Contents.

Pages 1-34 contain an elementary introduction on Matrices. It includes certain defin--itions and known theorems to which reference will be made in the course of the research commencing on p. 35.

Preface on Matrices.

We shall define a *Matrix* as being a set of numbers arranged in a rectangular array, and shall write it thus:-

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \qquad (1)$$

In this array, there are m rows and n columns, forming the m by n matrix (1).

When $m = n$, we get the square matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad (2)$$

This is a matrix of the n^{th} order, and it may be briefly referred to as A or (a_{ik}).

The numbers a_{ik} are called the "elements" of the matrix, a_{ik} being the one in row i and column k of the array.

They are regarded as having been drawn from some algebraic field specified in advance e.g. the field of positive integers, or of rational numbers, or of real numbers, or of complex numbers, and so on, and may be treated as either constants or variables.

A simple example of a matrix is provided by a set of co-ordinates in geometry: (x_1, y_1, z_1). This represents a matrix of a single row. Further, as $(1, 2, 5)$ and $(2, 1, 5)$ would denote two distinct points referred to three Cartesian Axes OX, OY, OZ, we agree to consider them as two distinct matrices.

In what follows we shall only be concerned with matrices like (2) so that the word "matrix" always means the <u>square</u> matrix. Rectangular matrices may be made square by affixing, where necessary, rows or columns of zeros. For example, we lay down the following equalities

$$\begin{pmatrix} 1 & 3 & 2 & 4 \\ 6 & 3 & 4 & 5 \\ 0 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 6 & 3 & 4 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $$\begin{pmatrix} 1 & 3 \\ 5 & 4 \\ 0 & 2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 3 & 0 \\ 5 & 4 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

always extending the rectangular array to the right or downwards when forming the square.

Cayley first saw the value of treating such an array of rows and columns as one number — a whole quantity or generalized number — which need seldom be decomposed into its elements or components, thus extending the underlying idea of analytical geometry and the method of vectors. In this way he was led to develop their properties on the basis of the first five laws of Algebra, excluding the sixth. Matrices can, in fact, be denoted by letters A, B, C which undergo two fundamental operations + and ×, and satisfy the laws :-

$$A + (B + C) = (A + B) + C,$$
$$(A \times B) \times C = A \times (B \times C),$$
$$A \times (B + C) = (A \times B) + (A \times C),$$
$$(A + B) \times C = (A \times C) + (B \times C),$$
$$A + B = B + A,$$

but not necessarily the 6th law,

$$AB = A \times B = BA.$$

Particular matrices may satisfy this 6th law, in which case they are said to be commutable with each other. Algebra so restricted is called "Linear Commutative Algebra".

Briefly, these laws are as follows:-

<u>Addition</u> is defined by the rule

$$A + B = (a_{ik} + b_{ik}), \quad \text{where } A = (a_{ik}),$$

$B = (b_{ik})$.

Thus, two matrices are added together by adding their corresponding elements, e.g.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} f & g \\ h & k \end{pmatrix} = \begin{pmatrix} a+f & b+g \\ c+h & d+k \end{pmatrix}.$$

<u>Subtraction</u> is effected by defining

$A - B$ as $(a_{ik} - b_{ik})$.

By considering $A+A$, $A+A+A$, and

Turnbull, <u>The Theory of Determinants, Matrices and Invariants</u>. p. 34, par. 5; p. 57, par. 1.

so on, multiplication of A by a scalar number λ may be defined by

$$\lambda A = (\lambda a_{ik})$$

e.g. $$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

<u>Linear combination</u> of matrices with <u>scalar coefficients</u> is defined, e.g., by

$$\lambda A + \mu B + \nu C = (\lambda a_{ik} + \mu b_{ik} + \nu c_{ik}).$$

<u>Multiplication</u> is defined by the following rule :-
From two matrices $A\ (a_{ik})$ and $B\ (b_{ik})$, each with n rows and columns, we form a new matrix $C\ (c_{ik})$ of the same order n by taking

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}.$$

e.g., if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$,

$$C = AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$ so that

in forming the product AB we multiply rows of A into columns of B.

Two matrices being called "equal" or "identical" when their corresponding elements are equal, AB is in general a different matrix from BA, for

$$BA = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

In the product AB, B is said to be pre-multiplied by A and A to be post-multiplied by B.

It can be seen now that $ABC \ldots K = A(BC \ldots K) = (AB)(C \ldots K)$, — in fact that multiplication of any finite number of matrices gives the same final result, irrespective of the grouping of the factors, <u>provided the order be retained.</u>

The rule for multiplication appears natural enough when we consider the matrix as typifying a linear transformation.

Suppose we take 3 variables x_1, x_2, x_3 and transform them into new variables y_1, y_2, y_3 by means of the first set of equations below, typified by the matrix of the coefficients, $(a_{11}\, a_{22} \cdots a_{nn})$. Further, suppose the y's are then changed into z's by means of the second set of equations.

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}\right\} \cdots (1)$$

,

$$\left.\begin{aligned} z_1 &= b_{11}y_1 + b_{12}y_2 + b_{13}y_3 \\ z_2 &= b_{21}y_1 + b_{22}y_2 + b_{23}y_3 \\ z_3 &= b_{31}y_1 + b_{32}y_2 + b_{33}y_3 \end{aligned}\right\} \cdots (2)$$

.

Then the z's are given in terms of the y's which themselves are given in terms of the x's. Now the z's might have been given directly in terms of the x's by the equations:-

$$z_1 = (b_{11}a_{11} + b_{12}a_{21} + b_{13}a_{31})x_1 + (b_{11}a_{12} + b_{12}a_{22} + b_{13}a_{32})x_2 + (b_{11}a_{13} + b_{12}a_{23} + b_{13}a_{33})x_3 ,$$

$$z_2 = (b_{21}a_{11} + b_{22}a_{21} + b_{23}a_{31})x_1 + (b_{21}a_{12} + b_{22}a_{22} + b_{23}a_{32})x_2$$
$$+ (b_{21}a_{13} + b_{22}a_{23} + b_{23}a_{33})x_3 ,$$

$$z_3 = (b_{31}a_{11} + b_{32}a_{21} + b_{33}a_{31})x_1 + (b_{31}a_{12} + b_{32}a_{22} + b_{33}a_{32})x_2$$
$$+ (b_{31}a_{13} + b_{32}a_{23} + b_{33}a_{33})x_3 ,$$

corresponding to a matrix

$$\begin{pmatrix} b_{11}a_{11}+b_{12}a_{21}+b_{13}a_{31} & b_{11}a_{12}+b_{12}a_{22}+b_{13}a_{32} & b_{11}a_{13}+b_{12}a_{23}+b_{13}a_{33} \\ b_{21}a_{11}+b_{22}a_{21}+b_{23}a_{31} & b_{21}a_{12}+b_{22}a_{22}+b_{23}a_{32} & b_{21}a_{13}+b_{22}a_{23}+b_{23}a_{33} \\ b_{31}a_{11}+b_{32}a_{21}+b_{33}a_{31} & b_{31}a_{12}+b_{32}a_{22}+b_{33}a_{32} & b_{31}a_{13}+b_{32}a_{23}+b_{33}a_{33} \end{pmatrix}$$

The last matrix, according to the rule for multiplication, is the product matrix BA, in which the <u>order</u> of the factor matrices must be carefully noted. The operation of transforming the x's into y's might have been symbolised by

$$y = Ax$$

and that of transforming the y's into z's by

$$z = By .$$

Then $By = B(Ax)$, which we agree to

write $(BA)x$.

We thus see that matrices differ from ordinary numbers in that they do not obey the commutative law of multiplication, which important difference causes a greater complexity in their algebra.

In every system of numbers there are two peculiar numbers with unique properties — zero and unity. We define the <u>zero matrix</u> or the <u>null matrix</u> to be such that all its elements are zero. When it either pre- or post-multiplies any matrix A, the resulting product is the zero matrix.

The <u>unit matrix</u>, written "$\underline{I}$" (in German books, "E") has unit elements in the principal diagonal, but zeros everywhere else.

It is easy to see that

$$\underline{I}A = A\underline{I} = A$$

for any matrix A.

We now have all the material for the algebra of matrices!

Associated with each matrix A is the "determinant of the matrix", $|A|$. Many of the properties of A depend on those of $|A|$. When $|A| = 0$, A will be called a "vacuous" matrix, or (following Bôcher and others) a "singular" matrix. This "singularity" of A is graded into n ranks, denoted by $0, 1, 2, \dots (n-1)$, n denoting the rank of the non-singular matrix. If all the determinants of order $(r+1)$ formed from a matrix are zero, and one at least of the determinants of order r is not zero, then the matrix is said to be of "rank" r. The order minus the rank, i.e. $n-r$, was called by Sylvester the "nullity" of the matrix. e.g., for a 3rd order matrix these ranks are illustrated by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}.$$

A is of rank 0, B is of rank 1, C is of rank 2, D is of rank 3.

Since the rule for multiplying

matrices is also one of the rules for multi-plying determinants, it follows that the determinant of the product of a number of matrices is equal in value to the product of the determinants of the separ-ate matrices, or

$$|A. B. C \ldots K| = |A|.|B|.|C| \ldots |K|.$$

A set of equations of linear trans-formation, e.g.

$$y_1 = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n,$$

$$y_2 = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$y_n = a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n.$$

can be solved for the x's, provided $|A| \neq 0$. Denoting the cofactor of a_{pq} in $|A|$ by A_{pq}, then we can show that

$$x_1 = \frac{A_{11}}{|A|} y_1 + \frac{A_{21}}{|A|} y_2 + \cdots + \frac{A_{n1}}{|A|} y_n,$$

$$x_2 = \frac{A_{12}}{|A|} y_1 + \frac{A_{22}}{|A|} y_2 + \cdots + \frac{A_{n2}}{|A|} y_n,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_n = \frac{A_{1n}}{|A|} y_1 + \frac{A_{2n}}{|A|} y_2 + \cdots + \frac{A_{nn}}{|A|} y_n.$$

The matrix $\left(\frac{A_{11}}{|A|}, \frac{A_{22}}{|A|}, \ldots \frac{A_{nn}}{|A|}\right)$ of the above system is called the <u>reciprocal</u> of A, and it is convenient to denote it by A^{-1}. The name is justifiable because the following relations can be proved to be true, by means of a well-known theorem in determinants due to Cauchy:-

$$AA^{-1} = A^{-1}A = I \text{, the unit matrix.}$$

So, symbolically, equations of linear transformation

$$y = Ax$$

imply $\quad x = A^{-1}y$.

The reciprocal of a product of matrices, e.g. ABC, is the product of their reciprocals in the reversed order: for

$$ABCC^{-1}B^{-1}A^{-1} = ABB^{-1}A^{-1} = AA^{-1} = \underline{I};$$

and $(ABC)(ABC)^{-1} = \underline{I}$.

$$\therefore \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

Since the multiplication of matrices is not in general commutative, it is necessary to distinguish the corresponding two kinds of division; thus the quotient of B by A may be either $A^{-1}B$ or BA^{-1}. It is possible to give a unique meaning to this quotient only when $AB = BA$, for then $A^{-1}BA = B$, and so $A^{-1}B = BA^{-1}$, which may then be written $\frac{B}{A}$. e.g. the fractional notation is legitimate in $\frac{1-A}{1+A}$ for we can prove at once that

$$(1+A)^{-1}(1-A) = (1-A)(1+A)^{-1}.$$

Again, corresponding to a polynomial in ordinary algebra,

$$g(x) = c_0 x^0 + c_1 x^1 + c_2 x^2 + \cdots + c_n x^n$$

we may have a matrix, or matrix poly-nomial, obtained by summing the separate matrix-terms of

$$g(A) = c_0 A^0 + c_1 A^1 + c_2 A^2 + \cdots + c_n A^n,$$

where A is a given matrix and $c_0, c_1, \ldots c_n$ are scalar numbers.

Since $A^r A^s = A^{r+s} = A^s A^r$,

we see that any two polynomials in A, say $g(A)$ and $h(A)$, are commut-able with each other, $\therefore \frac{g(A)}{h(A)}$, which is called a <u>rational function</u> of the matrix, has a unique meaning so long as $h(A)$ is not a singular matrix.

Because, if A be any matrix, and $|A|$ vanishes, we cannot form the reciprocal matrix A^{-1}, any more than we can deal with the scalar number a^{-1} when a is zero.

Although the matrix has no reciprocal, there still exists the matrix of elements A_{qp}

i.e. $$\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

which has certain properties analagous to those of the reciprocal matrix, and it is called the <u>Adjugate Matrix</u>.

The product of A and its adjug--ate gives

$$\begin{pmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{pmatrix}$$

$$= |A|\,\underline{I}.$$

References.

Turnbull, *The Theory of Determinants, Matrices and Invariants*, Blackie.
Ch. I ; Ch. III, pars. 5 and 6 ; Ch. IV ;
Ch. V, par. 1.

Dickson, *Modern Algebraic Theories*, (Chicago, 1926),
Ch. III.
Linear Algebras, Cambridge Tract,
No. 16, 1914.

Cullis, *Matrices and Determinoids*, (Cam-
-bridge, 1913), Vol. I.

Characteristic Function: Characteristic Equation: Latent Roots and Cayley - Hamilton Theorem.

If we have a matrix A of the nth order, the matrix $\lambda I - A$, where λ is a scalar, plays an important part in this theory.

The determinant of $\lambda I - A$, $|\lambda I - A|$, is a polynomial in λ of degree n, called the characteristic function of A. When equated to zero it gives rise to the characteristic equation, the n roots of which, say $\lambda_1, \lambda_2, \ldots \lambda_n$, were called by Sylvester the latent roots of A.

The properties of A are intimately related to its latent roots — their multiplicities, their absolute values, whether any vanish, and so on — very much as the properties of polynomials are related to the roots of the algebraic equation obtained by equating the polynomial to zero.

The fundamental properties of matrices are not many, one of the chief being that any matrix A, say,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

satisfies a characteristic equation (the Cayley-Hamilton equation) of its own order, namely :-

$$\begin{vmatrix} a_{11}-A & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-A & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-A \end{vmatrix} = 0. \qquad *$$

For a second order matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the characteristic equation is a quadratic

$$f(\lambda) = \begin{vmatrix} \lambda-a & -b \\ -c & \lambda-d \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

If we construct the corresponding function

$f(A)$ of the matrix A itself by evaluating $A^2 - (a+d)\,A + (ad - bc)\,I$ as a second order matrix, we obtain the result that all the elements of this matrix are zero.*

* The Theory of Determinants, Matrices, and Invariants, Turnbull, pp. 98-100.
Modern Algebraic Theories, Dickson, p. 47, par. 25.

The Fundamental Reduction of A to the form BCB^{-1}, where C has the same Latent Roots as A.

When the latent roots of a matrix A of order n are all different, say $\lambda_1, \lambda_2, \ldots \lambda_n$, an important theorem is that A can be expressed as a product BCB^{-1} where B is non-singular and C is a purely diagonal matrix,

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \lambda_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \lambda_n \end{pmatrix},$$

obviously having the same characteristic equation and latent roots as A.* C may be called the "normal form" of A.

[The case of the <u>singular</u> matrix A will be omitted from the following discussion as it would require special treatment.]

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* Turnbull, <u>The Theory of Determinants, Matrices, and Invariants.</u> p. 296.

When the latent roots of the matrix A are not all distinct, resolution in the precise form BCB^{-1} is not always possible. The original investigation of this case was due to Weierstrass. We will just mention the main indications.

The form of C now depends on what are called the "elementary divisors" of the characteristic determinant of A, $|\lambda \underline{I} - A|$. Let two matrices P and Q be called <u>similar</u> when they are connected by a relation

$$P = HQH^{-1},$$

H being non-singular. We must enquire what relations of invariancy exist between P and Q.

In the first place, the ranks of P and Q are the same. For it can be proved that the rank of a product of matrices cannot exceed the least rank in its factors,* so if Q is of order n and rank r, the rank of P cannot be greater than that of Q.

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* Dickson, <u>Modern Algebraic Theories</u>, (Chicago, 1926), pp. 49-51. Turnbull, <u>The Theory of Determinants, Matrices and Invariants</u>, (Blackie, 1928), p. 84, par. 7.

Similarly, since $Q = H^{-1} P H$, the rank of Q cannot be greater than that of P. Hence similar matrices have the same rank.
Secondly, P and Q have the same <u>characteristic or latent function</u>, for

$$\lambda \underline{I} - P = H \lambda H^{-1} - H Q H^{-1}$$
$$= H(\lambda \underline{I} - Q) H^{-1}$$

and so

$$|\lambda \underline{I} - P| = |\lambda \underline{I} - Q|.$$

Lastly, P and Q have the same "elementary divisors" which will now be defined.

Examine the determinant $|\lambda \underline{I} - P|$. When evaluated it gives the characteristic function, a polynomial $f(\lambda)$ with possibly sets of multiple roots,

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_K)(\lambda - \lambda_K) \cdots (\lambda - \lambda_r).$$

Let us suppose the root λ_k is of multiplicity p, so $(\lambda - \lambda_K)$ occurs to the power p in the whole determinant under examination. We next examine all the first minors of $|\lambda \underline{I} - P|$ and find the smallest power of $(\lambda - \lambda_K)$ occurring in any of these, i.e. the power of $(\lambda-\lambda_K)$ in the <u>highest common factor</u> of these minors. Let this power be p_1. Then examine the second minors in the same way, the power found being p_2, and so on. We thus obtain a sequence of exponents pertaining to the root λ_K, namely

$$p, \; p_1, \; p_2, \; \dots \; p_{n-1},$$

which are called the "exponents of the <u>invariant factors</u> of the characteristic determinant of A belonging to the root λ_K." The first differences of these p's, —

$$e_1 = p - p_1, \quad e_2 = p_1 - p_2, \quad e_3 = p_2 - p_3, \; \dots \text{—}$$

are called the "exponents of the <u>elementary divisors</u> of the characteristic determinant of A."

The part of Q referring to λ_K may now be formulated.

Supposing $p = 7$ and $e_1 = 3$, $e_2 = 2$, $e_3 = 1$, $e_4 = 1$, it is

$$\begin{pmatrix} \lambda_K & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_K & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_K & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_K \end{pmatrix}$$

In this case, as in that of distinct latent roots, the zeros occur below the principal diagonal, and the latent roots in the diagonal. The diagonal submatrices are enclosed by dotted lines. They are of orders e_1, e_2, e_3, e_4 or 3, 2, 1, 1, and we notice that there are 1's in the super-principal diagonal of each submatrix, with zeros everywhere else.

The parts of $\mathfrak{g}$ for the other roots are constructed according to the exponents of their elementary divisors, just as in the case of λ_K, and by joining these together the complete matrix Q or "normal form" of P is obtained.

All the minors of order r in $|\lambda \underline{I} - P|$ will, when evaluated, be polynomials in λ of degrees up to the rth. Let the highest common factor of these polynomials, when resolved into factors, be

$$(\lambda-\lambda_1)^{p_{r,1}} (\lambda-\lambda_2)^{p_{r,2}} \; . \; . \; . \; . \; (\lambda-\lambda_k)^{p_{r,k}} ,$$

$\lambda_1, \lambda_2, \ldots . \lambda_k$ denoting the latent roots of P. We then obtain evaluations like this :-

Det. $|\lambda \underline{I} - P|$:- $(\lambda-\lambda_1)^{p_{n,1}} (\lambda-\lambda_2)^{p_{n,2}} \ldots (\lambda-\lambda_k)^{p_{n,k}}$,

H.C.F of 1st Minors :- $(\lambda-\lambda_1)^{p_{n-1,1}} (\lambda-\lambda_2)^{p_{n-1,2}} \ldots (\lambda-\lambda_k)^{p_{n-1,k}}$,

H.C.F. of 2nd Minors:- $(\lambda-\lambda_1)^{p_{n-2,1}} (\lambda-\lambda_2)^{p_{n-2,2}} \ldots (\lambda-\lambda_k)^{p_{n-2,k}}$,

and so on.

Corresponding to each latent root, e.g. λ_1 , we obtain a sequence of exponents for the successive descending orders of minors, e.g. $p_{n,1}$, $p_{n-1,1}$, ... giving

$$e_{n,1} = p_{n,1} - p_{n-1,1} ,$$
$$e_{n,2} = p_{n-1,1} - p_{n-2,1} , \text{ etc.}$$

These e's can be shown to be the same for Q as for P, in fact they are invariant for similar matrices.

The polynomials

$$(\lambda-\lambda_1)^{e_{r,1}} (\lambda-\lambda_2)^{e_{r,2}} \cdots (\lambda-\lambda_k)^{e_{r,k}}$$

are called the "elementary divisors" of $|\lambda \underline{I} - A|$, $(r = n, n-1, n-2, \ldots 2, 1)$.

The proof of the invariancy in question follows from the determinantal theorem that the rth minor of the product of two determinants is <u>linear</u> in each of the rth minors of the factors,* so that the highest common factor polynomials will be preserved under pre- and post-multiplication by H and H^{-1}.

A normal form Q which will have the same elementary divisors as P, can

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* Dickson, <u>Modern Algebraic Theories</u>, p. 49, Th. 5.

then be constructed thus:-

Juxtapose diagonally submatrices as follows:-

1) Submatrix of order $e_{n,1}$, with λ_1's in the principal diagonal, 1's (or any constants, because, by a simple transformation, such constants may all be replaced by 1's) in the super-principal diagonal, zeros elsewhere.

Below it annex

2) A similarly constructed submatrix of order $e_{n-1,1}$; and so on for all exponents and all latent roots.

A matrix of order n is thus obtained which has zeros below the principal diagonal, the latent roots of P in the diagonal, a distribution of units and zeros in the superprincipal diagonal, and the rest of its elements zeros. This is the required normal form Q; and it will be readily found by inspection that the exponents of the elementary divisors are the same as those of P.

Suppose e.g. that A has latent roots λ_1 (tripled), λ_2 (doubled), and λ_3, the exponents for λ_1 being 2, 1, and those for λ_2 being $1, 1$. Then the shape assumed by Q is:-

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$

The successive submatrices are indicated by dotted enclosures.

Any matrix can be brought into this "normal form" by properly chosen transformations, and consequently any matrix is similar to some matrix which is in normal form.

The problem of determining the necessary and sufficient conditions for the equivalence of two families of non-singular bilinear forms was first solved by Weierstrass in 1868. Since then, the sub-ject has been dealt with by Frobenius, Kronecker, Weyr, Bromwich and Bôcher. In 1910, Hawkes gave a direct and practically self-contained derivation of these necessary and sufficient conditions by a method which is simpler in its application than are the classical methods. The simplicity rests in referring the general question of equivalence of families back to that of the "similarity"* of numerical matrices, this question of similarity being settled by the derivation of a normal form in which a complete system of invariants of a class of matrices is displayed:-

I The Rank of any matrix, A, of the class.

II The Characteristic Equation of A.

III The degrees of the Elementary Divisors of the matrix $\lambda \underline{I} - A$.

Hawkes shows that if the rank and latent roots of a matrix are known, every feature of the normal form can be fixed except this arrangement of 1's in the super-principal diagonal. He then shows that these 1's group themselves in a way determined by the values of the exponents of the elementary divisors.

Starting with a general matrix,

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* Frobenius defines the matrices P and Q as being <u>similar</u> (ähnlich) when a matrix H exists such that $P = HQH^{-1}$. see Crelle, 84, p. 21.

he first employs certain transformations to remove all the elements under the leading diagonal and replace them by zeros, while any repeated latent roots are grouped together along this diagonal.
The matrix A then takes the form

$$A_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 & - & - & - & 0 \\ 0 & a_2 & 0 & 0 & - & - & - & 0 \\ 0 & 0 & a_3 & 0 & - & - & - & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & a_K \end{pmatrix}$$

where a_j represents a submatrix, all of whose roots are equal and all its elements <u>below</u> the principal diagonal are zero. There are no elements extraneous to these submatrices.

To complete the reduction, a matrix like

$$a_j = \begin{pmatrix} \lambda & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & \lambda & a_{23} & \cdots & a_{2m} \\ 0 & 0 & \lambda & \cdots & a_{3m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix},$$

all of whose latent roots are equal, is dis-cussed. All these elements above the leading diagonal, except certain ones which can always be put in the super-principal diagonal, can be replaced by zeros by using not more than three main types of transformation. This is done for each submatrix a_j, and the required normal form is then obtained by joining up these sub-matrices.

The matrix to be reduced need not necessarily have rank n; the character-istic equation may have zero roots without affecting the method of reduction. Also, the transformation assumes the char-acteristic equation to have been solved. It may be irreducible in the domain of rational numbers so the process is then an irrational one.

If there are not multiple roots, no reduction is necessary, as each submat-rix a_j consists of a single element, and the equivalence turns on the identity of the characteristic equations themselves, which is determined by a rational process.

The reduction of a matrix to normal form is of fundamental importance in view of the fact that if

$$A = BCB^{-1},$$

$$A^2 = BCB^{-1}.BCB^{-1} = BC^2B^{-1}.$$

Thus A^2 has the same B as A.

Also, if the latent roots of A are all distinct, e.g.

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

then $$C^2 = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix},$$

$\therefore$ the latent roots of A^2 are the squares of the latent roots of A.

This result also follows when the latent roots are not all distinct because of the zeros under the leading diagonal of C.

e.g. if C is of the form

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} \lambda_1^2 & 2\lambda_1 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & \lambda_3^2 & 2\lambda_3 \\ 0 & 0 & 0 & \lambda_3^2 \end{pmatrix},$$

being a matrix of the same form as C and having its latent roots the squares of those of C.

Similarly, $A^m = BC^mB^{-1}$, and, more generally,

$$\phi(A) = B\,\phi(C)\,B^{-1}$$

where $\phi(x)$ is any polynomial in x.

Again, if $f(x)$ is a rational function, namely $\dfrac{\phi(x)}{\psi(x)}$,

$$f(A) = \frac{\phi(A)}{\psi(A)}$$

$$= \frac{B\,\phi(C)\,B^{-1}}{B\,\psi(C)\,B^{-1}}$$

$$= B\,\phi(C)\,B^{-1}\,B\,\frac{1}{\psi(C)}\,B^{-1}$$

$$= B\,\frac{\phi(C)}{\psi(C)}\,B^{-1}$$

$$= B\,f(C)\,B^{-1}.$$

We thus prove Sylvester's Law of Latency:-

<u>The latent roots of any rational function of a matrix are respectively the same functions of its latent roots.</u>

References.

Dickson, Modern Algebraic Theories, (Chicago, 1926), Chap. V.

Bôcher, Higher Algebra (New York, 1919), Chap. XX.

Cullis, Matrices and Determinoids, (Cambridge, 1926), III, p. 342.

Bromwich, Quadratic Forms and their Classification by means of Invariant-Factors, Cambridge Tract, No. 3. (1906), Chap. III.

Hawkes, The Reduction of Families of Bilinear Forms, Amer. Journ. of Maths, 1910, pp. 101-114.

The definition of compound matrices.

In order to complete the definitions required for what follows, we must now explain what is meant by a compound matrix.

Let A be any matrix of order n. Consider all those minors of A which are of order m — they are in number $\binom{n}{m}^2$ * — and arrange them so that in any one row (or column) there stand those minors which are contained in the same m rows (or columns) of A. The matrix thus formed is called the m^{th} compound matrix of A, written $A^{(m)}$.

The corresponding determinant, $|A^{(m)}|$, was first considered by Cauchy**; and the principal known theorem regarding compound determinants is due to Sylvester.*** It asserts that the mth compound of any determinant D is equal to $D^{\binom{n-1}{m-1}}$.

* $\binom{n}{m}$ means the number of combinations of n things taken m at a time.

** Journal de l'Ec. Pol., cah. 17, p. 93.

*** Phil. Mag. (4) (1851), pp. 295, 415.

As a specific case, let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

Then the 2nd compound matrix of A is

$$A^{(2)} = \begin{pmatrix}
\begin{vmatrix} a_1 a_2 \\ b_1 b_2 \end{vmatrix} & \begin{vmatrix} a_1 a_3 \\ b_1 b_3 \end{vmatrix} & \begin{vmatrix} a_1 a_4 \\ b_1 b_4 \end{vmatrix} & \begin{vmatrix} a_2 a_3 \\ b_2 b_3 \end{vmatrix} & \begin{vmatrix} a_2 a_4 \\ b_2 b_4 \end{vmatrix} & \begin{vmatrix} a_3 a_4 \\ b_3 b_4 \end{vmatrix} \\
\begin{vmatrix} a_1 a_2 \\ c_1 c_2 \end{vmatrix} & \begin{vmatrix} a_1 a_3 \\ c_1 c_3 \end{vmatrix} & \begin{vmatrix} a_1 a_4 \\ c_1 c_4 \end{vmatrix} & \begin{vmatrix} a_2 a_3 \\ c_2 c_3 \end{vmatrix} & \begin{vmatrix} a_2 a_4 \\ c_2 c_4 \end{vmatrix} & \begin{vmatrix} a_3 a_4 \\ c_3 c_4 \end{vmatrix} \\
\begin{vmatrix} a_1 a_2 \\ d_1 d_2 \end{vmatrix} & \begin{vmatrix} a_1 a_3 \\ d_1 d_3 \end{vmatrix} & \begin{vmatrix} a_1 a_4 \\ d_1 d_4 \end{vmatrix} & \begin{vmatrix} a_2 a_3 \\ d_2 d_3 \end{vmatrix} & \begin{vmatrix} a_2 a_4 \\ d_2 d_4 \end{vmatrix} & \begin{vmatrix} a_3 a_4 \\ d_3 d_4 \end{vmatrix} \\
\begin{vmatrix} b_1 b_2 \\ c_1 c_2 \end{vmatrix} & \begin{vmatrix} b_1 b_3 \\ c_1 c_3 \end{vmatrix} & \begin{vmatrix} b_1 b_4 \\ c_1 c_4 \end{vmatrix} & \begin{vmatrix} b_2 b_3 \\ c_2 c_3 \end{vmatrix} & \begin{vmatrix} b_2 b_4 \\ c_2 c_4 \end{vmatrix} & \begin{vmatrix} b_3 b_4 \\ c_3 c_4 \end{vmatrix} \\
\begin{vmatrix} b_1 b_2 \\ d_1 d_2 \end{vmatrix} & \begin{vmatrix} b_1 b_3 \\ d_1 d_3 \end{vmatrix} & \begin{vmatrix} b_1 b_4 \\ d_1 d_4 \end{vmatrix} & \begin{vmatrix} b_2 b_3 \\ d_2 d_3 \end{vmatrix} & \begin{vmatrix} b_2 b_4 \\ d_2 d_4 \end{vmatrix} & \begin{vmatrix} b_3 b_4 \\ d_3 d_4 \end{vmatrix} \\
\begin{vmatrix} c_1 c_2 \\ d_1 d_2 \end{vmatrix} & \begin{vmatrix} c_1 c_3 \\ d_1 d_3 \end{vmatrix} & \begin{vmatrix} c_1 c_4 \\ d_1 d_4 \end{vmatrix} & \begin{vmatrix} c_2 c_3 \\ d_2 d_3 \end{vmatrix} & \begin{vmatrix} c_2 c_4 \\ d_2 d_4 \end{vmatrix} & \begin{vmatrix} c_3 c_4 \\ d_3 d_4 \end{vmatrix}
\end{pmatrix}$$

maintaining the ascending order of letters and suffices.

When A is of order n, and we form its m^{th} compound, it follows from Sylvester's theorem that

$$|A^{(1)}| = |A|$$

$$|A^{(2)}| = |A|^{\binom{n-1}{1}}$$

$$|A^{(3)}| = |A|^{\binom{n-1}{2}}$$

$$\vdots \qquad \vdots$$

$$|A^{(n-1)}| = |A|^{\binom{n-1}{n-2}} = |A|^{n-1}$$

$$|A^{(n)}| = |A|^{\binom{n-1}{n-1}} = |A|.$$

Thus, if $n = 5$, $m = 2$, this set gives

$$|A|, \ |A|^4, \ |A|^6, \ |A|^4, \ |A|.$$

An important theorem, due to Rados (1891), states that <u>the latent roots of the m^{th} compound of a matrix A are the products, m at a time, of the latent roots of A.</u>*

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* Zur Theorie der Adjungirten Substitutionen, Math. Ann. 48 (1897), 417–424.

* Proc. Edin. Math. Soc., Vol. XXXV (Part 1), Session 1916–17.

This statement remains true even when the latent roots of A are not all distinct.

Many writers — Rados, Petr, Metzler, Whittaker and Aitken — have looked into the question of the latent roots; but it appears that the much more important question of the elementary divisors (the specification of which completely determines the invariance properties of matrices with respect to pre- and post- multiplication by B and B^{-1}) has not been treated. The present investigation is intended to remedy this deficiency.

If we regard the elements in columns of a matrix as point co-ordinates in a space, then the minors of order 2 form line co-ordinates, those of order 3 plane co-ordinates, and in general the m^{th} compound matrix specifies "compound" or "Grassmann" co-ordinates. Consequently it is of interest to study the relations of latent roots, elementary divisors, and invariance generally, of compound matrices with respect to the original matrix A.

The problem is this :-

Given a matrix A in the most general form BCB^{-1}, so that the elementary divisors of $|\lambda I - A|$ are known. Find the specification of the elementary divisors of the m^{th} compound matrix of A.

It is known that the m^{th} compound of the product of any number of matrices is identical with the product of the m^{th} compounds of these matrices.* The corresponding result for powers of a single matrix is evident.

Thus $\quad A = BCB^{-1} \quad$ implies

$$A^{(m)} = B^{(m)} \; C^{(m)} \; \left(B^{-1}\right)^{(m)}$$

$$= B^{(m)} \; C^{(m)} \left(B^{(m)}\right)^{-1},$$

and so the specification of the elementary divisors of $A^{(m)}$ becomes that of $C^{(m)}$, and therefore we can, in our investigation of the problem, consider the simpler form C instead of A. We therefore proceed

* See Muir's History of the Theory of Determinants, Vol. 1, p. 121.
and On the Latent Roots of Certain Matrices, by A. C. Aitken. Proc. Edin. Math. Soc. Series 2 - Vol. 1 - Part 2. 1928.

to review all the different possible forms of $C^{(m)}$.

Consider, for example, the case of compound matrices, where the original matrix A is of the 4th order. The different possible forms of C are :-

$$(1)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad (2)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad (3)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$$(4)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}, \quad (5)\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \quad (6)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$$(7)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad (8)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad (9)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix},$$

$$(10)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}, \quad (11)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \quad (12)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

$$(13)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \quad \text{and} \quad (14)\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}.$$

These could be written thus:-

(1) $\{(1),(1),(1),(1)\}$, (2) $\{(1,1),(1),(1)\}$, (3) $\{(1,1,1),(1)\}$,

(4) $\{(1,1),(1,1)\}$, (5) $\{(1,1,1,1)\}$, (6) $\{(2),(1),(1)\}$,

(7) $\{(2,1),(1)\}$, (8) $\{(3),(1)\}$, (9) $\{(2),(1,1)\}$,

(10) $\{(2),(2)\}$, (11) $\{(4)\}$, (12) $\{(3,1)\}$,

(13) $\{(2,2)\}$, (14) $\{(2,1,1)\}$,

there being a bracket, ( ), for each distinct latent root, containing the exponents of the elementary divisors for that root.

If we restrict ourselves to a single latent root λ_1, we have 5 types to consider, nos. (11), (12), (13), (14) and (5), these groupings corresponding to the partitions of the integer 4, namely
$[4], [3,1], [2,2], [2,1,1], [1,1,1,1]$.
When $\lambda_2, \lambda_3, \lambda_4$ are introduced, 9 more cases appear.

We will examine the various compounds of determinants of orders 1-6, taking all possible cases into con-

-sideration, with a view to tabulating their respective normal forms.

The 2nd compounds, $C^{(2)}$, of these 4th order matrices are matrices of the 6th order. The order of row-groups from which minors of C are chosen will, of course, be the same as the order of column-groups, so that the principal diagonal of the 2nd compound will contain the latent roots of $C^{(2)}$ exclusively. Using Sylvester's ordering of row and column suffixes in their formation i.e.

12, 13, 14, 23, 24, 34,

the 2nd compound of no. (14) is

$$C^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

In this case the problem is to examine the minors of the determinant $|\Lambda \underline{I} - C^{(2)}|$ or

$$\begin{vmatrix} \Lambda-\lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda-\lambda_1^2 & 0 & -\lambda_1 & 0 & 0 \\ 0 & 0 & \Lambda-\lambda_1^2 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 & \Lambda-\lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda-\lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda-\lambda_1^2 \end{vmatrix}$$

with a view to finding what powers of $(\Lambda-\lambda_1^2)$ occur therein, and thus determine the exponents of the elementary divisors.

It is found that the H.C.F of the 1st minors of this determinant contains $(\Lambda-\lambda_1^2)^4$; the H.C.F of the 2nd minors contains $(\Lambda-\lambda_1^2)^2$, the H.C.F of the 3rd minors contains $(\Lambda-\lambda_1^2)^1$ and the H.C.F^s of the 4th and 5th minors respectively do not contain $(\Lambda-\lambda_1^2)$. Thus, with the notation given on pp. 22 & 23, $n=4$, $m=2$ and $p=6$, $p_1=4$, $p_2=2$, $p_3=1$, $p_4=0$, $p_5=0$, so that

$e_1=2$, $e_2=2$, $e_3=1$, $e_4=1$, $e_5=0$,

and the normal form of $C^{(2)}$ then is

$$\begin{pmatrix} \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix},$$

which is of the form

$$\{(2, 2, 1, 1)\}.$$

Again, taking no. (10), and using the same ordering of row and column suffices as before,

$$C^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & \lambda_1 & \lambda_3 & 1 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{pmatrix}.$$

On examining the minors of $|\Lambda I - C^{(2)}|$, it is found that for both $(\Lambda - \lambda_1^2)$ and $(\Lambda - \lambda_3^2)$

$p = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = 0$,

so $e_1 = 1$, $e_2 = e_3 = e_4 = e_5 = 0$.

For $(\Lambda - \lambda_1\lambda_3)$,

$p = 4$, $p_1 = 1$, $p_2 = p_3 = p_4 = p_5 = 0$

so $e_1 = 3$, $e_2 = 1$, $e_3 = e_4 = e_5 = 0$.

Hence the normal form of $C^{(2)}$ in this case is

$$\begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{pmatrix},$$

which is of the form $\{\overset{\lambda_1^2}{(1)}, \overset{\lambda_1\lambda_3}{(3,1)}, \overset{\lambda_3^2}{(1)}\}$.

Various other compounds, formed by this row and column ordering, along with their normal forms obtained from an actual examination of the minors of $|\Lambda I - C^{(m)}|$, are given in Appendix A, pp. 173 - 185.

Now, the values of the exponents of the elementary divisors of $|\Lambda I - C^{(m)}|$ are not at once apparent in these compounds. Whatever ordering is adopted, [there are ${}_nC_m!$ possible orderings] $C^{(m)}$ is, unfortunately, never in normal form as it stands. The general formulation of elementary divisors is thus rendered difficult.

All that can be done is to adopt an ordering which will give the compound in a form approaching the normal, and from which any superfluous elements can be most readily removed, remembering that the elements occurring in the super-principal diagonal, other than zeros, will be constants, not necessarily units, and that the elements above them again are entirely arbitrary.

The simplest case upon

which to experiment is that in which all the latent roots are the same, 1's occurr-ing throughout the super-principal diagonal, with zeros everywhere else. We shall begin, therefore, with a matrix of the nth order, in this nor-mal form. Using Sylvester's notation, it could be written

$$C = \begin{pmatrix} 1\,2\,3\,4\,\cdots\,n \\ 1\,2\,3\,4\,\cdots\,n \end{pmatrix},$$

and its determinant

$$|C| = \begin{vmatrix} 1\,2\,3\,4\,\cdots\,n \\ 1\,2\,3\,4\,\cdots\,n \end{vmatrix}.$$

When investigating its mth compound, it is necessary to study the minors of order m of this determinant $|C|$. For example, those of order 3 which are picked from the first 3 rows, will be denoted by

$$\begin{vmatrix} 1\,2\,3 \\ 1\,2\,3 \end{vmatrix}, \begin{vmatrix} 1\,2\,3 \\ 1\,2\,4 \end{vmatrix}, \begin{vmatrix} 1\,2\,3 \\ 1\,3\,4 \end{vmatrix}, \begin{vmatrix} 1\,2\,3 \\ 1\,3\,5 \end{vmatrix}, \cdots$$ and so on.

An examination of these in the present case shows that where an upper row symbol is less than the lower column

column symbol under it by more than 1, the minor in question vanishes, e.g.

$$\begin{vmatrix} 123 \\ 125 \end{vmatrix} = 0 , \quad \begin{vmatrix} 123 \\ 145 \end{vmatrix} = 0.$$

It is also found that whenever any upper symbol exceeds the lower one, the minor vanishes, e.g.

$$\begin{vmatrix} 124 \\ 123 \end{vmatrix} = 0, \quad \begin{vmatrix} 135 \\ 124 \end{vmatrix} = 0, \quad \text{etc.}$$

In fact $\begin{pmatrix} i\,j\,k \\ i'\,j'\,k' \end{pmatrix} = \lambda^{(1+i'-i)+(1+j'-j)+(1+k'-k)}$

provided that $i-i', j-j', k-k' = 1$ or 0.
In all other cases $\begin{pmatrix} i\,j\,k \\ i'\,j'\,k' \end{pmatrix} = 0.$

This follows from the form of C. Thus, e.g.

$$\begin{pmatrix} 123 \\ 123 \end{pmatrix} = \lambda^3, \quad \begin{pmatrix} 123 \\ 234 \end{pmatrix} = 1, \quad \begin{pmatrix} 235 \\ 123 \end{pmatrix} = 0.$$

Consider, now, the formation of the m^{th} compound, $C^{(m)}$.
The order of row-groups from which minors of C are chosen will, of course, be the same as the order of column-groups, so that the principal diagonal of the mth compound will contain elements

λ^m exclusively, easily seen to be the latent roots of $C^{(m)}$. There being nCm possible selections of m from $1234 \cdots n$, there can be $(nCm)!$ orders of row-indices in $C^{(m)}$. The one considered by Sylvester in his work on compound determinants — and the most important — is that in which each group contains at least one symbol greater than the corresponding symbol in the preceding group. For example, when $n=5$, $m=3$, it would be

123, 124, 125, 134, 135, 145, 234, 235, 245, 345.

With this ordering, $C^{(m)}$ would be represented umbrally by

$$\begin{pmatrix} |123| & |124| & |125| & \cdot & \cdot & \cdot & \cdot & \cdot & |245| & |345| \\ |123| & |124| & |125| & \cdot & \cdot & \cdot & \cdot & \cdot & |245| & |345| \end{pmatrix},$$

and, in view of the observations on vanishing minors on p. 48, all the elements below the principal diagonal of $C^{(m)}$ are zeros — hence the characteristic determinant, $|C^{(m)} - xI| = (\lambda^m - x)^{nCm}$.

Since columns are ordered as rows, we may at this stage represent $C^{(m)}$ by, e.g.

$$\left(|123| \ |124| \ \cdots\cdots \ |345| \right),$$

or any other ordering.

By finding what power of $(\lambda^m - x)$ occurs in the H.C.F of all the first minors of this characteristic determinant, we obtain the first exponent of the elementary divisors of $|C^{(m)} - xI|$, say, $e_1^{(m)}$. For this purpose, the following new terms will be introduced.

In the expansion of any minor of order r of a determinant there are $r!$ terms, each being the continued product of r elements of that minor, no two of which stand in the same row or column. Let us say that the r elements of any such term lie on a track of the rth order in the determinant. There are $(nCr)^2 r!$ tracks of order r in a determinant of order n.

It is now clear that what must be done from the census of first minors of $|C^{(m)} - xI|$ is to find the track which has most non-diagonal elements and no zero elements. In doing this it can be shown that we need consider only elements to the immediate right of the principal diagonal, i.e. elements represented umbrally by $\left| {j \atop j+1} \right|$. The maximum number of these elements which are non-zero must now be found.

In view of the observations on vanishing minors given on p. 48, this could be done by forming the longest chain of row-groups in which any symbol does not exceed the correspond-ing symbol in the preceding group by more than 1. Evidently such an ordering is obtained by beginning with $1\,2\,3 \cdots m$, and increasing the symbols one at a time, by 1, without making two of them similar, until we reach $n-m+1,\ n-m+2, \ldots n$. There are various ways of doing this, e.g. we might increase <u>in turn</u> a different symbol every time, but the number of increm-ents in each case is evidently

$$\left\{ \overline{n-m+1} + \overline{n-m+2} + \cdots + n \right\} - \left\{ 1 + 2 + \cdots + m \right\}$$

or $\frac{m}{2}\left\{ \overline{2n-m+1} - \overline{m+1} \right\}$ i.e. $m(n-m)$, therefore such a chain contains $\{m(n-m)+1\}$

groups. Then a zero must appear so that, at last, recourse must be had to the principal diagonal. This is bound to happen because the remainder of the row-groups can be left in the relative order of p. 49, in which case the minor comprised by the remaining rows and columns will have nothing but zeros below its diagonal.

The result is, that of the terms in the first minors of $|C^{(m)} - xI|$, the least power of $(\lambda^m - x)$ occurs in a term which substitutes $m(n-m)$ factors independ-ent of x for the diagonal factors $(\lambda^m - x)$. Consequently, the exponent of $(\lambda^m - x)$ in the H.C.F. of first minors is less by $m(n-m)+1$ than that in the determinant. Hence, the first result obtained is

$$e_I^{(m)} = m(n-m)+1.$$

The question now arises – what is the second selection? Supposing the first chain has been formed by in-creasing, in turn, a different symbol every time, we proceed to form out of the remaining groups a second maximum chain on the unit-step principle, and leave the remaining groups in the original order. This chain will be formed by progressing from $(1, 2, 3, \ldots \overline{m-1}, m+2)$ to $(n-m-1, n-m+2, \ldots n-1, n)$, again by unit steps, involving $\{m(n-m)-3\}$ groups.

If the number of selections in this second chain is $e_2^{(m)}$, it is found that the maximum run of non-diagonal elements comprises $(e_1^{(m)}-1)+(e_2^{(m)}-1)$ super-diagonal elements. Then, as before, recourse must be had to diagonal elements. It follows that the H.C.F. of the second minors has the exponent of $(\lambda^m - x)$ less by $e_2^{(m)}$ than that of the first minors. So $e_2^{(m)}$ is the second exponent of elementary divisors and the second result obtained is

$$e_2^{(m)} = m(n-m)-3.$$

Continuing this process, we can reduce the problem to that of forming chains of row-group symbols on the unit-step principle. The later selections will generally be more difficult, although they can be written down readily enough in particular cases.

Whilst it may not be possible to find algebraic expressions for $e_r^{(m)}$ in terms of n, m and r, a prescription for their evaluation by a practical combinatory method in any particular case has been arrived at.

Consider this method in reference to some particular cases, e.g. the 3rd compounds of matrices of various orders.

A sequential association of columns with columns and the corresponding rows with rows is tried when selecting minors to form these compounds, the order being chosen so as to keep the super-principal diagonals devoid of zeros as long as possible.

) When $0 = \{(3)\}$, or $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, rows and columns are numbered in the original matrix by 1, 2, 3, thus:-

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \end{array}.$$

The choice of minors in the compound is ordered $[(123)]$, giving $c^{(3)} = \lambda^3$.

2) When $c = \{(4)\}$, or $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$,

$c^{(3)}$ is formed with row-and-column ordering $[(123), (124), (134), (234)]$.
This ordering happens to be the same as the fundamental one, and gives

$$c^{(3)} = \begin{pmatrix} \lambda^3 & \lambda^2 & \lambda & 1 \\ 0 & \lambda^3 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^3 & \lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}.$$

The normal form of $c^{(3)}$ is

$$\begin{pmatrix} \lambda^3 & 1 & 0 & 0 \\ 0 & \lambda^3 & 1 & 0 \\ 0 & 0 & \lambda^3 & 1 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}.$$ (see Appendix A, p. 176.)

There are present, therefore, superfluous elements above the super-principal diagonal which must be replaced by zeros.

3) When $c = \{(5)\}$, or $\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$,

it is found that there are 5 ways of going from 123 to 345 by unit steps, namely

$[(123), (124), (134), (234), (235), (245), (345).]$, (1)

$[(123), (124), (125), (135), (145), (245), (345).]$, (2)

$[(123), (124), (134), (135), (145), (245), (345)]$, (3)

$[(123), (124), (125), (135), (235), (245), (345)]$, (4)

$[(123), (124), (134), (135), (235), (245), (345)]$. (5)

Each combination, such as (234) in (1), arises from the preceding, e.g. (134), by the simple addition of a unit to one of its members. It is easily seen from the form of matrix being dealt with that if the addition is more than 1 undesired zeros would come in, for some λ or 1, instead of leaping to a λ or 1, would leap to the zero beyond:-

$$\begin{matrix} \cdots & \lambda & 1 & 0 & 0 & \cdots \\ \cdots & 0 & \lambda & 1 & 0 & \cdots \\ \cdots & \cdot & \cdot & \cdot & \cdot & \cdots \end{matrix}$$

Hence the first choice of minors is from

(123) by increments of a unit to some member each time, until the minor comprising the <u>last</u> 3 rows and columns, (456), is reached.

The number of increments is $3(5-3)$, or 6, in each case, so the number of minors in the first selection is $3(5-3)+1$ ie e_1.

The remaining groups, in Sylvester's ordering, are respectively

[125, 135, 145], (1)

[134, 234, 235], (2)

[125, 234, 235], (3)

[134, 145, 234], (4)

[125, 145, 234]. (5),

but only (1) and (2) give the longest possible chain on the unit-step principle, i.e. one consisting of $m(n-m)-3$, or e_2, groups.

On this principle they could be arranged so:-

$[(125),\ (135),(145)]$, (1)

$[(134),\ (234),\ (235)]$, (2)

$[(234),\ (235);\ (125)]$, (3)

$[(134),\ (234);\ (145)]$, (4)

$[(125);\ (145);\ (234)]$. (5)

Consequently, the 3rd compound of $\{(5)\}$, when maximum chains are formed, would be $\{(7,3)\}$, thus agreeing with the result given in Appendix A, p. 179.

The compounds formed with the above 5 orderings are respectively :-

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda^3
\end{array}\right) \quad (1),$$

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\
0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3
\end{array}\right) \qquad (2) \quad ,$$

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda & 1 & \lambda^2 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda & 1 & \lambda^3 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & \lambda^3 & 0 \\
0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & \lambda & \lambda^3
\end{array}\right) \qquad (3) \quad ,$$

$$\begin{pmatrix} \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda^3 \end{pmatrix} \quad (4)$$

$$\begin{pmatrix} \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & 0 & \lambda \\ 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda & \lambda^2 \\ 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & \lambda^3 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^3 \end{pmatrix} . \quad (5)$$

In every case there are zeros below the

leading diagonals of the various submatrices; but again superfluous elements occur above the super-principal diagonals and also extraneous elements which cause (1) and (2) to differ from the normal form previously found, i.e.

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{array}\right),$$

although they resemble it much more closely than does the compound derived from the fundamental ordering

$[(123), (124), (125), (134), (135), (145), (234), (235), (245), (345).]$,

viz.

$$\begin{pmatrix}
\lambda^3 & \lambda^2 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda^2 & \lambda & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & \lambda^3 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & \lambda^2 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{pmatrix}.$$

(see Appendix A, p. 179.).

On forming a variety of compounds of matrices of type $\{(n)\}$ on this principle, it is noticed that the formation of maximum chains gives the same values for e_1, e_2, e_3, etc., as those obtained by the other method, (see Appendix A), but that unwanted elements appear in every case.

It will now be shown that by utilizing a process similar to one employed by Hawkes these unwanted elements can be removed from compounds obtained by the "maximum chain" rule.

When other groupings are used, the matrix either takes the "maximum chain" form in the course of the removal of unwanted elements by this process, or other Hawke's transformations have to be employed before the elements can be removed, which also produce the form obtained by the "maximum chain" rule.

This process will now be described.

Two quite simple types of operation must be performed on these compounds, one equivalent to pre-multiplying by a matrix H and the other to post-multiplying by H^{-1}, which will, of course, not alter the normal form. One or two examples will illustrate these operations of the removal of elements.

<u>Ex. 1.</u>

The second compound of $\{(3)\}$ or $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ is formed from the ordering $[(12), (13), (23)]$ giving

$$\begin{pmatrix} \lambda^2 & \lambda & \underline{1} \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}.$$

It is proposed to eradicate the 1 in the top right-hand corner, by means of the super-principal diagonal λ in the same row. Consider the operation (column 3 $-\frac{1}{\lambda}$ column 2), where, as is usual in determinantal work, <u>the column (or row) to be modified is mentioned first.</u> Now, since matrix multiplication is row-by-column, it may readily be seen that this operation is equivalent to *post* - multiplying the compound by

$$H^{-1} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{\lambda} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $$H \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/\lambda \\ 0 & 0 & 1 \end{pmatrix},$$ $\therefore$ the necessary

pre - multiplication by H is equivalent to the operation row $2 + \frac{1}{\lambda}$ row 3.

In fact, any pair of operations, (row p + k row q), (col. q − k col. p), leave the normal form unchanged, and may thus be called *Complementary* operations.

The operation col. $3 - \frac{1}{\lambda}$ col. 2 will remove the $\underline{1}$ from this 2nd compound, but such an operation must be preceded by another one, i.e. row $2 + \frac{1}{\lambda}$ row 3.

Performing these, we obtain

(1) $\begin{pmatrix} \lambda^2 & \lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}$ and then (2) $\begin{pmatrix} \lambda^2 & \lambda & 0 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}$, i.e.

the 2nd compound is now reduced to

normal form.
Here the $\underline{\underline{1}}$ has been eliminated by means of an operation on columns, but the operation row 1 $- \frac{1}{\lambda}$ row 2, where the super-diagonal λ in row 2 is used instead, would have done equally well, having necessarily to be succeeded by col. 2 $+ \frac{1}{\lambda}$ col. 1.
The two steps would then be

$$\begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda^2 & \lambda & 0 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}.$$

It always will be that a non-zero element in the super-principal diagonal is utilized to remove other elements which are unwanted.

<u>Ex. 2.</u> As a further example, consider the 2nd compound of $\{(4)\}$ or $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$

and choose the ordering $[12, 13, 23, 24, 34;\ 14]$.
Then

$$C^{(2)} \text{ is } \left(\begin{array}{ccccc|c} \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 1 & 0 & \underline{\underline{\lambda}} \\ 0 & 0 & \lambda^2 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ \hline 0 & 0 & 0 & \underline{\underline{\lambda}} & 0 & \lambda^2 \end{array}\right)$$

The underlined elements are to be removed. The $\underline{1}$ in the first row can be removed by the operations row $2 + \frac{1}{\lambda}$ row 3, col. $3 - \frac{1}{\lambda}$ col. 2, giving

$$\left(\begin{array}{ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & \underline{2} & \frac{1}{\lambda} & \lambda \\
0 & 0 & \lambda^2 & \lambda & 1 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & \lambda & 0 & \lambda^2
\end{array}\right)$$

Next, to get rid of the underlined 2, perform row $3 + \frac{2}{\lambda}$ row 4, col $4 - \frac{2}{\lambda}$ col 3. This gives

$$\left(\begin{array}{ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & \underline{\underline{\frac{1}{\lambda}}} & \lambda \\
0 & 0 & \lambda^2 & \lambda & 3 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & \lambda & 0 & \lambda^2
\end{array}\right).$$

Next, row $3 + \frac{1}{\lambda^2}$ row 5, col.$5 - \frac{1}{\lambda^2}$ col 3. gives

$$\left(\begin{array}{ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & \lambda \\
0 & 0 & \lambda^2 & \lambda & \underline{\underline{3}} & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & \lambda & 0 & \lambda^2
\end{array}\right),$$

and row 4 + $\frac{3}{\lambda}$ row 5, col. 5 $- \frac{3}{\lambda}$ col 4. gives

$$\left(\begin{array}{ccccc:c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & \underline{\lambda} \\
0 & 0 & \lambda^2 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hdashline
0 & 0 & 0 & \lambda & -3 & \lambda^2
\end{array}\right).$$

Now all the superfluous elements in the leading submatrix of order 5 have been removed and it is easily seen that this can be done in the general case also. Observe that the process stops as soon as a zero appears in the super-principal diagonal, as it does in row 5.

To get rid of the 3 remaining extraneous elements, first use row 3 + row 6, col. 6 − col. 3 to remove the underlined λ.

This gives

$$\left(\begin{array}{ccccc:c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 2\lambda & -3 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & \underline{\lambda} & -3 & \lambda^2
\end{array}\right).$$

The troublesome λ below the diagonal can be cleared by a <u>row</u> operation, row $6 - \frac{1}{2}$ row 3, followed by col. $3 + \frac{1}{2}$ col. 6, giving

$$\left(\begin{array}{ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 2\lambda & \underline{\underline{-3}} & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & 0 & -\frac{3}{2} & \lambda^2
\end{array}\right).$$

Row $4 - \frac{3}{2\lambda}$ row 5, col. $5 + \frac{3}{2\lambda}$ col 4 removes the -3 in row 3, giving

$$\left(\begin{array}{ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 2\lambda & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & 0 & \underline{\underline{-\frac{3}{2}}} & \lambda^2
\end{array}\right),$$

and, lastly, row $6 + \frac{3}{2\lambda}$ row 4, col. $4 - \frac{3}{2\lambda}$ col. 6 gives

$$\left(\begin{array}{ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 2\lambda & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda^2
\end{array}\right)$$

The matrix is now reduced to a normal form, with exponents 5 and 1 agreeing with the result in Appendix A, p. 175.

Various compounds formed on the unit-step principle and reduced to normal form in this way are given in Appendix C.

In those examples, any elements above the leading diagonals can be removed by columnar operations, each preceded by the necessary complementary row operation, while any below these diagonals can be removed by row operations, each followed by the complementary column operation.

A few more points should be noted :-

Consider the 2nd compound of $\{(4)\}$, studying in detail its reduction. With ordering $[12, 13, 23, 24, 34;\ 14]$ it was

$$\left(\begin{array}{ccccc|c} \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 1 & 0 & \lambda \\ 0 & 0 & \lambda^2 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ \hline 0 & 0 & 0 & \lambda & 0 & \lambda^2 \end{array}\right)$$

By operations like row $2 + \frac{1}{\lambda}$ row 3, col. 3 $- \frac{1}{\lambda}$ col. 2, the 1's can easily be removed first. The matrix so reduced was found to be

$$\left(\begin{array}{ccccc:c} \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 0 & 0 & \underline{\lambda} \\ 0 & 0 & \lambda^2 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ \hdashline 0 & 0 & 0 & \lambda & -3 & \lambda^2 \end{array}\right).$$

It should be noticed that the -3 in the last row has appeared <u>to the right of the λ</u>, as the result of an operation col. 5 $- \frac{3}{\lambda}$ col. 4. This will always be so, since when removing elements above the diagonal it is the <u>later</u> column mentioned, (5), which is being modified.

We now come to the crucial point of the matter. The underlined λ is going to be removed by a col. 6 $-$ col. 3 operation, which must, of course, be preceded by a row 3 + row 6 one. The effect of row 3 + row 6 is to make row 3

$$0 \quad 0 \quad \lambda^2 \quad 2\lambda \quad -3 \;\vdots\; \lambda^2,$$

i.e. new elements have appeared in it,

but only to the right of the diagonal element. They can, of course, be removed by typical operations.

The appearance of new elements so conveniently only to the right of the diagonal is due to a fact relating to elements occurring in blocks outside the diagonal submatrices. If in one of these blocks there is a column of the matrix, say the jth., having its lowest placed non-zero element, within the block, in the kth row, then, in the jth row of the matrix, in the conjugate block, no non-zero element will appear in a column before the $(k+2)$th and may even appear later.

This can be seen in the second compound on p. 68, where the λ in column 6 is in the 2nd row; that in row 6 is in the $(2+2)$, or 4th column.

The reasons for this are simple ones, a proof being easily obtained by consideration of

(1) The method of naming the elements of the compound umbrally,

(2) The conditions under which these elements vanish (see p. 48),

and
(3) The fact that rows and columns have been ordered by a unit-step principle.

Compare, for example, the 5 elements above λ^2 in column 6, namely, $\binom{12}{14}, \binom{13}{14}, \binom{23}{14}, \binom{24}{14}, \binom{34}{14}$, with the 5 conjugate elements in row 6, $\binom{14}{12}, \binom{14}{13}, \binom{14}{23}, \binom{14}{24}, \binom{14}{34}$.

The first appearance of a non-zero element in the $(k+2)$th column is important for this reason:-

In removing a non-zero element in the jth column and the kth row by means of a super-diagonal non-zero element [which will be in the $(k+1)$th column], the $(k+1)$th row will be modified by an operation like row $(k+1)$ + λ row j. But, if the first non-zero element in row j is in the $(k+2)$th place, at the earliest, then the operation will affect only elements of row $(k+1)$ to the right of the diagonal. It might, for example, increase the super-diagonal element, which would not matter. The new elements in row $(k+1)$ could then be removed.

It is essential that the ordering of rows and columns should be such that when the end of a submatrix is reached, there must be nothing but zeros to the right of the corner element, in that row.

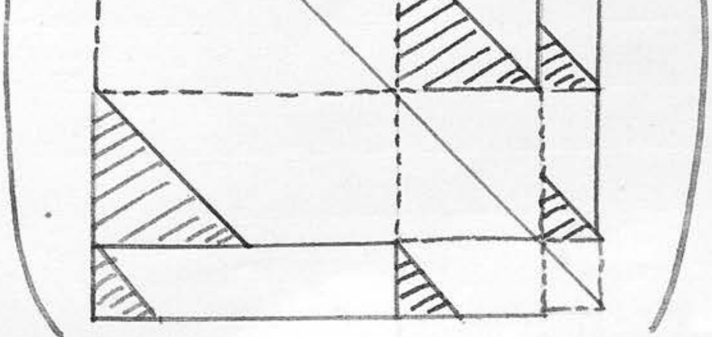
$$\begin{array}{c|cccc} \lambda^2 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

This being so, all elements to the right of the principal diagonal, except super-diagonal ones, can be cleared away, non-zero superdiagonal elements being used to effect the reduction.

The last step is to remove any unwanted elements which may occur <u>below</u> the diagonal, by the use of operations like

row q $-\lambda$ row p, col. $p + \lambda$ col. q.

To sum up, the foregoing remarks along with a study of the examples in Appendix C show that the effect of a Hawkes transformation on a non-zero extraneous element above the diagonal is to annihilate it at the expense of introducing (so long as the element in question is not on the right edge of the indicated rectangles, when it will simply disappear) new elements to the "south-east" of it. Hence in these transformations the extraneous elements are drafted in a "south-east" direction and finally removed. Similarly, in the case of extraneous elements below the diagonal, there is a Hawkes transformation which moves them "north-west". Accordingly, so long as there are no extraneous elements in the "critical triangles" (shaded in the diagram) we may replace elements by zeros and obtain a normal form without going through the reduction process. An element in a critical triangle cannot be removed for lack (at some stage) of the necessary super-diagonal element to complete the removal.



It is found that the arrangement of grouped indices on a unit-step principle (so as to secure maximum runs) gives the compound in a form from which unwanted elements can at once be removed. Although the triangles may be clear at the start, an element may appear in a critical triangle during reduction, unless maximum runs are formed. Then another type of Hawkes transformation must be employed to get such an element into another position where there is a convenient super-diagonal element wherewith to remove it.

Having explained the method of removing super-fluous elements, let us return to the discussion of the case $\{(n)\}$, considering first the 5 possible 3rd compounds of $\{(5)\}$ given on pp. 58-60.

It has been found that in compounds (1) and (2) where there are <u>maximum</u> chains and exponents $(7, 3, 0, \dots)$, any superfluous elements can easily be removed by the given process. [see Appendix C. pp. 207-210].

In case (4), the ordering used was $[(123), (124), (125), (135), (235), (245), (345); (134), (234); (145)]$ giving the compound

$$\left(\begin{array}{ccccccc|cc|c}
\lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda^3
\end{array}\right)$$

with exponents $(7, 2, 1, 0, \dots)$.

There are present no elements in the

critical triangles, but it will be discovered that, during the process of removal of unwanted elements, an element <u>appears</u> in row 9, column 10, and causes this matrix also to take the normal form $(7, 3, 0, \ldots)$. Thus, after the operations

row 10 − row 5, col. 5 + col. 10;
row 10 + $\frac{1}{\lambda}$ row 6, col. 6 − $\frac{1}{\lambda}$ col. 10;
row 9 − $\frac{1}{2}$ row 4, col. 4 + $\frac{1}{2}$ col. 9;
row 9 − $\frac{1}{\lambda}$ row 5, col. 5 + $\frac{1}{\lambda}$ col. 9;
row 8 − $\frac{3}{2}$ row 3, col. 3 + $\frac{3}{2}$ col. 8;
row 8 − $\frac{3}{4\lambda}$ row 4, col. 4 + $\frac{3}{4\lambda}$ col. 8

the compound becomes

$$\left(\begin{array}{ccccccc:ccc}
\lambda^3 & \lambda^2 & \frac{3}{2}\lambda & \frac{5}{4} & \frac{1}{\lambda} & 0 & 0 & \lambda & 1 & 0 \\
0 & \lambda^3 & \frac{5\lambda^2}{2} & \frac{5\lambda}{2} & 2 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & 2\lambda^2 & 0 & 0 & 0 & 0 & \lambda^2 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & -\lambda^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{array}\right).$$

The elements above the diagonal can now be removed without affecting the lower half of the matrix and the normal form is $(7, 3, 0, ..)$.

In case (3) the ordering used was $[(123), (124), (134), (135), (145), (245), (345); (234), (235); (125)]$, and gave exponents $(7, 2, 1, 0, \ldots)$ and compound

$$\left(\begin{array}{ccccccc|cc|c}
\lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda & 1 & \lambda^2 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda & 1 & \lambda^3 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & \lambda^3 & 0 \\
\hline
0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & \lambda & \lambda^3
\end{array}\right)$$

This gives an instance of an extraneous element being unremovable through lack of the necessary super-diagonal element, for, in removing elements below the diagonal by operations

row 8 $- \frac{1}{\lambda}$ row 5, col. 5 $+ \frac{1}{\lambda}$ col. 8;

row 8 $- \frac{1}{\lambda^2}$ row 6, col. 6 $+ \frac{1}{\lambda^2}$ col. 8;

row 9 − row 5, col. 5 + col. 9;
row 9 − $\frac{1}{\lambda}$ row 6, col. 6 + $\frac{1}{\lambda}$ col. 9;
row 10 − row 3, col. 3 + col. 10;
row 10 + $\frac{1}{\lambda}$ row 4, col. 4 − $\frac{1}{\lambda}$ col. 10;
row 8 − $\frac{1}{2}$ row 4, col. 4 + $\frac{1}{2}$ col. 8;
row 10 + $\frac{1}{3}$ row 3, col. 3 − $\frac{1}{3}$ col. 10;
row 10 − $\frac{5}{3\lambda}$ row 8, col. 8 + $\frac{5}{3\lambda}$ col. 10;
row 10 − $\frac{1}{2\lambda}$ row 4, col. 4 + $\frac{1}{2\lambda}$ col. 10,

the compound becomes

$$\left(\begin{array}{ccccccc|cc|c}
\lambda^3 & \lambda^2 & \lambda & \frac{1}{2} & \frac{1}{\lambda} & \frac{1}{\lambda^2} & 0 & 1 & 0 & 0 \\
0 & \lambda^3 & \frac{5\lambda^2}{3} & \lambda & 2 & \frac{2}{\lambda} & 0 & \frac{8\lambda}{3} & 1 & \lambda^2 \\
0 & 0 & \lambda^3 & \frac{3\lambda^2}{2} & 3\lambda & 3 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & \lambda^3 & 2\lambda^2 & 2\lambda & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\lambda^2}{3} & 0 & \lambda^3
\end{array}\right).$$

Notice that an element $-\frac{2\lambda^2}{3}$ has appeared in a critical triangle, and cannot be removed in the usual way.

Any two rows (say rows p and q) of a matrix A may be interchanged without altering its normal form, provided that this operation is followed by the inter-change of the corresponding columns, for these operations are respectively equivalent to pre- and post- multiplication of A by H and H^{-1}, where matrix H has its element $h_{pq} = h_{qp} = 1 = h_{kk}$, $(k \neq p, q)$, and all the other elements are zero. H^{-1} will thus be $= H$.

e.g. let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$.

By interchanging rows 2 and 3, it becomes

$$\begin{pmatrix} a & b & c \\ g & h & k \\ d & e & f \end{pmatrix}.$$

Then, interchange of columns 2 and 3 gives

$$\begin{pmatrix} a & c & b \\ g & k & h \\ d & f & e \end{pmatrix}.$$

Now, pre-multiplication of A by $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ gives $\begin{pmatrix} a & b & c \\ g & h & k \\ d & e & f \end{pmatrix}$, corresponding to an interchange of rows 2 and 3. Post-multiplication of the latter matrix by H^{-1} ie $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ gives $\begin{pmatrix} a & c & b \\ g & k & h \\ d & f & e \end{pmatrix}$, corresponding to an interchange of columns 2 and 3.

Similarly, by interchanging (1) rows 1 and 4, (2) cols. 1 and 4 in

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{pmatrix},$$

it becomes $\begin{pmatrix} q & n & p & m \\ h & f & g & e \\ l & j & k & i \\ d & b & c & a \end{pmatrix}$,

the same result being obtained from pre-multiplication of A by H =

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and post - multiplication by this matrix.

(See Appendix D, p. 232)

Thus, by interchanging (1) rows 8 and 10, cols. 8 and 10; (2) rows 9 and 10, cols. 9 and 10 in the compound under discussion, it becomes

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & \lambda^2 & \lambda & \frac{1}{2} & \frac{1}{\lambda} & \frac{1}{\lambda^2} & 0 & 0 & 1 & 0 \\
0 & \lambda^3 & \frac{5\lambda^2}{3} & \lambda & 2 & \frac{2}{\lambda} & 0 & \lambda^2 & \frac{8\lambda}{3} & 1 \\
0 & 0 & \lambda^3 & \frac{3\lambda^2}{2} & 3\lambda & 3 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & \lambda^3 & 2\lambda^2 & 2\lambda & 0 & 0 & 0 & \lambda^2 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & -\frac{2\lambda^2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{array}\right).$$

The elements above the diagonal can at once be removed and the normal form is $(7, 3, 0, \ldots)$.

Lastly, case (5) was formed by

using ordering $[(123), (124), (134), (135), (235), (245), (345);$ (125) ; (145) ; $(234)]$ and gave exponents $(7, 1, 1, 1, 0, \dots)$ and compound

$$\begin{pmatrix}
\lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & 0 & \lambda \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda & \lambda^2 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \boxed{\lambda^3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & \boxed{\lambda^3} & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & \boxed{\lambda^3}
\end{pmatrix}.$$

After the operations :-

row 8 − row 3, col. 3 + col. 8 ; row 8 + $\frac{1}{\lambda^2}$ row 5, col. 5 − $\frac{1}{\lambda^2}$ col. 8;
row 8 − $\frac{1}{\lambda^3}$ row 6, col. 6 + $\frac{1}{\lambda^3}$ col. 8; row 9 − row 5, col. 5 + col. 9;
row 9 + $\frac{1}{\lambda}$ row 6, col. 6 − $\frac{1}{\lambda}$ col. 9; row 10 − $\frac{1}{2}$ row 4, col. 4 + $\frac{1}{2}$ col. 10;
row 10 − $\frac{1}{\lambda}$ row 5, col. 5 + $\frac{1}{\lambda}$ col. 10; row 8 + $\frac{1}{3}$ row 3, col. 3 − $\frac{1}{3}$ col. 8;
row 8 + $\frac{1}{2\lambda}$ row 4, col. 4 − $\frac{1}{2\lambda}$ col. 8; row 8 − $\frac{1}{\lambda^2}$ row 5, col. 5 + $\frac{1}{\lambda^2}$ col. 8;
row 8 + $\frac{1}{\lambda^3}$ row 6, col. 6 − $\frac{1}{\lambda^3}$ col. 8

it will become

$$\begin{pmatrix}
\lambda^3 & \lambda & \lambda & \frac{1}{2} & \frac{1}{\lambda} & 0 & 0 & 0 & 0 & 1 \\
0 & \lambda^3 & \frac{5}{3}\lambda^2 & \lambda & 2 & 0 & 0 & \lambda^2 & 0 & \lambda \\
0 & 0 & \lambda^3 & \frac{3\lambda^2}{2} & 3\lambda & 0 & 0 & 0 & \lambda & \lambda^2 \\
0 & 0 & 0 & \lambda^3 & 2\lambda^2 & 0 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & -\frac{\lambda}{6} & -\frac{2\lambda^2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda^2}{2} & \lambda^3
\end{pmatrix}.$$

Removal of $-\frac{\lambda^2}{2}$ from row 10, col. 9 will cause an element to appear in row 10, col. 8 i.e. in a critical triangle. To avoid this, interchange rows 9 and 10, then cols. 9 and 10. The compound will then become

$$\begin{pmatrix}
\lambda^3 & \lambda^2 & \lambda & \frac{1}{2} & \frac{1}{\lambda} & 0 & 0 & 0 & 1 & 0 \\
0 & \lambda^3 & \frac{5}{3}\lambda^2 & \lambda & 2 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & \lambda^3 & \frac{3\lambda^2}{2} & 3\lambda & 0 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & \lambda^3 & 2\lambda^2 & 0 & 0 & 0 & 0 & \lambda^2 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & -\frac{2\lambda^2}{3} & -\frac{\lambda}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & -\lambda^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{pmatrix}$$

i.e. it will have exponents $(7, 3, 0, \cdots)$ as before.

When forming chains for the 3rd compound of matrix {(6)}, there are 42 ways of going from (123) to (456) for the first selection. The ordering (123), (124), (134), (234), (235), (245), (345), (346), (356), (456) occurs twice among these, and, if chosen, leaves (125), (126), (135), (136), (145), (146), (156), (236), (246), (256) in Sylvester's ordering.

A second maximum chain can then be formed from these remaining groups in 8 ways :-

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	125	125	125	125	125	125	125	125
	126	126	126	135	135	135	135	135
	136	136	136	136	136	136	145	145
	146	146	236	146	146	236	146	146
	156	246	246	156	246	246	156	246
	256	256	256	256	256	256	256	256
leaving	135	135	135	126	126	126	126	136
	145	145	145	145	145	145	136	156
	236	156	146	236	156	146	236	236
	246	236	156	246	236	156	246	126
			*				*	,

but only 2 * out of the 8 lead to a 3rd <u>maximum</u> chain of 4, formed on the unit-step principle. This suggests (10, 6, 4, 0, ...) as exponents of the elementary divisors in this case; and it is found that the matrix formed with ordering (3) can be reduced to a normal form with these exponents. If other orderings which do not give <u>maximum</u> chains be used, it is found that an interchange of rows, followed by an interchange of columns, is necessary, as in case $\{(5)\}$, before extraneous elements can be removed.

Appendix B contains various compounds of $\{(n)\}$ formed on this principle.

The orderings for 2nd compounds are as follow :-

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
12	12	12	12	12
	13	13	13	13
	23	23	23	23
		24	24	24
		34	34	34
		14	35	35
			45	45
			14	46
			15	56
			25	14
				15
				25
				26
				36
				16, and so on.

Note that the chains for $n=6$ follow on those for $n=5$, by the addition of groups involving the new integer 6, i.e. 46, 56 to the first chain; 26, 36 to the second and 16 to the third.

Similarly, the orderings for 3rd compounds are

$n=3$	$n=4$	$n=5$	$n=6$
123	123	123	123
	124	124	124
	134	134	134
	234	234	234
		235	235
		245	245
		345	345
			346
			356
			456
		125	125
		135	135
		145	145
			146
			156
			256
			126
			136
			236
			246

, and so on.

Thence the following table is obtained :-

$m = 2$

n	e_1	e_2	e_3	e_4	e_5	e_6		
* 3	3							
* 4	5	1						
* 5	7	3						
* 6	9	5	1					
* 7	11	7	3					
8	13	9	5	1				
9	15	11	7	3				
10	17	13	9	5	1			
11	19	15	11	7	3			

$m = 3$

n	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
* 3	1									
* 4	4									
* 5	7	3								
* 6	10	6	4							
7	13	9	7	5	1					
8	16	12	10	8	6	4				
9	19	15	13	11	9	7	7	3		
10	22	18	16	14	12	10	10	7	6	5
11	25	21	19	17	15	13	13	10	6	8_{11}

$m = 4.$

n	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
* 4	1							
* 5	5							
* 6	9	5	1					
* 7	13	9	7	5	1			
8	17	13	9					
9	21	17						
10	25	21						
11	29	25						

$m = 5$

n	e_1	e_2	e_3	e_4				
* 5	1							
* 6	6							
* 7	11	7	3					
8	16	12	10	8	6	4		
9	21	17						
10	26	22						
11	31	27						

$m = 6$

n	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
* 6	1								
* 7	7								
8	13	9	5	1					
9	19	15	13	11	9	7	7	3	
10	25	21							
11	31	27							
12	37	33							

$m = 7$

n	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
* 7	1									
* 8	8									
9	15	11	7	3						
10	22	18	16	14	12	10	10	7	6	5
11	29	25								

$m = 8$

n	e_1	e_2	e_3	e_4					
* 8	1								
* 9	9								
10	17	13	9	5	1				
11	25	21	19	17	15	13	13	10	8, 8 5 1

$m = 9$

n	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
* 9	1									
* 10	10									
11	19	15	11	7	3					

[The entries marked * were obtained (1) by reducing the compound to normal form and (2) by the formation of maximum chains. The others were obtained from (2) only.]

It has already been seen that e_1 is a determinate polynomial in m and n, namely $m(n-m)+1$, and that $e_2 = m(n-m)-3$.

In order to find $e_3, e_4, \ldots$ in terms of m and n, their numerical values were tabulated for several values of n with m kept fixed (at eq. 5, 6 or 7), and afterwards for some values of m with n fixed, but the method of finite differences failed to give a determinate polynomial in m and n for e_3.

Having considered the particular case $\{(n)\}$, we now pass to the general case, where compounds of matrices such as

$$C \equiv \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau \end{pmatrix}$$

are to be dealt with, i.e. where the orders of submatrices with non-zero super-principal diagonals are quite general, and where possibly $\lambda = \mu$, etc.

It will be found that the idea of unit-step is the clue to this general case also. For example, forming the 2nd compound of matrix

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

which is of type $\{(2,1,1)\}$, by Sylvester's ordering,

$$c^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix},$$

whence $e_1 = 2,\ e_2 = 2,\ e_3 = 1,\ e_4 = 1,\ e_5 = 0$
and the normal form is

$$\begin{pmatrix} \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}$$

(see Appendix A, p. 182.)

or $\{(2, 2, 1, 1)\}$.

Using bars to distinguish the submat-rices, write the original matrix so —

$$(1\ 2,\ \bar{3},\ \bar{\bar{4}}).$$

Observing the unit-step principle within the bounds of a sub-matrix, select groups of two indices. Obviously they fall into

$$\{(1\bar{3}, 2\bar{3}), (1\bar{\bar{4}}, 2\bar{\bar{4}}), (12), (\bar{3}\bar{\bar{4}})\}.$$

Notice the unit-step principle in the first two brackets.

Now, this matrix of the 6th order is of the type $\{(2, 2, 1, 1)\}$, thus agreeing with the previous result.

Had the original matrix been

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \text{ or } \{(3,1)\},$$

it could be written so –

$$(1\,2\,3, \bar{4})$$

when the ordering for $C^{(2)}$ would be

$$\{(12, 13, 23), (1\bar{4}, 2\bar{4}, 3\bar{4})\}$$

i.e. $C^{(2)}$ would be of the type $\{(3), (3)\}$.

Similarly,

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \text{ or } \{(2, 2)\},$$

could be written $(12, \bar{3}\bar{4})$. and forming the 2nd compound the ordering for $C^{(2)}$

would be

$$\left\{ (1\bar{3}, 2\bar{3}, 2\bar{4}), (1\bar{4}) \; ; \; (12) \; ; \; (\bar{3}\bar{4}) \right\}$$

i.e. $C^{(2)}$ would be of the type $\{(3,1),(1),(1)\}$,

both the foregoing results agreeing with those in Appendix A, pp. 179 + 181.

It will now be found that, if elements of compounds are represented umbrally as before, the step of a symbol from one submatrix to the next, whether the step is unitary or not, implies a vanishing minor. For example, if symbol 3 belongs to one submatrix, 4 to another, although $4-3=1$ minors like

$$\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{vmatrix}, \quad \text{etc. will vanish.}$$

Because of this, when considering tracks and chains, each submatrix must be regarded as possessing individuality, and may be distinguished by placing a particular sign, or group of signs, over its symbols. A number of suitable signs can be devised, such as the straight bar used above.

When forming chains, care must be taken to make each chain homo-

-geneous in its types of symbol (e.g. each of its members might have 2 symbols with one bar and 1 with two bars) whilst the unit-step principle is maintained.

With these precautions, tracks can be investigated by the method already stated (see p. 50) when it will be found that the number of groups in each chain gives the corresponding exponent.

e.g. (i) The 2nd compound of

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \mu \end{pmatrix} \quad \text{or} \quad \left\{ (12), (\bar{3}\bar{4}) \right\}$$

would be formed by ordering:-

	A chain of	Latent root.
12,	1,	λ^2
$\{1\bar{3}, 1\bar{4}, 2\bar{4}$,	3,	$\lambda\mu\}$
$2\bar{3}$,	1,	$\lambda\mu\}$
$\bar{3}\bar{4}$.	1.	μ^2

which gives the specification of latent roots and associated exponents.

The compound,

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\mu & \lambda & \underline{1} & \mu & 0 \\ 0 & 0 & \lambda\mu & \mu & 0 & 0 \\ 0 & 0 & 0 & \lambda\mu & 0 & 0 \\ 0 & 0 & 0 & \lambda & \lambda\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2 \end{pmatrix},$$

is in a suitable ordering of rows and columns.

Remove the underlined 1 by operations row 3 + $\frac{1}{\lambda}$ row 4, col. 4 − $\frac{1}{\lambda}$ col. 3. These give

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\mu & \lambda & 0 & \underline{\underline{\mu}} & 0 \\ 0 & 0 & \lambda\mu & \mu & 0 & 0 \\ 0 & 0 & 0 & \lambda\mu & 0 & 0 \\ 0 & 0 & 0 & \lambda & \lambda\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2 \end{pmatrix}.$$

Now remove the underlined μ by row 3 + $\frac{\mu}{\lambda}$ row 5, col. 5 − $\frac{\mu}{\lambda}$ col. 3. We get

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\mu & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda\mu & 2\mu & 0 & 0 \\ 0 & 0 & 0 & \lambda\mu & 0 & 0 \\ 0 & 0 & 0 & \underline{\underline{\lambda}} & \lambda\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2 \end{pmatrix}.$$

Lastly, the underlined λ can be removed by row 5 $- \dfrac{\lambda}{2\mu}$ row 3, col. 3 $+ \dfrac{\lambda}{2\mu}$ col. 5.

Then the matrix takes the normal form

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\mu & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda\mu & 2\mu & 0 & 0 \\ 0 & 0 & 0 & \lambda\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2 \end{pmatrix},$$

or $$\left\{ \overset{\lambda^2}{(1)}\,,\ \overset{\lambda\mu}{(3,1)}\,,\ \overset{\mu^2}{(1)} \right\}.$$

(ii)

Similarly, the 2nd compound of

$$C \equiv \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \nu \end{pmatrix} \quad \text{or} \quad (123, \bar{4}\bar{5}, \bar{\bar{6}}\bar{\bar{7}})$$

is formed from

Ordering	A chain of	Latent root.
$12, 13, 23$	3	λ^2
$1\bar{4}, 1\bar{5}, 2\bar{5}, 3\bar{5}.$	4	$\lambda\mu$
$2\bar{4}, 3\bar{4}$	2	$\lambda\mu$
$1\bar{\bar{6}}, 1\bar{\bar{7}}, 2\bar{\bar{7}}, 3\bar{\bar{7}}.$	4	$\lambda\nu$
$2\bar{\bar{6}}, 3\bar{\bar{6}}.$	2	$\lambda\nu$
$\bar{4}\bar{\bar{6}}, \bar{4}\bar{\bar{7}}, \bar{5}\bar{\bar{7}}.$	3	$\mu\nu$
$\bar{5}\bar{\bar{6}}$	1	$\mu\nu$
$\bar{4}\bar{5}$	1	μ^2
$\bar{\bar{6}}\bar{\bar{7}}$	1	ν^2

It is :-

$$\left(\begin{array}{ccccccccccccccccccccc}
\lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda\mu & \lambda & 1 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 1 & \lambda\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \lambda\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & \lambda & 1 & 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & \lambda\nu & \nu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \lambda\nu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & \mu & 1 & \nu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & \nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & \mu\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu^2
\end{array}\right)$$

Operations :-

row 20 $-\ \frac{\mu}{\nu}$ row 18, col. 18 $+\ \frac{\mu}{\nu}$ col. 20 ;

row 15 $-\ \frac{\lambda}{\nu}$ row 12, col. 12 $+\ \frac{\lambda}{\nu}$ col. 15 ;

row 14 $-\ \frac{2\lambda}{\nu}$ row 11, col. 11 $+\ \frac{2\lambda}{\nu}$ col. 14 ;

row 14 $- \frac{1}{\nu}$ row 12, col. 12 $+ \frac{1}{\nu}$ col. 14;
row 9 $- \frac{\lambda}{\mu}$ row 6, col. 6 $+ \frac{\lambda}{\mu}$ col. 9;

row 8 $- \frac{2\lambda}{\mu}$ row 5, col. 5 $+ \frac{2\lambda}{\mu}$ col. 8;

row 8 $- \frac{1}{\mu}$ row 6, col. 6 $+ \frac{1}{\mu}$ col. 8

will remove all elements <u>under</u> the leading diagonal. The compound becomes

$$\left(\begin{array}{ccc|cccc|cc|cccc|cc|c|ccc|c|c}
\lambda^2 & \lambda & 1 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
0 & \lambda^2 & \lambda & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&\lambda^2 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
\hline
0&0&0 & \lambda\mu & 3\lambda & 2 & 0 & \mu & 0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0 & \lambda\mu & \mu & 0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&\lambda\mu & \mu & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&0&\lambda\mu & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
\hline
0&0&0 & 0&0&0&0 & \lambda\mu & \mu & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&0&0 & 0 & \lambda\mu & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0&0\\
\hline
0&0&0 & 0&0&0&0 & 0&0 & \lambda\nu & 3\lambda & 2 & 0 & \nu & 0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&0&0 & 0&0 & 0 & \lambda\nu & \nu & 0 & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&0&0 & 0&0 & 0&0&\lambda\nu & \nu & 0&0 & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&\lambda\nu & 0&0 & 0 & 0&0&0 & 0&0\\
\hline
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & \lambda\nu & \nu & 0 & 0&0&0 & 0&0\\
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0 & \lambda\nu & 0 & 0&0&0 & 0&0\\
\hline
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & \mu^2 & 0&0&0 & 0&0\\
\hline
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & \mu\nu & 3\mu & 1 & \nu & 0\\
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0 & \mu\nu & \nu & 0&0\\
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&\mu\nu & 0&0\\
\hline
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & \mu\nu & 0\\
\hline
0&0&0 & 0&0&0&0 & 0&0 & 0&0&0&0 & 0&0 & 0 & 0&0&0 & 0 & \nu^2
\end{array}\right)$$

Then operations:-

row 18 + $\frac{\nu}{2\mu}$ row 20, Col. 20 − $\frac{\nu}{2\mu}$ col. 18;

row 18 + $\frac{1}{2\mu}$ row 19, Col. 19 − $\frac{1}{2\mu}$ Col. 18;

row 11 + $\frac{\nu}{3\lambda}$ row 14, Col. 14 − $\frac{\nu}{3\lambda}$ col. 11;

row 12 + $\frac{\nu}{3\lambda}$ row 15, Col. 15 − $\frac{\nu}{3\lambda}$ Col. 12;

row 10 − $\frac{2}{\nu}$ row 11, Col. 11 + $\frac{2}{\nu}$ Col. 10;

row 1 − $\frac{1}{\lambda}$ row 2, Col. 2 + $\frac{1}{\lambda}$ Col. 1;

row 5 + $\frac{\mu}{3\lambda}$ row 8, Col. 8 − $\frac{\mu}{3\lambda}$ Col. 5;

row 6 + $\frac{\mu}{3\lambda}$ row 9, Col. 9 − $\frac{\mu}{3\lambda}$ Col. 6;

row 4 − $\frac{2}{\mu}$ row 5, Col. 5 + $\frac{2}{\mu}$ Col. 4

give

(over

$$\begin{pmatrix}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda\mu & 3\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\mu & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & 3\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & \nu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda\nu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & \nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu^2
\end{pmatrix}$$

and the matrix is now reduced to normal form

$$\left\{ \overset{\lambda^2}{(3)} \; ; \; \overset{\lambda\mu}{(4,2)} \; ; \; \overset{\lambda\nu}{(4,2)} \; ; \; \overset{\mu^2}{(1)} \; ; \; \overset{\mu\nu}{(3,1)} \; ; \; \overset{\nu^2}{(1)} \right\}.$$

Examples showing the method of the removal of elements from various compounds are given in Appendix C. In some cases, a start has been made on the last row and the elements below the diagonal have been removed first. The matrix is then in the form discussed by Hawkes, i.e. it is composed of submatrices, each of which has all its roots equal and all the elements below the principal diagonal are zeros. Consequently, if an element $a_{ij} \neq 0$, and at the same time $a_{j-1,j} \neq 0$, we may trans-form the submatrix by operation $H^{-1}AH$, where, in H, $h_{kk} = 1$ and all the other elements are zero excepting $h_{i,j-1}$, which has the value $a_{ij} / a_{j-1,j}$.

This transformation replaces a_{ij} by zero, but it does not affect the remainder of the submatrix except certain elements in the first $i-1$ rows.

The probable validity of the rule of chains of maximum length, constructed on a unit-step-index principle, is seen from the results obtained in Appendix C. There, the various compounds of all possible matrices of orders 1 to 5 are formed according to this rule. It is then shown that in every case operations like (row p + r row q, col. q − r col. p) are sufficient to reduce the compound to normal form, no interchange of rows and columns being necessary.

While algebraic expressions for $e_r^{(m)}$ in terms of m, n and r have been arrived at only in the cases of e_1 and e_2 for compounds of matrices of type $\{(n)\}$, a prescription has been found for their evaluation (by a practical combinatory method) in any particular case.

Using this method we will now classify and give tables for the 2nd, 3rd, 4th, ... compounds (in normal form) of matrices in ascending order and of different normal specifications.

3rd order matrices.

The possible matrices are:-

$$(1)\ \begin{pmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 1\\ 0 & 0 & \lambda_1\end{pmatrix};\quad (2)\ \begin{pmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_1\end{pmatrix};\quad (3)\ \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_1\end{pmatrix};$$

$$(4)\ \begin{pmatrix}\lambda_1 & 1 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{pmatrix};\quad (5)\ \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{pmatrix};\quad (6)\ \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}.$$

(1) is of type $\{(3)\}$; (2) and (4) are of types $\{(2,1)\}$ and $\{(2);(1)\}$; (3), (5) and (6) are of types $\{(1,1,1)\}$, $\{(1,1);(1)\}$ and $\{(1);(1);(1)\}$.

As any compound of (2) could be derived from the same compound of (4) by writing λ_1 for λ_2, we will consider these to be of the same type, and write them $(2,1,0)$, to indicate their exponents, or, $(12,\bar{3})$, when intending to form chains. Similarly (3), (5) and (6) can all be written $(1,1,1)$ or $(1,\bar{2},\bar{\bar{3}})$.
Thus there are these three types to consider:-

(1) $(3,0,0)$ or (123)
(2) $(2,1,0)$ or $(12,\bar{3})$
(3) $(1,1,1)$ or $(1,\bar{2},\bar{\bar{3}})$.

1\. Type $(3,0,0)$ or $(\overset{\lambda_1}{123})$

Ordering for 2nd Compound.	A chain of	Latent Root.
$[12, 13, 23]$	3	λ_1^2

2\. Type $(2,1,0)$ or $(\overset{\lambda_1}{12}, \overset{\lambda_3}{\bar{\bar{3}}})$

Ordering for 2nd compound.	A chain of	Latent Root.
$[12;$	1	λ_1^2
$1\bar{3}, 2\bar{3}.]$	2	$\lambda_1\lambda_3$

3\. Type $(1,1,1)$ or $(\underset{\lambda_1}{1}, \underset{\lambda_2}{\bar{2}}, \underset{\lambda_3}{\bar{\bar{3}}})$

Ordering for 2nd compound.	A chain of	Latent Root.
$[1\bar{2};$	1	$\lambda_1\lambda_2$
$1\bar{\bar{3}};$	1	$\lambda_1\lambda_3$
$\bar{2}\bar{\bar{3}}]$	1	$\lambda_2\lambda_3.$

The 14 possible matrices of 4th order have already been given on p. 41.
Just as the 6 possible 3rd order matrices were reduced to three types, these 14 may be reduced on the same principle to the following five types:-

I, (1) is of type (4, 0, 0, 0);
II (2) and (6) are of type (3, 1, 0, 0);
III (5) and (9) are of type (2, 2, 0, 0);
IV (3), (7), (10) and (12) are of type (2, 1, 1, 0).
V (4), (8), (11), (13) and (14) are of type (1, 1, 1, 1).

λ_1

1. ## Type $(4,0,0,0)$ or (1234)

Ordering for 2nd compound.	A chain of	Latent Root.
[12, 13, 23, 24, 34;	5	λ_1^2
14]	1	λ_1^2
Ordering for 3rd Compound.		
[123, 124, 134, 234]	4	λ_1^3

2. ## Type $(3,1,0,0)$ or $(123, \bar{4})$

$\lambda_1 . \lambda_4$

Ordering for 2nd compound	A chain of	Latent Root.
[12, 13, 23;	3	λ_1^2
$1\bar{4}$, $2\bar{4}$, $3\bar{4}$]	3	$\lambda_1\lambda_4$
Ordering for 3rd compound.		
[$12\bar{4}$, $13\bar{4}$, $23\bar{4}$;	3	$\lambda_1^2\lambda_4$
123]	1	λ_1^3

3. ## Type $(2,2,0,0)$ or $(12, \bar{3}\bar{4})$

$\lambda_1 \quad \lambda_3$

Ordering for 2nd compound.		A chain of	Latent Root.
[$1\bar{3}$, $1\bar{4}$, $2\bar{4}$;	or [$1\bar{3}$, $2\bar{3}$, $2\bar{4}$;	3	$\lambda_1\lambda_3$
$2\bar{3}$;	$1\bar{4}$;	1	$\lambda_1\lambda_3$
12;	12;	1	λ_1^2
$\bar{3}\bar{4}$]	$\bar{3}\bar{4}$]	1	λ_3^2

Ordering for 3rd compound.	A chain of	Latent Root.
[$12\bar{3}$, $12\bar{4}$;	2	$\lambda_1^2\lambda_3$
$1\bar{3}\bar{4}$; $2\bar{3}\bar{4}$]	2	$\lambda_1\lambda_3^2$

4\. Type $(2,1,1,0)$ or $(12,\bar{3},\bar{\bar{4}})$

Ordering for 2nd compound	A chain of	Latent Root.
[$1\bar{3}$, $2\bar{3}$;	2	$\lambda_1\lambda_3$
$1\bar{\bar{4}}$, $2\bar{\bar{4}}$;	2	$\lambda_1\lambda_4$
12 ;	1	λ_1^2
$\bar{3}\bar{\bar{4}}$.]	1	$\lambda_3\lambda_4$
Ordering for 3rd compound.		
[$1\bar{3}\bar{\bar{4}}$, $2\bar{3}\bar{\bar{4}}$;	2	$\lambda_1\lambda_3\lambda_4$
$12\bar{3}$;	1	$\lambda_1^2\lambda_3$
$12\bar{\bar{4}}$]	1	$\lambda_1^2\lambda_4$

5\. Type $(1,1,1,1)$ or $(1,\bar{2},\bar{\bar{3}},\bar{\bar{\bar{4}}})$.

Ordering for 2nd compound.	A chain of	Latent Root.
[$1\bar{2}$;	1	$\lambda_1\lambda_2$
$1\bar{\bar{3}}$;	1	$\lambda_1\lambda_3$
$1\bar{\bar{\bar{4}}}$;	1	$\lambda_1\lambda_4$
$\bar{2}\bar{\bar{3}}$;	1	$\lambda_2\lambda_3$
$\bar{2}\bar{\bar{\bar{4}}}$;	1	$\lambda_2\lambda_4$
$\bar{\bar{3}}\bar{\bar{\bar{4}}}$.]	1	$\lambda_3\lambda_4$

Ordering for 3rd compound.	A chain of	Latent Root.
$[\ 1\bar{2}\bar{\bar{3}};$	1	$\lambda_1\lambda_2\lambda_3$
$1\bar{2}\bar{\bar{\bar{4}}};$	1	$\lambda_1\lambda_2\lambda_4$
$1\bar{\bar{3}}\bar{\bar{\bar{4}}};$	1	$\lambda_1\lambda_3\lambda_4$
$\bar{2}\bar{\bar{3}}\bar{\bar{\bar{4}}}\ ]$	1	$\lambda_2\lambda_3\lambda_4$

The types of 5th order matrices are :-

(1) $(5, 0, 0, 0, 0)$ or (12345);

(2) $(4, 1, 0, 0, 0)$ or $(1234, \bar{5})$;

(3) $(3, 2, 0, 0, 0)$ or $(123, \bar{4}\bar{5})$;

(4) $(3, 1, 1, 0, 0)$ or $(123, \bar{4}, \bar{\bar{5}})$;

(5) $(2, 2, 1, 0, 0)$ or $(12, \bar{3}\bar{4}, \bar{\bar{5}})$;

(6) $(2, 1, 1, 0, 0)$ or $(12, \bar{3}, \bar{\bar{4}}, \bar{\bar{\bar{5}}})$;

(7) $(1, 1, 1, 1, 1)$ or $(1, \bar{2}, \bar{\bar{3}}, \bar{\bar{\bar{4}}}, \bar{\bar{\bar{\bar{5}}}})$.

1. Type (5,0,0,0,0) or (12345).

Ordering for 2nd compound.	A chain of	Latent Root.
[12, 13, 23, 24, 34, 35, 45;	7	λ_1^2
14, 15, 25]	3	λ_1^2
Ordering for 3rd compound.		
[123, 124, 134, 234, 235, 245, 345;	7	λ_1^3
125, 135, 145]	3	λ_1^3
Ordering for 4th compound.		
[1234, 1235, 1245, 1345, 2345]	5	λ_1^4

2. Type (4,1,0,0,0) or (1234, $\bar{5}$).

Ordering for 2nd compound	A chain of	Latent Root.
[12, 13, 23, 24, 34;	5	λ_1^2
14;	1	λ_1^2
1$\bar{5}$, 2$\bar{5}$, 3$\bar{5}$, 4$\bar{5}$]	4	$\lambda_1\lambda_5$
Ordering for 3rd compound.		
[123, 124, 134, 234;	4	λ_1^3
12$\bar{5}$, 13$\bar{5}$, 23$\bar{5}$, 24$\bar{5}$, 34$\bar{5}$;	5	$\lambda_1^2\lambda_5$
14$\bar{5}$]	1	$\lambda_1^2\lambda_5$

Ordering for 4th compound	A chain of	Latent Root.
[1234 ;	1	λ_1^4
$123\bar{5}$, $124\bar{5}$, $134\bar{5}$, $234\bar{5}$]	4	$\lambda_1^3\lambda_5$

3. ## Type (3,2,0,0,0) or (123, $\bar{4}\bar{5}$)

Ordering for 2nd compound		A chain of	Latent Root.
[12, 13, 23 ;		3	λ_1^2
$\bar{4}\bar{5}$;		1	λ_4^2
$1\bar{4}$, $2\bar{4}$, $3\bar{4}$, $3\bar{5}$;	or $1\bar{4}$, $1\bar{5}$, $2\bar{5}$, $3\bar{5}$;	4	$\lambda_1\lambda_4$
$1\bar{5}$, $2\bar{5}$]	$2\bar{4}$, $3\bar{4}$]	2	$\lambda_1\lambda_4$

Ordering for 3rd compound.			
[$1\bar{4}\bar{5}$, $2\bar{4}\bar{5}$, $3\bar{4}\bar{5}$;		3	$\lambda_1\lambda_4^2$
123 ;		1	λ_1^3
$12\bar{4}$, $13\bar{4}$, $23\bar{4}$, $23\bar{5}$;	or $12\bar{4}$, $12\bar{5}$, $13\bar{5}$, $23\bar{5}$;	4	$\lambda_1^2\lambda_4$
$12\bar{5}$, $13\bar{5}$.]	$13\bar{4}$, $23\bar{4}$]	2	$\lambda_1^2\lambda_4$

Ordering for 4th compound.		
[$123\bar{4}$, $123\bar{5}$;	2	$\lambda_1^3\lambda_4$
$12\bar{4}\bar{5}$, $13\bar{4}\bar{5}$, $23\bar{4}\bar{5}$]	3	$\lambda_1^2\lambda_4^2$

4. ## Type $(3,1,1,0,0)$ or $(123, \bar{4}, \bar{\bar{5}})$

Ordering for 2nd compound	A chain of	Latent Root.
$[12, 13, 23;$	3	λ_1^2
$1\bar{4}, 2\bar{4}, 3\bar{4};$	3	$\lambda_1\lambda_4$
$1\bar{\bar{5}}, 2\bar{\bar{5}}, 3\bar{\bar{5}};$	3	$\lambda_1\lambda_5$
$\bar{4}\bar{\bar{5}}]$	1	$\lambda_4\lambda_5$

Ordering for 3rd compound		
$[123;$	1	λ_1^3
$12\bar{4}, 13\bar{4}, 23\bar{4};$	3	$\lambda_1^2\lambda_4$
$12\bar{\bar{5}}, 13\bar{\bar{5}}, 23\bar{\bar{5}};$	3	$\lambda_1^2\lambda_5$
$1\bar{4}\bar{\bar{5}}, 2\bar{4}\bar{\bar{5}}, 3\bar{4}\bar{\bar{5}}]$	3	$\lambda_1\lambda_4\lambda_5$

ordering for 4th compound.		
$[123\bar{4};$	1	$\lambda_1^3\lambda_4$
$123\bar{\bar{5}};$	1	$\lambda_1^3\lambda_5$
$12\bar{4}\bar{\bar{5}}, 13\bar{4}\bar{\bar{5}}, 23\bar{4}\bar{\bar{5}}]$	3	$\lambda_1^2\lambda_4\lambda_5.$

5. ## Type $(2,2,1,0,0)$ or $(12, \bar{3}\bar{4}, \bar{\bar{5}})$.

Ordering for 2nd compound		A chain of	Latent Root.
$[12;$ or	$[12;$	1	λ_1^2
$1\bar{3}, 2\bar{3}, 2\bar{4};$	$1\bar{3}, 1\bar{4}, 2\bar{4};$	3	$\lambda_1\lambda_3$
$1\bar{\bar{5}}, 2\bar{\bar{5}};$	$1\bar{\bar{5}}, 2\bar{\bar{5}};$	2	$\lambda_1\lambda_5$
$\bar{3}\bar{\bar{5}}, \bar{4}\bar{\bar{5}};$	$\bar{3}\bar{\bar{5}}, \bar{4}\bar{\bar{5}};$	2	$\lambda_3\lambda_5$
$1\bar{4};$	$2\bar{3};$	1	$\lambda_1\lambda_4$
$\bar{3}\bar{4}]$	$\bar{3}\bar{4}]$	1	$\lambda_3\lambda_4$

Ordering for 3rd compound		A chain of	Latent Root.
$[12\bar{3},\ 12\bar{4};$		2	$\lambda_1^2\lambda_3$
$1\bar{3}\bar{4},\ 2\bar{3}\bar{4};$		2	$\lambda_1\lambda_3^2$
$12\bar{\bar{5}};$		1	$\lambda_1^2\lambda_5$
$\bar{3}\bar{4}\bar{\bar{5}};$		1	$\lambda_3^2\lambda_5$
$1\bar{3}\bar{\bar{5}},\ 2\bar{3}\bar{\bar{5}},\ 2\bar{4}\bar{\bar{5}};$	or $1\bar{3}\bar{\bar{5}},\ 1\bar{4}\bar{\bar{5}},\ 2\bar{4}\bar{\bar{5}}$	3	$\lambda_1\lambda_3\lambda_5$
$1\bar{4}\bar{\bar{5}}\ ]$	$2\bar{3}\bar{\bar{5}}\ ]$	1	$\lambda_1\lambda_3\lambda_5$

Ordering for 4th compound		
$[12\bar{3}\bar{4};$	1	$\lambda_1^2\lambda_3^2$
$12\bar{3}\bar{\bar{5}},\ 12\bar{4}\bar{\bar{5}};$	2	$\lambda_1^2\lambda_3\lambda_5$
$1\bar{3}\bar{4}\bar{\bar{5}},\ 2\bar{3}\bar{4}\bar{\bar{5}}\ ]$	2	$\lambda_1\lambda_3^2\lambda_5$

6. <u>Type $(2, 1, 1, 1, 0)$ or $(12, \bar{3}, \bar{\bar{4}}, \bar{\bar{\bar{5}}})$.</u>

Ordering for 2nd compound.	A chain of	Latent Root.
$[\ 12;$	1	λ_1^2
$1\bar{3},\ 2\bar{3};$	2	$\lambda_1\lambda_3$
$1\bar{\bar{4}},\ 2\bar{\bar{4}};$	2	$\lambda_1\lambda_4$
$1\bar{\bar{\bar{5}}},\ 2\bar{\bar{\bar{5}}};$	2	$\lambda_1\lambda_5$
$\bar{3}\bar{\bar{4}};$	1	$\lambda_3\lambda_4$
$\bar{3}\bar{\bar{\bar{5}}};$	1	$\lambda_3\lambda_5$
$\bar{\bar{4}}\bar{\bar{\bar{5}}}\ ]$	1	$\lambda_4\lambda_5$

Ordering for 3rd compound	A chain of	Latent Root.
$[\,1 2 \bar{3}\,;$	1	$\lambda_1^2\lambda_3$
$1 2 \bar{\bar{4}}\,;$	1	$\lambda_1^2\lambda_4$
$1 2 \bar{\bar{\bar{5}}}\,;$	1	$\lambda_1^2\lambda_5$
$1\bar{3}\bar{\bar{4}}\,,\quad 2\bar{3}\bar{\bar{4}}\,;$	2	$\lambda_1\lambda_3\lambda_4$
$1\bar{3}\bar{\bar{\bar{5}}}\,,\quad 2\bar{3}\bar{\bar{\bar{5}}}\,;$	2	$\lambda_1\lambda_3\lambda_5$
$1\bar{\bar{4}}\bar{\bar{\bar{5}}}\,,\quad 2\bar{\bar{4}}\bar{\bar{\bar{5}}}\,;$	2	$\lambda_1\lambda_4\lambda_5$
$\bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}}\,]$	1	$\lambda_3\lambda_4\lambda_5$
Ordering for 4th compound		
$[\,1 2 \bar{3}\bar{\bar{4}}\,;$	1	$\lambda_1^2\lambda_3\lambda_4$.
$1 2 \bar{3}\bar{\bar{\bar{5}}}\,;$	1	$\lambda_1^2\lambda_3\lambda_5$
$1 2 \bar{\bar{4}}\bar{\bar{\bar{5}}}\,;$	1	$\lambda_1^2\lambda_4\lambda_5$
$1\bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}}\,,\quad 2\bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}}\,]$	2	$\lambda_1\lambda_3\lambda_4\lambda_5$

7. Type $(1,1,1,1,1)$ or $(1,\bar{2},\bar{\bar{3}},\bar{\bar{\bar{4}}},\bar{\bar{\bar{\bar{5}}}})$.

Ordering for 2nd compound	A chain of	Latent Root.
$[\,1\bar{2}\,;$	1	$\lambda_1\lambda_2$
$1\bar{\bar{3}}\,;$	1	$\lambda_1\lambda_3$
$1\bar{\bar{\bar{4}}}\,;$	1	$\lambda_1\lambda_4$
$1\bar{\bar{\bar{\bar{5}}}}\,;$	1	$\lambda_1\lambda_5$
$\bar{2}\bar{\bar{3}}\,;$	1	$\lambda_2\lambda_3$
$\bar{2}\bar{\bar{\bar{4}}}\,;$	1	$\lambda_2\lambda_4$
$\bar{2}\bar{\bar{\bar{\bar{5}}}}\,;$	1	$\lambda_2\lambda_5$
$\bar{\bar{3}}\bar{\bar{\bar{4}}}\,;$	1	$\lambda_3\lambda_4$
$\bar{\bar{3}}\bar{\bar{\bar{\bar{5}}}}\,;$	1	$\lambda_3\lambda_5$
$\bar{\bar{\bar{4}}}\bar{\bar{\bar{\bar{5}}}}\,]$	1	$\lambda_4\lambda_5$

Ordering for 3rd compound	A chain of	Latent Roots
[123;	1	$\lambda_1\lambda_2\lambda_3$
124;	1	$\lambda_1\lambda_2\lambda_4$
125;	1	$\lambda_1\lambda_2\lambda_5$
134;	1	$\lambda_1\lambda_3\lambda_4$
135;	1	$\lambda_1\lambda_3\lambda_5$
145;	1	$\lambda_1\lambda_4\lambda_5$
234;	1	$\lambda_2\lambda_3\lambda_4$
235;	1	$\lambda_2\lambda_3\lambda_5$
245;	1	$\lambda_2\lambda_4\lambda_5$
345.]	1	$\lambda_3\lambda_4\lambda_5$
Ordering for 4 th compound.		
[1234;	1	$\lambda_1\lambda_2\lambda_3\lambda_4$
1235;	1	$\lambda_1\lambda_2\lambda_3\lambda_5$
1245;	1	$\lambda_1\lambda_2\lambda_4\lambda_5$
1345;	1	$\lambda_1\lambda_3\lambda_4\lambda_5$
2345]	1	$\lambda_2\lambda_3\lambda_4\lambda_5$

The types of 6th order matrices are:-

(1) $(6,0,0,0,0,0)$ or $(1\,2\,3\,4\,5\,6)$;

(2) $(5,1,0,0,0,0)$ or $(1\,2\,3\,4\,5,\bar{6})$;

(3) $(4,2,0,0,0,0)$ or $(1\,2\,3\,4,\bar{5}\,\bar{6})$;

(4) $(4,1,1,0,0,0)$ or $(1\,2\,3\,4,\bar{5},\bar{\bar{6}})$;

(5) $(3,3,0,0,0,0)$ or $(1\,2\,3,\bar{4}\,\bar{5}\,\bar{6})$;

(6) $(3,2,1,0,0,0)$ or $(1\,2\,3,\bar{4}\,\bar{5},\bar{\bar{6}})$;

(7) $(3,1,1,1,0,0)$ or $(1\,2\,3,\bar{4},\bar{\bar{5}},\bar{\bar{\bar{6}}})$;

(8) $(2,2,2,0,0,0)$ or $(1\,2,\bar{3}\,\bar{4},\bar{\bar{5}}\,\bar{\bar{6}})$;

(9) $(2,2,1,1,0,0)$ or $(1\,2,\bar{3}\,\bar{4},\bar{\bar{5}}\,\bar{\bar{\bar{6}}})$;

(10) $(2,1,1,1,1,0)$ or $(1\,2,\bar{3},\bar{\bar{4}},\bar{\bar{\bar{5}}},\bar{\bar{\bar{\bar{6}}}})$;

(11) $(1,1,1,1,1,1)$ or $(1,\bar{2},\bar{\bar{3}},\bar{\bar{\bar{4}}},\bar{\bar{\bar{\bar{5}}}},\bar{\bar{\bar{\bar{\bar{6}}}}})$.

1. Type $(6,0,0,0,0,0)$ or (123456)

Ordering for 2nd compound.	A chain of	Latent Root
[12, 13, 23, 24, 34, 35, 45, 46, 56;	9	λ_1^2
14, 15, 25, 26, 36;	5	λ_1^2
16]	1	λ_1^2
Ordering for 3rd compound.		
[123, 124, 134, 234, 235, 245, 345, 346, 356, 456;	10	λ_1^3
125, 135, 145, 146, 156, 256;	6	λ_1^3
126, 136, 236, 246]	4	λ_1^3
Ordering for 4th compound.		
[1234, 1235, 1245, 1345, 2345, 2346, 2356, 2456, 3456;	9	λ_1^4
1236, 1246, 1256, 1356, 1456;	5	λ_1^4
1346]	1	λ_1^4
Ordering for 5th compound.		
[12345, 12346, 12356, 12456, 13456, 23456]	6	λ_1^5

Type $(5,1,0,0,0,0)$ or $(12345,\bar{6})$.

Ordering for 2nd compound.	A chain of	Latent Root
[12, 13, 23, 24, 34, 35, 45;	7	λ_1^2
14, 15, 25;	3	λ_1^2
$1\bar{6}$, $2\bar{6}$, $3\bar{6}$, $4\bar{6}$, $5\bar{6}$]	5	$\lambda_1\lambda_6$

Ordering for 3rd compound.	A chain of	Latent Roo
$[123, 124, 134, 234, 235, 245, 345;$	7	λ_1^3
$125, 135, 145;$	3	λ_1^3
$12\bar{6}, 13\bar{6}, 23\bar{6}, 24\bar{6}, 34\bar{6}, 35\bar{6}, 45\bar{6};$	7	$\lambda_1^2\lambda_6$
$14\bar{6}, 15\bar{6}, 25\bar{6}]$	3	$\lambda_1^2\lambda_6$
Ordering for 4th compound		
$[1234, 1235, 1245, 1345, 2345;$	5	λ_1^4
$123\bar{6}, 124\bar{6}, 134\bar{6}, 234\bar{6}, 235\bar{6}, 245\bar{6}, 345\bar{6};$	7	$\lambda_1^3\lambda_6$
$125\bar{6}, 135\bar{6}, 145\bar{6}]$	3	$\lambda_1^3\lambda_6$
Ordering for 5th compound.		
$[12345;$	1	λ_1^5
$1234\bar{6}, 1235\bar{6}, 1245\bar{6}, 1345\bar{6}, 2345\bar{6}]$	5	$\lambda_1^4\lambda_6$

3. <u>Type (4, 2, 0, 0, 0, 0) or $(1234, \bar{5}\bar{6})$.</u>

Ordering for 2nd compound			A chain of	Latent Root
$[12, 13, 23, 24, 34;$			5	λ_1^2
$14;$			1	λ_1^2
$1\bar{5}, 1\bar{6}, 2\bar{6}, 3\bar{6}, 4\bar{6};$	or	$1\bar{5}, 2\bar{5}, 3\bar{5}\ 4\bar{5}, 4\bar{6};$	5	$\lambda_1\lambda_5$
$2\bar{5}, 3\bar{5}, 4\bar{5};$		$1\bar{6}, 2\bar{6}, 3\bar{6}$	3	$\lambda_1\lambda_5$
$\bar{5}\bar{6}]$			1	λ_5^2

Ordering for 3rd compound	A chain of	Latent Roots
[123, 124, 134, 234;	4	λ_1^3
$12\bar{5}$, $12\bar{6}$, $13\bar{6}$, $23\bar{6}$, $24\bar{6}$, $34\bar{6}$;	6	$\lambda_1^2\lambda_5$
$13\bar{5}$, $23\bar{5}$, $24\bar{5}$, $34\bar{5}$;	4	$\lambda_1^2\lambda_5$
$14\bar{5}$, $14\bar{6}$;	2	$\lambda_1^2\lambda_5$
$1\bar{5}\bar{6}$, $2\bar{5}\bar{6}$, $3\bar{5}\bar{6}$, $4\bar{5}\bar{6}$]	4	$\lambda_1\lambda_5^2$

There are 3 other orderings possible, all leading to chains of the same lengths as above and to the same latent roots.

Ordering for 4th compound.		
[1234;	1	λ_1^4
$123\bar{5}$, $123\bar{6}$, $124\bar{6}$, $134\bar{6}$, $234\bar{6}$;	5	$\lambda_1^3\lambda_5$
$124\bar{5}$, $134\bar{5}$, $234\bar{5}$;	3	$\lambda_1^3\lambda_5$
$12\bar{5}\bar{6}$, $13\bar{5}\bar{6}$, $23\bar{5}\bar{6}$, $24\bar{5}\bar{6}$, $34\bar{5}\bar{6}$;	5	$\lambda_1^2\lambda_5^2$
$14\bar{5}\bar{6}$]	1	$\lambda_1^2\lambda_5^2$.

One other ordering is possible and gives the same result.

Ordering for 5th compound.		
[$123\bar{5}\bar{6}$, $124\bar{5}\bar{6}$, $134\bar{5}\bar{6}$ $234\bar{5}\bar{6}$;	4	$\lambda_1^3\lambda_5^2$
$1234\bar{5}$, $1234\bar{6}$]	2	$\lambda_1^4\lambda_5$

4 Type $(4,1,1,0,0,0)$ or $(1234, \bar{5}, \bar{\bar{6}})$

Ordering for 2nd compounds	A chain of	Latent Root
$[12, 13, 23, 24, 34;$	5	λ_1^2
$14;$	1	λ_1^2
$1\bar{5}, 2\bar{5}, 3\bar{5}, 4\bar{5};$	4	$\lambda_1\lambda_5$
$1\bar{\bar{6}}, 2\bar{\bar{6}}, 3\bar{\bar{6}}, 4\bar{\bar{6}};$	4	$\lambda_1\lambda_6$
$\bar{5}\bar{\bar{6}}]$	1	$\lambda_5\lambda_6$
Ordering for 3rd compounds		
$[123, 124, 134, 234;$	4	λ_1^3
$12\bar{5}, 13\bar{5}, 23\bar{5}, 24\bar{5}, 34\bar{5};$	5	$\lambda_1^2\lambda_5$
$14\bar{5};$	1	$\lambda_1^2\lambda_5$
$12\bar{\bar{6}}, 13\bar{\bar{6}}, 23\bar{\bar{6}}, 24\bar{\bar{6}}, 34\bar{\bar{6}};$	5	$\lambda_1^2\lambda_6$
$14\bar{\bar{6}};$	1	$\lambda_1^2\lambda_6$
$1\bar{5}\bar{\bar{6}}, 2\bar{5}\bar{\bar{6}}, 3\bar{5}\bar{\bar{6}}, 4\bar{5}\bar{\bar{6}}]$	4	$\lambda_1\lambda_5\lambda_6$
One other ordering is possible and gives the same result.		
Ordering for 4th compound.		
$[1234;$	1	λ_1^4
$123\bar{5}, 124\bar{5}, 134\bar{5}, 234\bar{5};$	4	$\lambda_1^3\lambda_5$
$123\bar{\bar{6}}, 124\bar{\bar{6}}, 134\bar{\bar{6}}, 234\bar{\bar{6}};$	4	$\lambda_1^3\lambda_6$
$12\bar{5}\bar{\bar{6}}, 13\bar{5}\bar{\bar{6}}, 23\bar{5}\bar{\bar{6}}, 24\bar{5}\bar{\bar{6}}, 34\bar{5}\bar{\bar{6}};$	5	$\lambda_1^2\lambda_5\lambda_6$
$14\bar{5}\bar{\bar{6}}]$	1	$\lambda_1^2\lambda_5\lambda_6$

Ordering for 5 th compound	A chain of	Latent Root
[$1234\bar{5}$;	1	$\lambda_1^4\lambda_5$
$1234\bar{\bar{6}}$;	1	$\lambda_1^4\lambda_6$
$123\bar{5}\bar{\bar{6}}$, $124\bar{5}\bar{\bar{6}}$, $134\bar{5}\bar{\bar{6}}$, $234\bar{5}\bar{\bar{6}}$]	4	$\lambda_1^3\lambda_5\lambda_6$

5. <u>Type (3,3,0,0,0,0) or $(123, \bar{4}\bar{5}\bar{6})$.</u>

Ordering for 2 nd compound.	A chain of	Latent Root
[12, 13, 23;	3	λ_1^2
$1\bar{4}$, $1\bar{5}$, $1\bar{6}$, $2\bar{6}$, $3\bar{6}$;	5	$\lambda_1\lambda_4$
$2\bar{4}$, $2\bar{5}$, $3\bar{5}$;	3	$\lambda_1\lambda_4$
$3\bar{4}$;	1	$\lambda_1\lambda_4$
$\bar{4}\bar{5}$, $\bar{4}\bar{6}$, $\bar{5}\bar{6}$,]	3	λ_4^2

Three other orderings are possible, and they all give the same result.

Ordering for 3 rd compound.		
[123;	1	λ_1^3
$\bar{4}\bar{5}\bar{6}$;	1	λ_4^3
$12\bar{4}$, $12\bar{5}$, $12\bar{6}$, $13\bar{6}$, $23\bar{6}$;	5	$\lambda_1^2\lambda_4$
$13\bar{4}$, $13\bar{5}$, $23\bar{5}$;	3	$\lambda_1^2\lambda_4$
$23\bar{4}$;	1	$\lambda_1^2\lambda_4$
$1\bar{4}\bar{5}$, $1\bar{4}\bar{6}$, $1\bar{5}\bar{6}$, $2\bar{5}\bar{6}$, $3\bar{5}\bar{6}$;	5	$\lambda_1\lambda_4^2$
$2\bar{4}\bar{5}$, $2\bar{4}\bar{\bar{6}}$, $3\bar{4}\bar{6}$;	3	$\lambda_1\lambda_4^2$
$3\bar{4}\bar{5}$]	1	$\lambda_1\lambda_4^2$

ordering for 3rd compound	A chain of	Latent Root.
Three other orderings are possible and they all give the same result.		

Ordering for 4th compound.		
[$123\bar{4}$, $123\bar{5}$, $123\bar{6}$;	3	$\lambda_1^3\lambda_4$
$12\bar{4}\bar{5}$, $12\bar{4}\bar{6}$, $12\bar{5}\bar{6}$, $13\bar{5}\bar{6}$, $23\bar{5}\bar{6}$;	5	$\lambda_1^2\lambda_4^2$
$13\bar{4}\bar{5}$, $13\bar{4}\bar{6}$, $23\bar{4}\bar{6}$;	3	$\lambda_1^2\lambda_4^2$
$23\bar{4}\bar{5}$;	1	$\lambda_1^2\lambda_4^2$
$1\bar{4}\bar{5}\bar{6}$, $2\bar{4}\bar{5}\bar{6}$, $3\bar{4}\bar{5}\bar{6}$.]	3	$\lambda_1\lambda_4^3$
other orderings give same result.		

Ordering for 5th compound.		
[$123\bar{4}\bar{5}$, $123\bar{4}\bar{6}$, $123\bar{5}\bar{6}$;	3	$\lambda_1^3\lambda_4^2$
$12\bar{4}\bar{5}\bar{6}$, $13\bar{4}\bar{5}\bar{6}$, $23\bar{4}\bar{5}\bar{6}$].	3	$\lambda_1\lambda_4^3$

Type (3,2,1,0,0,0) or (123, $\bar{4}\bar{5}$, $\bar{\bar{6}}$).

Ordering for 2nd compound	A chain of	Latent Root.
[12, 13, 23;	3	λ_1^2
$1\bar{4}$, $1\bar{5}$, $2\bar{5}$, $3\bar{5}$;	4	$\lambda_1\lambda_4$
$2\bar{4}$, $3\bar{4}$;	2	$\lambda_1\lambda_4$
$1\bar{6}$, $2\bar{\bar{6}}$, $3\bar{\bar{6}}$;	3	$\lambda_1\lambda_6$
$\bar{4}\bar{\bar{6}}$, $\bar{5}\bar{\bar{6}}$;	2	$\lambda_4\lambda_6$
$\bar{4}\bar{5}$.]	1	λ_4^2

Ordering for 2nd compounds	A chain of	Latent Root.
There is one other ordering possible which gives the same result.		
Ordering for 3rd compounds.		
$[123;$	1	λ_1^3
$12\bar{4},\ 12\bar{5},\ 13\bar{5},\ 23\bar{5};$	4	$\lambda_1^2\lambda_4$
$13\bar{4},\ 23\bar{4};$	2	$\lambda_1^2\lambda_4$
$1\bar{4}\bar{\bar{6}},\ 1\bar{5}\bar{\bar{6}},\ 2\bar{5}\bar{\bar{6}},\ 3\bar{5}\bar{\bar{6}};$	4	$\lambda_1\lambda_4\lambda_6$
$2\bar{4}\bar{\bar{6}},\ 3\bar{4}\bar{\bar{6}};$	2	$\lambda_1\lambda_4\lambda_6$
$1\bar{4}\bar{5},\ 2\bar{4}\bar{5},\ 3\bar{4}\bar{5};$	3	$\lambda_1\lambda_4^2$
$12\bar{\bar{6}},\ 13\bar{\bar{6}},\ 23\bar{\bar{6}};$	3	$\lambda_1^2\lambda_6$
$\bar{4}\bar{5}\bar{\bar{6}}]$	1	$\lambda_4^2\lambda_6.$
Other orderings give same result.		
Ordering for 4th compounds.		
$[123\bar{4},\ 123\bar{5};$	2	$\lambda_1^3\lambda_4$
$123\bar{\bar{6}};$	1	$\lambda_1^3\lambda_6$
$12\bar{4}\bar{5},\ 13\bar{4}\bar{5},\ 23\bar{4}\bar{5};$	3	$\lambda_1^2\lambda_4^2$
$12\bar{4}\bar{\bar{6}},\ 12\bar{5}\bar{\bar{6}},\ 13\bar{5}\bar{\bar{6}},\ 23\bar{5}\bar{\bar{6}};$	4	$\lambda_1^2\lambda_4\lambda_6$
$13\bar{4}\bar{\bar{6}},\ 23\bar{4}\bar{\bar{6}};$	2	$\lambda_1^2\lambda_4\lambda_6$
$1\bar{4}\bar{5}\bar{\bar{6}},\ 2\bar{4}\bar{5}\bar{\bar{6}},\ 3\bar{4}\bar{5}\bar{\bar{6}}].$	3	$\lambda_1\lambda_4^2\lambda_6$
Other orderings give same result.		

Ordering for 5 th compound.	A chain of	Latent Root.
$[12\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}};\ 13\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}},\ 23\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}};$	3	$\lambda_1^2\lambda_4^2\lambda_6$
$123\bar{4}\bar{\bar{\bar{6}}},\ 123\bar{\bar{5}}\bar{\bar{\bar{6}}};$	2	$\lambda_1^3\lambda_4\lambda_6$
$123\bar{4}\bar{\bar{5}}]$	1	$\lambda_1^3\lambda_4^2$

7. <u>Type $(3,1,1,1,0,0)$ or $(123, \bar{4}, \bar{\bar{5}}, \bar{\bar{\bar{6}}})$.</u>

Ordering for 2 nd compound	A chain of	Latent Root.
$[12, 13, 23;$	3	λ_1^2
$1\bar{4}, 2\bar{4}, 3\bar{4};$	3	$\lambda_1\lambda_4$
$1\bar{\bar{5}}, 2\bar{\bar{5}}, 3\bar{\bar{5}};$	3	$\lambda_1\lambda_5$
$1\bar{\bar{\bar{6}}}, 2\bar{\bar{\bar{6}}}, 3\bar{\bar{\bar{6}}};$	3	$\lambda_1\lambda_6$
$\bar{4}\bar{\bar{5}};$	1	$\lambda_4\lambda_5$
$\bar{4}\bar{\bar{\bar{6}}};$	1	$\lambda_4\lambda_6$
$\bar{\bar{5}}\bar{\bar{\bar{6}}}]$	1	$\lambda_5\lambda_6$

Ordering for 3rd compound.		
$[123;$	1	λ_1^3
$12\bar{4}, 13\bar{4}, 23\bar{4};$	3	$\lambda_1^2\lambda_4$
$12\bar{\bar{5}}, 13\bar{\bar{5}}, 23\bar{\bar{5}};$	3	$\lambda_1^2\lambda_5$
$12\bar{\bar{\bar{6}}}, 13\bar{\bar{\bar{6}}}, 23\bar{\bar{\bar{6}}};$	3	$\lambda_1^2\lambda_6$
$1\bar{4}\bar{\bar{5}}, 2\bar{4}\bar{\bar{5}}, 3\bar{4}\bar{\bar{5}};$	3	$\lambda_1\lambda_4\lambda_5$
$1\bar{4}\bar{\bar{\bar{6}}}, 2\bar{4}\bar{\bar{\bar{6}}}, 3\bar{4}\bar{\bar{\bar{6}}};$	3	$\lambda_1\lambda_4\lambda_6$
$1\bar{\bar{5}}\bar{\bar{\bar{6}}}, 2\bar{\bar{5}}\bar{\bar{\bar{6}}}, 3\bar{\bar{5}}\bar{\bar{\bar{6}}};$	3	$\lambda_1\lambda_5\lambda_6$
$\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}]$	1	$\lambda_4\lambda_5\lambda_6$

Ordering for 4th compound.	A chain of	Latent Root.
[$123\bar{4}$;	1	$\lambda_1^3\lambda_4$
$123\bar{\bar{5}}$;	1	$\lambda_1^3\lambda_5$
$123\bar{\bar{\bar{6}}}$;	1	$\lambda_1^3\lambda_6$
$12\bar{4}\bar{\bar{5}}$, $13\bar{4}\bar{\bar{5}}$, $23\bar{4}\bar{\bar{5}}$;	3	$\lambda_1^2\lambda_4\lambda_5$
$12\bar{4}\bar{\bar{\bar{6}}}$, $13\bar{4}\bar{\bar{\bar{6}}}$, $23\bar{4}\bar{\bar{\bar{6}}}$;	3	$\lambda_1^2\lambda_4\lambda_6$
$12\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $13\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $23\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	3	$\lambda_1^2\lambda_5\lambda_6$
$1\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $2\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $3\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$]	3	$\lambda_1\lambda_4\lambda_5\lambda_6$

Ordering for 5th compound		
[$123\bar{4}\bar{\bar{5}}$;	1	$\lambda_1^3\lambda_4\lambda_5$
$123\bar{4}\bar{\bar{\bar{6}}}$;	1	$\lambda_1^3\lambda_4\lambda_6$
$123\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	1	$\lambda_1^3\lambda_5\lambda_6$
$12\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $13\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $23\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$]	3	$\lambda_1^2\lambda_4\lambda_5\lambda_6$

8. <u>Type $(2, 2, 2, 0, 0, 0)$ or $(12, \bar{3}\bar{4}, \bar{\bar{5}}\bar{\bar{6}})$.</u>

Ordering for 2nd compound	A chain of	Latent Root.
[12 ;	1	λ_1^2
$1\bar{3}$, $1\bar{4}$, $2\bar{4}$;	3	$\lambda_1\lambda_3$
$2\bar{3}$;	1	$\lambda_1\lambda_3$
$1\bar{\bar{5}}$, $1\bar{\bar{6}}$, $2\bar{\bar{6}}$;	3	$\lambda_1\lambda_5$
$2\bar{\bar{5}}$;	1	$\lambda_1\lambda_5$
$\bar{3}\bar{4}$;	1	λ_3^2
$\bar{3}\bar{\bar{5}}$, $\bar{3}\bar{\bar{6}}$, $\bar{4}\bar{\bar{6}}$;	3	$\lambda_3\lambda_5$
$\bar{4}\bar{\bar{5}}$;	1	$\lambda_3\lambda_5$
$\bar{\bar{5}}\bar{\bar{6}}$]	1	λ_5^2

Ordering for 3rd compound	A chain of	Latent Root.
$[12\bar{3},\ 12\bar{4};$	2	$\lambda_1^2\lambda_3$
$12\bar{\bar{5}},\ 12\bar{\bar{6}};$	2	$\lambda_1^2\lambda_5$
$1\bar{3}\bar{4},\ 2\bar{3}\bar{4};$	2	$\lambda_1\lambda_3^2$
$1\bar{\bar{5}}\bar{\bar{6}},\ 2\bar{\bar{5}}\bar{\bar{6}};$	2	$\lambda_1\lambda_5^2$
$\bar{3}\bar{4}\bar{\bar{5}},\ \bar{3}\bar{4}\bar{\bar{6}};$	2	$\lambda_3^2\lambda_5$
$\bar{3}\bar{\bar{5}}\bar{\bar{6}},\ \bar{4}\bar{\bar{5}}\bar{\bar{6}};$	2	$\lambda_3\lambda_5^2$
$1\bar{3}\bar{\bar{5}},\ 1\bar{3}\bar{\bar{6}},\ 2\bar{3}\bar{\bar{6}},\ 2\bar{4}\bar{\bar{6}};$	4	$\lambda_1\lambda_3\lambda_5$
$1\bar{4}\bar{\bar{5}},\ 1\bar{4}\bar{\bar{6}};$	2	$\lambda_1\lambda_3\lambda_5$
$2\bar{3}\bar{\bar{5}},\ 2\bar{4}\bar{\bar{5}}.]$	2	$\lambda_1\lambda_3\lambda_5$
Other orderings give same result.		
Ordering for 4th compound.		
$[12\bar{3}\bar{4};$	1	$\lambda_1^2\lambda_3^2$
$12\bar{3}\bar{\bar{5}},\ 12\bar{4}\bar{\bar{5}},\ 12\bar{4}\bar{\bar{6}};$	3	$\lambda_1^2\lambda_3\lambda_5$
$12\bar{3}\bar{\bar{6}};$	1	$\lambda_1^2\lambda_3\lambda_5$
$12\bar{\bar{5}}\bar{\bar{6}};$	1	$\lambda_1^2\lambda_5^2$
$1\bar{3}\bar{4}\bar{\bar{5}},\ 2\bar{3}\bar{4}\bar{\bar{5}},\ 2\bar{3}\bar{4}\bar{\bar{6}};$	3	$\lambda_1\lambda_3^2\lambda_5$
$1\bar{3}\bar{4}\bar{\bar{6}};$	1	$\lambda_1\lambda_3^2\lambda_5$
$1\bar{3}\bar{\bar{5}}\bar{\bar{6}},\ 2\bar{3}\bar{\bar{5}}\bar{\bar{6}},\ 2\bar{4}\bar{\bar{5}}\bar{\bar{6}};$	3	$\lambda_1\lambda_3\lambda_5^2$
$1\bar{4}\bar{\bar{5}}\bar{\bar{6}};$	1	$\lambda_1\lambda_3\lambda_5^2$
$\bar{3}\bar{4}\bar{\bar{5}}\bar{\bar{6}}]$	1	$\lambda_3^2\lambda_5^2$

Ordering for 5th compound	A chain of	Latent Root.
[$1 2 \bar{3} \bar{4} \bar{\bar{5}}$, $1 2 \bar{3} \bar{4} \bar{\bar{6}}$;	2	$\lambda_1^2 \lambda_3^2 \lambda_5$
$1 2 \bar{3} \bar{\bar{5}} \bar{\bar{6}}$, $1 2 \bar{4} \bar{\bar{5}} \bar{\bar{6}}$;	2	$\lambda_1^2 \lambda_3 \lambda_5^2$
$1 \bar{3} \bar{4} \bar{\bar{5}} \bar{\bar{6}}$, $2 \bar{3} \bar{4} \bar{\bar{5}} \bar{\bar{6}}$]	2	$\lambda_1 \lambda_3^2 \lambda_5^2$

9. Type $(2,2,1,1,0,0)$ or $(12,\ \bar{3}\bar{4},\ \bar{\bar{5}},\bar{\bar{6}})$.

Ordering for 2nd compound	A chain of	Latent Root.
[12;	1	λ_1^2
$1\bar{3}$, $2\bar{3}$, $2\bar{4}$;	3	$\lambda_1\lambda_3$
$1\bar{4}$;	1	$\lambda_1\lambda_3$
$1\bar{\bar{5}}$, $2\bar{\bar{5}}$;	2	$\lambda_1\lambda_5$
$1\bar{\bar{6}}$, $2\bar{\bar{6}}$;	2	$\lambda_1\lambda_6$
$\bar{3}\bar{4}$;	1	λ_3^2
$\bar{3}\bar{\bar{5}}$, $\bar{4}\bar{\bar{5}}$;	2	$\lambda_3\lambda_5$
$\bar{3}\bar{\bar{6}}$, $\bar{4}\bar{\bar{6}}$;	2	$\lambda_3\lambda_6$
$\bar{\bar{5}}\bar{\bar{6}}$]	1	$\lambda_5\lambda_6$

Ordering for 3rd compounds.		
[$12\bar{3}$, $12\bar{4}$;	2	$\lambda_1^2\lambda_3$
$12\bar{\bar{5}}$;	1	$\lambda_1^2\lambda_5$
$12\bar{\bar{6}}$;	1	$\lambda_1^2\lambda_6$
$1\bar{3}\bar{4}$, $2\bar{3}\bar{4}$;	2	$\lambda_1\lambda_3^2$
$1\bar{3}\bar{\bar{5}}$, $1\bar{4}\bar{\bar{5}}$, $2\bar{4}\bar{\bar{5}}$;	3	$\lambda_1\lambda_3\lambda_5$
$2\bar{3}\bar{\bar{5}}$;	1	$\lambda_1\lambda_3\lambda_5$

Ordering for 3rd compound : contd.	A chain of	Latent Root:
$1\bar{3}\bar{\bar{\bar{6}}}$, $1\bar{4}\bar{\bar{\bar{6}}}$, $2\bar{4}\bar{\bar{\bar{6}}}$;	3	$\lambda_1\lambda_3\lambda_6$
$2\bar{3}\bar{\bar{\bar{6}}}$;	1	$\lambda_1\lambda_3\lambda_6$
$1\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $2\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	2	$\lambda_1\lambda_5\lambda_6$
$\bar{3}\bar{4}\bar{\bar{5}}$;	1	$\lambda_3^2\lambda_5$
$\bar{3}\bar{4}\bar{\bar{\bar{6}}}$;	1	$\lambda_3^2\lambda_6$
$\bar{3}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$]	2	$\lambda_3\lambda_5\lambda_6$
Ordering for 4th compounds.		
[$12\bar{3}\bar{4}$;	1	$\lambda_1^2\lambda_3^2$
$12\bar{3}\bar{\bar{5}}$, $12\bar{4}\bar{\bar{5}}$;	2	$\lambda_1^2\lambda_3\lambda_5$
$12\bar{3}\bar{\bar{\bar{6}}}$, $12\bar{4}\bar{\bar{\bar{6}}}$;	2	$\lambda_1^2\lambda_3\lambda_6$
$12\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	1	$\lambda_1^2\lambda_5\lambda_6$
$1\bar{3}\bar{4}\bar{\bar{5}}$, $2\bar{3}\bar{4}\bar{\bar{5}}$;	2	$\lambda_1\lambda_3^2\lambda_5$
$1\bar{3}\bar{4}\bar{\bar{\bar{6}}}$, $2\bar{3}\bar{4}\bar{\bar{\bar{6}}}$;	2	$\lambda_1\lambda_3^2\lambda_6$
$1\bar{3}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $1\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $2\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	3	$\lambda_1\lambda_3\lambda_5\lambda_6$
$2\bar{3}\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	1	$\lambda_1\lambda_3\lambda_5\lambda_6$
$\bar{3}\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$]	1	$\lambda_3^2\lambda_5\lambda_6$
Ordering for 5th compound.		
[$12\bar{3}\bar{4}\bar{\bar{5}}$;	1	$\lambda_1^2\lambda_3^2\lambda_5$
$12\bar{3}\bar{4}\bar{\bar{\bar{6}}}$;	1	$\lambda_1^2\lambda_3^2\lambda_6$
$12\bar{3}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $12\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$;	2	$\lambda_1^2\lambda_3\lambda_5\lambda_6$
$1\bar{3}\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$, $2\bar{3}\bar{4}\bar{\bar{5}}\bar{\bar{\bar{6}}}$]	2	$\lambda_1\lambda_3^2\lambda_5\lambda_6$

10. Type (2, 1, 1, 1, 1, 0) or $(12, \bar{3}, \bar{\bar{4}}, \bar{\bar{\bar{5}}}, \bar{\bar{\bar{\bar{6}}}})$.

Ordering for 2nd compound.	A chain of	Latent Root.
$[12;$	1	λ_1^2
$1\bar{3},\ 2\bar{3};$	2	$\lambda_1\lambda_3$
$1\bar{\bar{4}},\ 2\bar{\bar{4}};$	2	$\lambda_1\lambda_4$
$1\bar{\bar{\bar{5}}},\ 2\bar{\bar{\bar{5}}};$	2	$\lambda_1\lambda_5$
$1\bar{\bar{\bar{\bar{6}}}},\ 2\bar{\bar{\bar{\bar{6}}}};$	2	$\lambda_1\lambda_6$
$\bar{3}\bar{\bar{4}};$	1	$\lambda_3\lambda_4$
$\bar{3}\bar{\bar{\bar{5}}};$	1	$\lambda_3\lambda_5$
$\bar{3}\bar{\bar{\bar{\bar{6}}}};$	1	$\lambda_3\lambda_6$
$\bar{\bar{4}}\bar{\bar{\bar{5}}};$	1	$\lambda_4\lambda_5$
$\bar{\bar{4}}\bar{\bar{\bar{\bar{6}}}};$	1	$\lambda_4\lambda_6$
$\bar{\bar{\bar{5}}}\bar{\bar{\bar{\bar{6}}}}]$	1	$\lambda_5\lambda_6$

Ordering for 3rd compound.		
$[12\bar{3};$	1	$\lambda_1^2\lambda_3$
$12\bar{\bar{4}};$	1	$\lambda_1^2\lambda_4$
$12\bar{\bar{\bar{5}}};$	1	$\lambda_1^2\lambda_5$
$12\bar{\bar{\bar{\bar{6}}}};$	1	$\lambda_1^2\lambda_6$
$1\bar{3}\bar{\bar{4}},\ 2\bar{3}\bar{\bar{4}};$	2	$\lambda_1\lambda_3\lambda_4$
$1\bar{3}\bar{\bar{\bar{5}}},\ 2\bar{3}\bar{\bar{\bar{5}}};$	2	$\lambda_1\lambda_3\lambda_5$
$1\bar{3}\bar{\bar{\bar{\bar{6}}}},\ 2\bar{3}\bar{\bar{\bar{\bar{6}}}};$	2	$\lambda_1\lambda_3\lambda_6$
$1\bar{\bar{4}}\bar{\bar{\bar{5}}},\ 2\bar{\bar{4}}\bar{\bar{\bar{5}}};$	2	$\lambda_1\lambda_4\lambda_5$
$1\bar{\bar{4}}\bar{\bar{\bar{\bar{6}}}},\ 2\bar{\bar{4}}\bar{\bar{\bar{\bar{6}}}};$	2	$\lambda_1\lambda_4\lambda_6$
$1\bar{\bar{\bar{5}}}\bar{\bar{\bar{\bar{6}}}},\ 2\bar{\bar{\bar{5}}}\bar{\bar{\bar{\bar{6}}}};$	2	$\lambda_1\lambda_5\lambda_6$
$\bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}};$	1	$\lambda_3\lambda_4\lambda_5$
$\bar{3}\bar{\bar{4}}\bar{\bar{\bar{\bar{6}}}};$	1	$\lambda_3\lambda_4\lambda_6$
$\bar{3}\bar{\bar{\bar{5}}}\bar{\bar{\bar{\bar{6}}}};$	1	$\lambda_3\lambda_5\lambda_6$
$\bar{\bar{4}}\bar{\bar{\bar{5}}}\bar{\bar{\bar{\bar{6}}}}]$	1	$\lambda_4\lambda_5\lambda_6$

Ordering for 4 th compound.	A chain of	Latent Root.
$[1\,2\,\bar{3}\,\bar{\bar{4}}\,;$	1	$\lambda_1^2\lambda_3\lambda_4$
$1\,2\,\bar{3}\,\bar{\bar{\bar{5}}}\,;$	1	$\lambda_1^2\lambda_3\lambda_5$
$1\,2\,\bar{3}\,\bar{\bar{\bar{\bar{6}}}}\,;$	1	$\lambda_1^2\lambda_3\lambda_6$
$1\,2\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,;$	1	$\lambda_1^2\lambda_4\lambda_5$
$1\,2\,\bar{\bar{4}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	1	$\lambda_1^2\lambda_4\lambda_6$
$1\,2\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	1	$\lambda_1^2\lambda_5\lambda_6$
$1\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,,\quad 2\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,;$	2	$\lambda_1\lambda_3\lambda_4\lambda_5$
$1\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{\bar{6}}}}\,,\quad 2\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	2	$\lambda_1\lambda_3\lambda_4\lambda_6$
$1\,\bar{3}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,,\quad 2\,\bar{3}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	2	$\lambda_1\lambda_3\lambda_5\lambda_6$
$1\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,,\quad 2\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	2	$\lambda_1\lambda_4\lambda_5\lambda_6$
$\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,]$	1	$\lambda_3\lambda_4\lambda_5\lambda_6$

Ordering for 5 th compound.		
$[\,1\,2\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,;$	1	$\lambda_1^2\lambda_3\lambda_4\lambda_5$
$1\,2\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	1	$\lambda_1^2\lambda_3\lambda_4\lambda_6$
$1\,2\,\bar{3}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	1	$\lambda_1^2\lambda_3\lambda_5\lambda_6$
$1\,2\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,;$	1	$\lambda_1^2\lambda_4\lambda_5\lambda_6$
$1\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,;\quad 2\,\bar{3}\,\bar{\bar{4}}\,\bar{\bar{\bar{5}}}\,\bar{\bar{\bar{\bar{6}}}}\,]$	2	$\lambda_1\lambda_3\lambda_4\lambda_5\lambda_6$

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Type (1,1,1,1,1,1) or $(1\,\overline{2}\,\overline{\overline{3}}\,\overline{\overline{\overline{4}}}\,\overline{\overline{\overline{\overline{5}}}}\,\overline{\overline{\overline{\overline{\overline{6}}}}})$.

Ordering for 2nd compound	A chain of	Latent Root.
[12;	1	$\lambda_1\lambda_2$
13;	1	$\lambda_1\lambda_3$
14;	1	$\lambda_1\lambda_4$
15;	1	$\lambda_1\lambda_5$
16;	1	$\lambda_1\lambda_6$
23;	1	$\lambda_2\lambda_3$
24;	1	$\lambda_2\lambda_4$
25;	1	$\lambda_2\lambda_5$
26;	1	$\lambda_2\lambda_6$
34;	1	$\lambda_3\lambda_4$
35;	1	$\lambda_3\lambda_5$
36;	1	$\lambda_3\lambda_6$
45;	1	$\lambda_4\lambda_5$
46;	1	$\lambda_4\lambda_6$
56]	1	$\lambda_5\lambda_6$.
Ordering for 3rd compound		
[123;	1	$\lambda_1\lambda_2\lambda_3$
124;	1	$\lambda_1\lambda_2\lambda_4$
125;	1	$\lambda_1\lambda_2\lambda_5$
126;	1	$\lambda_1\lambda_2\lambda_6$
134;	1	$\lambda_1\lambda_3\lambda_4$
135;	1	$\lambda_1\lambda_3\lambda_5$
136;	1	$\lambda_1\lambda_3\lambda_6$

Ordering for 3rd compound	A chain of	Latent Root.
[145;	1	$\lambda_1\lambda_4\lambda_5$
146;	1	$\lambda_1\lambda_4\lambda_6$
156;	1	$\lambda_1\lambda_5\lambda_6$
234;	1	$\lambda_2\lambda_3\lambda_4$
235;	1	$\lambda_2\lambda_3\lambda_5$
236;	1	$\lambda_2\lambda_3\lambda_6$
245;	1	$\lambda_2\lambda_4\lambda_5$
246;	1	$\lambda_2\lambda_4\lambda_6$
256;	1	$\lambda_2\lambda_5\lambda_6$
345;	1	$\lambda_3\lambda_4\lambda_5$
346;	1	$\lambda_3\lambda_4\lambda_6$
356;	1	$\lambda_3\lambda_5\lambda_6$
456]	1	$\lambda_4\lambda_5\lambda_6$

Ordering for 4th compound.		
[1234;	1	$\lambda_1\lambda_2\lambda_3\lambda_4$
1235;	1	$\lambda_1\lambda_2\lambda_3\lambda_5$
1236;	1	$\lambda_1\lambda_2\lambda_3\lambda_6$
1245;	1	$\lambda_1\lambda_2\lambda_4\lambda_5$
1246;	1	$\lambda_1\lambda_2\lambda_4\lambda_6$
1256;	1	$\lambda_1\lambda_2\lambda_5\lambda_6$
1345;	1	$\lambda_1\lambda_3\lambda_4\lambda_5$
1346;	1	$\lambda_1\lambda_3\lambda_4\lambda_6$
1356;	1	$\lambda_1\lambda_3\lambda_5\lambda_6$

	A chain of	Latent Root.
1456;	1	$\lambda_1\lambda_4\lambda_5\lambda_6$
2345;	1	$\lambda_2\lambda_3\lambda_4\lambda_5$
2346;	1	$\lambda_2\lambda_3\lambda_4\lambda_6$
2356;	1	$\lambda_2\lambda_3\lambda_5\lambda_6$
2456;	1	$\lambda_2\lambda_4\lambda_5\lambda_6$
3456]	1	$\lambda_3\lambda_4\lambda_5\lambda_6$

Ordering for 5th compound.

[12345;	1	$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$
12346;	1	$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6$
12356;	1	$\lambda_1\lambda_2\lambda_3\lambda_5\lambda_6$
12456;	1	$\lambda_1\lambda_2\lambda_4\lambda_5\lambda_6$
13456;	1	$\lambda_1\lambda_3\lambda_4\lambda_5\lambda_6$
23456]	1	$\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6$

Tables of exponents of elementary divisors are given on the next two pages.

Compound.

Type	2nd	3rd	4th	5th	6th
(3, 0, 0)	(3, 0, 0)	1			
(2, 1, 0)	(2, 1, 0)	1			
(1, 1, 1)	(1, 1, 1)	1			
(4, 0, 0, 0)	(5, 1, 0, 0, 0, 0)	(4, 0, 0, 0)	1		
(3, 1, 0, 0)	(3, 3, 0, 0, 0, 0)	(3, 1, 0, 0)	1		
(2, 2, 0, 0)	(3, 1, 1, 1, 0, 0)	(2, 2, 0, 0)	1		
(2, 1, 1, 0)	(2, 2, 1, 1, 0, 0)	(2, 1, 1, 0)	1		
(1, 1, 1, 1)	(1, 1, 1, 1, 1, 1)	(1, 1, 1, 1)	1		
(5, 0, 0, 0, 0)	(7, 3, 0, 0 …)	(7, 3, 0, 0 …)	(5, 0, 0, 0, 0)	1	
(4, 1, 0, 0, 0)	(5, 4, 1, 0, …)	(5, 4, 1, 0, …)	(4, 1, 0, 0, 0)	1	
(3, 2, 0, 0, 0)	(4, 3, 2, 1, 0, ..)	(4, 3, 2, 1, 0..)	(3, 2, 0, 0, 0)	1	
(3, 1, 1, 0, 0)	(3, 3, 3, 1, 0, ..)	(3, 3, 3, 1, 0..)	(3, 1, 1, 0, 0)	1	
(2, 2, 1, 0, 0)	(3, 2, 2, 1, 1, 1, ..)	(3, 2, 2, 1, 1, 1..)	(2, 2, 1, 0, 0)	1	
(2, 1, 1, 1, 0)	(2, 2, 2, 1, 1, 1, 1, 0..)	(2, 2, 2, 1, 1, 1, 1..)	(2, 1, 1, 1, 0)	1	
(1, 1, 1, 1, 1)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	(1, 1, 1, 1, 1)	1	
(6, 0, 0, 0, 0, 0)	(9, 5, 1, 0, …)	(10, 6, 4, 0, …)	(9, 5, 1, 0…)	(6, 0, 0, 0, 0, 0)	1
(5, 1, 0, 0, 0, 0)	(7, 5, 3, 0, …)	(7, 7, 3, 3, 0, ..)	(7, 5, 3, 0, ..)	(5, 1, 0, 0, 0, 0)	1
(4, 2, 0, 0, 0, 0)	(5, 5, 3, 1, 1, 0, ..)	(6, 4, 4, 4, 2, 0..)	(5, 5, 3, 1, 1, 0..)	(4, 2, 0..)	1
(4, 1, 1, 0, 0, 0)	(5, 4, 4, 1, 1, 0, ..)	(5, 5, 4, 4, 1, 1..)	(5, 4, 4, 1, 1, 0..)	(4, 1, 1, 0..)	1
(3, 3, 0, 0, 0, 0)	(5, 3, 3, 3, 1, 0..)	(5, 5, 3, 3, 1, 1, 1, 1)	(5, 3, 3, 3, 1, 0..)	(3, 3, 0..)	1
(3, 2, 1, 0, 0, 0)	(4, 3, 3, 2, 2, 1, ..)	(4, 4, 3, 3, 2, 2, 1, 1)	(4, 3, 3, 2, 2, 1)	(3, 2, 1, ..)	1

Compound.

Type.	2nd	3rd	4th	5th	6th.
(3,1,1,1,0,0)	(3,3,3,3,1,1,1)	(3,3,3,3,3,3,1,1)	(3,3,3,3,1,1,1,)	(3,1,1,1,..)	
(2,2,2,0,0,0)	(3,3,3,1,1,1,1,1,1)	(4,2,2,2,2,2,2,2)	(3,3,3,1,1,1,1,1,1)	(2,2,2,0..)	
(2,2,1,1,0,0)	(3,2,2,2,2,1,1,1,1)	(3,3,2,2,2,2,1,1,1,1,1,1)	(3,2,2,2,2,1,1,1,1)	(2,2,1,1,..)	
(2,1,1,1,1,0)	(2,2,2,2,1,1,1,1,1,1,1)	(2,2,2,2,2,2,1,1,1,1,1,1,1,1)	(2,2,2,2,1,1,1,1,1,1,1)	(2,1,1,1,1..)	
(1,1,1,1,1,1)	(1,1 ... 1,1) 15	(1,1 . . . . . 1,1) 20	(1,1 . . . 1,1) 15	(1,1,1,1,1,1)	

The Normal Form of the Adjugate Matrix.

The adjugate matrix has already been defined in the Preface, p. 14.
It is obtained by replacing each element a_{pq} of a matrix A by A_{qp}, where A_{pq} denotes the cofactor of a_{pq} in A.

Thus the adjugate of $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is $\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$.

It will now be shown that <u>the specification of the exponents of the elementary divisors of the adjugate is the same as for the matrix itself.</u>

e.g. Matrix of type $\{(4)\}$ is $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$.

Its adjugate is

$$\begin{pmatrix} \lambda^3 & -\lambda^2 & \underline{\lambda} & \underline{-1} \\ 0 & \lambda^3 & -\lambda^2 & \underline{\lambda} \\ 0 & 0 & \lambda^3 & -\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}.$$

The underlined elements can be removed by

operations $\quad$ row $1 + \frac{1}{\lambda}$ row 2, $\quad$ col. $2 - \frac{1}{\lambda}$ col. 1 ;
row $3 - \frac{1}{\lambda}$ row 4, $\quad$ col. $4 + \frac{1}{\lambda}$ col. 3

giving, as its normal form,

$$\begin{pmatrix} \lambda^3 & -\lambda^2 & 0 & 0 \\ 0 & \lambda^3 & -\lambda^2 & 0 \\ 0 & 0 & \lambda^3 & -\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}.$$

This is also of type $\{(4)\}$ in λ^3.

Matrix of type $\{(2), (2), (1)\}$ or $(12, \bar{3}\bar{4}, \bar{\bar{5}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3^2\lambda_5 & -\lambda_3^2\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3^2\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_3\lambda_5 & -\lambda_1^2\lambda_5 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3^2 \end{pmatrix};$$

which also is of type $\{(2), (2), (1)\}$.

Matrix of type $\{(4);(2)\}$ or $(1234,\bar{5}\bar{6})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\left(\begin{array}{cccc|cc} \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & \underline{\lambda_1\lambda_5^2} & \underline{-\lambda_5^2} & 0 & 0 \\ 0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & \underline{\lambda_1\lambda_5^2} & 0 & 0 \\ 0 & 0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 & -\lambda_1^4 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 \end{array}\right)$$

The underlined elements can be removed by operations :-

row 2 + $\frac{1}{\lambda_1}$ row 3, col. 3 − $\frac{1}{\lambda_1}$ col. 2;

row 1 − $\frac{1}{\lambda_1^2}$ row 3, col. 3 + $\frac{1}{\lambda_1^2}$ col. 1;

row 1 + $\frac{2}{\lambda_1}$ row 2, col. 2 − $\frac{2}{\lambda_1}$ col. 1

giving the normal form

$$\left(\begin{array}{cccc|cc}
\lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^3\lambda_5^2 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 & -\lambda_1^4 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5
\end{array}\right)$$

which is of type $\{(4);(2)\}$.

In appendix E are given all the possible types of matrices of orders 3, 4, 5 and 6, along with their adjugates and the reduction of them to normal form. It will be seen that in every case the specification of the elementary divisors of an adjugate is the same as that of the matrix from which it is formed. The reason is this :-

Writing the adjugates of matrices of orders 3, 4, 5, 6 in umbral notation, they are respectively :-

$$(1)\quad \begin{pmatrix} \begin{vmatrix}23\\23\end{vmatrix} & -\begin{vmatrix}13\\23\end{vmatrix} & \begin{vmatrix}12\\23\end{vmatrix} \\ -\begin{vmatrix}23\\13\end{vmatrix} & \begin{vmatrix}13\\13\end{vmatrix} & -\begin{vmatrix}12\\13\end{vmatrix} \\ \begin{vmatrix}23\\12\end{vmatrix} & -\begin{vmatrix}13\\12\end{vmatrix} & \begin{vmatrix}12\\12\end{vmatrix} \end{pmatrix},$$

$$(2)\quad \begin{pmatrix} \begin{vmatrix}234\\234\end{vmatrix} & -\begin{vmatrix}134\\234\end{vmatrix} & \begin{vmatrix}124\\234\end{vmatrix} & -\begin{vmatrix}123\\234\end{vmatrix} \\ -\begin{vmatrix}234\\134\end{vmatrix} & \begin{vmatrix}134\\134\end{vmatrix} & -\begin{vmatrix}124\\134\end{vmatrix} & \begin{vmatrix}123\\134\end{vmatrix} \\ \begin{vmatrix}234\\124\end{vmatrix} & -\begin{vmatrix}134\\124\end{vmatrix} & \begin{vmatrix}124\\124\end{vmatrix} & -\begin{vmatrix}123\\124\end{vmatrix} \\ -\begin{vmatrix}234\\123\end{vmatrix} & \begin{vmatrix}134\\123\end{vmatrix} & -\begin{vmatrix}124\\123\end{vmatrix} & \begin{vmatrix}123\\123\end{vmatrix} \end{pmatrix},$$

$$(3)\quad \left(\begin{matrix}
\begin{vmatrix}2345\\2345\end{vmatrix} & -\begin{vmatrix}1345\\2345\end{vmatrix} & \begin{vmatrix}1245\\2345\end{vmatrix} & -\begin{vmatrix}1235\\2345\end{vmatrix} & \begin{vmatrix}1234\\2345\end{vmatrix} \\
-\begin{vmatrix}2345\\1345\end{vmatrix} & \begin{vmatrix}1345\\1345\end{vmatrix} & -\begin{vmatrix}1245\\1345\end{vmatrix} & \begin{vmatrix}1235\\1345\end{vmatrix} & -\begin{vmatrix}1234\\1345\end{vmatrix} \\
\begin{vmatrix}2345\\1245\end{vmatrix} & -\begin{vmatrix}1345\\1245\end{vmatrix} & \begin{vmatrix}1245\\1245\end{vmatrix} & -\begin{vmatrix}1235\\1245\end{vmatrix} & \begin{vmatrix}1234\\1245\end{vmatrix} \\
-\begin{vmatrix}2345\\1235\end{vmatrix} & \begin{vmatrix}1345\\1235\end{vmatrix} & -\begin{vmatrix}1245\\1235\end{vmatrix} & \begin{vmatrix}1235\\1235\end{vmatrix} & -\begin{vmatrix}1234\\1235\end{vmatrix} \\
\begin{vmatrix}2345\\1234\end{vmatrix} & -\begin{vmatrix}1345\\1234\end{vmatrix} & \begin{vmatrix}1245\\1234\end{vmatrix} & -\begin{vmatrix}1235\\1234\end{vmatrix} & \begin{vmatrix}1234\\1234\end{vmatrix}
\end{matrix}\right). \text{ and}$$

$$(4)\quad \left(\begin{matrix}
\begin{vmatrix}23456\\23456\end{vmatrix} & -\begin{vmatrix}13456\\23456\end{vmatrix} & \begin{vmatrix}12456\\23456\end{vmatrix} & -\begin{vmatrix}12356\\23456\end{vmatrix} & \begin{vmatrix}12346\\23456\end{vmatrix} & -\begin{vmatrix}12345\\23456\end{vmatrix} \\
-\begin{vmatrix}23456\\13456\end{vmatrix} & \begin{vmatrix}13456\\13456\end{vmatrix} & -\begin{vmatrix}12456\\13456\end{vmatrix} & \begin{vmatrix}12356\\13456\end{vmatrix} & -\begin{vmatrix}12346\\13456\end{vmatrix} & \begin{vmatrix}12345\\13456\end{vmatrix} \\
\begin{vmatrix}23456\\12456\end{vmatrix} & -\begin{vmatrix}13456\\12456\end{vmatrix} & \begin{vmatrix}12456\\12456\end{vmatrix} & -\begin{vmatrix}12356\\12456\end{vmatrix} & \begin{vmatrix}12346\\12456\end{vmatrix} & -\begin{vmatrix}12345\\12456\end{vmatrix} \\
-\begin{vmatrix}23456\\12356\end{vmatrix} & \begin{vmatrix}13456\\12356\end{vmatrix} & -\begin{vmatrix}12456\\12356\end{vmatrix} & \begin{vmatrix}12356\\12356\end{vmatrix} & -\begin{vmatrix}12346\\12356\end{vmatrix} & \begin{vmatrix}12345\\12356\end{vmatrix} \\
\begin{vmatrix}23456\\12346\end{vmatrix} & -\begin{vmatrix}13456\\12346\end{vmatrix} & \begin{vmatrix}12456\\12346\end{vmatrix} & -\begin{vmatrix}12356\\12346\end{vmatrix} & \begin{vmatrix}12346\\12346\end{vmatrix} & -\begin{vmatrix}12345\\12346\end{vmatrix} \\
-\begin{vmatrix}23456\\12345\end{vmatrix} & \begin{vmatrix}13456\\12345\end{vmatrix} & -\begin{vmatrix}12456\\12345\end{vmatrix} & \begin{vmatrix}12356\\12345\end{vmatrix} & -\begin{vmatrix}12346\\12345\end{vmatrix} & \begin{vmatrix}12345\\12345\end{vmatrix}
\end{matrix}\right).$$

By studying the foregoing whilst remembering the conditions for the vanishing of minors stated on ~~pp~~p. 48, the following points will be noticed.

1) All the elements below the leading diagonals of these adjugates will be zeros.

2) When zeros appear in the super-principal diagonal, it is owing to a unit-step from one sub-matrix to another.

e.g. Consider the 6th order matrix of type $\{(3), (2), (1)\}$ or $(123, \bar{4}\bar{5}, \bar{\bar{6}})$.
The super-principal diagonal is formed from 12345, 12346, 12356, 12456, 13456, 23456, its elements being the minors

$$\begin{vmatrix} 12345 \\ 12346 \end{vmatrix}, \quad \begin{vmatrix} 12346 \\ 12356 \end{vmatrix}, \quad \begin{vmatrix} 12356 \\ 12456 \end{vmatrix}, \quad \begin{vmatrix} 12456 \\ 13456 \end{vmatrix}, \quad \begin{vmatrix} 13456 \\ 23456 \end{vmatrix}$$

but written in the reverse order.
The steps are $\therefore$ from 1 to 2, 2 to 3, 3 to $\bar{4}$, $\bar{4}$ to $\bar{5}$, $\bar{5}$ to $\bar{\bar{6}}$, and the super-principal diagonal will be of form a b o d o, a, b, d denoting non-zero elements.

3) When a super-principal diagonal element becomes zero, so do (1) all the elements vertically above it, (2) those in the same horizontal line and to the right of it, and (3) all the

elements in the rectangular block to the right of this vertical and above this horizontal line; because, the step which causes the superprincipal diagonal element to vanish also occurs in all these elements.

It follows that the elements to be removed occur only in the submatrices themselves and not in the blocks beyond them, on the right. Vertically below each of these elements is a super-principal diagonal element so that the unwanted element can be drafted in a north-westerly direction and finally removed.

In the general case, the steps to be considered in the umbral notation of super-diagonal elements are:-
1 to 2, 2 to 3, ... $\overline{n-1}$ to n.

It will readily be seen, from condition (2) for the appearance of a zero in the super-principal diagonal, that the exponents of the invariant factors will be the same for both the original matrix and its adjugate.

It follows that the <u>Reciprocal Matrix A^{-1}</u>, being the adjugate of

matrix A divided by $|A|I$, will also have the same normal form as A.
Thence we see that all positive and negative integer powers of A are of that normal form.

The Principle of Duality.

In the algebra of determinants and matrices, there are always present complementary, reciprocal or dual theorems. The results given on pp. 139 + 140 indicate the existence of a principle of duality which will now be discussed.

If the original matrix is of the n th order, the specification of the m th compound, $c^{(m)}$, is that of the $(n-m)$ th compound, $c^{(n-m)}$, a matrix of the same order as $c^{(m)}$.

For example, there are two ways of forming the 2 nd compound of $\{(5)\}$ on the unit-step principle, so as to secure maximum chains, namely with ordering

(a) 12, 13, 23, 24, 34, 35, 45; 14, 15, 25

or

(b) 12, 13, 14, 15, 25, 35, 45; 23, 24, 34,

so that the exponents are $(7, 3, 0, ..)$.

In forming the 3 rd compound of this matrix, the two possible orderings were found to be (a′) 123, 124, 125, 135, 145, 245, 345; 134, 234, 235

and

(b′) 123, 124, 134, 234, 235, 245, 345; 125, 135, 145,

the exponents being again (7, 3, 0, ...).

It will be noticed that for every choice in the mth compound, there is a <u>complementary</u> choice in the $(n-m)$th, and thus the schemes of exponents will be identical. In fact, the law is:-

Write the numbers not in any bracket of the mth compound in their reverse order; then replace each of these by its deflect from $(n+1)$. This gives the corresponding bracket for the $(n-m)$th compound.

Thus, in (a) above, replace (12) by (6−5, 6−4, 6−3) or (123); replace (13) by (6−5, 6−4, 6−2) or (124), and so on. This gives 123, 124, 125, 135, 145, 245, 345; 134, 234, 235 ie (a′).

Similarly, (b) gives 123, 124, 134, 234, 235, 245, 345; 125, 135, 145 or (b′).

Further, when forming the first chain for the 3rd compound of $\{(6)\}$, there are 42 ways of going from (123) to (456) by unit-steps, but only 12 of these lead to <u>maximum</u> chains and hence to the exponents $(10, 6, 4, 0, \ldots)$. In this instance $m = n - m$ and six of these are the complements of the remaining six.

In the following table (a), (b), ... represent possible orderings while (a′), (b′), ... represent their complementaries:-

(a)	(b)	(c)	(d)	(e)	(f)
123	123	123	123	123	123
124	124	124	124	124	124
134	134	134	125	125	125
234	234	234	126	135	135
235	235	235	136	145	145
245	245	236	236	146	245
345	345	246	246	156	345
346	346	346	346	256	346
356	356	356	356	356	356
456.	456	456	456	456	456

125	125	125	134	134	134
126	135	126	135	234	234
136	145	136	145	235	235
236	146	146	146	245	236
246	156	156	156	345	246
256	256	256	256	346	256

135	126	135	234	126	126
145	136	145	235	136	136
146	236	245	245	236	146
156.	246	345	345	246	156

(a′)	(b′)	(c′)	(d′)	(e′)	(f′)
123	123	123	123	123	123
124	124	124	124	124	124
125	125	125	134	134	134
126	126	126	234	135	135
136	136	136	235	145	145
146	146	236	236	245	146
156	156	246	246	345	156
256	256	256	256	346	256
356	356	356	356	356	356
456	456	456	456	456	456

134	134	134	125	125	125
234	135	234	135	126	126
235	145	235	145	136	136
236	245	245	245	146	236
246	345	345	345	156	246
346	346	346	346	256	346

135	234	135	126	234	234
145	235	145	136	235	235
245	236	146	146	236	245
345	246	156	156	246	345

Let us introduce a term used by Muir, "bi-complementary sets", defined thus: e.g.
Consider (123) in relation to (12345). The integers not in (123) are (45), and the complements of these from (5+1) or 6 give (12). Then (12) and (123) are called "bi-complementary sets" of indices and matrices formed from (a) and (a′), given on pp. 150 & 151, are called "<u>bi-complementary or dual compounds.</u>"

The first row, formed on the unit-step principle, of the 2nd compound of the general matrix of order 5, $\binom{12345}{12345}$, is

$$\left(\begin{vmatrix} 12 \\ 12 \end{vmatrix}, \begin{vmatrix} 12 \\ 13 \end{vmatrix}, \begin{vmatrix} 12 \\ 23 \end{vmatrix}, \begin{vmatrix} 12 \\ 24 \end{vmatrix}, \begin{vmatrix} 12 \\ 34 \end{vmatrix}, \begin{vmatrix} 12 \\ 35 \end{vmatrix}, \begin{vmatrix} 12 \\ 45 \end{vmatrix} \; ; \; \begin{vmatrix} 12 \\ 14 \end{vmatrix}, \begin{vmatrix} 12 \\ 15 \end{vmatrix}, \begin{vmatrix} 12 \\ 25 \end{vmatrix} \right).$$

Writing bi-complementaries everywhere, this becomes

$$\left(\begin{vmatrix} 123 \\ 123 \end{vmatrix}, \begin{vmatrix} 123 \\ 124 \end{vmatrix}, \begin{vmatrix} 123 \\ 125 \end{vmatrix}, \begin{vmatrix} 123 \\ 135 \end{vmatrix}, \begin{vmatrix} 123 \\ 145 \end{vmatrix}, \begin{vmatrix} 123 \\ 245 \end{vmatrix}, \begin{vmatrix} 123 \\ 345 \end{vmatrix} \; ; \; \begin{vmatrix} 123 \\ 134 \end{vmatrix}, \begin{vmatrix} 123 \\ 234 \end{vmatrix}, \begin{vmatrix} 123 \\ 235 \end{vmatrix} \right),$$

i.e. the first row of the complementary or 3rd compound in a unit-step order, (though not in the same sequence).

As will be seen from the results obtained in Appendix C, the specification of the exponents for the mth compound is the same as that for the $(n-m)$th, not only in the case $\{(n)\}$ just considered, but in the general case also.

eg. the 2nd compound of $(123\bar{4}\bar{\bar{5}})$ is formed from ordering

$[12, 13, 23;\ \ 1\bar{4}, 2\bar{4}, 3\bar{4};\ \ 1\bar{\bar{5}}, 2\bar{\bar{5}}, 3\bar{\bar{5}};\ \ \bar{4}\bar{\bar{5}}]$,

giving chains of 3, 3, 3 and 1.

Its 3rd compound is formed from ordering $[123;\ \ 12\bar{4}, 13\bar{4}, 23\bar{4};\ \ 12\bar{\bar{5}}, 13\bar{\bar{5}}, 23\bar{\bar{5}};\ \ 1\bar{4}\bar{\bar{5}}, 2\bar{4}\bar{\bar{5}}, 3\bar{4}\bar{\bar{5}}]$, also giving chains of 3, 3, 3 and 1. It will be observed that the ordering for the 3rd compound can be got from that of the 2nd by replacing each of its groups by the three integers (in ascending order of magnitude) remaining in the set 1, 2, 3, 4, 5. eg. replacing (12) by $(3\bar{4}\bar{\bar{5}})$, and so on.

The proof of this theorem for the general case can be obtained by the same reasoning as was used when considering the normal form of the adjugate. Just as the adjugate is obtained from the matrix (a_{pq}) by replacing each of its elements by A_{qp}, so the $(n-m)$th compound of (C) can be obtained from the mth by replacing every minor of order m in it by its complement in the original matrix, and then writing rows as columns. Then, multiplying together these two complementary matrices, row by column, the elements in this product are all determinants of order n, because they are Laplace's expansions of these determinants. The leading diagonal elements are all equal to $|C|^n$, and the others vanish.

To illustrate this, let $n = 5$; $C = \{(n)\}$ $C^{(2)}$ is formed from ordering [12, 13, 23, 24, 34, 35, 45; 14, 15, 25] and gives

$$\left(\begin{array}{ccccccc|ccc}
\lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^2
\end{array}\right)$$

$c^{(3)}$ is formed from ordering

[345, 245, 145, 135, 125, 124, 123; 235, 234, 134],

giving

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda^2 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & \lambda^2 & \lambda^3 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda^3 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & \lambda^2 & \lambda^3 & 0 & 1 & \lambda & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda^3 & 0 & 1 & \lambda \\
\hline
\lambda & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 \\
1 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda^3 & 0 \\
0 & 1 & \lambda & \lambda^2 & 0 & 0 & 0 & \lambda & \lambda^2 & \lambda^3
\end{array}\right)$$

Writing rows as columns in this matrix and giving to each of its elements the necessary + or − sign so that it becomes the complement of the corresponding element of $c^{(2)}$, we form $c^{(3)'}$. The product $c^{(2)} \cdot c^{(3)'}$ is then

$$\begin{pmatrix}
\lambda^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^5
\end{pmatrix} \quad \text{or} \quad |c|\,E.$$

*

Thus, just as the product of a matrix and its adjugate is $\begin{pmatrix} |A| & 0 & 0 & \cdot \\ 0 & |A| & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$, so

the product of two complementary compounds is found to be a similar diagonal matrix.

It follows therefore, that the compounds stand to each other in the relation of matrix and adjugate and so have the same specification of exponents of elementary divisors.

* Muir. *History of the Theory of Determinants.*
p. 118:- The product of two complementary compounds is a power of the original determinant.

Possible applications of the results obtained in the foregoing research.

I. Application to the Numerical Evaluation of Latent Roots by Matrix - Squaring.

In practice it is often necessary to evaluate the latent roots of a matrix, eq. in finding periods of oscillations.

A well-known problem of Lagrange in dynamics requires the solution of the algebraic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-\lambda & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-\lambda & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}-\lambda \end{vmatrix} = 0,$$

where the determinant $|a_{pq}|$ is axi-symmetric, and the elementary divisors linear with respect to the latent roots.

In general we may wish to approximate as closely as possible to the latent roots of a matrix A.

When these latent roots are all distinct, say $\lambda_1, \lambda_2, \dots \lambda_n$, A may be resolved into normal form BCB^{-1}, where

$$C \equiv \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

B being non-singular; and so

$$A^k = BC^kB^{-1}$$

$$= B \begin{pmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{pmatrix} B^{-1}.$$

Thus each element of A^k is linear in $\lambda_1^k, \lambda_2^k, \dots \lambda_n^k$.

If λ_1 is the greatest of the latent roots, then, by increasing k, terms in λ_1^k will tend to preponderate over terms in $\lambda_2^k, \lambda_3^k, \dots \lambda_n^k$. Hence the limit of the quotient obtained by dividing any element in A^{k+1} by the corresponding element in A^k will be λ_1.

The calculation can be carried out conveniently by a process of successive matrix-squaring, being thus analogous to the similar process of "root-squaring" in the solution of algebraic equations.

Suppose that it is required to solve the equation

$$\begin{vmatrix} 11-\lambda & -6 & 2 \\ -6 & 10-\lambda & -4 \\ 2 & -4 & 6-\lambda \end{vmatrix} = 0,$$

eq. in finding the axes of a quadric.

Here $A = \begin{pmatrix} 11 & -6 & 2 \\ -6 & 10 & -4 \\ 2 & -4 & 6 \end{pmatrix}$ and we find

$$A^2 = \begin{pmatrix} 161 & -134 & 58 \\ -134 & 152 & -76 \\ 58 & -76 & 56 \end{pmatrix},$$

$$A^4 = \begin{pmatrix} 47{,}241 & -46{,}350 & 22{,}770 \\ -46{,}350 & 46{,}836 & -23{,}580 \\ 22{,}770 & -23{,}580 & 12{,}276 \end{pmatrix},$$

$$A^8 = \begin{pmatrix} 4{,}898{,}507{,}481 & -4{,}897{,}385{,}550 & 2{,}448{,}135{,}090 \\ -4{,}897{,}385{,}550 & 4{,}897{,}949{,}796 & -2{,}449{,}250{,}460 \\ 2{,}448{,}135{,}090 & -2{,}449{,}250{,}460 & 1{,}225{,}189{,}476 \end{pmatrix}.$$

A stage is ultimately reached where the quotient formed by dividing any element in one of these powers by the correspond-ing element in the preceding power tends towards a fixed quantity. Thus an approximation to λ_1^4 is

$$\frac{4{,}898{,}507{,}481}{47{,}241} \quad \text{or} \quad 103{,}692,$$

hence λ_1 is $17 \cdot 945$.
[The actual value of λ_1 is 18].

Not only the greatest root but <u>all</u> the roots may be found approximately by matrix-powering. By taking minors of successively higher orders, λ_1, $\lambda_1\lambda_2$, $\lambda_1\lambda_2\lambda_3, \ldots \ldots \lambda_1\lambda_2 \cdots \lambda_n$ and hence $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ will be found. because, as is already known, when the latent roots of a matrix are all different or when the element-ary divisors are linear, i.e. there are no 1's in the super-principal diagonal of the normal form, the ratio of any minor of order m in A^{t+1} to the corresponding one in A^t is $\lambda_1 \lambda_2 \dots \lambda_m$, being the product of the m greatest roots. The proof depends upon the facts that

$$\left(A^{(m)}\right)^t = \left(A^t\right)^{(m)},$$

and
$$\left(BCB^{-1}\right)^{(m)} = B^{(m)} C^{(m)} \left(B^{-1}\right)^{(m)},$$

C being entirely diagonal.

When 1's appear in the super-principal diagonal of C the case is more complicated. Somewhat extensive modifications must then be introduced into this method, necessitating consideration, based on the results of the foregoing investigation, of the elements of $\left(C^{(m)}\right)^t$ which depend on the normal form of $A^{(m)}$.

II. There is another possible application in connection with Aitken's extension of the Bernoulli method of approximating to the roots of an algebraic equation.

Let the difference equation or recurrence relation, corresponding to an algebraic equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad \ldots\ (1),$$

be $$a_n X_n + a_{n-1} X_{n-1} + \cdots + a_1 X + a_0 = 0 \quad \ldots (2).$$

Bernoulli found that, beginning with arbitrary $X_0, X_1, \ldots X_{n-1}$, and successively using relation (2) to find $X_n, X_{n+1}, X_{n+2}, \ldots$, the greatest root of (1), x_n, is $\mathop{\mathcal{L}}_{t \to \infty} \dfrac{X_{t+1}}{X_t}$.

In the paper referred to *, Aitken shows that, having found the greatest root, these X's need not be discarded, but can be used again to find the other roots in a determinantal and arithmetical manner. He has since observed that

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* Proc. Royal Soc.y of Edinburgh. Session 1925–1926. Vol. XLVI – Part III – (No. 25), On Bernoulli's Numerical Solution of Algebraic Equations. By A. C. Aitken, D.Sc.

his discovery is related to the method of squaring or powering a matrix of which (1) is the characteristic equation, namely, the matrix

$$A = \begin{pmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

taking $a_n = 1$ which can of course be done.

When this matrix is raised to a high power t by matrix-squaring, and the result is multiplied by A to obtain A^{t+1}, the limit of any minor of order m in $|A^{t+1}|$ to the one formed from the same rows and columns of $|A|^t$ is in general the product of the m greatest roots of the equation (1). But it is shown in this paper, that if there are equalities among the roots, the sequences do not approach their limits in so conven-

-ient and rapid a manner, although sequences derived from them (by a process which is really finite differencing) converge more rapidly.

e.g. let us solve

$$x^3 - 3x^2 + 3x - 1 = 0$$

by the extended Bernoulli method. Setting $X_0 = X_1 = 0$, $X_2 = 1$ we find X_3, X_4, etc. to be 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ... and, proceeding according to the method given in Aitken's paper, we have

X'	X''	X'''
0	0	0
1	1	1
3	3	1
6	6	1
10	10	1
15	15	1
21	21	1
28	28	1
36	36	1
45	45	1
55	55	1
66	66	1

, each column being obtained by squaring the entry on the same horizontal line in the preceding column, subtracting the product of the entries immediately above and below it, and then dividing the result so obtained by the corresponding entry in the column preceding that again.

Denoting the roots by x_1, x_2, x_3, the ratio $\frac{X'''_{t+1}}{X'''_t} = 1$, is, as it should be, the product $x_1 x_2 x_3$.

The ratios $\frac{X''_{t+1}}{X''_t}$ and $\frac{X'_{t+1}}{X'_t}$, in the case of distinct roots, give $x_1 x_2$ and x_1 respectively; but, while they are not 1 in this case, we observe that $\frac{\Delta^2 X'_{t+1}}{\Delta^2 X'_t}$ and $\frac{\Delta^2 X''_{t+1}}{\Delta^2 X''_t}$ are both 1.

If, similarly, we construct a sequence for the equation

$$x^4 - 4x^3 + 6x^2 - 4x + 1 = 0,$$

we get

X'	X''	X'''	X^{iv}
1	1	1	1
4	6	4	1
10	20	10	1
20	50	20	1
35	105	35	1
56	196	56	1
84	336	84	1
120	540	120	1
165	825	165	1
⋮	⋮	⋮	⋮

;

whence $$\frac{X_{t+1}^{iv}}{X_t^{iv}} = 1, \quad \frac{\Delta^3 X'''_{t+1}}{\Delta^3 X'''_t} = 1,$$

$$\frac{\Delta^4 X''_{t+1}}{\Delta^4 X''_t} = 1, \quad \frac{\Delta^3 X'_{t+1}}{\Delta^3 X'_t} = 1.$$

Also, for the equation,

$$x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1 = 0,$$

X'	X''	X'''	X^{iv}	X^{v}
1	1	1	1	1
5	10	10	5	1
15	50	50	15	1
35	175	175	35	1
70	490	490	70	1
126	1176	1176	126	1
210	2520	2520	210	1
⋮	⋮	⋮	⋮	⋮

whence

$$\frac{X^{v}_{t+1}}{X^{v}_t} = 1, \quad \frac{\Delta^4 X^{iv}_{t+1}}{\Delta^4 X^{iv}_t} = 1,$$

$$\frac{\Delta^6 X'''_{t+1}}{\Delta^6 X'''_t} = 1, \quad \frac{\Delta^6 X''_{t+1}}{\Delta^6 X''_t} = 1,$$

$$\frac{\Delta^4 X'_{t+1}}{\Delta^4 X'_t} = 1.$$

These results show that when the multiplicity, n, of the root 1 is 3, $\Delta^3 X' = 0$, $\Delta^3 X'' = 0$; when $n = 4$, $\Delta^4 X' = 0$, $\Delta^5 X'' = 0$, $\Delta^4 X''' = 0$; and when $n = 5$, $\Delta^5 X' = 0$, $\Delta^7 X'' = 0$, $\Delta^7 X''' = 0$, $\Delta^5 X^{iv} = 0$. Hence it will be found that corresponding to an n-fold root, the $[m(n-m)+1]$ th finite differences of the entries in column m vanish. Thus the appropriate order of differencing for the sequence $X^{(n)}$ would be $m(n-m)$.

The reason is this:-

The solution of an algebraic equation is also the problem of finding the latent roots of a matrix; and, if it has multiple latent roots, these determine a submatrix in the normal form, with 1's in the super-principal diagonal. The process outlined in the extended Bernoulli method is probably equivalent to finding the latent roots of the m th compound of this matrix, giving a submatrix like

$$\begin{array}{cccccc} \lambda_1\lambda_2\cdots\lambda_m & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1\lambda_2\cdots\lambda_m & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1\lambda_2\cdots\lambda_m & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_1\lambda_2\cdots\lambda_m \end{array}.$$

e.g. take the case where the matrix has 3 latent roots all equal to λ. Its normal form is then

$$C = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{and} \quad C^3 = \begin{pmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{pmatrix},$$

$$C^4 = \begin{pmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{pmatrix}.$$

BC^3B^{-1} would have a function like $a\lambda^3 + 3b\lambda^2 + 3c\lambda$ as an element. BC^4B^{-1} would have a function like $a\lambda^4 + 4b\lambda^3 + 6c\lambda^2$ as the corresponding element: and, although

$$\frac{a\lambda^4 + 4b\lambda^3 + 6c\lambda^2}{a\lambda^3 + 3b\lambda^2 + 3c\lambda}$$ and similar quotients may not converge very rapidly to λ,

if we difference numerators and denominators a number of times corresponding to the number of terms less 1, i.e. the order of the submatrix less 1, we shall reduce them to their first terms, i.e to a form like $\dfrac{f\lambda^{k+1}}{f\lambda^{k}}$ or λ.

e.g. $x^3 - 3x^2 + 3x - 1 = 0$ is the characteristic equation of the matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ -3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

The elements in row 1, col. 1 of $A, A^2, A^3, \dots A^9$ are respectively 3, 6, 10, 15, 21, 28, 36, 45, 55, and their 2nd differences are all equal to 1.

Knowledge of the value of the highest exponent of the elementary divisors of the m th compound of $(n, 0, 0, \dots)$ will therefore have considerable bearing on the theory of algebraic equations.

Appendix A.

The evaluation of the exponents of elementary divisors of certain matrices by actual examination of their minors.

Appendix A.

Using Sylvester's ordering of rows and columns in forming compounds, the various minors were actually examined in certain cases and the following results obtained:-

(1)
$$C = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} ; \quad C^{(2)} = \begin{pmatrix} \lambda_1\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4 \end{pmatrix}.$$

$C^{(2)}$ is in normal form.

(2)
$$C = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} ; \quad C^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

$C^{(2)}$ is in normal form.

$$c^{(3)} = \begin{pmatrix} \lambda_1^3 & 0 & 0 & 0 \\ 0 & \lambda_1^3 & 0 & 0 \\ 0 & 0 & \lambda_1^3 & 0 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

$c^{(3)}$ is in normal form.

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(3)

$$c = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} ; \quad c^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & \lambda_1 & \lambda_3 & 1 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_3 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{pmatrix}.$$

<u>For λ_1^2</u> :- $\quad p = 1, \; p_1 = p_2 = p_3 = p_4 = p_5 = 0$

$\therefore \quad e_1 = 1, \; e_2 = e_3 = e_4 = e_5 = 0.$

<u>So for λ_3^2.</u>

<u>For $\lambda_1\lambda_3$</u> :- $\quad p = 4, \; p_1 = 1, \; p_2 = p_3 = p_4 = p_5 = 0.$

$\therefore \quad e_1 = 3, \; e_2 = 1, \; e_3 = e_4 = e_5 = 0.$

and $\therefore$ normal form is

$$\begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{pmatrix}$$

$$C^{(3)} = \begin{pmatrix} \lambda_1^2\lambda_3 & \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2\lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3^2 & \lambda_3^2 \\ 0 & 0 & 0 & \lambda_1\lambda_3^2 \end{pmatrix}$$

For both $\lambda_1^2\lambda_3$ and $\lambda_1\lambda_3^2$

$p = 2, \quad p_1 = p_2 = p_3 = 0$

$\therefore \quad e_1 = 2, \quad e_2 = e_3 = 0.$

and normal form is

$$\left(\begin{array}{cc|cc} \lambda_1^2\lambda_3 & 1 & 0 & 0 \\ 0 & \lambda_1^2\lambda_3 & 0 & 0 \\ \hline 0 & 0 & \lambda_1\lambda_3^2 & 1 \\ 0 & 0 & 0 & \lambda_1\lambda_3^2 \end{array}\right).$$

---

(4) $C = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$; $\quad C^{(2)} = \begin{pmatrix} \lambda_1^2 & \lambda_1 & 0 & 1 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$

$p = 6, \quad p_1 = 1, \quad p_2 = p_3 = p_4 = p_5 = 0.$

$\therefore \quad e_1 = 5, \quad e_2 = 1, \quad e_3 = e_4 = e_5 = 0.$ and normal

form is
$$\left(\begin{array}{ccccc:c} \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{array}\right)$$

$$c^{(3)} = \begin{pmatrix} \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

$p = 4, \quad p_1 = p_2 = p_3 = 0 \quad \therefore$
$e_1 = 4, \quad e_2 = e_3 = 0.$
and normal form is
$$\begin{pmatrix} \lambda_1^3 & 1 & 0 & 0 \\ 0 & \lambda_1^3 & 1 & 0 \\ 0 & 0 & \lambda_1^3 & 1 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

5) $c = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}; \quad c^{(2)} = \begin{pmatrix} \lambda_1^2 & \lambda_1 & 1 \\ 0 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}.$

$p = 3,\ p_1 = 0,\ p_2 = 0.$ $\therefore$ $e_1 = 3,\ e_2 = 0$ and normal form is

$$\begin{pmatrix} \lambda_1^2 & 1 & 0 \\ 0 & \lambda_1^2 & 1 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

(6)

$$C = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{pmatrix}; \quad C^{(4)} = \begin{pmatrix} \lambda_1^4 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ 0 & \lambda_1^4 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^4 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^4 & \lambda_1^3 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix}.$$

$p = 5, \quad p_1 = p_2 = p_3 = p_4 = 0.$

$\therefore \quad e_1 = 5, \quad e_2 = e_3 = e_4 = 0.$

and normal form is

$$\begin{pmatrix} \lambda_1^4 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1^4 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1^4 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1^4 & 1 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix}.$$

(over

$$C^{(2)} = \begin{pmatrix}
\lambda_1^2 & \lambda_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & \lambda_1 & 0 & \lambda_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & \lambda_1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{pmatrix}$$

$p = 10, \quad p_1 = 3, \quad p_2 = p_3 = \cdots = p_q = 0.$

$\therefore \; e_1 = 7, \quad e_2 = 3, \quad e_3 = e_4 = \cdots = e_q = 0.$

and normal form is

$$\left(\begin{array}{ccccccc|ccc}
\lambda_1^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{array}\right)$$

$$C^{(3)} = \begin{pmatrix} \lambda_1^3 & \lambda_1^2 & 0 & \lambda_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1^2 & \lambda_1 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1^3 & 0 & \lambda_1^2 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & \lambda_1^2 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & 0 & \lambda_1^2 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}$$

$p = 10, \quad p_1 = 3, \quad p_2 = p_3 = \cdots = p_9 = 0$

$\therefore \; e_1 = 7, \; e_2 = 3, \; e_3 = e_4 = \cdots = e_9 = 0.$

and normal form is similar to that of $C^{(2)}$, but with λ_1^3, in leading diagonal.

(7.)

$$C = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}; \quad C^{(2)} = \begin{pmatrix} \lambda_1^2 & \lambda_1 & 0 & 1 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

$p = 6, \quad p_1 = 3, \quad p_2 = p_3 = p_4 = p_5 = 0.$

$\therefore \; e_1 = 3, \; e_2 = 3, \; e_3 = e_4 = e_5 = 0.$

and normal form is

$$\begin{pmatrix} \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}$$

$$C^{(3)} = \begin{pmatrix} \lambda_1^3 & 0 & 0 & 0 \\ 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

$p = 4, \quad p_1 = 1, \quad p_2 = p_3 = 0$

$\therefore \quad e_1 = 3, \quad e_2 = 1, \quad e_3 = 0$

and normal form is

$$\begin{pmatrix} \lambda_1^3 & 1 & 0 & 0 \\ 0 & \lambda_1^3 & 1 & 0 \\ 0 & 0 & \lambda_1^3 & 0 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}$$

8.
$$C = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}; \quad C^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

$p = 6$, $p_1 = 3$, $p_2 = 2$, $p_3 = 1$, $p_4 = p_5 = 0$.
$\therefore$ $e_1 = 3$, $e_2 = 1$, $e_3 = 1$, $e_4 = 1$, $e_5 = 0$
and normal form is

$$\begin{pmatrix} \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

$$C^{(3)} = \begin{pmatrix} \lambda_1^3 & \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^3 & 0 & 0 \\ 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}$$

$p = 4$, $p_1 = 2$, $p_2 = p_3 = 0$.
$\therefore$ $e_1 = 2$, $e_2 = 2$, $e_3 = 0$.

and normal form is

$$\left(\begin{array}{cc|cc} \lambda_1^3 & 1 & 0 & 0 \\ 0 & \lambda_1^3 & 0 & 0 \\ \hline 0 & 0 & \lambda_1^3 & 1 \\ 0 & 0 & 0 & \lambda_1^3 \end{array}\right)$$

9. $$c = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}; \quad c^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

$p = 6$, $p_1 = 4$, $p_2 = 2$, $p_3 = 1$, $p_4 = 0$, $p_5 = 0$.

$\therefore$ $e_1 = 2$, $e_2 = 2$, $e_3 = 1$, $e_4 = 1$, $e_5 = 0$.

$\therefore$ normal form is

$$\begin{pmatrix} \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}$$

$$C^{(3)} = \begin{pmatrix} \lambda_1^3 & 0 & 0 & 0 \\ 0 & \lambda_1^3 & 0 & 0 \\ 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

$p = 4$, $p_1 = 2$, $p_2 = 1$, $p_3 = 0$.
$e_1 = 2$, $e_2 = 1$, $e_3 = 1$.

$\therefore$ Normal form is
$$\begin{pmatrix} \lambda_1^3 & 1 & 0 & 0 \\ 0 & \lambda_1^3 & 0 & 0 \\ 0 & 0 & \lambda_1^3 & 0 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}$$

10.
$$C = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} ; \quad C^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$$

$p = 3$, $p_1 = 1$, $p_2 = p_3 = 0$
$\therefore$ $e_1 = 2$, $e_2 = 1$, $e_3 = 0$
and normal form is
$$\begin{pmatrix} \lambda_1^2 & 1 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}.$$

11. $$c = \begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{pmatrix} ; \quad c^{(4)} = \begin{pmatrix} \lambda_1^4 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^4 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^4 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^4 & 0 \\ 0 & 0 & 0 & \lambda_1^3 & \lambda_1^4 \end{pmatrix}$$

$p = 5, \quad p_1 = 3, \quad p_2 = 2, \quad p_3 = 1, \quad p_4 = 0$

$\therefore \; e_1 = 2, \quad e_2 = 1, \quad e_3 = 1, \quad e_4 = 0.$

and normal form is

$$\begin{pmatrix} \lambda_1^4 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1^4 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^4 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix}$$

$$c^{(2)} = \begin{pmatrix} \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{pmatrix}$$

$p = 10$, $p_1 = 8$, $p_2 = 6$, $p_3 = 4$, $p_4 = 3$, $p_5 = 2$, $p_6 = 1$, $p_7 = 0$, $p_8 = 0$, $p_9 = 0$.

$\therefore$ $e_1 = 2$, $e_2 = 2$, $e_3 = 2$, $e_4 = 1$, $e_5 = 1$, $e_6 = 1$,

$e_7 = 1$, $e_8 = 0$, $e_9 = 0$.

and normal form is

$$\begin{pmatrix}
\lambda_1^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{pmatrix}$$

Appendix B.

Examples of Compounds of $\{(n)\}$ formed on the "Unit-Step" Principle.

Appendix B.

Various compounds of $\{(n)\}$ formed on the unit-step principle.

I. 2nd compounds.

1. $$C = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} ; \qquad C^{(2)} = \begin{pmatrix} \lambda^2 & \lambda & 1 \\ 0 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^2 \end{pmatrix},$$

row and column ordering being $[12, 13, 23]$.

2. $$C = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} ; \qquad C^{(2)} = \left(\begin{array}{ccccc:c} \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 1 & 0 & \lambda \\ 0 & 0 & \lambda^2 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ \hdashline 0 & 0 & 0 & \lambda & 0 & \lambda^2 \end{array}\right),$$

row and column ordering being $[12, 13, 23, 24, 34; 14]$,

or $$\left(\begin{array}{ccccc:c} \lambda^2 & \lambda & 0 & 0 & 0 & 1 \\ 0 & \lambda^2 & \lambda & 1 & 0 & \lambda \\ 0 & 0 & \lambda^2 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ \hdashline 0 & 0 & 0 & \lambda & 1 & \lambda^2 \end{array}\right),$$ row and column ordering being $[12, 13, 14, 24, 34; 23]$.

3. $$C = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}; \quad C^{(2)} = \left(\begin{array}{ccccccc|ccc} \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^2 \end{array}\right),$$

row and column ordering being

$[12, 13, 23, 24, 34, 35, 45;\ 14, 15, 25]$,

or $$\left(\begin{array}{ccccccc|ccc} \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & \lambda^2 \end{array}\right),$$ row and column ordering being

$[12, 13, 14, 15, 25, 35, 45;\ 23, 24, 34]$.

4. $$C = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix};$$

$$C^{(2)} = \left(\begin{array}{ccccccccc|ccccc|c}
\lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \lambda^2
\end{array}\right),$$

row and column ordering being
[12, 13, 23, 24, 34, 35, 45, 46, 56; 14, 15, 25, 26, 36; 16]

5.

$$C = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix};$$

$$C^{(2)} = \left(\begin{array}{ccccccccccc|ccccccc|ccc}
\lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^2
\end{array}\right),$$

row and column ordering being [12, 13, 23, 24, 34, 35, 45, 46, 56, 57, 67; 14, 15, 25, 26, 36, 37, 47; 16, 17, 27.]

II   3rd compounds.

1. $$C = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \; ; \qquad C^{(3)} = \left(\lambda^3\right),$$ row and column ordering being [123].

---

2. $$C = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \; ; \qquad C^{(3)} = \begin{pmatrix} \lambda^3 & \lambda^2 & \lambda & 1 \\ 0 & \lambda^3 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^3 & \lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix},$$

row and column ordering being
[123, 124, 134, 234].

---

3. $$C = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \; ; \qquad C^{(3)} = \left(\begin{array}{ccccccc|ccc} \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda \\ 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda^3 \end{array}\right),$$

row and column ordering being
[123, 124, 134, 234, 235, 245, 345; 125, 135, 145], or

$$\left(\begin{array}{ccccccc|ccc}
\lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\
0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3
\end{array}\right),$$

row and column ordering being [123, 124, 125, 135, 145, 245, 345; 134, 234, 235].

4.

$$C = \begin{pmatrix}
\lambda & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.$$

Row and column ordering being

[123, 124, 134, 234, 235, 245, 345, 346, 356, 456; 125, 126, 136, 236, 246, 256; 135, 145, 146, 156],

(over)

$$C^{(3)} = \left(\begin{array}{cccccccccc|cccccc|cccc}
\lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{array}\right)$$

4 th compounds.

1\. III. $c = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$; $c^{(4)} = (\lambda^4)$, row and column ordering being [1234].

2\. $c = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$; $c^{(4)} = \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 \\ 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda \\ 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 \\ 0 & 0 & 0 & \lambda^4 & \lambda^3 \\ 0 & 0 & 0 & 0 & \lambda^4 \end{pmatrix}$,

row and column ordering being [1234, 1235, 1245, 1345, 2345].

3\.

$$c = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Row and column ordering being

[1234, 1235, 1245, 1345, 2345, 2346, 2356, 2456, 3456; 1236, 1246, 1346, 1356, 1456; 1256] ,

$$
c^{(4)} = \left(\begin{array}{ccccccccc|ccccc|c}
\lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & \lambda^2 \\
0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 & \lambda^3 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & \lambda^4
\end{array}\right).
$$

If formed from row and column ordering

[1234, 1235, 1236, 1246, 1256, 1356, 1456, 2456, 3456;
1245, 1345, 1346, 2346, 2356; 2345],

(over)

$$C^{(4)} = \left(\begin{array}{ccccccccc|ccccc|c}
\lambda^4 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 1 \\
0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & \lambda \\
0 & 0 & \lambda^4 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & 1 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & \lambda & \lambda^3 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^4 & \lambda^3 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^4 & \lambda^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & \lambda^4 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda^4
\end{array}\right)$$

Appendix C.

Reduction of Matrices to Normal Form.

Appendix C.

Reduction to Normal Form of Matrices of orders 3, 4, 5 and 6.

3rd order matrices.

1. Type $\{(3)\}$ or (123) is $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[12, 13, 23]$ is $\begin{pmatrix} \lambda_1^2 & \lambda_1 & 1 \\ 0 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$. The 1 can be removed by operations row 1 $- \frac{1}{\lambda_1}$ row 2, col. 2 $+ \frac{1}{\lambda_1}$ col. 1, giving

$$\begin{pmatrix} \lambda_1^2 & \lambda_1 & 0 \\ 0 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^2 \end{pmatrix},$$ which is of type $\{(3)\}$.

2. Type $\{(2),(1)\}$ or $(12, \bar{3})$ is $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[1\bar{3}, 2\bar{3}\,;\, 12]$ is $\begin{pmatrix} \lambda_1\lambda_3 & \lambda_3 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$, which is of type $\{(2), (1)\}$.

3. <u>Type $\{(1), (1), (1)\}$ or $(1, \bar{2}, \bar{\bar{3}})$</u> is $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[1\bar{2}, 1\bar{3}, 2\bar{3}]$ is $\begin{pmatrix} \lambda_1\lambda_2 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_2\lambda_3 \end{pmatrix}$, which is of type $\{(1), (1), (1)\}$.

Note that when $n = 3$, $m = 2$, $(n-m) = 1$, the specification of the elementary divisors for the m th compound is the same as that for the $(n-m)$th.

II.

4th order matrices.

1. Type $\{(4)\}$ or (1234) is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[12, 13, 23, 24, 34;\ 14]$ is

$$\left(\begin{array}{ccccc|c} \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 & 1 & 0 & \lambda_1 \\ 0 & 0 & \lambda_1^2 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 0 & \lambda_1^2 \end{array}\right).$$

Operations:-

row $2 + \frac{1}{\lambda}$ row 3, col. $3 - \frac{1}{\lambda}$ col. 2;
row $3 + \frac{2}{\lambda}$ row 4, col. $4 - \frac{2}{\lambda}$ col. 3;
row $4 + \frac{3}{\lambda}$ row 5, col. $5 - \frac{3}{\lambda}$ col. 4;
row 3 + row 6, col. 6 − col. 3;
row $6 - \frac{1}{2}$ row 3, col. $3 + \frac{1}{2}$ col. 6;
row $4 - \frac{3}{2\lambda}$ row 5, col. $5 + \frac{3}{2\lambda}$ col. 4;
row $6 + \frac{3}{2\lambda}$ row 4, col. $4 - \frac{3}{2\lambda}$ col. 6

give

$$\left(\begin{array}{ccccc|c} \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 2\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_1^2 \end{array}\right),$$

which is of type $\{(5), (1)\}$.

Its 3rd compound, formed with ordering [123, 124, 134, 234] is

$$\begin{pmatrix} \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

Operations:-
row 2 + $\frac{1}{\lambda}$ row 3, col. 3 − $\frac{1}{\lambda}$ col. 2;
row 3 + $\frac{2}{\lambda}$ row 4, col. 4 − $\frac{2}{\lambda}$ col. 3;
row 2 + $\frac{1}{\lambda^2}$ row 4, col. 4 − $\frac{1}{\lambda^2}$ col. 2

give

$$\begin{pmatrix} \lambda_1^3 & \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^3 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix},$$ which is of type $\{(4)\}$.

---

2. <u>Type $\{(3),(1)\}$ or $(123, \bar{4})$</u> is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$.

Its 2nd compound, formed with ordering [12, 13, 23; $1\bar{4}$, $2\bar{4}$, $3\bar{4}$] is

$$\left(\begin{array}{ccc|ccc} \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 \end{array}\right).$$

Operations:-
row 1 − $\frac{1}{\lambda_1}$ row 2, col. 2 + $\frac{1}{\lambda_1}$ col. 1
replace the 1 by 0
so that the matrix is of type $\{(3), (3)\}$.

Its 3rd compound, formed with ordering $[12\bar{4},\ 13\bar{4},\ 23\bar{4};\ 123]$ is

$$\left(\begin{array}{ccc|c} \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & \lambda_4 & 0 \\ 0 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4 & 0 \\ \hline 0 & 0 & 0 & \lambda_1^3 \end{array}\right).$$

Operations row $1 - \frac{1}{\lambda_1}$ row 2, col. $2 + \frac{1}{\lambda_1}$ col. 1 replace the λ_4 by 0, so that the matrix is of type $\{(3), (1)\}$.

---

3. <u>Type $\{(2),(2)\}$ or $(12, \bar{3}\bar{4})$</u> is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[1\bar{3}, 1\bar{4},\ 2\bar{4};\ 2\bar{3};\ 12;\ \bar{3}\bar{4}]$ is

$$\left(\begin{array}{cccc|c|c} \lambda_1\lambda_3 & \lambda_1 & 1 & \lambda_3 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & \lambda_1\lambda_3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{array}\right).$$

Operations :-
row $1 - \frac{1}{\lambda_3}$ row 2, col. $2 + \frac{1}{\lambda_3}$ col. 1;
row $2 + \frac{\lambda_3}{\lambda_1}$ row 4, col. $4 - \frac{\lambda_3}{\lambda_1}$ col. 2

give

$$\begin{pmatrix} \lambda_1\lambda_3 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 2\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & \lambda_1\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{pmatrix}$$ and $\text{row}_4 - \dfrac{\lambda_1}{2\lambda_3}\,\text{row}\,2$,

$\text{col.}\,2 + \dfrac{\lambda_1}{2\lambda_3}\,\text{col.}\,4$

gives

$$\begin{pmatrix} \lambda_1\lambda_3 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 2\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{pmatrix}$$, which is of type $\{(3,1);(1);(1)\}$.

Its 3rd compound, formed with ordering $[12\bar{3}, 12\bar{4};\ 1\bar{3}\bar{4}, 2\bar{3}\bar{4}]$ is

$$\left(\begin{array}{cc|cc} \lambda_1^2\lambda_3 & \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2\lambda_3 & 0 & 0 \\ \hline 0 & 0 & \lambda_1\lambda_3^2 & \lambda_3^2 \\ 0 & 0 & 0 & \lambda_1\lambda_3^2 \end{array}\right)$$, which is of type $\{(2), (2)\}$.

4. Type $\{(2),(1),(1)\}$ or $(12, \bar{3}, \bar{\bar{4}})$. is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[1\bar{3}, 2\bar{3};\ 1\bar{\bar{4}}, 2\bar{\bar{4}};\ 12;\ \bar{3}\bar{\bar{4}}]$ is

$$\begin{pmatrix} \lambda_1\lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4 \end{pmatrix},$$

which is of type $\{(2), (2), (1), (1)\}$.

Its 3rd compound, formed with ordering $[1\bar{3}\bar{\bar{4}}, 2\bar{3}\bar{\bar{4}};\ 12\bar{3};\ 12\bar{\bar{4}}]$ is

$$\begin{pmatrix} \lambda_1\lambda_3\lambda_4 & \lambda_3\lambda_4 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_4 \end{pmatrix},$$

which is of type $\{(2), (1), (1)\}$.

5. <u>Type $\{(1), (1), (1), (1)\}$ or $(1, \bar{2}, \bar{\bar{3}}, \bar{\bar{\bar{4}}})$</u> is $\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$.

Its 2nd compound, formed with ordering $1\bar{2}$; $1\bar{\bar{3}}$; $1\bar{\bar{\bar{4}}}$; $\bar{2}\bar{\bar{3}}$; $\bar{2}\bar{\bar{\bar{4}}}$; $\bar{\bar{3}}\bar{\bar{\bar{4}}}$ is

$$\begin{pmatrix} \lambda_1\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2\lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4 \end{pmatrix},$$

which is of type $\{(1), (1), (1), (1), (1), (1)\}$.

Its 3rd compound, formed with ordering $1\bar{2}\bar{\bar{3}}$; $1\bar{2}\bar{\bar{\bar{4}}}$; $1\bar{\bar{3}}\bar{\bar{\bar{4}}}$; $\bar{2}\bar{\bar{3}}\bar{\bar{\bar{4}}}$ is

$$\begin{pmatrix} \lambda_1\lambda_2\lambda_3 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_2\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_3\lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_2\lambda_3\lambda_4 \end{pmatrix},$$

which is of type $\{(1), (1), (1), (1)\}$.

Note that when $n = 4$, $m = 3$, $n - m = 1$, the specification of the elementary divisors for the m th compound is the same as that for the $(n-m)$ th.

III. 5th order matrices.

1. Type $\{(5)\}$ or (12345) is
$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{pmatrix}.$$

Its 2nd compound, formed with ordering $[12, 13, 23, 24, 34, 35, 45;\ 14, 15, 25]$ is

$$\left(\begin{array}{ccccccc|ccc}
\lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & \lambda_1 \\
0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 \\
0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & \lambda_1^2
\end{array}\right)$$

Operations :-

row 10 − row 5, col. 5 + col. 10;

row 10 + $\frac{1}{\lambda}$ row 6, col. 6 − $\frac{1}{\lambda}$ col. 10;

row 9 − $\frac{1}{2}$ row 4, col. 4 + $\frac{1}{2}$ col. 9;

row 9 + $\frac{1}{\lambda}$ row 5, col. 5 − $\frac{1}{\lambda}$ col. 9;

row 9 $-\frac{1}{\lambda^2}$ row 6, col. 6 $+\frac{1}{\lambda^2}$ col. 9;
row 8 $-\frac{3}{2}$ row 3, col. 3 $+\frac{3}{2}$ col. 8;
row 8 $+\frac{3}{4\lambda}$ row 4, col. 4 $-\frac{3}{4\lambda}$ col. 8;
row 9 $+\frac{7}{4\lambda}$ row 10, col. 10 $-\frac{7}{4\lambda}$ col. 9;

row 6 $+\frac{1}{\lambda}$ row 7, col. 7 $-\frac{1}{\lambda}$ col. 6;
row 5 $+\frac{1}{2}$ row 10, col. 10 $-\frac{1}{2}$ col. 5;
row 4 $-\frac{1}{2\lambda}$ row 10, col. 10 $+\frac{1}{2\lambda}$ col. 4;
row 3 $-\frac{1}{2\lambda}$ row 4, col. 4 $+\frac{1}{2\lambda}$ col. 3;
row 3 $+\frac{1}{20\lambda^2}$ row 10, col. 10 $-\frac{1}{20\lambda^2}$ col. 3;

row 2 $-\frac{3}{2\lambda}$ row 3, col. 3 $+\frac{3}{2\lambda}$ col. 2;

[The elements in row 1, except those in the principal and super-principal diagonals, can now be replaced by zeros, without altering any other row.]

row 3 $+\frac{2}{5}$ row 8, col. 8 $-\frac{2}{5}$ col. 3;
row 4 $+\frac{2}{5}$ row 9, col. 9 $-\frac{2}{5}$ col. 4;
row 5 $+\frac{1}{10}$ row 10, col. 10 $-\frac{1}{10}$ col. 5

give

$$\left(\begin{array}{ccccccc:ccc}
\lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \frac{5\lambda_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & 2\lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{array}\right),$$

which is of type $\{(7), (3)\}$.

Its 3rd compound, formed with ordering
[123, 124, 134, 234, 235, 245, 345 ; 125, 135, 145] is

$$\left(\begin{array}{ccccccc|ccc}
\lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 \\
0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & \lambda_1^2 & \lambda_1 \\
0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & \lambda_1^3 & \lambda_1^2 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & \lambda_1^3
\end{array}\right).$$

Operations:-

row 10 − row 5, col. 5 + col. 10;
row 10 + $\frac{1}{\lambda}$ row 6, col. 6 − $\frac{1}{\lambda}$ col. 10;
row 9 − 2 row 4, col. 4 + 2 col. 9;
row 9 + $\frac{2}{\lambda}$ row 5, col 5 − $\frac{2}{\lambda}$ col. 9;
row 8 + $\frac{1}{\lambda}$ row 4, col. 4 − $\frac{1}{\lambda}$ col. 8;
row 8 − $\frac{1}{\lambda^2}$ row 5, col. 5 + $\frac{1}{\lambda^2}$ col. 8;
row 8 − $\frac{2}{3}$ row 3, col. 3 + $\frac{2}{3}$ col. 8;
row 9 − $\frac{2}{\lambda}$ row 10, col. 10 + $\frac{2}{\lambda}$ col. 9;
row 6 + $\frac{1}{\lambda}$ row 7, col. 7 − $\frac{1}{\lambda}$ col. 6;
row 4 − $\frac{1}{\lambda}$ row 5, col. 5 + $\frac{1}{\lambda}$ col. 4;

[The 3rd, 4th and 5th elements in row 1

can now be replaced by zeros without altering any other row.]

row 3 $-\frac{3}{\lambda}$ row 4; col. 4 $+\frac{3}{\lambda}$ col. 3;

row 3 $-$ 3 row 8, col. 8 $+$ 3 col. 3;

row 3 $-\frac{3}{\lambda}$ row 9, col. 9 $+\frac{3}{\lambda}$ col. 3;

row 2 $-\frac{7}{3\lambda}$ row 3, col. 3 $+\frac{7}{3\lambda}$ col. 2;

row 2 $-\frac{2}{\lambda^2}$ row 4, col. 4 $+\frac{2}{\lambda^2}$ col. 2.

[Two elements in row 1 can be replaced by zeros.]

row 2 $-\frac{2}{\lambda^2}$ row 9, col. 9 $+\frac{2}{\lambda^2}$ col. 2

[One element in row 1 can be replaced by zero.]

row 3 $+\frac{18}{5\lambda}$ row 9, col. 9 $-\frac{18}{5\lambda}$ col. 3;

The remaining extraneous elements in row 2 can be replaced by zeros, giving, finally,

$$\left(\begin{array}{ccccccc:ccc}
\lambda_1^3 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^3 & \frac{5\lambda_1^2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^3 & 3\lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \frac{\lambda_1^2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3
\end{array}\right),$$

which is of type $\{(7), (3)\}$.

Its 4th compound, formed with ordering [1234, 1235, 1245, 1345, 2345], is

$$\begin{pmatrix} \lambda_1^4 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ 0 & \lambda_1^4 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\ 0 & 0 & \lambda_1^4 & \lambda_1^3 & \lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^4 & \lambda_1^3 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix}.$$

Operations :-

row 1 $-\frac{1}{\lambda_1}$ row 2, col. 2 $+\frac{1}{\lambda_1}$ col. 1,

row 3 $+\frac{1}{\lambda_1^2}$ row 5, col. 5 $-\frac{1}{\lambda_1^2}$ col. 3,

row 3 $+\frac{1}{\lambda_1}$ row 4, col. 4 $-\frac{1}{\lambda_1}$ col. 3;

row 4 $+\frac{2}{\lambda_1}$ row 5, col. 5 $-\frac{2}{\lambda_1}$ col. 4;

give

$$\begin{pmatrix} \lambda_1^4 & \lambda_1^3 & 0 & 0 & 0 \\ 0 & \lambda_1^4 & \lambda_1^3 & 0 & 0 \\ 0 & 0 & \lambda_1^4 & \lambda_1^3 & 0 \\ 0 & 0 & 0 & \lambda_1^4 & \lambda_1^3 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix},$$ which is of type $\{(5)\}$.

2. <u>Type $\{(4), (1)\}$ or $(1234, \bar{5})$</u> is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[12, 13, 23, 24, 34;\ 1\bar{5}, 2\bar{5}, 3\bar{5}, 4\bar{5};\ 14]$ is

$$\left(\begin{array}{ccccc|cccc|c}
\lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & \lambda_1 \\
0 & 0 & \lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & 0 \\
\hline
0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{array}\right).$$

Operations:-

row 3 $- \frac{1}{\lambda_1}$ row 4, col. 4 $+ \frac{1}{\lambda_1}$ col. 3;
row 2 $- \frac{2}{\lambda_1}$ row 3, col. 3 $+ \frac{2}{\lambda_1}$ col. 2;
row 1 $- \frac{1}{\lambda_1^2}$ row 3, col. 3 $+ \frac{1}{\lambda_1^2}$ col. 1;
row 1 $- \frac{3}{\lambda_1}$ row 2, col. 2 $+ \frac{3}{\lambda_1}$ col. 1;
row 2 $- \frac{3}{\lambda_1}$ row 10, col. 10 $+ \frac{3}{\lambda_1}$ col. 2;
row 3 + row 10, col. 10 $-$ col. 3;
row 2 $+ \frac{3}{2\lambda_1}$ row 3, col. 3 $- \frac{3}{2\lambda_1}$ col. 2;
row 1 $+ \frac{3}{2\lambda_1}$ row 2, col. 2 $- \frac{3}{2\lambda_1}$ col. 1;
row 10 $- \frac{1}{2}$ row 3, col. 3 $+ \frac{1}{2}$ col. 10

give

$$\left(\begin{array}{ccccc|cccc|c}
\lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & 2\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{array}\right),$$

which is of type $\{(5,1),(4)\}$.

Its 3rd compound, formed with ordering $[12\bar{5},\ 13\bar{5},\ 23\bar{5},\ 24\bar{5},\ 34\bar{5};\ 14\bar{5};\ 123,\ 124,\ 134,\ 234]$ is

$$\left(\begin{array}{ccccc|c|cccc}
\lambda_1^2\lambda_5 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & \lambda_5 & 0 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1\lambda_5 & 0 & \lambda_1^2\lambda_5 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3
\end{array}\right)$$

Operations:-

row 3 $-\frac{1}{\lambda_1}$ row 4, col. 4 $+\frac{1}{\lambda_1}$ col. 3;

row 2 $-\frac{2}{\lambda_1}$ row 3, col. 3 $+\frac{2}{\lambda_1}$ col. 2;

row 1 $-\frac{1}{\lambda_1^2}$ row 3, col. 3 $+\frac{1}{\lambda_1^2}$ col. 1;

row 1 $-\frac{3}{\lambda_1}$ row 2, col. 2 $+\frac{3}{\lambda_1}$ col. 1;

row 2 $-\frac{3}{\lambda_1}$ row 6, col. 6 $+\frac{3}{\lambda_1}$ col. 2;

row 3 + row 6, col. 6 $-$ col. 3;

row 2 $+\frac{3}{2\lambda_1}$ row 3, col. 3 $-\frac{3}{2\lambda_1}$ col. 2;

row 1 $+\frac{3}{2\lambda_1}$ row 2, col. 2 $-\frac{3}{2\lambda_1}$ col. 1;

row 6 $-\frac{1}{2}$ row 3, col. 3 $+\frac{1}{2}$ col. 6

give

$$\left(\begin{array}{ccccc|c|cccc}
\lambda_1^2\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_5 & 2\lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 & \lambda_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3 & \lambda_1^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3
\end{array}\right).$$

Then operations

row 8 $-\frac{1}{\lambda_1}$ row 9, col. 9 $+\frac{1}{\lambda_1}$ col. 8;

row 7 $- \frac{2}{\lambda_1}$ row 8, col. 8 $+ \frac{2}{\lambda_1}$ col. 7;

row 7 $- \frac{1}{\lambda_1^2}$ row 9, col. 9 $+ \frac{1}{\lambda_1^2}$ col. 7

replace the 3 underlined elements by zeros, and the matrix is of type $\{(5,1);(4)\}$.

Its 4th compound, formed with ordering $[123\bar{5},\ 124\bar{5},\ 134\bar{5},\ 234\bar{5};\ 1234]$ is

$$\left(\begin{array}{cccc|c}
\lambda_1^3\lambda_5 & \lambda_1^2\lambda_5 & \underline{\lambda_1\lambda_5} & \underline{\lambda_5} & 0 \\
0 & \lambda_1^3\lambda_5 & \lambda_1^2\lambda_5 & \underline{\lambda_1\lambda_5} & 0 \\
0 & 0 & \lambda_1^3\lambda_5 & \lambda_1^2\lambda_5 & 0 \\
0 & 0 & 0 & \lambda_1^3\lambda_5 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda_1^4
\end{array}\right).$$

Operations:-

row 2 $- \frac{1}{\lambda_1}$ row 3, col. 3 $+ \frac{1}{\lambda_1}$ col. 2;

row 1 $- \frac{2}{\lambda_1}$ row 2, col. 2 $+ \frac{2}{\lambda_1}$ col. 1;

row 1 $- \frac{1}{\lambda_1^2}$ row 3, col. 3 $+ \frac{1}{\lambda_1^2}$ col. 1

replace the underlined elements by zeros and matrix is of type $\{(4),(1)\}$.

3. Type $\{(3),(2)\}$ or $(123\bar{4}\bar{5})$ is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 1 \\ 0 & 0 & 0 & 0 & \lambda_4 \end{pmatrix}$.

Its 2nd compound, formed with ordering $[12, 13, 23;\ 1\bar{4}, 2\bar{4}, 3\bar{4}, 3\bar{5};\ 1\bar{5}, 2\bar{5};\ \bar{4}\bar{5}]$ is

$$\left(\begin{array}{ccc|cccc|cc|c}
\lambda_1^2 & \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 & \lambda_1 & 1 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 1 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & \lambda_1\lambda_4 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4^2
\end{array}\right).$$

Operations :-

row 1 $-\frac{1}{\lambda_1}$ row 2, col. 2 $+\frac{1}{\lambda_1}$ col. 1;

row 6 $+\frac{1}{\lambda_4}$ row 7, col. 7 $-\frac{1}{\lambda_4}$ col. 6;

row 5 $+\frac{\lambda_1}{\lambda_4}$ row 8, col. 8 $-\frac{\lambda_1}{\lambda_4}$ col. 5;

row 5 $+\frac{1}{\lambda_4}$ row 9, col. 9 $-\frac{1}{\lambda_4}$ col. 5;

row 6 $+\frac{2\lambda_1}{\lambda_4}$ row 9, col. 9 $-\frac{2\lambda_1}{\lambda_4}$ col. 6;

$$\text{row } 6 + \frac{1}{\lambda_4}\text{ row } 7, \qquad \text{col. } 7 - \frac{1}{\lambda_4}\text{ col. } 6;$$
$$\text{row } 9 - \frac{\lambda_4}{3\lambda_1}\text{ row } 6, \qquad \text{col. } 6 + \frac{\lambda_4}{3\lambda_1}\text{ col. } 9;$$
$$\text{row } 8 - \frac{\lambda_4}{3\lambda_1}\text{ row } 5, \qquad \text{col. } 5 + \frac{\lambda_4}{3\lambda_1}\text{ col. } 8.$$

give
$$\left(\begin{array}{ccc|cccc|cc|c}
\lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & 3\lambda_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4^2
\end{array}\right),$$

which is of type $\{(4,2);(3);(1)\}$.

Its 3rd compound, formed with ordering $[12\bar{4}, 13\bar{4}, 23\bar{4}, 23\bar{5};\ 12\bar{5}, 13\bar{5};\ 1\bar{4}\bar{5}, 2\bar{4}\bar{5}, 3\bar{4}\bar{5};\ 123]$ is

(over

$$\begin{pmatrix}
\lambda_1^2\lambda_4 & \lambda_1\lambda_4 & \lambda_4 & 1 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & \lambda_1 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_4 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_4 & 0 & \lambda_1^2\lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4^2 & \lambda_4^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4^2 & \lambda_4^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3
\end{pmatrix}.$$

Operations:-

row 1 $- \frac{1}{\lambda_1}$ row 2, col. 2 $+ \frac{1}{\lambda_1}$ col. 1;

row 2 $- \frac{1}{\lambda_1}$ row 3, col. 3 $+ \frac{1}{\lambda_1}$ col. 2;

row 1 $- \frac{1}{\lambda_1}$ row 2, col. 2 $+ \frac{1}{\lambda_1}$ col. 1;

row 2 $+ \frac{\lambda_1}{\lambda_4}$ row 5, col. 5 $- \frac{\lambda_1}{\lambda_4}$ col. 2;

row 2 $- \frac{1}{\lambda_4}$ row 6, col. 6 $+ \frac{1}{\lambda_4}$ col. 2;

row 3 $+ \frac{2\lambda_1}{\lambda_4}$ row 6, col. 6 $- \frac{2\lambda_1}{\lambda_4}$ col. 3;

row 6 $+ \frac{1}{\lambda_1}$ row 4, col. 4 $- \frac{1}{\lambda_1}$ col. 6;

row 6 $- \frac{\lambda_4}{3\lambda_1}$ row 3, col. 3 $+ \frac{\lambda_4}{3\lambda_1}$ col. 6;

row 5 $- \frac{\lambda_4}{3\lambda_1}$ row 2, col. 2 $+ \frac{\lambda_4}{3\lambda_1}$ col. 5

give

$$\left(\begin{array}{cccc|cccccc}
\lambda_1^2\lambda_4 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_4 & 3\lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4^2 & \lambda_4^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4^2 & \lambda_4^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^3
\end{array}\right),$$

which is of type $\{(4,2),(3),(1)\}$.

Its 4th compound, formed with ordering $[12\bar{4}\bar{5},\ 13\bar{4}\bar{5},\ 23\bar{4}\bar{5};\ 123\bar{4},\ 123\bar{5}]$ is

$$\left(\begin{array}{ccc|cc}
\lambda_1^2\lambda_4^2 & \lambda_1\lambda_4^2 & \underline{\lambda_4^2} & 0 & 0 \\
0 & \lambda_1^2\lambda_4^2 & \lambda_1\lambda_4^2 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_4^2 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1^3\lambda_4 & \lambda_1^3 \\
0 & 0 & 0 & 0 & \lambda_1^3\lambda_4
\end{array}\right).$$

Operations:- row 1 $-\frac{1}{\lambda_1}$ row 2, col. 2 $+\frac{1}{\lambda_1}$ col. 1 replace the underlined element by zero, and matrix is of type $\{(3),(2)\}$.

4. Type $\{(3), (1), (1)\}$ or $(123, \bar{4}, \bar{\bar{5}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its 2nd compound, formed with ordering $[12, 13, 23; \quad 1\bar{4}, 2\bar{4}, 3\bar{4}; \quad 1\bar{\bar{5}}, 2\bar{\bar{5}}, 3\bar{\bar{5}}; \quad \bar{4}\bar{\bar{5}}]$.

is

$$\left(\begin{array}{ccc|ccc|ccc|c}
\lambda_1^2 & \lambda_1 & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4\lambda_5
\end{array}\right)$$

Operations:-

row 1 $- \frac{1}{\lambda_1}$ row 2, col. 2 $+ \frac{1}{\lambda_1}$ col. 1

remove the underlined 1 and matrix is of type $\{(3), (3), (3), (1)\}$.

Its 3rd compound, formed with ordering
$[123;\ 12\bar{4},\ 13\bar{4},\ 23\bar{4};\ 12\bar{\bar{5}},\ 13\bar{\bar{5}},\ 23\bar{\bar{5}};\ 1\bar{4}\bar{\bar{5}},\ 2\bar{4}\bar{\bar{5}},\ 3\bar{4}\bar{\bar{5}}]$ is

$$\begin{pmatrix}
\lambda_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & \underline{\lambda_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_4 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & \underline{\lambda_5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4\lambda_5 & \lambda_4\lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4\lambda_5 & \lambda_4\lambda_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4\lambda_5
\end{pmatrix}.$$

The two underlined elements can be removed by operations :-
row 2 $-\ \frac{1}{\lambda_1}$ row 3, col. 3 $+\ \frac{1}{\lambda_1}$ col. 2 ;
row 5 $-\ \frac{1}{\lambda_1}$ row 6, col. 6 $+\ \frac{1}{\lambda_1}$ col. 5.
and matrix is of type $\{(3), (3), (3), (1)\}$.

Its 4 th compound, formed with ordering
$[12\bar{4}\bar{\bar{5}},\ 13\bar{4}\bar{\bar{5}},\ 23\bar{4}\bar{\bar{5}};\ 123\bar{4};\ 123\bar{\bar{5}}]$ is

(over

$$\begin{pmatrix} \lambda_1^2\lambda_4\lambda_5 & \lambda_1\lambda_4\lambda_5 & \underline{\lambda_4\lambda_5} & 0 & 0 \\ 0 & \lambda_1^2\lambda_4\lambda_5 & \lambda_1\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & 0 & \boxed{\lambda_1^2\lambda_3\lambda_4} & 0 \\ 0 & 0 & 0 & 0 & \boxed{\lambda_1^2\lambda_3\lambda_5} \end{pmatrix}$$

The underlined element can be removed by row 1 $- \frac{1}{\lambda_1}$ row 2, col. 2 $+ \frac{1}{\lambda_1}$ col. 1 and matrix is of type $\{(3), (1), (1)\}$.

<u>Type $\{(2), (2), (1)\}$ or $(12, \bar{3}\bar{4}, \bar{\bar{5}})$</u> is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its 2 nd compound, formed with ordering $[1\bar{3},\ 2\bar{3},\ 2\bar{4};\ 1\bar{\bar{5}},\ 2\bar{\bar{5}};\ \bar{3}\bar{\bar{5}},\ \bar{4}\bar{\bar{5}};\ 1\bar{4};\ \bar{3}\bar{4};\ 12]$

is

(over

$$\begin{pmatrix}
\lambda_1\lambda_3 & \lambda_3 & \underline{1} & 0 & 0 & 0 & 0 & \underline{\lambda_1} & 0 & 0 \\
0 & \lambda_1\lambda_3 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_5 & \lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_5 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{pmatrix}.$$

The underlined elements in rows 1+8 can be re-moved by operations :-

$$\text{row } 2 + \frac{\lambda_1}{\lambda_3}\ \text{row } 8, \quad \text{col. } 8 - \frac{\lambda_1}{\lambda_3}\ \text{col. } 2;$$

$$\text{row } 8 - \frac{\lambda_3}{2\lambda_1}\ \text{row } 2, \quad \text{col. } 2 + \frac{\lambda_3}{2\lambda_1}\ \text{col. } 8.$$

The matrix becomes

$$\begin{pmatrix}
\lambda_1\lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_3 & 2\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_5 & \lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2
\end{pmatrix},$$

which is of type $\{(3,1);(2);(2);(1);(1)\}$.

Its 3rd compound, formed with ordering $[1\bar{3}\bar{\bar{5}},\ 2\bar{3}\bar{\bar{5}},\ 2\bar{4}\bar{\bar{5}};\ 1\bar{4}\bar{\bar{5}};\ 12\bar{3},\ 12\bar{4};\ 1\bar{3}\bar{4},\ 2\bar{3}\bar{4};\ 12\bar{\bar{5}};\ \bar{3}\bar{4}\bar{\bar{\bar{5}}}]$, is

$$\left(\begin{array}{ccc|c|cc|cc|c|c}
\lambda_1\lambda_3\lambda_5 & \lambda_3\lambda_5 & \lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_3\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \lambda_3\lambda_5 & \lambda_1\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2\lambda_3 & \lambda_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3^2 & \lambda_3^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3^2\lambda_5
\end{array}\right)$$

Operations:-

row 1 $-\frac{1}{\lambda_1}$ row 2, col. 2 $+\frac{1}{\lambda_1}$ col. 1;

row 4 $-\frac{\lambda_3}{\lambda_1}$ row 2, col. 2 $+\frac{\lambda_3}{\lambda_1}$ col. 4;

row 2 $+\frac{\lambda_1}{2\lambda_3}$ row 4, col. 4 $-\frac{\lambda_1}{2\lambda_3}$ col. 2

give

$$\left(\begin{array}{ccc|ccccccc}
\lambda_1\lambda_3\lambda_5 & 2\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_3\lambda_5 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2\lambda_3 & \lambda_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3^2 & \lambda_3^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3^2\lambda_5
\end{array}\right),$$

which is of type $\{(3,1);(2);(2);(1);(1)\}$.

Its 4 th compound, formed with ordering $[1\,2\,\bar{3}\,\bar{\bar{5}},\ 1\,2\,\bar{4}\,\bar{\bar{5}};\ 1\,\bar{3}\,\bar{4}\,\bar{\bar{5}},\ 2\,\bar{3}\,\bar{4}\,\bar{\bar{5}};\ 1\,2\,\bar{3}\,\bar{4}]$ is

$$\left(\begin{array}{cc|cc|c}
\lambda_1^2\lambda_3\lambda_5 & \lambda_1^2\lambda_5 & 0 & 0 & 0 \\
0 & \lambda_1^2\lambda_3\lambda_5 & 0 & 0 & 0 \\
\hline
0 & 0 & \lambda_1\lambda_3^2\lambda_5 & \lambda_3^2\lambda_5 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_3^2\lambda_5 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda_1^2\lambda_3^2
\end{array}\right),$$

which is of type $\{(2);(2);(1)\}$.

Type $\{(2),(1),(1),(1)\}$ or $(12,\ \bar{3},\ \bar{\bar{4}},\ \bar{\bar{\bar{5}}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its 2nd compound, formed with ordering $[1\bar{3},\ 2\bar{3};\quad 1\bar{\bar{4}},\ 2\bar{\bar{4}};\quad 1\bar{\bar{\bar{5}}},\ 2\bar{\bar{\bar{5}}};\quad 12;\ \bar{3}\bar{\bar{4}};\ \bar{3}\bar{\bar{\bar{5}}};\ \bar{\bar{4}}\bar{\bar{\bar{5}}}]$ is

$$\begin{pmatrix} \lambda_1\lambda_3 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & \lambda_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4\lambda_5 \end{pmatrix},$$

which is of type $\{(2),(2),(2),(1),(1),(1),(1)\}$.

Its 3rd compound, formed with ordering $[1\bar{3}\bar{\bar{4}},\ 2\bar{3}\bar{\bar{4}};\ 1\bar{3}\bar{\bar{\bar{5}}},\ 2\bar{3}\bar{\bar{\bar{5}}};\ 1\bar{\bar{4}}\bar{\bar{\bar{5}}},\ 2\bar{\bar{4}}\bar{\bar{\bar{5}}};\ 12\bar{3};\ 12\bar{\bar{4}};\ 12\bar{\bar{\bar{5}}};\ \bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}}]$ is

$$\begin{pmatrix}
\lambda_1\lambda_3\lambda_4 & \lambda_3\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_3\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_3\lambda_5 & \lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_4\lambda_5 & \lambda_4\lambda_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4\lambda_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4\lambda_5
\end{pmatrix},$$

which is of type $\{(2); (2); (2); (1); (1); (1); (1)\}$.

Its 4th compound, formed with ordering $[1\bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}},\ 2\bar{3}\bar{\bar{4}}\bar{\bar{\bar{5}}};\ 12\bar{3}\bar{\bar{4}};\ 12\bar{3}\bar{\bar{\bar{5}}};\ 12\bar{\bar{4}}\bar{\bar{\bar{5}}}]$ is

$$\begin{pmatrix}
\lambda_1\lambda_3\lambda_4\lambda_5 & \lambda_3\lambda_4\lambda_5 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_3\lambda_4\lambda_5 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^2\lambda_3\lambda_4 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^2\lambda_4\lambda_5
\end{pmatrix},$$

which is of type $\{(2), (1), (1), (1)\}$.

1. Type $\{(1);(1);(1);(1);(1)\}$ or $(1, \bar{2}, \bar{\bar{3}}, \bar{\bar{\bar{4}}}, \bar{\bar{\bar{\bar{5}}}})$ is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its 2nd compound, formed with ordering $[12;\ 13;\ 14;\ 15;\ 23;\ 24;\ 25;\ 34;\ 35]$ is

$$\begin{pmatrix} \lambda_1\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2\lambda_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2\lambda_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4\lambda_5 \end{pmatrix},$$

which is of type $\{(1);(1);(1);(1),(1);(1);(1);(1);(1);(1)\}$

Its 3rd compound, formed with ordering [123; 124; 125; 134; 135; 145; 234; 235; 245; 345] is

$$\begin{pmatrix}
\lambda_1\lambda_2\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_2\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_2\lambda_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_3\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1\lambda_3\lambda_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_4\lambda_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_2\lambda_3\lambda_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2\lambda_3\lambda_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2\lambda_4\lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\lambda_4\lambda_5
\end{pmatrix},$$

which is of type $\{(1); (1); (1); (1); (1); (1); (1); (1); (1); (1)\}$.

Its 4th compound, formed with ordering [1234; 1235; 1245; 1345; 2345] is

$$\begin{pmatrix}
\lambda_1\lambda_2\lambda_3\lambda_4 & 0 & 0 & 0 & 0 \\
0 & \lambda_1\lambda_2\lambda_3\lambda_5 & 0 & 0 & 0 \\
0 & 0 & \lambda_1\lambda_2\lambda_4\lambda_5 & 0 & 0 \\
0 & 0 & 0 & \lambda_1\lambda_3\lambda_4\lambda_5 & 0 \\
0 & 0 & 0 & 0 & \lambda_2\lambda_3\lambda_4\lambda_5
\end{pmatrix},$$

which is of type $\{(1), (1); (1); (1); (1)\}$.

Note that when

$n = 5, \quad m = 4, \quad n-m = 1;$

$n = 5, \quad m = 3, \quad n-m = 2;$

$n = 5, \quad m = 2, \quad n-m = 3;$

$n = 5, \quad m = 1, \quad n-m = 4,$

the specification of the elementary divisors for the m th compound is the same as that for the $(n-m)$ th.

Appendix D.

Examples of Reduction to Normal form by interchange of 2 rows, followed by interchange of the corresponding 2 columns, already mentioned on pp. 81-83.

The method of an interchange of rows p and q, followed by an interchange of columns p and q, already mentioned on pp. 81-83, can be used to bring certain compounds (formed with Sylvester's row and column ordering) into normal form: eg. consider a matrix

$$C = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$ which is of type $\{(2),(1),(1)\}$ or $(12\bar{3}\bar{\bar{4}})$.

Its 2nd compound, formed with the fundamental ordering $[12, 13, 14, 23, 24, 34]$, is

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & \underline{\lambda} & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & \underline{\lambda} & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}.$$

By interchanging pairs of rows and then the corresponding pairs of columns, bring the underlined λ's into the super-principal diagonal, thus :-

$$\left.\begin{array}{l} \text{Interchange rows } 1 \text{ and } 2; \\ \quad\text{"} \qquad \text{cols. } 1 \text{ " } 2 \end{array}\right\}.$$

The matrix becomes

$$\begin{pmatrix} \lambda^2 & 0 & 0 & \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}.$$

Then by interchange of rows 2 and 4, cols. 2 and 4, it becomes

$$\begin{pmatrix} \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}.$$

Lastly, by interchange of rows 4 and 5, cols. 4 and 5, the matrix becomes

$$\begin{pmatrix} \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}$$

and is now in its normal form and of type $\{(2), (2), (1), (1)\}$.

Ex. 2. $\quad C = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$, of type $\{(3, 1).\}$

$$C^{(2)} = \begin{pmatrix} \lambda^2 & \lambda & 0 & 1 & 0 & 0 \\ 0 & \lambda^2 & 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}.$$

By interchange of rows 3 and 4, cols. 3 and 4, $C^{(2)}$ becomes

$$\left(\begin{array}{ccc:ccc} \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{array}\right),$$

which is of type $\{(3, 3)\}$.

Ex. 3. The 2nd compound of $C = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$,

which is of type $\{(2,1,1,1)\}$, is

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}.$$

By interchange of rows 1 and 2, cols. 1 and 2 it becomes

$$\begin{pmatrix} \lambda^2 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 \end{pmatrix}.$$

By interchange of rows 2 and 5, cols. 2 and 5 it becomes

$$\begin{pmatrix}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2
\end{pmatrix},$$

and interchanging (1) rows 4 and 6, cols. 4 and 6 (2) rows 5 and 6, cols. 5 and 6 (3) rows 6 and 7, cols. 6 and 7 it becomes

$$\begin{pmatrix}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2
\end{pmatrix},$$

which is of type $\{(2, 2, 2, 1, 1, 1, 1)\}$.

<u>Ex. 4.</u> The 2nd compound of $\{(6)\}$, formed with ordering $[12, 13, 14, 15, 16, 23, 24, 25. 26, 34, 35, 36, 45, 46, 56]$ is

$$\begin{pmatrix}
\lambda^2 & \lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2
\end{pmatrix}.$$

By interchange of

rows 6 and 9,	cols. 6 and 9;
" 7 " 12,	" 7 " 12;
" 8 " 14,	" 8 " 14;
" 9 " 15,	" 9 " 15

the matrix becomes

$$
\left(\begin{array}{ccccccccc|cccccc}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \lambda \\
0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 1 & 0 \\
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & \lambda^2 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & \lambda^2 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \lambda & 0 & 0 & \lambda^2
\end{array}\right)
$$

By interchange of

rows	10	and	15,	cols.	10	and	15,
"	11	"	12,	"	11	"	12,
"	12	"	15,	"	12	"	15,
"	13	"	14,	"	13	"	14,
"	14	"	15,	"	14	"	15,
"	13	"	15,	"	13	"	15,
"	13	"	14,	"	13	"	14

it becomes

$$\left(\begin{array}{ccccccccc|ccccc|c}
\lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \lambda^2
\end{array}\right)$$

The extraneous elements can now be removed by the usual row and column operations and matrix is of type $\{(9,5,1,0\ldots)\}$.

<u>Ex. 5.</u> The 3 rd compound of $\{(6)\}$, formed from ordering
[123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456] is

$$
\begin{pmatrix}
\lambda^3 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda & 0 & 0 & \lambda^2 & \lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda^2 & \lambda & 0 & 0 & \lambda^2 & \lambda & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda^2 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda^2 & \lambda & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & \lambda^2 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3
\end{pmatrix}.
$$

By interchange of

rows 5 and 7, cols. 5 and 7;
" 6 " 9, " 6 " 9;
" 7 " 10, " 7 " 10;
" 8 " 16, " 8 " 16;
" 9 " 19, " 9 " 19;
" 10 " 20, " 10 " 20

the matrix becomes

$$\left(\begin{array}{cccccccccc|cccccccccc}
\lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \\
0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & \lambda^2 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda^2 & \lambda & 0 & \lambda & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & \lambda^2 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & \lambda^2 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & \lambda^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & \lambda & 1 & \lambda^2 & 0 & 0 & \lambda^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 1 & 0 & \lambda & 0 & 0 & \lambda^2 & \lambda^3
\end{array}\right)$$

By interchange of

rows 16	and	18,		cols. 16	and	18;
" 17	"	20,		" 17	"	20;
" 18	"	19,		" 18	"	19;

rows 14 and 19, cols. 14 and 19;
" 14 " 18, " 14 " 18;
" 13 " 14, " 13 " 14;
" 13 " 17, " 13 " 17;
" 12 " 13, " 12 " 13;
" 11 " 12, " 11 " 12.

the matrice becomes

$$\left(\begin{array}{cccccccccc|cccccc|cccc}
\lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 1 & 0 & 0 & \lambda^2 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^3 & \lambda^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & \lambda^3 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & \lambda^3 & \lambda^2 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & \lambda & 0 & 0 & \lambda^3 & \lambda^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & \lambda^2 & 0 & 0 & 0 & \lambda^3
\end{array}\right)$$

The extraneous elements can all be removed and the normal form of the matrix is $\{(10, 6, 4, 0, \ldots)\}$.

Appendix E.

The adjugates of matrices of orders 3, 4, 5 and 6, together with their normal forms.

Appendix E.

The adjugates of matrices of orders 3, 4, 5 and 6, together with their normal forms.

I Matrices of order 3.

1. Matrix of type $\{(3)\}$ or $(1\,2\,3)$ is $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.

Its adjugate is $\begin{pmatrix} \lambda^2 & -\lambda & \underline{1} \\ 0 & \lambda^2 & -\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix}$.

Remove underlined 1 by operations
row 1 + $\frac{1}{\lambda}$ row 2, col. 2 - $\frac{1}{\lambda}$ col. 1.

Then normal form is

$$\begin{pmatrix} \lambda^2 & -\lambda & 0 \\ 0 & \lambda^2 & -\lambda \\ 0 & 0 & \lambda^2 \end{pmatrix},$$

which is also of type $\{(3)\}$.

2. Matrix of type $\{(2), (1)\}$ or $(12, \bar{3})$ is $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

Its adjugate is $\begin{pmatrix} \lambda_1\lambda_3 & -\lambda_3 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$,

which is also of type $\{(2), (1)\}$.

3. Matrix of type $\{(1),(1),(1)\}$ or $(1,\bar{2},\bar{\bar{3}})$ is $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

Its adjugate is
$$\begin{pmatrix} \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix},$$
which is also of type $\{(1),(1),(1)\}$.

II. Matrices of order 4.

1. Matrix of type $\{(4)\}$ or (1234) is $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$.

Its adjugate is
$$\begin{pmatrix} \lambda^3 & -\lambda^2 & \underline{\lambda} & \underline{-1} \\ 0 & \lambda^3 & -\lambda^2 & \underline{\lambda} \\ 0 & 0 & \lambda^3 & -\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix}.$$

The underlined elements can be removed by operations:-

row $1 + \frac{1}{\lambda}$ row 2, col. $2 - \frac{1}{\lambda}$ col. 1;

row $3 - \frac{1}{\lambda}$ row 4, col. $4 + \frac{1}{\lambda}$ col. 3.

Then normal form is
$$\begin{pmatrix} \lambda^3 & -\lambda^2 & 0 & 0 \\ 0 & \lambda^3 & -\lambda^2 & 0 \\ 0 & 0 & \lambda^3 & -\lambda^2 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix},$$
which is also of type $\{(4)\}$.

2. Matrix of type $\{(3),(1)\}$ or $(123, \bar{4})$ is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$.

Its adjugate is
$$\begin{pmatrix} \lambda_1^2\lambda_4 & -\lambda_1\lambda_4 & \underline{\lambda_4} & 0 \\ 0 & \lambda_1^2\lambda_4 & -\lambda_1\lambda_4 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix}.$$

The λ_4 may be removed from row 1 by
row 1 + $\frac{1}{\lambda_1}$ row 2, col. 2 − $\frac{1}{\lambda_1}$ col. 1.

Then normal form is
$$\begin{pmatrix} \lambda_1^2\lambda_4 & -\lambda_1\lambda_4 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4 & -\lambda_1\lambda_4 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_1^3 \end{pmatrix},$$
which is also of type $\{(3),(1)\}$.

---

3. Matrix of type $\{(2),(2)\}$ or $(12, \bar{3}\bar{4})$ is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$.

Its adjugate is
$$\begin{pmatrix} \lambda_1\lambda_3^2 & -\lambda_3^2 & 0 & 0 \\ 0 & \lambda_1\lambda_3^2 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_3 & -\lambda_1^2 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3 \end{pmatrix},$$
which is also of type $\{(2),(2)\}$.

---

4. Matrix of type $\{(2), (1), (1)\}$ or $(12, \bar{3}, \bar{\bar{4}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3\lambda_4 & -\lambda_3\lambda_4 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3 \end{pmatrix},$$

which is also of type $\{(2), (1), (1)\}$.

5. Matrix of type $\{(1),(1),(1),(1)\}$ or $(1, \bar{2}, \bar{\bar{3}}, \bar{\bar{\bar{4}}})$ is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_2\lambda_3\lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_2\lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_2\lambda_3 \end{pmatrix},$$

which is also of type $\{(1), (1), (1), (1)\}$.

III. Matrices of order 5.

1\. Matrix of type $\{(5)\}$ or (12345) is $\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$.

Its adjugate is

$$\begin{pmatrix} \lambda^4 & -\lambda^3 & \underline{\lambda^2} & \underline{-\lambda} & \underline{1} \\ 0 & \lambda^4 & -\lambda^3 & \underline{\lambda^2} & \underline{-\lambda} \\ 0 & 0 & \lambda^4 & -\lambda^3 & \underline{\lambda^2} \\ 0 & 0 & 0 & \lambda^4 & -\lambda^3 \\ 0 & 0 & 0 & 0 & \lambda^4 \end{pmatrix}$$

All the underlined elements can be removed by the following operations:-

row 3 $+ \frac{1}{\lambda}$ row 4, col. 4 $- \frac{1}{\lambda}$ col. 3;
row 2 $- \frac{1}{\lambda^2}$ row 4, col. 4 $+ \frac{1}{\lambda^2}$ col. 2;
row 2 $+ \frac{2}{\lambda}$ row 3, col. 3 $- \frac{2}{\lambda}$ col. 2;
row 1 $+ \frac{1}{\lambda^3}$ row 4, col. 4 $- \frac{1}{\lambda^3}$ col. 1;
row 1 $- \frac{4}{\lambda^2}$ row 3, col. 3 $+ \frac{4}{\lambda^2}$ col. 1,
row 1 $- \frac{1}{\lambda}$ row 2, col. 2 $+ \frac{1}{\lambda}$ col. 1.

Then normal form is

$$\begin{pmatrix} \lambda^4 & -\lambda^3 & 0 & 0 & 0 \\ 0 & \lambda^4 & -\lambda^3 & 0 & 0 \\ 0 & 0 & \lambda^4 & -\lambda^3 & 0 \\ 0 & 0 & 0 & \lambda^4 & -\lambda^3 \\ 0 & 0 & 0 & 0 & \lambda^4 \end{pmatrix},$$

which is also of type $\{(5)\}$.

2. Matrix of type $\{(4),(1)\}$ or $(1234, \bar{5})$ is $\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}$.

Its adjugate is

$$\begin{pmatrix} \lambda_1^3\lambda_5 & -\lambda_1^2\lambda_5 & \underline{\lambda_1\lambda_5} & \underline{-\lambda_5} & 0 \\ 0 & \lambda_1^3\lambda_5 & -\lambda_1^2\lambda_5 & \underline{\lambda_1\lambda_5} & 0 \\ 0 & 0 & \lambda_1^3\lambda_5 & -\lambda_1^2\lambda_5 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix}.$$

The underlined elements can be removed by means of the following operations:–

row 2 + $\frac{1}{\lambda_1}$ row 3, col. 3 − $\frac{1}{\lambda_1}$ col. 2;

row 1 − $\frac{1}{\lambda_1^2}$ row 3, col. 3 + $\frac{1}{\lambda_1^2}$ col. 1;

row 1 + $\frac{2}{\lambda_1}$ row 2, col. 2 − $\frac{2}{\lambda_1}$ col. 1.

Then normal form is

$$\begin{pmatrix} \lambda_1^3\lambda_5 & -\lambda_1^2\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_1^3\lambda_5 & -\lambda_1^2\lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_1^3\lambda_5 & -\lambda_1^2\lambda_5 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4 \end{pmatrix},$$

which is also of type $\{(4),(1)\}$.

3. Matrix of type $\{(3), (2)\}$ or $(123, \bar{4}\bar{5})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 1 \\ 0 & 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^2\lambda_4^2 & -\lambda_1\lambda_4^2 & \underline{\lambda_4^2} & 0 & 0 \\ 0 & \lambda_1^2\lambda_4^2 & -\lambda_1\lambda_4^2 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_4 & -\lambda_1^3 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4 \end{pmatrix}$$

The underlined element can be removed by row $1 + \frac{1}{\lambda_1}$ row 2, col. $2 - \frac{1}{\lambda_1}$ col. 1.

Then normal form is

$$\begin{pmatrix} \lambda_1^2\lambda_4^2 & -\lambda_1\lambda_4^2 & 0 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4^2 & -\lambda_1\lambda_4^2 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_4 & -\lambda_1^3 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4 \end{pmatrix},$$

which is also of type $\{(3), (2)\}$.

Matrix of type $\{(3),(1),(1)\}$ or $(123, \bar{4}, \bar{\bar{5}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^2\lambda_4\lambda_5 & -\lambda_1\lambda_4\lambda_5 & \underline{\lambda_4\lambda_5} & 0 & 0 \\ 0 & \lambda_1^2\lambda_4\lambda_5 & -\lambda_1\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4 \end{pmatrix}.$$

The underlined element can be removed by
row $1 + \frac{1}{\lambda_1}$ row 2, col. $2 - \frac{1}{\lambda_1}$ col. 1.

Then normal form is

$$\begin{pmatrix} \lambda_1^2\lambda_4\lambda_5 & -\lambda_1\lambda_4\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4\lambda_5 & -\lambda_1\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4 \end{pmatrix},$$

which is also of type $\{(3),(1),(1)\}$.

5. Matrix of type $\{(2),(2),(1)\}$ or $(12, \bar{3}\bar{4}, \bar{\bar{5}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3^2\lambda_5 & -\lambda_3^2\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3^2\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_3\lambda_5 & -\lambda_1^2\lambda_5 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3^2 \end{pmatrix},$$

which is also of type $\{(2),(2),(1)\}$.

---

6. Matrix of type $\{(2),(1),(1),(1)\}$ or $(12, \bar{3}, \bar{\bar{4}}, \bar{\bar{\bar{5}}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3\lambda_4\lambda_5 & -\lambda_3\lambda_4\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_4 \end{pmatrix},$$

which is also of type $\{(2),(1),(1),(1)\}$.

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7. Matrix of type $\{(1),(1),(1),(1),(1)\}$ or $(1,\bar{2},\bar{\bar{3}},\bar{\bar{\bar{4}}},\bar{\bar{\bar{\bar{5}}}})$ is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_2\lambda_3\lambda_4\lambda_5 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4\lambda_5 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_2\lambda_4\lambda_5 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_2\lambda_3\lambda_5 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_2\lambda_3\lambda_4 \end{pmatrix},$$

which is also of type $\{(1),(1),(1),(1),(1)\}$.

IV Matrices of order 6.

1. Matrix of type $\{(6)\}$ or (123456) is
$$\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Its adjugate is
$$\begin{pmatrix} \lambda^5 & -\lambda^4 & \underline{\lambda}^3 & -\underline{\lambda}^2 & \underline{\lambda} & -\underline{1} \\ 0 & \lambda^5 & -\lambda^4 & \underline{\lambda}^3 & -\underline{\lambda}^2 & \underline{\lambda} \\ 0 & 0 & \lambda^5 & -\lambda^4 & \underline{\lambda}^3 & -\underline{\lambda}^2 \\ 0 & 0 & 0 & \lambda^5 & -\lambda^4 & \underline{\lambda}^3 \\ 0 & 0 & 0 & 0 & \lambda^5 & -\lambda^4 \\ 0 & 0 & 0 & 0 & 0 & \lambda^5 \end{pmatrix}.$$

The underlined elements can be removed by means of the following operations:-

row 4 + $\frac{1}{\lambda}$ row 5, col. 5 − $\frac{1}{\lambda}$ col. 4;
row 3 − $\frac{1}{\lambda^2}$ row 5, col. 5 + $\frac{1}{\lambda^2}$ col. 3;
row 3 + $\frac{2}{\lambda}$ row 4, col. 4 − $\frac{2}{\lambda}$ col. 3;
row 2 + $\frac{1}{\lambda^3}$ row 5, col. 5 − $\frac{1}{\lambda^3}$ col. 2;
row 2 − $\frac{3}{\lambda^2}$ row 4, col. 4 + $\frac{3}{\lambda^2}$ col. 2;
row 2 + $\frac{3}{\lambda}$ row 3, col. 3 − $\frac{3}{\lambda}$ col. 2;
row 1 − $\frac{1}{\lambda^4}$ row 5, col. 5 + $\frac{1}{\lambda^4}$ col. 1;

$$\text{row } 1 + \frac{4}{\lambda^3}\text{ row } 4, \qquad \text{col. } 4 - \frac{4}{\lambda^3}\text{ col. } 1;$$
$$\text{row } 1 - \frac{6}{\lambda^2}\text{ row } 3, \qquad \text{col. } 3 + \frac{6}{\lambda^2}\text{ col. } 1;$$
$$\text{row } 1 + \frac{4}{\lambda}\text{ row } 2, \qquad \text{col. } 2 - \frac{4}{\lambda}\text{ col. } 1.$$

Then normal form is

$$\begin{pmatrix} \lambda^5 & -\lambda^4 & 0 & 0 & 0 & 0 \\ 0 & \lambda^5 & -\lambda^4 & 0 & 0 & 0 \\ 0 & 0 & \lambda^5 & -\lambda^4 & 0 & 0 \\ 0 & 0 & 0 & \lambda^5 & -\lambda^4 & 0 \\ 0 & 0 & 0 & 0 & \lambda^5 & -\lambda^4 \\ 0 & 0 & 0 & 0 & 0 & \lambda^5 \end{pmatrix},$$

which is also of type $\{(6)\}$.

---

2. Matrix of type $\{(5), (1)\}$ or $(12345, \bar{6})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & \underline{\lambda_1^2\lambda_6} & \underline{-\lambda_1\lambda_6} & \underline{\lambda_6} & 0 \\ 0 & \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & \underline{\lambda_1^2\lambda_6} & \underline{-\lambda_1\lambda_6} & 0 \\ 0 & 0 & \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & \underline{\lambda_1^2\lambda_6} & 0 \\ 0 & 0 & 0 & \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^5 \end{pmatrix}.$$

The underlined elements can be removed by means of operations :-

row 3 $+ \frac{1}{\lambda_1}$ row 4, col. 4 $- \frac{1}{\lambda_1}$ col. 3;
row 2 $- \frac{1}{\lambda_1^2}$ row 4, col. 4 $+ \frac{1}{\lambda_1^2}$ col. 2;
row 2 $+ \frac{2}{\lambda_1}$ row 3, col. 3 $- \frac{2}{\lambda_1}$ col. 2;
row 1 $+ \frac{1}{\lambda_1^3}$ row 4, col. 4 $- \frac{1}{\lambda_1^3}$ col. 1;
row 1 $- \frac{3}{\lambda_1^2}$ row 3, col. 3 $+ \frac{3}{\lambda_1^2}$ col. 1;
row 1 $- \frac{1}{\lambda_1}$ row 2, col. 2 $+ \frac{1}{\lambda_1}$ col. 1.

Then normal form is

$$\begin{pmatrix} \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^4\lambda_6 & -\lambda_1^3\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^5 \end{pmatrix},$$

which is also of type $\{(5), (1)\}$.

3. Matrix of type $\{(4),(2)\}$ or $(1234, \bar{5}\bar{6})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & \underline{\lambda_1\lambda_5^2} & \underline{-\lambda_5^2} & 0 & 0 \\ 0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & \underline{\lambda_1\lambda_5^2} & 0 & 0 \\ 0 & 0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 & -\lambda_1^4 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 \end{pmatrix}.$$

The underlined elements can be removed by operations:-

row 2 + $\frac{1}{\lambda_1}$ row 3, col. 3 − $\frac{1}{\lambda_1}$ col. 2;

row 1 − $\frac{1}{\lambda_1^2}$ row 3, col. 3 + $\frac{1}{\lambda_1^2}$ col. 1;

row 1 + $\frac{2}{\lambda_1}$ row 2, col. 2 − $\frac{2}{\lambda_1}$ col. 1.

Then normal form is

$$\begin{pmatrix}
\lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 & 0 & 0 \\
0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 & 0 \\
0 & 0 & \lambda_1^3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^3\lambda_5^2 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 & -\lambda_1^4 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5
\end{pmatrix},$$

which is also of type $\{(4),(2)\}$.

---

4. Matrix of type $\{(4),(1),(1)\}$ or $(1234, \bar{5}, \bar{\bar{6}})$ is

$$\begin{pmatrix}
\lambda_1 & 1 & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 1 & 0 & 0 & 0 \\
0 & 0 & \lambda_1 & 1 & 0 & 0 \\
0 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_6
\end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix}
\lambda_1^3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & \underline{\lambda_1\lambda_5\lambda_6} & \underline{-\lambda_5\lambda_6} & 0 & 0 \\
0 & \lambda_1^3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & \underline{\lambda_1\lambda_5\lambda_6} & 0 & 0 \\
0 & 0 & \lambda_1^3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & 0 & 0 \\
0 & 0 & 0 & \lambda_1^3\lambda_5\lambda_6 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_1^4\lambda_6 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5
\end{pmatrix}.$$

The underlined elements can be removed by operations:-

row 2 + $\frac{1}{\lambda_1}$ row 3, col. 3 − $\frac{1}{\lambda_1}$ col. 2;

row 1 − $\frac{1}{\lambda_1^2}$ row 3, col. 3 + $\frac{1}{\lambda_1^2}$ col. 1;

row 1 − $\frac{2}{\lambda_1}$ row 2, col. 2 + $\frac{2}{\lambda_1}$ col. 1

Then normal form is

$$\begin{pmatrix} \lambda_1^3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^4\lambda_5 \end{pmatrix},$$

which is also of type $\{(4),(1),(1)\}$.

---

5. Matrix of type $\{(3),(3)\}$ or $(123;\bar{4}\bar{5}\bar{6})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^2\lambda_4^3 & -\lambda_1\lambda_4^3 & \underline{\lambda_4}^3 & 0 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4^3 & -\lambda_1\lambda_4^3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_4^2 & -\lambda_1^3\lambda_4 & \underline{\lambda_1}^3 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4^2 & -\lambda_1^3\lambda_4 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4^2 \end{pmatrix}.$$

The underlined elements can be removed by operations :-

row 1 + $\frac{1}{\lambda_1}$ row 2, col. 2 − $\frac{1}{\lambda_1}$ col. 1;

row 4 + $\frac{1}{\lambda_4}$ row 5, col. 5 − $\frac{1}{\lambda_1}$ col. 4.

Then normal form is

$$\begin{pmatrix} \lambda_1^2\lambda_4^3 & -\lambda_1\lambda_4^3 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4^3 & -\lambda_1\lambda_4^3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_4^2 & -\lambda_1^3\lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4^2 & -\lambda_1^3\lambda_4 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4^2 \end{pmatrix},$$

which is also of type $\{(3), (3)\}$.

6. Matrix of type $\{(3),(2),(1)\}$ or $(123, \bar{4}\bar{5}, \bar{\bar{6}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix},$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^2\lambda_4^2\lambda_6 & -\lambda_1\lambda_4^2\lambda_6 & \underline{\lambda_4^2\lambda_6} & 0 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4^2\lambda_6 & -\lambda_1\lambda_4^2\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4^2\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_4\lambda_6 & -\lambda_1^3\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4^2 \end{pmatrix}.$$

The underlined element can be removed by operations row 1 + $\frac{1}{\lambda_1}$ row 2, col. 2 $- \frac{1}{\lambda_1}$ col. 1, which leave the rest of the matrix unaltered. Then normal form is of type $\{(3),(2),(1)\}$.

7. Matrix of type $\{(3), (1), (1), (1)\}$ or $(123, \bar{4}, \bar{\bar{5}}, \bar{\bar{\bar{6}}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1^2\lambda_4\lambda_5\lambda_6 & -\lambda_1\lambda_4\lambda_5\lambda_6 & \underline{\lambda_4\lambda_5\lambda_6} & 0 & 0 & 0 \\ 0 & \lambda_1^2\lambda_4\lambda_5\lambda_6 & -\lambda_1\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^3\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^3\lambda_4\lambda_5 \end{pmatrix}.$$

The underlined element can be removed by operations:- row $1 + \frac{1}{\lambda_1}$ row 2, col. 2 $- \frac{1}{\lambda_1}$ col. 1, which leave the rest of the matrix unaltered.
Then normal form is also of type $\{(3), (1), (1), (1)\}$.

8. Matrix of type $\{(2);(2);(2)\}$ or $(12,\ \bar{3}\bar{4},\ \bar{\bar{5}}\bar{\bar{6}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3^2\lambda_5^2 & -\lambda_3^2\lambda_5^2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3^2\lambda_5^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_3\lambda_5^2 & -\lambda_1^2\lambda_5^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3^2\lambda_5 & -\lambda_1\lambda_3^2 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_3^2\lambda_5 \end{pmatrix},$$

which is also of type $\{(2);(2);(2)\}$.

---

9. Matrix of type $\{(2),(2),(1),(1)\}$ or $(12,\bar{3}\bar{4},\bar{\bar{5}}\bar{\bar{\bar{6}}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3^2\lambda_5\lambda_6 & -\lambda_3^2\lambda_5\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3^2\lambda_5\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_3\lambda_5\lambda_6 & -\lambda_1^2\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3^2\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3^2\lambda_5 \end{pmatrix},$$

which is also of type $\{(2), (2), (1), (1)\}$.

10. Matrix of type $\{(2), (1), (1), (1), (1).\}$ or $(12, \bar{3}, \bar{\bar{4}}, \bar{\bar{\bar{5}}}, \bar{\bar{\bar{\bar{6}}}})$ is

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_1\lambda_3\lambda_4\lambda_5\lambda_6 & -\lambda_3\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1^2\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1^2\lambda_3\lambda_4\lambda_5 \end{pmatrix},$$

which is also of type $\{(2), (1), (1), (1), (1)\}$.

---

11. Matrix of type $\{(1), (1), (1), (1), (1), (1).\}$ or $(1, \bar{2}, \bar{\bar{3}}, \bar{\bar{\bar{4}}}, \bar{\bar{\bar{\bar{5}}}}, \bar{\bar{\bar{\bar{\bar{6}}}}})$ is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_6 \end{pmatrix}.$$

Its adjugate is

$$\begin{pmatrix} \lambda_2\lambda_3\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_3\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_2\lambda_4\lambda_5\lambda_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_2\lambda_3\lambda_5\lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_2\lambda_3\lambda_4\lambda_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \end{pmatrix},$$

which is also of type $\{(1),(1),(1),(1),(1),(1)\}$.