbf{w}(a^*) = 1$ and $\mathbf{w}(\omega_i) = -2$ for all $i \in Q_0$.

has a structure of Lie algebra, where the Lie bracket is defined by taking the commutator Lie bracket, coming from the associative algebra structure on $\mathcal{A}_{\tilde{Q}, \tilde{W}}^{T', \chi}$. Furthermore, there is an isomorphism of $H_{T'}(\text{pt})$ algebras

$$\mathbf{U}_{H_{T'}(\text{pt})}(\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}, T'}) \simeq \mathcal{A}_{\tilde{Q}, \tilde{W}}^{T', \chi}.$$

We call $\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}, T'}$ the T' -deformed affinized BPS Lie algebra.

6.3 Heis Lie algebra action

The action of the tautological bundle on the vector space $\mathcal{A}_{Q, W}^{\psi}$ increases the cohomological degree by 2. We can describe the other half of this action, which acts by decreasing the cohomological degree. This structure has been exploited in [Joy21] as a translation operator in the construction of vertex algebras and by [Dav22b] for the calculation of the affinized BPS Lie algebra for the Jordan quiver. We consider the action by scaling by the automorphisms of $\mathfrak{M}_{\mathbf{d}}^{T, \zeta}(Q)$. More precisely, we have a morphism

$$\text{act} : \text{BC}_{\mathbf{u}}^* \times \mathfrak{M}_{\mathbf{d}}^{T, \zeta}(Q) \rightarrow \mathfrak{M}_{\mathbf{d}}^{T, \zeta}(Q)$$

defined by sending the tautological line bundle \mathcal{L} and family of representations \mathcal{F} to $\mathcal{F} \otimes \mathcal{L}$. Doing the pullback (3.2.2), gives a map of sheaves

$${}^p\Phi_{\text{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T, \zeta-\text{ss}}(Q)}) \rightarrow \text{act}_*(\mathbb{Q}_{\text{BC}_{\mathbf{u}}^*} \boxtimes {}^p\Phi_{\text{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T, \zeta-\text{ss}, \psi}(Q)})). \quad (6.17)$$

After shifting and taking global sections, this gives a map of cohomologically graded vector spaces

$$\text{act}^* : H(\mathfrak{M}_{\mathbf{d}}^{T, \zeta-\text{ss}}(Q), {}^p\Phi_{\text{Tr}(W_{\mathbf{d}})}) \rightarrow H(\text{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes H(\mathfrak{M}_{\mathbf{d}}^{T, \zeta-\text{ss}}(Q), {}^p\Phi_{\text{Tr}(W_{\mathbf{d}})}).$$

In particular, for any $\alpha \in \mathcal{A}_{Q, W, \mathbf{d}}^{T, \zeta, \psi}$ we have

$$\text{act}^*(\alpha) = \sum_{n \geq 0} u^n \otimes \alpha_{(n)}.$$

We define

$$\partial : \mathcal{A}_{Q, W}^{T, \zeta, \psi} \rightarrow \mathcal{A}_{Q, W}^{T, \zeta, \psi}$$

by $\alpha \mapsto \alpha_{(1)}$. Since the morphism act^* preserves the cohomological degree, ∂ is an operator which decreases the cohomological degree by -2 . We then have

Proposition 6.3.1. [Dav22a] *The morphisms*

$$\begin{aligned} \partial &: \mathcal{A}_{Q, W}^{T, \zeta, \psi} \rightarrow \mathcal{A}_{Q, W}^{T, \zeta, \psi} \\ u &: \mathcal{A}_{Q, W}^{T, \zeta, \psi} \rightarrow \mathcal{A}_{Q, W}^{T, \zeta, \psi} \end{aligned}$$

define derivations on $\mathcal{A}_{Q,W}^{T,\zeta,\psi}$ i.e.

$$\begin{aligned}\partial(a \star b) &= (\partial a) \star b + a \star (\partial b) \\ u(a \star b) &= (u \cdot a) \star b + a \star (u \cdot b).\end{aligned}$$

Furthermore, for any $\alpha \in \mathcal{A}_{Q,W,d}^{T,\zeta,\psi}$, we have $[\partial, u]\alpha = |\mathbf{d}|\alpha$.

We now show that the operators ∂ and u interact with the perverse filtration \mathfrak{P} on $\mathcal{A}_{Q,W}^{T,\zeta,\psi}$ in a nice way.

Proposition 6.3.2. *The action u increases the perverse degree by 2 while ∂ decreases the perverse degree by 2, i.e.*

$$\begin{aligned}u \cdot \mathfrak{P}^i &\subset \mathfrak{P}^{i+2} \\ \partial \cdot \mathfrak{P}^i &\subset \mathfrak{P}^{i-2}\end{aligned}$$

Proof. Applying $\mathrm{JH}_{\mathbf{d}}^{T,\zeta}$ to morphism (6.17), gives a morphism of complexes in $\mathrm{D}^b(\mathcal{M}_{\mathbf{d}}^{T,\zeta-ss}(Q))$

$$a : (\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}}) \rightarrow (\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathrm{act}_*(\mathbb{Q}_{\mathrm{BC}_{\mathbf{u}}^*} \boxtimes \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}})).$$

After taking semisimplification, the action map collapses, and so we have the following commutative square:

$$\begin{array}{ccc} \mathrm{BC}_{\mathbf{u}}^* \times \mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q) & \xrightarrow{\mathrm{act}} & \mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q) \\ \downarrow \pi_2 & & \downarrow \mathrm{JH}_{\mathbf{d}}^{T,\zeta} \\ \mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q) & \xrightarrow{\mathrm{JH}_{\mathbf{d}}^{T,\zeta}} & \mathcal{M}_{\mathbf{d}}^{T,\zeta-ss}(Q) \end{array}$$

where π_2 is the projection to the second component. Thus we have

$$(\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathrm{act}_*(\mathbb{Q}_{\mathrm{BC}_{\mathbf{u}}^*} \boxtimes \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}})) \simeq \mathrm{H}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes (\mathrm{JH}_{\mathbf{d}}^{T,\zeta-ss})_* \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}}).$$

But since

$$\tau^{\leq n}(\mathrm{H}^{2k}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes (\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}})) = \tau^{\leq n-2k}((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}})),$$

the morphism $\tau^{\leq n}(a)$ factors through

$$\bigoplus_{k \geq 0} \mathrm{H}^{2k}(\mathrm{BC}_{\mathbf{u}}^*, \mathbb{Q}) \otimes \tau^{n-2k}((\mathrm{JH}_{\mathbf{d}}^{T,\zeta})_* \mathbb{P}\Phi_{\mathrm{Tr}(W_{\mathbf{d}})}(\mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^{T,\zeta-ss}(Q)}^{\mathrm{vir}})).$$

So for any $\alpha \in \mathfrak{P}^i$, $\mathrm{act}^*(\alpha) = \sum_{n \geq 0} u^n \alpha_n$ where $\alpha_n \in \mathfrak{P}^{i-2n}$ and thus in particular $\alpha_1 \in \mathfrak{P}^{\leq i-2}$. Finally, the action of u increases the perverse degree, which is exactly [DM20][Lemma 5.8] applied to the determinant bundle. \square

The above argument also shows that if $\alpha_{\mathbf{d}} \in \mathfrak{g}_{Q,W,\mathbf{d}}^{\mathrm{BPS},T,\zeta}$ then $\partial(\alpha_{\mathbf{d}}) = 0$, since

the perverse filtration starts from degree 1 (Section 4.1.5). Now, by repetitively applying the Proposition 6.3.2, it follows that

Proposition 6.3.3. *Let $\alpha_{\mathbf{d}} \in \mathfrak{g}_{Q,W,\mathbf{d}}^{\text{BPS},T,\zeta}$, then $u^n \alpha_{\mathbf{d}} \in \mathfrak{P}^{2n+1}$ and we have $\partial(u^n \alpha_{\mathbf{d}}) = n|\mathbf{d}|u^{n-1} \alpha_{\mathbf{d}}$.*

The action of u, ∂ can be interpreted as an action of the 3-dimensional Heisenberg **Heis** on $\mathcal{A}_{Q,W}^{T,\zeta}$, as considered in [Dav22a].

Definition 6.3.4. Let **Heis** be the Lie algebra over \mathbb{Q} having basis p, q, c such that

$$[q, p] = c,$$

c is central. We give **Heis** a ‘cohomological’ grading by setting elements p, q, c as elements of cohomological degree 2, $-2, 0$ respectively. Then M is said to be a graded **Heis** module, if $M = \bigoplus_{i \in \mathbb{Z}} M_{2i}$ and the **Heis** action respects the grading, i.e. $p(M_{2i}) \subset M_{2i+2}, q(M_{2i}) \subset M_{2i-2}, c(M_{2i}) \subset M_{2i}$.

If a graded **Heis** module \mathfrak{L} has a Lie algebra structure, then it is said to be a **Heis** Lie algebra if p and q act by Lie algebra derivations on \mathfrak{L} . In many cases, a much smaller subspace controls **Heis** Lie algebras. For any **Heis** Lie algebra \mathfrak{L} , let $\mathfrak{L}^{\leq 0} = \bigoplus_{i \leq 0} \mathfrak{L}_{2i}$ be the the subspace of non-positive even graded elements. Clearly $\mathfrak{L}^{\leq 0} \subset \mathfrak{L}$ forms a Lie subalgebra.

Definition 6.3.5 (Negatively determined **Heis** Lie algebras). We say that a **Heis** Lie algebra \mathfrak{L} is negatively determined if $\mathfrak{L}_i = 0$ for all odd i and the map $q : \mathfrak{L}_{2i} \rightarrow \mathfrak{L}_{2i-2}$ is an isomorphism of vector spaces for any $i > 0$.

If we have two negatively determined Lie algebras, then a morphism between them is also negatively determined.

Proposition 6.3.6. *Let \mathfrak{L} and \mathfrak{H} be two negatively determined **Heis** Lie algebras such that there is a homomorphism of Lie algebras*

$$\phi^{\leq 0} : \mathfrak{L}^{\leq 0} \rightarrow \mathfrak{H}^{\leq 0}$$

*which commutes with the action of **Heis**, i.e., where for any $\alpha_{-2i} \in \mathfrak{L}_{-2i}, i > 0$, $\phi^{\leq 0}(p\alpha_{-2i}) = p\phi^{\leq 0}(\alpha_{-2i})$, for any $\alpha_{-2i+2} \in \mathfrak{L}_{-2i+2}, i > 0$, $\phi^{\leq 0}(q\alpha_{-2i+2}) = q\phi^{\leq 0}(\alpha_{-2i+2})$ and for any $\alpha_0 \in \mathfrak{L}_0, \phi^{\leq 0}(c\alpha_0) = c\phi^{\leq 0}(\alpha_0)$. Then, the map $\phi^{\leq 0}$ extends uniquely to map $\phi : \mathfrak{L} \rightarrow \mathfrak{H}$ of graded **Heis** modules. Furthermore, ϕ is a Lie algebra homomorphism if and only if $\phi([\alpha_{2i}, \phi_{2j}]) = [\phi(\alpha_{2i}), \phi(\alpha_{2j})]$ for $i + j \leq 0$.*

Proof. For any element $\alpha_{2i} \in \mathfrak{L}_{2i}$ with $i > 0$, we define $\phi(\alpha_{2i})$ to be the unique element in \mathfrak{H}_{2i} such that $q^i \phi(\alpha_{2i}) = \phi^{\leq 0}(q^i \alpha_{2i})$ and for any $\alpha_{2i} \in \mathfrak{L}_{2i}$ with $i \leq 0$, we set $\phi(\alpha_{2i}) = \phi^{\leq 0}(\alpha_{2i})$. The map is well defined by the injectivity of q on positive components.

The morphism ϕ is a map of **Heis** modules since when $i = 1$, any $\alpha_2 \in \mathfrak{L}_2$ satisfies $q\phi(\alpha_2) = \phi^{\leq 0}(q\alpha_2)$ and if $i > 1$, then $\phi(q\alpha_{2i})$ is defined to be unique element in \mathfrak{H} such that $q^{i-1} \phi(q\alpha_{2i}) = \phi^{\leq 0}(q^i \alpha_{2i})$, which is clearly satisfied by $q\phi(\alpha_{2i})$. Similarly, since $q^i c\phi(\alpha_{2i}) = cq^i \phi(\alpha_{2i}) = c\phi^{\leq 0}(q^i \alpha_{2i}) = \phi^{\leq 0}(cq^i \alpha_{2i}) = \phi^{\leq 0}(q^i c\alpha_{2i})$

implies that $\phi(c\alpha_{2i}) = c\phi(\alpha_{2i})$. Finally successive application of $[q, p] = c$ implies that for any $\alpha \in \mathfrak{H}$ or \mathfrak{L} , $q^{i+1}p\alpha = pq^{i+1}\alpha + (i+1)q^i c\alpha$. So $q^{i+1}(p\phi(\alpha_{2i})) = (pq^{i+1} + (i+1)q^i c)\phi(\alpha_{2i}) = pq\phi^{\leq 0}(q^i\alpha_{2i}) + (i+1)c\phi^{\leq 0}(q^i\alpha_{2i}) = \phi^{\leq 0}((pq^{i+1} + (i+1)cq^i)\alpha_{2i}) = \phi^{\leq 0}(q^{i+1}p\alpha_{2i})$. Thus $p\phi(\alpha_{2i}) = \phi(p\alpha_{2i})$.

It suffices to check that it is indeed a Lie algebra homomorphism if $\phi([\alpha_{2i}, \beta_{2j}]) = [\phi(\alpha_{2i}), \phi(\beta_{2j})]$ for $i+j \leq 0$. For this, we consider $\alpha_{2i} \in \mathfrak{L}_{2i}$ and $\beta_{2j} \in \mathfrak{L}_{2j}$. If $i+j \leq 0$, then $\phi([\alpha_{2i}, \beta_{2j}]) = ([\phi(\alpha_{2i}), \phi(\beta_{2j})])$ by the assumption. So, we can assume $i+j > 0$. Then $\phi([\alpha_{2i}, \beta_{2j}])$ is the unique element such that

$$q^{i+j}\phi([\alpha_{2i}, \beta_{2j}]) = \phi^{\leq 0}(q^{i+j}[\alpha_{2i}, \beta_{2j}]) = \sum_{r=0}^{i+j} \binom{i+j}{r} \phi^{\leq 0}[q^r\alpha_{2i}, q^{i+j-r}\beta_{2j}]$$

By assumption, we have

$$\begin{aligned} \sum_{r=0}^{i+j} \binom{i+j}{r} \phi^{\leq 0}[q^r\alpha_{2i}, q^{i+j-r}\beta_{2j}] &= \sum_{r=0}^{i+j} \binom{i+j}{r} [\phi(q^r\alpha_{2i}), \phi(q^{i+j-r}\beta_{2j})] \\ &= \sum_{r=0}^{i+j} \binom{i+j}{r} [q^r\phi(\alpha_{2i}), q^{i+j-r}\phi(\beta_{2j})] \\ &= q^{i+j}[\phi(\alpha_{2i}), \phi(\beta_{2j})]. \end{aligned}$$

and thus $\phi([\alpha_{2i}, \beta_{2j}]) = [\phi(\alpha_{2i}), \phi(\beta_{2j})]$ for all i, j and we are done. \square

Definition 6.3.7. We say that a **Heis** module M is integrable if it admits a **Heis** module decomposition $M = \bigoplus_{n \in \mathbb{Z}} M^{[n]}$ such that each the central element c acts on $M^{[n]}$ by scaling with n . We correspondingly say that M is integrable **Heis** Lie algebra if it is integrable **Heis** module, **Heis** Lie algebra, and if the Lie algebra structure is also \mathbb{Z} graded (i.e $[M^{[n]}, M^{[m]}] \subset M^{[n+m]}$).

An example of an integrable **Heis** Lie algebra is as follows.

Definition 6.3.8. For any Lie algebra \mathfrak{g} we define $\mathfrak{g}[D] := \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{Q}[D]$ to be a Lie algebra given by basis αD^n where $n \geq 0$ and $\alpha \in \mathfrak{g}$ with the relations $[\alpha D^n, \alpha' D^m] = [\alpha, \alpha'] D^{n+m}$ where $\alpha, \alpha' \in \mathfrak{g}$ and $n, m \in \mathbb{Z}_{\geq 0}$.

Proposition 6.3.9. *Let \mathfrak{g} be a \mathbb{Z} graded Lie algebra. Then*

$$\mathfrak{g}[D] = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m[D]$$

*is negatively determined integrable **Heis** Lie algebra defined by $q(\alpha_m D^n) = n\alpha_m D^{n-1}$, $c(\alpha_m D^n) = m\alpha_m D^n$ and $p(\alpha_m D^n) = m\alpha_m D^{n+1}$ for any $\alpha_m \in \mathfrak{g}_m$.*

We then have the following consequence of Proposition 6.3.6.

Proposition 6.3.10. *Let \mathfrak{L} be a negatively determined integrable **Heis** Lie algebra such that $\mathfrak{L}_i = 0$ for all $i < 0$. Then there is an isomorphism of Lie algebras $\mathfrak{L}_0[D] \simeq \mathfrak{L}$ where $\mathfrak{L}_0 \subset \mathfrak{L}$ is the degree 0 Lie subalgebra.*

Proof. Since \mathfrak{L} is integral, it admits a decomposition $\mathfrak{L} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{L}^{[m]}$ such that $\mathfrak{L}_0 = \bigoplus_{m \in \mathbb{Z}} \mathfrak{L}_0^{[m]}$ is \mathbb{Z} graded Lie algebra. By Proposition 6.3.9, the Lie algebra $\mathfrak{L}_0[D]$ is integrable **Heis** Lie algebras. Note that $\mathfrak{L}_0[D]$ and \mathfrak{L} are both negatively determined Lie algebras. Thus by Proposition 6.3.6, the morphism $\text{id} : \mathfrak{L}_0 \rightarrow \mathfrak{L}_0$ extends to a morphism of Lie algebras $\phi : \mathfrak{L}_0[D] \rightarrow \mathfrak{L}$. This morphism is an injection, since for a non zero $\alpha \in \mathfrak{L}_0$, $q^i(\phi(\alpha D^i)) = \alpha \neq 0$. Similarly it is surjection since for any $\alpha_{2i} \in \mathfrak{L}_{2i}$, $\phi((q^i \alpha_{2i}) D^i) = \alpha_{2i}$. \square

Next, we see that affinized BPS Lie algebras are also negatively determined integrable **Heis** Lie algebras.

Proposition 6.3.11. *For any quiver Q , the affinized BPS Lie algebra $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is negatively determined integrable **Heis** Lie algebra.*

Proof. From Proposition 6.3.2, it follows that

$$\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}} = \bigoplus_{i \in \mathbb{Z}, \mathbf{d} \in \mathbb{N}^{Q_0}} \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i} = \bigoplus_{m \in \mathbb{Z}} \left(\bigoplus_{i \in \mathbb{Z}, \mathbf{d} \in \mathbb{N}^{Q_0}, |\mathbf{d}|=m} \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i} \right)$$

is integrable **Heis** Lie algebra. The cohomology degrees of the BPS Lie algebra $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is determined by the Kac polynomial (4.2.3) and in particular, it implies that $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, i} = 0$ if $i > 0$ and for all odd integers $i \in \mathbb{Z}$. But then since there is an isomorphism of cohomologically graded vector spaces $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}} \simeq \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}[u]$, $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is spanned by $u^n \cdot \alpha$ where $\alpha \in \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ and in particular if the cohomological degree of $u^n \cdot \alpha > 0$ then $n \geq 1$. For a fixed dimension vector \mathbf{d} , the derivation ∂ is a morphism of vector spaces $\partial_{\mathbf{d}}^i : \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i} \rightarrow \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i-2}$. When i is an even positive integer then by Proposition 6.3.3, the map $\partial_{\mathbf{d}}^i$ is surjective since it sends the elements $u^n \alpha$ for $n > 0$ to $n|\mathbf{d}|u^{n-1} \alpha$. But by above argument or Equation 6.16, $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i}$ and $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, i-2}$ has the same dimension when $i > 0$ and thus ∂ is in fact isomorphism and we are done. \square

6.4 Spherical subalgebra of CoHA

We define the spherical part of a \mathbb{N}^{Q_0} graded algebra as a subalgebra generated by the smallest possible dimension vectors.

Definition 6.4.1 (Spherical Generation). For any quiver Q , let \mathcal{A} be a \mathbb{N}^{Q_0} graded algebra over a ring R . Then we say that \mathcal{A} is spherically generated if \mathcal{A} is generated as an R algebra by $\langle \mathcal{A}_{\delta_i} \mid i \in Q_0 \rangle$. The spherical part of algebra \mathcal{A} is the subalgebra generated by $\langle \mathcal{A}_{\delta_i} \mid i \in Q_0 \rangle$ and is denoted by \mathcal{SA} .

We then have the following interesting consequence of the above abstraction. We determine $\mathcal{SA}_{\widetilde{Q}, \widetilde{W}}$ for any quiver without loops.

Theorem 6.4.2. *For any quiver Q , let Q' be the subquiver formed after removing all the loops of Q . Then*

1. The subvector space $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0} \otimes \text{H}(\text{BC}_{\mathbf{u}}^*, \mathbb{Q}) \subset \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is closed under the Lie bracket and we have a Lie algebra isomorphism

$$\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0} \otimes \text{H}(\text{BC}_{\mathbf{u}}^*, \mathbb{Q}) \simeq \mathfrak{n}_{Q'}^+[D].$$

2. For any Q , there is an embedding of algebras

$$\mathbf{U}(\mathfrak{n}_{Q'}^+[D]) \hookrightarrow \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^X$$

which is an isomorphism if and only if the quiver Q is of finite type.

3. For quiver Q without loops, the image of above embedding is the spherical subalgebra $\mathcal{SA}_{\widetilde{Q}, \widetilde{W}}^X$, so that we have an isomorphism of algebras

$$\mathbf{U}(\mathfrak{n}_{Q'}^+[D]) \simeq \mathcal{SA}_{\widetilde{Q}, \widetilde{W}}^X.$$

Proof. For part (1), we note that given $\alpha^{(n)} = u^n \cdot \alpha$ and $\beta^{(m)} := u^m \cdot \beta$ in $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0} \otimes \text{H}(\text{BC}_{\mathbf{u}}^*, \mathbb{Q})$ where $\alpha, \beta \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0}$, we have

$$[\alpha^{(n)}, \beta^{(m)}] = \sum_{i \geq 0} u^{n+m+i} \gamma_i \quad (6.18)$$

for some $\gamma_i \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, -2i}$, since $[\alpha^{(n)}, \beta^{(m)}]$ is of cohomological degree $2n + 2m$ and $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is all supported in non-positive cohomological degrees (see proof of Proposition 6.3.11). We claim that almost all the terms in the above summation are superfluous, i.e., we claim that $\gamma_i = 0$ for all $i > 0$. Since the summation in (6.18) is a finite sum, let $N > 0$ be the smallest such that $\gamma_i = 0$ for all $i > N$. We apply the derivation ∂ , $n + m + N$ times to the Equation 6.18 and using Proposition 6.3.3 gives that

$$0 = (n + m + N)! (|\mathbf{d}|)^{n+m+N} \gamma_N$$

where \mathbf{d} is the dimension vector on which $[\alpha^{(n)}, \beta^{(m)}]$ are supported. Thus $\gamma_N = 0$. Contradicting the minimality of N . Thus $N = 0$ and the claim follows.

Thus $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0} \otimes \text{H}(\text{BC}_{\mathbf{u}}^*, \mathbb{Q}) \subset \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ is closed under the Lie bracket. It is a negatively determined integrable **Heis** Lie algebra which starts from degree 0. Thus by Proposition 6.3.10, there is an isomorphism of Lie algebras $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0} \otimes \text{H}(\text{BC}_{\mathbf{u}}^*, \mathbb{Q}) \simeq \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0}[D]$. Further, by the Proposition 4.2.5, there is an isomorphism of Lie algebras $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, 0} \simeq \mathfrak{n}_{Q'}^+$ and so we are done. Part (2) follows from the fact that Kac polynomials are constant if and only if Q is of finite type. Finally for part (3), we observe that the subalgebra generated by $\mathcal{A}_{\widetilde{Q}, \widetilde{W}, \delta_i}^X$ is exactly the universal enveloping algebra of the Lie subalgebra of $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ generated by $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \delta_i}^{\text{BPS}}$ for $i \in Q_0$. But then for a quiver without loops, we have $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}, \delta_i}^{\text{BPS}} = \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \delta_i}^{\text{BPS}, 0}$ and so we are done by the above argument. □

In the light of above theorem, one might be tempted to conjecture that infact the affinized BPS Lie algebra are trivial extension of the Lie algebra structure on $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$, i.e. there is an isomorphism of Lie algebras $\widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}} \simeq \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}[D]$. However, this breaks down already in the case of the Jordan quiver.

Let $W_{1+\infty}^+$ be the positive half of $W_{1+\infty}$ Lie algebra, which is a universal central extension of the Lie algebra of differential operators on \mathbb{C}^* . The Lie algebra $W_{1+\infty}^+$ is spanned by $z^m D^n$ where $m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0}$ where we have set $D = zd/dz$. The Lie bracket is given by

$$[z^m D^a, z^n D^b] = z^{m+n}((D+n)^a D^b - D^a (D+m)^b).$$

There is a filtration F on $W_{1+\infty}^+$ such that the associated graded Lie algebra $\text{Gr}_F(W_{1+\infty}^+)$ is spanned by $z^m D^n$ where $m \in \mathbb{Z}_{\geq 1}, n \in \mathbb{Z}_{\geq 0}$ such that the Lie bracket is given by

$$[z^m D^a, z^n D^b] = (an - bm)z^{m+n} D^{a+b-1}.$$

Then it is proved in [Dav22a] that

Theorem 6.4.3. *There is an isomorphism of Lie algebras*

$$\widehat{\mathfrak{g}}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}}^{\text{BPS}} \simeq \text{Gr}_F(W_{1+\infty}^+).$$

Note that in Example 18, we calculated that $\mathfrak{g}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}}^{\text{BPS}} \simeq \oplus_{n \geq 1} \mathbb{Q}[2]$ is commutative Lie algebra. Thus, the trivial extension $\mathfrak{g}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}_{\text{Jor}}}^{\text{BPS}}[D]$ is also commutative, but from the above theorem, the affinized BPS Lie algebra is highly non-trivial. In our next theorem, we show that this can be made true, but at the cost of passing through the associated graded algebra with respect to the Less Perverse filtration.

6.5 Less Perverse Associated Graded Subalgebra

We provide another application for the above abstraction. In Section 4.2.2, we defined Less perverse filtration \mathfrak{L}^\bullet of the CoHA. Our next theorem completely determines $\text{Gr}_{\mathfrak{L}^\bullet}(\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^\times) \simeq \text{Gr}_{\mathfrak{L}^\bullet}(\mathcal{A}_{\Pi_Q}^\times)$ for arbitrary quiver Q .

Theorem 6.5.1. *There is an isomorphism of algebras*

$$\text{Gr}_{\mathfrak{L}^\bullet}(\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^\times) \simeq \mathbf{U}(\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}[D])$$

where $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}[D]$ is trivial extension of BPS Lie algebra $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$.

Proof. We claim that

$$\begin{aligned} u \cdot \mathfrak{L}^i &\subset \mathfrak{L}^{i+2} \\ \partial \cdot \mathfrak{L}^i &\subset \mathfrak{L}^{i-2} \end{aligned}$$

In Proposition 6.3.2, we checked that this is true in the case of Perverse filtration for the action of ∂ ; the same argument works for Less Perverse filtration. The action of u is given by the cup product with $c_1(\text{Det}(\mathcal{L}))$ where \mathcal{L} is the tautological bundle on $\mathfrak{M}_d(\tilde{Q})$. Thus, by [BD23][Proposition 5.7], the action of u increases the Less perverse filtered degree by 2, proving the claim.

Since the Less perverse filtration starts from degree 0, we have

$$\text{Gr}_{\mathcal{L}^\bullet}(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x) = \bigoplus_{i \geq 0} \text{Gr}_{\mathcal{L}^\bullet}^i(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x)$$

where $\text{Gr}_{\mathcal{L}^\bullet}^i(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x) = \mathcal{L}^i(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x) / \mathcal{L}^{i-1}(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x)$ and $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}} \in \text{Gr}_{\mathcal{L}^\bullet}^0(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x)$ since $\text{Gr}_{\mathcal{L}^\bullet}^0(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x) = \text{H}(\mathcal{M}(\tilde{Q}), \text{Sym}(\mathcal{BPS}_{\Pi_Q}))$. Let $\alpha, \beta \in \mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}$. Then since $\partial\alpha = 0$, it follows that $u^n \cdot \alpha \in \text{Gr}_{\mathcal{L}^\bullet}^{2i}(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x)$ and we have splitting

$$\text{Gr}_{\mathcal{L}^\bullet}(\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}) = \bigoplus_{i \geq 0} \text{Gr}_{\mathcal{L}^\bullet}^{2i}(\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}})$$

Thus, in the associated graded algebra, $\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}$ is negatively determined integrable **Heis** Lie algebra with no strictly negative terms (Here we are using less perverse grading then the cohomological grading) and thus by Proposition 6.3.10, there is an isomorphism of Lie algebras

$$\text{Gr}_{\mathcal{L}^\bullet}(\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}) \simeq \text{Gr}_{\mathcal{L}^\bullet}^0(\widehat{\mathfrak{g}}_{\tilde{Q}, \tilde{W}}^{\text{BPS}})[D] = \mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}[D].$$

Thus we have morphism of Lie algebras $\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}[D] \rightarrow \text{Gr}_{\mathcal{L}^\bullet}(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x)$, where the Lie algebra structure on the right comes from associative algebra structure. This induces morphism of algebras

$$\mathbf{U}(\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}[D]) \rightarrow \text{Gr}_{\mathcal{L}^\bullet}(\mathcal{A}_{\tilde{Q}, \tilde{W}}^x)$$

which is clearly an isomorphism by PBW theorem. □

Chapter 7

Nakajima Quiver Varieties for Cyclic Quivers

In this chapter, we will consider the action of cohomological Hall algebras on the cohomology of Nakajima quiver varieties. These actions have been defined in [YZ18b], [Dav22b], [SV20] in the case of stability condition ζ^- , which demands that the image of the framed vector generates the entire representation (Section 7.4.1). When the quiver is cyclic, this gives an action of CoHA on the cohomology of the equivariant Hilbert Scheme. However, we can do similar construction for a different stability condition (Section 7.4.10) ζ^n which allows us to define action of a subalgebra of CoHA on the cohomology of Hilbert scheme of points on the resolution of Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_K$ (Section 7.4.15). We study these two actions by comparing them with the work of Nakajima, where they defined the action of Kac-Moody Lie algebra on the cohomology of the equivariant Hilbert scheme and the action of Heisenberg Lie algebra on the cohomology of the Hilbert scheme of points.

7.1 Nakajima quiver variety

Given any quiver Q , fix an identification of the vertex set Q_0 with the set $[0, 1, \dots, K]$ where $K = |Q_0| - 1$. Let $\mathbf{f} \in \mathbb{N}^{Q_0}$ be a dimension vector, called the framing vector. Then we consider the space of representations:

$$\mathrm{Rep}_{\mathbf{d}, \mathbf{f}}(Q) := \prod_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\mathbf{d}(s(a))}, \mathbb{C}^{\mathbf{d}(t(a))}) \times \prod_{i \in Q_0} \mathrm{Hom}(\mathbb{C}^{\mathbf{f}_i}, \mathbb{C}^{\mathbf{d}_i}).$$

Then the cotangent bundle $T^* \mathrm{Rep}_{\mathbf{d}, \mathbf{f}}(Q)$ carries the action of $\mathrm{GL}_{\mathbf{d}}$ which preserves the symplectic form. So we have the moment map

$$\mu_{\mathbf{d}, \mathbf{f}} : T^* \mathrm{Rep}_{\mathbf{d}, \mathbf{f}}(Q) \rightarrow \mathfrak{gl}_{\mathbf{d}}. \quad (7.1)$$

Then Nakajima quiver variety is defined by taking the GIT quotient of $\mu_{\mathbf{d}, \mathbf{f}}^{-1}(0)$. By a trick due to Crawley-Boevey [Cra01], we can look at the space $T^* \mathrm{Rep}_{\mathbf{d}, \mathbf{f}}(Q)$ as a representation of another quiver $Q_{\mathbf{f}}$ of dimension $(\mathbf{d}, 1)$. $Q_{\mathbf{f}}$ is defined by adding a vertex ∞ to the vertex set Q_0 , and an arrows $\alpha_{i, n}$ for all $i \in Q_0, 1 \leq n \leq \mathbf{f}_i$

where the source $s(\alpha_{i,n}) = \infty$ and $t(\alpha_{i,n}) = i$. For any dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ and number $n \in \mathbb{N}$, let $(\mathbf{d}, n) \in \mathbb{N}^{Q_f}$ be the dimension vector where we denote by $(\mathbf{d}, n)_\infty = n$ and $(\mathbf{d}, n)_i = \mathbf{d}_i$ for all $i \in Q_0$. Here to get a representation of quiver Q_f from $\text{Rep}_{\mathbf{d},\mathbf{f}}(Q)$, we are essentially identifying $\text{Hom}(\mathbb{C}^{\mathbf{f}_i}, \mathbb{C}^{\mathbf{d}_i})$ with $\text{Hom}(\mathbb{C}, \mathbb{C}^{\mathbf{d}_i})^{\mathbf{f}_i}$.

Note that the moment map for $(\mathbf{d}, 1)$ dimensional representation of the preprojective algebra gives a morphism

$$\mu_{Q_f,(\mathbf{d},1)} : \text{Rep}_{(\mathbf{d},1)}(\overline{Q_f}) \rightarrow \mathfrak{gl}_{(\mathbf{d},1)}$$

The morphism 7.1 is the composition of $\mu_{Q_f,(\mathbf{d},1)}$ and the projection $\mathfrak{gl}_{(\mathbf{d},1)} \rightarrow \mathfrak{gl}_{\mathbf{d}}$.

$$\mu_{\mathbf{d},\mathbf{f}} : \text{Rep}_{(\mathbf{d},1)}(\overline{Q_f}) \longrightarrow \mathfrak{gl}_{(\mathbf{d},1)} \xrightarrow{\pi} \mathfrak{gl}_{\mathbf{d}}$$

So we have an extra relation

$$\sum_{i \in Q_0, 1 \leq n \leq \mathbf{f}_i} \rho(\alpha_{i,n}^*) \rho(\alpha_{i,n}) = 0$$

in $\mu_{Q_f,(\mathbf{d},1)}^{-1}(0)$. However, since the dimension of the vertex ∞ is 1, this is same condition as having

$$\sum_{i \in Q_0, 1 \leq n \leq \mathbf{f}_i} \text{Tr}(\rho(\alpha_{i,n}^*) \rho(\alpha_{i,n})) = 0. \quad (7.2)$$

However this is superflous since the defining relations for $\mu_{(\mathbf{d},\mathbf{f})}^{-1}(0)$ are $\sum_{a \in Q_0} [\rho(a), \rho(a^*)] + \sum_{i \in Q_0, 1 \leq n \leq \mathbf{f}_i} \rho(\alpha_{i,n}) \rho(\alpha_{i,n}^*) = 0$ but then $\text{Tr}([A, B]) = 0$ so this implies that

$$\sum_{i \in Q_0, 1 \leq n \leq \mathbf{f}_i} \text{Tr}(\rho(\alpha_{i,n}) \rho(\alpha_{i,n}^*)) = 0$$

which is same as equation 7.2. Thus there are essentially no new equations and so

$$\mu_{\mathbf{d},\mathbf{f}}^{-1}(0) = \mu_{Q_f,(\mathbf{d},1)}^{-1}(0).$$

So from now on we will working with later point of view. Taking the affine GIT quotient defines

Definition 7.1.1 (Affine Nakajima Quiver Variety). The affine Nakajima quiver variety is defined to be the affine quotient

$$\mathbb{N}_{\mathbf{d},\mathbf{f}}^0(Q) = \mathbb{C}[\mu_{Q_f,(\mathbf{d},1)}^{-1}(0)]^{\text{GL}_{\mathbf{d}}}.$$

Remark 7.1.2. Given any quiver Q with two-dimensional vectors \mathbf{d} and framing \mathbf{f} , it determines a $3d \mathcal{N} = 4$ quiver gauge theory, and its Higgs Branch is identified with the affine Nakajima quiver variety.

Example 24 ([Nak16]). Let Q be the A_1 quiver. Let $\mathbf{d} = n$ and $\mathbf{f} = r$. Then $\mu_{Q_r,(n,1)}^{-1}(0) = (X, Y) \subset \text{Hom}(\mathbb{C}^r, \mathbb{C}^n) \times \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ such that $XY = 0$. Note that YX is GL_n invariant and satisfies $(YX)^2 = 0$. Infact one can show that

$$N_{A_1}^0(n, r) = \{A \in \mathrm{GL}_r \mid A^2 = 0, \mathrm{rank}(A) \leq n\}.$$

Example 25. When Q is the Jordan quiver. Then the affine Nakajima quiver variety $N_{(n,1)}^0(Q_{\mathrm{Jor}})$ is isomorphic to $\mathrm{Sym}^n(\mathbb{C}^2)$.

The varieties $N_{\mathbf{d},\mathbf{f}}^0(Q)$ are singular. There is a way to define an open subset of $\mu_{Q_{\mathbf{f}},(\mathbf{d},1)}^{-1}(0)$ where the action of $\mathrm{GL}_{\mathbf{d}}$ is free. Given a dimension vector \mathbf{d} and a stability condition $\zeta \in \mathbb{Q}^{Q_0}$, we extend this to a stability condition $\hat{\zeta} \in \mathbb{Q}^{(Q_{\mathbf{f}})_0}$ by setting $\zeta_{\infty} = -\sum \mathbf{d}_i \zeta_i$. Assume that $\hat{\zeta}$ is generic for dimension vector $(\mathbf{d}, 1)$. This defines an open subset $\mu^{-1}(0)_{Q_{\mathbf{f}},(\mathbf{d},1)}^{\zeta\text{-ss}} \subset \mu_{Q_{\mathbf{f}},(\mathbf{d},1)}^{-1}(0)$ of $\hat{\zeta}$ stable representations.

Example 26 (ζ^{∞} stability condition). For any quiver Q , let $\zeta \in \mathbb{Q}^{Q_0}$ be any tuple where $\zeta_i < 0$ for all $i \in Q_0$. Then $\hat{\zeta}$ is generic for any dimension $(\mathbf{d}, 1)$ since $\chi_{\hat{\zeta}}(\mathbf{d}', 0) = \sum \zeta_i \mathbf{d}'_i \neq 0$ for all $(\mathbf{d}', 0) < (\mathbf{d}, 1)$ unless $\mathbf{d}'_i = 0$ and $\chi_{\hat{\zeta}}(\mathbf{d}', 1) = \sum_{i \in Q_0} \zeta_i (\mathbf{d}'_i - \mathbf{d}_i)$ unless all $(\mathbf{d}', 1) = (\mathbf{d}, 1)$. Infact since $\chi_{\hat{\zeta}}(\mathbf{d}', 1)$ is always positive unless $\mathbf{d}' = \mathbf{d}$ implies that for any ζ stable representation of quiver Q , it cannot be stable unless $\mathbf{d}' = \mathbf{d}$. Also, $\chi_{\hat{\zeta}}(\mathbf{d}', 0)$ is always negative, so it doesn't impose any condition on the stability. Thus, a representation of ζ is semistable iff it is generated by the image of a vector at the vertex ∞ . We shall denote this stability condition by ζ^{∞} .

Definition 7.1.3 (Nakajima quiver variety). Given quiver Q , dimension vector \mathbf{d} , framing vector \mathbf{f} and generic stability condition ζ , the Nakajima quiver variety is defined to be the quotient

$$N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q) = \mu_{Q_{\mathbf{f}},(\mathbf{d},1)}^{-1}(0)^{\zeta\text{-ss}} / \mathrm{GL}_{\mathbf{d}}$$

We have the following fundamental theorem of Nakajima.

Theorem 7.1.4. *The open subset $\mu^{-1}(0)_{Q_{\mathbf{f}},(\mathbf{d},1)}^{\zeta\text{-ss}}$ is a non-singular complex variety of dimension $\dim(\mathrm{Rep}_{(\mathbf{d},1)}(Q_{\mathbf{f}})) - \dim(\mathrm{GL}_{\mathbf{d}}) = \sum_{a \in Q_0} (\mathbf{d}_{s(a)} \mathbf{d}_{t(a)}) + 2 \sum_{i \in Q_0} \mathbf{d}_i \mathbf{f}_i - \sum_{i \in Q_0} \mathbf{d}_i^2$ and the Nakajima quiver variety $N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q)$ is a non-singular complex variety of dimension $\dim(\mathrm{Rep}_{(\mathbf{d},1)}(Q_{\mathbf{f}})) - 2 \dim(\mathrm{GL}_{\mathbf{d}}) = 2 \sum_{a \in Q_0} (\mathbf{d}_{s(a)} \mathbf{d}_{t(a)}) + 2 \sum_{i \in Q_0} \mathbf{d}_i \mathbf{f}_i - 2 \sum_{i \in Q_0} \mathbf{d}_i^2 = -2\chi_Q(\mathbf{d}, \mathbf{d}) + 2\mathbf{d} \cdot \mathbf{f}$.*

Note that for any weighting $\mathbf{w} : (\overline{Q_{\mathbf{f}}})_1 \rightarrow \mathbb{Z}^r$ for which $\sum_{a \in Q_{\mathbf{f}}} [a, a^*]$ is homogenous. Then the corresponding T action preserves $\mu_{Q_{\mathbf{f}},(\mathbf{d},1)}^{-1}(0)$ and thus defines an action on Nakajima quiver variety. By the definition of GIT, there exists a projective morphism

$$q : N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q) \rightarrow N_{\mathbf{d},\mathbf{f}}^0(Q) \tag{7.3}$$

which is a symplectic resolution whenever surjective.

Example 27. Let Q_{Jor} be the Jordan quiver. Let $\mathbf{d} = n$ be the dimension vector and $\mathbf{f} = r$ be the framing. Then the quiver $\overline{Q_{\mathbf{f}}}$ is the ADHM quiver. For $\zeta = -1$, we have an isomorphism $N_{n,r}^{\zeta}(Q_{\mathrm{Jor}}) \simeq M_{r,n}$ where $M_{r,n}$ is the moduli of instantons. More precisely, it is space of the pairs (\mathcal{F}, ϕ) where \mathcal{F} is torsion free sheaf on $\mathbb{C}\mathbb{P}^2$ of rank r , second chern class $c_2(\mathcal{F}) = n$ and $\phi : \mathcal{F} |_{l_{\infty}} \simeq \mathcal{O}_{l_{\infty}}^r$ is the framing at the line at infinity $l_{\infty} = [0 : z_1 : z_2]$.

When $r = 1$, this space coincides with the Hilbert scheme of points on a plane, and the affine Nakajima quiver variety $N_{n,1}^0(Q_{\text{Jor}})$ is isomorphic to $\text{Sym}^n(\mathbb{C}^2)$. The GIT map is just the Hilbert-Chow morphism $\text{Hilb}^n(\mathbb{C}^2) \rightarrow \text{Sym}^n(\mathbb{C}^2)$.

By definition, Nakajima quiver varieties come with non-trivial tautological bundles for every vertex:

Definition 7.1.5 (Tautological Bundles on Nakajima quiver Variety). For each vertex i of the quiver, we define tautological bundle \mathcal{V}_i to be the vector bundle associated with the GL_{d_i} principal bundle

$$(\mu_{Q_{\mathbf{f}},(\mathbf{d},1)}^{-1}(0) \cap \text{Rep}_{(\mathbf{d},1)}^{\hat{\zeta}^{-\text{ss}}}(\overline{Q_{\mathbf{f}}})) / \text{GL}_{\mathbf{d}}^i \rightarrow N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q)$$

where by $\text{GL}_{\mathbf{d}}^i$ we mean $\prod_{j \neq i} \text{GL}_{d_j}$, and the action on $\mu_{(\mathbf{d},1)}^{-1}(0) \cap \text{Rep}_{(\mathbf{d},1)}^{\hat{\zeta}^{-\text{ss}}}(\overline{Q_{\mathbf{f}}}) / \text{GL}_{\mathbf{d}}^i$ is given by the injection $\prod_{j \neq i} \text{GL}_{d_j} \rightarrow \text{GL}_{\mathbf{d}}$.

Later, we will also consider topologically trivial vector bundles \mathcal{W}_i of rank \mathbf{f}_i defined by $N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q) \times \mathbb{C}^{\mathbf{f}_i} \rightarrow N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q)$. Nakajima quiver variety carries an action of the torus T , and the equivariant Chern class of \mathcal{W}_i will then carry non-trivial information.

In Section 2.1.1, we defined the Lusztig nilpotent stack of preprojective algebra representations. Analogously, we can define

Definition 7.1.6. Given any quiver Q , dimension \mathbf{d} , framing \mathbf{f} and stability condition ζ , we define nilpotent quiver variety $L_{\mathbf{d},\mathbf{f}}^{\zeta}(Q)$ as the closed subvariety $q^{-1}(0)$ where $q : N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q) \rightarrow N_{\mathbf{d},\mathbf{f}}^0(Q)$ is the GIT projective morphism defined in equation 7.3.

Note that since q is a projective morphism, $L_{\mathbf{d},\mathbf{f}}^{\zeta}(Q)$ is always projective.

7.2 Critical locus description of Nakajima quiver varieties

In [Dav22b], Davison provides a critical locus description of the Nakajima quiver varieties $N_{\mathbf{d},\mathbf{f}}^{\zeta}(Q)$. Their method doesn't work for an arbitrary generic stability condition ζ . In this section, using dimension reduction, we will show that the statement is true at the level of cohomology.

Given framing \mathbf{f} , we consider the tripled framed quiver $\widetilde{Q}_{\mathbf{f}}$. This is the usual doubled framed quiver $\overline{Q}_{\mathbf{f}}$ that we used to define the Nakajima quiver varieties, but with an extra loop at every vertex, including a loop at the framing vertex. We then consider the canonical cubic potential

$$\widetilde{W}_{\mathbf{f}} = \left(\sum_{i \in (Q_{\mathbf{f}})_0} \omega_i \right) \left(\sum_{a \in (Q_{\mathbf{f}})_1} [a, a^*] \right).$$

Assume that $\hat{\zeta}$ is generic for dimension vector $(\mathbf{d}, 1)$. Then the 3d Nakajima quiver variety is defined to be smooth variety

$$\mathcal{M}_{\mathbf{d}, \mathbf{f}}^{\zeta, 3d}(Q) := \text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\tilde{Q}_{\mathbf{f}}) / \text{GL}_{\mathbf{d}},$$

where $\text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\tilde{Q}_{\mathbf{f}}) \subset \text{Rep}_{(\mathbf{d}, 1)}(\tilde{Q}_{\mathbf{f}})$ the open subset of ζ semistable representations. We can similarly define $\text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}}) \subset \text{Rep}_{(\mathbf{d}, 1)}(\overline{Q}_{\mathbf{f}})$ the open subset of $\hat{\zeta}$ stable $\overline{Q}_{\mathbf{f}}$ representations. Let $\text{Tr}^{\zeta}(\tilde{W}_{\mathbf{f}})$ denote the restriction of the trace function. We then have

Lemma 7.2.1. *Let $\pi_{Q_{\mathbf{f}}, \mathbf{d}} : \text{Rep}_{(\mathbf{d}, 1)}(\tilde{Q}_{\mathbf{f}}) \rightarrow \text{Rep}_{(\mathbf{d}, 1)}(\overline{Q}_{\mathbf{f}})$ be the projection map. Then*

$$\text{crit}(\text{Tr}^{\zeta}(\tilde{W}_{\mathbf{f}})) \cap \pi_{Q_{\mathbf{f}}, \mathbf{d}}^{-1}(\text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}})) = \text{crit}(\text{Tr}^{\zeta}(\tilde{W}_{\mathbf{f}})) \cap \text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\tilde{Q}_{\mathbf{f}})$$

Proof. This proof uses same argument as in [Dav23c][Lemma 6.3]. Let $\rho \in \text{crit}(\text{Tr}^{\zeta}(\tilde{W}_{\mathbf{f}})) \cap \text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\tilde{Q}_{\mathbf{f}}) \setminus \text{crit}(\text{Tr}^{\zeta}(\tilde{W}_{\mathbf{f}})) \cap \pi_{Q_{\mathbf{f}}, \mathbf{d}}^{-1}(\text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}}))$. Then since $\text{Jac}(\tilde{Q}_{\mathbf{f}}, \tilde{W}_{\mathbf{f}}) \simeq \Pi_{Q_{\mathbf{f}}}[\omega]$, we have that $\rho = (\rho', f)$ where ρ' is a representation of $\Pi_{Q_{\mathbf{f}}}$ with an endomorphism $f \in \text{End}_{\Pi_{Q_{\mathbf{f}}}}(\rho')$. Our assumption on ρ is equivalent to ρ' being not a semistable representation of $\Pi_{Q_{\mathbf{f}}}$. But then by the existence of Harder-Narasimhan filtrations, there exists a filtration of ρ' as a $\Pi_{Q_{\mathbf{f}}}$ module

$$0 = \rho'_0 \subset \rho'_1 \subset \rho'_2 \cdots \subset \rho_n = \rho',$$

such that the slopes $\mu(\rho'_i/\rho'_{i-1})$ are strictly decreasing. Each of ρ'_i/ρ'_{i-1} is $\hat{\zeta}$ semistable. Consider the restriction $f|_{\rho'_1} : \rho'_1 \rightarrow \rho'_n$. Since $\mu(\rho'_1) > \mu(\rho'_n/\rho'_{n-1})$, $f|_{\rho'_1} : \rho'_1 \rightarrow \rho'_n \rightarrow \rho'_n/\rho'_{n-1}$ is the 0 morphism. Thus $f|_{\rho'_1}$ factors through ρ'_{n-1} . Since $\mu(\rho'_1) > \mu(\rho'_i/\rho'_{i-1})$ for all i , we may continue this argument inductively to conclude that ρ'_1 is invariant under f . Thus $(\rho'_1, f|_{\rho'_1})$ forms a sub representation of ρ with higher slope, contradicting the semistability of ρ . \square

Note that

$$\pi_{Q_{\mathbf{f}}, \mathbf{d}}^{-1}(\text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}})) = \text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}}) \times \bigoplus_{i \in Q_{\mathbf{f}}} \text{Hom}(\mathbb{C}^{\mathbf{d}_i}, \mathbb{C}^{\mathbf{d}_i})$$

is a $\text{GL}_{\mathbf{d}}$ equivariant decomposition. Thus, by the above lemma and the dimensional reduction theorem (3.3) applied in the case when

$$\begin{aligned} \overline{X} &= \pi_{Q_{\mathbf{f}}, \mathbf{d}}^{-1}(\text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}})) \\ X &= \text{Rep}_{(\mathbf{d}, 1)}^{\hat{\zeta}\text{-ss}}(\overline{Q}_{\mathbf{f}}) \\ \mathbb{A}^n &= \bigoplus_{i \in Q_{\mathbf{f}}} \text{Hom}(\mathbb{C}^{\mathbf{d}_i}, \mathbb{C}^{\mathbf{d}_i}), \end{aligned}$$

it follows that

$$\text{DR} : \text{H}_c^T(\mathcal{M}_{\mathbf{d}, \mathbf{f}}^{\zeta, 3d}(Q), {}^p\Phi_{\text{Tr}^{\zeta}(\tilde{W}_{\mathbf{f}})} \mathbb{Q}_{\mathcal{M}_{\mathbf{d}, \mathbf{f}}^{\zeta}(Q)}[2(\mathbf{d}, 1) \cdot (\mathbf{d}, 1)]) \simeq \text{H}_c^T(\text{N}_{\mathbf{d}, \mathbf{f}}^{\zeta}(Q), \mathbb{Q}).$$

Thus, by applying Verdier duality, since $\dim(\mathcal{M}_{\mathbf{d},\mathbf{f}}^{\zeta,3d}(Q) - \dim(\mathbb{N}_Q^\zeta(\mathbf{d},\mathbf{f}) - 2(\sum_{i \in Q_{\mathbf{f}}} \mathbf{d}_i^2) = -1$, it follows that

Proposition 7.2.2. *Let $\zeta \in \mathbb{Q}^{Q_0}$ by any stability condition such that $\hat{\zeta}$ is generic for dimension vector $(\mathbf{d}, 1)$. Then we have a natural isomorphism of graded vector spaces*

$$(-1)^{\binom{\mathbf{d}}{2}} \mathrm{DR} : \mathrm{H}^T(\mathcal{M}_{\mathbf{d},\mathbf{f}}^{\zeta,3d}(Q), {}^p\Phi_{\mathrm{Tr}^\zeta(\widetilde{W}_{\mathbf{f}})} \mathcal{IC}_{\mathcal{M}_{\mathbf{d},\mathbf{f}}^{\zeta,3d}(Q)}[-1]) \simeq \mathrm{H}^T(\mathbb{N}_{\mathbf{d},\mathbf{f}}^\zeta(Q), \mathbb{Q}^{\mathrm{vir}}). \quad (7.4)$$

Note that here we have twisted dimension reduction isomorphism by the sign $(-1)^{\binom{\mathbf{d}}{2}}$, this is so that we have compatibility with the action of preprojective algebra, as we shall later see.

7.3 CoHA Action

In this section, we will construct the action of COHA on the Nakajima quiver variety for the stability condition ζ^∞ . This is done in [Dav22a] in the formulation we explain next and in [YZ18a], [SV20] for preprojective CoHA's. We will also see the compatibility with the cup product of the tautological line bundle. Later, we will see how these ideas can be used to define an action of a subalgebra of CoHA for a different stability condition.

Given two dimension vectors $\mathbf{d}_1, \mathbf{d}_2$ and framing \mathbf{f} , let

$$\mathcal{M}_{(\mathbf{d}_1, \mathbf{d}_2, \mathbf{f})}^{\zeta^\infty, 3d}(Q) = \{0 \rightarrow (\rho_1)_{(\mathbf{d}_1, 0)} \rightarrow (\rho_2)_{(\mathbf{d}_1 + \mathbf{d}_2, 1)}^{\zeta^\infty - \mathrm{ss}} \rightarrow (\rho_3)_{(\mathbf{d}_2, 1)} \rightarrow 0\}$$

denote the stack of short exact sequences where ρ_2 is ζ^∞ semistable. We then have the usual correspondence diagram

$$\mathfrak{M}_{\mathbf{d}_1}(\widetilde{Q}) \times \mathcal{M}_{\mathbf{d}_2, \mathbf{f}}^{\zeta^\infty, 3d}(Q) \xleftarrow{\pi_1 \times \pi_3} \mathcal{M}_{(\mathbf{d}_1, \mathbf{d}_2, \mathbf{f})}^{\zeta^\infty, 3d}(Q) \xrightarrow{\pi_2} \mathcal{M}_{\mathbf{d}_1 + \mathbf{d}_2, \mathbf{f}}^{\zeta^\infty, 3d}(Q),$$

where the morphism $\pi_1 \times \pi_3$ is projection to (ρ_1, ρ_3) and π_2 is projection to ρ_2 . π_2 is a proper map, since the ζ^∞ stability condition implies that every $(\mathbf{d}_2, 1)$ dimensional quotient of a $(\mathbf{d}_1 + \mathbf{d}_2, 1)$ dimensional ζ^∞ semistable representation is also semistable. Then, by doing push-pull along vanishing cycle cohomology, it follows that we have a left action of $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}$ on

$$\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{H}(\mathcal{M}_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty, 3d}(Q), {}^p\Phi_{\mathrm{Tr}^\zeta(\widetilde{W}_{\mathbf{f}})} \mathcal{IC}_{\mathcal{M}_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty - \mathrm{ss}}(Q)}[-1]) \simeq \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{H}(\mathbb{N}_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}^{\mathrm{vir}}) =: \mathbb{N}_{\mathbf{f}}^{\zeta^\infty}(Q)$$

Here, the second isomorphism is defined in Proposition 7.2.2.

Proposition 7.3.1. *For any quiver Q and framing \mathbf{f} , the cohomological Hall algebra $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^X$ acts on the cohomology of Nakajima quiver variety $\mathbb{N}_{\mathbf{f}}^{\zeta^\infty}(Q)$.*

The proof that the above defines an action is the same for associativity of cohomological Hall algebra as proved in [KS11]. We denote this action by \bullet .

Since we have defined the action of CoHA, there is an action of $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ on $N_{\mathbf{f}}^{\zeta^\infty}(Q)$. In theorem 5.1.6, we saw that $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ can be realized as a positive Half of the Generalized Kac Moody Lie algebra. Infact the action of $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ can be lifted to action of full $\mathfrak{g}_Q^{\text{GKM}}$.

Theorem 7.3.2 ([DHM24], Theorem 10.5). *The $\mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}$ action on $N_{\mathbf{f}}^{\zeta^\infty}(Q)$ extends to $\mathfrak{g}_Q^{\text{GKM}}$ action and there is decomposition of $\mathfrak{g}_Q^{\text{GKM}}$ modules*

$$N_{\mathbf{f}}^{\zeta^\infty}(Q) \simeq \bigoplus_{(\mathbf{d}, 1) \in \Phi_{\Pi_{Q_{\mathbf{f}}}}^+} H^*(\mathcal{IC}(\mathcal{M}_{\Pi_{Q_{\mathbf{f}}}, (\mathbf{d}, 1)})) \otimes L_{\mathfrak{g}_Q^{\text{GKM}}, \hat{\mathbf{d}}}$$

where $\hat{\mathbf{d}} \in \text{Hom}(\mathbb{Z}^{Q_0}, \mathbb{Z})$ given by $\mathbf{d}' \mapsto (\mathbf{d}, \mathbf{d}')_{\Pi_Q} - \mathbf{f} \cdot \mathbf{d}'$.

Example 28. $(0, 1)$ is always a real positive root. Thus $L_{\mathfrak{g}_Q^{\text{GKM}}, \hat{\mathbf{0}}}$ where $\hat{\mathbf{0}} : \mathbf{d}' \mapsto \mathbf{f} \cdot \mathbf{d}'$ is always a summand in $N_{\mathbf{f}}^{\zeta^\infty}(Q)$.

Example 29 (Jordan Quiver). Let Q be the Jordan quiver and let $\mathbf{f} = 1$. Then in Example 27, we showed that $N_{(n, 1)}^{\zeta^\infty}(Q_{\text{Jor}}) \simeq \text{Hilb}^n(\mathbb{C}^2)$. We claim that only simple roots of form $(n, 1) \in \Phi_{\Pi_{Q_1}}^+$ are $(0, 1)$. This is simply because $((n, 1), (n, 1))_{\Pi_{Q_1}} = 2(1 - n)$. But then $2 - ((n, 1), (n, 1))_{\Pi_{Q_1}} = 2n$. So if $n \geq 1$, we may write $(n, 1) = (0, 1) + (1, 1) + \dots + (1, 1)$ where $(1, 1)$ is repeated n times and each of $2 - ((1, 1), (1, 1))_{\Pi_{Q_1}} = 2$, while $2 - ((0, 1), (0, 1))_{\Pi_{Q_1}} = 2$ contradicting the primitivity of $(n, 1)$. Thus, only simple roots are $(0, 1)$. But then, by the above theorem

$$N_1^{\zeta^\infty} \simeq L_{\mathfrak{gl}(1), -1},$$

recovering theorem of Nakajima-Grojnowski.

On the other hand, there is the action of preprojective CoHA $\mathcal{A}_{\Pi_Q}^X$ on $N_{\mathbf{f}}^{\zeta^\infty}(Q)$ defined in [YZ18b][Section 5] and [SV20][Section 5.6]. However, it is shown in [YZ20a][Theorem A] that these actions are compatible under the twisted dimension reduction morphism $\widetilde{\text{DR}}$ in Section 4.2.1.

Although we will not be pursuing this until Chapter 12, the perspective we want to give is that $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{\mathcal{N}} \simeq \mathcal{A}_{\Pi_Q}^{\mathcal{N}}$ should be thought of as the opposite half of the CoHA in the following way. First, it is shown in [SV20] that we can lift the above action to the case of nilpotent quiver variety (One can also prove using the above construction).

Proposition 7.3.3 ([SV23]). *For any quiver Q and framing \mathbf{f} , the nilpotent cohomological Hall algebra $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{\mathcal{N}, \chi}$ acts on the cohomology of nilpotent Nakajima quiver variety¹*

$$L_{\mathbf{f}}^{\zeta^\infty, \mathcal{N}}(Q) := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} H^{\text{BM}}(L_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}[2(\mathbf{d}, \mathbf{d}) - 2\mathbf{f} \cdot \mathbf{d}]).$$

¹Note that the nilpotent quiver variety $L_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q)$ aren't always smooth.

Now, we claim that there is duality

$$H(N_{\mathbf{d},\mathbf{f}}^{\zeta_\infty}(Q), \mathbb{Q}^{\text{vir}}) \simeq H^{\text{BM}}(L_{\mathbf{d},\mathbf{f}}^{\zeta_\infty}(Q), \mathbb{Q}[2(\mathbf{d}, \mathbf{d}) - 2\mathbf{f} \cdot \mathbf{d}])^*$$

This is simply because the varieties $N_{\mathbf{d},\mathbf{f}}^{\zeta_\infty}(Q)$ and $L_{\mathbf{d},\mathbf{f}}^{\zeta_\infty}(Q)$ are homotopic. But $L_{\mathbf{d},\mathbf{f}}^{\zeta_\infty}(Q)$ is also a projective variety. Hence, its usual cohomology is the same as the compactly supported cohomology, which is Poincaré dual to the Borel-Moore homology. Thus by Proposition 7.3.3, we get a *right* action of $\mathcal{A}_{\tilde{Q},\tilde{W}}^{\mathcal{N}}$ on $N_{\mathbf{f}}^{\zeta_\infty}(Q)$.

Remark 7.3.4. Note that there is addition action of $\text{GL}_{\mathbf{f}}$ on $\mathcal{M}_{\mathbf{d},\mathbf{d}}^{\zeta_\infty,3d}(Q)$ for any quiver Q which reparametrize the framing of $\tilde{Q}_{\mathbf{f}}$. Then the action of CoHA also works equivariantly. We get a left action of $\mathcal{A}_{\tilde{Q},\tilde{W}}^T$ on

$$\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} H^{T \times \text{GL}_{\mathbf{f}}} (N_{\mathbf{d},\mathbf{f}}^{\zeta_\infty}(Q), \mathbb{Q}^{\text{vir}}).$$

and similarly a right action of $\mathcal{A}_{\tilde{Q},\tilde{W}}^{T,\tilde{\mathcal{N}}}$. In the case when T scales the symplectic form non-trivially, due to a theorem of [BD23] and [SV23], these CoHA can be realized as positive and negative half of MO Yangian respectively, which implies that after taking direct sum across all framings, these actions are faithful (See Chapter 12). However, faithfulness is conjectural when T acts trivially on the symplectic form.

Compatibility with Cup product

Note that, in the same way, we constructed the action of tautological bundles on $\mathcal{A}_{Q,W}$ in Section 4.1.3, we can construct analogous action of the tautological line bundle $\otimes_{i \in Q_0} \mathcal{V}_i$ on $N_{\mathbf{f}}^{\zeta_\infty}(Q)$ by $u \cdot v := \sum_{i \in Q_0} c_1(\mathcal{V}_i) \cup v$. The following proposition will be useful later to understand the induced action of the affinized BPS Lie algebra.

Proposition 7.3.5. *The action of $\mathcal{A}_{\tilde{Q},\tilde{W}}$ on $N_{\mathbf{f}}^{\zeta_\infty}(Q)$ satisfies*

$$(u \cdot \alpha) \bullet v = u \cdot (\alpha \bullet v) - \alpha \bullet (u \cdot v)$$

Proof. We have

$$\begin{aligned} u \cdot (\alpha \bullet v) &= \sum_{i \in Q_0} c_1(\mathcal{V}_i) \cup ((\pi_2)_*(\pi_1 \times \pi_3)^*(\alpha \otimes v)) \\ &= \sum_{i \in Q_0} (\pi_2)_*((\pi_2)^*(c_1(\mathcal{V}_i)) \cup (\pi_1 \times \pi_3)^*(\alpha \otimes v)) \\ &= \sum_{i \in Q_0} (\pi_2)_*((\pi_1 \times \pi_3)^*((c_1(\mathcal{V}_i) \otimes 1 + 1 \otimes c_1(\mathcal{V}_i)) \cdot (\alpha \otimes v))) \\ &= \sum_{i \in Q_0} (u \cdot \alpha) \bullet v + \alpha \bullet (u \cdot v) \end{aligned}$$

where the third equality is true since for a short exact sequence $0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0$, $c_1(L_2) = c_1(L_1) + c_1(L_3)$. \square

7.4 Kleinian singularity and McKay correspondence

Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. The McKay correspondence associates to every finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ a Dynkin graph of affine type ADE. Let $\rho_0, \rho_1, \dots, \rho_{d-1}$ be the set of all the isomorphism classes of irreducible representations of the group G where ρ_0 is assumed to be the trivial representation. Since $G \subset \mathrm{SL}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C})$, we have the tautological representation of G on \mathbb{C}^2 . Let a_{ij} be the multiplicities in the decomposition of the tensor product $\rho_i \otimes \mathbb{C}^2$ into the sum of irreducible representations, that is

$$\rho_i \otimes \mathbb{C}^2 \simeq \bigoplus_{j=0}^{d-1} \rho_j^{a_{ij}}.$$

Then the graph with the set of vertices $0, 1, \dots, d-1$ and a_{ij} edges between the vertices i and j is the corresponding McKay graph. We now consider the finite cyclic group of order $K+1$ $\mathbb{Z}_{K+1} = \langle \omega \rangle \subset \mathrm{SL}_2(\mathbb{C})$ embedded in $\mathrm{SL}_2(\mathbb{C})$ by $\omega \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$ where ω is a primitive $K+1$ -th root of unity. The McKay graph corresponding to the group \mathbb{Z}_{K+1} gives the doubled cyclic quiver $\overline{Q^K}$.

On the other hand, we may consider the affine quotient $\mathbb{C}^2/\mathbb{Z}_{K+1}$. It is a singular space with a singularity at the origin. Interestingly, the Nakajima quiver variety associated with the quiver cyclic quiver Q^K gives a minimal resolution of this singularity. In the case we are interested in, these varieties can be understood as equivariant Hilbert schemes.

7.4.1 Equivariant Hilbert scheme

Let $\mathrm{Hilb}(\mathbb{C}^2)$ denote the Hilbert scheme of points on \mathbb{C}^2 . Since \mathbb{Z}_{K+1} acts on \mathbb{C}^2 , we have an induced action of \mathbb{Z}_{K+1} on $\mathrm{Hilb}(\mathbb{C}^2)$. For any finite group G , let $R(G)$ be the monoid of the isomorphism classes of finite-dimensional representations of the group G . For every representation $V \in R(\mathbb{Z}_{K+1})$, let

$$\mathrm{Hilb}^V(\mathbb{C}^2) = \{J \subset \mathbb{C}[x, y], \mathbb{C}[x, y]/J \simeq_{\mathbb{Z}_{K+1}} V\}$$

where $\simeq_{\mathbb{Z}_{K+1}}$ denote isomorphism as representations of the finite group \mathbb{Z}_{K+1} . This gives a decomposition into components

$$(\mathrm{Hilb}(\mathbb{C}^2))^{\mathbb{Z}_{K+1}} = \bigoplus_{V \in R(\mathbb{Z}_{K+1})} \mathrm{Hilb}^V(\mathbb{C}^2). \quad (7.5)$$

where $(\mathrm{Hilb}(\mathbb{C}^2))^{\mathbb{Z}_{K+1}}$ is the fixed point locus of the \mathbb{Z}_{K+1} action on $\mathrm{Hilb}(\mathbb{C}^2)$. Note that since \mathbb{Z}_{K+1} is abelian, all the irreducible representations are 1-dimensional.

Thus for every representation V , $V \simeq \bigoplus \rho_i^{\oplus \mathbf{d}_i}$ as \mathbb{Z}_{K+1} representations, where ρ_i is the 1 dimensional representation of weight i , given by $v \mapsto \omega^i v$. This gives an isomorphism of monoids $\mathbf{d} : R(\mathbb{Z}_{K+1}) \mapsto \mathbb{N}^{K+1}$.

Example 30. Consider the ideal I of the functions which vanish at the points of a free \mathbb{Z}_{K+1} -orbit. The orbit consists of $K + 1$ elements and the corresponding \mathbb{Z}_{K+1} -module $\mathbb{C}[x, y]/I$ is a regular representation ρ_{reg} of \mathbb{Z}_{K+1} . In fact $\mathbf{d}(\rho_{\text{reg}}) = \underbrace{(1, 1, \dots, 1)}_{K+1 \text{ times}} = \delta$. The action of \mathbb{Z}_{K+1} on \mathbb{C}^2 is free outside the origin.

The restriction of the Hilbert-Chow map (Proposition 7.4.2) gives the minimal resolution $\text{Hilb}^{\rho_{\text{reg}}}(\mathbb{C}^2) \rightarrow \mathbb{C}^2/\mathbb{Z}_{K+1}$.

Tautological bundles

By the way we have defined them, $\text{Hilb}^V(\mathbb{C}^2)$ is an \mathbb{Z}_{K+1} invariant subvariety of $\text{Hilb}^{\dim(V)}(\mathbb{C}^2)$. Denote by $\mathcal{O}_{\mathbb{C}^2}^n$, the tautological rank $\dim(V)$ bundle on the Hilbert scheme $\text{Hilb}^{\dim(V)}(\mathbb{C}^2)$ (See Definition 7.4.7). Consider the restriction $(\mathcal{O}_{\mathbb{C}^2}^{\dim(V)})|_{\text{Hilb}^V(\mathbb{C}^2)}$ over $\text{Hilb}^V(\mathbb{C}^2)$. Then, each fibre carries a structure of representation of G . We thus have

$$(\mathcal{O}_{\mathbb{C}^2}^n)|_{\text{Hilb}^V(\mathbb{C}^2)} \simeq \bigoplus_{k \in \mathbb{Z}_{K+1}} \mathcal{H}_k \otimes \rho_k,$$

giving tautological bundles \mathcal{H}_k on $\text{Hilb}^V(\mathbb{C}^2)$. Note that we can equivalently write $\mathcal{H}_k = \text{Hom}(\rho_i, (\mathcal{O}_{\mathbb{C}^2}^n)|_{\text{Hilb}^V(\mathbb{C}^2)})$

Nakajima quiver variety description

The equivariant Hilbert scheme is an example of a Nakajima quiver variety.

Proposition 7.4.1 ([Nak02]). *We have an isomorphism of smooth connected algebraic varieties*

$$\text{Hilb}^V(\mathbb{C}^2) \simeq \mathbb{N}_{\mathbf{d}(V), \delta_0}^{\zeta^\infty}(Q^K)$$

and so their dimension is $2\mathbf{d}_0(V) - \sum_{i \in \mathbb{Z}_{K+1}} (\mathbf{d}_i(V) - \mathbf{d}_{i+1}(V))^2$.

On closed points, given an ideal J , the above isomorphism sends J to a representation of $\overline{Q}_{\delta_0}^K$, given by

$$V_i = \text{Hom}_{\mathbb{Z}_{K+1}}(\rho_i, \mathbb{C}[z_1, z_2]/J)$$

for $i \in [0, K]$ and $V_\infty = \mathbb{C}$ and the morphisms between these vector spaces are given by composition with the morphisms $z_1 : \mathbb{C}[z_1, z_2]/J \rightarrow \mathbb{C}[z_1, z_2]/J$ and $z_2 : \mathbb{C}[z_1, z_2]/J \rightarrow \mathbb{C}[z_1, z_2]/J$ given by multiplication by z_1 and z_2 respectively. In particular, this means that the above isomorphism sends the tautological bundles \mathcal{H}_i on $\text{Hilb}^V(\mathbb{C}^2)$ to the tautological bundles \mathcal{V}_i on the Nakajima quiver variety $\mathbb{N}_{\mathbf{d}(V), \delta_0}^{\zeta^\infty}(Q^K)$.

Convention 4. Given any smooth variety X and an irreducible closed subvariety V , let

$$[V] \in \mathbb{H}^{2(\dim(X) - \dim(V))}(X)$$

be the fundamental class [AF24][Appendix A].

Nakajima action

We now briefly explain the action of the positive half \mathfrak{n}_{K+1}^+ of the affine Kac-Moody Lie algebra $\widetilde{\mathfrak{sl}}_{K+1}$ on

$$N_{\delta_0}^{\zeta^\infty}(Q^K) \simeq \bigoplus_{V \in R(\mathbb{Z}_{K+1})} H(\mathrm{Hilb}^V(\mathbb{C}^2), \mathbb{Q}^{\mathrm{vir}})$$

due to [Nak98]². For each i and V , the Hecke correspondence is defined to be the subvariety

$$\mathfrak{B}_i(V) := \{(I_1, I_2) \in \mathrm{Hilb}^{V-\rho_i}(\mathbb{C}^2) \times \mathrm{Hilb}^V(\mathbb{C}^2) \mid I_2 \subset I_1\} \subset \mathrm{Hilb}^{V-\rho_i}(\mathbb{C}^2) \times \mathrm{Hilb}^V(\mathbb{C}^2).$$

Note that x, y act trivially on I_1/I_2 . By [Nak98][Section 5], $\mathfrak{B}_i(V)$ are known to be smooth Lagrangian submanifolds inside the product $\mathrm{Hilb}^{V-\rho_i}(\mathbb{C}^2) \times \mathrm{Hilb}^V(\mathbb{C}^2)$. Let e_i be the standard Chevalley generators of \mathfrak{n}_{K+1}^+ as in (4.2.2). We consider the usual correspondence diagram

$$\begin{array}{ccc} \mathfrak{B}_i(V) & \hookrightarrow & \mathrm{Hilb}^{V-\rho_i}(\mathbb{C}^2) \times \mathrm{Hilb}^V(\mathbb{C}^2) \\ & & \swarrow p_1 \quad \searrow p_2 \\ \mathrm{Hilb}^{V-\rho_i}(\mathbb{C}^2) & & \mathrm{Hilb}^V(\mathbb{C}^2) \end{array}$$

Let $v \in H(\mathrm{Hilb}^{V-\rho_i}(\mathbb{C}^2), \mathbb{Q}^{\mathrm{vir}})$. The action of \mathfrak{n}_{K+1}^+ on $N_{\delta_0}^{\zeta^\infty}(Q^K)$ is defined by

$$e_i(v) := (-1)^{d_i(V-\rho_i)+d_{i+1}(V-\rho_i)} (p_2)_*(p_1^*(v) \cup [\mathfrak{B}_i(V)]). \quad (7.6)$$

Strictly speaking, in the original work, Nakajima constructed operators on a Lagrangian subvariety of $\mathrm{Hilb}^V(\mathbb{C}^2)$, which is dual to the cohomology of $\mathrm{Hilb}^V(\mathbb{C}^2)$. This formulation is present in the work of Nagao in [Nag09][Section 5.3.1]. Later in Chapter 12, we will see how these operators are generalized to the action of Yangians on the Equivariant cohomology of these varieties by Varagnolo [Var00].

Minimal resolution as Nakajima quiver variety

Theorem 7.4.2 ([Kro89]). *The affine Nakajima quiver variety $N_{\delta, \delta_0}^0(Q^K)$ is isomorphic to the affine quotient $\mathbb{C}^2/\mathbb{Z}_{K+1}$ and the quiver variety $N_{\delta, \delta_0}^{\zeta^\infty}(Q) \simeq \mathrm{Hilb}^{\rho_{\mathrm{reg}}}(\mathbb{C}^2)$ is the minimal resolution via the GIT map³*

$$\pi: N_{(\delta, \delta_0)}^{\zeta^\infty}(Q^K) \rightarrow N_{(\delta, \delta_0)}^0(Q^K) \simeq \mathbb{C}^2/\mathbb{Z}_{K+1}.$$

The exceptional fibre $\pi^{-1}(0)$ components can be understood as Nakajima correspondences defined in the previous section. Consider $\mathfrak{B}_i(\rho_{\mathrm{reg}})$ for $i \neq 0$. Then by

²There is an action for general Nakajima quiver variety for quiver without loops. Here we only focus on a special case.

³There is more general result for any finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Proposition 7.4.1, the dimension of $\text{Hilb}^{\rho_{\text{reg}} - \rho_i}(\mathbb{C}^2)$ is 0 and thus $\text{Hilb}^{\rho_{\text{reg}} - \rho_i}(\mathbb{C}^2) = \{\text{pt}_i\}$, for some point $\text{pt}_i \in \text{Hilb}^{\rho_{\text{reg}} - \rho_i}(\mathbb{C}^2)$. Thus by definition, we may view $\mathfrak{B}_i(\rho_{\text{reg}})$ as a subvariety of $\text{Hilb}^{\rho_{\text{reg}}}(\mathbb{C}^2)$:

$$\mathfrak{B}_i(\rho_{\text{reg}}) = \{\text{pt}_i \subset I\} \subset \text{Hilb}^{\rho_{\text{reg}}} \simeq S_K \quad (7.7)$$

The subvarieties $\mathfrak{B}_i(\rho_{\text{reg}})$ are smooth Lagrangian manifold inside the surface S_K . These are isomorphic to \mathbb{P}^1 [Nak01b][Proposition 6.2]. In [Nak01b][Example 6.3], it is shown that the exceptional fiber $\pi^{-1}(0)$ is the union of $\mathfrak{B}_i(\rho_{\text{reg}})$. For any representation ρ of $\overline{Q_{\delta_0}^K}$ of dimension $(\mathbf{d}, 1)$ and for any arrow $a \in Q_{\delta_0}^K$, let $\rho_a : \mathbb{C}^{s(a)} \rightarrow \mathbb{C}^{t(a)}$ be the corresponding linear map and $\rho_a^* : \mathbb{C}^{t(a)} \rightarrow \mathbb{C}^{s(a)}$ be the map in the opposite direction.

Example 31. We can observe that $\mathfrak{B}_i(\rho_{\text{reg}})$ correspond to the isomorphism classes of representations of dimension δ where $\rho_a = 1, \rho_a^* = 0$ where $a : j-1 \rightarrow j$ for $j = 1, \dots, i-1$; $\rho_a = 0, \rho_a^* = 1$ where $a : j-1 \rightarrow j$ for $j = i+2, \dots, K+1$ and $\rho_a = 1, \rho_a^* = 0$ where $a : \infty \rightarrow 0$, while the choice of the maps $\rho_{i-1 \rightarrow i}$ and $\rho_{i \rightarrow i+1}^*$ correspond to \mathbb{P}^1 (See Figure 7.1 for $K = 3, i = 2$).

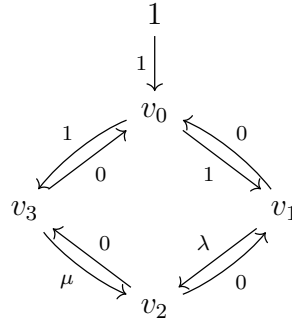


Figure 7.1: $\mathfrak{B}_i(\delta)$ for $K = 3, i = 2$

The cohomology $H(S_K, \mathbb{Q})$ is spanned by $[S_K] \in H^0(S_K, \mathbb{Q})$ and irreducible components of exceptional fibers $[\mathfrak{B}_i(\rho_{\text{reg}})] \in H^2(S_K, \mathbb{Q})$. On the other hand, since Nakajima quiver varieties are tautologically generated [MN18], it follows that the Chern class of the tautological bundles $c_1(\mathcal{R}_i)$ also spans $H^2(S_K, \mathbb{Q})$. For any finite subgroup $G \subset \text{SL}_2(\mathbb{C})$, the geometric McKay correspondence [GV83] gives a duality between the irreducible components of the exceptional fiber $\pi : S_G \rightarrow \mathbb{C}^2/G$ and the tautological bundles associated to the irreducible representation of the group G . Reinterpreting this in terms of the Nakajima quiver variety gives

Proposition 7.4.3. *For all $i \in \{1, \dots, K\}$, we have*

$$[\mathfrak{B}_i(\rho_{\text{reg}})] \simeq \sum_{k=1}^K c_{ik} c_1(\mathcal{V}_k),$$

where $c_{ik} \in \text{End}_K(\mathbb{Z})$ is the Cartan matrix for the A_K quiver, i.e. $c_{ik} = 2\delta_{ik} - \delta_{i,k-1} - \delta_{i,k+1}$ where $1 \leq i, j \leq K$.

Proof. For each edge i where $i \neq 0$, we have tautological morphisms $\rho_{i,i+1} : \mathcal{V}_i \rightarrow \mathcal{V}_{i+1}$ and $\rho_{i-1,i}^* : \mathcal{V}_i \rightarrow \mathcal{V}_{i-1}$. So the preprojective relations give rise to a complex of tautological bundles

$$\mathcal{V}_i \xrightarrow{\rho_{i-1,i}^* \oplus \rho_{i,i+1}} \mathcal{V}_{i-1} \oplus \mathcal{V}_{i+1} \xrightarrow{\rho_{i-1,i} - \rho_{i,i+1}^*} \mathcal{V}_i$$

Thus we have a map between vector bundles

$$\mathcal{V}_i \xrightarrow{\rho_{i-1,i}^* \oplus \rho_{i,i+1}} \text{Ker}(\rho_{i-1,i} - \rho_{i,i+1}^*).$$

Consider the locus Z_i where the above map is not an isomorphism. It follows from Example 31 or [Nak01b][Example 6.3] that $Z_i = \mathfrak{B}_i(\rho_{\text{reg}})$. But note that the morphism

$$\mathcal{V}_{i-1} \oplus \mathcal{V}_{i+1} \xrightarrow{\rho_{i-1,i} - \rho_{i,i+1}^*} \mathcal{V}_i$$

is a surjection. Since it is not a surjection, then it must be 0. But then this gives a subrepresentation of \overline{Q}^K which is not ∞ generated, contradicting the stability. Thus, this morphism is a surjection. But this implies that the fundamental class $[\mathfrak{B}_i(\rho_{\text{reg}})] = 2c_1(\mathcal{V}_i) - c_1(\mathcal{V}_{i+1}) - c_1(\mathcal{V}_{i-1})$. \square

By Proposition 7.3.1, we have action of $\mathfrak{g}_{\overline{Q}^K, \widetilde{W}^K}^{\text{BPS}}$ on

$$N_{\delta_0}^{\zeta^\infty}(Q^K) = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \text{H}(\mathcal{M}_{\mathbf{d}, \text{fr}}^{\zeta^- - \text{ss}}(Q), {}^p\Phi_{\text{Tr}(\widetilde{W}_t)} \mathcal{IC}_{\mathcal{M}_{\mathbf{d}, \text{fr}}^{\zeta^- - \text{ss}}(Q^K)}[1]) \simeq \bigoplus_{V \in R(\mathbb{Z}_{K+1})} \text{H}(\text{Hilb}^V(\mathbb{C}^2), \mathbb{Q}^{\text{vir}}).$$

The induced action of the $\mathfrak{n}_{Q^K}^+ \subset \mathfrak{g}_{\overline{Q}^K, \widetilde{W}^K}^{\text{BPS}}$ on $N_{\delta_0}^{\zeta^\infty}(Q^K)$ can be identified with Nakajima's operators e_i . We have

Proposition 7.4.4. *The action of generator $\alpha_i \in \mathfrak{g}_{\overline{Q}^K, \widetilde{W}^K, \delta_i}^{\text{BPS}}$*

$$\alpha_i : \text{H}(\text{Hilb}^{V-\rho_i}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}}) \rightarrow \text{H}(\text{Hilb}^V(\mathbb{C}^2), \mathbb{Q}^{\text{vir}}),$$

coincides with the action of Nakajima operator e_i .

Proof. It is shown in [YZ20a][Theorem 5.1], that dimension reduction from $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^X \simeq \mathcal{A}_{\Pi_Q}^X$ is compatible with action on Nakajima quiver variety. Furthermore, in [YZ18b][Theorem 5.6], it is shown that the action of $x_{i,1}^0 \in \mathbb{C}[x_{i,1}] = \mathcal{A}_{\Pi_Q, \delta_i}^X$ coincides with the action of Nakajima operators. \square

We also consider the action of any non-zero $\gamma_\delta \in \mathfrak{g}_{\overline{Q}^K, \widetilde{W}^K, \delta}^{\text{BPS}, -2}$, this make sense since by Proposition 5.0.2, $\mathfrak{g}_{\overline{Q}^K, \widetilde{W}^K, \delta}^{\text{BPS}, -2}$ is a 1 dimensional vector space and $u \cdot \gamma_\delta$ on $N_{\delta_0}^{\zeta^\infty}(Q^K)$. Choose a non zero element $|0\rangle \in \text{H}(\text{Hilb}^0(\mathbb{C}^2), \mathbb{Q}^{\text{vir}}) \subset N_{\delta_0}^{\zeta^\infty}(Q^K)$. We then have

Proposition 7.4.5. *For some non-zero $\lambda' \in \mathbb{Q}$, we have an equality*

$$\gamma_\delta |0\rangle = \lambda' [S_K]$$

and

$$(u \cdot \gamma_\delta)|0\rangle = \lambda' \left(\sum_{i=1}^K c_1(\mathcal{R}_i) \right) = \lambda' \left(\sum_{i=1}^K \sum_{j=1}^K A_{ij} [\mathfrak{B}_j(\rho_{reg})] \right)$$

where $A = C^{-1}$, is the inverse of the Cartan matrix of A_K .

Proof. The action of γ_δ gives a map

$$\gamma_\delta: \mathrm{H}(\mathrm{Hilb}^0(\mathbb{C}^2), \mathbb{Q}) \rightarrow \mathrm{H}(\mathrm{Hilb}^{\rho_{reg}}, \mathbb{Q}^{\mathrm{vir}}) \simeq \mathbb{Q}[2] \oplus \mathbb{Q}^K[0]$$

Since the element γ_δ is of cohomological degree -2 , we have $\gamma_\delta|0\rangle = \lambda'[S]$ for some λ' . We claim that $\lambda' \neq 0$.

By Theorem 5.1.6, the action of $\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{BPS}}$ lifts to an action of double $\mathfrak{g}_{Q^K}^{\mathrm{GKM}}$. By Example 28, $N_{\delta_0}^{\infty}(Q^K)$ contains the Lowest weight module $L_{\mathfrak{g}_{Q^K}, \hat{\mathfrak{o}}}$ of $\mathfrak{g}_{Q^K}^{\mathrm{GKM}}$ where $\hat{\mathfrak{o}} : \mathbf{d}' \mapsto -\delta_0 \cdot \mathbf{d}'$. In Example 22, we calculated $\mathfrak{g}_{Q^K}^{\mathrm{GKM}}$ and in particular we can identify γ_δ with Chevalley generator e_δ . We also have annihilation operator f_δ and element h_δ in the Cartan. We can identify the vacuum element $|0\rangle$ as $1 \otimes 1 \in L_{\mathfrak{g}_{Q^K}, \hat{\mathfrak{o}}}$. Since $L_{\mathfrak{g}_{Q^K}, \hat{\mathfrak{o}}}$ is a Lowest weight module, by definition $f_\delta|0\rangle = 0$ and $h_\delta|0\rangle = -\delta_i \cdot \delta|0\rangle = -|0\rangle$. But then $e_\delta|0\rangle$ can't be 0, since otherwise $h_\delta|0\rangle = [e_\delta, f_\delta]|0\rangle = 0$. Thus $e_\delta|0\rangle \neq 0$ and hence $\gamma_\delta|0\rangle \neq 0$ and thus $\lambda' \neq 0$.

Next, by Proposition 7.3.5, it follows that $(u \cdot \gamma_\delta) \star [0] = u \cdot ([S]) = c_1(\mathcal{R}) = \sum_{i=1}^K c_1(\mathcal{R}_i)$. Then, the claim follows from Proposition 7.4.3. \square

The inverse of the Cartan matrix of A_K can be described explicitly. We note down the matrix.

Proposition 7.4.6. *The inverse of the Cartan matrix of A_K is given by Matrix $A_{k,j} = \min(k, j) - \frac{kj}{K+1}$.*

7.4.2 Hilbert schemes

For the minimal resolution $S_K \rightarrow \mathbb{C}^2/\mathbb{Z}_{K+1}$, we consider the $\mathbb{N} \times \mathbb{Z}$ graded vector space

$$V_K := \bigoplus_{n \in \mathbb{N}} \mathrm{H}^{\mathrm{BM}}(\mathrm{Hilb}^n(S_K), \mathbb{Q}^{\mathrm{vir}})$$

and introduce a similarly defined action of a subalgebra of $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}$ and study their relationship with Nakajima's operators. Note that since $\mathrm{Hilb}^n(S_K)$ is smooth, there is an isomorphism between the Borel-Moore homology and the usual cohomology $\mathrm{H}^{\mathrm{BM}}(\mathrm{Hilb}^n(S_K), \mathbb{Q}^{\mathrm{vir}}) \simeq \mathrm{H}(\mathrm{Hilb}^n(S_K), \mathbb{Q}^{\mathrm{vir}})$.

Nakajima operators

In [Nak97], for any quasi-projective surface S , the author constructs the action of certain operators $p_m(\alpha)$ for $m \in \mathbb{Z}$ and $\alpha \in H(S, \mathbb{Q})$ on

$$V(S) := \bigoplus_{n \in \mathbb{N}} H^{\text{BM}}(\text{Hilb}^n(S), \mathbb{Q}^{\text{vir}})$$

such that they satisfy infinite dimensional Heisenberg Lie algebra type relations. We explain the construction for $m > 0$, since that's what we will be using later. Let

$$\text{Hilb}^{n, n+m}(S) = \{I \subset J, \text{Supp}(I/J) = \text{pt} \in S\} \subset S \times \text{Hilb}^{n+m}(S) \times \text{Hilb}^n(S).$$

Note that, we have a map $s : \text{Hilb}^{n, n+m}(S) \rightarrow S$ given by taking $s(I \subset J) = \text{Supp}(I/J)$. For any $I \subset \{1, 2, 3\}$, let π_I be the projection maps from $S \times \text{Hilb}^{n+m}(S) \times \text{Hilb}^n(S)$ to components in I . Then

$$p_m(\alpha) : H^{\text{BM}}(\text{Hilb}^n(S), \mathbb{Q}^{\text{vir}}) \rightarrow H^{\text{BM}}(\text{Hilb}^{n+m}(S), \mathbb{Q}^{\text{vir}})$$

is defined to be $p_m(\alpha)(c) = (\pi_2)_*((\pi_{1,3})^*((PD([S]) \cap \alpha) \otimes c) \cap [\text{Hilb}^{n, n+m}(S)])$ where $PD : H(S, \mathbb{Q}^{\text{vir}}) \simeq H^{\text{BM}}(S, \mathbb{Q}^{\text{vir}})$ is the Poincare duality map. The action of $p_m(\alpha)$ for $m < 0$ is defined similarly. Furthermore, the action of operators $p_n(\alpha)$ for any n and α is faithful on $V(S)$ [Nak97].

Lehn and Lin-Qing-Wang relations

In [Leh99], the author constructs operators on the Hilbert schemes of points on surfaces by combining Nakajima's operators and the action of the Chern classes of tautological bundles.

Definition 7.4.7 (Tautological bundles on Hilbert scheme of points). Let F be any locally free sheaf on the smooth quasi-projective surface S . For any $n \geq 0$, let $Z_S^n \subset \text{Hilb}^n(S) \times S$ be the universal family. The associated tautological bundle $F^{[n]}$ is defined to be $(\pi_1)_*(\mathcal{O}_{Z_S^n} \otimes (\pi_2)^*F)$ where π_1, π_2 are the projection maps from $\text{Hilb}^n(S) \times S$. It is a locally free sheaf of rank rn on $\text{Hilb}^n(S)$ where the rank of F is r . By definition, the fiber $F^{[n]}(Z)$ for Z a point in $\text{Hilb}^n(S)$ can be naturally identified with $H^0(Z, F \otimes \mathcal{O}_Z)$. When the context is clear, we sometimes abuse notation and refer to the associated tautological bundle $F^{[n]}$ as F itself.

Definition 7.4.8 (Lehn operators). Given cohomology class $\alpha \in H(S, \mathbb{Q})$, let $p_n(\alpha)$ be the Nakajima operator. Then we define an operator $p_n^1(\alpha) = [\partial, p_n(\alpha)]$ on $V(S)$ where $\partial \cdot v = c_1(\mathcal{O}_S^{[m]}) \cup v$ for any $v \in H^{\text{BM}}(\text{Hilb}^m(S), \mathbb{Q}^{\text{vir}}) \subset V(S)$ and more generally we define

$$p_n^k(\alpha) := \text{ad}_{\partial}^k(p_n(\alpha)) = \underbrace{[\partial, [\partial, [\dots, [\partial, p_n(\alpha)]]]]}_{k \text{ times}}.$$

For a smooth projective surface S , the relations between the operators $[p_n^{k_n}(\alpha), p_m^{k_m}(\beta)]$ have been computed in [Leh99][Theorem 3.10] and

[LQW02][Theorem 5.5]. The formula involves contributions of the first Chern of the canonical bundle $c_1(\mathcal{K}_S)$ and the Euler class $c_2(\mathcal{T}_S)$. However, the surface S_K is only *quasi*-projective in our setting. However, due to the result of [MN18], the cohomology of S_K is still pure. Furthermore, S_K is Calabi-Yau, i.e. $\mathcal{K}_{S_K} = \mathcal{O}_{S_K}$. Let $i : S_K \hookrightarrow \overline{S_K}$ be a smooth compactification. Then the pullback $H^*(\text{Hilb}^n(\overline{S_K}), \mathbb{Q}) \rightarrow H^*(\text{Hilb}^n(S_K), \mathbb{Q})$ is a surjection of cohomology rings. Furthermore, the Chern classes of tautological bundles are supported on the pure cohomology, so that $c_i(V) \cup \alpha = c_i(V)|_{S_K} \cup \alpha$. Since $i^*(c_1(\mathcal{K}_{\overline{S_K}})) = c_1(\mathcal{K}_{S_K}) = c_1(\mathcal{O}_{S_K}) = 0$ and $i^*(c_2(\mathcal{T}_{\overline{S_K}})) = 0$ by cohomological degree reasons, we conclude from [Leh99][Theorem 4.2] and [LQW02][Theorem 5.5, 4.2, Definition 5.1] that:

Proposition 7.4.9. *For any $v \in V_K$, we have:*

1. *For any line bundle F on S_K , we have:*

$$[c_1(F), p_1(\alpha)]v = p_1^1(\alpha)v + p_1(c_1(F)\alpha)v.$$

2. *For any $n_1, n_2 \geq 1$, $a_1, a_2 \geq 0$ and $\alpha, \beta \in H(S_K, \mathbb{Q})$, we have:*

$$[p_{n_1}^{a_1}(\alpha), p_{n_2}^{a_2}(\beta)]v = n_1^{a_1} n_2^{a_2} \frac{a_2 n_1 - n_2 a_1}{(n_1 + n_2)^{a_1 + a_2 - 1}} p_{n_1 + n_2}^{a_1 + a_2 - 1}(\alpha\beta)v.$$

CoHA subalgebra action

We now consider a subalgebra

$$\mathcal{A}_{\overline{Q^K}, \overline{W^K}}^{\mathfrak{S}} := \bigoplus_{n \in \mathbb{Z}_{\geq 0 \cdot \delta}} \mathcal{A}_{\overline{Q^K}, \overline{W^K}, n \cdot \delta}$$

of $\mathcal{A}_{\overline{Q^K}, \overline{W^K}}$. We show that there is an action of $\mathcal{A}_{\overline{Q^K}, \overline{W^K}}^{\mathfrak{S}}$ on V_K . To do so, we use the work in [KV00], which describes $\text{Hilb}^n(S_K)$ as the Nakajima quiver variety on a cyclic quiver with a different stability condition than what we considered before.

Definition 7.4.10 (Kuznetsov's stability condition). For any $n \geq 1$, given a dimension vector $n \cdot \delta$, we consider the stability condition ζ^n on the cyclic quiver Q^K defined by $\zeta_i^n = -2n$ for all $i = 1, \dots, K$ and $\zeta_0^n = 2Kn - 1$.

Note that this stability condition lies in a chamber different from ζ^- [Cra21][Theorem 4.2]. We now show that ζ^n stability fits the criterion to apply the Proposition 7.2.2.

Proposition 7.4.11. *The stability condition $\hat{\zeta}^n$ is generic for the dimension $(n \cdot \delta, 1)$.*

Proof. We have to show that $\mu_{\hat{\zeta}^n}(\mathbf{d}') \neq \mu_{\hat{\zeta}^n}((n\delta, 1))$ for any $\mathbf{d}' < (n\delta, 1)$.

- Assume that $\mathbf{d}' = (\mathbf{d}, 0)$ for some $\mathbf{d} < n\delta$. Then we have $\sum_{i=0}^K \zeta_i^n \mathbf{d}_i = 0$. Thus

$$(2Kn - 1)\mathbf{d}_0 = 2n \left(\sum_{i=1}^K \mathbf{d}_i \right). \quad (7.8)$$

This implies that $2n$ divides \mathbf{d}_0 , which is not possible since $\mathbf{d}_0 \leq n$ unless $\mathbf{d}_0 = 0$. But if $\mathbf{d}_0 = 0$ then by the equation 7.8, $\sum_{i=1}^K \mathbf{d}_i = 0$ and so each of $\mathbf{d}_i = 0$.

- Now assume $\dim(\rho') = (\mathbf{d}, 1)$ for some $\mathbf{d} < n\delta$. Then again, by the same argument, we see that it implies $\mathbf{d} = n \cdot \delta$, which contradicts that ρ' is a subrepresentation.

□

The reason why we consider this peculiar stability condition is due to the following theorem.

Proposition 7.4.12. [Kuz07] [Theorem 4.3] *For any $n \geq 0$, there is a canonical isomorphism of varieties*

$$g_n : \text{Hilb}^n(S_K) \simeq \mathbb{N}_{n,\delta,\delta_0}^{\zeta^n}(Q^K).$$

Let $\mathcal{V}_i^{\zeta^n}$ be the tautological bundles on the Nakajima quiver variety $\mathbb{N}_{n,\delta,\delta_0}^{\zeta^n}(Q^K)$. Note that since $S_K \simeq \mathbb{N}_{\delta,\delta_0}^{\zeta^\infty}(Q^K)$ there are tautological bundles \mathcal{V}_i on S_K , which by Definition 7.4.7 gives tautological bundles $[\mathcal{V}_i]^{[n]}$ on $\text{Hilb}^n(S_K)$. We then have

Proposition 7.4.13. *Under the above isomorphism, the tautological vector bundles $\mathcal{V}_i^{\zeta^n}$ on $\mathbb{N}_{n,\delta,\delta_0}^{\zeta^n}(Q^K)$ are pulled back to vector bundles $[\mathcal{V}_i]^{[n]}$ on $\text{Hilb}^n(S_K)$.*

Proof. This is really in the construction of the isomorphism. Let Z be a closed point in $\text{Hilb}^n(S_K)$. It is a length n subscheme in S_K . Let \mathcal{O}_Z be its structure sheaf. Then the map g_n associates a closed point on $\mathbb{N}_{n,\delta,\delta_0}^{\zeta^n}(Q^K)$ by assigning vector space $V_i(Z) := \mathbb{H}^0(S_K, \mathcal{V}_i \otimes \mathcal{O}_Z)$ to the vertex i where $i = 0, \dots, K$, $\mathbb{H}^0(S_K, \mathcal{O}_{S_K})$ to the vertex ∞ . Since the fiber of $(\mathcal{V}_i)^{[n]}$ on any point Z is exactly $\mathbb{H}^0(S_K, \mathcal{V}_i \otimes \mathcal{O}_Z)$ the statement follows. □

By Proposition 7.2.2, there is an isomorphism of graded vector spaces:

$$\bigoplus_{n \geq 0} \mathbb{H}(\mathcal{M}_{n,\delta,\delta_0}^{\zeta^n, 3d}(Q^K), {}^p\Phi_{\text{Tr}^{\zeta^n}(\widetilde{W}_{\delta_0})} \mathcal{IC}_{\mathcal{M}_{n,\delta,\delta_0}^{\zeta^n, 3d}(Q^K)}[-1]) \simeq \bigoplus_{n \geq 0} \mathbb{H}^{\text{BM}}(\mathbb{N}_{n,\delta,\delta_0}^{\zeta^n}(Q^K), \mathbb{Q}^{\text{vir}}) \simeq V_K.$$

We now mimic the construction as in Section 7.3. For any $n, m \geq 0$, we consider the stack of short exact sequences

$$\mathcal{M}_{n,\delta,m\cdot\delta,\delta_0}^{\zeta^{n+m}, 3d}(Q^K) = \{0 \rightarrow (\rho_1)_{(n,\delta,0)} \rightarrow (\rho_2)_{((m+n)\cdot\delta,1)}^{\zeta^{n+m}-\text{ss}} \rightarrow (\rho_3)_{(m\cdot\delta,0)} \rightarrow 0\},$$

where ρ_1, ρ_2, ρ_3 are representations of \widetilde{Q}_{δ_0} of dimension $(n \cdot \delta, 0)$, $((m+n) \cdot \delta, 1)$, $(m \cdot \delta, 1)$ respectively such that ρ_2 is ζ^{n+m} -semistable. We then have

Lemma 7.4.14. *Let ρ be a representation of dimension $((n+m) \cdot \delta, 1)$ such that ρ is ζ^{n+m} -semistable then any quotient ρ' of dimension $(m \cdot \delta, 1)$ is ζ^m -semistable.*

Proof. Let $\pi : \rho \rightarrow \rho'$ be the surjection. Let $V \subset \rho'$ be a non-zero proper subrepresentation such that $V \neq \rho'$. Define $\chi_m((\mathbf{d}, a)) = (\sum_{i=0}^K \zeta_i^m \mathbf{d}_i) + \zeta_\infty^m a$. Recall (Section 2.0.9) that the slope of the representation V is defined as $\mu_{\zeta^m}(V) = \chi_m((\mathbf{d}, a)) / (\sum_{i=1}^{i=K} \mathbf{d}_i + a)$. Thus, it suffices to show that $\chi_m(V) \leq 0$. We consider the representation $\pi^{-1}(V)$.

- Assume that $\dim(V) = (\mathbf{d}, 1)$. Then $\dim(\pi^{-1}(V)) = (\mathbf{d} + n \cdot \delta, 1)$ and is a non-zero proper subrepresentation of ρ . Since ρ is ζ^{n+m} -semistable, it follows that $\chi_{n+m}(\pi^{-1}(V)) < 0$. So

$$\begin{aligned} 0 > \chi_{n+m}(\pi^{-1}(V)) &= \sum_{i=0}^K (\zeta_i^{n+m}(\mathbf{d}_i + n) - \zeta_i^{m+n}(m + n)) \\ &= \sum_{i=1}^K (-2(n+m)(\mathbf{d}_i - m)) + (2K(m+n) - 1)(\mathbf{d}_0 - m) \\ &= -2(n+m) \left(\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) \right) + m - \mathbf{d}_0 \end{aligned}$$

But then since $V \subset \rho'$ is a proper subrepresentation, $\mathbf{d} < m \cdot \delta$. The above inequality implies that

$$\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) > (m - \mathbf{d}_0) / 2(n+m) > 0$$

But since $\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0)$ is an integer, above inequality implies that in fact $\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) \geq 1$. So

$$\begin{aligned} \chi_m(V) &= \sum_{i=0}^K \zeta_i^m (\mathbf{d}_i - m) \\ &= (-2m) \sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) + m - \mathbf{d}_0 \leq -m - \mathbf{d}_0 < 0. \end{aligned}$$

- Now assume that $\dim(V) = (\mathbf{d}, 0)$. Then again

$$\begin{aligned} 0 > \chi_{n+m}(\pi^{-1}(V)) &= \sum_{i=0}^K \zeta_i^{n+m} (\mathbf{d}_i + n) \\ &= \sum_{i=1}^K (-2(n+m)(\mathbf{d}_i + n)) + (2K(n+m) - 1)(\mathbf{d}_0 + n) \\ &= -2(n+m) \left(\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) \right) - n - \mathbf{d}_0 \end{aligned}$$

and so $\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) > -(n + \mathbf{d}_0) / (2(n+m)) \geq -(n+m) / (2(n+m)) > -1/2$, since $\mathbf{d}_0 \leq n$. But then since $\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0)$ is an integer, in fact we have

$\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) \geq 0$. Now

$$\chi_m(V) = \sum_{i=0}^K \zeta_i^m(\mathbf{d}_i) = -2m \left(\sum_{i=1}^K (\mathbf{d}_i - \mathbf{d}_0) \right) - \mathbf{d}_0 \leq 0.$$

So for every non-zero proper subrepresentation $V \subset \rho'$, $\chi_m(V) \leq 0$ and so ρ' is ζ^m -semistable. \square

Thus we have the usual correspondence

$$\mathfrak{M}_{n,\delta}(Q^K) \times \mathcal{M}_{m,\delta,\delta_0}^{\zeta^m,3d}(Q^K) \xleftarrow{\pi_1 \times \pi_3} \mathcal{M}_{n,\delta,m,\delta,\delta_0}^{\zeta^{n+m},3d}(Q^K) \xrightarrow{\pi_2} \mathcal{M}_{(n+m),\delta,\delta_0}^{\zeta^{n+m},3d}(Q^K).$$

The map $\pi_1 \times \pi_3$ is smooth since it is a fibre bundle. Unlike the ζ^- stability condition, it is not entirely obvious why the morphism π_2 is a proper map. However, by the previous lemma, fibres of the map π_2 are Quot schemes, and so, in particular, the morphism π_2 is proper. Thus again, just as in the definition of cohomological Hall algebra, we do the pushforward (3.2.4) along π_2 and the pullback (3.2.2) along $\pi_1 \times \pi_3$ in the vanishing cycle cohomology and finally apply the Thom-Sebastiani isomorphism (3.2.1) to get that

Proposition 7.4.15. *There is a left action \blacksquare of $\mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathfrak{S}}$ on the cohomologically graded vector space V_K . It satisfies $(u \cdot \alpha) \blacksquare v = u \cdot (\alpha \blacksquare v) - \alpha \blacksquare (u \cdot v)$ where action of u on $v \in \mathrm{H}^{\mathrm{BM}}(\mathrm{Hilb}^n(S_K), \mathbb{Q}^{\mathrm{vir}}) \subset V_K$ is given by cup product with $\sum_{i=0}^K c_1(\mathcal{V}_i^{[n]})$.*

Proof. The associativity of the action can be proven in the same way as the associativity of the cohomological Hall algebra in [KS11]. The action of u is induced by the determinant line on the Nakajima quiver variety realization in Proposition 7.4.12. Thus, the claim follows from Proposition 7.4.13. \square

Thus we have an induced left action of $\mathfrak{g}_{Q^K, \widetilde{W^K}}^{\mathfrak{S}}$ on V_K where

$$\mathfrak{g}_{Q^K, \widetilde{W^K}}^{\mathfrak{S}} := \bigoplus_{n \geq 1} \mathfrak{g}_{Q^K, \widetilde{W^K}, n, \delta}^{\mathrm{BPS}}.$$

Proposition 7.4.16. *The action of $\mathfrak{g}_{Q^K, \widetilde{W^K}}^{\mathfrak{S}}$ on the graded vector space V_K is faithful.*

Proof. We first claim that there is isomorphism of graded vector spaces

$$\mathrm{H}^{\mathrm{BM}}(\mathrm{Hilb}^n(S_K), \mathbb{Q}^{\mathrm{vir}}) \simeq \mathfrak{g}_{Q_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (n,\delta,1)}^{\mathrm{BPS}}.$$

We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_{(n,\delta,1)}^{\zeta^n-\mathrm{ss}}(\widetilde{Q}_{\delta_0}^K) & \hookrightarrow & \mathfrak{M}_{(n,\delta,1)}(\widetilde{Q}_{\delta_0}^K) \\ \downarrow \mathrm{JH}_{(n,\delta,1)}^{\zeta^n} & & \downarrow \mathrm{JH}_{(n,\delta,1)} \\ \mathcal{M}_{n,\delta,\delta_0}^{\zeta^n,3d}(Q^K) & \xrightarrow{q_{(n,\delta,1)}^{\zeta^n}} & \mathcal{M}_{(n,\delta,1)}(\widetilde{Q}_{\delta_0}^K), \end{array}$$

where q is the affinization map and JH are the Jordan-Holder maps. Note that $\text{JH}_{(n\cdot\delta,1)}^{\zeta^n}$ is a pt/\mathbb{C}^* fibre bundle and thus we have

$$(\text{JH}_{(n\cdot\delta,1)}^{\zeta^n})^* \mathbb{P}\Phi_{\text{Tr}^{\zeta^n}(\widetilde{W}_{\delta_0}^K)} \mathcal{IC}_{\mathfrak{M}_{(n\cdot\delta,1)}^{\zeta^n-\text{ss}}(\widetilde{Q}_{\delta_0}^K)} \simeq \mathbb{P}\Phi_{\text{Tr}^{\zeta^n}(\widetilde{W}_{\delta_0}^K)} \mathcal{IC}_{\mathcal{M}_{n\cdot\delta,\delta_0}^{\zeta^n,3d}(Q^K)}[-1] \otimes \text{H}(\text{BC}_{\mathfrak{u}}^*, \mathbb{Q}).$$

So, in particular,

$$\mathcal{BPS}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K}^{\zeta^n} = \mathbb{P}'_{\tau \leq 1}((\text{JH}_{(n\cdot\delta,1)}^{\zeta^n})^* \mathbb{P}\Phi_{\text{Tr}^{\zeta^n}(\widetilde{W}_{\delta_0}^K)} \mathcal{IC}_{\mathfrak{M}_{(n\cdot\delta,1)}^{\zeta^n-\text{ss}}(\widetilde{Q}_{\delta_0}^K)}[1]) \simeq \mathbb{P}\Phi_{\text{Tr}^{\zeta^n}(\widetilde{W}_{\delta_0}^K)} \mathcal{IC}_{\mathcal{M}_{n\cdot\delta,\delta_0}^{\zeta^n,3d}(Q^K)}.$$

By the theorem of Toda (Proposition 4.1.4), we have a natural isomorphism

$$\text{H}(\mathcal{M}_{n\cdot\delta,\delta_0}^{\zeta^n,3d}(Q^K), \mathcal{BPS}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K}^{\zeta^n}[-1]) \simeq \mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (n\cdot\delta,1)}^{\text{BPS}}.$$

Finally, by Proposition 7.2.2, there is an isomorphism

$$\text{H}(\mathcal{M}_{n\cdot\delta,\delta_0}^{\zeta^n,3d}(Q^K), \mathbb{P}\Phi_{\text{Tr}^{\zeta^n}(\widetilde{W}_{\delta_0}^K)} \mathcal{IC}_{\mathcal{M}_{n\cdot\delta,\delta_0}^{\zeta^n}(Q^K)}[-1]) \simeq \text{H}^{\text{BM}}(\text{N}_{(n\cdot\delta,\delta_0)}^{\zeta^n}(Q^K), \mathbb{Q}^{\text{vir}}).$$

and thus the claim follows from Proposition 7.4.12.

So we may identify the action of $\mathfrak{g}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathfrak{S}}$ on $V_K \simeq \bigoplus_{n \geq 0} \mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (n\cdot\delta,1)}^{\text{BPS}}$ by the natural identification $\mathfrak{g}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathfrak{S}} \simeq \bigoplus_{n \geq 0} \mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (n\cdot\delta,0)}^{\text{BPS}}$. Let $\alpha \in \mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (m\cdot\delta,0)}^{\text{BPS}}$ for some $m \geq 0$. By Theorem 5.1.6, we may identify $\mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K}^{\text{BPS}}$ as a positive Half of the generalized Kac Moody Lie algebra $\mathfrak{g}_{\Pi_{Q_{\delta_0}^K}}^{\text{GKM}}$. Note that $(0\cdot\delta, 1)$ is a primitive positive root of $\mathfrak{g}_{\Pi_{Q_{\delta_0}^K}}^{\text{GKM}}$, since the Symmetrized Euler form $((0\cdot\delta, 1), (0\cdot\delta, 1))_{\Pi_{Q_{\delta_0}^K}} = 2$. Thus there exist Chevalley generators for dimension $(0\cdot\delta, 1)$ in the subspace $\bigoplus_{n \geq 0} \mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (n\cdot\delta,1)}^{\text{BPS}} \subset \mathfrak{g}_{\Pi_{Q_{\delta_0}^K}}^{\text{GKM}}$. We denote them by e_∞, f_∞ and h_∞ where $e_\infty \in \mathfrak{g}_{\widetilde{Q}_{\delta_0}^K, \widetilde{W}_{\delta_0}^K, (0\cdot\delta,1)}^{\text{BPS}}$. We claim that $[\alpha, e_\infty] \neq 0$, which proves the assertion. Note that $[\alpha, f_\infty] = 0$ as there are no elements in dimension $(m\cdot\delta, -1)$. Also by definition of Generalized Kac Moody Lie algebra, $[\alpha, h_\infty] = ((m\cdot\delta, 0), (0\cdot\delta, 1))_{\Pi_{Q_{\delta_0}^K}} = -h_\infty \neq 0$. But then if $[\alpha, e_\infty] = 0$, then $[\alpha, [f_\infty, e_\infty]] = 0 = [\alpha, h_\infty] = 0$ which we checked is non-zero. \square

Proposition 7.4.17. *Up to a non-zero scalar, the operator γ_δ acts on V_K as the Nakajima raising operator $p_1([S_K])$.*

Proof. Applying dimension reduction (3.3) by forgetting loops, i.e. the morphism of representation spaces $\text{Rep}_\delta(\widetilde{Q}) = \text{Rep}_\delta(\widetilde{Q}^K) \times \text{Rep}_\delta(L^K) \rightarrow \text{Rep}_\delta(\widetilde{Q}^K)$, where L^K is the quiver obtained by removing all the arrows a and a^* from the tripled quiver \widetilde{Q}^K , The action of $\mathfrak{g}_{\widetilde{Q}^K, \widetilde{W}^K, \delta}^{\text{BPS}}$ on V_K gives a map of cohomologically graded vector spaces

$$a_{1,n} : \text{H}^{\text{BM}}(\mathfrak{M}_\delta(\Pi_{Q^K}), \mathbb{Q}^{\text{vir}}) \times \text{H}^{\text{BM}}(\text{Hilb}^n(S_K), \mathbb{Q}^{\text{vir}}) \rightarrow \text{H}^{\text{BM}}(\text{Hilb}^{n+1}(S_K), \mathbb{Q}^{\text{vir}}).$$

We consider the stability condition ζ^1 (Definition 7.4.10) so that $\mathfrak{M}_\delta^{\zeta^1}(\Pi_{Q^K}) \simeq \text{Coh}_{\text{pt}}(S) \simeq S/\mathbb{C}^*$. Then by Toda's theorem (Proposition 4.1.4), $\mathfrak{g}_{\widehat{Q^K}, \widehat{W^K}, \delta}^{\text{BPS}, \zeta^1} \simeq \mathfrak{g}_{\widehat{Q^K}, \widehat{W^K}, \delta}^{\text{BPS}}$, we have an isomorphism of vector spaces $\mathfrak{g}_{\widehat{Q^K}, \widehat{W^K}, \delta}^{\text{BPS}} \simeq \text{H}^{\text{BM}}(S_K, \mathbb{Q}^{\text{vir}})$. By observing the cohomological degree, we can identify γ_δ with $[S_K] \in \text{H}^{\text{BM}}(S_K, \mathbb{Q}^{\text{vir}})$. Let

$$\mathcal{M}_{(n, n+1)}^{\text{st}}(\Pi_Q) := \{0 \rightarrow (\rho_1)_\delta \rightarrow (\rho_2)_{(n+1)\cdot\delta, 1} \rightarrow (\rho_3)_{(n\cdot\delta, 1)} \rightarrow 0\}$$

be the variety of short exact sequences on representations of $\Pi_{Q_{\delta_0}}$ such that ρ_1 is ζ^1 -semistable, ρ_2 is ζ^{n+1} -semistable and ρ_3 is ζ^n stable. Given a point $(Z_1 \subset Z_2) \in \text{Hilb}^{n, n+1}(S_K)$, it induces a short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_s \rightarrow \mathcal{O}_{Z_2} \rightarrow \mathcal{O}_{Z_1} \rightarrow 0$$

for some $s \in S_K$ which equivalently after applying g from Proposition 7.4.12, gives an exact sequence of representations

$$0 \rightarrow \widehat{g_1(s)} \rightarrow g_{n+1}(Z_2) \rightarrow g_n(Z_1) \rightarrow 0$$

where $\widehat{g_1(s)}$ is the kernel of the surjection $g_1(s) \rightarrow \rho_0$ where ρ_0 is unique $(\mathbf{0}, 1)$ dimensional representation. The same can be done the other way, i.e., given a short exact sequence $0 \rightarrow \rho_1 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow 0$ in $\mathcal{M}_{(n, n+1)}^{\text{st}}(\Pi_{Q^K})$, we can apply the inverse of isomorphism g to get a closed point $g_n^{-1}(\rho_3) \subset g_{n+1}^{-1}(\rho_2)$ of $\text{Hilb}^{n, n+1}(S_K)$. Thus, it follows that $\text{Hilb}^{n, n+1}(S_K) \simeq \mathcal{M}_{(n, n+1)}^{\text{st}}(\Pi_Q)$. We then consider the commutative diagram

$$\begin{array}{ccccc} S_K \times \text{Hilb}^n(S_K) & \xleftarrow{p_1 \times p_3} & \mathcal{M}_{(n, n+1)}^{\text{st}}(\Pi_Q) \simeq \text{Hilb}^{n, n+1}(S_K) & \xrightarrow{p_2} & \text{Hilb}^{n+1}(S_K) \\ & \searrow \pi_{1,3} & \downarrow i & \nearrow \pi_2 & \\ & & S_K \times \text{Hilb}^{n+1}(S_K) \times \text{Hilb}^n(S_K) & & \end{array}$$

where the morphism $p_1 \times p_3$ is projection to (ρ_1, ρ_3) and p_2 is projection to ρ_2 . In particular, $p_1 \times p_3$ is already quasi-smooth, and p_2 is proper. However, since dimension reduction is compatible with the action of the cohomological Hall algebra of preprojective algebra on Nakajima quiver varieties [YZ20a]⁴, we have

$$\begin{aligned} a_{1,n}([S_K] \otimes \alpha) &= (p_2)_*(p_1 \times p_3)^!([S_K] \otimes \alpha) = (\pi_2)_*(\pi_{1,3}^*([S_K] \otimes \alpha) \cap [\text{Hilb}^{n, n+1}(S_K)]) \\ &= p_1([S_K])\alpha, \end{aligned}$$

as we wanted. □

⁴In particular, since $\pi_1 \times \pi_3$ is smooth, we use the compatibility between dimension reduction with pullback and pushforward.

Chapter 8

Affinized BPS Lie algebra for Cyclic Quivers

8.1 Integral Matrix $\mathcal{W}_{1+\infty}$ Lie algebras

In the introduction, we defined the associative algebras $D_{\hbar}(\mathbb{C}^*)$ and $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$ for any $K \geq 1$ (Definition 1.2.1). We further defined a subspace $\mathcal{W}_K \subset (D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{\pm 1}]$. We check that

Proposition 8.1.1. *The subspace $\mathcal{W}_K \subset (D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K) \otimes_{\mathbb{C}[\hbar]} \mathbb{C}[\hbar^{-1}]$ forms a Lie subalgebra.*

Proof. We recall that \mathcal{W}_K is the $\mathbb{C}[\hbar]$ linear subspace spanned by $T_{m,a}(X) = z^m D^a X$ where $m \in \mathbb{Z}, a \geq 0$, and $t_{m,a} = T_{m,a}(1)/\hbar$ where $1 \in \mathfrak{gl}_K$ is the identity matrix. Since $[D, z] = \hbar z$ and $[D, z^{-1}] = -\hbar z^{-1}$ it follows inductively that

$$[z^m D^a, z^n D^b] = z^{m+n} ((D + n\hbar)^a D^b - D^a (D + m\hbar)^b). \quad (8.1)$$

Since the expression on the right is of the form $\hbar \cdot$ (some linear expression in $t_{p,q}$), it shows that the Lie brackets $[t_{m,a}, t_{n,b}]$ are closed. We can similarly check for the Lie brackets of type $[t_{m,a}, T_{n,b}(X)]$ since

$$[z^m D^a, z^n D^b X] = z^{m+n} ((D + n\hbar)^a D^b - D^a (D + m\hbar)^b) X. \quad (8.2)$$

The other Lie brackets are clearly closed since $D_{\hbar}(\mathbb{C}^*) \otimes \mathfrak{gl}_K$ forms a Lie algebra. \square

We now define the positive half of \mathcal{W}_K , by mimicking the definition of standard positive half of the affine Lie algebra.

Definition 8.1.2 (Positive half of \mathcal{W}_K). The positive half $(\mathcal{W}_K)^+$ is defined to be $\mathbb{C}[\hbar]$ linear subalgebra generated by $T_{k,a}(X)$ where $k \geq 1, a \geq 0, X \in \mathfrak{gl}_K$ or $k = 0, a \geq 0, X \in \mathfrak{n}_K$ where $\mathfrak{n}_K \subset \mathfrak{gl}_K$ is the Lie subalgebra generated by upper diagonal matrices and $t_{k,a}$ where $k \geq 1, a \geq 0$.

We will also consider the classical limit of \mathcal{W}_K . Note, however, that the way it is defined, we cannot directly set $\hbar = 0$ since $t_{n,a} = T_{n,a}(1)/\hbar$. But using relations

(8.1) and (8.2) and more generally

$$[z^m D^a X, z^n D^b Y] = z^{m+n} (D + n\hbar)^a D^b XY - z^{m+n} D^a (D + m\hbar)^b YX, \quad (8.3)$$

for any $X, Y \in \mathfrak{gl}_K$, it follows that

Proposition 8.1.3. *Let \widetilde{W}_K be the Lie algebra generated by $T_{k,a}(X)$ where $X \in \mathfrak{sl}_K$ and $t_{k,a}$ where $k \in \mathbb{Z}$ and $a \geq 0$ with the relations*

$$[t_{m,a}, T_{n,b}(X)] = (na - mb)T_{m+n,a+b-1}(X) \quad (8.4)$$

$$[t_{m,a}, t_{n,b}] = (na - mb)t_{m+n,a+b-1} \quad (8.5)$$

$$[T_{m,a}(X), T_{n,b}(Y)] = T_{m+n,a+b}([X, Y]) \quad (8.6)$$

Then there is an isomorphism of Lie algebras

$$(\mathcal{W}_K)/(\hbar = 0) \simeq \widetilde{W}_K$$

Recall the Lie algebra $\mathfrak{po}(T^*(\mathbb{C}^*)) \times (\mathcal{O}(T^*(\mathbb{C}^*)))$ defined in Section 1.2.1. We note that

$$\mathfrak{po}(T^*(\mathbb{C}^*)) \times (\mathcal{O}(T^*(\mathbb{C}^*)) \otimes \mathfrak{sl}_K) \simeq \widetilde{W}_K,$$

since $\mathfrak{po}(T^*(\mathbb{C}^*))$, the Lie algebra of the functions on the cotangent bundle $T^*(\mathbb{C}^*)$ is generated by $D = z \frac{\partial}{\partial z}$ and $z^{\pm 1}$ with the Poisson bracket, determined by $\{D, z\} = z$. It is spanned by $z^k D^a$ where $k \in \mathbb{Z}$, $a \geq 0$ and the morphism

$$A: \mathfrak{po}(T^*(\mathbb{C}^*)) \times (\mathcal{O}(T^*(\mathbb{C}^*))) \rightarrow \widetilde{W}_K,$$

given by $A(z^{k_1} D^{a_1}, z^{k_2} D^{a_2} X) \mapsto t_{k_1, a_1} + T_{k_2, a_2}(X)$ is an isomorphism of Lie algebras. We now define the positive half of \widetilde{W}_K in the same way as before.

Definition 8.1.4 (Positive half of \widetilde{W}_K). Let $\widetilde{W}_K^+ \subset \widetilde{W}_K$ be the Lie subalgebra spanned by $T_{k,a}(X)$ where $k \geq 1, a \geq 0, X \in \mathfrak{sl}_K$ or $k = 0, a \geq 0, X \in \mathfrak{n}_K$ where $\mathfrak{n}_K \subset \mathfrak{sl}_K$ is the Lie subalgebra generated by upper diagonal matrices and $t_{k,a}$ where $k \geq 1, a \geq 0$.

We now show that \widetilde{W}_K is, in fact, a Lie algebra in the category of **Heis** modules. Let

$$H_K = \sum (i - 1/2 - K/2) E_{i,i}$$

be a diagonal element inside \mathfrak{sl}_K . We defined it so that $[H_K, E_{i,j}] = (i - j)E_{i,j}$ for $E_{i,j} \in \mathfrak{sl}_K$.

Definition 8.1.5 (Grading on \widetilde{W}_K^+). The Lie algebra \widetilde{W}_K^+ has two gradings. We define the *cohomological grading* by $\text{CG}(T_{m,a}(X)) = 2a$, $\text{CG}(t_{m,a}) = 2a - 2$ and *dimension grading* induced by $\dim(T_{m,a}(E_{i,j})) = Km + j - i$ and $\dim(t_{m,a}) = Km$.

Proposition 8.1.6. *There is an action of **Heis** on \widetilde{W}_K^+ by derivations, given by*

$$\begin{aligned} p &\rightarrow \left[\frac{t_{0,2}}{2} - \frac{T_{0,1}(H_K)}{K}, - \right] \\ q &\rightarrow \partial_D \end{aligned}$$

where $\partial_D(T_{m,a}(X)) = aKT_{m,a-1}(X)$, $\partial_D(t_{m,a}) = aKt_{m,a-1}$. For every graded piece of fixed dimension \dim , the central charge is the dimension \dim , and the Lie algebra \widetilde{W}_K^+ is negatively determined.

Proof. Since p acts as a commutator, it is a derivation. The same can be checked for q using Lie algebra relations (8.4). On the other hand,

$$\begin{aligned} [q, p](T_{m,a}(E_{i,j})) &= (mK + (j - i))T_{m,a}(E_{i,j}) \\ [q, p](t_{m,a}) &= mKt_{m,a} \end{aligned}$$

The negatively determined assumption holds since the **Heis** grading is the cohomological grading and ∂_D is injective for $a \geq 1$. □

Next, we see that \widetilde{W}_K^+ is generated by a much smaller set of generators.

Proposition 8.1.7. *The Lie algebra \widetilde{W}_K^+ is generated as a Lie algebra by the elements $t_{1,a}$ for $a \geq 0$, $T_{0,a}(E_{i,i+1})$ for $i = 1, 2, \dots, K - 1; a \geq 0$ and $T_{1,a}(E_{1,K}), a \geq 0$.*

Proof. Since $[t_{1,0}, t_{n,b}] = (-b)t_{n+1,b-1}$, the elements $t_{n,a}, n \geq 1; a \geq 0$ are in the subalgebra generated by $t_{1,a}$ where $a \geq 0$. The elements $E_{i,i+1}$ for $i \in [1, K - 1]$ and $zE_{K,1}$ are the generators of the positive half of the affine Lie algebra $\widetilde{\mathfrak{sl}}_K$, which is $\mathfrak{n}_K \oplus z\mathbb{Q}[z]\mathfrak{sl}_K$. Since $[T_{m,a}(X), T_{n,b}(Y)] = T_{m+n,a+b}([X, Y])$, it follows that all $T_{k,a}(X)$ where $k \geq 1, a \geq 0, X \in \mathfrak{sl}_K$ or $k = 0, a \geq 0, X \in \mathfrak{n}_K$ are in the algebra generated by $T_{0,a}(E_{i,i+1})$ for $i = 1, 2, \dots, K - 1; a \geq 0$ and $T_{1,a}(E_{1,K}), a \geq 0$ and so we are done. □

8.2 Proof of Theorem A

We are ready to calculate the affinized BPS Lie algebra $\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$ for $K \geq 1$. The affinized BPS Lie algebra for the cyclic quiver Q^K is spanned by $\alpha_{\mathbf{d}}^{(n)}$ where $\alpha_{\mathbf{d}} \in \mathfrak{g}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$. It suffices to understand the Lie bracket between these generators. By Proposition 5.0.2, we have an isomorphism of Lie algebras $\mathfrak{g}_{Q^K, \widetilde{W}^K}^{\text{BPS}} \simeq \mathfrak{n}_{Q^K}^+ \oplus s\mathbb{Q}[s]$. Let $\mathfrak{rc}_K \subset \widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$ be the subvector space spanned by the elements $\alpha_{\mathbf{d}}^{(n)}$ for $n \geq 0$, where $\alpha_{\mathbf{d}} \in \mathfrak{g}_{Q^K, \widetilde{W}^K}^{\text{BPS}, 0} \simeq \mathfrak{n}_{Q^K}^+$. It follows from Theorem 6.4.2 that

Proposition 8.2.1. *We have an isomorphism of Lie algebras*

$$\mathfrak{rc}_K \simeq \mathfrak{n}_{Q^K}^+[D].$$

Next, let $\mathfrak{im}_K \subset \widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$ be the subvector space generated by the elements $\alpha_{m,\delta}^{(n)}$ where $\alpha_{m,\delta}$ spans $\mathfrak{g}_{Q^K, \widetilde{W}^K}^{\text{BPS}, -2} \simeq \mathbb{C}$. Since $\mathfrak{g}_{Q^K, \widetilde{W}^K}^{\text{BPS}, -2}$ is one dimensional, $\alpha_{m,\delta}$ is uniquely determined up to a non-zero scalar.

Proposition 8.2.2. *For any $n \geq 1$, there exist some non-zero scalar $\lambda_n \in \mathbb{Q}$ such that we have an equality*

$$[\alpha_\delta^{(1)}, \alpha_{n \cdot \delta}] = \lambda_n \alpha_{(n+1) \cdot \delta}.$$

Proof. Note that the Lie bracket $[\alpha_\delta^{(1)}, \alpha_{n \cdot \delta}]$ is an element of cohomological degree -2 in the dimension $(n+1) \cdot \delta$. Since $\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K, (n+1) \cdot \delta}^{\text{BPS}, -2} \simeq \mathbb{Q}$ is a one dimensional vector space, it is either 0 or up to a non-zero scalar, it is $\alpha_{(n+1) \cdot \delta}$. We claim that it is non-zero. We consider the action \blacksquare on the cohomology of Hilbert scheme of S_K constructed in Proposition 7.4.15 and show that the Lie bracket $[\alpha_\delta^{(1)}, \alpha_{n \cdot \delta}]$ acts non-trivially. We claim that for any $n \geq 1$, up to a factor of a non-zero scalar, there is an equality

$$\alpha_{n \cdot \delta} \blacksquare v = p_n([S_K])v$$

where $p_n([S_K])$ is Nakajima's operator (See Section 7.4.2) and $v \in V_K$. The case when $n = 1$ is proved in Proposition 7.4.17. Then by Proposition 7.4.9

$$\begin{aligned} \alpha_\delta^{(1)} \blacksquare v &= \sum_{i=0}^K [c_1(\mathcal{R}_i), p_1([S_K])]v = \sum_{i=0}^K (p_1^1([S_K])v + p_1(c_1(\mathcal{R}_i))v) \\ &= (K+1)p_1^1([S_K])v + \sum_{i=0}^K p_1(c_1(\mathcal{R}_i))v \end{aligned} \quad (8.7)$$

where for each i , \mathcal{R}_i is the tautological bundle on S_K . But then, assuming the claim for $n = N$, we have

$$\begin{aligned} [\alpha_\delta^{(1)}, \alpha_{N \cdot \delta}] \blacksquare v &= (K+1)[p_1^1([S_K]), p_N([S_K])]v + \sum_{i=0}^K [p_1(c_1(\mathcal{R}_i)), p_N([S_K])]v \\ &= -(N)(K+1)p_{N+1}([S_K])v, \end{aligned}$$

The second equality is again a consequence of Proposition 7.4.9. But then for any n , there exists some $v \in V_K$ such that $p_{N+1}([S_K])v \neq 0$, since Nakajima's operators act faithfully (see Section 7.4.2), and thus $[\alpha_\delta^{(1)}, \alpha_{N \cdot \delta}] \neq 0$ and thus up to a factor of non-zero scalar $[\alpha_\delta^{(1)}, \alpha_{N \cdot \delta}] = \alpha_{(n+1) \cdot \delta}$ and both the claim and proposition follows by induction. \square

From now on, we fix γ_δ as some renormalization of α_δ , so that for the vacuum vector $|0\rangle = [0] \in \mathbb{H}^{\text{BM}}(\text{Hilb}^0, \mathbb{Q}_0) \in \mathbb{N}_{\delta_0}^{\zeta_\infty}(Q^K)$ we have $\gamma_\delta \bullet |0\rangle = [S_K]$. This is allowed by Proposition 7.4.5. Then we have $\gamma_\delta^{(r)} = u^r \cdot \gamma_\delta$. Finally for all $k \geq 1$, we fix

$$\gamma_{k \cdot \delta} := \mathbf{ad}_{\gamma_\delta^{(1)}}^{k-1}(\gamma_\delta) \quad (8.8)$$

which by the previous proposition, is just the renormalization of $\alpha_{k \cdot \delta}$ and it spans

$\mathfrak{g}_{\overline{Q^K}, \overline{W^K}}^{\text{BPS}, -2}$. Similarly, we have

$$\gamma_{k,\delta}^{(r)} := u^r \mathbf{ad}_{\gamma_\delta^{(1)}}^{k-1}(\gamma_\delta^{(0)}).$$

Now we study how an element of \mathfrak{im}_K interacts with an element of \mathfrak{re}_K . By $[a_1, a_2, a_3, \dots, a_N]$ we mean the iterated Lie bracket $[a_1, [a_2, [a_3, [\dots, a_N]]]]$.

Proposition 8.2.3. *For any $i \in [0, K]$, the Lie bracket $[\alpha_i, \gamma_\delta^{(1)}] \neq 0$.*

Proof. We consider the action of $\mathcal{A}_{\overline{Q^K}, \overline{W^K}}$ on $N_{\delta_0}^{\zeta_\infty}(Q^K)$ as constructed in Section 7.3. We show that $[\alpha_i, \gamma_\delta^{(1)}] \bullet |0\rangle \neq 0$. By Proposition 7.4.4, we can identify the action of α_i with the action of the generator e_i of the affine Lie algebra $\tilde{\mathfrak{sl}}_{K+1}$ on $N_{\delta_0}^{\zeta_\infty}(Q^K)$, due to Nakajima (Section 7.4.1).

Note that $\text{Hilb}^{\rho_0}(\mathbb{C}^2) = \text{pt}$ as $I = \langle x, y \rangle$ is the only \mathbb{Z}_{K+1} equivariant ideal with $\mathbb{C}[x, y]/I \simeq \rho_0$. Thus $\mathfrak{B}_0(\rho_{\text{reg}}) = \text{Hilb}^{\rho_0}(\mathbb{C}^2)$. Let $[0_{[0,1]}] \in \text{H}(\text{Hilb}^{\rho_0}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}})$ be the fundamental class of the point. Thus by (7.6), we have

$$\alpha_0 \bullet |0\rangle = [0_{[0,1]}].$$

Similarly, since the closed points of $\text{Hilb}^1(\mathbb{C}^2)$ are just the maximal ideals $(x - t_1, y - t_2)$, only \mathbb{Z}_{K+1} equivariant point is the ideal (x, y) . It follows that $\text{Hilb}^{\rho_i}(\mathbb{C}^2) = 0$ for all $i \neq 0$ and so by (7.6), for all $i \neq 0$ we have

$$\alpha_i \bullet |0\rangle = 0.$$

Now for any $i \neq K$, we consider the equivariant Hilbert scheme $\text{Hilb}^{\oplus_{j=0}^i \rho_j}(\mathbb{C}^2)$. By Proposition 7.4.1, $\text{Hilb}^{\oplus_{j=0}^i \rho_j}(\mathbb{C}^2)$ is 0 dimensional connected scheme and thus just a point. Let $[0_{[0,i]}] \in \text{H}(\text{Hilb}^{\oplus_{j=0}^i \rho_j}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}})$ be the fundamental class of the point. Consider $\alpha_1 \bullet (\alpha_0 \bullet |0\rangle)$. Again since each of the schemes $\text{Hilb}^{\rho_0}(\mathbb{C}^2)$ and $\text{Hilb}^{\rho_0 \oplus \rho_1}(\mathbb{C}^2)$ are points and $\rho_0 \subset \rho_0 \oplus \rho_1$, it follows that $\mathfrak{B}_1(\rho_0) = \text{Hilb}^{\rho_0 \oplus \rho_1}(\mathbb{C}^2)$. We thus have by (7.6) that

$$\alpha_1 \bullet (\alpha_0 \bullet |0\rangle) = [0_{[0,2]}].$$

We may continue this argument succesively to calculate $\alpha_{i-1} \bullet (\alpha_{i-2} \bullet (\dots \alpha_0 \bullet |0\rangle))$ for any $i \in [1, K]$. By associativity of the action of the $\mathfrak{n}_{Q^K}^+$, we conclude that for any $i \in [1, K]$,

$$(\alpha_{i-1} \star \alpha_{i-2} \star \dots \star \alpha_0) \bullet |0\rangle = [0_{[0,i]}] \in \text{H}(\text{Hilb}^{\oplus_{j=0}^i \rho_j}(\mathbb{C}^2), \mathbb{Q}).$$

Now consider the equivariant Hilbert scheme $\text{Hilb}^{\oplus_{j=0}^i \rho_j \oplus \rho_K}(\mathbb{C}^2)$. Again, by Proposition 7.4.1, it is a 0-dimensional connected scheme and hence a point. Let $[0_{[K,i+1]}] \in \text{H}(\text{Hilb}^{\oplus_{j=0}^i \rho_j \oplus \rho_K}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}})$ be the fundamental class of the point. Then again by (7.6), we have

$$\alpha_K \bullet ((\alpha_{i-1} \star \alpha_{i-2} \star \dots \star \alpha_0) \bullet |0\rangle) = (-1)[0_{[K,i+1]}] \in \text{H}(\text{Hilb}^{\oplus_{j=0}^i \rho_j \oplus \rho_K}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}}).$$

Note that the sign comes from the sign in the formula (7.6). For any $i \in [1, K]$, we

now consider $\text{Hilb}^{(\oplus_{j=0, j \neq i}^K \rho_j)}(\mathbb{C}^2)$. Again, $\text{Hilb}^{(\oplus_{j=0, j \neq i}^K \rho_j)}(\mathbb{C}^2)$ is a point by Proposition 7.4.1. Let $[0_{[i+1, K]}] \in \text{H}(\text{Hilb}^{(\oplus_{j=0, j \neq i}^K \rho_j)}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}})$ be the fundamental class of the point. Then by the same argument as above, we conclude for any $i \in [1, K]$, that

$$(\alpha_{i+1} \star \alpha_{i+2} \star \cdots \star \alpha_K \star \alpha_{i-1} \star \cdots \star \alpha_0) \bullet |0\rangle = (-1)^{K-i} [0_{[i+1, K]}] \in \text{H}(\text{Hilb}^{(\oplus_{j=0, j \neq i}^K \rho_j)}(\mathbb{C}^2), \mathbb{Q}^{\text{vir}}).$$

Note that $\rho_{\text{reg}} - \rho_i = \oplus_{j=0, j \neq i}^K \rho_j$. By (7.6), for $i \in [1, K]$, we have $\alpha_i \bullet [0_{[i+1, K]}] = (-1)[\mathfrak{B}_i(\rho_{\text{reg}})]$, where by 7.7, we may view $\mathfrak{B}_i(\rho_{\text{reg}})$ as a Lagrangian inside the surface S_K . In particular $[\mathfrak{B}_i(\rho_{\text{reg}})] \in \text{H}(S_K, \mathbb{Q}^{\text{vir}})$. We thus have for $i \in [1, K]$,

$$(\alpha_i \star \alpha_{i+1} \cdots \star \alpha_K \star \alpha_{i-1} \cdots \star \alpha_0) \bullet |0\rangle = (-1)^{K+1-i} [\mathfrak{B}_i(\rho_{\text{reg}})].$$

But since $\alpha_i \bullet |0\rangle = 0$ for $i \neq 0$, we can write above as

$$[\alpha_i, \alpha_{i+1} \cdots, \alpha_K, \alpha_{i-1} \cdots, \alpha_0] \bullet |0\rangle = (-1)^{K+1-i} [\mathfrak{B}_i(\rho_{\text{reg}})]. \quad (8.9)$$

Now for $i \neq 0$, we consider the commutator $[\alpha_i, \gamma_\delta^{(1)}]$. Again since $\alpha_i \bullet |0\rangle = 0$, we have by Proposition 7.4.5 that

$$[\alpha_i, \gamma_\delta^{(1)}] |0\rangle = \alpha_i \bullet \gamma_\delta^{(1)} |0\rangle = \alpha_i \star \left(\sum_j \left(\sum_k A_{kj} \right) [\mathfrak{B}_j(\rho_{\text{reg}})] \right).$$

But by the Equation 8.9, this is the same as

$$[\alpha_i, \gamma_\delta^{(1)}] |0\rangle = \sum_j (-1)^{K+1-j} \sum_k A_{kj} [\alpha_i; \mathfrak{B}_j] \bullet |0\rangle,$$

where $[\alpha_i, \mathfrak{B}_j]$ represents the commutator

$$[\alpha_i, \alpha_j, \alpha_{j+1} \cdots, \alpha_K, \alpha_{j-1} \cdots, \alpha_0].$$

and $A_{k,j}$ are the entries of the inverse of the Cartan matrix of A_K . It is easy to calculate the inverse of the Cartan matrix of A_K . We have $A_{kj} = \min(k, j) - \frac{kj}{K+1}$, which shows that $S_j = \sum_k A_{kj} = j(K+1-j)/2$. Similarly, since $\mathfrak{n}_{Q^K}^+ \simeq (\mathfrak{sl}_{K+1}[z, z^{-1}])^+$ after identifying e_i with $E_{i, i+1}$ and e_0 with $zE_{K+1, 1}$; we may identify $[\alpha_i, \mathfrak{B}_i] = 2((-1)^{K-i} z E_{i, i+1})$ while $[\alpha_i, \mathfrak{B}_{i-1}] = [\alpha_i, \mathfrak{B}_{i+1}] = (-1)^{K-i} z E_{i, i+1}$ and $[\alpha_i, \mathfrak{B}_j] = 0$, otherwise. In particular $[\alpha_i, \mathfrak{B}_i] = 2[\alpha_i, \mathfrak{B}_{i-1}] = 2[\alpha_i, \mathfrak{B}_{i+1}]$. Thus we have

$$[\alpha_i, \gamma_\delta^{(1)}] \bullet |0\rangle = (-1)^{K+1-i} \left(S_i - \frac{S_{i-1} + S_{i+1}}{2} \right) [\alpha_i, \mathfrak{B}_i] \bullet |0\rangle = (-1)^{(K+1-i)} / 2 [\alpha_i, \mathfrak{B}_i] \bullet |0\rangle.$$

But $[\alpha_i, \mathfrak{B}_i] \bullet |0\rangle \neq 0$ by the same argument for faithfulness as in Proposition 7.4.5. This in particular means that $[\alpha_i, \gamma_\delta^{(1)}] \neq 0$. By symmetry, we also get the same relation for $i = 0$, since we can construct the representation by adding the

framing to vertex 1 instead of 0. Thus we have

$$[\alpha_0, \gamma_\delta^{(1)}] \bullet |0\rangle = \frac{(-1)^K}{2} [\alpha_0, \alpha_0, \alpha_1, \dots, \alpha_K] \bullet |0\rangle \neq 0$$

and so we are done. \square

Note that $[\alpha_i, \gamma_\delta^{(1)}]$ is an element of cohomological degree 0, in dimension $\delta + \delta_i$. From proposition 8.2.3, it follows that $[\alpha_i, \gamma_\delta^{(1)}] = \lambda[\alpha_i, \mathcal{B}_i]$ for some non-zero λ , since $\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}, \delta + \delta_i}^{\text{BPS}, 0}$ is 1 dimensional. But then, the above calculation even precisely determines λ , and so finally, we have

$$[\alpha_i, \gamma_\delta^{(1)}] = \frac{(-1)^{K+1-i}}{2} [\alpha_i, \mathcal{B}_i]. \quad (8.10)$$

Furthermore, since $[\alpha_i, \gamma_\delta^{(0)}] = 0$, we have $[\gamma_\delta^{(0)}, \alpha_i^{(1)}] = \frac{(-1)^{K+1-i}}{2} [\alpha_i, \mathcal{B}_i]$.

Corollary 8.2.3.1 (Strong rationality for Kleinian singularity). *For any $\mathbf{d} \in \mathbb{N}^{Q_0}$, the commutator*

$$[\gamma_\delta^{(1)}, -] : \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^X \rightarrow \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^X$$

maps $\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}, \mathbf{d}}^{\text{BPS}}$ isomorphically to $\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}, \mathbf{d} + \delta}^{\text{BPS}}$.

Proof. The Lie algebra generated by α_i where $i \in [0, K]$ is isomorphic to the Lie algebra $\mathfrak{n}_{Q^K}^+ \simeq (\mathfrak{sl}_{K+1}[z, z^{-1}])^+$ by Proposition 4.2.5. The isomorphism is given by the identification α_i with $E_{i, i+1}$ for $i \neq 0$ and α_0 with $zE_{K+1, 1}$. Thus after this identification, equation (8.10) implies that

$$[\gamma_\delta^{(1)}, E_{i, i+1}] = zE_{i, i+1}, \quad [\gamma_\delta^{(1)}, zE_{K+1, 1}] = z^2E_{K+1, 1}$$

But then one can write $\alpha_{\mathbf{d}} \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, 0}$ as linear combination of commutators of $E_{i, i+1}$ and zE_{K+1} . Thus if $\alpha_{\mathbf{d}} \neq 0$ then $[\gamma_\delta^{(1)}, \alpha_{\mathbf{d}}] \neq 0$. While for $\alpha_{\mathbf{d}} \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}, -2}$, $[\gamma_\delta^{(1)}, \alpha_{\mathbf{d}}] \neq 0$ is Proposition 8.2.2. Since $\gamma_\delta^{(1)}$ is an element of cohomological degree 0, for any $\alpha_{\mathbf{d}} \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}}$, $[\gamma_\delta^{(1)}, \alpha_{\mathbf{d}}] \in \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d} + \delta}^{\text{BPS}}$. So we have checked that the morphism

$$[\gamma_\delta^{(1)}, -] : \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}} \rightarrow \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d} + \delta}^{\text{BPS}}$$

is injective. Since the dimensions are the same, it is an isomorphism. \square

Proposition 8.2.4. *The subspace \mathfrak{im}_K forms a Lie subalgebra of $\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}}$ and is generated by $\gamma_\delta^{(r)}$ for $r \geq 0$.*

Proof. The claim about generation follows from the definition of $\gamma_{k, \delta}^{(r)}$ and the fact that u acts by derivation. It suffices to show that for $k \geq 1, n, m \geq 0$, the Lie bracket $[\gamma_\delta^{(n)}, \gamma_{k, \delta}^{(m)}] \in \mathfrak{im}_K$. Since for all $r \geq 1$, $\gamma_{r, \delta}^{(n)}$ is of cohomological degree

$2n - 2$, there must exist $\lambda_k^{n,m} \in \mathbb{Q}$ and $\mu_k^{n,m} \in \mathbb{Q}$ such that

$$[\gamma_\delta^{(n)}, \gamma_{k\cdot\delta}^{(m)}] = \lambda_k^{n,m} \gamma_{(1+k)\cdot\delta}^{(n+m-1)} + \mu_k^{n,m} \beta_{(1+k)\cdot\delta}^{(n+m-2)} \quad (8.11)$$

where $\beta_{(1+k)\cdot\delta} \in \mathfrak{g}_{\widetilde{Q^k, \widetilde{W^k}}^{\text{BPS},0}}$. It suffices to show that $\mu_k^{n,m} = 0$. Note that showing this for $m = 0, n \geq 0$, i.e. to show that $\mu_k^{n,0} = 0$ for all $n \geq 0$ is enough, since if for some $i \geq 0$, $[\gamma_\delta^{(n)}, \gamma_{k\cdot\delta}^{(i)}] \in \mathfrak{im}_K$ then applying u gives that $[\gamma_\delta^{(n+1)}, \gamma_{k\cdot\delta}^{(i)}] + [\gamma_\delta^{(n)}, \gamma_{k\cdot\delta}^{(i+1)}] \in \mathfrak{im}_K$. So if $[\gamma_\delta^{(n+1)}, \gamma_{k\cdot\delta}^{(i)}] \in \mathfrak{im}_K$ then so does $[\gamma_\delta^{(n)}, \gamma_{k\cdot\delta}^{(i+1)}]$. We consider the action \blacksquare of $\mathcal{A}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathfrak{S}}$ on V_K as constructed in Proposition 7.4.15. We then have

Claim: For any $m \geq 1$, there exist a non-zero scalar $\lambda_m'' \in \mathbb{Q}$, such that for any $v \in V_K$, we have the equality

$$\gamma_{m\cdot\delta}^{(n)} \blacksquare v = \lambda_m'' \left((K+1)p_m^n([S_K])v + mn \sum p_m^{n-1}(c_1(\mathcal{R}_i))v \right) \quad (8.12)$$

Proof: We use the same argument as in Proposition 8.2.2. We identify the action $\gamma_\delta \blacksquare v$ with $p_1([S_K])v$, and then the result follows from the repetitive use of formulas in Proposition 7.4.9 applied to the definition of $\gamma_{m\cdot\delta}^{(n)}$ (equation 8.8). \square It follows from the above claim (8.12), equation (8.11), and Proposition 7.4.9, that up to a factor of a non-zero scalar, for all $v \in V_K$, we have

$$[\gamma_\delta^{(n)}, \gamma_{k\cdot\delta}^{(0)}] \blacksquare v = \gamma_{(1+k)\cdot\delta}^{(n-1)} \blacksquare v$$

To conclude, it suffices to show that for some $v \in V_K$, $u^{n-2}\alpha_{(1+k)\cdot\delta} \blacksquare v \neq 0$. By Proposition 7.4.16, the action of $\mathfrak{g}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathfrak{S}}$ on V_K is faithful, thus the statement that there exists $v \in V_K$ such that $u^{n-2}\alpha_{(1+k)\cdot\delta} \blacksquare v \neq 0$ is true for $n = 2$. Now, assume otherwise for some $n > 2$. Then for all v , $[\gamma_\delta^0, u^{n-2}\alpha_{n\cdot\delta}] \blacksquare v = 0$. But by Proposition 8.2.3.1, $[\gamma_\delta, u^{n-2}\alpha_{k\cdot\delta}] = u^{n-3}\alpha'_{(k+1)\cdot\delta}$ for some $\alpha'_{(k+1)\cdot\delta}$. Thus action of $u^{n-3}\alpha'_{(k+1)\cdot\delta}$ isn't faithful either. Thus, we inductively conclude that action of some non-zero $\beta_{(k+n-2)\cdot\delta} \in \mathfrak{g}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathfrak{S}}$ on V_K isn't faithful, which is a contradiction to Proposition 7.4.16. \square

Note that as vector spaces $\mathfrak{im}_K + \mathfrak{re}_K = \widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}}$. Thus it remains to understand $[\mathfrak{im}_K, \mathfrak{re}_K]$.

Proposition 8.2.5. *We have*

$$[\mathfrak{im}_K, \mathfrak{re}_K] \subset \mathfrak{re}_K$$

and is determined by $[\gamma_\delta^{(n)}, \alpha_i^{(m)}]$, $n \geq 0$ and $i \in [0, K], m \geq 0$.

Proof. Since \mathfrak{im} is generated by $\gamma_\delta^r, r \geq 0$ it suffices to show that $[\gamma_\delta^r, \mathfrak{re}] \subset \mathfrak{re}$. But \mathfrak{re} is also generated by e_i^n by the Proposition 8.2.1 so, in fact we need to consider the commutators between $[\gamma_\delta^r, e_i^n]$. The commutator $[\gamma_\delta^r, f_i^n]$ is an element of dimension $\delta + \delta_i$ in $\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}}$. Since \mathfrak{im}_K only contains elements whose dimension vector is a multiple of δ , the commutator must be in \mathfrak{re}_K . \square

We now sieve back everything together. Recall the Lie algebra \widetilde{W}_{K+1}^+ from Definition 8.1.4. By Proposition 8.1.6, \widetilde{W}_{K+1}^+ is a negatively determined **Heis** Lie algebra. We consider the Lie subalgebra $\widetilde{W}_{K+1}^{+,\leq 0} \subset \widetilde{W}_{K+1}^+$. It is a Lie subalgebra spanned by $T_{0,0}(X)$ where $X \in \mathfrak{n}_{K+1}^+$, $T_{m+1,0}(X)$ where $\in \mathfrak{gl}_{K+1}$, $m \geq 0$ and $t_{n,1}, t_{n,0}$ for all $n \geq 1$. Similarly, by Proposition 6.3.11, $\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$ is a negatively determined **Heis** Lie algebra. Let $(\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}})^{\leq 0} \subset \widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$ be the non positive degree subspace of $\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}}$. Clearly it is generated by $\alpha_i^{(0)}$, $i \in [0, K]$ and $\gamma_{n,\delta}^{(0)}, \gamma_{n,\delta}^{(1)}$ for $n \geq 1$.

Theorem 8.2.6. *For any $K \geq 1$, the morphism $F : (\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}})^{\leq 0} \rightarrow \widetilde{W}_{K+1}^{+,\leq 0}$ given by*

$$\begin{aligned} \gamma_{n,\delta}^{(0)} &\rightarrow (-1)^n (n-1)! (K+1)^n t_{n,0} \\ \gamma_{n,\delta}^{(1)} &\rightarrow (-1)^n (n)! (K+1)^{n-1} ((K+1)t_{n,1} - T_{n,0}(H_{K+1})) \\ \alpha_i^{(0)} &\rightarrow T_{0,0}(E_{i,i+1}), \forall i \in [1, K] \\ \alpha_0^{(0)} &\rightarrow T_{1,0}(E_{K+1,1}) \end{aligned}$$

extends to an isomorphism of Lie algebras

$$\widehat{F} : \widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}} \rightarrow \widetilde{W}_{K+1}^+.$$

Proof. We check that F satisfies the conditions in Proposition 6.3.6. We first check the compatibility with **Heis** actions. We have

$$\begin{aligned} F(u \cdot \gamma_{n,\delta}^{(0)}) &= F(\gamma_{n,\delta}^{(1)}) \\ &= (-1)^n (n)! (K+1)^{n-1} ((K+1)t_{n,1} - T_{n,0}(H_{K+1})) \\ &= \left[\frac{t_{0,2}}{2} - \frac{T_{0,1}(H_{K+1})}{K+1}, (-1)^n (n-1)! (K+1)^n t_{n,0} \right] \\ &= pF(\gamma_{n,\delta}^{(0)}), \end{aligned}$$

where the last equality is the consequence of the definition of p in Proposition 8.1.6 and similarly $F(\partial \cdot \gamma_{n,\delta}^{(1)}) = nF(\gamma_{n,\delta}^{(0)}) = qF(\gamma_{n,\delta}^{(1)})$ by Proposition 8.1.6. We now check that F is a morphism of Lie algebras. Since the structure constants between $[\gamma_{n,\delta}^{(1)}, \gamma_{m,\delta}^{(0)}]$ and $[\gamma_{n,\delta}^{(1)}, \gamma_{m,\delta}^{(1)}]$ for $m, n > 1$ are determined by relations of $[\gamma_\delta^{(1)}, \gamma_{m,\delta}^{(0)}]$ and $[\gamma_\delta^{(1)}, \gamma_{m,\delta}^{(1)}]$, we only consider the latter. Clearly $F([\gamma_\delta^{(0)}, \gamma_{n,\delta}^{(0)}]) = 0 = -(-1)^{n+1} [(K+1)t_{1,0}, (n-1)! (K+1)^n t_{n,0}] = 0$. Next, we see that

$$\begin{aligned} [F(\gamma_\delta^{(1)}), F(\gamma_{n,\delta}^{(0)})] &= [(-1)((K+1)t_{1,1} - T_{1,0}(H_{K+1})), (-1)^n (n-1)! (K+1)^n t_{n,0}] \\ &= (-1)^{n+1} n! (K+1)^{n+1} t_{n+1,0} \\ &= F(\gamma_{(n+1),\delta}^{(0)}) \\ &= F([\gamma_\delta^{(1)}, \gamma_{n,\delta}^{(0)}]) \end{aligned}$$

where the last equality is Equation (8.8). The same argument holds for the

relation $[\gamma_{n,\delta}^{(1)}, \gamma_\delta^{(0)}] = \gamma_{(n+1),\delta}$. On the other hand via Proposition 8.2.4 we have relation $[\gamma_\delta^{(1)}, \gamma_{n,\delta}^{(1)}] = (n-1)/(n+1)\gamma_{(n+1),\delta}^{(1)}$ whose compatibility can be checked in the same way. We now consider relations among $\alpha_i^{(0)}$ and $\gamma_{n,\delta}$. For $i \neq 0$, we have

$$\begin{aligned} [F(\gamma_\delta^{(1)}), F(\alpha_i^{(0)})] &= [(-1)((K+1)t_{1,1} - T_{1,0}(H_{K+1})), T_{0,0}(E_{i,i+1})] \\ &= T_{1,0}(E_{i,i+1}) \\ &= \frac{(-1)^{K-i}}{2} [T_{0,0}(E_{i,i+1}), T_{0,0}(E_{i,i+1}), \dots, T_{0,0}(E_{K,K+1}), T_{0,0}(E_{i-1,i}), \dots, T_{1,0}(E_{K+1,1})] \\ &= F([\gamma_\delta^{(1)}, \alpha_i^{(0)}]) \end{aligned}$$

via Equation 8.10. The same argument holds when $n > 1$ and when $i = 0$. In the same way, for $i \neq 0$

$$\begin{aligned} [F(\gamma_{n,\delta}^{(0)}), F(\alpha_i^{(0)})] &= [(-1)^n(n-1)!(K+1)^n t_{n,0}, T_{0,0}(E_{i,i+1})] \\ &= 0 = F([\gamma_{n,\delta}^{(0)}, \alpha_i^{(0)}]) \end{aligned}$$

and for $i = 0$, we have

$$\begin{aligned} [F(\gamma_{n,\delta}^{(0)}), F(\alpha_0^{(0)})] &= [(-1)^n(n-1)!(K+1)^n t_{n,0}, T_{1,0}(E_{K+1,1})] \\ &= 0 = F([\gamma_{n,\delta}^{(0)}, \alpha_0^{(0)}]) \end{aligned}$$

where the last equalities are because of Proposition 5.0.2. Finally, the relations between $[\alpha_i^{(0)}, \alpha_j^{(0)}]$ are also satisfied since by Proposition 5.0.2, the Lie algebra generated by $\alpha_i^{(0)}$ is exactly the positive half \mathfrak{n}_{K+1}^+ and the above map is just the identification of \mathfrak{n}_{K+1}^+ with $\widetilde{\mathfrak{sl}}_{K+1}^+$.

By Proposition 6.3.6, F extends to a map of **Heis** modules \hat{F} . To check if the map extends to the whole Lie algebra, it suffices to check the relations for $[\gamma_{n,\delta}, \gamma_{m,\delta}^{(2)}]$, $[\gamma_{n,\delta}^{(1)}, \gamma_{m,\delta}^{(1)}]$, $[\gamma_{n,\delta}, \alpha_i^{(1)}]$. Since by definition, it is a map of **Heis** modules, this boils down to checking the relation for $[\gamma_\delta^{(2)}, \gamma_{n,\delta}]$. But again it can be checked by definition that

$$\hat{F}(\gamma_\delta^{(2)}) = (-1)((K+1)t_{1,2} - 2T_{1,1}(H_{K+1}))$$

and so

$$\begin{aligned} [\hat{F}(\gamma_\delta^{(2)}), \hat{F}(\gamma_{n,\delta})] &= [(-1)((K+1)t_{1,2} - 2T_{1,1}(H_{K+1})), (-1)^n(n-1)!(K+1)^n t_{n,0}] \\ &= (-1)^{n+1}(n)!(K+1)^n((K+1)t_{n+1,1} - T_{n+1,0}(H_{K+1})) \\ &= 2/(n+1)F(\gamma_{(n+1),\delta}^0) \\ &= \hat{F}([\gamma_\delta^{(2)}, \gamma_{n,\delta}]). \end{aligned}$$

where the last equality follows from Proposition 8.2.4. Thus \hat{F} is a map of Lie algebras. By Proposition 8.1.7, the map \hat{F} is surjective. The map \hat{F} preserves cohomological grading and total dimensions. We note by Proposition 5.0.2, that

$$\dim((\widehat{\mathfrak{g}}_{Q^K, \widetilde{W}^K}^{\text{BPS}, i})_{|\mathbf{d}|}) = \begin{cases} 1 & \text{if } i = -2, |\mathbf{d}| \equiv 0 \pmod{K+1} \\ K+1 & \text{if } i \in 2\mathbb{Z}_{\geq 0}, |\mathbf{d}| > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\dim((\widetilde{W}_{K+1}^+)_{\dim}^{\text{CG}}) = \begin{cases} 1 & \text{if } \text{CG} = -2, \dim \equiv 0 \pmod{K+1} \\ K+1 & \text{if } \text{CG} \in 2\mathbb{Z}_{\geq 0}, \dim > 0 \\ 0 & \text{otherwise} \end{cases}$$

Since they are the same, it follows that the morphism \widehat{F} is, in fact, an isomorphism of Lie algebras. \square

Chapter 9

Application: CoHA of Kleinian surface

We recall $\pi : S_K \rightarrow \mathbb{C}^2/\mathbb{Z}_{K+1}$ is the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_{K+1}$. By [KV00][1.4,3.4], there is an equivalence of derived categories

$$\Psi : D^b(\text{Coh}(S_K)) \rightarrow D^b(\Pi_{Q^K} - \text{Mod})$$

which restricts to an equivalence

$$\Psi : D^b(\text{Coh}_{\text{ps}}(S_K)) \rightarrow D^b(\Pi_{Q^K} - \text{mod})$$

where $\text{Coh}_{\text{ps}}(S_K)$ is the abelian category of compactly supported coherent sheaves on S_K . This equivalence doesn't preserve the heart given by the usual bounded derived t structures on the derived category. It is known that the abelian subcategory on the left, which is mapped to the representation category of the finite dimensional representations of the preprojective algebra of the cyclic quiver Π_{Q^K} is the abelian category of compactly supported perverse coherent sheaves associated to $\pi : S_K \rightarrow \mathbb{C}^2/\mathbb{Z}_{K+1}$ [Van04][3.1].

Recall that for $i \in [1, K]$, $\mathfrak{B}_i(\rho_{\text{reg}})$ are the irreducible components of the exceptional fiber $\pi^{-1}(0)$ (Section 7.4.1). Let $\omega \in \text{Pic}_{\mathbb{Q}}(S_K) := \text{Pic}(S_K) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a \mathbb{Q} polarization of S_K . By polarization, we mean that $\langle \omega, \mathfrak{B}_i(\rho_{\text{reg}}) \rangle > 0$ for all $i \in [1, K]$, where $\langle -, - \rangle$ is the intersection pairing on the Picard group of S_K . The slope of a sheaf $\mathcal{F} \in \text{Coh}_{\text{ps}}(S_K)$ is defined to be

$$\mu_{\omega}(\mathcal{F}) := \begin{cases} \frac{\chi(\mathcal{F})}{\text{ch}_1(\mathcal{F}) \cdot \omega} & \text{if } \text{ch}_1(\mathcal{F}) \cdot \omega \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

For any $\mu \in \mathbb{Q}_{>} \cup \infty$, let $\text{Coh}_{\mu}^{\omega}(S_K)$ be the classical truncation of the derived stack of ω -semistable properly supported coherent sheaves of S_K of slope μ . Then in [PS23][2.3.2] and [DPS23][3.1], the authors define a cohomological Hall algebra $\mathcal{A}_{\mu}^{\zeta}(S_K)$ on the Borel-Moore homology $H^{\text{BM}}(\text{Coh}_{\mu}^{\omega}(S_K), \mathbb{Q})$ of the stack $\text{Coh}_{\mu}^{\omega}(S_K)$. Furthermore, it is shown that under the above derived equivalence, this algebra is isomorphic to the semistable cohomological Hall algebra of the preprojective algebra of the cyclic quiver. More precisely,

Proposition 9.0.1. [DPS23] [Corollary 3.4] Let $\zeta_i = \langle \omega, \mathfrak{B}_i(\rho_{reg}) \rangle$ for $1 \leq i \leq K$ and

$$\zeta_0 := \begin{cases} \frac{1}{\mu} - \sum_{i=1}^K \zeta_i & , \text{ if } \mu \neq \infty \\ - \sum_{i=1}^K \zeta_i & , \text{ if } \mu = \infty \end{cases}.$$

The derived equivalence Ψ induces an algebra isomorphism

$$\mathcal{A}_\mu^\omega(S_K) \simeq \mathcal{A}_{\Pi_{Q^K}, 0}^\zeta.$$

where $\mathcal{A}_{\Pi_{Q^K}, 0}^\zeta$ is the ζ -semistable, slope 0 preprojective cohomological Hall algebra of the cyclic quiver Q^K .

In what follows, we will calculate the algebra $\mathcal{A}_{\Pi_{Q^K}, 0}^\zeta$, which by the dimension reduction theorem, [Dav22a][Section 3.2.2] is isomorphic to the algebra $\mathcal{A}_{Q^K, \widetilde{W}^K, 0}^{\zeta, \chi}$. We first consider a more general setting. Given any symmetric quiver Q , stability condition ζ and slope μ , consider the Lie subalgebra

$$\widehat{\mathfrak{g}}_{\Pi_Q, \mu}^{\text{BPS}, \zeta} := \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \{0\}} \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}}[u] \subset \widehat{\mathfrak{g}}_{\widetilde{Q}, \widetilde{W}}^{\text{BPS}}.$$

We then consider its universal enveloping algebra

$$\mathcal{C}_{\widetilde{Q}, \widetilde{W}, \mu}^\zeta := \mathbf{U}(\widehat{\mathfrak{g}}_{\Pi_Q, \mu}^{\text{BPS}, \zeta}) \subset \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^\chi.$$

The following proposition gives a general way to realize the algebra $\mathcal{A}_{Q^K, \widetilde{W}^K, \mu}^{\zeta, \chi}$ as a subalgebra of $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K, \mu}^\chi$ explicitly.

Proposition 9.0.2. *There is an isomorphism of algebras*

$$\mathcal{C}_{\widetilde{Q}, \widetilde{W}, \mu}^\zeta \simeq \mathcal{A}_{\widetilde{Q}, \widetilde{W}, \mu}^{\zeta, \chi}.$$

Proof. For every $\mathbf{d} \in \Lambda_\mu^\zeta$, we have open inclusion $\mathfrak{M}_{\mathbf{d}}^\zeta(\widetilde{Q}) \hookrightarrow \mathfrak{M}_{\mathbf{d}}(\widetilde{Q})$, which after restriction, induces a map of algebras $(i_\mu^\zeta)^*: \mathcal{C}_{\widetilde{Q}, \widetilde{W}, \mu}^\zeta \rightarrow \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^\zeta$ such that

$$(i_\mu^\zeta)^*_{|\mathfrak{g}^{\text{BPS}}}: \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \{0\}} \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}} \rightarrow \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \{0\}} \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, \zeta}$$

is an isomorphism by Toda's theorem (Equation 4.1.5). Now since the cup product with the tautological bundle commutes with the open inclusion, it follows that the morphism

$$(i_\mu^\zeta)^*: \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \{0\}} \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}}[u] \rightarrow \bigoplus_{\mathbf{d} \in \Lambda_\mu^\zeta \setminus \{0\}} \mathfrak{g}_{\widetilde{Q}, \widetilde{W}, \mathbf{d}}^{\text{BPS}, \zeta}[u]$$

is an isomorphism of Lie algebras. But then by the PBW theorem (4.1.1) for $\mathcal{A}_{\widetilde{Q}, \widetilde{W}, \mu}^{\zeta, \chi}$, it follows that in fact $(i_\mu^\zeta)^*$ is an isomorphism. \square

We now consider the algebras $\mathcal{A}_\mu^\omega(S_K) \simeq \mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K, 0}^{\zeta, \chi}$. When $\mu = \infty$, the only ω -semistable properly supported coherent sheaves on S_K are the 0 dimensional sheaves. The cohomological Hall algebra $\mathcal{A}(S)$ of 0-dimensional sheaves on any quasi-projective surface S is studied and explicitly calculated in [Mel+23]. We calculate this algebra for $S = S_K$ using Proposition 9.0.2. Note that for $\zeta_i > 0$ and $\zeta_0 = -\sum \zeta_i$, having the slope $\mu_\zeta(\mathbf{d}) = 0$ implies that $\mathbf{d}_i = \mathbf{d}_0 = n$ for some $n \geq 0$. Thus the cohomological Hall algebra is the universal enveloping algebra of the Lie subalgebra $\bigoplus_{n \geq 1} \mathfrak{g}_{\mathbb{Q}^K, \widetilde{W}^K, n \cdot \delta}^{\text{BPS}}[u] \subset \widehat{\mathfrak{g}}_{\Pi_K}^{\text{BPS}}$. So by Theorem 8.2.6, it follows that

Corollary 9.0.2.1. *Let $W(S_K) \subset \widetilde{W}_{K+1}$ be the Lie subalgebra on the subspace spanned by $T_{n,a}(H_i)$ where $i \in [1, K], n \geq 1, a \geq 0$ and $t_{n,a}$ where $n \geq 1, a \geq 0$ with the relations*

$$\begin{aligned} [t_{m,a}, t_{n,b}] &= (na - mb)t_{m+n, a+b-1} \\ [t_{m,a}, T_{n,b}(H_i)] &= (na - mb)T_{m+n, a+b-1}(H_i) \\ [T_{m,a}(H_i), T_{n,b}(H_j)] &= T_{m+n, a+b}([H_i, H_j]) = 0 \end{aligned}$$

Then we have an isomorphism of algebras

$$\mathcal{A}(S_K) \simeq \mathbf{U}(W(S_K)).$$

When $\mu \neq \infty$, then $\mu_\zeta(n \cdot \delta) \neq 0$ and in particular the Lie subalgebra $\bigoplus_{\mathbf{d} \in A_0^\zeta} \mathfrak{g}_{\mathbb{Q}^K, \widetilde{W}, \mathbf{d}}^{\text{BPS}}[u]$ is always a Lie subalgebra of the real part $\mathfrak{re}_K \subset \widehat{\mathfrak{g}}_{\mathbb{Q}^K, \widetilde{W}^K}^{\text{BPS}}$. Given the stability condition $\zeta = \{\zeta_i \mid i \in [0, K]\}$, we are interested in representations of slope 0. We solve for dimension vectors \mathbf{d} , where \mathbf{d} is a real positive root of the affine Lie algebra $\widetilde{\mathfrak{sl}}_{K+1}$ which satisfies the equality

$$\mu_\zeta(\mathbf{d}) = \sum_{i=1}^K \zeta_i(\mathbf{d}_i - \mathbf{d}_0) + \mathbf{d}_0/\mu = 0. \quad (9.1)$$

Since $\zeta_i > 0$, any solution would be of type

$$n \cdot \delta + [K+1-j, i+j] = \underbrace{(n+1, \dots, n+1)}_{i \text{ times}}, n, \dots, n, \underbrace{(n+1, \dots, n+1)}_{j \text{ times}}$$

for $i > 0$. Note that if $\mathbf{d}', \mathbf{d}''$ satisfies the equation (9.1) then so does $\mathbf{d}' + \mathbf{d}''$. Since $\mathfrak{g}_{\mathbb{Q}^K, \widetilde{W}, n \cdot \delta + [K+1-j, i+j]}^{\text{BPS}}$ is 1 dimensional, we can identify the generator with $T_{n+1,0}(E_{K+2-j, i+1})$ when $j > 0$ and with $T_{n,0}(E_{1, i+1})$ when $j = 0$.

Definition 9.0.3. Given ω a \mathbb{Q} -polarization of S_K and slope $\mu \in \mathbb{Q}_{>}$, let $\zeta_i = \langle \omega, \mathfrak{B}_i(\rho_{\text{reg}}) \rangle$ for $1 \leq i \leq K$ and $\zeta_0 = 1/\mu - \sum_{i=1}^K \zeta_i$. Let $\mathbf{L} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{> 0} \times \mathbb{Z}_{\geq 0}$ be the subset of tuples (n, i, j) such that $\mathbf{d} = n \cdot \delta + [K+1-j, i+j]$ satisfies equation (9.1). Let $W_\mu^\omega(S_K) \subset \widetilde{W}_{K+1}$ be the Lie algebra spanned by $T_{n+1, a}(E_{K+2-j, i})$ for all $(n, i, j) \in \mathbf{L}, j > 0, a \in \mathbb{Z}_{\geq 0}$ and $T_{n,0}(E_{1, i+1})$ for all $(n, i, 0) \in \mathbf{L}, a \in \mathbb{Z}_{\geq 0}$ satisfying

$$[T_{m,a}(X), T_{n,b}(Y)] = T_{m+n, a+b}([X, Y]).$$

for any $X, Y \in \mathfrak{gl}_{K+1}$.

We then have by Theorem 8.2.6, Proposition 9.0.2 and Proposition 9.0.1 that

Corollary 9.0.3.1. *Let ω be any \mathbb{Q} polarization of S_K and let $\mu \in \mathbb{Q}_{>0}$ be any slope. Then there is an isomorphism of algebras*

$$\mathcal{A}_\mu^\omega(S_K) \simeq \mathbf{U}(W_\mu^\omega(S_K)).$$

Chapter 10

Affine Yangians and deformed CoHA

We will now consider cohomological Hall algebras $\mathcal{A}_{\tilde{Q}, \tilde{W}}^T$ in the case when the torus T doesn't preserve the symplectic form, i.e., in the case when the construction of the affinized BPS Lie algebra doesn't work. In this case, we get a deformation of the universal enveloping algebra $\mathbf{U}(\mathfrak{g}_{\tilde{Q}, \tilde{W}}^{\text{BPS}}[u])$, i.e., Yangians. We calculate these algebras by embedding them into cohomological Hall algebras $\mathcal{A}_{\tilde{Q}}^T$, which carry a description using the Shuffle product.

10.1 Injection to shuffle algebra

Given any T , with $W = 0$, the cohomological Hall algebra can be explicitly described in terms of the shuffle product. Since $\text{Rep}_{\mathbf{d}}(Q)$ is homotopic to a point, we have as a graded vector space

$$\mathcal{A}_{\tilde{Q}}^T = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}_T(\text{pt})[x_{i,n} | i \in Q_0, 1 \leq n \leq \mathbf{d}_i]^{S_{\mathbf{d}}}$$

where $S_{\mathbf{d}} := \prod S_{\mathbf{d}_i}$ is the product of symmetric groups and $S_{\mathbf{d}_i}$ acts by permuting the variables $x_{i,1}, \dots, x_{i,\mathbf{d}_i}$. Then it is proved in [KS11][Theorem 2.2]) that¹

Proposition 10.1.1. *Let $f \in \mathcal{A}_{\tilde{Q}, \mathbf{d}'}$ and $g \in \mathcal{A}_{\tilde{Q}, \mathbf{d}''}$ then the product is given by*

$$f(x_{0,1}, \dots, x_{|Q_0|-1, \mathbf{d}'_{|Q_0|-1}}) \star g(x_{0,1}, \dots, x_{|Q_0|-1, \mathbf{d}''_{|Q_0|-1}}) = \sum_{\sigma \in \text{Sh}_{\mathbf{d}', \mathbf{d}''}} \sigma(f(x_{0,1}, \dots, x_{|Q_0|-1, \mathbf{d}'_{|Q_0|-1}})g(x_{0, \mathbf{d}'_0+1}, \dots, x_{|Q_0|-1, \mathbf{d}'_{|Q_0|-1} + \mathbf{d}''_{|Q_0|-1}})) K_Q(\mathbf{x}_{\mathbf{d}'}; \mathbf{x}_{\mathbf{d}''}), \quad (10.1)$$

where $K_Q(\mathbf{x}_{\mathbf{d}'}; \mathbf{x}_{\mathbf{d}''})$ is called the shuffle kernel, and it depends on the quiver pre-

¹Originally this is proved only in the case when T is trivial by localization formula, but then the same proof works in the case when there is T equivariance.

cisely by

$$K_Q(\mathbf{x}_{\mathbf{d}'}; \mathbf{x}_{\mathbf{d}''}) := \frac{\prod_{a \in Q_1} \prod_{\substack{1 \leq m \leq \mathbf{d}'_{s(a)} \\ \mathbf{d}'_{t(a)} < n \leq \mathbf{d}'_{t(a)} + \mathbf{d}''_{t(a)}}} (x_{t(a),n} - x_{s(a),m} + \mathbf{t}(a))}{\prod_{i \in Q_0} \prod_{\substack{1 \leq m \leq \mathbf{d}'_i \\ \mathbf{d}'_i < n \leq \mathbf{d}'_i + \mathbf{d}''_i}} (x_{i,n} - x_{i,m})} \quad (10.2)$$

and $\text{Sh}_{\mathbf{d}', \mathbf{d}''} \subset S_{\mathbf{d}' + \mathbf{d}''}$ is defined to be the subset of permutations $(\sigma_i) \in S_{\mathbf{d}' + \mathbf{d}''}$ where $i \in Q_0$ such that

$$\sigma_i(1) < \cdots < \sigma_i(\mathbf{d}'_i) \text{ and } \sigma_i(\mathbf{d}'_i + 1) < \cdots < \sigma_i(\mathbf{d}'_i + \mathbf{d}''_i)$$

This product is often referred to as the shuffle product.

Remark 10.1.2. For tripled cyclic quiver \tilde{Q}_K , also incorporating the twist χ in (4.1.4) the shuffle kernel can be written as

$$K_{\tilde{Q}^K}(x_{\mathbf{d}'}; x_{\mathbf{d}''}) := (-1)^{\sum_{i \in \mathbb{Z}_{K+1}} \mathbf{d}'_i \mathbf{d}''_i - \mathbf{d}'_i \mathbf{d}''_{i+1}} \prod_{i, j \in \mathbb{Z}_{K+1}} \prod_{\substack{1 \leq m \leq \mathbf{d}'_i \\ \mathbf{d}'_i < n < \mathbf{d}'_i + \mathbf{d}''_j}} \bar{\omega}_{i,j}^{\text{CoHA}}(x_{i,m}, x_{j,n}) \quad (10.3)$$

where

$$\bar{\omega}_{i,j}^{\text{CoHA}}(z, w) = (w - z + t_1)^{\delta_{j,i+1}} (w - z + t_2)^{\delta_{j,i-1}} \left(\frac{w - z - t_1 - t_2}{w - z} \right)^{\delta_{j,i}} \quad (10.4)$$

In [BT19][Section 5.2], a similarly defined algebra structure on the ring of Laurent symmetric functions is considered as an additive shuffle algebra of cyclic type. Let $\text{Sh}_{\tilde{Q}^K}^{\text{BT}} = \bigoplus_{\mathbf{d} \in \mathbb{N}^{K+1}} \text{Sh}_{\tilde{Q}^K, \mathbf{d}}^{\text{BT}}$ be a \mathbb{N}^{K+1} graded vector space, where

$$\text{Sh}_{\tilde{Q}^K, \mathbf{d}}^{\text{BT}} := \mathbb{C}(t_1, t_2)(x_{i,n} | i \in \mathbb{Z}_{K+1}, 1 \leq n \leq \mathbf{d}_i)^{S_{\mathbf{d}}}$$

This vector space is endowed with a associative algebra product \star_{BT} given by

$$f(x_{0,1}, \dots, x_{K, \mathbf{d}'_K}) \star_{\text{BT}} g(x_{0,1}, \dots, x_{K, \mathbf{d}''_K}) := \text{Sym} \left(f(x_{0,1}, \dots, x_{K, \mathbf{d}'_K}) g(x_{0, \mathbf{d}'_0 + 1}, \dots, x_{K, \mathbf{d}'_K + \mathbf{d}''_K}) \prod_{i, j \in \mathbb{Z}_{K+1}} \left(\prod_{\substack{1 \leq m \leq \mathbf{d}'_i \\ \mathbf{d}'_i < n < \mathbf{d}'_i + \mathbf{d}''_j}} \bar{\omega}_{i,j}(x_{i,m}, x_{j,n}) \right) \right)$$

where².

$$\bar{\omega}_{i,j}(z, w) = \left(\frac{w - z + t_1}{w - z} \right)^{\delta_{j,i+1}} \left(\frac{w - z + t_2}{w - z} \right)^{\delta_{j,i-1}} \left(\frac{w - z - t_1 - t_2}{w - z} \right)^{\delta_{j,i}}.$$

²We do the substitution $\bar{h}_1 = -t_2, \bar{h}_2 = t_1 + t_2, \bar{h}_3 = -t_1$

and for a function $f \in \mathbb{C}(t_1, t_2)(x_{i,n} | i \in \mathbb{Z}_{K+1}, 1 \leq n \leq \mathbf{d}_i)$, it's symmetrization is defined to be

$$\text{Sym}(f) := \left(\prod_{i \in \mathbb{Z}_{K+1}} \frac{1}{\mathbf{d}_i!} \right) \left(\sum_{\sigma \in S_{\mathbf{d}}} \sigma(f(x_{0,1}, \dots, x_{K, \mathbf{d}_K})) \right).$$

Let $\mathcal{A}_{Q^K, \text{Loc}}^{\mathbb{T}, \chi}$ be the same vector space as $\text{Sh}_{Q^K}^{\text{BT}}$ but with the algebra product induced by the algebra structure on $\mathcal{A}_{Q^K}^{\mathbb{T}, \chi}$. Then the morphism

$$S : \mathcal{A}_{Q^K, \text{Loc}}^T \rightarrow \text{Sh}_{Q^K}^{\text{BT}}$$

given by

$$S(f(x_{\mathbf{d}})) = (-1)^{\sum_{i \in \mathbb{Z}_{K+1}} \binom{\mathbf{d}_i}{2} + \mathbf{d}_i \mathbf{d}_{i+1}} f(x_{\mathbf{d}}) \left(\prod_{i \in \mathbb{Z}_{K+1}} \frac{1}{\mathbf{d}_i!} \left(\prod_{\substack{1 \leq n \leq \mathbf{d}_i \\ 1 \leq m \leq \mathbf{d}_{i+1}}} \frac{1}{x_{i,n} - x_{i+1,m}} \right) \right)$$

is an algebra isomorphism and so in particular $\mathcal{A}_{Q^K}^{\mathbb{T}, \chi} \hookrightarrow \mathcal{A}_{Q^K, \text{Loc}}^{\mathbb{T}, \chi} \rightarrow \text{Sh}_{Q^K}^{\text{BT}}$ is an injection of algebras.

Homomorphism to shuffle algebra

Given any quiver Q with potential W and a torus T preserving the potential W , assume that there exist a weighting $\mathbf{w}' : Q_1 \rightarrow \mathbb{Z}_{\geq 0}$, for which W is homogeneous and of strictly positive weight. Let $Q' \subset Q$ be the subquiver consisting only those edges a for which $\mathbf{w}'(a) = 0$. Then there is a homomorphism from the cohomological Hall algebra $\mathcal{A}_{Q, W}^{T, \chi}$ to $\mathcal{A}_Q^{T, \chi}$. Let $i_{\mathbf{d}, T} : Z_{\mathbf{d}}^T(Q) := \text{Tr}(W)_{\mathbf{d}}^{-1}(0) \rightarrow \mathfrak{M}_{\mathbf{d}}^T(Q)$ be the inclusion where $\text{Tr}(W)_{\mathbf{d}} : \mathfrak{M}_{\mathbf{d}}^T(Q) \rightarrow \mathbb{C}$. By considering the torus action associated with the weighting \mathbf{w}' , one can construct a homotopy between $Z_{\mathbf{d}}^T(Q)$ and $\mathfrak{M}_{\mathbf{d}}^T(Q')$. The vanishing cycle functor comes with a natural map

$${}^p\Phi_W \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(Q)}^{\text{vir}} \rightarrow (i_{\mathbf{d}, T})_*(i_{\mathbf{d}, T})^* \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(Q)}^{\text{vir}}$$

which after taking global sections, gives map of graded vector spaces

$$i_{\mathbf{d}, T} : \mathcal{A}_{Q, W, \mathbf{d}}^T \rightarrow \text{H}(Z_{\mathbf{d}}^T(Q), (i_{\mathbf{d}})^* \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(Q)}^{\text{vir}}) \simeq \mathcal{A}_{Q, \mathbf{d}}^T.$$

Here, the second isomorphism is simply because both $\mathfrak{M}_{\mathbf{d}}^T(Q')$ and $\mathfrak{M}_{\mathbf{d}}^T(Q)$ are homotopic to a point. This gives a map of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded vector spaces

$$i^T : \mathcal{A}_{Q, W}^{T, \psi} \rightarrow \mathcal{A}_Q^{T, \psi}.$$

Then it is proved in [BD23][Prop 4.4] that

Proposition 10.1.3. *The morphism*

$$i^T : \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \psi} \rightarrow \mathcal{A}_Q^{T, \psi}.$$

is a homomorphism of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded algebras.

Remark 10.1.4. We remark that Proposition 10.1.3 can be considered as the generalization of the morphism to shuffle algebras considered by Schiffmann-Vasserot in [SV13] for Jordan quivers, and later generalized by Yang-Zhao in [YZ18a]. Let $\mathbf{w} : \overline{Q}^1 \rightarrow \mathbb{Z}^r$ such that $\sum[a, a^*]$ is homogenous and let T be torus defined by weighing $\mathbf{w}' : \widetilde{Q}_1 \rightarrow \mathbb{Z}^r$ uniquely extended from \mathbf{w} such that potential \widetilde{W} is of weight 0. Then they consider another shuffle algebra structure $\text{Sh}_Q^{T, SV}$ on the vector space \mathcal{A}_Q^T . In their algebra, the shuffle kernel is given by

$$\frac{\left(\prod_{a \in \overline{Q}_1} \prod_{\substack{1 \leq m \leq \mathbf{d}'_{s(a)} \\ \mathbf{d}'_{t(a)} < n \leq \mathbf{d}'_{t(a)} + \mathbf{d}'_{t(a)}}} (x_{t(a), n} - x_{s(a), m} + \mathbf{t}(a)) \right) \left(\prod_{i \in Q_0} \prod_{\substack{1 \leq m \leq \mathbf{d}'_i \\ \mathbf{d}'_i < n \leq \mathbf{d}'_i + \mathbf{d}''_i}} (x_{i, m} - x_{i, n} + \mathbf{t}(a) + \mathbf{t}(a^*)) \right)}{\prod_{i \in Q_0} \prod_{\substack{1 \leq m \leq \mathbf{d}'_i \\ \mathbf{d}'_i < n \leq \mathbf{d}'_i + \mathbf{d}''_i}} (x_{i, n} - x_{i, m})}$$

which is the same as

$$(-1)^{\mathbf{d}' \cdot \mathbf{d}''} K_{\overline{Q}}(\mathbf{x}_{\mathbf{d}'}; \mathbf{x}_{\mathbf{d}''}).$$

Thus, there is an isomorphism of algebras:

$$\mathcal{A}_Q^T \rightarrow \text{Sh}_Q^{T, SV}$$

given by $f \mapsto (-1)^{\binom{\mathbf{d}}{2}} f$.

We now show that this morphism respects the action of **Heis** constructed in Proposition 6.3.1.

Proposition 10.1.5. *The morphism i^T is a morphism of **Heis** modules.*

Proof. Since the diagram

$$\begin{array}{ccc} \mathfrak{M}_{\mathbf{d}}^T(Q) & \xrightarrow{\det \times \text{id}} & \text{BC}_{\mathbf{u}}^* \times \mathfrak{M}_{\mathbf{d}}^T(Q) \\ i_{\mathbf{d}}^T \uparrow & & \uparrow \text{id} \times i_{\mathbf{d}}^T \\ Z_{\mathbf{d}}^T(Q) & \xrightarrow{\det \times \text{id}} & \text{BC}_{\mathbf{u}}^* \times Z_{\mathbf{d}}^T(Q) \end{array}$$

commutes, it follows from the naturality of the morphism (3.1) that the diagram

$$\begin{array}{ccc} {}^p\Phi_{0 \boxplus W_{\mathbf{d}}} \mathbb{Q}_{\text{BC}_{\mathbf{u}}^* \times \mathfrak{M}_{\mathbf{d}}^T(Q)} & \longrightarrow & (\text{Det} \times \text{id})_* {}^p\Phi_{W_{\mathbf{d}}} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}}^T(Q)} \\ \downarrow & & \downarrow \\ (\text{id} \times i_{\mathbf{d}}^T)_* \mathbb{Q}_{\text{BC}_{\mathbf{u}}^* \times Z_{\mathbf{d}}^T(Q)} & \longrightarrow & (i_{\mathbf{d}}^T)_* \mathbb{Q}_{Z_{\mathbf{d}}^T(Q)} \end{array}$$

commutes. By taking global sections, it follows that the action by u commutes. The compatibility with the ∂_u action follows in the same way since the diagram

$$\begin{array}{ccc} \mathrm{BC}_u^* \times \mathfrak{M}_d^T(Q) & \xrightarrow{\mathrm{act}} & \mathfrak{M}_d^T(Q) \\ \mathrm{id} \times i_d^T \uparrow & & \uparrow i_d^T \\ \mathrm{BC}_u^* \times Z_d^T(Q) & \xrightarrow{\mathrm{act}} & Z_d^T(Q) \end{array}$$

commutes as $\mathrm{Tr}(W)(\rho) = 0 \implies \mathrm{Tr}(W)(\lambda\rho) = 0$ for any $\lambda \in \mathbb{C}$. □

The morphism i^T is not, in general, an injection. We can already see this case for tripled cyclic quivers.

Not injection for small torus

Since $\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{BPS}}$ is non-trivial by Proposition 5, it follows that neither of algebras $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^\chi, \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{C}^*, \chi}$ are commutative. However by the shuffle formula above, $\mathcal{A}_{\widetilde{Q^K}}^\chi$ and $\mathcal{A}_{\widetilde{Q^K}}^{\mathrm{C}^*, \chi}$ are commutative. This, in particular, means that the algebra morphisms i and i^{C^*} cannot be injective.

10.1.1 Injection for generic torus

However for the torus \mathbb{T} , it is shown in [Dav23c][Theorem 10.2] and [SV13] that $H(\mathfrak{M}_d^{\mathbb{T}}(\widetilde{Q^K})^{\mathrm{p}\Phi}_{\mathrm{Tr}(\widetilde{W^K})})$ is a torsion free $H_{\mathbb{T} \times \mathrm{GL}_d}$ module, which in particular implies that

Proposition 10.1.6. *The natural morphism*

$$i^{\mathbb{T}} : \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi} \rightarrow \mathcal{A}_{\widetilde{Q^K}}^{\mathbb{T}, \chi}$$

is an injection of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded algebras.

10.2 Cyclic quivers and Affine Yangians

We now study the \mathbb{T} deformed cohomological Hall algebra $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi}$. By Proposition 4.2.6, we have an isomorphism of cohomologically graded vector spaces

$$\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi} \simeq \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^\chi \otimes \mathbb{C}[t_1, t_2]$$

This morphism restricts to an isomorphism of cohomologically graded $H_{\mathbb{T}}(\mathrm{pt})$ modules $\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{BPS}, \mathbb{T}} \simeq \mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{BPS}} \otimes \mathbb{C}[t_1, t_2]$. Thus for every element $\alpha_i \in \mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{BPS}}, i \in [0, K]$ and $\gamma_{n\cdot\delta} \in \mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathrm{BPS}}, n \geq 1$, we have a unique (up to scalar), non-canonical lift $\tilde{\alpha}_i := \alpha_i \otimes 1$ and $\tilde{\gamma}_{n\cdot\delta} := \gamma_{n\cdot\delta} \otimes 1$, since both $H^{-2}(\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}, n\cdot\delta}^{\mathrm{BPS}, \mathbb{T}})$ and $H^0(\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}, \delta_i}^{\mathrm{BPS}, \mathbb{T}})$ are 1 dimensional vector spaces. We then have

Proposition 10.2.1. *The $\mathbb{C}[t_1, t_2]$ algebra $\mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}}}^{\mathbb{T}, \chi}$ is generated as a $\mathbb{C}[t_1, t_2]$ algebra by $u^k \cdot \widetilde{\alpha}_i$ for all $i \in [0, K], k \geq 0$ and $u^k \cdot \widetilde{\gamma}_\delta$ for all $k \geq 0$.*

Proof. There is an isomorphism of Lie algebras

$$\mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}}}^{\text{BPS}, \mathbb{T}} \simeq \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}}}^{\text{BPS}} \otimes \mathbb{C}[t_1, t_2]$$

by [DHM24][Corollary 11.9]. For every $\alpha \in \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}}}^{\text{BPS}}$ we set $\widetilde{\alpha} = \alpha \otimes 1 \in \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}}}^{\text{BPS}, \mathbb{T}}$ via this isomorphism and $\widetilde{\alpha}^{(m_i)} := u^m \cdot \widetilde{\alpha}$. By Theorem 8.2.6, we may identify $\mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}}}^{\text{BPS}}$ as a Lie subalgebra of \widetilde{W}_{K+1} , spanned by $T_{k,0}(X)$ where $k \geq 1, X \in \mathfrak{sl}_K$ or $k = 0, X \in \mathfrak{n}_K$ where $\mathfrak{n}_K \subset \mathfrak{sl}_K$ is the Lie subalgebra generated by upper diagonal matrices and $t_{k,0}$ where $k \geq 1$. For any $k \geq 1$, up to a factor of non-zero scalar, $t_{k,0}$ can be written as $t_{k,0} = \mathbf{ad}_{t_{1,1}}^{k-1}(t_{1,0})$. Similarly, each of $T_{k,0}(X)$ where $k \geq 1, X \in \mathfrak{sl}_K$ or $k = 0, X \in \mathfrak{n}_K$ can be written as successive Lie brackets on $T_{0,0}(E_{i,i+1}), i \in [1, K]$ and $T_{1,0}(E_{K+1,1})$. Thus every basis element $\alpha_{\mathbf{d}} \in \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}, \mathbf{d}}}^{\text{BPS}}$ can be written as a linear combination of products of $\alpha_0^{(0)}, \alpha_1^{(0)}, \dots, \alpha_K^{(0)}, \gamma_\delta^{(1)}, \gamma_\delta^{(0)}$, i.e. there exists an expression $g_{\alpha_{\mathbf{d}}}$ such that

$$g_{\alpha_{\mathbf{d}}}(\alpha_0^{(0)}, \alpha_1^{(0)}, \dots, \alpha_K^{(0)}, \gamma_\delta^{(0)}, \gamma_\delta^{(1)}) = \alpha_{\mathbf{d}}.$$

We fix such an expression for every basis element $\alpha_{\mathbf{d}} \in \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}, \mathbf{d}}}^{\text{BPS}}$. We then define a map of $\mathbb{C}[t_1, t_2]$ modules

$$\psi_{\mathbf{d}} : \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}, \mathbf{d}}}^{\text{BPS}} \otimes \mathbb{C}[t_1, t_2] \otimes \mathbb{C}[u] \rightarrow \mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}}}^{\mathbb{T}, \chi}$$

given by

$$\psi_{\mathbf{d}}(\alpha_{\mathbf{d}} \otimes f(t_1, t_2) \otimes u^m) = f(t_1, t_2) u^m \cdot g_{\alpha_{\mathbf{d}}}(\widetilde{\alpha}_0^{(0)}, \widetilde{\alpha}_1^{(0)}, \dots, \widetilde{\alpha}_K^{(0)}, \widetilde{\gamma}_\delta^{(0)}, \widetilde{\gamma}_\delta^{(1)})$$

and thus we have map of free $\mathbb{C}[t_1, t_2]$ modules

$$\Psi : \text{Sym} \left(\bigoplus_{\mathbf{d}} \mathfrak{g}_{\widetilde{Q^K, \widetilde{W^K}, \mathbf{d}}}^{\text{BPS}} \otimes \mathbb{C}[t_1, t_2] \otimes \mathbb{C}[u] \right) \rightarrow \mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}}}^{\mathbb{T}}$$

However, $\Psi \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}[t_1, t_2]/(t_1, t_2)$ is an isomorphism by the PBW theorem (Theorem 4.1.1) and thus Ψ is an isomorphism of $\mathbb{C}[t_1, t_2]$ modules by the graded Nakayama lemma and so the statement follows. \square

Image in the Shuffle algebra

We now determine the image of $\mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}}}^{\mathbb{T}, \chi}$ under the map $i^{\mathbb{T}}$. By Proposition 10.2.1, it suffices to understand the image of $\widetilde{\alpha}_i^{(k)}, \widetilde{\gamma}_\delta^{(k)}$ for all $k \geq 0$.

Proposition 10.2.2. *The image $i^{\mathbb{T}}(\mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}}}^{\mathbb{T}, \chi}) \subset \mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}}}^{\mathbb{T}, \chi}$ is the $\mathbb{C}[t_1, t_2]$ subalgebra generated by $x_{i,1}^r \in \mathcal{A}_{\widetilde{Q^K, \widetilde{W^K}, \delta_i}}^{\mathbb{T}, \chi}$ for all $r \geq 0, i \in [0, \dots, K]$ and $(t_1 +$*

$t_2)^K(x_{0,1} + x_{1,1} + \cdots + x_{K,1})^r \in \mathcal{A}_{\mathbb{Q}^K, \delta}^{\mathbb{T}, \chi}$ for all $r \geq 0$. We have $i^{\mathbb{T}}(\tilde{\alpha}_i^{(r)}) = x_{i,1}^r$ and $i^{\mathbb{T}}(\tilde{\gamma}_\delta^{(r)}) = \lambda(t_1 + t_2)^K(x_{0,1} + x_{1,1} + \cdots + x_{K,1})^r$ for some non-zero scalar λ .

Proof. Since $i_{\delta_i}^{\mathbb{T}} : \text{CoHA}_{(0, \dots, 1, \dots, 0)}^T \simeq \mathbb{C}[t_1, t_2][x_{i,1}]$ and $\tilde{\alpha}_i = (i^{\mathbb{T}})^{-1}(1)$, Proposition 10.1.5 implies that

$$i^{\mathbb{T}}(\tilde{\alpha}_i^{(k)}) = x_{i,1}^k$$

for all $k \geq 0$. Let $i^{\mathbb{T}}(\tilde{\gamma}_\delta^0) = f$ where $f \in \mathbb{Q}[t_1, t_2][x_{0,1}, x_{1,1}, x_{2,1}, \dots, x_{K,1}]$. Then by the shuffle formula (10.1), it follows that

$$\tilde{\gamma}_\delta^0 \star \tilde{\alpha}_i = f(x_{0,1}, x_{1,1}, \dots, x_{i,1}, \dots, x_{K,1}) \frac{(x_{i,2} - x_{i,1} - t_1 - t_2)(x_{i,2} - x_{i-1,1} + t_1)(x_{i,2} - x_{i+1,1} + t_2)}{(x_{i,2} - x_{i,1})} + \quad (10.5)$$

$$f(x_{0,1}, x_{1,1}, \dots, x_{i,2}, \dots, x_{K,1}) \frac{(x_{i,1} - x_{i,2} - t_1 - t_2)(x_{i,1} - x_{i-1,1} + t_1)(x_{i,1} - x_{i+1,1} + t_2)}{(x_{i,1} - x_{i,2})}$$

and

$$\tilde{\alpha}_i \star \tilde{\gamma}_\delta^0 = f(x_{0,1}, \dots, x_{i,2}, \dots, x_{K,1}) \frac{(x_{i,2} - x_{i,1} - t_1 - t_2)(x_{i+1,1} - x_{i,1} + t_1)(x_{i-1,1} - x_{i,1} + t_2)}{(x_{i,2} - x_{i,1})} + \quad (10.6)$$

$$f(x_{0,1}, \dots, x_{i,1}, \dots, x_{K,1}) \frac{(x_{i,1} - x_{i,2} - t_1 - t_2)(x_{i+1,1} - x_{i,2} + t_1)(x_{i-1,1} - x_{i,2} + t_2)}{(x_{i,1} - x_{i,2})}$$

Since $\tilde{\gamma}_\delta^{(0)}$ is of cohomological degree -2 and $\tilde{\alpha}_i^{(0)}$ is of cohomological degree 0 , the Lie bracket $[\tilde{\gamma}_\delta^0, \tilde{\alpha}_i] = 0$ is of cohomological degree -2 . Since $H^{-2}(\mathfrak{g}_{\mathbb{Q}^K, \widehat{W}^K, \delta + \delta_i}^{\text{BPS}, \mathbb{T}}) = 0$, we must have $[\tilde{\gamma}_\delta^0, \tilde{\alpha}_i] = 0$. Since

$$\frac{(x_{i,2} - x_{i,1} + t_3)(x_{i,2} - x_{i-1,1} + t_1)(x_{i,2} - x_{i+1,1} + t_2)}{(x_{i,2} - x_{i,1})} - \frac{(x_{i,1} - x_{i,2} + t_3)(x_{i+1,1} - x_{i,2} + t_1)(x_{i-1,1} - x_{i,2} + t_2)}{(x_{i,1} - x_{i,2})} =$$

$$\frac{(x_{i,2} - x_{i,1} + t_3)(x_{i+1,1} - x_{i,1} + t_1)(x_{i-1,1} - x_{i,1} + t_2)}{(x_{i,2} - x_{i,1})} - \frac{(x_{i,1} - x_{i,2} + t_3)(x_{i,1} - x_{i-1,1} + t_1)(x_{i,1} - x_{i+1,1} + t_2)}{(x_{i,1} - x_{i,2})},$$

equation (10.5), (10.6) and the condition that $\tilde{\alpha}_i \star \tilde{\gamma}_\delta^0 = \tilde{\gamma}_\delta^0 \star \tilde{\alpha}_i$ implies that for all $i \in \{0, 1, \dots, K\}$ we must have

$$f(x_{0,1}, \dots, x_{i-1,1}, x_{i,1}, x_{i+1,1}, \dots, x_{K,1}) = f(x_{0,1}, \dots, x_{i-1,1}, x_{i,2}, x_{i+1,1}, \dots, x_{K,1})$$

This means that for any i , f is independent of the variable $x_{i,1}$, i.e. $f \in \mathbb{C}[t_1, t_2]$. Since $\tilde{\gamma}_\delta^{(0)}$ is of cohomological degree -2 , this implies that the degree of polynomial $f(t_1, t_2)$ must be K . Now $[\gamma_\delta^0, \alpha_i^{(1)}]$ is an element of cohomological degree 0 in the dimension vector $\delta_i + \delta$. Since the $\mathbb{C}[t_1, t_2]$ action only increases the cohomological degree, it means that up to a scalar, the only non-zero term of cohomological degree 0 in dimension $\delta_i + \delta$ in $\mathfrak{g}_{\mathbb{Q}^K, \widehat{W}^K}^{\text{BPS}, \mathbb{T}}$ is $[\tilde{\alpha}_i, \tilde{\alpha}_i, \dots, \tilde{\alpha}_K, \tilde{\alpha}_{i-1}, \dots, \tilde{\alpha}_0]$ (Proposition 5.0.2). Again by the shuffle product description of $\mathcal{A}_{\mathbb{Q}^K}^{\mathbb{T}, \chi}$, we can explicitly calculate $[i^{\mathbb{T}}(\tilde{\alpha}_i), i^{\mathbb{T}}(\tilde{\alpha}_i), \dots, i^{\mathbb{T}}(\tilde{\alpha}_K), i^{\mathbb{T}}(\tilde{\alpha}_{i-1}), \dots, i^{\mathbb{T}}(\tilde{\alpha}_0)]$. We see that for some non-zero constant $\lambda_i \in \mathbb{Q}$ we have

$$[i^\mathbb{T}(\tilde{\alpha}_i), i^\mathbb{T}(\tilde{\alpha}_i), \dots, i^\mathbb{T}(\tilde{\alpha}_K), i^\mathbb{T}(\tilde{\alpha}_{i-1}), \dots, i^\mathbb{T}(\tilde{\alpha}_0)] = \lambda_i \frac{(t_0 + t_1)^K [i^\mathbb{T}(\tilde{\gamma}_\delta^{(0)}), i^\mathbb{T}(\tilde{\alpha}_i^{(1)})]}{f(t_1, t_2)} \quad (10.7)$$

Now Proposition 8.2.3 implies that there exist some $\mu_i \neq 0$ such that in $\mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$,

$$[\tilde{\gamma}_\delta^{(0)}, \tilde{\alpha}_i^{(1)}] = \mu_i [\tilde{\alpha}_i, \tilde{\alpha}_i, \dots, \tilde{\alpha}_K, \tilde{\alpha}_{i-1}, \dots, \tilde{\alpha}_0] + (t_1, t_2) i(\mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \chi}).$$

This implies

$$\left(\frac{f(t_1, t_2)}{\lambda_i (t_0 + t_1)^K} - \mu_i \right) [\tilde{\alpha}_i, \tilde{\alpha}_i, \dots, \tilde{\alpha}_K, \tilde{\alpha}_{i-1}, \dots, \tilde{\alpha}_0] \in (t_1, t_2) i(\mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \chi}).$$

By the PBW theorem (Proposition 4.1.1), the only element of cohomological degree -2 in dimension vector $\delta + \delta_i$ could be $\tilde{\gamma}_\delta^{(0)} \star \tilde{\alpha}_i^{(0)}$. We again apply the shuffle product formula to calculate $i^\mathbb{T}(\tilde{\gamma}_\delta^{(0)}) \star i^\mathbb{T}(\tilde{\alpha}_i^{(0)})$. We see that

$$[i^\mathbb{T}(\tilde{\alpha}_i), i^\mathbb{T}(\tilde{\alpha}_i), \dots, i^\mathbb{T}(\tilde{\alpha}_K), i^\mathbb{T}(\tilde{\alpha}_{i-1}), \dots, i^\mathbb{T}(\tilde{\alpha}_0)] \notin (t_1, t_2) \cdot i^\mathbb{T}(\tilde{\gamma}_\delta^{(0)}) \star i^\mathbb{T}(\tilde{\alpha}_i^{(0)})$$

and thus $[\tilde{\alpha}_i, \tilde{\alpha}_i, \dots, \tilde{\alpha}_K, \tilde{\alpha}_{i-1}, \dots, \tilde{\alpha}_0] \notin (t_1, t_2) i(\mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \chi})$. But $[\tilde{\alpha}_i, \tilde{\alpha}_i, \dots, \tilde{\alpha}_K, \tilde{\alpha}_{i-1}, \dots, \tilde{\alpha}_0] \neq 0$ by the calculation of $\mathfrak{g}_{\mathbb{Q}^K, \widetilde{W}^K}^{\text{BPS}, \mathbb{T}}$ in Proposition 5.0.2. Thus, we must have

$$f(t_1, t_2) = \lambda_i \mu_i (t_0 + t_1)^K.$$

So $i^\mathbb{T}(\gamma_\delta^{(0)}) = \lambda (t_0 + t_1)^K$ up to a non-zero scalar λ . Then by Proposition 10.1.5, $i^\mathbb{T}(\gamma_\delta^{(r)}) = \lambda (t_0 + t_1)^K (x_{01} + x_{11} + \dots + x_{K,1})^r$ and we are done. \square

This already allows us to identify a commutative subalgebra inside the image $i^\mathbb{T}(\mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K}^{\mathbb{T}}) \subset \mathcal{A}_{\mathbb{Q}^K}^{\mathbb{T}}$.

Corollary 10.2.2.1. *Let $e = \sum_{i \in [0, K]} x_{i,1} \in \mathcal{A}_{\mathbb{Q}^K, \delta}^{\mathbb{T}, \chi}$. For each $n \geq 1$, we define*

$$L_n := \underbrace{[e, [e, [\dots [e, 1]]]]}_{n-1 \text{ times}} \in \mathcal{A}_{\mathbb{Q}^K, n, \delta}^{\mathbb{T}, \chi}$$

Then, the subalgebra generated by $L_n, n \geq 1$ is a polynomial subalgebra isomorphic to $\mathbb{C}[L_1, L_2, \dots]$.

Proof. Consider the Lie subalgebra generated by $\tilde{\gamma}_{n, \delta}, n \geq 1$. Then, by Proposition 5.0.2, it is a trivial Lie algebra, and the subalgebra generated by this Lie algebra is a polynomial subalgebra inside $\mathcal{A}_{\mathbb{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$. Let us call this algebra A^K . We show that the image of this under $i^\mathbb{T}$ up to a factor of polynomials in $\mathbb{C}[t_1, t_2]$ is exactly the subalgebra generated by the above generators. For any $n \geq 1$ and $\tilde{\gamma}_{n, \delta} \in \mathbb{H}^{-2}(\mathfrak{g}_{\mathbb{Q}^K, \widetilde{W}^K, n, \delta}^{\text{BPS}, \mathbb{T}})$ consider the commutator $[\tilde{\gamma}_\delta^{(1)}, \tilde{\gamma}_{n, \delta}]$. Then since

$\tilde{\gamma}_\delta^{(1)} \in \mathfrak{P}^{\leq 3}$ and $\tilde{\gamma}_{n \cdot \delta} \in \mathfrak{P}^{\leq 1}$, it follows that $[\tilde{\gamma}_\delta^{(1)}, \tilde{\gamma}_{n \cdot \delta}] \in \mathfrak{P}^{\leq 3}$ where we recall that $\mathfrak{P}^{\leq i}$ denote the perverse filtration defined in Section 4.1.5. Thus, it should be a linear combination of elements of the form $\mathfrak{P}^{\leq 1}, \mathfrak{P}^{\leq 1} \star \mathfrak{P}^{\leq 1}$ or $u \cdot \mathfrak{P}^{\leq 1}$. However minimal cohomological degree of $u \cdot \mathfrak{P}^{\leq 1}$ is 0 and thus it follows that we must have

$$[\tilde{\gamma}_\delta^{(1)}, \tilde{\gamma}_{n \cdot \delta}] = \lambda \tilde{\gamma}_{(n+1) \cdot \delta} + \sum_{n_i+n_j=n+1} \lambda_{n_i, n_j} \tilde{\gamma}_{n_i \cdot \delta} \star \tilde{\alpha}_{n_j \cdot \delta} + \sum_{k_i+k_j=n+1} f_{k_i, k_j}(t_1, t_2) \tilde{\gamma}_{k_i \cdot \delta} \star \tilde{\gamma}_{k_j \cdot \delta} \quad (10.8)$$

where $\tilde{\alpha}_{n_j \cdot \delta} \in \mathfrak{n}_{Q^K}^- \subset H^0(\mathfrak{g}_{\widetilde{Q^K}, \widetilde{W^K}, n_j \cdot \delta}^{\text{BPS}})$ for some $\lambda, \lambda_{n_i, n_j} \in \mathbb{C}$ and linear functions $f_{k_i, k_j}(t_1, t_2)$. We claim that each of $\lambda_{n_i, n_j} = 0$. For any $\tilde{\alpha}_{n_j \cdot \delta}$, by Proposition 5.0.2, there exist $\tilde{\alpha}_j$ in dimension δ_j such that $[\tilde{\alpha}_{n_j \cdot \delta}, \tilde{\alpha}_j] \neq 0$. But then since $\tilde{\gamma}_{n \cdot \delta}$ commutes with $\mathfrak{n}_{Q^K}^-$, applying $[-, \tilde{\alpha}_j]$ to the above equation (10.8) gives that

$$0 = \sum_{n_i+n_j=n+1} \lambda_{n_i, n_j} \tilde{\gamma}_{n_i \cdot \delta} \star [\tilde{\alpha}_{n_j \cdot \delta}, \tilde{\alpha}_j].$$

But each of non-zero $\tilde{\gamma}_{n_i \cdot \delta} \star [\tilde{\alpha}_{n_j \cdot \delta}, \tilde{\alpha}_j]$ can be extended to a basis of $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}}$ by the PBW theorem (Proposition 4.1.1). Since the above equation gives a relation, it implies that $\lambda_{n_i, n_j} = 0$. Furthermore $\lambda \neq 0$ by Proposition 8.2.2. Thus it follows that we must have $[\tilde{\gamma}_\delta^{(1)}, \tilde{\gamma}_{n \cdot \delta}] \in A^K$. By equation 10.8, it follows that the algebra generated by $[\tilde{\gamma}_\delta^{(1)}, \tilde{\gamma}_{n \cdot \delta}], n \geq 1$ and $\tilde{\gamma}_\delta$ contains all the generators $\tilde{\gamma}_{n \cdot \delta}$ of A^K . But then, so does

$$\tilde{L}_n := \underbrace{[\tilde{\gamma}_\delta^{(1)}, [\tilde{\gamma}_\delta^{(1)}, [\dots [\tilde{\gamma}_\delta^{(1)}, \tilde{\gamma}_\delta]]]]}_{n-1 \text{ times}} \in A^K \subset \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}} \quad (10.9)$$

Thus algebra generated by $\tilde{L}_n, n \geq 1$ is A^K , the commutative subalgebra. Thus, we are done from Proposition 10.2.2. \square

Proposition 10.2.2 also gives a slight refinement of a particular case of [SV17] and [Neg23a] about spherical generation of localized cohomological Hall algebra. We note it down as the argument will be helpful later.

Corollary 10.2.2.2. *The $\mathbb{C}[t_1, t_2]$ linear algebra $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi}$ is not a spherically generated $\mathbb{C}[t_1, t_2]$ algebra, while the localized algebra $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi} \otimes \mathbb{C}[t_1^{-1}, t_2^{-1}]$ is spherically generated.*

Proof. The fact that $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi}$ is not spherically generated follows from the fact that the cohomological degree of $\tilde{\alpha}_i^{(r)} \geq 0$, while cohomological degree of $\tilde{\gamma}_\delta^0 = -2$. Since the product structure preserves cohomological grading, it is not possible to reach this element by taking products of $\tilde{\alpha}_i^{(r)}$, nor by multiplication with any element of $\mathbb{C}[t_1, t_2]$, since that would also increase cohomological degree.

On the other hand by Proposition 10.2.2, up to a factor of non-zero scalar, $i^{\mathbb{T}}(\gamma_\delta^{(0)}) = (t_1 + t_2)^K \in \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}, \delta}^{\mathbb{T}, \chi}$. By the argument above $i^{\mathbb{T}}(\gamma_\delta^{(0)})$ cannot be

written as the $\mathbb{C}[t_1, t_2]$ -linear combination of products of $i^\mathbb{T}(\tilde{\alpha}_i^{(r)})$. However, we can write it as a $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ -linear combination of the products of $i^\mathbb{T}(\tilde{\alpha}_i^{(r)})$. Let

$$\mathcal{Y}_2 = \tilde{\alpha}_0^{(0)}\tilde{\alpha}_1^{(0)} + \tilde{\alpha}_1^{(0)}\tilde{\alpha}_0^{(0)} \quad (10.10)$$

$$\mathcal{Z}_2 = [\tilde{\alpha}_0^{(0)}, \tilde{\alpha}_1^{(1)}] + [\tilde{\alpha}_1^{(0)}, \tilde{\alpha}_0^{(1)}] \quad (10.11)$$

for $K = 1$ and

$$\mathcal{Y}_{K+1} = \sum_{i \in \mathbb{Z}_{K+1}} \tilde{\alpha}_i^{(0)} \cdot \left([\tilde{\alpha}_{i+1}^{(0)}, [\tilde{\alpha}_{i+2}^{(0)}, \dots, [\tilde{\alpha}_{i+K-1}^{(0)}, \tilde{\alpha}_{i+K}^{(0)}]]] \right) \quad (10.12)$$

$$\mathcal{Z}_{K+1} = \sum_{i \in \mathbb{Z}_{K+1}} [\tilde{\alpha}_i^{(0)}, [\tilde{\alpha}_{i+1}^{(1)}, [\tilde{\alpha}_{i+2}^{(0)}, [\dots, [\tilde{\alpha}_{i+K-1}^{(0)}, \tilde{\alpha}_{i+K}^{(0)}]]]]] \quad (10.13)$$

for $K \geq 2$.

Note that second term in \mathcal{Z}_{K+1} is $\tilde{\alpha}_{i+1}^{(1)}$. Using the shuffle product formula (10.1), it follows that for

$$i^\mathbb{T}(\mathcal{Y}_2) = (-2)((x_{1,1} - x_{0,1})^2 + t_1 t_2)$$

and

$$i^\mathbb{T}(\mathcal{Y}_{K+1}) = (-1)^K (t_1 + t_2)^{K-1} \sum_{i \in \mathbb{Z}_{K+1}} (x_{i+1,1} x_{i-1,1} - 2x_{i+1,1} x_{i,1} + x_{i,1}^2 + t_1 t_2)$$

for $K \geq 2$. Similarly, we calculate that

$$i^\mathbb{T}(\mathcal{Z}_2) = (-2)(t_1 + t_2)(x_{1,1} - x_{0,1})^2$$

and

$$i^\mathbb{T}(\mathcal{Z}_{K+1}) = (-1)^K (t_1 + t_2)^K \sum_{i \in \mathbb{Z}_{K+1}} (x_{i+1,1} x_{i-1,1} - 2x_{i+1,1} x_{i,1} + x_{i,1}^2)$$

for $K \geq 2$.

Thus for all $K \geq 1$, it follows that

$$(t_1 + t_2) i^\mathbb{T}(\mathcal{Y}_{K+1}) - i^\mathbb{T}(\mathcal{Z}_{K+1}) = (-1)^K (K + 1) (t_1 + t_2)^K (t_1 t_2)$$

and thus it follows that

$$\frac{1}{(-1)^K (K + 1)} \left(\left(\frac{t_1 + t_2}{t_1 t_2} \right) i^\mathbb{T}(\mathcal{Y}_{K+1}) - \left(\frac{1}{t_1 t_2} \right) i^\mathbb{T}(\mathcal{Z}_{K+1}) \right) = (t_1 + t_2)^K$$

Thus by Proposition 10.1.5, it follows that for any $K \geq 1$, we can write $\tilde{\gamma}_\delta^r$ as a non-zero scalar multiple of

$$\mathcal{K}_{K+1}^r := u^r \cdot \left(\frac{(t_1 + t_2) \mathcal{Y}_{K+1} - \mathcal{Z}_{K+1}}{t_1 t_2} \right) \quad (10.14)$$

□

Now that we know the image of $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, X}$ in the shuffle algebra, we show that the image is an integral form of an affine Yangian.

10.2.1 Affine Yangian

Let $n \geq 2$. Let $\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(n))$ be the affine Yangian of \mathfrak{gl}_n considered in [BT19], which is a slight reparametrization and is isomorphic to the affine Yangian defined by [Gua07]. The definition of the affine Yangian for $n = 2$ was first given in [Kod19].

Definition 10.2.3. $\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n))$ is a $\mathbb{C}[\hbar_1, \hbar_2]$ algebra generated by the elements $X_{i,r}^+$ where $i = 0, \dots, n-1$ and $r \geq 0$ such that for all $i \in [0, n-1]$ and $r, s \geq 0$ we have relations

$$[X_{i,r+1}^+, X_{i+1,s}^+] - [X_{i,r}^+, X_{i+1,s+1}^+] = (\hbar_1 + \hbar_2/2)[X_{i,r}^+, X_{i+1,s}^+] - \hbar_2/2\{X_{i,r}^+, X_{i+1,s}^+\} \quad (10.15a)$$

$$[X_{i,r+1}^+, X_{i-1,s}^+] - [X_{i,r}^+, X_{i-1,s+1}^+] = -(\hbar_1 + \hbar_2/2)[X_{i,r}^+, X_{i-1,s}^+] - \hbar_2/2\{X_{i,r}^+, X_{i-1,s}^+\} \quad (10.15b)$$

$$[X_{i,r+1}^+, X_{i,s}^+] - [X_{i,r}^+, X_{i,s+1}^+] = \hbar_2(X_{i,r}^+ X_{i,s}^+ + X_{i,s}^+ X_{i,r}^+) \quad (10.15c)$$

$$[X_{i,r}^+, X_{j,s}^+] = 0 \quad \forall |i-j| > 1 \quad (10.15d)$$

$$\sum_{\sigma \in S_2} [X_{i,r_{\sigma(1)}}^+, [X_{i,r_{\sigma(2)}}^+, X_{i+1,s}^+]] = 0 \quad (10.15e)$$

when $n \geq 2$ and

$$[X_{i,r+1}^+, X_{i,s}^+] - [X_{i,r}^+, X_{i,s+1}^+] = \hbar_2\{X_{i,r}^+, X_{i,s}^+\} \quad (10.16a)$$

$$[X_{i,r+2}^+, X_{i+1,s}^+] - 2[X_{i,r+1}^+, X_{i+1,s+1}^+] + [X_{i,r}^+, X_{i+1,s+2}^+] = \hbar_1(\hbar_1 + \hbar_2)[X_{i,r}^+, X_{j,s}^+] - \hbar_2(\{X_{i,r+1}^+, X_{i+1,s}^+\} - \{X_{i,r}^+, X_{i+1,s+1}^+\}) \quad (10.16b)$$

$$\text{Sym}_{r_1, r_2, r_3}[X_{i,r_1}^+, [X_{i,r_2}^+, [X_{i,r_3}^+, X_{i+1,s}^+]]] = 0 \quad (10.16c)$$

when $n = 2$.

Note that we can give a *dimension* grading to the algebra $\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n))$ by declaring $\mathbf{dim}(X_{i,r}^+) = \delta_i$ where $\delta_i = (0, \dots, 1, \dots, 0) \in \mathbb{N}^n$. We also have a *cohomological* grading given by declaring $\text{CG}(X_{i,r}^+) = 2r$ and $\text{CG}(\hbar_1) = \text{CG}(\hbar_2) = 2$. Then the size of this algebra is known due to earlier work. We have

Proposition 10.2.4. *There is an isomorphism of free $\mathbb{C}[\hbar_1, \hbar_2]$ modules*

$$\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n)) \simeq \text{Sym}(\mathcal{L}_n) \otimes \mathbb{C}[\hbar_1, \hbar_2]$$

where \mathcal{L}_n is $\mathbb{N}^n \times \mathbb{Z}$ graded vector space of graded dimension

$$\dim((\mathcal{L}_n)_{\mathbf{dim}}^{\text{CG}}) = \begin{cases} n-1 & \text{if } \text{CG} = 0, \mathbf{dim} \in \mathbb{Z}_{\geq 1} \cdot \delta \\ n & \text{if } \text{CG} \geq 1, \mathbf{dim} \in \mathbb{Z}_{\geq 1} \cdot \delta \\ 1 & \text{if } \text{CG} \in \mathbb{Z}_{\geq 0}, \mathbf{dim} \text{ is a positive real root of } \widetilde{\mathfrak{sl}}_n \\ 0 & \text{otherwise} \end{cases} \quad (10.17)$$

Proof. We define a filtration F of $\check{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n))$ by assigning $X_{i,r}^+$ degree $2r$ and rest 0. Then the associated graded algebra $\text{Gr}_F(\check{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n))) \simeq \mathbf{U}(\mathcal{L}_n) \otimes \mathbb{C}[\hbar_1, \hbar_2]$ where \mathcal{L} is Lie algebra generated by relations \mathcal{L}_n is a Lie algebra over \mathbb{C} , generated by $X_{i,r}^+, r \geq 0, i \in [0, n-1]$ with the defining relations

$$\begin{aligned} [X_{i,r+1}^+, X_{i,s}^+] - [X_{i,r}^+, X_{i,s+1}^+] &= 0 \\ [X_{i,r+2}^+, X_{i+1,s}^+] - 2[X_{i,r+1}^+, X_{i+1,s+1}^+] + [X_{i,r}^+, X_{i+1,s+2}^+] &= 0 \\ \text{Sym}_{r_1, r_2, r_3}[X_{i,r_1}^+, [X_{i,r_2}^+, [X_{i,r_3}^+, X_{i+1,s}^+]]] &= 0 \end{aligned}$$

when $n = 2$ and

$$\begin{aligned} [X_{i,r+1}^+, X_{i+1,s}^+] &= [X_{i,r}^+, X_{i+1,s+1}^+] \\ [X_{i,r+1}^+, X_{i-1,s}^+] &= [X_{i,r}^+, X_{i-1,s+1}^+] \\ [X_{i,r+1}^+, X_{i,s}^+] &= [X_{i,r}^+, X_{i,s+1}^+] \\ [X_{i,r}^+, X_{j,s}^+] &= 0 \quad \forall |i-j| > 1 \\ \text{Sym}_{r_1, r_2}[X_{i,r_1}^+, [X_{i,r_2}^+, X_{i\pm 1,s}^+]] &= 0 \end{aligned}$$

when $n > 2$.

Given any commutative algebra A , we can consider the Lie algebra $\mathfrak{sl}_n(A)$, defined by the Lie bracket $[X \otimes a, Y \otimes b] = [X, Y] \otimes ab$ where $X, Y \in \mathfrak{sl}_n$ and $a, b \in A$ are arbitrary elements. Let $\widehat{\mathfrak{sl}}_n(A)$ be the universal central extension of $\mathfrak{sl}_n(A)$ which is as vector spaces $\mathfrak{sl}_n(A) \oplus \Omega^1(A)/dA$, where $\Omega^1(A)/dA$ is the space of 1 forms on the affine variety $\text{Spec}(A)$, modulo the exact forms (We refer to [MRY90] and [Gua07] for details). The Lie algebra is defined by

$$[z_1 \otimes a_1, z_2 \otimes a_2] = [z_1, z_2] \otimes a_1 \cdot a_2 + \langle z_1, z_2 \rangle a_2 da_1$$

where $\langle \cdot, \cdot \rangle$ is the Killing form. We consider the setting when $A = \mathbb{C}[u^{\pm 1}, v]$. Let $\mathfrak{b}_n^+ \subset \widehat{\mathfrak{sl}}_n[u^{\pm 1}, v]$ be the positive half. It can be identified as the Lie algebra spanned by $Xu^a v^b$ where $X \in \mathfrak{sl}_n, a \geq 1, b \geq 0$ and Xv^b where $X \in \mathfrak{n}_n^+$ and $K_b[a]$ where $a, b \geq 1$, such that they satisfy the relations

$$[Xu^{m_1} v^{n_1}, Yu^{m_2} v^{n_2}] = [X, Y] \otimes u^{m_1+m_2} v^{n_1+n_2} + \langle X, Y \rangle (m_2 n_1 - m_1 n_2) K_{m_1+m_2}[n_1+n_2],$$

with $K_b[a]$ central.

For $n > 2$, by [Gua07][Lemma 3.4] and for $n = 2$, by [Dia+25][Proposition IV.3.10]; we have an isomorphism of Lie algebras $\mathcal{L}_n \simeq \mathfrak{b}_n^+$ given by $X_{i,r}^+ \mapsto$

$E_{i,i+1}v^r$ and $X_{0,r}^+ \mapsto E_{n,1}wv^r$. The advantage is that now we have an explicit spanning set of the Lie algebra \mathcal{L}_n . The dim and CG grading makes \mathcal{L}_n a $\mathbb{N}^n \times \mathbb{Z}$ graded vector spaces. Computing these graded dimensions proves the above theorem. \square

Proposition 10.2.5. *Assume $K \geq 1$. Then we have a homomorphism of algebras³*

$$\Phi : \check{\mathcal{Y}}_{-t_2, t_1+t_2}^+ (\mathfrak{gl}(K+1)) \rightarrow \mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathbb{T}, \chi}$$

given by the assignment $X_{i,s}^+ \mapsto \widetilde{\alpha}_i^{(s)}$.

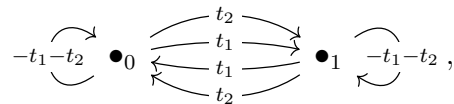
Proof. By Proposition 10.1.6, there is an injection of algebras $i^{\mathbb{T}} : \mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathbb{T}, \chi} \hookrightarrow \mathcal{A}_{Q^K}^{\mathbb{T}, \chi}$. Since $i^{\mathbb{T}}(\Phi(X_{i,s}^+)) = x_{i,1}^s$. We can use the explicit shuffle product (Equation 10.2) on $\mathcal{A}_{Q^K}^{\mathbb{T}, \chi}$ to check that the defining relations (10.2.3) of the positive half of the affine Yangian are satisfied.

Assume $K > 1$. Then by the Shuffle formula, we have

$$\begin{aligned} [x_{i,1}^r, x_{i+1,1}^s] &= -(t_1 + t_2)x_{i,1}^r x_{i+1,1}^s \\ \{x_{i,1}^r, x_{i+1,1}^s\} &= x_{i,1}^r x_{i+1,1}^s (2(x_{i,1} - x_{i+1,1}) + t_2 - t_1) \\ [x_{i,1}^r \cdot x_{i,1}^s] &= 2(t_1 + t_2) \left(\frac{x_{i,2}^s x_{i,1}^r - x_{i,2}^r x_{i,1}^s}{x_{i,2} - x_{i,1}} \right) \\ \{x_{i,1}^r \cdot x_{i,1}^s\} &= -2(x_{i,2}^r x_{i,1}^s + x_{i,1}^r x_{i,2}^s) \\ [x_{i,1}^r, x_{j,1}^s] &= 0 \quad \forall \quad |i - j| > 1 \end{aligned}$$

Thus when $\hbar_1 = -t_2, \hbar_2 = t_1 + t_2$, the Relations 10.15a, 10.15b, 10.15c, 10.15d are clearly satisfied. Checking Serre relation 10.15e is a similar but lengthier computation.

When $K = 1$. The shuffle formula for quiver



gives

$$\begin{aligned} [x_{i,1}^r, x_{i+1,1}^s] &= -2x_{i,1}^r x_{i+1,1}^s (t_1 + t_2)(x_{i+1,1} - x_{i,1}) \\ \{x_{i,1}^r, x_{i+1,1}^s\} &= -2x_{i,1}^r x_{i+1,1}^s ((x_{i+1,1} - x_{i,1})^2 + t_1 t_2) \\ [x_{i,1}^r \cdot x_{i,1}^s] &= 2(t_1 + t_2) \left(\frac{x_{i,2}^s x_{i,1}^r - x_{i,2}^r x_{i,1}^s}{x_{i,2} - x_{i,1}} \right) \\ \{x_{i,1}^r \cdot x_{i,1}^s\} &= -2(x_{i,2}^r x_{i,1}^s + x_{i,1}^r x_{i,2}^s) \end{aligned}$$

³A 1-parameter version of this proposition is also proved in [YZ18a] for any loop-free quiver Q .

So

$$[X_{i,r+2}^+, X_{i+1,s}^+] - 2[X_{i,r+1}^+, X_{i+1,s+1}^+] + [X_{i,r}^+, X_{i+1,s+2}^+] \mapsto 2(t_1+t_2)(x_{i,1}^r x_{i+1,1}^s)(x_{i,1} - x_{i+1,1})^3$$

and similarly, we have

$$\begin{aligned} -t_1 t_2 [X_{i,r}^+, X_{j,s}^+] + (t_1 + t_2) (\{X_{i,r+1}^+, X_{i+1,s}^+\} - \{X_{i,r}^+, X_{i+1,s+1}^+\}) \mapsto \\ 2(t_1 + t_2) x_{i,1}^r x_{i+1,1}^s (x_{i,1} - x_{i+1,1})^3. \end{aligned}$$

Thus Relation 10.16b is satisfied.

Similarly,

$$\begin{aligned} [X_{i,r+1}^+, X_{i,s}^+] - [X_{i,r}^+, X_{i,s+1}^+] \mapsto 2(t_1 + t_2) \frac{x_{i,2}^s x_{i,1}^{r+1} - x_{i,2}^{r+1} x_{i,1}^s - x_{i,2}^{s+1} x_{i,1}^r + x_{i,2}^r x_{i,1}^{s+1}}{x_{i,2} - x_{i,1}} \\ = 2(t_1 + t_2)(-1)(x_{i,2}^r x_{i,1}^s + x_{i,1}^r x_{i,2}^s) \end{aligned}$$

But

$$(t_1 + t_2) \{X_{i,r}^+, X_{i,s}^+\} \mapsto -2(t_1 + t_2)(x_{i,2}^r x_{i,1}^s + x_{i,1}^r x_{i,2}^s).$$

So the Relation 10.16a is also satisfied. We can similarly check the Serre relation 10.16c. \square

We now show that in fact the morphism Φ is an injection⁴.

Proposition 10.2.6. *Assume $K \geq 1$. Then we have an injection of algebras:*

$$\Phi : \check{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1)) \rightarrow \mathcal{A}_{Q^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$$

given by the assignment $X_{i,s}^+ \mapsto \widetilde{\alpha}_i^{(s)}$.

Proof. We show that, in fact, we have an isomorphism

$$\Phi : \check{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1)) \simeq \mathcal{S}\mathcal{A}_{Q^K, \widetilde{W}^K}^{\mathbb{T}, \chi} \hookrightarrow \mathcal{A}_{Q^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$$

Clearly by definition, Φ is a surjection on its image. The spherical subalgebra $\mathcal{S}\mathcal{A}_{Q^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$ is cohomologically and dimensionally graded. It is shown in [SV20][Theorem 5.18] that $\mathcal{S}\mathcal{A}_{Q^K, \widetilde{W}^K}^{\mathbb{T}, \chi}$ is isomorphic to the nilpotent cohomological Hall algebra $\mathcal{A}_{\Pi_Q}^{\mathbb{T}, \mathcal{N}, \chi} \simeq \mathcal{A}_{Q, \widetilde{W}}^{\mathbb{T}, \widetilde{\mathcal{N}}, \chi}$. We now compute the graded dimension of nilpotent CoHA. By Theorem 4.2.8, we have an isomorphism of free $\mathbb{C}[t_1, t_2]$ modules

$$\mathcal{A}_{Q, \widetilde{W}}^{\mathbb{T}, \widetilde{\mathcal{N}}} \simeq \mathcal{A}_{Q, \widetilde{W}}^{\widetilde{\mathcal{N}}} \otimes \mathbb{C}[t_1, t_2].$$

In Example 10.18, we proved that there is an isomorphism of graded vector spaces:

$$\mathcal{A}_{Q, \widetilde{W}}^{\widetilde{\mathcal{N}}} \simeq \text{Sym}(\text{H}(\mathcal{M}_{\mathbf{d}}^{\mathcal{N}_{\overline{Q}}}(\overline{Q}), i^! \mathcal{BPS}_{\Pi_Q})[u]).$$

⁴In an earlier version of the paper, for $K > 1$, we proved this by comparing graded dimension after taking Perverse filtration.

and

$$\sum_i (-1)^i \dim(\mathrm{H}^i(\mathcal{M}_{\mathbf{d}}^{\mathcal{N}_{\overline{Q}}}(\overline{Q}), i^! \mathcal{BPS}_{\Pi_Q})) q^{i/2} = a_{Q, \mathbf{d}}(q). \quad (10.18)$$

and thus by calculation of Kac polynomial for cyclic quiver in Equation 5.0.1, it follows that

$$\dim(\mathrm{H}(\mathcal{M}_{\mathbf{d}}^{\mathcal{N}_{\overline{Q}}}(\overline{Q}), i^! \mathcal{BPS}_{\Pi_Q}[u]^i) = \begin{cases} K & \text{if } i = 0, \mathbf{d} \in \mathbb{Z}_{\geq 1} \cdot \delta \\ K + 1 & \text{if } i \geq 1, \mathbf{d} \in \mathbb{Z}_{\geq 1} \cdot \delta \\ 1 & \text{if } i \in \mathbb{Z}_{\geq 0}, \mathbf{d} \text{ is a positive real root of } \widetilde{\mathfrak{sl}}_{K+1} \\ 0 & \text{otherwise} \end{cases} \quad (10.19)$$

which matches the graded dimensions of the affine Yangian in Proposition 10.2.4. Thus, the morphism Φ is an injection. \square

Remark 10.2.7. From the above argument, it follows that there is an isomorphism of $\mathbb{C}[t_1, t_2]$ linear algebras $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1)) \simeq \mathcal{A}_{\overline{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \widetilde{\mathcal{N}}, \chi}$. This statement is now also proved in [Dia+25][Theorem IV.3.11].

We now identify the integral form precisely.

Lemma 10.2.8. *There exists a derivation $T : \ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n)) \rightarrow \ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n))$ such that $T(X_{i,r}^+) = X_{i,r+1}^+$*

Proof. We define $T(X_{i,r}^+) = X_{i,r+1}^+$ and extend it by $T(ab) := T(a)b + aT(b)$ for $a, b \in \ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n))$ and note that it satisfies the relations in Definition 10.2.3 and hence is well defined⁵. \square

Definition 10.2.9. Assume $n \geq 2$. Let $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), +, \text{CoHA}} \subset \ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(n)) \otimes_{\mathbb{C}[\hbar_1, \hbar_2]} \mathbb{C}[\hbar_1^{\pm 1}, (\hbar_1 + \hbar_2)^{\pm 1}]$ be a $\mathbb{C}[\hbar_1, \hbar_2]$ linear sub algebra generated by $X_{i,r}^+$ and $\mathfrak{K}_n^{(r)}$ for $r \geq 0$ such that for all $r \geq 0$, we have an extra relation

$$(\hbar_1)(\hbar_1 + \hbar_2) \mathfrak{K}_n^{(r)} = \mathrm{T}^r(\mathfrak{Z}_n - \hbar_2 \mathfrak{Y}_n)$$

where

$$\begin{aligned} \mathfrak{Y}_2 &:= X_{0,0}^+ X_{1,0}^+ + X_{1,0}^+ X_{0,0}^+ \\ \mathfrak{Z}_2 &:= [X_{0,0}^+, X_{1,1}^+] + [X_{1,0}^+, X_{0,1}^+] \end{aligned}$$

and

$$\begin{aligned} \mathfrak{Y}_n &:= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} X_{i,0}^+ \cdot ([X_{i+1,0}^+, [X_{i+2,0}^+, \cdots, [X_{i+n-2,0}^+, X_{i+n-1,0}^+]]]) \\ \mathfrak{Z}_n &:= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} [X_{i,0}^+, [X_{i+1,1}^+, [X_{i+2,0}^+, [\cdots, [X_{i+n-2,0}^+, X_{i+n-1,0}^+]]]]]. \end{aligned}$$

⁵Alternatively one can use [GNW18][Lemma 2.22] to construct such a operator in $\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^{(0)}(\mathfrak{gl}(n))$.

for $n \geq 2$. Note that second term in the sum defining \mathfrak{Z}_n is $X_{i+1,1}^+$.

Note that these extra elements are defined purely via cohomological Hall algebras, but quite remarkably they satisfy interesting algebraic simplification when $\hbar_2 = 0$. See Proposition 11.0.2 in particular.

Theorem 10.2.10. *We have an isomorphism of $\mathbb{C}[t_1, t_2]$ algebras*

$$\tilde{\Phi} : \mathcal{Y}_{-t_2, t_1+t_2}^{(K+1), +, \text{CoHA}} \rightarrow \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi},$$

given by the assignment $X_{i,s}^+ \mapsto \tilde{\alpha}_i^{(s)}$ and $\mathfrak{R}_{K+1}^{(r)} \mapsto \tilde{\gamma}_\delta^{(r)}$.

Proof. The map is well defined by Proposition 10.2.6, Equation 10.14, and Proposition 10.2.2. By Corollary 10.2.1, the morphism $\tilde{\Phi}$ is a surjection. The algebra $\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, \chi}$ is a free $\mathbb{C}[t_1, t_2]$ module by Theorem 4.2.7. After localizing, $\tilde{\Phi} \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)$ is just the morphism $\Phi \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)$ in the Proposition 10.2.6, which is an injection. So $\tilde{\Phi}$ is an injection, and we are done. \square

Chapter 11

Deformed Affinized BPS Lie algebra for Cyclic Quivers

We recall that in Section 6.1, we gave $\mathcal{A}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathbb{C}^*, \chi}$ a cocommutative coproduct by considering the eigenvalues of $\omega := \sum \omega_i$, and we concluded that there is an isomorphism of algebras

$$\mathcal{A}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathbb{C}^*, \chi} \simeq \mathbf{U}_{\mathbb{C}[t]}(\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*}),$$

where $\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*}$ is a $\mathbb{C}[t]$ linear Lie algebra, called the \mathbb{C}^* deformed affinized BPS Lie algebra. Previous results allows us to calculate the Lie algebra $\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*}$ explicitly. Since $\mathfrak{g}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*} \simeq \mathfrak{g}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}} \otimes \mathbb{C}[t]$ as graded vector spaces, let $\hat{\alpha}_i$ and $\hat{\gamma}_\delta$ be unique non-canonical lift of $\alpha_i, \gamma_\delta \in \mathfrak{g}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}}$. We then have

Proposition 11.0.1. *The $\mathbb{C}[t]$ linear Lie algebra $\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*}$ is generated by $u^k \cdot \hat{\alpha}_i$ for all $i \in [0, K], k \geq 0$ and $u^k \cdot \hat{\gamma}_\delta$ for all $k \geq 0$.*

Proof. Note that since $\mathcal{A}_{\widetilde{Q, \widetilde{W}}}^{\mathbb{C}^*, \chi}$ is a free $\mathbb{C}[t]$ module, the isomorphism of algebras $\mathcal{A}_{\widetilde{Q, \widetilde{W}}}^{\mathbb{C}^*, \chi} \simeq \mathbf{U}_{\mathbb{C}[t]}(\widehat{\mathfrak{g}}_{\widetilde{Q, \widetilde{W}}}^{\text{BPS}, \mathbb{C}^*})$ implies that $\widehat{\mathfrak{g}}_{\widetilde{Q, \widetilde{W}}}^{\text{BPS}, \mathbb{C}^*}$ is a free $\mathbb{C}[t]$ module. We define a map of free $\mathbb{C}[t]$ modules

$$J : \widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}} \otimes \mathbb{C}[t] \rightarrow \widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*},$$

defined by $J(\alpha_i^{(k)} f(t)) := u^k \cdot \hat{\alpha}_i$ and $J(\gamma_\delta^{(k)} f(t)) = f(t) u^k \cdot \hat{\gamma}_\delta$ and extending to $\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}, \mathbb{C}^*}$ by the Lie algebra relations on $\widehat{\mathfrak{g}}_{\widetilde{Q^k, \widetilde{W^k}}}^{\text{BPS}}$ (Theorem 8.2.6). It suffices to show that the morphism J is surjective. Let $m_{\mathbb{C}^*} = (t) \subset \mathbb{C}[t]$ be the maximal ideal. Then the morphism $J/(m_{\mathbb{C}^*})$ is an isomorphism of $\mathbb{N}^{Q_0} \times \mathbb{Z}$ graded modules, where each graded piece is finite-dimensional and thus J is an isomorphism of vector spaces and so in particular surjective. \square

We now calculate $\mathcal{A}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathbb{C}^*, \chi}$. Since there is an isomorphism of algebras

$$\mathcal{A}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathbb{C}^*, \chi} \simeq \mathcal{A}_{\widetilde{Q^k, \widetilde{W^k}}}^{\mathbb{T}, \chi} \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}[t_1, t_2]/(t_1 + t_2),$$

it follows by Theorem 10.2.10, that there is an isomorphism of algebras

$$(\mathcal{Y}_{-t_2, t_1+t_2}^{(K+1),+, \text{CoHA}})_{t_1+t_2=0} \simeq \mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{C}^*, \chi}.$$

But setting $t_2 = \hbar$, yields

$$(\mathcal{Y}_{-t_2, t_1+t_2}^{(K+1),+, \text{CoHA}})_{t_1+t_2=0} \simeq \mathbf{U}_{\mathbb{C}[\hbar]}(S_{K+1}) \quad (11.1)$$

and an isomorphism of Lie algebras

$$\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}, \mathbb{C}^*} \simeq S_{K+1}, \quad (11.2)$$

where S_{K+1} is a $\mathbb{C}[\hbar]$ linear Lie algebra generated by $X_{i,r}^+$, $r \geq 0$, $i \in [0, K]$ and \mathfrak{K}_{K+1}^r , $r \geq 0$ with the defining relations

$$[X_{i,r+1}^+, X_{i+1,s}^+] - [X_{i,r}^+, X_{i+1,s+1}^+] = -\hbar[X_{i,r}^+, X_{i+1,s}^+] \quad (11.3)$$

$$[X_{i,r+1}^+, X_{i-1,s}^+] - [X_{i,r}^+, X_{i-1,s+1}^+] = \hbar[X_{i,r}^+, X_{i-1,s}^+] \quad (11.4)$$

$$[X_{i,r+1}^+, X_{i,s}^+] - [X_{i,r}^+, X_{i,s+1}^+] = 0 \quad (11.5)$$

$$[X_{i,r}^+, X_{j,s}^+] = 0 \quad \forall |i-j| > 1 \quad (11.6)$$

$$\sum_{\sigma \in S_2} [X_{i,r_{\sigma(1)}}^+, [X_{i,r_{\sigma(2)}}^+, X_{i+1,s}^+]] = 0 \quad (11.7)$$

$$\mathbf{T}^r(\mathfrak{Z}_{K+1}) = \hbar^2 \mathfrak{K}_{K+1}^{(r)} \quad (11.8)$$

when $K > 1$ and

$$[X_{i,r+1}^+, X_{i,s}^+] - [X_{i,r}^+, X_{i,s+1}^+] = 0 \quad (11.9a)$$

$$[X_{i,r+2}^+, X_{i+1,s}^+] - 2[X_{i,r+1}^+, X_{i+1,s+1}^+] + [X_{i,r}^+, X_{i+1,s+2}^+] = \hbar^2[X_{i,r}^+, X_{j,s}^+] \quad (11.9b)$$

$$\text{Sym}_{r_1, r_2, r_3} [X_{i,r_1}^+, [X_{i,r_2}^+, [X_{i,r_3}^+, X_{i+1,s}^+]]] = 0 \quad (11.9c)$$

$$\mathbf{T}^r(\mathfrak{Z}_2) = \hbar^2 \mathfrak{K}_2^{(r)} \quad (11.9d)$$

when $K = 2$.

The isomorphism of the $\mathbb{C}[\hbar]$ linear Lie algebras $\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}, \mathbb{C}^*} \rightarrow S_{K+1}$ is given by $u^r \cdot \hat{\alpha}_i \mapsto X_{i,r}^+$ for all $i \in [0, K]$, $r \geq 0$ and $u^r \cdot \hat{\gamma}_\delta \mapsto \mathfrak{K}_{K+1}^{(r)}$, $r \geq 0$. The Lie algebra S_{K+1} can be described more explicitly. Recall the Lie algebra $(\mathcal{W}_{K+1})^+$ from Section 8.1. We consider a derivation of $(\mathcal{W}_{K+1})^+$, defined by $\mathbf{ad}_{P_{K+1}} := [P_{K+1}, \cdot]$ where

$$P_{K+1} = \frac{(K+1)^2}{2} t_{0,2} + (K+1)T_{0,1}(H'_{K+1}) + (K+1)\hbar T_{0,0}(H''_{K+1})$$

such that $H'_{K+1} = \sum_{i=1}^{K+1} (K+3/2-i)E_{i,i}$ and $H''_{K+1} = \sum_{i=1}^{K+1} \mu_i E_{i,i}$ for any μ_i satisfying $\mu_i - \mu_{i+1} = (1 - i/(K+1))$ and $\sum \mu_i = 0$. They are chosen so that $[H'_{K+1}, E_{i,i+1}] = E_{i,i+1}$ for all $i \in [1, K]$, $[H'_{K+1}, E_{K+1,1}] = -K$ and $[H''_{K+1}, E_{i,i+1}] = (1 - i/(K+1))E_{i,i+1}$ for all $i \in [1, K]$. This might look adhoc, but important property is that they satisfy equations 11.10.

Proposition 11.0.2. *We have an isomorphism of $\mathbb{C}[\hbar]$ linear Lie algebras*

$$G: S_{K+1} \rightarrow (\mathcal{W}_{K+1})^+$$

given by

$$\begin{aligned} X_{0,r}^+ &\mapsto \mathbf{ad}_{P_{K+1}}^r(T_{1,0}(E_{K+1,1})) \\ X_{i,r}^+ &\mapsto \mathbf{ad}_{P_{K+1}}^r(T_{0,0}(E_{i,i+1})) \\ \mathfrak{K}_{K+1}^{(r)} &\mapsto \mathbf{ad}_{P_{K+1}}^r(t_{1,0}) \end{aligned}$$

Proof. We see that in terms of matrices

$$\mathbf{ad}_{P_{K+1}}^r(T_{1,0}(E_{K+1,1})) = (-1)^r (K+1)^r D^r z E_{K+1,1} \quad (11.10)$$

$$\mathbf{ad}_{P_{K+1}}^r(T_{0,0}(E_{i,i+1})) = (-1)^r (K+1)^r (D + (1 - i/(K+1))\hbar)^r E_{i,i+1} \quad (11.11)$$

It can be checked explicitly that the defining relations (11.3)(11.4)(11.5) (11.6) (11.7) and similarly the ones for $K = 1$ of S_{K+1} are satisfied. Recall that we defined

$$\mathfrak{Z}_{K+1} = \sum_{i \in \mathbb{Z}/(K+1)\mathbb{Z}} [X_{i,0}^+, [X_{i+1,1}^+, [X_{i+2,0}^+, [\dots, [X_{i+K-1,0}^+, X_{i+K,0}^+]]]]]$$

Let $R_i = [X_{i,0}^+, [X_{i+1,1}^+, [X_{i+2,0}^+, [\dots, [X_{i+K-1,0}^+, X_{i+K,0}^+]]]]]$. Then When $i = 0, K$, we calculate that

$$G(R_i) = (-1)(K+1) \left(zD + \left(2 - \frac{i+1}{K+1}\right)\hbar z \right) (E_{i,i} - E_{i+1,i+1})$$

When $i = 0, K$, we calculate that

$$\begin{aligned} G(R_0) &= (-1)(K+1) \left(zD(E_{K+1,K+1} - E_{1,1}) + \frac{\hbar z K(E_{K+1,K+1} - E_{1,1})}{K+1} - \hbar z E_{1,1} \right) \\ G(R_K) &= (-1)(K+1) (zD(E_{K,K} - E_{K+1,K+1}) + \hbar z(E_{K,K} - E_{K+1,K+1})) \end{aligned}$$

Then summing up them up gives

$$G(\mathfrak{Z}_{K+1}) = \sum_{i=0}^K G(R_i) = \hbar z \text{Id}$$

Thus $G(\mathfrak{Z}_{K+1}) = \hbar^2 t_{1,0}$, thus relations 11.8 and 11.9d are satisfied for $r = 0$. But note that $G(T(X_{i,r}^+)) = G(X_{i,r+1}^+) = \mathbf{ad}_{P_{K+1}}(G(X_{i,r}))$ thus these relations are also satisfied for $r > 1$. It is a surjective map of Lie algebras since $(\mathcal{W}_{K+1})^+$ is generated by $t_{1,r}, r \geq 0$ and $T_{0,r}(E_{i,i+1}), T_{1,r}(E_{K+1,1})$, all of which are inductively in the image of G . To show the injection, we note that G respects both cohomological and dimension grading on S_{K+1} , induced by the isomorphism $\widehat{\mathfrak{g}}_{\widehat{Q^K}, \widehat{W^K}}^{\text{BPS}, \mathbb{C}^*} \simeq S_{K+1}$. It is easy to check by Proposition 5.0.2 that the graded dimensions of S_{K+1} and

$(\mathcal{W}_{K+1})^+$ are finite dimensional and are the same. Thus G is, in fact, an isomorphism. Alternatively, it is an injective map after localizing with $\mathbb{C}[\hbar^{-1}]$ since then $G[\hbar^{-1}]$ is the positive half of the isomorphism of Lie algebras considered in [Tsy17a][Theorem 2.2]. \square

Combining Proposition 11.0.2 and Equation (11.2) implies

Theorem 11.0.3. *There is an isomorphism of of $\mathbb{C}[\hbar]$ linear Lie algebras*

$$\widehat{\mathfrak{g}}_{\widetilde{Q^K}, \widetilde{W^K}}^{\text{BPS}, \mathbb{C}^*} \simeq (\mathcal{W}_{K+1})^+$$

Chapter 12

Application: Maulik-Okounkov Yangian for Cyclic Quiver

Given any quiver Q in [MO19] and torus T which scales the natural symplectic form on the Nakajima quiver varieties associated with quiver Q , Maulik and Okounkov defined an associative algebra $H_T(\text{pt})$ linear algebra $\mathbf{Y}_Q^{T, \text{MO}}$ called the Maulik-Okounkov Yangian. They further conjectured a strong relationship between the Bethe Subalgebra of these algebras and the equivariant Quantum cohomology ring of the Nakajima quiver varieties, which makes calculating these $H_T(\text{pt})$ linear algebras a worthwhile investment. In the latter half of [MO19], the authors calculate these algebras for the Jordan quiver and realise them as the affine Yangian of $\mathfrak{gl}(1)$, which are algebras that were defined via Cohomological Hall algebras in [AS13]. In [Zhu24], it is proved that, after localization, the K-theoretic version of the MO Yangian is isomorphic to quantum toroidal algebras. Thus, after taking additive degeneration, it is natural to expect the relation between the affine Yangian and MO Yangian for cyclic quivers. In this chapter, we calculate $H_T(\text{pt})$ linear algebras $\mathbf{Y}_Q^{T, \text{MO}}$ in the case of cyclic quivers. We note the emphasis on $H_T(\text{pt})$ linearity, as the Nakajima quiver varieties are not proper, so after localizing, i.e., $\mathbf{Y}_Q^{T, \text{MO}} \otimes_{H_T(\text{pt})} \text{Frac}(H_T(\text{pt}))$ forgets a lot of the geometry of Nakajima quiver varieties.

12.0.1 Maulik-Okounkov Yangian

Following [SV23], we first quickly recall the definition of the MO Yangian. Let Q be any quiver and Let T be any torus, defined by weighting $\mathbf{w} : \overline{Q}_1 \rightarrow \mathbb{Z}^r$ such that $\rho = \sum_{a \in Q_1} [a, a^*]$ is homogenous. We recall that the torus associated to this is given by $\text{Hom}(\mathbb{Z}^r, \mathbb{C}^*)$. In particular for ρ , we get a homomorphism $T \rightarrow \mathbb{C}^*$ given by $t \mapsto t(\mathbf{w}(\rho))$, which by taking pullback, defines a cohomology class $\mathbf{t}(\rho) \in H_T(\text{pt})$. We define $\hbar := \mathbf{t}(\rho)$ and we assume that $\hbar \neq 0$ ¹. Given any framing $\mathbf{f} \in \mathbb{N}^{Q_0}$, we can extend weighting \mathbf{w}' to a weighting $\mathbf{w} : \overline{Q}_{\mathbf{f}1} \rightarrow \mathbb{Z}^r$

¹The assumption $\hbar \neq 0$ is crucial as only in this case MO Yangians are defined.

defined by $\mathbf{w}'(a) = \mathbf{w}(a)$ for all $a \in \overline{Q}_1$ and $\mathbf{w}'(\alpha_{i,m}) = \mathbf{w}(\rho)$, $\mathbf{w}'(\alpha_{i,m}^*) = 0$. Let

$$N_{\mathbf{f}}^{\widetilde{\mathrm{GL}}_{\mathbf{f}}, \zeta^\infty}(Q) := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} H^{\widetilde{\mathrm{GL}}_{\mathbf{f}}}(N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}^{\mathrm{vir}}).$$

be the equivariant cohomology of Nakajima quiver varieties, where $\widetilde{\mathrm{GL}}_{\mathbf{f}} = \mathrm{GL}_{\mathbf{f}} \times T$, and $\mathrm{GL}_{\mathbf{f}} = \prod_{i \in Q_0} \mathrm{GL}_{f_i}$ is the gauge group which reparametrize the framing. Then in [MO19], by developing theory of Stable envelopes, authors define certain morphisms

$$R_{\mathbf{f}_1, \mathbf{f}_2}(Q) \in \mathrm{End}_{H_{\widetilde{\mathrm{GL}}_{\mathbf{f}}}}(N_{\mathbf{f}_1}^{\widetilde{\mathrm{GL}}_{\mathbf{f}_1}, \zeta^\infty}(Q) \otimes_{H_T(\mathrm{pt})} N_{\mathbf{f}_2}^{\widetilde{\mathrm{GL}}_{\mathbf{f}_2}, \zeta^\infty}(Q)) \otimes_{H_{\widetilde{\mathrm{GL}}_{\mathbf{f}}}(\mathrm{pt})} \mathrm{Frac}(H_{\widetilde{\mathrm{GL}}_{\mathbf{f}}}(\mathrm{pt}))$$

for any choice of framing $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{N}^{Q_0}$, where $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$.

By the way the theory is structured, these morphisms automatically satisfy the Yang-Baxter equations. They prove that for any $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3$, we have

$$R_{\mathbf{f}_1, \mathbf{f}_2}^{12}(Q) R_{\mathbf{f}_1, \mathbf{f}_3}^{13}(Q) R_{\mathbf{f}_2, \mathbf{f}_3}^{23}(Q) = R_{\mathbf{f}_2, \mathbf{f}_3}^{23}(Q) R_{\mathbf{f}_1, \mathbf{f}_3}^{13}(Q) R_{\mathbf{f}_1, \mathbf{f}_2}^{12}(Q)$$

Let S be any ring containing \mathbb{Q} . Given a collection of S modules F_i and spectral R matrices $R_{ij} \in \mathrm{End}(F_i \otimes F_j)(u)$, satisfying the Yang-Baxter Equation, there is a general way called the FRT formalism, named after Faddeev, Reshetikhin, and Takhtadzhyan, which constructs a quasi-triangular Hopf algebra which naturally acts on $F_i \otimes S(u)$. This was exploited by Maulik and Okounkov to construct the Maulik-Okounkov Yangian.

For any quiver Q , let

$$A_Q^{T, \zeta^\infty} = \bigoplus_{\mathbf{f} \in \mathbb{N}^{Q_0}} \mathrm{End}_{H_{\widetilde{\mathrm{GL}}_{\mathbf{f}}}}(N_{\mathbf{f}}^{\widetilde{\mathrm{GL}}_{\mathbf{f}}, \zeta^\infty}(Q)).$$

Then the MO Yangians are defined to be a subalgebra

$$\mathbf{Y}_Q^{T, \mathrm{MO}} \subset A_Q^{T, \zeta^\infty}$$

defined by certain operators using above R matrices, as explained in [MO19][Section 5.2.6]. Due to the work of Botta-Davison in [BD23] and Schiffmann-Vasserot in [SV13], there is now an understanding of this subalgebra in terms of cohomological Hall algebras, which we now recall.

12.0.2 Relation with Cohomological Hall algebras

For any Nakajima quiver variety $N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q)$, we have tautological bundles \mathcal{V}_i and \mathcal{W}_i as defined in Section 4.1.3. These defines operators $\mathrm{ch}_l(\mathcal{V}_i), \mathrm{ch}_l(\mathcal{W}_i)$ acting by cup product on there cohomology. We see them as elements inside A_Q^T . In Section 7.3.1, we saw that there is action of $\mathcal{A}_{\widetilde{Q}, \widetilde{W}}^T$ and $\mathcal{A}_{\widetilde{Q}, \widetilde{N}}^T$ on $N_{\mathbf{f}}^{\widetilde{\mathrm{GL}}_{\mathbf{f}}, \zeta^\infty}(Q)$, and

so we have morphisms

$$\rho_Q: \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \chi} \rightarrow A_Q^{T, \zeta^\infty} \quad (12.1)$$

$$\rho_Q^{\mathcal{N}}: \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \widetilde{\mathcal{N}}, \chi} \rightarrow A_Q^{T, \zeta^\infty} \quad (12.2)$$

Assume that the weighting $\mathbf{w} : \overline{Q}_1 \rightarrow \mathbb{Z}^r$ is defined so that it factors through weighting $\mathbf{w}'' : \overline{Q}_1 \rightarrow \mathbb{Z}^2$ given by $\mathbf{w}''(a) = (1, 0)$, $\mathbf{w}''(a^*) = (0, 1)$. Then it is proved in [BD23][Corollary 1.4] that the morphism ρ_Q is an injection, while the same is proved in [SV17] for $\rho_Q^{\mathcal{N}}$. Combining Theorem 4.13 and Lemma 3.18 of [SV23], and Corollary 1.4 of [BD23], it follows that

Theorem 12.0.1. *The Maulik-Okounkov Yangian can be identified by the subalgebra of the cohomology ring of Nakajima quiver variety generated by the image of $\rho_Q, \rho_Q^{\mathcal{N}}$ and tautological operators $\text{ch}_k(\mathcal{V}_i)$ and $\text{ch}_k(\mathcal{W}_i)$. i.e we have*

$$\mathbf{Y}_Q^{T, \text{MO}} = \langle \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^T, \mathcal{A}_{\widetilde{Q}, \widetilde{W}}^{T, \widetilde{\mathcal{N}}}, \text{ch}_k(\mathcal{V}_i), \text{ch}_k(\mathcal{W}_i) \rangle \subset A_Q^{T, \zeta^\infty}$$

as $H_T(\text{pt})$ algebras.

We now specialize to the case when $Q = Q^K$ is a cyclic quiver and torus $T = \mathbb{T}$. By Remark 10.2.7, we may identify $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \widetilde{\mathcal{N}}, \chi}$ with the positive Half of the affine Yangian $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1))$. However, in the view that the action on the Nakajima quiver variety is the right action, it is more natural to identify the nilpotent CoHA with the opposite algebra of the affine Yangian, i.e., with the negative half. This statement is also proved in [Dia+25][Theorem III.7.3]. We have

Proposition 12.0.2. *Let $\ddot{\mathcal{Y}}_{h_1, h_2}^-(\mathfrak{gl}(n))$ be the negative Half of the affine Yangian, i.e. the subalgebra $\langle X_{i,r}^- \mid i \in \mathbb{Z}/n\mathbb{Z}, r \geq 0 \rangle$ of $\mathcal{Y}_{h_1, h_2}^{(n), \text{CoHA}}$ defined in Definition 1.2.2. Then for any $K \geq 1$, we have an isomorphism of algebras $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathbb{T}, \widetilde{\mathcal{N}}} \simeq \ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1))$ such that for any framing \mathbf{f} , this isomorphism induces left action of $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^-(\mathfrak{gl}(K+1))$ on $N_{\mathbf{f}}^{\widetilde{\text{GL}}_{\mathbf{f}}, \zeta^\infty}(Q^K)$.*

In Theorem 10.2.10, we proved that $\mathcal{A}_{\widetilde{Q}^K, \widetilde{W}^K}^{\mathbb{T}}$ is isomorphic to an integral form $\mathcal{Y}_{-t_2, t_1+t_2}^{+, (K+1), \text{CoHA}}$ of the affine Yangian and thus inducing action of the integral form on $N_{\mathbf{f}}^{\widetilde{\text{GL}}_{\mathbf{f}}, \zeta^\infty}(Q^K)$ for any framing \mathbf{f} . It suffices to understand how the positive and negative halves glue, which we can do by the work of Varagnolo.

12.0.3 Varagnolo Action of Yangian

We consider the action of the full affine Yangian on the Nakajima quiver variety, due to Varagnolo [Var00], which is based on work of Nakajima in the context of equivariant K theory. In the case of two-parameter deformation and $K = 1$, this action was also considered in [Kod19] and in [Dia+25]. The action is defined by convolution, in a similar way to Nakajima's action of Kac-Moody Lie algebras we considered earlier.

Let \mathbf{f} be any framing vector. Given two dimension vectors \mathbf{d}_1 and \mathbf{d}_2 ,
Let $\mathbf{d}_2 = \mathbf{d}_1 + \delta_i$ for some $i \in Q_0$. Then we define subvariety $C_i^+(\mathbf{d}, \mathbf{f}) \subset N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q) \times N_{\mathbf{d}+\delta_i, \mathbf{f}}^{\zeta^\infty}(Q)$ given by pairs $(\rho_1(a), \rho_1(a^*), \rho_1(\alpha_{i,l}), \rho_1(\alpha_{i,l}^*))$ and $(\rho_2(a), \rho_2(a^*), \rho_2(\alpha_{i,l}), \rho_2(\alpha_{i,l}^*))$ such that the representation ρ_2 is quotient of representation ρ_1 . It is proved in [Nak01a] that these subvarieties are Lagrangian and smooth. We can then define line bundle \mathcal{L}_i on $C_i^+(\mathbf{d}, \mathbf{f})$ by $p_2^* \mathcal{V}_i / p_1^* \mathcal{V}_i$ where $p_1 : C_i^+(\mathbf{d}, \mathbf{f}) \rightarrow N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q)$ and $p_2 : C_i^+(\mathbf{d}, \mathbf{f}) \rightarrow N_{\mathbf{d}+\delta_i, \mathbf{f}}^{\zeta^\infty}(Q)$ are the projection maps and \mathcal{V}_i are the tautological bundles at vertex i . Then for any \mathbf{d} , one define morphisms $x_{i,r}^+ : H^{\widetilde{\text{GL}}_f}(N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}^{\text{vir}}) \rightarrow H^{\widetilde{\text{GL}}_f}(N_{\mathbf{d}+\delta_i, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}^{\text{vir}})$ by $(-1)^{(\delta_i, \mathbf{d})} (p_2)_* ((c_1(\mathcal{L}_i)^r \cup p_1^*(\alpha)))$ and $x_{i,r}^- : H^{\widetilde{\text{GL}}_f}(N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}^{\text{vir}}) \rightarrow H^{\widetilde{\text{GL}}_f}(N_{\mathbf{d}-\delta_i, \mathbf{f}}^{\zeta^\infty}(Q), \mathbb{Q}^{\text{vir}})$ by $(-1)^{(\delta_i, \mathbf{d}-\delta_i)} (p_1)_* ((c_1(\mathcal{L}_i)^r \cup p_2^*(\alpha)))$.
Given any (\mathbf{d}, \mathbf{f}) and $i \in [0, K]$, $s \in \mathbb{Z}_{\geq 0}$. let $h_{i,s}$ be the element in the tautological ring of $N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q)$ defined by

$$\text{Res}_{z=0} \left(z^s \left(-1 + \frac{\lambda_{-1/z}(\mathcal{F}_i(\mathbf{d}, \mathbf{f}))}{\lambda_{-1/z}(q_1 q_2 \mathcal{F}_i(\mathbf{d}, \mathbf{f}))} \right) \right) / (t_1 + t_2)$$

where $\mathcal{F}_i(\mathbf{d}, \mathbf{f})$ is the element in the equivariant K theory $K^{T \times \text{GL}_f}(N_{\mathbf{d}, \mathbf{f}}^{\zeta^\infty}(Q))$ given by

$$\mathcal{F}_i(\mathbf{d}, \mathbf{f}) = (q_1 q_2)^{-1} \mathcal{W}_i - (1 + (q_1 q_2)^{-1}) \mathcal{V}_i + q_1^{-1} \mathcal{V}_{i-1} + q_2^{-1} \mathcal{V}_{i+1}$$

Here q_1, q_2 are line bundles associated to action of \mathbb{T} , i.e $c_1(q_1) = t_1$ and $c_1(q_2) = t_2$ and $\lambda_{1/z}(V)$ is the equivariant chern polynomial $\lambda_{1/z}(V) = \sum_{i \geq 0} c_i(V) / z^i$. Then we have

Theorem 12.0.3 ([Var00]). *For any framing \mathbf{f} , there is an action of affine Yangian $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}(\mathfrak{gl}(n))$ on $N_{\mathbf{f}}^{\widetilde{\text{GL}}_f, \zeta^\infty}(Q^K)$, given by $X_{i,r}^\pm \mapsto x_{i,r}^\pm$ and $H_{i,s} \mapsto h_{i,s} \cup$*

Thanks to Theorem [YZ18a][Theorem 5.6], we can see the operators $x_{i,r}^\pm$ as an action of the spherical part of CoHA.

Proposition 12.0.4. *Under the morphism $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1)) \rightarrow \mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathbb{T}, \chi}$, the Varagnolo's action of $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^+(\mathfrak{gl}(K+1))$ on the $N_{\mathbf{f}}^{\widetilde{\text{GL}}_f, \zeta^\infty}(Q^K)$ coincides with $\mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathbb{T}, \chi}$ action defined in Section 7.3. Similarly the Varagnolo's action of $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^-(\mathfrak{gl}(K+1))$ can be identified with the action of $\mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathbb{T}, \mathcal{N}, \chi}$ under the isomorphism $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^-(\mathfrak{gl}(K+1)) \rightarrow \mathcal{A}_{Q^K, \widetilde{W^K}}^{\mathbb{T}, \mathcal{N}, \chi}$.*

Furthermore, by [Dia+25][Theorem III.7.8], the subalgebra $\langle \text{ch}_k(\mathcal{V}_i), \text{ch}_k(\mathcal{W}_i) \rangle \subset A_{Q^K}^{\mathbb{T}}$, generated by tautological classes is isomorphic to the subalgebra $\langle h_{i,s}, \text{ch}_k(\mathcal{W}_i) \rangle \subset A_{Q^K}^{\mathbb{T}}$. This is simply because universal polynomials exist that relate the Chern polynomial and Chern characters.

Let $\overline{\mathcal{Y}}_{\hbar_1, \hbar_2}^{n, \text{CoHA}}$ be the subalgebra of $\mathcal{Y}_{\hbar_1, \hbar_2}^{n, \text{CoHA}}$ which is generated by generators $\langle \mathfrak{K}_+^{(r)}, X_{i,r}^\pm, H_{i,r} \mid i \in \mathbb{Z}/n\mathbb{Z}, r \geq 0 \rangle$ (So we only exclude the generators $\mathfrak{K}_-^{(r)}$ for $r \geq 0$). Let $\overline{\mathcal{Y}}_{\hbar_1, \hbar_2}^{n, \text{CoHA}, \mathfrak{c}} = \overline{\mathcal{Y}}_{\hbar_1, \hbar_2}^{n, \text{CoHA}}[\langle k_{i,r} \rangle]$ be the extended algebra, where we have added central elements $\langle k_{i,r} \mid i \in \mathbb{Z}/n\mathbb{Z}, r \geq 0 \rangle$ to the list of generators.

Since the cup product with tautological classes $\text{ch}_k(\mathcal{W}_i)$ commutes, it follows from Theorem 12.0.1 that there is a $\mathbb{C}[t_1, t_2]$ linear algebra surjection

$$\Psi : \overline{\mathcal{Y}}_{-t_2, t_1+t_2}^{K+1, \text{CoHA}, \epsilon} \rightarrow Y_{Q^K}^{\text{MO}, \mathbb{T}} \subset A_{Q^K}^{\mathbb{T}}$$

where the morphism sends $k_{i,r} \mapsto \text{ch}_r(\mathcal{W}_i)$.

Theorem 12.0.5. *The morphism Ψ is an isomorphism of $\mathbb{C}[t_1, t_2]$ linear algebras.*

Proof. It suffices to show that this morphism is an injection. The affine Yangian is a free $\mathbb{C}[t_1, t_2]$ module, and so does the Maulik-Okounkov Yangian by [MO19][Theorem 5.5.1]. Thus, it suffices to show that:

$$\Psi : (\overline{\mathcal{Y}}_{-t_2, t_1+t_2}^{K+1, \text{CoHA}, \epsilon})_{\mathbf{K}} \rightarrow (Y_{Q^K}^{\text{MO}, \mathbb{T}})_{\mathbf{K}}$$

is an isomorphism, where for any algebra A , $A_{\mathbf{K}}$ denote the localization $A \otimes_{\mathbb{C}[t_1, t_2]} \mathbb{C}(t_1, t_2)$. Clearly $(\overline{\mathcal{Y}}_{-t_2, t_1+t_2}^{K+1, \text{CoHA}, \epsilon})_{\mathbf{K}} = (\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{\epsilon}(K+1))_{\mathbf{K}}$, where $\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{\epsilon}(K+1)$ denote the extended affine Yangian (We add central elements $k_{i,r}, i \in [0, K], r \geq 0$). Let $(\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{\ddot{0}, \epsilon}(K+1))_{\mathbf{K}}$ be the subalgebra generated by generators $H_{i,s}$ and $k_{i,l}$. Then it shown in [Dia+25][Theorem III.5.6], that the extended affine Yangian $(\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{\epsilon}(K+1))_{\mathbf{K}}$ has a triangular decomposition, i.e the morphism

$$m : (\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{+}(K+1))_{\mathbf{K}} \otimes (\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{\ddot{0}, \epsilon}(K+1))_{\mathbf{K}} \otimes (\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{-}(K+1))_{\mathbf{K}} \rightarrow (\ddot{\mathcal{Y}}_{-t_2, t_1+t_2}^{\epsilon}(K+1))_{\mathbf{K}}$$

is an isomorphism of vector spaces. It is proved in [SV23] that the Maulik-Okounkov Yangian also admits a triangular decomposition, and so we have

$$m : (Y_{Q^K}^{\text{MO}, \mathbb{T}, +})_{\mathbf{K}} \otimes (Y_{Q^K}^{\text{MO}, \mathbb{T}, 0})_{\mathbf{K}} \otimes (Y_{Q^K}^{\text{MO}, \mathbb{T}, -})_{\mathbf{K}} \rightarrow (Y_{Q^K}^{\text{MO}, \mathbb{T}})_{\mathbf{K}},$$

such that there is isomorphism of algebras $(Y_{Q^K}^{\text{MO}, \mathbb{T}, +})_{\mathbf{K}} \simeq (\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, +})_{\mathbf{K}}$ and $(Y_{Q^K}^{\text{MO}, \mathbb{T}, -})_{\mathbf{K}} \simeq (\mathcal{A}_{\widetilde{Q^K}, \widetilde{W^K}}^{\mathbb{T}, -})_{\mathbf{K}}$ and $(Y_{Q^K}^{\text{MO}, \mathbb{T}, 0})_{\mathbf{K}} \simeq \langle \text{ch}_k(\mathcal{V}_i), \text{ch}_k(\mathcal{W}_i) \rangle$. Thus, the morphism $\Psi_{\mathbf{K}}$ restricted to each of the parts of the triangular decomposition is an isomorphism, and so we are done by the following lemma. \square

Lemma 12.0.6. *Suppose $F : A \rightarrow B$ a morphism of algebras where both A and B admits a triangular decomposition $A^+ \otimes A^0 \otimes A^- \xrightarrow{m} A$, $B^+ \otimes B^0 \otimes B^- \xrightarrow{m} B$. Suppose that F restricts to isomorphism of algebras $F^+ : A^+ \rightarrow B^+$, $F^0 : A^0 \rightarrow B^0$ and $F^- : A^- \rightarrow B^-$ then F is an isomorphism of algebras.*

Proof. We have commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ m_A \uparrow & & \uparrow m_B \\ A^+ \otimes A^0 \otimes A^- & \xrightarrow{F^+ \otimes F^0 \otimes F^-} & B^+ \otimes B^0 \otimes B^- \end{array}$$

and thus F is isomorphism of vector spaces. Since F is a morphism of algebras, we are done. \square

Chapter 13

Relation with algebras of Gaiotto-Rapčák-Zhou

In [Rap+23], a procedure to do the Drinfeld double of the spherical part of CoHA is explained. Then, a certain quotient of the Drinfeld double gives rise to *reduced* Drinfeld double [YZ18a][Section 4.1]. Mimicking the same construction of reduced Drinfeld double to $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), +, \text{CoHA}}$ yields the algebra $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), \text{CoHA}}$ for $n > 2$, defined in Definition 1.2.2¹. Then in an analogous way to [Gua07], we define

Definition 13.0.1 (CoHA Loop Yangian). Let $\mathcal{L}_{\hbar_1, \hbar_2}^{(n), \text{CoHA}}$ be the $\mathbb{C}[\hbar_1, \hbar_2]$ algebra, defined to be the quotient $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), \text{CoHA}} / (\mathbf{c})$ where $\mathbf{c} := \sum_{i=0}^{n-1} H_{i,0}$

We then have a doubled version of the Proposition 11.0.2, which follows from work of Tsymbaliuk [Tsy17a] who calculated the classical limit of affine Yangians.

Proposition 13.0.2. *We have an isomorphism of $\mathbb{C}[\hbar]$ linear algebras*

$$\mathcal{L}_{\hbar, \hbar_2}^{(n), \text{CoHA}} / (\hbar_2) \simeq \mathbf{U}_{\mathbb{C}[\hbar]}(\mathcal{W}_n).$$

Proof. We use the morphism considered in [Tsy17a][Theorem 2.2]² whose positive half is the morphism we considered in Theorem 11.0.2 i.e., we define a morphism of algebras

$$\Psi : \mathcal{L}_{\hbar, \hbar_2}^{(n), \text{CoHA}} / (\hbar_2) \rightarrow \mathbf{U}_{\mathbb{C}[\hbar]}(\mathcal{W}_n)$$

¹More precisely, we apply the construction to the localized algebra (which is spherically generated due to Proposition 10.2.2.2 and 10.2.6) and take the minimal subalgebra containing both $\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), +, \text{CoHA}}$ and $(\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), -, \text{CoHA}}) := (\mathcal{Y}_{\hbar_1, \hbar_2}^{(n), +, \text{CoHA}})^{\text{op}}$.

²The codomain of the morphism considered in [Tsy17a][Theorem 2.2] is a central extension, however in the proof, they consider the morphism induced after quotienting by the central extension.

which sends

$$\begin{aligned}
X_{0,r}^+ &\mapsto n^r D^r z \otimes E_{n,1} \\
X_{i,r}^+ &\mapsto n^r (D + (1 - i/n)\hbar)^r \otimes E_{i,i+1} \\
X_{0,r}^- &\mapsto n^r (z^{-1} D^r) \otimes E_{1,n} \\
X_{i,r}^- &\mapsto n^r (D + (1 - i/n)\hbar)^r \otimes E_{i+1,i} \\
H_{0,r} &\mapsto n^r (D^r \otimes E_{n,n} - (D + \hbar)^r \otimes E_{1,1}) \\
H_{i,r} &\mapsto n^r (D + (1 - i/n)\hbar)^r \otimes (E_{i,i} - E_{i+1,i+1}).
\end{aligned}$$

We haven't defined image of $\mathfrak{K}_{\pm}^{(r)}$, but note that we can still calculate image of elements \mathfrak{Z}_n , and we can check that this map must send $\mathfrak{K}_+^{(0)}$ to $\frac{z}{\hbar} = T_{1,0}(1)/\hbar = t_{1,0}$ while $\mathfrak{K}_-^{(0)}$ to $-\frac{z^{-1}}{\hbar} = -T_{-1,0}(1)/\hbar = -t_{-1,0}$ which allows to extend define using the same trick as in proof of Proposition 11.0.2. After inverting \hbar , this map is an isomorphism by [Tsy17a][Theorem 2.2] while it is a surjection, since again $D_{\hbar}(\mathbb{C}^*) \otimes \widetilde{\mathfrak{gl}}_n$ is generated by $t_{1,r}, t_{-1,r}, T_{-1,r}(E_{1,n}), T_{0,r}(E_{i,i+1}), T_{0,r}(E_{i+1,i}), T_{1,r}(E_{n,1})$ all of which can be easily seen to be in the image. \square

We consider the $\mathbb{C}[\hbar_1, \hbar_2]$ linear algebras³ Y^K and its loop version L^K , as defined in [GRZ23][Definition 7.01, 6.02]. By [GRZ23][Remark 6.18], the algebra L^K satisfies

$$L^K/\hbar_2 \simeq \mathbf{U}_{\mathbb{C}[\hbar]}(\mathcal{W}_K),$$

which by Propostion 13.0.2 is also satisfied by $\mathcal{L}_{\hbar_1, \hbar_2}^{(n), \text{CoHA}}$. They also show that the algebras Y^K and L^K are the integral forms of the affine Yangian $\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K))$ and the loop Yangian $\check{\mathcal{L}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K)) := \ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(K))/\mathbf{c}$ respectively. For $K = 1$, Y^K and L^K are in fact isomorphic to the affine Yangian $\ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}(\mathfrak{gl}(1))$ of $\widehat{\mathfrak{gl}(1)}$ and its loop version respectively [GRZ23][Theorem 17]. Furthermore, it is known due to ([Dav22a]+ [Rap+23]) and [SV13] that

$$\mathcal{A}_{\widetilde{Q}_{\text{Jor}}, \widetilde{W}}^{\mathbb{T}} \simeq \ddot{\mathcal{Y}}_{\hbar_1, \hbar_2}^+(\mathfrak{gl}(1)).$$

We thus expect that there is an isomorphism of algebras for all $K > 2$,

$$Y^K \simeq \mathcal{Y}_{\hbar_1, \hbar_2}^{(K), \text{CoHA}}$$

³We are switching ϵ_1 to \hbar_2 and ϵ_2 to \hbar_1 .

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