Local and Global Well-posedness for Nonlinear Dirac Type Equations

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Abstract

We investigate the local and global well-posedness of a variety of nonlinear Dirac type equations with null structure on $\mathbb{R}^{1+1}$. In particular, we prove global existence in L^2 for a nonlinear Dirac equation known as the Thirring model. Local existence in H^s for $s > 0$, and global existence for $s > \frac{1}{2}$, has recently been proven by Selberg-Tesfahun where they used $X^{s,b}$ spaces together with a type of null form estimate. In contrast, motivated by the recent work of Machihara-Nakanishi-Tsugawa, we prove local existence in the scale invariant class L^2 by using null coordinates. Moreover, again using null coordinates, we prove almost optimal local well-posedness for the Chern-Simons-Dirac equation which extends recent work of Huh. To prove global well-posedness for the Thirring model, we introduce a decomposition which shows the solution is linear (up to gauge transforms in $U(1)$), with an error term that can be controlled in L^∞. This decomposition is also applied to prove global existence for the Chern-Simons-Dirac equation.

This thesis also contains a study of bilinear estimates in $X^{s,b}_{\pm}(\mathbb{R}^2)$ spaces. These estimates are often used in the theory of nonlinear Dirac equations on $\mathbb{R}^{1+1}$. We prove estimates that are optimal up to endpoints by using dyadic decomposition together with some simplifications due to Tao. As an application, by using the I-method of Colliander-Keel-Staffilani-Takaoka-Tao, we extend the work of Tesfahun on global existence below the charge class for the Dirac-Klein-Gordon equation on $\mathbb{R}^{1+1}$.

The final result contained in this thesis concerns the space-time Monopole equation. Recent work of Czubak showed that the space-time Monopole equation is locally well-posed in the Coulomb gauge for small initial data in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{4}$. Here we show that the Monopole equation has null structure in Lorenz gauge, and use this to prove local well-posedness for large initial data in $H^s(\mathbb{R}^2)$ with $s > \frac{1}{4}$.

Declaration

I do hereby declare that this thesis was composed by myself and that the work described within is my own, except where explicitly stated otherwise.

Timothy Candy
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Notation

We write $a \lesssim b$ if there is some constant C, independent of the variables under consideration, such that $a \leqslant Cb$. If we wish to make explicit that the constant C depends on δ we write $a \lesssim_\delta b$. Occasionally we write $a \ll b$ if $C < 1$. We use $a \approx b$ to denote the inequalities $a \lesssim b$ and $b \lesssim a$. We let $\mathbb{1}_\Omega$ denote the characteristic function of the set $\Omega \subset \mathbb{R}^d$, although we occasionally abuse notation and write $\mathbb{1}_{|x|\approx N}$ instead of $\mathbb{1}_{\{|x|\approx N\}}$. If A is a matrix, we use $A^\dagger$ to denote the conjugate transpose.

We define ∂_μ to be the partial derivative in the x_μ direction, and let $x_0 = t$. The space-time gradient is denoted by $\partial = (\partial_0, \partial_1, ..., \partial_d) = (\partial_t, \partial_1, ..., \partial_d)$, and the usual gradient is $\nabla = (\partial_1, ..., \partial_d)$. The Einstein summation convention is in effect, in other words repeated Greek indices are summed over $\mu = 0, ..., d$, and repeated Roman indices are summed over $i = 1, ..., d$. Indices are raised and lowered with respect to the metric $g = \text{ diag } (-1, 1, ..., 1)$. Thus

$$\Box = \partial_\mu \partial^\mu = -\partial_t^2 + \Delta, \qquad \Delta = \partial_i \partial^i = \partial_1^2 + ... + \partial_d^2.$$

For a measurable set $\Omega \subset \mathbb{R}^d$ and $1 \leqslant p \leqslant \infty$ we let $L^p(\Omega)$ denote the usual Lebesgue space of p-integrable functions. We also use the mixed version $L^p_t L^q_x(I \times \Omega)$ with norm

$$\|u\|_{L^p_t L^q_x(I\times\Omega)} = \left(\int_I \left(\int_\Omega |u(t,x)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}}$$

where $I \subset \mathbb{R}$ and we make the obvious modification if $p, q = \infty$. Occasionally we write $L^p(\mathbb{R}^d) = L^p$ when we can do so without causing confusion. This comment also applies to the other function spaces which appear throughout this paper. We let C_0^∞ denote the space of smooth functions with compact support and use $\mathcal{S}$ to denote the Schwartz class of smooth functions with rapidly decreasing derivatives. If X is a metric space and $I \subset \mathbb{R}$ is an interval, then $C(I, X)$ denotes the set of continuous functions from I into X.

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is denoted by $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx$. We use the notation $\widetilde{f}(\tau, \xi)$ for the space-time Fourier transform of a function $f(t,x)$ on $\mathbb{R}^{1+d}$. We write $\mathcal{F}_y f$ for the Fourier transform of the function f with respect to the variable y. If $1 < p < \infty$ and $s \in \mathbb{R}$, then we define the standard Sobolev space $W^{s,p}$ as the completion of $\mathcal{S}$ using the norm

$$\|f\|_{W^{s,p}} = \|\Lambda^s f\|_{L^p}$$

where we define the Fourier multiplier Λ^s as $\widehat{\Lambda^s f}(\xi) = \langle \xi \rangle^s \widehat{f}$. In the special case $p = 2$ we let $H^s = W^{s,2}$. For $s > -\frac{n}{p}$, we define the homogeneous variant via the norm

$$\|f\|_{\dot{W}^{s,p}} = \||\nabla|^s f\|_{L^p}$$

where $|\nabla|^s$ denotes the Fourier multiplier $\widehat{|\nabla|^s f} = |\xi|^s \widehat{f}$. We also define the Sobolev spaces in the case $p = 1$ by restricting $s \in \mathbb{N}$ and defining $W^{s,1}$ via the norm

$$\|f\|_{W^{s,1}} = \sum_{|\kappa| \leqslant s} \|\partial^\kappa f\|_{L^1}.$$

Chapter 1

Introduction

This thesis contains a study of the low regularity theory for a number of nonlinear partial differential equations (PDEs) arising in relativistic quantum mechanics. Namely, the Thirring Model, the Chern-Simons-Dirac equation, the Dirac-Klein-Gordon equation, and the space-time Monopole equation. We do not attempt to present a complete theory for such equations, rather we focus on the problem of lowering the required regularity to ensure that local and global well-posedness holds.

There are a number of ways to motivate the study of PDE at low regularities. For instance, many PDEs arise in mathematical physics (in particular, all the equations studied in this thesis) and physically relevant quantities such as the charge (L^2 norm), energy (H^1 norm), and momentum (roughly the $H^{\frac{1}{2}}$ norm) occur at low regularities. Thus it is of interest to develop an deeper understanding of how these equations behave in these regularity classes. Another motivation is more mathematical in nature. Looking for solutions with little regularity is often substantially harder than finding smooth solutions. Consequently a large number of mathematical tools have to be developed to deal with the additional difficulties that working in low regularities brings. A number of these tools have subsequently been found to either substantially simplify existing results at higher regularities, or in fact even lead to new results in the smooth regularity class. An example[1] of this can be found in the work of Colliander-Keel-Staffilani-Takaoka-Tao where a crucial component in their study of the nonlinear Schrödinger equation at low regularities was the discovery of a new Morawetz-type inequality [24]. Subsequently this new inequality was used to prove the global existence of large solutions to the energy critical nonlinear Schrodinger equation, a fact that was new even for generic smooth initial data [25].

The nonlinear Dirac equations we consider in this thesis can all be thought of as nonlinear wave equations with null structure. The presence of null structure allows improved local and global well-posedness results for nonlinear wave equations at low regularities. To illustrate this, consider the following example of nonlinear wave equation on $\mathbb{R}^{1+3}$

$$\Box u = B(\partial u, \partial u) \tag{1.1}$$

[1] This example comes from an excellent interview with T. Tao by the Clay Mathematics Institute, which can be found at *http://www.claymath.org/interviews/tao.php*.

with initial data $u(0) = f \in H^s(\mathbb{R}^3)$, $\partial_t u(0) = g \in H^{s-1}(\mathbb{R}^3)$, where B is some bilinear form. Equation (1.1) is invariant under the scaling

$$u(t,x) \mapsto u(\lambda t, \lambda x)$$

and so the scale invariant data space is $\dot{H}^{\frac{3}{2}}$. Local well-posedness in the high regularity case, $s > \frac{3}{2} + 1$, follows from the standard energy inequality, together with the fact that H^s is an algebra for $s > \frac{3}{2}$. This classical result can be improved to $s > 2$ by using Strichartz estimates[2], and is due to Ponce-Sideris [69]. The requirement $s > 2$ is in fact optimal due to the counterexamples of Lindblad in [57, 58] who considered various special cases of the nonlinearity B. Thus the nonlinear wave equation (1.1) is in general only well-posed for initial data in $H^{2+\epsilon} \times H^{1+\epsilon}$, which is $\frac{1}{2}$ a derivative above the scale invariant regularity $\frac{3}{2}$.

If we now consider the case where $B(\partial u, \partial u) = Q_0(u,u) \equiv \partial_t u^2 - |\nabla u|^2$ in (1.1), then in light of the counterexamples of Lindblad, we might expect that the optimal local well-posedness result is again $s > 2$. However, a remarkable breakthrough due to Klainerman-Machedon [49] showed that, if B is a linear combination of the *null forms* Q_0 and $Q_{ij}(u,v) = \partial_j u \partial_i v - \partial_j v \partial_i u$, then local well-posedness holds with initial data $(f,g) \in H^2 \times H^1$. This was then lowered to $s > \frac{3}{2}$ in a series of subsequent papers [51, 52]; see also the work of Zhou [88]. The improvement comes from the fact that if u, v are solutions to the homogeneous wave equation, then from the point of view of space-time estimates, the product Q_{ij} behaves significantly better than the general product $\partial u \partial v$. Homogeneous bilinear estimates for null forms are now well understood due to Foschi-Klainerman [41] for the L^2 case, and more recently Lee-Vargas [56], and Lee-Rogers-Vargas [55] in the L^p case. It is a remarkable fact that null forms appear naturally in a number of important PDEs in mathematical physics. For example the Dirac-Klein-Gordon equation, the Maxwell-Dirac equation, and the Yang-Mills equation all exhibit null structure when viewed in the appropriate way [53]. Of course finding null structure is often far from trivial.

We now turn our attention to the Dirac equation on $\mathbb{R}^{1+1}$

$$-i\gamma^\mu \partial_\mu \psi + m\psi = 0 \tag{1.2}$$

where we regard ψ as a vector in $\mathbb{C}^2$, and the matrices γ^μ are defined by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

The Dirac equation is one of the fundamental equations of relativistic quantum mechanics and is used to model particles of spin $\frac{1}{2}$, such as electrons. It was introduced by Paul Dirac as a relativistic version of the Schrödinger equation; see Section 1.1 for a brief derivation, or [83, 85] for a more in-depth look at the properties of the Dirac equation.

The Dirac equation is the Euler-Lagrange equation associated to the Lagrangian

$$\mathcal{L} = -i\overline{\psi}\gamma^\mu \partial_\mu \psi + m\overline{\psi}\psi$$

[2]Strichartz estimates give control over the $L^p_t L^q_x$ norm, see for instance [47] for optimal estimates of this form. Strichartz estimates can be proven for any dispersive PDE and have proven highly useful in the local and global theory of such equations.

with the corresponding stress-energy tensor

$$T^{\alpha}_{\beta} = -i\overline{\psi}\gamma^{\alpha}\partial_{\beta}\psi - \delta^{\alpha}_{\beta}\mathcal{L}$$

where $\overline{\psi} = \psi^{\dagger}\gamma^{0}$ is the Dirac adjoint. For a solution ψ to (1.2), by computing the divergence of T^{α}_{β} we obtain the conservation of the energy

$$E = \int_{\mathbb{R}^{1+1}} T^{0}_{0} dx = \int_{\mathbb{R}^{1+1}} -i\overline{\psi}\gamma^{0}\partial_{0}\psi \, dx = \int_{\mathbb{R}^{1+1}} \overline{\psi}(i\gamma^{1}\partial_{1}\psi - m\psi) \, dx$$

as well as the momentum

$$M = \int_{\mathbb{R}^{1+1}} T^{0}_{1} dx = -i\int_{\mathbb{R}^{1+1}} \overline{\psi}\gamma^{0}\partial_{1}\psi \, dx.$$

The invariance of solutions under the change of phase $e^{i\omega t}$ leads to the conservation of charge

$$Q = \int_{\mathbb{R}^{1+1}} \psi^{\dagger}\psi \, dx.$$

The energy E, and momentum M, are not coercive, in particular there exist negative energy solutions to the Dirac equation. Thus the energy and momentum do not play a role in the global theory of the Dirac equation. On the other hand, the charge Q is coercive (as is just the L^2 norm of the solution) and it will prove to be a crucial component of the global well-posedness arguments we present in this thesis.

A simple calculation shows that any solution ψ to (1.2) is also a solution to the Klein-Gordon equation

$$\Box\psi - m\psi = 0.$$

Thus it is natural to expect that the presence of null structure will prove to be crucial to any low regularity well-posedness theory for the Dirac equation. To understand the form null structure may take for the Dirac equation, consider a solution ψ to (1.2) on $\mathbb{R}^{1+1}$ with $m = 0$ and $\psi(0)^T = (f, g)^T$. We claim that the product[3] $\overline{\psi}\psi$ is a null form where $\overline{\psi} = \psi^{\dagger}\gamma^{0}$ and $\dagger$ denotes the conjugate transpose. To see this, let ϕ be a solution to

$$\gamma^{\mu}\partial_{\mu}\phi = \psi.$$

Then $\Box\phi = 0$ and if we write $\phi = (\phi_1, \phi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$, a computation shows that

$$\overline{\psi}\psi = 2\Re\big(\psi_1\overline{\psi}_2\big) = 2\Re\Big((\partial_t - \partial_x)\phi_2(\partial_t + \partial_x)\overline{\phi}_1\Big) = 2\Re\Big(Q_0(\phi_2, \overline{\phi}_1) + Q_{01}(\phi_2, \overline{\phi}_1)\Big)$$

where $Q_0(u, v) = \partial_t u\partial_t v - \partial_x u\partial_x v$ and $Q_{01}(u, v) = \partial_t u\partial_x v - \partial_x u\partial_t v$ are the classical null forms. Thus the term $\overline{\psi}\psi$ is a null form. The problem with this approach to uncovering null forms is that it involves solving an auxiliary equation, which can often lead to a loss of information.

An alternative approach to attempting to find classical null forms is to just estimate the product $\overline{\psi}\psi$ directly. To this end note that we can write the solution to (1.2) as $\psi_1(t, x) =$

[3]This type of product will appear frequently in the Dirac equations we consider in this thesis, and is closely related to the symmetry of the Lorentz group.

$f(x-t)$ and $\psi_2(t,x) = g(x+t)$. Since $\overline{\psi}\psi = 2\Re\big(\psi_1\overline{\psi_2}\big)$, we see that

$$\|\overline{\psi}\psi\|_{L^2([-T,T]\times\mathbb{R})} \leqslant 2\|f(x-t)\overline{g}(x+t)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2}\|g\|_{L^2}. \tag{1.4}$$

On the other hand, if we consider the product $|\psi|^2$, then by the Sobolev embedding theorem,

$$\||\psi|^2\|_{L^2([-T,T]\times\mathbb{R})} \lesssim T^{\frac{1}{2}}\big\||f|^2+|g|^2\big\|_{L^2} \lesssim T^{\frac{1}{2}}\Big(\|f\|^2_{H^{\frac{1}{4}}} + \|g\|^2_{H^{\frac{1}{4}}}\Big). \tag{1.5}$$

Thus to estimate the product $|\psi|^2$ we require $\frac{1}{4}$ derivatives, while from (1.4), the null form $\overline{\psi}\psi$ can be controlled by L^2. Consequently, we expect that the nonlinear Dirac equation with the nonlinearity $\overline{\psi}\psi$ will behave better at low regularities than the corresponding equation with a generic quadratic nonlinearity such as $|\psi|^2$. This is indeed the case; see for instance the results in [12, 75].

To prove low regularity existence for the nonlinear Dirac equations with null structure, we need to find ways to exploit null form estimates of the form (1.4). One way to do this is to use the $X^{s,b}$ spaces of Bourgain-Klainerman-Machedon. These types of spaces were first used by Bourgain in the context of the nonlinear Schrödinger equation [8, 9]. Subsequently[4] versions of these spaces adapted to the wave equation were introduced by Klainerman-Machedon in [49, 51]. Let $\|u\|_{X^{s,b}_{\pm}} = \big\|\langle\tau\pm\xi\rangle^b\langle\xi\rangle^s\widetilde{u}(\tau,\xi)\big\|_{L^2_{\tau,\xi}(\mathbb{R}^{1+1})}$. Then by the transference principle (see [31, Lemma 4] or [79, Lemma 2.9]), we obtain for all $\psi\in\mathcal{S}(\mathbb{R}^2)$

$$\|\overline{\psi}\psi\|_{L^2_{t,x}} \lesssim \|\psi_1\|_{X^{0,b}_{+}}\|\psi_2\|_{X^{0,b}_{-}} \tag{1.6}$$

for any $b>\frac{1}{2}$. This type of null form estimate is a crucial component in the low regularity theory of the nonlinear Dirac equation on $\mathbb{R}^{1+1}$ and has been used by a number of authors; see for instance [17, 60, 67, 68, 75] or Chapter 4 for more estimates of this type. An alternative approach that has proven successful [19, 48, 63] is to use product Sobolev spaces based on the null coordinates $\alpha = x+t$, $\beta = x-t$; see Chapter 2 and Chapter 3.

We now give a brief outline of the main results contained in this thesis.

Global Well-posedness for the Thirring model

The Thirring model is given by

$$\begin{aligned} -i\gamma^\mu\partial_\mu\psi + m\psi &= \lambda(\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi \\ \psi(0) &= \psi_0 \end{aligned} \tag{1.7}$$

where ψ is a $\mathbb{C}^2$-valued function of $(t,x)\in\mathbb{R}^{1+1}$, and $m,\lambda\in\mathbb{R}$. The Dirac matrices γ^μ are defined in (1.3), and for a vector valued function ψ we let $\overline{\psi} = \psi^\dagger\gamma^0$, where $\psi^\dagger$ denotes the conjugate transpose. The Thirring model corresponds to the Lagrangian

$$\mathcal{L}_{TM} = -i\overline{\psi}\gamma^\mu\partial_\mu\psi + m\overline{\psi}\psi - \frac{\lambda}{2}\big(\overline{\psi}\gamma^\mu\psi\big)\big(\overline{\psi}\gamma_\mu\psi\big)$$

[4]Though similar spaces were used earlier by Rauch-Reed [70] and Beals [4] in the context of singularity propagation for wave equations.

and has the associated conserved quantities:

$$E_{TM} = \int_{\mathbb{R}} \overline{\psi}(i\gamma^1\partial_1\psi - m\psi) + \lambda\big(\overline{\psi}\gamma^\mu\psi\big)\big(\overline{\psi}\gamma_\mu\psi\big)\,dx, \qquad \text{(energy)}$$

$$M_{TM} = -i\int_{\mathbb{R}} \overline{\psi}\gamma^0\partial_1\psi\,dx, \qquad \text{(momentum)}$$

$$Q_{TM} = \int_{\mathbb{R}} \psi^\dagger\psi\,dx = \|\psi\|_{L^2}^2. \qquad \text{(charge)}$$

As in the the discussion of the free Dirac equation, the only coercive quantity is the charge, and the energy and momentum will not play a role. The Thirring model was first introduced by Walter Thirring [84] to model the self interaction of a Dirac field and subsequently was heavily studied in the physics literature. In particular it was found to be completely integrable [2, 54] and in the massive case $m \neq 0$ have (multi) soliton solutions [3, 66] of the form

$$\psi(t,x) = \begin{pmatrix} f_\omega(x)e^{i\omega t} \\ g_\omega(x)e^{-i\omega t}\end{pmatrix}.$$

Note that using Lorentz invariance, $(t,x) \mapsto (\frac{t-cx}{\sqrt{1-c^2}}, \frac{x-ct}{\sqrt{1-c^2}})$, we obtain a family of traveling solitons.

A natural question to ask is: if we take initial data with finite charge, $\psi_0 \in L^2(\mathbb{R})$, is the Thirring model (1.7) globally well-posed? From a purely mathematical point of view, this is an interesting question as the Thirring model is L^2 critical and additionally satisfies conservation of charge $\|\psi(t)\|_{L^2_x} = \|\psi_0\|_{L^2_x}$.

Local well-posedness (LWP) for the Thirring model is known for $s > 0$ due to recent work of Selberg-Tesfahun [75]. Progress on the question of global existence was first made by Delgado [35] for initial data in H^s, $s \geqslant 1$. The general idea is as follows. Let $\psi = (u,v)^T$. Assuming ψ is a solution to (1.7), a computation shows that

$$\begin{aligned}(\partial_t + \partial_x)|u|^2 &= 2m\Im(u\overline{v}) \\ (\partial_t - \partial_x)|v|^2 &= -2m\Im(u\overline{v}).\end{aligned} \tag{1.8}$$

Therefore, by integrating and taking L^∞_x norms of both sides, we obtain

$$\|\psi(t)\|_{L^\infty_x}^2 \leqslant \|\psi(0)\|_{L^\infty_x}^2 + 2m\int_0^t \|\psi(t')\|_{L^\infty_x}^2\,dt'.$$

Applying Gronwall's inequality then gives the upper bound $\|\psi(t)\|_{L^\infty_x}^2 \lesssim \|\psi(0)\|_{L^\infty_x}^2 e^{2mt}$. As the time of existence can be controlled by the L^∞ norm, this bound is enough to prove global existence in the high regularity case $s > \frac{1}{2}$ due to the embedding $H^s \subset L^\infty$ [75].

In Chapter 2, we first show that the Thirring model is locally well-posed in the charge class, improving the recent work of Selberg-Tesfahun [75]. Secondly we prove that global well-posedness (GWP) also holds from initial data in the charge class L^2, and moreover that any additional regularity is retained. We remark that, as the Thirring model is L^2 critical, the time of existence of the local well-posedness result will depend on the profile of the initial data. Thus the conservation of charge does not give sufficient control over the dynamics to rule out the possibility of concentration of charge at a point. Instead we have to exploit the structure of the system by using a novel decomposition, together with a more refined version of the argument of Delgado given above.

Note: The results in Chapter 2 have appeared in

[18] T. Candy, *Global existence for an L^2 critical nonlinear Dirac equation in one dimension*, Adv. Differential Equations **16** (2011), no. 7-8, 643–666.

Local and Global Well-posedness for the Chern-Simons-Dirac system on $\mathbb{R}^{1+1}$

The Chern-Simons-Dirac (CSD) system is given by

$$\begin{aligned} -i\gamma^\mu D_\mu \psi + m\psi &= 0 \\ \partial_t A_1 - \partial_x A_0 &= \overline{\psi}\psi \\ \partial_t A_0 - \partial_x A_1 &= 0 \end{aligned} \tag{1.9}$$

with initial data $\psi(0) = f$, $A(0) = a$, where the spinor ψ is a $\mathbb{C}^2$-valued field, the gauge components A_μ are real-valued, the covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$, and as in the case of the Thirring model, we let $\overline{\psi} = \psi^\dagger\gamma^0$. The Chern-Simons action on $\mathbb{R}^{1+2}$ was introduced in [22]. Subsequently, it was proposed as an alternative gauge field theory to the standard Maxwell theory of electrodynamics on Minkowski space $\mathbb{R}^{1+2}$ [36]. The $\mathbb{R}^{1+1}$ system (1.9) we study in Chapter 3 was introduced in [46]. Note that the second equation in (1.9) is the one dimension analogue of the Chern-Simons action, while the last equation is the standard Lorenz gauge condition. Various properties of the Chern-Simons action have been studied previously, namely the existence of vortex solutions, as well as its topological properties; we refer the reader to [38, 43] for more information. More recently, a number of results have appeared studying the well-posedness theory of the various Chern-Simons theories [13, 44, 45, 46, 76].

In Chapter 3 we prove that the system (1.9) is locally well-posed provided we have $f \in H^s$, $a \in H^r$ with $\frac{-1}{2} < r \leqslant s \leqslant r+1$. In addition, we use the decomposition introduced in Chapter 2 to prove global well-posedness for the CSD equation provided $s \geqslant 0$. This extends the work of Huh [46] who proved local well-posedness in the case $s = r = 0$ and global well-posedness if $s = r = 1$. Note that solutions to (1.9) are invariant under the scaling $(\psi, A) \mapsto \lambda\big(\psi, A\big)(\lambda t, \lambda x)$. Hence the scale invariant space is $\dot{H}^{-\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ and so the local well-posedness result we prove in Chapter 3 is essentially optimal, except possibly at the endpoint $s = \frac{-1}{2}$.

Note: The results in Chapter 3 are joint work with Nikolaos Bournaveas and Shuji Machihara and have appeared in

[15] N. Bournaveas, T. Candy, and S. Machihara, *Local and global well-posedness for the Chern-Simons-Dirac system in one dimension*, To appear in Differential and Integral Equations (2012), arXiv:1110.6345.

Global well-posedness for the Dirac-Klein-Gordon equation on $\mathbb{R}^{1+1}$

The Dirac-Klein-Gordon (DKG) equation is given by

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi \\ \big(-\Box + m^2\big)\phi &= \overline{\psi}\psi \end{aligned} \tag{1.10}$$

with initial data

$$\psi(0) = \psi_0 \in H^s, \qquad \phi(0) = \phi_0 \in H^r, \qquad \partial_t \phi(0) = \phi_1 \in H^{r-1} \tag{1.11}$$

for some values of $s, r \in \mathbb{R}$. The DKG equation models the interaction of a meson field, ϕ, with the fermion ψ via the Yukawa interaction [7]. There are a number of previous results on the low regularity existence theory for the DKG equation [11, 17, 16, 20, 21, 40, 61, 63, 67, 72, 73, 82]. We describe these results in more detail in the introduction to Chapter 4. It is known from the work of Machihara-Nakanishi-Tsugawa that local well-posedness holds provided $s > \frac{-1}{2}$, $|s| \leqslant r \leqslant s+1$, and moreover that this region is essentially sharp [63]. In terms of global well-posedness, the best result is due to Tesfahun who showed global well-posedness provided[5] $\frac{-1}{8} < s < 0$, $s + \sqrt{s^2 - s} < r \leqslant s + 1$ [82]. The proof of global existence follows from the the conservation of charge, $\|\psi(t)\|_{L^2_x} = \|\psi_0\|_{L^2_x}$, together with the I-method of Colliander-Keel-Staffilani-Takaoka-Tao. However, there is a complication because there is no conservation law for the scalar ϕ. Thus an additional induction on free waves argument of Selberg [72] is needed to complete the proof. A related idea was used by Colliander-Holmer-Tzirakis to prove GWP for the Zakharov and Klein-Gordon-Schrdinger systems [26].

In Chapter 4 we improve the result of Tesfahun and prove global well-posedness for

$$-\frac{1}{6} < s < 0, \qquad s - \frac{1}{4} + \sqrt{\Big(s - \frac{1}{4}\Big)^2 - s} < r \leqslant s + 1.$$

The proof follows the argument of Tesfahun and makes use of the I-method, together with the induction on free waves argument introduced by Selberg [72]. The main new contribution is a study of bilinear estimates of the form

$$\|\psi_1 \psi_2\|_{X^{-s,-b}_{\pm}} \lesssim \|\psi_1\|_{X^{s_1,b_1}_{-}} \|\psi\|_{X^{s_2,b_2}_{+}} \tag{1.12}$$

where we define $\|u\|_{X^{s,b}_{\pm}} = \big\| \langle \tau \pm \xi \rangle^b \langle \xi \rangle^s \widetilde{u}(\tau, \xi)\big\|_{L^2_{\tau,\xi}}$. We prove optimal (up to endpoints) conditions on the exponents s, b, s_j, b_j such that the estimate (1.12) holds. The proof uses a dyadic decomposition, together with a number of simplifications due to Tao [78].

Local well-posedness for the space-time Monopole equation

The space-time Monopole equation is

$$F_A = *D_A \phi \tag{1.13}$$

where F_A is the curvature of a one-form connection $A = A_\alpha dx^\alpha$, D_A is a covariant derivative of the Higgs field ϕ, and $*$ is the Hodge star operator with respect to the Minkowski metric

[5] Note that global well-posedness for $s \geqslant 0$ follows from the persistence of regularity result proved in [72].

diag(-1, 1, 1) on $\mathbb{R}^{1+2}$. The components of the connection $A = A_\alpha dx^\alpha$, and the Higgs field ϕ, are maps from $\mathbb{R}^{1+2}$ into $\mathfrak{g}$

$$A_\alpha : \mathbb{R}^{1+2} \to \mathfrak{g}, \qquad \phi : \mathbb{R}^{1+2} \to \mathfrak{g},$$

where $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\cdot,\cdot]$. For simplicity we will always assume $\mathfrak{g}$ is the Lie algebra of a matrix Lie group such as $SO(d)$ or $SU(d)$. The curvature F_A of the connection A, and the covariant derivative $D_A\phi$ are given by

$$F_A = \frac{1}{2}\big(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]\big)dx^\alpha \wedge dx^\beta, \qquad D_A\phi = (\partial_\alpha \phi + [A_\alpha, \phi])dx^\alpha.$$

The space-time Monopole equation is an example of a non-abelian gauge field theory and can be derived by dimensional reduction from the anti-selfdual Yang-Mills equations; see for instance [29] or [65]. It was first introduced by Ward in [86] as a hyperbolic analogue of the Bogomolny equations, or magnetic monopole equations, which describe a point source of magnetic charge. The space-time Monopole equation is an example of a completely integrable system and has an equivalent formulation as a Lax pair. The Lax pair formulation of (1.13), together with the inverse scattering transform, was used by Dai-Terng-Uhlenbeck in [29] to prove global existence and uniqueness up to a gauge transform from small initial data in $W^{2,1}(\mathbb{R}^2)$. The survey [29] also contains a number of other interesting results related to the space-time Monopole equation.

The space-time Monopole equation (1.13) is gauge invariant. More precisely, if (A, ϕ) is a solution to (1.13) then so is $(gAg^{-1} + gdg^{-1}, g\phi g^{-1})$ where the gauge transform $g : \mathbb{R}^{1+2} \to G$ is a smooth and compactly supported map into the Lie group G. Thus, to obtain a well-posed problem we need to specify a choice of gauge. Recently[6] Czubak [28], showed that the space-time Monopole equations in the Coulomb gauge are locally well-posed for small initial data in H^s with $s > \frac{1}{4}$.

In Chapter 5 we instead consider the Lorenz gauge condition

$$\partial_\alpha A^\alpha = 0$$

and prove that the Monopole equation is well-posed for large initial data in H^s for any $s > \frac{1}{4}$. The main component of the proof is to show that the Monopole equation also has null structure in the Lorenz gauge. To exploit this null structure, we are forced to rewrite the equation using certain projection operators. Once we write the Monopole equation in the right form, the well-posedness proof essentially follows from the null form estimates of Foschi-Klainerman [41].

Note: The results in Chapter 5 are joint work with Nikolaos Bournaveas and have appeared in

[14] N. Bournaveas and T. Candy, *Local well-posedness for the space-time Monopole equation in Lorenz gauge*, Nonlinear Diff. Equations and Applications **19** (2012), no. 1, 67–78.

1.1 Derivation of the Dirac equation

The Dirac equation is one of the fundamental equations of relativistic quantum mechanics and is used to model particles of spin $\frac{1}{2}$, such as electrons. It was introduced by Paul Dirac as a

[6]Though the result was obtain earlier in Czubak's PhD thesis [27].

relativistic version of the Schrödinger equation and can be motivated as follows. In Quantum mechanics, every observable corresponds to an operator. Thus for the energy E, and momentum $\mathbf{p}$ we have[7]

$$E \to i\partial_t, \qquad \mathbf{p} \to -i\nabla.$$

If we now consider the relativistic energy momentum relation $E = \sqrt{-\mathbf{p}^2 + m^2}$, and use the correspondence principle, we arrive at the Klein-Gordon equation

$$-\partial_t^2 \phi = -\Delta\phi + m^2\phi.$$

The Klein-Gordon equation initially led to a number of conceptual difficulties, in particular it predicted a negative probability density for the wave function. Consequently, the Klein-Gordon equation was discarded and a new approach was needed. The key idea of Dirac was to linearise the energy momentum relation by using matrices. In other words writing

$$E = \sqrt{-\mathbf{p}^2 + m^2} = \alpha \cdot \mathbf{p} + \beta m$$

where $\alpha = (\alpha_j)$ and we require $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}$, $\alpha_j\beta + \beta\alpha_j = 0$, and $\beta^2 = 1$. If we now apply the correspondence principle we obtain the equation $i\partial_t\psi = -i\alpha \cdot \nabla\psi + m\beta\psi$. To ensure the operator is self-adjoint, we also require the matrices α_j and β to be Hermitian. If we multiply by β, and let $\gamma^0 = \beta$, $\gamma^j = \beta\alpha_j$, then we arrive at the Dirac equation

$$-i\gamma^\mu \partial_\mu \psi + m\psi = 0$$

where we regard ψ as a vector in $\mathbb{C}^N$, the matrices γ^μ are required to satisfy the conditions

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I, \qquad (\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^j)^\dagger = -\gamma^j. \tag{1.14}$$

If we specialise to the case $d = 1$, then one representation of the Dirac matrices is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have not mentioned the more geometric interpretation of the Dirac equation and we refer the interested reader to [83] for a more geometric perspective. See also [85] for more on the physics of the Dirac equation.

1.2 Function Spaces

In this section we briefly define the main function spaces we use throughout this thesis, and state a few important results. The results are all relatively well known and for the most part we omit the proofs.

[7] For simplicity, we take $c = \hbar = 1$, where c is the speed of light and $\hbar$ is Planck's constant.

1.2.1 Sobolev Spaces on $\mathbb{R}^d$

Sobolev spaces are widely used in the theory of PDE, and seem to be the natural spaces in which to measure the smoothness of a distribution. The definition makes use of the Bessel potentials (see for instance [77])

$$\widehat{\Lambda^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi).$$

We now define the Sobolev spaces $W^{s,p}$ as follows.

Definition 1.2.1. *For $s \in \mathbb{R}$, $1 < p < \infty$, we define $W^{s,p}$ as the closure of $\mathcal{S}$ using the norm*

$$\|f\|_{W^{s,p}} = \|\Lambda^s f\|_{L^p}.$$

Similarly, for $s > -\frac{d}{p}$, we define the homogeneous variant by using the norm

$$\|f\|_{\dot{W}^{s,p}} = \||\nabla|^s f\|_{L^p}.$$

In the case $p = 1$, we define for $k \in \mathbb{N}$

$$\|f\|_{W^{k,1}} = \sum_{|\kappa| \leqslant k} \|\partial^\kappa f\|_{L^1}$$

where $\kappa = (\kappa_1, \ldots, \kappa_n)$ and $\partial^\kappa = \partial_1^{\kappa_1} \cdots \partial_n^{\kappa_n}$.

Note that the restriction $s > -\frac{d}{p}$ is needed for the multiplier $|\nabla|^s$ to be defined as a map from $\mathcal{S}$ to L^p. The Sobolev spaces are all Banach spaces (by definition in our case), and in the case $s \in \mathbb{N}$ we have

$$\|f\|_{W^{s,p}} \approx \|f\|_{L^p} + \sum_{|\kappa| = s} \|\partial^\kappa f\|_{L^p}.$$

In the special case $p = 2$, the Sobolev space $W^{s,2}$ forms a Hilbert space and we write $H^s = W^{s,2}$. One of the many alternative characterisations of H^s is the following, which will prove useful later.

Theorem 1.2.2. *Let $0 < s < 1$. Then*

$$\|f\|_{H^s}^2 \approx \|f\|_{L^2}^2 + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+y) - f(x)|^2}{|y|^{2s}} \frac{dy}{|y|^d} dx.$$

Proof. The equivalence of norms follows almost immediately from the observation that

$$|\xi|^{2s} \approx \int_{\mathbb{R}^d} \frac{\left|1 - e^{-i\xi \cdot y}\right|^2}{|y|^{2s+d}} dy$$

together with Plancheral and Fubini. □

There are versions of Theorem 1.2.2 in the case $p \neq 2$; see [6, Theorem 6.2.5].

Among the most useful properties of the Sobolev spaces is the following Sobolev embedding theorem (see the appendix in [79]).

Theorem 1.2.3 (Sobolev Embedding Theorem)**.** *Let $s > 0$ and $1 < p < q < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. Then for any $f \in C_0^\infty$*

$$\|f\|_{L^q} \lesssim \|f\|_{\dot{W}^{s,p}}.$$

The endpoint $q = \infty$ fails. This can be seen in the $p = 2$ case by using the Fourier transform together with the example $\widehat{f}(\xi) = \frac{\mathbb{1}_{|\xi|>1}}{\log(|\xi|)|\xi|^d}$. The essential point is that $\widehat{f} \notin L^1$, but $\langle \xi \rangle^{\frac{d}{2}} \widehat{f} \in L^2$. Despite this, if we spend ϵ derivatives more, then the endpoint $q = \infty$ does hold.

Theorem 1.2.4 (Sobolev Embedding Theorem, $q = \infty$)**.** *Let $s > 0$ and $1 < p < \infty$ with $s > \frac{d}{p}$. Then*

$$\|f\|_{L^\infty} \lesssim \|f\|_{W^{s,p}}.$$

An alternative approach is to replace the Sobolev norm $W^{s,p}$, with its Besov-Lipschitz counterpart $B^s_{p,q}$, see below. We also have the product inequality which can be found in, for instance, [81, Proposition 1.1, Chapter 2].

Theorem 1.2.5 (Sobolev Product Theorem)**.** *Let $s > 0$ and $1 < p < \infty$. Then*

$$\|fg\|_{W^{s,p}} \lesssim \|f\|_{L^{q_1}} \|g\|_{W^{s,r_1}} + \|g\|_{L^{q_2}} \|f\|_{W^{s,r_2}}$$

provided $\frac{1}{p} = \frac{1}{q_j} + \frac{1}{r_j}$, $1 < r_j < \infty$, and $1 < q_j \leqslant \infty$.

A simple consequence of Theorem 1.2.5 is that the L^2 based Sobolev space H^s is an algebra for $s > \frac{d}{2}$. This important observation was a crucial component in the high regularity theory of nonlinear wave equations, and can be used to give local well-posedness results for equations of the form

$$\Box u = |\partial u|^2$$

with initial data in $H^s \times H^{s-1}$, $s > \frac{d}{2} + 1$.

Finally we state a version of Hardy's inequality that can be found in the appendix to [79].

Theorem 1.2.6 (Hardy's Inequality)**.** *Let $0 < s < \frac{d}{2}$. Then*

$$\||x|^{-s} f\|_{L^2} \lesssim \|f\|_{\dot{H}^s}.$$

1.2.2 Sobolev Spaces on Domains

In this thesis we often consider the local existence problem on say $[0, T] \times \mathbb{R}$. This will require the use of function spaces defined on $[0, T] \times \mathbb{R}$, so we need to be able to define Sobolev spaces on domains. The following definitions will prove useful.

Definition 1.2.7. *Let B be a Banach space, and $E \subset B$ a closed subspace. Then we define the quotient space (or factor space in [87]) B/E as the set of equivalence classes $[x] = \{x+y \,|\, y \in E\}$ where $x \in B$. The quotient space B/E is a Banach space with norm*

$$\|[x]\|_{B/E} = \inf_{y \in [x]} \|y\|_B$$

see for instance [87, pg 59].

Definition 1.2.8. *Assume $f \in \mathcal{S}'(\mathbb{R}^d)$ and let $\Omega \subset \mathbb{R}^d$ be an open set. We say $f = 0$ on Ω if for every $\phi \in \mathcal{S}$ with supp $\phi \subset \Omega$, we have $f(\phi) = 0$.*

We can now define spaces on domains as follows.

Definition 1.2.9. *Let $X \subset \mathcal{S}'$ be a Banach space. Then for an open set Ω we define*

$$X(\Omega) = X/\{f \in X \,|\, f = 0 \text{ on } \Omega\}.$$

It is easy to see that $\{f \in X \,|\, f = 0 \text{ on } \Omega\}$ is a closed subspace and hence $X(\Omega)$ is also a Banach space.

We often abuse notation and simply write f instead of $[f]$ for the equivalence class of $f \in X(\Omega)$. The usefulness of this definition is that any estimates on $\mathbb{R}^d$ naturally extends to arbitrary open domains $\Omega \subset \mathbb{R}^d$. For instance if we have two Banach spaces X and Y and an estimate of the form

$$\|f\|_X \lesssim \|f\|_Y,$$

then we immediately get

$$\|f\|_{X(\Omega)} \lesssim \|f\|_{Y(\Omega)}.$$

Moreover, if X is a function space defined on $\mathbb{R}^{1+d}$, which is continuously embedded in $C\big(\mathbb{R}, H^s(\mathbb{R}^d)\big)$, then $X(S_T) \subset C\big([0,T], H^s(\mathbb{R}^d)\big)$ where $S_T = (0,T) \times \mathbb{R}^d$. Thus we can continuously extend functions in $X(S_T)$ to the boundary of the domain $(0,T) \times \mathbb{R}^d$. This observation is important when considering the $X^{s,b}$ spaces on domains. In particular it ensures that if $u \in X^{s,b}(S_T)$ for $b > \frac{1}{2}$, then both $u(0)$ and $u(T)$ are well-defined.

If we turn our attention to the Sobolev spaces H^s, then the characterisation in Theorem 1.2.2 gives a hint on how to construct an intrinsic definition of the local Sobolev spaces. We only state the following results in the one dimensional case. Similar results hold in higher dimensions and for $p \neq 2$; see for instance [1, 6, 80].

Theorem 1.2.10. *Let $0 < s < \frac{1}{2}$, $R > 0$, and $I_R = (-R, R)$.*

(i) Then

$$\|f\|^2_{H^s(I_R)} \approx_R \|f\|^2_{L^2(I_R)} + \int_{I_R}\int_{I_R} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}} dxdy$$

(ii) Take $I_R(x) = (x-R, x+R)$. Then

$$\|f\|^2_{H^s(\mathbb{R})} \lesssim \sum_{j\in\mathbb{Z}} \|f\|^2_{H^s(I_1(j))}$$

and

$$\sum_{j\in\mathbb{Z}} \|f\|^2_{H^s(I_2(j))} \lesssim \|f\|^2_{H^s(\mathbb{R})}.$$

(iii) If $\frac{1}{2} = \frac{1}{p} + s$ and $s < \frac{1}{q} < 1$ then we have

$$\big\||g|^2 f\big\|_{H^s(I_2)} \lesssim \|g\|^2_{L^\infty(I_2)} \|f\|_{H^s(I_2)} + \|g\|_{L^\infty(I_2)} \|g\|_{W^{1,q}(I_2)} \|f\|_{L^p(I_2)}.$$

Proof. For the readers convenience we sketch the proof. The inequalities are all well known, see for instance [80, page 169] for a more general version[8] of (i). Part (ii) is essentially a corollary of (i) while the last inequality (iii) is an application of Theorem 1.2.5.

We start by proving (i). For an interval $I_R = (-R, R)$ define

$$\|f\|^2_{\widetilde{H}^s(I_R)} = \|f\|^2_{L^2(I_R)} + \int_{I_R}\int_{I_R} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}} dydx.$$

[8]We should mention that the $W^{s,p}$ spaces defined in [80] do not agree with the Sobolev spaces defined in this thesis. More precisely in [80] the author takes $W^{s,p} = B^s_{p,p}$, where $B^s_{p,q}$ is the Besov-Lipschitz space. Thus the definitions only agree in the case $p = 2$.

We need to show that

$$\|f\|_{H^s(I_R)} \lesssim_R \|f\|_{\widetilde{H^s}(I_R)} \lesssim_R \|f\|_{H^s(I_R)}. \tag{1.15}$$

The second inequality is straight forward as take any extension $g \in H^s(\mathbb{R})$ of $f \in H^s(I_R)$. Then by Theorem 1.2.2 we have

$$\begin{aligned}\|f\|^2_{\widetilde{H^s}} &\lesssim \|g\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|g(x)-g(y)|^2}{|x-y|^{1+2s}} dxdy \\ &\lesssim \|g\|^2_{H^s}\end{aligned}$$

and so taking the infimum over all extensions g, we obtain the second inequality in (1.15).

The first inequality in (1.15) is harder to prove. For $f \in L^2(I_R)$ we follow [80] and define an extension $\mathcal{E}(f)$ of f by letting,

$$\mathcal{E}(f)(x) = \begin{cases} \rho(x)f(\pm 2R - x) & \pm x \geqslant R \\ f(x) & |x| < R \end{cases} \tag{1.16}$$

where $\rho \in C_0^\infty$ with $\rho(x) = 1$ for $|x| < R$, supp $\rho \subset \{|x| < 2R\}$, and $|\rho(x)| \leqslant 1$ for ever $x \in \mathbb{R}$. Then

$$\begin{aligned}\|f\|^2_{H^s(I_R)} \leqslant \|\mathcal{E}(f)\|^2_{H^s(\mathbb{R})} &\lesssim \|\mathcal{E}(f)\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\mathcal{E}(f)(x)-\mathcal{E}(f)(y)|^2}{|x-y|^{1+2s}} dxdy \\ &\lesssim \|f\|_{L^2(I_R)} + \int_{I_R}\int_{I_R} + 2\int_{I_R}\int_{y>R} + 2\int_{I_R}\int_{y<-R} + \int_{\mathbb{R}\setminus I_R}\int_{\mathbb{R}\setminus I_R}.\end{aligned}$$

The first integral term is obvious. For the second we note that for $|x| < R$ and $y > R$ we have

$$\begin{aligned}|\mathcal{E}(f)(x) - \mathcal{E}(f)(y)| &= |f(x) - \rho(y)f(2R-y)| \\ &\leqslant |f(x)||\rho(x)-\rho(y)| + |f(x) - f(2R-y)|\end{aligned}$$

and $|x - (2R-y)| \leqslant |x-y|$. Hence

$$\begin{aligned}\int_{I_R}\int_{y>R} \frac{|\mathcal{E}(f)(x)-\mathcal{E}(f)(y)|^2}{|x-y|^{1+2s}} dydx &\lesssim \int_{I_R}\int_R^{2R} \frac{|f(x)|^2|\rho(x)-\rho(y)|^2}{|x-y|^{1+2s}} dydx \\ &\quad + \int_{I_R}\int_R^{2R} \frac{|f(x)-f(2R-y)|^2}{|x-(2R-y)|^{1+2s}} dydx \\ &\lesssim_\rho \|f\|^2_{L^2(I_R)} \int_0^{3R} r^{1-2s} dr + \int_{I_R}\int_{I_R} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}} dydx \\ &\lesssim_{\rho,R} \|f\|^2_{\widetilde{H^s}(I_R)}.\end{aligned}$$

The other terms are handled similarly and so we obtain (1.15).

We now prove (ii). The second inequality follows from a simple application of (i) and so we will concentrate on the first. Since

$$\|f\|^2_{H^s} \lesssim \|f\|^2_{L^2} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}} dxdy$$

it suffices to prove

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}} dxdy \leqslant \sum_{j\in\mathbb{Z}} \|f\|^2_{H^s(I_1(j))}. \tag{1.17}$$

Now

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}dxdy &= \sum_{j\in\mathbb{Z}} \int_{I_{\frac{1}{2}}(j)}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}dxdy \\ &= \sum_{j\in\mathbb{Z}} \int_{I_{\frac{1}{2}}(j)}\int_{I_1(j)} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}dxdy \\ &\qquad + \int_{I_{\frac{1}{2}}(j)}\int_{|x-j|>1} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}dxdy.\end{aligned}$$

Part (i) allows us to control the first integral. For the second we have

$$\begin{aligned}\int_{I_{\frac{1}{2}}(j)}\int_{|x-j|>1} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}dxdy &\lesssim \int_{I_{\frac{1}{2}}(j)} |f(y)|^2dy + \int_{|x-j|>1} \frac{|f(x)|^2}{\big(2|x-j|-1\big)^{1+2s}}dx \\ &\lesssim \|f\|^2_{L^2(I_1(j))} + \sum_{|j-k|\geqslant 1} \frac{1}{|j-k|^{1+2s}}\|f\|^2_{L^2(I_1(k))}.\end{aligned}$$

Therefore, as $1+2s>1$, an application of Young's inequality for sequences gives (1.17) and so result follows.

Finally we come to the proof of (iii). We begin by fixing $s<\frac{1}{q}<1$. An application of Theorem 1.2.5 together with Sobolev embedding shows that,

$$\begin{aligned}\||g|^2f\|_{H^s(\mathbb{R})} &\lesssim \|g\|^2_{L^\infty(\mathbb{R})}\|f\|_{H^s(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})}\|g\|_{W^{s,r}(\mathbb{R})}\|f\|_{L^p(\mathbb{R})} \\ &\lesssim \|g\|^2_{L^\infty(\mathbb{R})}\|f\|_{H^s(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})}\|g\|_{W^{1,q}(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}\end{aligned}$$

where $r=\frac{1}{s}$. To obtain (iii) we make use of the extension operator $\mathcal{E}$ defined above. It is easy to see that $\mathcal{E}$ is bounded on L^r for every $1\leqslant r\leqslant\infty$. Moreover the proof of (i) shows that it is also bounded on H^s. Hence

$$\begin{aligned}\big\||g|^2f\big\|_{H^s(I_R)} &\leqslant \big\||\mathcal{E}(g)|^2\mathcal{E}(f)\big\|_{H^s(\mathbb{R})} \\ &\lesssim \|\mathcal{E}(g)\|^2_{L^\infty(\mathbb{R})}\|\mathcal{E}(f)\|_{H^s(\mathbb{R})} + \|\mathcal{E}(g)\|_{L^\infty(\mathbb{R})}\|\mathcal{E}(g)\|_{W^{1,q}(\mathbb{R})}\|\mathcal{E}(f)\|_{L^p(\mathbb{R})} \\ &\lesssim \|g\|^2_{L^\infty(I_R)}\|f\|_{H^s(I_R)} + \|g\|_{L^\infty(I_R)}\|\mathcal{E}(g)\|_{W^{1,q}(\mathbb{R})}\|f\|_{L^p(I_R)}\end{aligned}$$

and so it suffices to prove that

$$\|\mathcal{E}(g)\|_{W^{1,q}(\mathbb{R})} \lesssim \|g\|_{W^{1,q}(I_R)}.$$

However this follows easily using the characterisation

$$\|f\|_{W^{1,p}(\mathbb{R})} \approx \|f\|_{L^p(\mathbb{R})} + \|\partial_x f\|_{L^p(\mathbb{R})}.$$

Note that as a consequence of this we have

$$\|f\|_{W^{1,p}(I_R)} \approx \|f\|_{L^p(I_R)} + \|\partial_x f\|_{L^p(I_R)}.$$

□

The previous theorem will prove crucial in our study of the Thirring Model in Chapter 2.

1.2.3 Besov-Lipschitz Spaces

As we have seen, the Sobolev embedding in Theorem 1.2.3 fails at the endpoint $q = \infty$. One way to get around this is to use the approach in Theorem 1.2.4 and put ϵ more derivatives on the righthand side. Alternatively we can make use of the Besov-Lipschitz scale of spaces $B^s_{p,q}$. The definition of the Besov-Lipschitz spaces requires the following Littlewood-Paley decomposition. Let $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(r) = 1$ for $|r| < 1$ and supp $\phi \subset \{|r| < 2\}$. Then, for any dyadic number $N \in 2^{\mathbb{Z}}$ we define the Fourier multipliers $P_{<N}$, $P_{>N}$, and P_N by letting

$$\widehat{P_{<N}f}(\xi) = \phi\big(\frac{|\xi|}{N}\big)\widehat{f}(\xi), \qquad \widehat{P_{>N}f}(\xi) = \big(1-\phi\big(\frac{|\xi|}{N}\big)\big)\widehat{f}(\xi), \qquad \widehat{P_N f}(\xi) = \Big(\phi\big(\frac{|\xi|}{N}\big)-\phi\big(\frac{2|\xi|}{N}\big)\Big)\widehat{f}(\xi).$$

Definition 1.2.11. *Let $s \in \mathbb{R}$, $0 < p, q < \infty$. Then we define the Besov-Lipschitz space $B^s_{p,q}(\mathbb{R}^d)$ as the completion of $\mathcal{S}$ under the norm*

$$\|f\|^q_{B^s_{p,q}} = \|P_{<1}f\|^q_{L^p} + \sum_{N\geqslant 1} \Big(N^s \|P_N f\|_{L^p}\Big)^q$$

where the sum is over dyadic numbers $N \in 2^{\mathbb{N}}$.

It is well known that $H^s = B^s_{2,2}$; see for instance [1]. In fact we only make use of the case $p = 2$ in this thesis. The usefulness of the Besov-Lipschitz scale is that we can extend certain endpoint estimates that fail for the Sobolev spaces. An example is given by the following lemma.

Lemma 1.2.12. *Let $f \in \mathcal{S}$. Then*

$$\|f\|_{L^\infty} \lesssim \|f\|_{B^{\frac{d}{2}}_{2,1}}.$$

Proof. This follows by noting that

$$\|P_N f\|_{L^\infty} \lesssim \|\widehat{f}\|_{L^1(|\xi|\approx N)} \lesssim N^{\frac{d}{2}} \|P_N f\|_{L^2}.$$

□

Another example is the following endpoint version of Theorem 1.2.5.

Theorem 1.2.13. *Let $-\frac{d}{2} < s < \frac{d}{2}$. Then*

$$\|fg\|_{H^s} \lesssim \|f\|_{B^{\frac{d}{2}}_{2,1}} \|g\|_{H^s}. \tag{1.18}$$

Proof. As we are using L^2 norms, we can replace the smooth multiplier $\phi\big(\frac{\xi}{N}\big) - \phi\big(\frac{2\xi}{N}\big)$ with the sharp cutoff $\mathbb{1}_{|\xi|\approx N}$. Let

$$\widehat{f_N} = \widehat{P_N f} = \mathbb{1}_{\{|\xi|\sim N\}}\widehat{f}$$

for $N > 1$ with $\widehat{f_1} = \chi_{\{|\xi|\lesssim 1\}}\widehat{f}$. We also use the notation $\widehat{f}_{\ll N} = \mathbb{1}_{\{|\xi|\ll N\}}\widehat{f}$. To prove (1.18) we recall the characterisation

$$\|f\|^2_{H^s} \approx \sum_{N\in 2^{\mathbb{N}}} N^{2s}\|f_N\|^2_{L^2}.$$

as well as the Trichotomy formula

$$P_N(fg) \approx f_{\ll N}g_N + f_N g_{\ll N} + \sum_{M\geqslant N} P_N(f_M g_M)$$

where the sum is over dyadic numbers $M \in 2^{\mathbb{N}}$. We estimate each of these terms separately. For the first term we observe that

$$\|f_{\ll N} g_N\|_{L^2} \lesssim \|\widehat{f}_{\ll N}\|_{L^1} \|\widehat{g}_N\|_{L^2} \lesssim \Big(\sum_{M \ll N} M^{\frac{d}{2}} \|f_M\|_{L^2}\Big) \|g_N\|_{L^2}$$

and so

$$\begin{aligned} \sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_{\ll N} g_N\|_{L^2}^2 &\lesssim \sum_{N \in 2^{\mathbb{N}}} N^{2s} \Big(\sum_{M \ll N} M^{\frac{d}{2}} \|f_M\|_{L^2}\Big)^2 \|g_N\|_{L^2}^2 \\ &\lesssim \Big(\sum_{M \in 2^{\mathbb{N}}} M^{\frac{d}{2}} \|f_M\|_{L^2}\Big)^2 \sum_{N \in 2^{\mathbb{N}}} N^{2s} \|g_N\|_{L^2}^2 \approx \|f\|_{B^{\frac{d}{2}}_{2,1}}^2 \|g\|_{H^s}^2. \end{aligned}$$

To estimate the term $f_N g_{\ll N}$ a similar computation gives

$$\sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_N g_{\ll N}\|_{L^2}^2 \lesssim \sum_{N \in 2^{\mathbb{N}}} N^{2s} \Big(\sum_{M \ll N} M^{\frac{d}{2}} \|g_M\|_{L^2}\Big)^2 \|f_N\|_{L^2}^2.$$

Now since $s < \frac{d}{2}$, we have

$$\Big(\sum_{M \ll N} M^{\frac{d}{2}} \|g_M\|_{L^2}\Big)^2 \lesssim \Big(\sum_{M \ll N} M^{d-2s}\Big)\Big(\sum_{M \ll N} M^{2s} \|g_M\|_{L^2}^2\Big) \lesssim N^{d-2s} \|g\|_{H^s}$$

and therefore

$$\sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_N g_{\ll N}\|_{L^2}^2 \lesssim \|g\|_{H^s}^2 \sum_{N \in 2^{\mathbb{N}}} N^d \|f_N\|_{L^2}^2 \approx \|f\|_{H^{\frac{d}{2}}}^2 \|g\|_{H^s}^2 \lesssim \|f\|_{B^{\frac{d}{2}}_{2,1}}^2 \|g\|_{H^s}^2.$$

Finally, for the remaining term $\sum_{M > N} P_N(f_M g_M)$, we note that

$$\begin{aligned} \Big\| \sum_{M > N} P_N(f_M g_M)\Big\|_{L^2} &\lesssim \sum_{M > N} \|P_N(f_M g_M)\|_{L^2} \\ &\lesssim \sum_{M > N} N^{\frac{d}{2}} \|f_M\|_{L^2} \|g_M\|_{L^2} \\ &\lesssim N^{\frac{d}{2}} \Big(\sum_{M > N} M^{-2s} \|f_M\|_{L^2}^2 \Big)^{\frac{1}{2}} \|g\|_{H^s}. \end{aligned}$$

Hence, for $s > \frac{-d}{2}$,

$$\begin{aligned} \sum_{N \in 2^{\mathbb{N}}} N^{2s} \Big\| \sum_{M > N} P_N(f_M g_M)\Big\|_{L^2}^2 &\lesssim \|g\|_{H^s}^2 \sum_{N \in 2^{\mathbb{N}}} N^{2s+d} \sum_{M > N} M^{-2s} \|f_M\|_{L^2}^2 \\ &\lesssim \|g\|_{H^s}^2 \sum_{M \in 2^{\mathbb{N}}} M^{-2s} \|f_M\|_{L^2}^2 \sum_{N < M} N^{d+2s} \\ &\lesssim \|g\|_{H^s}^2 \sum_{M \in 2^{\mathbb{N}}} M^d \|f_M\|_{L^2}^2 \\ &\lesssim \|g\|_{H^s}^2 \|f\|_{B^{\frac{d}{2}}_{2,1}}^2 \end{aligned}$$

and so (1.18) follows. □

This has the following useful consequence.

Corollary 1.2.14. *Let* $\frac{-1}{2} < s < \frac{1}{2}$ *and* $0 < T < 1$*. Assume* $\rho \in B^{\frac{1}{2}}_{2,1}(\mathbb{R})$ *and let* $\rho_T(t) = \rho\big(\frac{t}{T}\big)$*. Then*

$$\|\rho_T(t)f(t)\|_{H^s_t(\mathbb{R})} \lesssim_\rho \|f\|_{H^s_t(\mathbb{R})}$$

with constant independent of T*.*

Proof. Apply Theorem 1.2.13 together with the estimate $\|\rho_T\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|\rho\|_{B^{\frac{1}{2}}_{2,1}}$ which can be deduced from, for instance, the characterisation

$$\|f\|_{B^{\frac{1}{2}}_{2,1}} \approx \|f\|_{L^2} + \int_{\mathbb{R}} \frac{\|f(x-t) - f(x)\|_{L^2_x}}{|t|^{\frac{1}{2}+1}} dt$$

(See [1, Theorem 7.47]). □

Chapter 2

Global Well-posedness for the Thirring Model

In this chapter we prove global existence from L^2 initial data for a nonlinear Dirac equation known as the Thirring model [84]. Local existence in H^s for $s > 0$, and global existence for $s > \frac{1}{2}$, has recently been proven by Selberg and Tesfahun in [75] where they used $X^{s,b}$ spaces together with a type of null form estimate. In contrast, motivated by the recent work of Machihara, Nakanishi, and Tsugawa [63], we first prove local existence in L^2 by using null coordinates, where the time of existence depends on the profile of the initial data. To extend this to a global existence result we need to rule out concentration of L^2 norm, or charge, at a point. This is done by decomposing the solution into an approximately linear component, and a component with improved integrability. We then prove global existence for all $s \geqslant 0$.

2.1 Introduction

We consider the nonlinear Dirac equation

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + m\psi &= \lambda(\overline{\psi}\gamma^\mu\psi)\gamma_\mu\psi \\ \psi(0) &= \psi_0 \end{aligned} \tag{2.1}$$

where ψ is a $\mathbb{C}^2$ valued function of $(t, x) \in \mathbb{R}^{1+1}$, and $m, \lambda \in \mathbb{R}$. The Dirac matrices γ^μ are as in (1.3) and for a vector valued function ψ we let $\overline{\psi} = \psi^\dagger\gamma^0$. The nonlinear Dirac equation (2.1) is also known as the Thirring model and describes the vector self-interaction of a Dirac field [84]. Classical solutions to (2.1) satisfy conservation of charge

$$\|\psi(t)\|^2_{L^2_x} = \|\psi_0\|^2_{L^2_x}.$$

The scale invariant space is the charge class L^2, thus the equation is L^2 critical and so we expect the global well-posedness result proved below to be sharp. However, we have no explicit counterexample to well-posedness for $s < 0$.

Let $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\psi_0 = \begin{pmatrix} f \\ g \end{pmatrix}$. Writing out equation (2.1) in terms of u and v we obtain the system

$$\begin{aligned} \partial_t u + \partial_x u &= -imv - i2\lambda|v|^2 u \\ \partial_t v - \partial_x v &= -imu - i2\lambda|u|^2 v \\ \big(u, v\big)^T(0) &= (f, g)^T \end{aligned} \tag{2.2}$$

where we take $f, g \in H^s$. In the classical case, $s \geqslant 1$, global existence was first proved by Delgado in [35] where he noticed that if (u, v) is a solution to (2.2), then $(|u|^2, |v|^2)$ satisfies a quadratic nonlinear Dirac equation (see the calculation leading to (1.8)). Thus, particularly for global in time problems, the nonlinearity is milder for the square of the solution. Together with Gronwall's inequality, Delgado used this quadratic nonlinear Dirac equation to obtain an a priori bound on the L^∞ norm of the solution. Since the time of existence can be shown to depend only on the L^∞ norm, an application of the Sobolev embedding theorem shows the solution exists globally in time.

More recently, Selberg-Tesfahun [75] used the $X^{s,b}$ spaces together with the null form type estimate[1]

$$\big\||v|^2 u\big\|_{X_+^{s,b-1+\epsilon}} \lesssim \|v\|^2_{X_-^{s,b}} \|u\|_{X_+^{s,b}} \tag{2.3}$$

to prove local existence in the almost critical case $s > 0$, where $X_+^{s,b}$ and $X_-^{s,b}$ are the $X^{s,b}$ spaces adapted to the linear propagators in (2.2). This estimate fails at the endpoint[2] $s = 0$ and so the approach using standard $X^{s,b}$ spaces seems limited to the case $s > 0$. We also mention that the paper [75] included global existence for $s > \frac{1}{2}$ by using the method of Delgado referred to above. Finally we remark that in [37] and [64] the closely related (though without null structure) nonlinearity $|u|^2 u$ was considered. Local well-posedness results for Dirac equations with quadratic nonlinearities have appeared in [12], [62], [60], and [75].

In the current chapter, we use null coordinates to prove global existence in H^s for all $s \geqslant 0$, similar to the method used in the recent work of Machihara, Nakanishi, and Tsugawa [63]. The use of null coordinates has certain advantages over using the $X^{s,b}$ framework as we can work exclusively in the spatial domain and make use of the embedding $W^{1,1} \subset L^\infty$. Furthermore the local existence component of the proof is surprisingly straightforward. Once we change into null coordinates we will be forced to localise in both space and time. In the L^2 case this is not an issue as the Dirac equation satisfies finite speed of propagation. However, when trying to extend the global existence result to $s > 0$, localising in both space and time will prove to be a little inconvenient and some technical results on localised Sobolev spaces will be required.

The time of existence of the local solution obtained below depends on the profile of the initial data. As a consequence, the conservation of charge property does not imply global existence. This is to be expected as we are dealing with an equation at a scale invariant regularity, see for instance [79] for a discussion related to the problem of proving global existence for the energy critical wave equation. Thus, to obtain a global in time result, we need to have some control over the profile of the solution. This is done by modifying the approach of Delgado. Note that in previous works, the method of Delgado gave L^∞ control of the solution provided the initial

[1] The term null form estimate is used for (2.3) as the inequality relies crucially on the structure of the nonlinear term. In particular if we replace $|v|^2 u$ with $|u|^2 u$ then this estimate fails. See the discussion leading to (1.6).

[2] This can be seen by letting u and v be the relevant homogeneous solutions.

data belonged to L^∞. Here however, we are working with low regularity solutions and have no L^∞ control over the initial data. Thus a new idea is required.

The way forward is to decompose our solution into two components. We show that the first of these components satisfies an essentially linear equation, while the second component can be controlled in L^∞, see Proposition 2.4.1. We remark that, since the Dirac equation in one dimension is roughly a coupled transport equation, the solution does not disperse[3]. Thus generically we should not expect the solution to have any better integrability than the initial data. Thus the fact that we can decompose our solution into a linear piece and an L^∞ piece is quite remarkable.

We now state the main result contained in this chapter.

Theorem 2.1.1. *Let $s \geqslant 0$ and $f, g \in H^s$. There exists a global solution $(u, v) \in C(\mathbb{R}, H^s)$ to (2.2) such that the charge is conserved, so*

$$\|u(t)\|^2_{L^2_x} + \|v(t)\|^2_{L^2_x} = \|f\|^2_{L^2_x} + \|g\|^2_{L^2_x}$$

for every $t \in \mathbb{R}$. Moreover, the solution is unique in a subspace of $C(\mathbb{R}, L^2_{loc})$ and we have continuous dependence on initial data.

The first step in the proof of Theorem 2.1.1 is the following local in time result.

Theorem 2.1.2. *Let $f, g \in L^2$. There exists $T > 0$ such that we have a solution $(u, v) \in C\big([-T, T], L^2\big)$ to (2.2). Moreover, the solution is unique in a subspace of $C\big([-T, T], L^2_{loc}\big)$ and we have continuous dependence on initial data.*

In Theorem 2.1.2 we require $T > 0$ to satisfy, for every $x \in \mathbb{R}$,

$$\int_{|x-y|<2T} |f|^2 + |g|^2 dy < \epsilon$$

for a small $\epsilon > 0$. Thus, as remarked above, conservation of charge does not immediately lead to global existence.

We now give a brief outline of this chapter. In Section 2.2 we introduce the function spaces we iterate in, as well the estimates we need for the proof of Theorem 2.1.2. Section 2.3 contains the proof of Theorem 2.1.2 and in Section 2.4 we prove Theorem 2.1.1 in the case $f, g \in L^2$. Finally, in Section 2.5 we extend the global result to $s > 0$.

2.2 Preliminaries

Let $I_R = (-R, R)$ and $\Omega_R = (-R, R) \times (-R, R)$ where $R > 0$, this notation is used throughout this chapter. Define the spaces Y_R and X_R as the completion of C^∞ using the norms

$$\|u\|_{Y_R} = \|u^*\|_{L^\infty_\alpha L^2_\beta(\Omega_R)} + \|\partial_\alpha u^*\|_{L^1_\alpha L^2_\beta(\Omega_R)}$$

[3]This is not quite true, since in the case $m > 0$ we do have some dispersion since solutions to the Thirring model are also solutions to a Klein-Gordon equation, and thus decay like the wave equation on $\mathbb{R}^{1+2}$. However, as we allow the case $m = 0$, we can not exploit any dispersive effects in our proof of global existence.

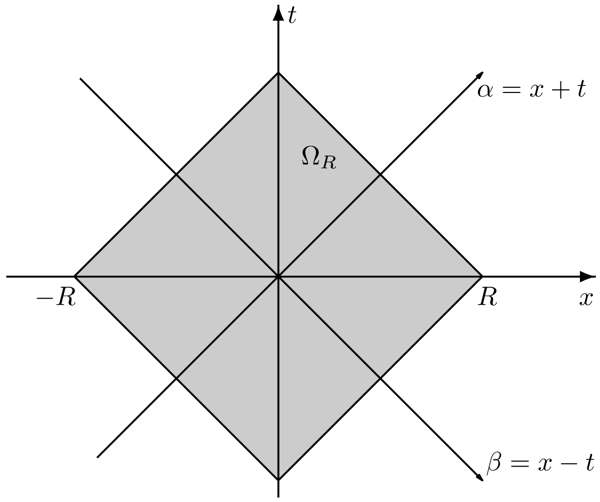

Figure 2.1: Null coordinates and the domain Ω_R.

and

$$\|v\|_{X_R} = \|v^*\|_{L^\infty_\beta L^2_\alpha(\Omega_R)} + \|\partial_\beta v^*\|_{L^1_\beta L^2_\alpha(\Omega_R)}$$

where $u(t,x)$ is a space time map into $\mathbb{C}$ and we define $u^*(\alpha,\beta) = u\big(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\big)$. We refer to the coordinates $(\alpha,\beta) = (x+t, x-t)$ as null coordinates, see Figure 2.1. Note that if we compare the Y_R and $X^{s,b}$ norms, it is easy to see what we gain by using null coordinates. Roughly speaking, for $X^{s,b}$ spaces we have $\frac{1}{2}$ a derivative in L^2 in the null direction, which fails to control L^∞. In our norms however, we have a full derivative in L^1, which does control L^∞, despite the fact that the norms $\|\cdot\|_{\dot{H}^{\frac{1}{2}}}$ and $\|\cdot\|_{\dot{W}^{1,1}}$ have the same scaling.

The Y_R and X_R norms are similar to those used in [63], where they used norms of the form $\|\cdot\|_{L^2_\alpha L^\infty_\beta}$. In fact the norms $\|\cdot\|_{L^2_\alpha L^\infty_\beta}$ would suffice to give the L^2 case of Theorem 2.1.1. However using $L^2_\alpha L^\infty_\beta$ type spaces gives no control over derivatives in the null directions, which is required in the persistence of regularity argument in Section 2.5. Thus we need to use the slightly stronger Y_R, X_R norms.

The first result we will need is the following energy type inequality.

Proposition 2.2.1. *Assume u is a solution to $\partial_t u + \partial_x u = F$ with $u(0) = f$, $f \in C^\infty(I_R)$, and $F \in C^\infty(\Omega_R)$. Then*

$$\|u\|_{Y_R} \leqslant \|f\|_{L^2(I_R)} + \|F^*\|_{L^1_\alpha L^2_\beta(\Omega_R)}.$$

Similarly, if v solves $\partial_t v - \partial_x v = G$ with $u(0) = g$ and $g, G \in C^\infty$, then

$$\|v\|_{X_R} \leqslant \|g\|_{L^2(I_R)} + \|G^*\|_{L^1_\beta L^2_\alpha(\Omega_R)}.$$

Proof. We only prove the first inequality as the second is almost identical. Write the solution u as

$$u(t,x) = f(x-t) + \int_0^t F(s, x-t+s)ds.$$

Then a simple change of variables gives

$$u^*(\alpha,\beta) = f(\beta) + \frac{1}{2}\int_\beta^\alpha F^*(s,\beta)ds.$$

Therefore the proposition follows from the definition of $\|\cdot\|_{Y_R}$ together with Minkowski's inequality.

□

The energy type inequality gains a full derivative in the relevant null direction, this gain of regularity will prove crucial and is a substitute for the null form estimates of the form (2.3) used in [75].

We will also require the following estimate, which is essentially the embedding $W^{1,1} \subset L^\infty$.

Lemma 2.2.2. *For any $R > 0$ we have*

$$\|u^*\|_{L^2_\beta L^\infty_\alpha(\Omega_R)} \leqslant \|u\|_{Y_R}$$

and

$$\|v^*\|_{L^2_\alpha L^\infty_\beta(\Omega_R)} \leqslant \|v\|_{X_R}.$$

Proof. Since C^∞ is dense in Y_R and X_R, it suffices to consider the case $u, v \in C^\infty$. Then for every $\alpha, \beta \in \Omega_R$

$$u^*(\alpha,\beta) = \int_0^\alpha \partial_\alpha u^*(\gamma,\beta)d\gamma + u^*(0,\beta).$$

Taking the supremum over α followed by the L^2 norm in β gives the inequality for u^*. The inequality for v^* is similar.

□

Corollary 2.2.3. *Let $0 < T < R$. Then we have the continuous embeddings $Y_R, X_R \subset C\big([-T,T], L^2(I_{R-T})\big)$.*

Proof. Write $u(t,x) = u^*(x+t, x-t)$. Since $(t,x) \in [-T,T] \times (-R+T, R-T)$ and $0 < T < R$ we have $|t+x| < R$ and $|x-t| < R$. Therefore

$$\begin{aligned}\|u\|_{L^\infty_t L^2_x(I_T \times I_{R-T})} &= \|u^*(x+t, x-t)\|_{L^\infty_t L^2_x(I_T \times I_{R-T})} \\ &\leqslant \|u^*(\alpha,\beta)\|_{L^2_\beta L^\infty_\alpha(\Omega_R)}\end{aligned}$$

and so the previous lemma gives

$$\|u\|_{L^\infty_t L^2_x(I_T \times I_{R-T})} \leqslant \|u\|_{Y_R}. \tag{2.4}$$

The L^2 continuity of $u(t)$ then follows from the uniform bound (2.4) together with the density of C^∞ in Y_R. The embedding $X_R \subset C\big([-T,T], L^2(I_{R-T})\big)$ follows from a similar application of Lemma 2.2.2. □

2.3 Local Existence

We deduce Theorem 2.1.2 from the following localised version via translation invariance.

Theorem 2.3.1. *Let $0 < R < \frac{1}{16|m|}$. There exists $\epsilon > 0$ depending only on λ such that if $f, g \in L^2(I_R)$ satisfy*

$$\|f\|_{L^2(I_R)} + \|g\|_{L^2(I_R)} < \epsilon, \tag{2.5}$$

then there exists a unique solution $(u, v) \in Y_R \times X_R$ to (2.2) such that

$$\|u\|_{Y_R} + \|v\|_{X_R} < 2\epsilon.$$

Moreover the solution map, mapping initial data satisfying the condition (2.5) to the solution $(u, v) \in Y_R \times X_R$, is Lipschitz continuous.

Proof. Let

$$\mathcal{X}_R = \big\{(u, v) \in Y_R \times X_R \big| \; \|u\|_{Y_R} + \|v\|_{X_R} < 2\epsilon \big\}$$

where $\epsilon > 0$ is a small constant to be fixed later. Define $N_R : \mathcal{X}_R \to \mathcal{X}_R$ by $N_R(u, v) = (u, v)$ where

$$u(t, x) = f(x - t) - \int_0^t \big(imv + i2\lambda|v|^2 u\big)(s, x - t + s)ds$$
$$v(t, x) = g(x + t) - \int_0^t \big(imu + i2\lambda|u|^2 v\big)(s, x + t - s)ds.$$

By Proposition 2.2.1 we have

$$\|u\|_{Y_R}+\|v\|_{X_R} \leqslant \epsilon+\|mv^*\|_{L^1_\alpha L^2_\beta(\Omega_R)}+\|mu^*\|_{L^1_\beta L^2_\alpha(\Omega_R)}+2\big\|\lambda|v^*|^2u^*\big\|_{L^1_\alpha L^2_\beta(\Omega_R)}+2\big\|\lambda|u^*|^2v^*\big\|_{L^1_\beta L^2_\alpha(\Omega_R)}.$$

An application of Hölder's inequality shows that

$$\begin{aligned}\|v^*\|_{L^1_\alpha L^2_\beta(\Omega_R)} + \|u^*\|_{L^1_\beta L^2_\alpha(\Omega_R)} &\leqslant 2R\|v^*\|_{L^\infty_\beta L^2_\alpha(\Omega_R)} + 2R\|u^*\|_{L^\infty_\alpha L^2_\beta(\Omega_R)}\\ &\leqslant 4R\epsilon.\end{aligned}$$

The nonlinear terms can be controlled by Hölder's inequality followed by Lemma 2.2.2, for instance

$$\begin{aligned}\big\||v^*|^2u^*\big\|_{L^1_\alpha L^2_\beta(\Omega_R)} &\leqslant \|v^*\|^2_{L^2_\alpha L^\infty_\beta(\Omega_R)}\|u^*\|_{L^\infty_\alpha L^2_\beta(\Omega_R)}\\ &\leqslant \|v\|^2_{X_R}\|u\|_{Y_R}\end{aligned}$$

the remaining term is similar. Combining these estimates we obtain

$$\|u\|_{Y_R} + \|v\|_{X_R} \leqslant \epsilon + \big(4R|m| + 16|\lambda|\epsilon^2\big)\epsilon.$$

Therefore provided $0 < R < \frac{1}{16|m|}$ and ϵ is sufficiently small (depending only on λ), we see that N_R is well defined. To show N_R is a contraction mapping follows by a similar application of Proposition 2.2.1. Hence we obtain existence. Continuous dependence on initial data in $\mathcal{X}_R$ is a simple corollary of the estimates used to deduce that N_R is a contraction mapping.

It only remains to prove uniqueness. Assume we have a solution $(u', v') \in Y_R \times X_R$ with initial data $(f, g) \in L^2(I_R)$ satisfying (2.5) and let (u, v) denote the solution constructed by the above fixed point argument with the same initial data (f, g). By choosing $R' \leqslant R$ sufficiently

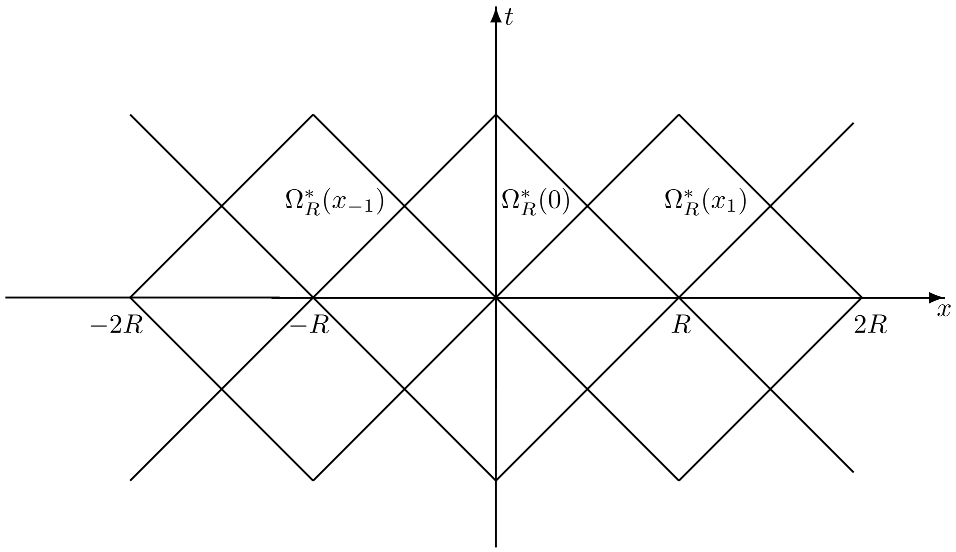

Figure 2.2: The regions $\Omega^*_R(x_j)$ in the proof of Theorem 2.1.2.

small we have

$$\|u'\|_{Y_{R'}} + \|v'\|_{X_{R'}} < 2\epsilon$$

and so $(u', v') \in \mathcal{X}_{R'}$. Note that we also have $(u, v) \in \mathcal{X}_R \subset \mathcal{X}_{R'}$. Thus, as there is a unique fixed point in $\mathcal{X}_{R'}$, we deduce that $(u', v') = (u, v)$ on $\Omega_{R'}$. Define

$$R_{max} = \sup \left\{ r \leqslant R \,\middle|\, \|u'\|_{Y_r} + \|v'\|_{X_r} < 2\epsilon \right\}$$

and suppose $R_{max} < R$. Then by the above argument we have $(u', v') = (u, v)$ on Ω_r for every $r < R_{max}$ and hence

$$\begin{aligned} \|u'\|_{Y_{R_{max}}} + \|v'\|_{X_{R_{max}}} &= \|u\|_{Y_{R_{max}}} + \|v\|_{X_{R_{max}}} \\ &\leqslant \|u\|_{Y_R} + \|v\|_{X_R} < 2\epsilon. \end{aligned}$$

Consequently $(u', v') \in \mathcal{X}_r$ for some $r > R_{max}$, contradicting the definition of R_{max}. Therefore we must have $R_{max} = R$ and so our solutions agree on Ω_R. Finally, we note that by uniqueness, the continuous dependence on initial data extends from $\mathcal{X}_R$ to $Y_R \times X_R$.

□

We can now prove Theorem 2.1.2 by using translation invariance and uniqueness.

Proof of Theorem 2.1.2. Assume $f, g \in L^2$ and let $I_R(x) = (x - R, x + R)$. Choose R sufficiently small so that

$$\sup_{j \in \mathbb{Z}} \Big(\|f\|_{L^2(I_R(jR))} + \|g\|_{L^2(I_R(jR))} \Big) < \epsilon \tag{2.6}$$

where $0 < R < \frac{1}{16|m|}$ and $\epsilon > 0$ is the constant in Theorem 2.3.1. By Theorem 2.3.1 and spatial invariance we then get a solution $(u_j, v_j) \in Y_{R,x_j} \times X_{R,x_j}$, where Y_{R,x_j} denotes the Y_R space centered at $x_j = jR$ with radius R, see Figure 2.2. Using uniqueness we can glue these solutions together to get a solution (u, v) on $\bigcup_{j \in \mathbb{Z}} \Omega^*_R(x_j)$ where

$$\Omega^*_R(x) = \left\{ (t, y) \,\middle|\, |t + y - x| < R,\ |t - y + x| < R \right\}.$$

Letting $T = \frac{R}{2}$ and noting that $|x_j - x_{j+1}| = R$ we have $(-T, T) \times \mathbb{R} \subset \bigcup_{j \in \mathbb{Z}} \Omega^*_R(x_j)$. Thus it only remains to prove that, firstly, $(u, v) \in C([-T, T], L^2)$ and secondly, that the solution map is continuous.

To this end assume (f_k, g_k) converges to (f, g) in L^2. By choosing $N > 0$ sufficiently large we can ensure that (f_k, g_k) satisfies (2.6) for every $k \geqslant N$ and $j \in \mathbb{Z}$. Using Theorem 2.3.1 and repeating the above argument we then get a solution (u_k, v_k) on $(-T, T) \times \mathbb{R}$. Moreover, the Lipschitz continuity of the localised solution map together with the embedding of Corollary 2.2.3 gives for every $|t| < T$

$$\int_{|x-x_j|\leqslant T} |u(t) - u_k(t)|^2 + |v(t) - v_k(t)|^2 dx \lesssim \int_{|x-x_j|\leqslant 2T} |f - f_k|^2 + |g - g_k|^2 dx.$$

Summing these inequalities over $j \in \mathbb{Z}$ we obtain

$$\|u - u_k\|_{L^\infty_t L^2_x} + \|v - v_k\|_{L^\infty_t L^2_x} \lesssim \|f - f_k\|_{L^2} + \|g - g_k\|_{L^2}$$

and so the solution map is continuous. It is also now easy to see that $(u, v) \in C([-T, T], L^2)$.

□

2.4 Global Existence

We start by showing global existence forward in time; existence backwards in time will then follow by a symmetry argument. Suppose we tried to iterate forwards the local in time result of Theorem 2.1.2. Then we would obtain a sequence of strictly increasing times $T_0 < T_1 < ...$ and a solution on $[0, T_j]$, where the size of each T_j would depend only on how small we needed to make R before

$$\sup_{y\in\mathbb{R}} \int_{|x-y|<R} |u(T_{j-1}, x)|^2 + |v(T_{j-1}, x)|^2 dx < \epsilon.$$

Thus, roughly speaking, provided we can ensure R does not shrink to zero, we would obtain global existence. Note that the usual conservation of charge property is not sufficient, as it does not prevent the charge from concentrating at a point. Instead we need to make use of the structure of the equation (2.2) via an argument similar to that of Delgado [35].

Proposition 2.4.1. *Let $2 \leqslant p \leqslant \infty$. Assume $(u, v) \in C^\infty$ is a solution to (2.2) on $[0, T] \times \mathbb{R}$ with initial data $f, g \in C^\infty_0$. Then there exists a decomposition*

$$(u, v) = (u_L, v_L) + (u_N, v_N)$$

such that

$$|u_L(t, x)| = |f(x - t)|, \qquad |v_L(t, x)| = |g(x + t)|,$$

and for every $0 \leqslant t \leqslant T$,

$$\|u_N(t)\|_{L^p_x} + \|v_N(t)\|_{L^p_x} \lesssim_{m,T} \|f\|_{L^2} + \|g\|_{L^2}.$$

Proof. Assume $(f, g) \in C^\infty_0$ and let u, v denote the corresponding (smooth) solutions to (2.2).

Let (u_N, v_N) be the solution to

$$\begin{aligned}\partial_t u_N + \partial_x u_N &= -imv - i2\lambda|v|^2 u_N\\ \partial_t v_N - \partial_x v_N &= -imu - i2\lambda|u|^2 v_N\end{aligned}$$

with $u_N(0) = v_N(0) = 0$ and let (u_L, v_L) be the solution to

$$\begin{aligned}\partial_t u_L + \partial_x u_L &= -i2\lambda|v|^2 u_L\\ \partial_t v_L - \partial_x v_L &= -i2\lambda|u|^2 v_L\end{aligned}$$

with initial data $u_L(0) = f$ and $v_L(0) = g$. Note that by uniqueness of smooth solutions we have $(u, v) = (u_L, v_L) + (u_N, v_N)$. A computation shows that

$$\begin{aligned}\partial_t|u_N|^2 + \partial_x|u_N|^2 &= 2m\Im(v\overline{u}_N)\\ \partial_t|v_N|^2 - \partial_x|v_N|^2 &= 2m\Im(u\overline{v}_N)\end{aligned}$$

and

$$\begin{aligned}\partial_t|u_L|^2 + \partial_x|u_L|^2 &= 0\\ \partial_t|v_L|^2 - \partial_x|v_L|^2 &= 0\end{aligned}$$

where $\Im(z)$ denotes the imaginary part of $z \in \mathbb{C}$. Thus we can write the solutions u_L and u_N as $|u_L| = |f(x - t)|$ and

$$|u_N(t, x)|^2 = 2m \int_0^t \Im(v\overline{u}_N)(s, x - t + s)ds. \tag{2.7}$$

Since $v = v_L + v_N$ and $|v_L(t, x)| = |g(x + t)|$ we have

$$\begin{aligned}\Big|\int_0^t \Im(\overline{v}u_N)(s, x - t + s)ds\Big| &\lesssim \int_0^t |v(s, x - t + s)|^2 ds + \int_0^t |u_N(s, x - t + s)|^2 ds\\ &\lesssim \int_0^t |g(2s + x - t)|^2 ds\\ &\qquad + \int_0^t |v_N(s, x - t + s)|^2 + |u_N(s, x - t + s)|^2 ds\\ &\lesssim \|g\|_{L^2}^2 + \int_0^t \|v_N(s)\|_{L_x^\infty}^2 + \|u_N(s)\|_{L_x^\infty}^2 ds.\end{aligned}$$

Taking the L_x^∞ norm of both sides of (2.7) we obtain

$$\|u_N(t)\|_{L_x^\infty}^2 \lesssim_m \|g\|_{L^2}^2 + \int_0^t \|u_N(s)\|_{L_x^\infty}^2 + \|v_N(s)\|_{L_x^\infty}^2 ds.$$

A similar argument gives

$$\|v_N(t)\|_{L_x^\infty}^2 \lesssim_m \|f\|_{L^2}^2 + \int_0^t \|u_N(s)\|_{L_x^\infty}^2 + \|v_N(s)\|_{L_x^\infty}^2 ds.$$

Therefore using Gronwall's inequality we see that for every $0 \leqslant t \leqslant T$ we have

$$\|u_N(t)\|_{L^\infty_x} + \|v_N(t)\|_{L^\infty_x} \lesssim_{m,T} \|f\|_{L^2} + \|g\|_{L^2}.$$

The finiteness of the L^2 norm follows by using conservation of charge

$$\begin{aligned}\|u_N(t)\|_{L^2_x} + \|v_N(t)\|_{L^2_x} &\leqslant \|u(t)\|_{L^2_x} + \|v(t)\|_{L^2_x} + \|u_L(t)\|_{L^2_x} + \|v_L(t)\|_{L^2_x} \\ &\lesssim \|f\|_{L^2_x} + \|g\|_{L^2_x}.\end{aligned}$$

Thus the result follows by interpolation. □

Remark 2.4.2. The equation for $u_L = (u_L, v_L)$ implies that we have

$$u_L(t,x) = (e^{iA_1} f(x-t), e^{iA_2} g(x+t))$$

for some real valued function $A = (A_1, A_2)$. Thus u_L is linear up to a gauge transform in $U(1)$. It should be possible to follow [64] and use this structure to deduce further properties of the evolution but we do not do so here.

The above proposition contains the decomposition alluded to in the introduction. Essentially the term u_L is linear while the remaining term, u_N, has vanishing initial data and more integrability than one would naively expect. This additional integrability will then allow us to rule out concentration of charge.

The proof of Theorem 2.1.1 in the case $s = 0$ is now straightforward.

Proof of Theorem 2.1.1 in the case $s = 0$. Suppose $f, g \in L^2$ and let $(u, v) \in C\big([0,T), L^2\big)$ be the corresponding solution to (2.2) where $[0,T)$ is the maximal forward time of existence. By Theorem 2.1.2 it suffices to prove that

$$\limsup_{t\to T} \sup_{x\in\mathbb{R}} \int_{|x-y|<4(T-t)} |u(t)|^2 + |v(t)|^2 dy = 0. \tag{2.8}$$

Since assuming (2.8) holds, there exists $0 < t^* < T$ such that $4(T - t^*) < \frac{1}{8|m|}$ and

$$\sup_{x\in\mathbb{R}} \int_{|x-y|<4(T-t^*)} |u(t^*)|^2 + |v(t^*)|^2 dy < \epsilon$$

where $\epsilon = \epsilon(\lambda)$ is the small constant from the proof of Theorem 2.1.2. Taking $(u(t^*), v(t^*))$ as new initial data, by Theorem 2.1.2 we can extend the solution to $[0,T) \cup [t^*, t^* + 2(T - t^*)]$. However, since $t^* + 2(T - t^*) > T$, this contradicts the assumptions that $[0,T)$ was the maximal forward time of existence. Therefore we must have $T = \infty$ and so solution exists globally in time.

We now prove (2.8). Since the solution depends continuously on the initial data, we may assume that $f, g \in C^\infty_0$. An application of Proposition 2.4.1 with $p = \infty$ shows that

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \int_{|y-x|<4(T-t)} & |u(t)|^2 + |v(t)|^2 dy \\
&\lesssim \sup_{x\in\mathbb{R}} \int_{|y-x|<4(T-t)} |f(y-t)|^2 + |g(y+t)|^2 dy \\
&\qquad + (T-t)\big(\|u_N(t)\|^2_{L^\infty_y} + \|v_N(t)\|^2_{L^\infty_y}\big) \\
&\lesssim_T \sup_{x\in\mathbb{R}} \int_{|y-x+T|<5(T-t)} |f(y)|^2 dy \\
&\qquad + \sup_{x\in\mathbb{R}} \int_{|y-x-T|<5(T-t)} |g(y)|^2 dy + (T-t)\big(\|f\|^2_{L^2} + \|g\|^2_{L^2}\big).
\end{aligned}$$

Hence letting t tend to T we obtain (2.8) and so we have global existence forward in time.

To obtain global existence backwards in time suppose (u, v) is a solution to (2.2) on $(-T, 0]$ and define $u'(t, x) = v(-t, x)$ and $v'(t, x) = u(-t, x)$. Then (u', v') solves (2.2) on $[0, T)$ with m and λ replaced with $-m$ and $-\lambda$. The forwards in time argument above then shows that we can extend (u', v') to $[0, \infty)$. Undoing the time reversal we see that we have a solution (u, v) on $(-\infty, 0]$. Therefore for every initial data $f, g \in L^2$ we have a global solution $(u, v) \in C(\mathbb{R}, L^2)$ to (2.2).

□

2.5 Persistence of Regularity

In this section we extend the global result for $s > \frac{1}{2}$ of Selberg and Tesfahun [75] to $s > 0$. This will complete the proof of Theorem 2.1.1. Ideally, since we already have global existence for $s = 0$, we would like to include the $s > 0$ result in the L^2 iteration scheme by using the standard persistence of regularity type arguments. However, since the Y_R, X_R norms contain derivatives in L^1, they do not interact very well with fractional derivatives. Consequently, the proof of global existence for $s > 0$ will be slightly more complicated than the L^2 case and some technical results on localised Sobolev spaces will be required. We remark that we still make no use of the $X^{s,b}$ type spaces, thus null coordinates can also be used for $s > 0$, see also [63].

The main result we prove in this section is the following.

Theorem 2.5.1. *Let $0 < s < \frac{1}{4}$ and $\frac{1}{2} = \frac{1}{p} + s$. There exists a small constant $0 < \epsilon^* < 1$ such that if $|m| < \epsilon^*$ and $f, g \in H^s$ satisfy*

$$\|f\|_{L^p} + \|g\|_{L^p} < \epsilon^*, \tag{2.9}$$

then there exists a solution $(u, v) \in C([-1, 1], H^s)$ solving (2.2). Moreover, the solution is unique in a subspace of $C([-1, 1], H^s)$ and depends continuously on the initial data.

The small mass assumption in Theorem 2.5.1 is required as the interval of existence is $[-1, 1]$. To motivate this consider the local existence result in L^2, Theorem 2.3.1, where we needed the time of existence[4] to satisfy $T \lesssim \frac{1}{|m|}$. Thus if $T = 1$ we have to take the mass, m, to be small. Note that for any mass m, and any initial data $f, g \in L^p$, by rescaling we can ensure

[4]Note that the size of the domain Ω_R was essentially the time of existence of a solution.

the conditions in Theorem 2.5.1 are satisfied.

Assuming Theorem 2.5.1 holds, the proof of Theorem 2.1.1 is straightforward.

Proof of Theorem 2.1.1 in the case $s > 0$. The persistence of regularity proven by Selberg and Tesfahun in [75], reduces the problem to showing global existence for $0 < s < \frac{1}{4}$. We now make use of a simple scaling argument. Take $f, g \in H^s$ and define $f_\tau = \tau^{\frac{1}{2}} f(\tau x)$, $g_\tau = \tau^{\frac{1}{2}} g(\tau x)$, $m' = \tau m$. By choosing τ sufficiently small we see that f, g, and m' satisfy the conditions of Theorem 2.5.1. Therefore we get a solution $(u_\tau, v_\tau) \in C([-1,1], H^s)$ to (2.2) with m replaced by m'. To undo the scaling we let $u(t,x) = \tau^{-\frac{1}{2}} u_\tau(\tau^{-1}t, \tau^{-1}x)$ and define v similarly. It is easy to see that (u, v) is a solution to (2.2) with $(u, v) \in C([-T, T], H^s)$ where T only depends on the size of some negative power of $\|f\|_{L^p} + \|g\|_{L^p}$. To conclude the proof we note that the decomposition in Proposition 2.4.1 shows that $\|u(t)\|_{L^p} + \|v(t)\|_{L^p} < C(t) < \infty$ for every $t \in \mathbb{R}$ where $C(t) \in L^\infty_{loc}(\mathbb{R})$. Therefore the solution must exist globally in time. □

We have reduced the proof of global existence for $s > 0$ to proving the local result in Theorem 2.5.1. The main tool to do this will again be the use of null coordinates together with a decomposition along the lines of Proposition 2.4.1.

We now present some results on localised Sobolev spaces that we require in the proof of Theorem 2.5.1. For more detail we refer the reader to Section 1.2. To start with note that any inequality for $W^{s,p}(\mathbb{R})$ implies a corresponding inequality for the localised space $W^{s,p}(I)$ for any open set $I \subset \mathbb{R}$. In particular, if $\frac{1}{q} \leqslant \frac{1}{p} + s$ and $1 < p \leqslant q < \infty$, we have Sobolev embedding

$$\|f\|_{L^p(I)} \lesssim \|f\|_{W^{s,q}(I)}$$

and if $0 < s < \frac{1}{2}$ and $y \in \mathbb{R}$ we have Hardy's inequality[5]

$$\left\| \frac{f(x)}{|x-y|^s} \right\|_{L^2_x(I)} \lesssim \|f\|_{\dot{H}^s(I)}.$$

We also make use of the following well known characterisation of localised Sobolev spaces.

Theorem 2.5.2. *Let $0 < s < \frac{1}{2}$.*

(i) Then

$$\|f\|^2_{H^s(I_R)} \approx_R \|f\|^2_{L^2(I_R)} + \int_{I_R} \int_{I_R} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2s}} dxdy$$

(ii) Take $I_R(x) = (x - R, x + R)$. Then

$$\|f\|^2_{H^s(\mathbb{R})} \lesssim \sum_{j \in \mathbb{Z}} \|f\|^2_{H^s(I_1(j))}$$

and

$$\sum_{j \in \mathbb{Z}} \|f\|^2_{H^s(I_2(j))} \lesssim \|f\|^2_{H^s(\mathbb{R})}.$$

[5] See for instance page 334 of [79] for a proof of Hardy's inequality on $\mathbb{R}^d$.

(iii) If $\frac{1}{2} = \frac{1}{p} + s$ *and* $s < \frac{1}{q} < 1$ *then we have*

$$\big\||g|^2 f\big\|_{H^s(I_2)} \lesssim \|g\|^2_{L^\infty(I_2)} \|f\|_{H^s(I_2)} + \|g\|_{L^\infty(I_2)} \|g\|_{W^{1,q}(I_2)} \|f\|_{L^p(I_2)}.$$

Proof. See Theorem 1.2.10. □

We also require the following estimates.

Proposition 2.5.3. *Let* $0 < s < \frac{1}{2}$.

(i) Then

$$\Big\| \int_{-2}^{x} F(t', x) dt' \Big\|_{H^s_x(I_2)} \lesssim \|F\|_{L^1_t H^s_x(\Omega_2)}.$$

(ii) We have

$$\|v^*\|_{L^1_\alpha H^s_\beta(\Omega_2)} \lesssim \|v\|_{X_2}$$

and

$$\|u^*\|_{L^1_\beta H^s_\alpha(\Omega_2)} \lesssim \|u\|_{Y_2}.$$

Proof. We start with (i). The characterisation in Theorem 2.5.2 shows that

$$\Big\| \int_{-2}^{x} F(t', x) dt' \Big\|^2_{H^s_x(I_2)} \lesssim \Big\| \int_{-2}^{x} F(t', x) dt' \Big\|^2_{L^2_x(I_2)} + \int_{I_2} \int_{I_2} \frac{\big| \int_{-2}^{x} F(t', x) dt' - \int_{-2}^{y} F(t', y) dt' \big|^2}{|x-y|^{1+2s}} dy dx.$$

The L^2 component is easily controlled by using Minkowski's inequality. For the remaining part we note that, by symmetry, we may assume $x > y$. Then using the inequality

$$\begin{aligned} \Big| \int_{-2}^{x} F(t', x) dt' - \int_{-2}^{y} F(t', y) dt' \Big| &\leqslant \int_{y}^{x} |F(t', x)| dt' + \int_{-2}^{y} |F(t', x) - F(t', y)| dt' \\ &\leqslant \int_{y}^{x} |F(t', x)| dt' + \|F(x) - F(y)\|_{L^1_t(I_2)} \end{aligned}$$

we reduce to estimating the integrals

$$\int_{I_2} \int_{I_2} \frac{\big(\int_y^x |F(t', x)| dt' \big)^2}{|x-y|^{1+2s}} dy dx$$

and

$$\int_{I_2} \int_{I_2} \frac{\|F(x) - F(y)\|^2_{L^1_t(I_2)}}{|x-y|^{1+2s}} dy dx.$$

The latter is again easily controlled by an application of Minkowski's inequality and so it only remains to estimate the former. To this end note that

$$\begin{aligned} \int_{I_2} \int_{I_2} \frac{\big(\int_y^x |F(t', x)| dt' \big)^2}{|x-y|^{1+2s}} dy dx &\leqslant \Big(\int_{I_2} \Big(\int_t^2 \int_{-2}^{t} \frac{|F(t,x)|^2}{|x-y|^{1+2s}} dy dx \Big)^{\frac{1}{2}} dt \Big)^2 \\ &\lesssim \Big(\int_{I_2} \Big(\int_t^2 \frac{|F(t,x)|^2}{|x-t|^{2s}} + \frac{|F(t,x)|^2}{|x+2|^{2s}} dx \Big)^{\frac{1}{2}} dt \Big)^2 \\ &\leqslant \Big(\int_{I_2} \Big\| \frac{F(t,x)}{|x+2|^s} \Big\|_{L^2_x(I_2)} + \Big\| \frac{F(t,x)}{|x-t|^s} \Big\|_{L^2_x(I_2)} dt \Big)^2 \\ &\lesssim \|F\|^2_{L^1_t H^s_x(\Omega_2)} \end{aligned}$$

where we needed $0 < s < \frac{1}{2}$ to apply Hardy's inequality.

To prove the first inequality in (ii) we note that by Hölder's inequality together with Lemma 2.2.2 it suffices to show that for all $\alpha \in I_2$

$$\|v^*\|_{H^s_\beta(I_2)} \lesssim \|v^*\|_{L^\infty_\beta(I_2)} + \|\partial_\beta v^*\|_{L^1_\beta(I_2)}.$$

To this end we note that

$$\begin{aligned}\|v^*\|^2_{H^s_\beta(I_2)} &\lesssim \|v^*\|^2_{L^2_\beta(I_2)} + \int_{I_2}\int_{I_2} \frac{|v^*(\sigma)-v^*(\gamma)|^2}{|\sigma-\gamma|^{1+2s}} d\sigma d\gamma \\ &\lesssim \|v^*\|^2_{L^\infty_\beta(I_2)} + \|v^*\|_{L^\infty_\beta(I_2)} \int_{I_2}\int_{I_2}\int_\sigma^\gamma \frac{|\partial_\beta v^*(\beta)|}{|\sigma-\gamma|^{1+2s}} d\beta d\sigma d\gamma\end{aligned}$$

where, as before, we may assume $\sigma < \gamma$. To control the integral term we just change the order of integration to obtain

$$\begin{aligned}\int_{I_2}\int_{I_2}\int_\sigma^\gamma \frac{|\partial_\beta v^*(\beta)|}{|\sigma-\gamma|^{1+2s}} d\beta d\sigma d\gamma &= \int_{I_2} |\partial_\beta v^*(\beta)| \int_\beta^2 \int_{-2}^\beta |\gamma-\sigma|^{-1-2s} d\gamma d\sigma d\beta \\ &\lesssim \|\partial_\beta v^*\|_{L^1(I_2)}\end{aligned}$$

since $0 < s < \frac{1}{2}$. Therefore the first inequality in (ii) follows. The second is similar and we omit the details. □

For the remainder of this chapter we fix p, q such that

$$\frac{1}{2} = \frac{1}{p} + s \tag{2.10}$$

and

$$\frac{1}{q} = 1 - 2s. \tag{2.11}$$

Note that $2q = p$ and for $s < \frac{1}{4}$, we have $2 < p < 4$ and $1 < q < 2$. Also by Sobolev embedding, $H^s(I) \subset L^p(I)$. Define the spaces Y^s_R and X^s_R by using the norms

$$\|u\|_{Y^s_R} = \|u^*\|_{L^\infty_\alpha H^s_\beta(\Omega_R)} + \|\partial_\alpha u^*\|_{L^q_\alpha L^2_\beta(\Omega_R)} + \|\partial_\alpha u^*\|_{L^1_\alpha H^s_\beta(\Omega_R)}$$

and

$$\|v\|_{X^s_R} = \|v^*\|_{L^\infty_\beta H^s_\alpha(\Omega_R)} + \|\partial_\beta v^*\|_{L^q_\beta L^2_\alpha(\Omega_R)} + \|\partial_\beta v^*\|_{L^1_\beta H^s_\alpha(\Omega_R)}.$$

Note that in the case $s = 0$ we simply have $Y^0_R = Y_R$. It is easy to see that if $f \in H^s$ then solutions to

$$\begin{aligned}\partial_t u + \partial_x u &= 0 \\ u(0) &= f\end{aligned}$$

lie in Y^s_R for any q. A similar comment applies for the space X^s_R. Furthermore, we have the following properties.

Proposition 2.5.4. *Let* $0 < s < \frac{1}{2}$.

(i) For any $0 < T < R$ we have the embeddings

$$\|u\|_{L^\infty_t H^s_x(I_T \times I_{R-T})} \lesssim \|u\|_{Y^s_R}$$

and

$$\|v\|_{L^\infty_t H^s_x(I_T \times I_{R-T})} \lesssim \|v\|_{X^s_R}.$$

(ii) Suppose $\partial_t u + \partial_x u = F$ with $u(0) = f$ and $f, F \in C^\infty_0$. Then

$$\|u\|_{Y^s_R} \lesssim \|f\|_{H^s(I_R)} + \|F^*\|_{L^q_\alpha L^2_\beta(\Omega_R)} + \|F^*\|_{L^1_\alpha H^s_\beta(\Omega_R)}.$$

Similarly, if $\partial_t v - \partial_x v = G$ with $v(0) = g$ and $g, G \in C^\infty_0$, then

$$\|v\|_{X^s_R} \lesssim \|g\|_{H^s(I_R)} + \|G^*\|_{L^q_\beta L^2_\alpha(\Omega_R)} + \|G^*\|_{L^1_\beta H^s_\alpha(\Omega_R)}.$$

Proof. We begin by proving (i). Write

$$u(t,x) = \int_0^{x+t} \partial_\alpha u^*(\gamma, x-t)d\gamma + u^*(0, x-t).$$

Since $(t,x) \in I_T \times I_{R-T}$ we have $|x-t| < R$ and so

$$\|u^*(0, x-t)\|_{H^s_x(I_{R-T})} \leqslant \|u^*(0,\beta)\|_{H^s_\beta(I_R)}.$$

Together with a slight modification of Proposition 2.5.3 we see that

$$\begin{aligned}
\|u(t,x)\|_{H^s_x(I_R)} &\leqslant \Big\| \int_0^{x+t} \partial_\alpha u^*(\gamma, x-t)d\gamma \Big\|_{H^s_x(I_{R-T})} + \|u^*(0,x-t)\|_{H^s_x(I_{R-T})} \\
&\leqslant \|\partial_\alpha u^*(\alpha, x-t)\|_{L^1_\alpha H^s_x(I_2 \times I_{R-T})} + \|u^*(0,\beta)\|_{H^s_\beta(I_R)} \\
&\leqslant \|u\|_{Y^s_R}.
\end{aligned}$$

The proof of the remaining estimate in (i) is similar.

To prove (ii) we follow the proof of Theorem 2.2.1 and write the solution u as

$$u(t,x) = u^*(\alpha,\beta) = f(\beta) + \frac{1}{2}\int_\beta^\alpha F^*(\gamma,\beta)d\gamma.$$

Applying Proposition 2.5.3 we obtain (ii) for u. The inequality for v is similar and we omit the details.

□

We also need the following version of the decomposition in Proposition 2.4.1.

Lemma 2.5.5. *Assume $f, g \in C^\infty_0$ and $|m| \leqslant 1$. Let $(u,v) \in C^\infty$ be the corresponding solution to (2.2). Then we can write $(u,v) = (u_L, v_L) + (u_N, v_N)$ with*

$$|u_L(t,x)| = |f(x-t)|, \qquad |v_L(t,x)| = |g(x+t)|,$$

and (u_N, v_N) *satisfies*

$$\|u_N^*\|_{L^\infty_{\alpha,\beta}(\Omega_2)} + \|v_N^*\|_{L^\infty_{\alpha,\beta}(\Omega_2)} \lesssim \|u\|_{Y_2} + \|v\|_{X_2}.$$

Proof. We begin by using the same decomposition as in Proposition 2.4.1,

$$(u, v) = (u_L, v_L) + (u_N, v_N),$$

where we recall that $|u_L^*(\alpha,\beta)| = |f(\beta)|$, $|v_L^*(\alpha,\beta)| = |g(\alpha)|$, and

$$|u_N^*(\alpha,\beta)|^2 = m\int_\beta^\alpha \Im(v\overline{u}_N)^*(\gamma,\beta)d\gamma, \qquad |v_N^*(\alpha,\beta)|^2 = m\int_\beta^\alpha \Im(u\overline{v}_N)^*(\alpha,\gamma)d\gamma.$$

Estimating u_N we get

$$\|u_N^*(\alpha)\|^2_{L^\infty_\beta(I_2)} \leqslant \|v^*\|^2_{L^2_\alpha L^\infty_\beta(\Omega_2)} + \int_{-2}^\alpha \|u_N^*(\gamma)\|^2_{L^\infty_\beta(I_2)} d\gamma$$

and so the estimate for u_N follows by an application of Gronwall's inequality together with Lemma 2.2.2. The estimate for v_N is similar.

□

The technical details are in place and we are now able to prove a local version of Theorem 2.5.1.

Theorem 2.5.6. *Let* $0 < s < \frac{1}{4}$ *and* $\frac{1}{2} = \frac{1}{p} + s$. *There exists* $0 < \epsilon^* < 1$ *such that for any* $|m| < \epsilon^*$ *and* $f, g \in C_0^\infty$ *with*

$$\|f\|_{L^p(\mathbb{R})} + \|g\|_{L^p(\mathbb{R})} < \epsilon^*,$$

the corresponding solution $(u, v) \in C^\infty$ *to (2.2) satisfies*

$$\|u\|_{L^\infty_t H^s_x(I_1\times I_1)} + \|v\|_{L^\infty_t H^s_x(I_1\times I_1)} \lesssim \|f\|_{H^s(I_2)} + \|g\|_{H^s(I_2)}$$

with constant independent of f, g, *and* m.

Proof. By Proposition 2.5.4 it suffices to prove

$$\|u\|_{Y_2^s} + \|v\|_{X_2^s} \lesssim \|f\|_{H^s(I_2)} + \|g\|_{H^s(I_2)}$$

Since $p > 2$, by taking $\epsilon^* < \epsilon$, where ϵ is the constant appearing in Theorem 2.3.1, we have a smooth solution[6] (u, v) such that

$$\|u\|_{Y_2} + \|v\|_{X_2} < 2\epsilon^*.$$

An application of Proposition 2.5.4 gives the estimate

$$\|u\|_{Y_2^s} \lesssim \|f\|_{H^s(I_2)} + \|mv^* + 2\lambda|v^*|^2u^*\|_{L^q_\alpha L^2_\beta(\Omega_2)} + \|mv^* + 2\lambda|v^*|^2u^*\|_{L^1_\alpha H^s_\beta(\Omega_2)}. \tag{2.12}$$

[6]Note that the classical smooth solution from the initial data f, g belongs to $Y_2 \times X_2$ and so agrees with the solution given by Theorem 2.3.1.

For the second term, noting that $2q = p$ and $1 < q < 2$, we have by Lemma 2.5.5

$$\begin{aligned}\|mv^* + \lambda|v^*|^2u^*\|_{L^q_\alpha L^2_\beta(\Omega_2)} &\lesssim \epsilon^*\|v^*\|_{L^2_\alpha L^\infty_\beta(\Omega_2)} + \big\||v^*|^2\big\|_{L^q_\alpha L^\infty_\beta(\Omega_2)}\|u\|_{L^\infty_\alpha L^2_\beta(\Omega_2)} \\ &\lesssim \epsilon^*\|v\|_{X_2} + \Big(\|g\|^2_{L^{2q}(I_2)} + \|v_N\|^2_{L^\infty_{\alpha,\beta}(\Omega_2)}\Big)\|u\|_{Y_2} \\ &\lesssim \epsilon^*\|v\|_{X_2} + \Big(\|g\|^2_{L^p(I_2)} + \|v\|^2_{X_2}\Big)\|u\|_{Y_2}.\end{aligned}$$

Thus taking $\epsilon^* > 0$ sufficiently small, we have

$$\|mv^* + \lambda|v^*|^2u^*\|_{L^q_\alpha L^2_\beta(\Omega_2)} \leqslant \frac{1}{8}\big(\|u\|_{Y_2^s} + \|v\|_{X_2^s}\big).$$

For the third term in (2.12) we need to estimate $\|v^*\|_{L^1_\alpha H^s_\beta(\Omega_2)}$ and $\big\||v^*|^2u^*\big\|_{L^1_\alpha H^s_\beta(\Omega_2)}$. We can control the first term by using (ii) in Proposition 2.5.3 while for the cubic term by Theorem 2.5.2 we have for all $\alpha \in I_2$

$$\big\||v^*|^2u^*\big\|_{H^s_\beta(I_2)} \lesssim \|v^*\|^2_{L^\infty_\beta(I_2)}\|u^*\|_{H^s_\beta(I_2)} + \|v^*\|_{L^\infty_\beta(I_2)}\|v^*\|_{W^{1,q}_\beta(I_2)}\|u^*\|_{L^p_\beta(I_2)}.$$

Therefore,

$$\begin{aligned}\big\||v^*|^2u^*\big\|_{L^1_\alpha H^s_\beta(\Omega_2)} &\lesssim \|v^*\|^2_{L^2_\alpha L^\infty_\beta(\Omega_2)}\|u^*\|_{L^\infty_\alpha H^s_\beta(\Omega_2)} + \|v^*\|_{L^2_\alpha L^\infty_\beta(\Omega_2)}\|v^*\|_{L^2_\alpha W^{1,q}_\beta(\Omega_2)}\|u^*\|_{L^\infty_\alpha L^p_\beta(\Omega_2)} \\ &\lesssim \|v\|^2_{X_2}\|u\|_{Y_2^s} + \|v\|_{X_2}\|v\|_{X_2^s}\big(\|f\|_{L^p(I_2)} + \|u\|_{Y_2}\big) \\ &\lesssim \big(\epsilon^*\big)^2\big(\|u\|_{Y_2^s} + \|v\|_{X_2^s}\big)\end{aligned}$$

where we used Lemma 2.5.5 together with the characterisation

$$\|f\|_{W^{1,p}(I_2)} \approx \|f\|_{L^p(I_2)} + \|\partial_x f\|_{L^p(I_2)}$$

which follows from the proof of Theorem 2.5.2. Thus provided we choose ϵ^* sufficiently small, we obtain

$$\|u\|_{Y_2^s} \leqslant C\|f\|_{H^s(I_2)} + \frac{1}{4}\big(\|u\|_{Y_2^s} + \|v\|_{X_2^s}\big).$$

A similar argument shows

$$\|v\|_{X_2^s} \leqslant C\|g\|_{H^s(I_2)} + \frac{1}{4}\big(\|u\|_{Y_2^s} + \|v\|_{X_2^s}\big)$$

and so result follows.

□

Finally we come to the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. Let $\epsilon^* > 0$ be the constant from Theorem 2.5.6. Assume $f, g \in H^s$ satisfy (2.9) and $|m| < \epsilon^*$. Choose a smooth approximating sequence $f_n, g_n \in C_0^\infty$ converging to f, g in H^s. Note that we may also assume f_n, g_n satisfy (2.9) for every $n \in \mathbb{N}$. Suppose for the moment that we had the estimate

$$\|u_n\|_{L^\infty_t H^s_x(I_1\times\mathbb{R})} + \|v_n\|_{L^\infty_t H^s_x(I_1\times\mathbb{R})} \lesssim \|f_n\|_{H^s(\mathbb{R})} + \|g_n\|_{H^s(\mathbb{R})} \tag{2.13}$$

with the constant independent of $n \in \mathbb{N}$. The continuous dependence on initial data proven by Selberg and Tesfahun in [75] implies that (u_n, v_n) converges to a solution $(u, v) \in C([-T^*, T^*], H^s)$ with possibly[7] $T^* < 1$. The uniform bound (2.13) on the interval $(-1, 1)$ implies we can repeat the H^s local existence result of [75] and extend the solution to (at least) the interval $[-1, 1]$. Thus we obtain $(u, v) \in C([-1, 1], H^s)$ as required.

It remains to prove (2.13). To this end assume $f, g \in C_0^\infty$ and let (u, v) be the corresponding smooth solution. Similar to the proof of Theorem 2.1.2 we take $I_R(x) = (x - R, x + R)$. Since the Dirac equation is invariant under translation by Theorem 2.5.6 we have for every $j \in \mathbb{Z}$

$$\|u\|_{L_t^\infty H_x^s(I_1 \times I_1(j))} + \|v\|_{L_t^\infty H_x^s(I_1 \times I_1(j))} \lesssim \|f\|_{H^s(I_2(j))} + \|g\|_{H^s(I_2(j))}.$$

Therefore, by (ii) in Theorem 2.5.2 we have for every $|t| \leqslant 1$

$$\begin{aligned}
\|u(t)\|_{H_x^s}^2 + \|v(t)\|_{H_x^s}^2 &\lesssim \sum_{j\in\mathbb{Z}} \Big(\|u(t)\|_{H_x^s(I_1(j))}^2 + \|v(t)\|_{H_x^s(I_1(j))}^2 \Big) \\
&\lesssim \sum_{j\in\mathbb{Z}} \Big(\|f\|_{H^s(I_2(j))}^2 + \|g\|_{H^s(I_2(j))}^2 \Big) \\
&\lesssim \|f\|_{H^s}^2 + \|g\|_{H^s}^2 .
\end{aligned}$$

Thus the inequality (2.13) holds and the result follows.

□

[7] The proof of local existence contained in [75] gives a time of existence T^* depending on the size of the initial data in H^s.

Chapter 3

Local and Global well-posedness for the Chern-Simons-Dirac System in One Dimension

We consider the Cauchy problem for the Chern-Simons-Dirac system on $\mathbb{R}^{1+1}$ with initial data in H^s. Almost optimal local well-posedness is obtained. Moreover, we show that the solution is global in time, provided that initial data for the spinor component has finite charge, or L^2 norm.

3.1 Introduction

The Chern-Simons action was first studied from a geometric point of view in [22]. Subsequently, it was proposed as an alternative gauge field theory to the standard Maxwell theory of electrodynamics on Minkowski space $\mathbb{R}^{1+2}$ [36]. As well as being of interest theoretically, it has also been successfully applied to explain phenomena in the physics of planar condensed matter, such as the fractional quantum Hall effect [59]. Recently, much progress has been made on the Cauchy problem for the Chern-Simons action coupled with various other field theories such as Chern-Simons-Higgs, [13, 45], and Chern-Simons-Dirac [45].

In this chapter, we consider the Cauchy problem for the Chern-Simons-Dirac (CSD) system in $\mathbb{R}^{1+1}$. This system was first studied by Huh in [46] as a simplified version of the more standard CSD system on $\mathbb{R}^{1+2}$. The CSD system on $\mathbb{R}^{1+1}$ is given by

$$\begin{aligned} -i\gamma^\mu D_\mu \psi + m\psi &= 0 \\ \partial_t A_1 - \partial_x A_0 &= \overline{\psi}\psi \\ \partial_t A_0 - \partial_x A_1 &= 0 \end{aligned} \tag{CSD}$$

with initial data $\psi(0) = f$, $A(0) = a$, where the spinor ψ is a $\mathbb{C}^2$ valued function of $(t, x) = (x_0, x_1) \in \mathbb{R}^{1+1}$ and the gauge components A_0 and A_1 of the gauge $A = (A_0, A_1)$ are real valued. The covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$ and as in the previous chapter, $\overline{\psi} = \psi^\dagger\gamma^0$. The matrices γ^α are defined in (1.3). Note that the second equation in (CSD) is one dimension analogue of the Chern-Simons action, while the last equation is the standard Lorenz gauge condition.

The system (CSD) is interesting from a mathematical point of view for a number of reasons. Firstly solutions to (CSD) satisfy conservation of charge, i.e. we have $\|\psi(t)\|_{L^2} = \|f\|_{L^2}$ for any $t \in \mathbb{R}$. This is similar to the Dirac-Klein-Gordon (DKG) equation where conservation of charge also holds. We remark that conservation of charge forms a crucial component in the study of global existence for DKG [72, 82], see also Chapter 4. On the other hand, conservation of charge fails for other quadratic Dirac equations which have been studied in the literature [12, 60, 62]. Secondly, there is substantial null structure in the nonlinear terms in (CSD), in the sense that (CSD) is roughly equivalent to a system of nonlinear wave equations of the form

$$\Box \Psi = Q(\Psi, \Psi)$$

where $Q(\Psi, \Psi)$ is a combination of the null forms $Q_{ij} = \partial_i \Psi_\mu \partial_j \Psi_\nu - \partial_j \Psi_\mu \partial_i \Psi_\nu$ and $Q_0 = g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi$. Moreover the structure of the equation means that in the mass free case $m = 0$, the spinor ψ can be explicitly solved in terms of the initial data ψ_0 and the gauge A. This idea was used in [46] to derive a number of interesting observations on the asymptotic behaviour of solutions to (CSD) as $t \to \infty$.

Currently the best known results for the Cauchy problem for (CSD) are due to Huh in [46] where it was shown that the (CSD) system is locally well-posed for initial data in the charge class $(f, a) \in L^2 \times L^2$, and globally well-posed for $(f, a) \in H^1 \times H^1$. To prove the local in time result, Huh rewrote (CSD) as a system of nonlinear wave equations and showed that the nonlinear terms contained null structure. The null form estimates of Klainerman and Machedon [49] then completed the proof.

In the current chapter we follow the approach used in Chapter 2. Instead of rewriting (CSD) as a wave equation, we factor the Dirac and Gauge components into null-coordinates $x \pm t$ and use Sobolev spaces adapted to these coordinates. In one space dimension, Sobolev spaces based on null coordinates seem to behave better than the closely related $X^{s,b}_{\pm}$ type spaces of Bourgain-Klainerman-Machedon which have been used in many other low-regularity results on Dirac equations in one dimension; see for instance the results in [19, 63].

The main result in this chapter is the following.

Theorem 3.1.1. *Let $\frac{-1}{2} < r \leqslant s \leqslant r+1$ and $(f, a) \in H^s \times H^r$. Then there exists $T > 0$ and a solution $(\psi, A) \in C\big([-T, T], H^s \times H^r\big)$ to (CSD). Moreover the solution depends continuously on the initial data, is unique in some subspace of $C\big([-T, T], H^s \times H^r\big)$, and any additional regularity persists in time*[1].

Remark 3.1.2. If we set $m = 0$, then solutions to (CSD) are invariant under the scaling $(\psi, A) \mapsto \frac{1}{\lambda}(\psi, A)\big(\frac{t}{\lambda}, \frac{x}{\lambda}\big)$. Hence the scale invariant space is $\dot{H}^{-\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$. Since we do not expect any well-posedness below the scaling regularity, the range of well-posedness in Theorem 3.1.1 is essentially optimal, except possibly at the endpoint $r = \frac{-1}{2}$. Moreover, it should be possible to show that (CSD) is ill-posed in some sense outside of the range given in Theorem 3.1.1 by using the techniques in [63], but we do not consider the problem of ill-posedness here.

[1] More precisely, if $(\psi_0, a_0) \in H^{s'} \times H^{r'}$ with $s' \geqslant s$, $r' \geqslant r$, and $r' \leqslant s' \leqslant r' + 1$, then we can conclude that $(\psi, A) \in C\big([-T, T], H^{s'} \times H^{r'}\big)$, where T only depends on the size of $\|f\|_{H^s} + \|a\|_{H^r}$.

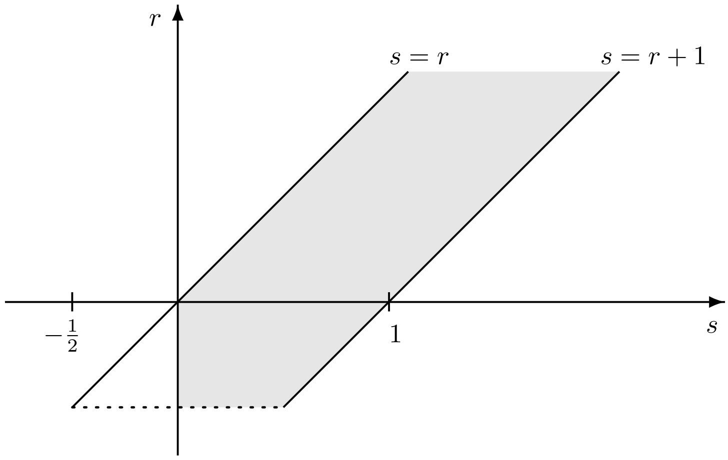

Figure 3.1: The domain of local/global well-posedness from Theorem 3.1.1 and Corollary 3.1.5. We have local existence inside the lines $s = r$ and $s = r + 1$ for $r > -\frac{1}{2}$. Global existence holds inside the shaded region.

Remark 3.1.3. The proof of Theorem 3.1.1 uses product Sobolev spaces based on the null coordinates $\alpha = x + t$, $\beta = x - t$. If instead we tried to prove Theorem 3.1.1 using the $X^{s,b}$ space techniques used in Chapter 4, we would require the estimate

$$\|uv\|_{X^{s,b}_{\pm}} \lesssim \|u\|_{X^{s,b}_{\pm}} \|v\|_{X^{s,b}_{\mp}} \tag{3.1}$$

where $\|w\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau \pm \xi\rangle^b \widetilde{w}\|_{L^2(\mathbb{R}^2)}$ is the standard $X^{s,b}$ space associated to the linear propagators in (CSD). However the estimate (3.1) requires the condition $s > -\frac{1}{4}$, as can be seen in Theorem 4.1.1 in Chapter 4. Hence Theorem 3.1.1 is $\frac{1}{4}$ of a derivative better than what can be obtained via standard $X^{s,b}$ spaces.

Remark 3.1.4. The natural scaling for the CSD equation shows that the spinor ψ and the gauge A scale the same way. Thus scaling suggests that the spinor ψ and the gauge A should have the same regularity, in other words we should take $(f, a) \in H^s \times H^s$. Theorem 3.1.1 shows that we can break this scaling by adding one derivative to the spinor component and still obtain well-posedness.

The local existence portion of Theorem 3.1.1 will follow by the standard iteration argument, using estimates contained in [63]. The proof of uniqueness is more difficult and does not follow directly from the existence proof, primarily because the spaces used to prove existence do not scale nicely on the domain $[-T, T] \times \mathbb{R}$. Instead we will need to prove a more precise version of an energy inequality from [63]. See Proposition 3.4.3 below. Finally the persistence of regularity is quite interesting as it allows both the regularity of the spinor, ψ, and the gauge, A, to be varied independently, provided that we remain in the region of well-posedness.

We now turn to the question of global well-posedness. In the case $s \geqslant 0$ we can exploit the conservation of charge together with a decomposition argument from [19] to obtain the following GWP region, see Figure 3.1.

Corollary 3.1.5. *Assume that $s \geqslant 0$ in Theorem 3.1.1. Then the local solution can be extended to a global solution $(\psi, A) \in C\big(\mathbb{R}, H^s \times H^r\big)$.*

Remark 3.1.6. An interesting question that remains open is that of GWP below $s = 0$. This is substantially more difficult than the case $s \geqslant 0$ as there is no conserved quantity below the charge class. However it may be possible to apply the I-method of Colliander-Keel-Staffilani-Takaoka-Tao [23] and develop a global well-posedness theory below the charge class. This has been done for the related Dirac-Klein-Gordon system, see Chapter 4. There are some obstructions to applying the method used in Chapter 4 to the GWP problem for the CSD equation below the charge class, see Remark 4.3.10 in Chapter 4.

We now give a brief outline of this Chapter. In Section 3.2 we gather together the estimates we require in the proof of Theorem 3.1.1. The local existence component of Theorem 3.1.1 is proven in Section 3.3. The proof of uniqueness is contained in Section 3.4. Finally in Section 3.5 we prove Corollary 3.1.5.

3.2 Estimates

We start by introducing the following notation from [63]. If $a, b, c \in \mathbb{R}$ we write $c \prec \{a, b\}$ to denote that either

$$a + b \geqslant 0, \qquad c \leqslant \min\{a, b\}, \qquad c < a + b - \frac{1}{2}$$

or

$$a + b > 0, \qquad c < \min\{a, b\}, \qquad c \leqslant a + b - \frac{1}{2}.$$

Note that the condition $c \prec \{a, b\}$ implies that the following product inequality for Sobolev spaces holds

$$\|fg\|_{H^c(\mathbb{R})} \lesssim \|f\|_{H^a(\mathbb{R})} \|g\|_{H^b(\mathbb{R})}.$$

This estimate, and other versions of it (see Lemma 3.2.3), will be used frequently throughout this chapter.

The main estimates we require in the proof of Theorem 3.1.1 have already been proven in [63]. Define

$$\|u\|_{Z^{s,b}_{\pm}} = \big\| \langle \tau \mp \xi \rangle^s \langle \tau \pm \xi \rangle^b \widetilde{\psi}(\tau, \xi) \big\|_{L^2_{\tau,\xi}}.$$

Note that $Z^{s,b}_{\pm}$ is just a product Sobolev space in the null directions $x \pm t$. The $Z^{s,b}_{\pm}$ space is enough to control the nonlinear terms in (CSD). However, for s close to $\frac{-1}{2}$, the space $Z^{s,b}_{\pm}$ is not contained inside $C(\mathbb{R}, H^s(\mathbb{R}))$. Thus to prove the local well-posedness result in Theorem 3.1.1, we need to add a component to control the $L^\infty_t H^s_x$ norm. To this end, following [63], we define the space $Y^{s,b}_{\pm}$ by using the norm

$$\|u\|_{Y^{s,b}_{\pm}} = \big\| \langle \xi \rangle^s \langle \tau \pm \xi \rangle^b \widetilde{u}(\tau, \xi) \big\|_{L^2_\xi L^1_\tau}.$$

It is easy to see that

$$\|u\|_{L^\infty_t H^s_x} \leqslant \|u\|_{Y^{s,0}_{\pm}}$$

and so $Z^{s,b}_{\pm} \bigcap Y^{s,0}_{\pm} \subset C\big(\mathbb{R}, H^s(\mathbb{R})\big)$. We remark that spaces of the form $Y^{s,b}_{\pm}$ have been used previously to augment the standard $X^{s,b}$ spaces for $b = \frac{1}{2}$ in the periodic case in [9]; see also [42].

The first result we need is the following energy type inequality.

Lemma 3.2.1 ([63] Lemma 3.2). *Let $s, b \in \mathbb{R}$ and $S = (-1,1) \times \mathbb{R}$. Suppose u is a solution to*

$$\partial_t u \pm \partial_x u = F$$
$$u(0) = f$$

on S. Then

$$\|u\|_{Z^{s,b}_{\pm}(S)} + \|u\|_{Y^{s,0}_{\pm}(S)} \lesssim \|f\|_{H^s} + \inf_{F'|_S = F} \left(\|F'\|_{Z^{s,b-1}_{\pm}} + \|F'\|_{Y^{s,-1}_{\pm}} \right) \tag{3.2}$$

where the infimum is over all $F' \in Z^{s,b-1}_{\pm} \cap Y^{s,-1}_{\pm}$ with $F' = F$ on S.

The previous energy inequality is sufficient to prove existence of solutions to (CSD), however to obtain uniqueness we require a slightly more refined version of Lemma 3.2.1 which we leave to Section 3.4.

To close the iteration argument we make use of the following nonlinear estimate contained in [63].

Lemma 3.2.2 ([63], Lemma 3.4). *Let $s_1, s_2, b_1, b_2, s \in \mathbb{R}$ and assume there exists $a_0, b_0 \in \mathbb{R}$ such that*

$$\begin{aligned} a_0 \prec \{s_1, b_2\}, \qquad b_0 \prec \{s_2, b_1\}, \qquad s \prec \{a_0, b_0 + 1\} \\ s_1 + b_1 > \frac{-1}{2}, \qquad s_2 + b_2 > \frac{-1}{2}. \end{aligned} \tag{3.3}$$

Then we have

$$\|uv\|_{Y^{s,-1}_{\pm}} \lesssim \|u\|_{Z^{s_1,b_1}_{\pm}} \|v\|_{Z^{s_2,b_2}_{\mp}}.$$

We also have the following modification of the well known product estimates for Sobolev spaces.

Lemma 3.2.3 ([63]). *Assume $s \prec \{s_1, b_2\}$ and $b \prec \{b_1, s_2\}$. Then*

$$\|uv\|_{Z^{s,b}_{\pm}} \lesssim \|u\|_{Z^{s_1,b_1}_{\pm}} \|v\|_{Z^{s_2,b_2}_{\mp}}.$$

Proof. It suffices to prove $\|\psi\phi\|_{H^s_x H^b_t} \lesssim \|\psi\|_{H^{s_1}_x H^{b_1}_t} \|\phi\|_{H^{s_2}_x H^{b_2}_t}$ which is equivalent to showing that

$$\begin{aligned} \int_{\tau_1+\tau_2=\tau_3} \int_{\xi_1+\xi_2=\xi_3} \widetilde{\psi}(\tau_1,\xi_1)\widetilde{\phi}(\tau_2,\xi_2)\widetilde{\eta}(\tau_3,\xi_3) d\sigma(\xi) d\sigma(\tau) \\ \lesssim \|\psi\|_{H^{s_1}_x H^{b_1}_t} \|\phi\|_{H^{s_2}_x H^{b_2}_t} \|\eta\|_{H^{-s}_x H^{-b}_t} \end{aligned} \tag{3.4}$$

where $d\sigma$ is the surface measure on $\{x_1 + x_2 = x_3\}$. Recall the one dimensional product estimate can be written as

$$\int_{\xi_1+\xi_2=\xi_3} \widetilde{f}_1(\xi_1)\widetilde{f}_2(\xi_2)\widetilde{f}_3(\xi_3) d\sigma(\xi) \lesssim \|f_1\|_{H^{s_1}} \|f_2\|_{H^{s_2}} \|f_3\|_{H^{-s}} \tag{3.5}$$

where we require the condition $s \prec \{s_1, s_2\}$. The estimate (3.4) now follows from two applications of (3.5)

$$\begin{aligned}\int_{\tau_1+\tau_2=\tau_3}\int_{\xi_1+\xi_2=\xi_3} &\widetilde{\psi}(\tau_1,\xi_1)\widetilde{\phi}(\tau_2,\xi_2)\widetilde{\eta}(\tau_3,\xi_3)d\sigma(\xi)d\sigma(\tau)\\ &\lesssim \int_{\tau_1+\tau_2=\tau_3} \big\|\langle\xi\rangle^{s_1}\widetilde{\psi}(\tau_1)\big\|_{L^2_\xi}\big\|\langle\xi\rangle^{s_2}\widetilde{\phi}(\tau_2)\big\|_{L^2_\xi}\big\|\langle\xi\rangle^{-s}\widetilde{\eta}(\tau_3)\big\|_{L^2_\xi}d\sigma(\tau)\\ &\lesssim \|\psi\|_{H^{s_1}_x H^{b_1}_t}\|\phi\|_{H^{s_2}_x H^{b_2}_t}\|\eta\|_{H^{-s}_x H^{-b}_t}.\end{aligned}$$

□

Finally we require the following lemma which will help simplify the arguments leading to uniqueness.

Lemma 3.2.4. *Let* $\frac{-1}{2} < s < \frac{1}{2}$ *and* $0 < T < 1$. *Assume* $\rho \in B^{\frac{1}{2}}_{2,1}$ *and let* $\rho_T(t) = \rho\big(\frac{t}{T}\big)$. *Then*

$$\|\rho_T(t)u\|_{Z^{s,0}_\pm} \lesssim_\rho \|u\|_{Z^{s,0}_\pm} \tag{3.6}$$

with constant independent of T.

Proof. A change of variables shows that

$$\begin{aligned}\|\rho_T(t)\psi\|_{Z^{s,0}_\pm} &= \Big\|\langle\tau\mp\xi\rangle^s\int\widehat{\rho_T}(\lambda)\widetilde{\psi}(\tau-\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}}\\ &= \Big\|\langle\tau\rangle^s\int\widehat{\rho_T}(\lambda)\widetilde{\psi}(\tau\pm\xi-\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}}\end{aligned}$$

and hence (3.6) follows from the estimate

$$\|\rho_T(t)f(t)\|_{H^s_t} \lesssim_\rho \|f\|_{H^s_t}, \tag{3.7}$$

see Corollary 1.2.14.

□

3.3 Local Existence

We start by noting that if we let $\psi = (u_+, u_-)^T$, and $A_\pm = A_0 \mp A_1$, we can rewrite (CSD) in the form

$$\begin{aligned} i(\partial_t u_+ + \partial_x u_+) &= mu_- - A_- u_+\\ i(\partial_t u_- - \partial_x u_-) &= mu_+ - A_+ u_-\\ u_\pm(0) &= f_\pm\end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \partial_t A_+ + \partial_x A_+ &= -2\Re(u_+\overline{u}_-)\\ \partial_t A_- - \partial_x A_- &= 2\Re(u_+\overline{u}_-)\\ A_\pm(0) &= a_\pm\end{aligned} \tag{3.9}$$

where $f_\pm = f_1 \pm f_2$, $a_\pm = a_0 \mp a_1$, and we use $\Re(z)$ to denote the real part of $z \in \mathbb{C}$. The formulation (3.8), (3.9) is much easier to work with than (CSD) as the null structure is more

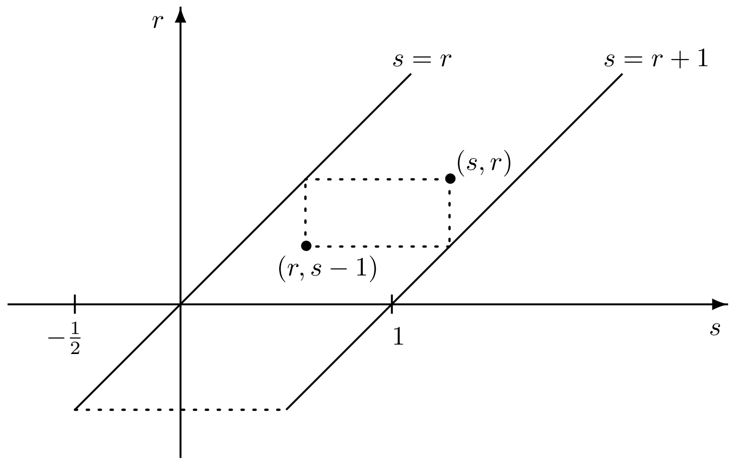

Figure 3.2: The time of existence given by the rescaled version of Theorem 3.3.1 at regularity $H^s \times H^r$, only depends on the size of the initial data at the regularity $H^r \times H^{s-1}$ (provided $s - 1 > \frac{-1}{2}$).

apparent. Namely all the nonlinear terms involve products of the form $\psi_+ \phi_-$ which behave far better than the product $\psi_+ \phi_+$, see for instance the estimates in Chapter 4. The fact that the nonlinear terms in (3.8) and (3.9) are all $+-$ products is a reflection of the null structure present in the (CSD) system.

We deduce Theorem 3.1.1 from the following.

Theorem 3.3.1. *Let $\frac{-1}{2} < r \leqslant s \leqslant r+1$ and assume $f \in H^s$, $a \in H^r$. Choose $r^* > \frac{-1}{2}$ with $s - 1 \leqslant r^* \leqslant r$. Then there exists $\epsilon > 0$ such that if $|m| < \epsilon$ and*

$$\|f\|_{H^r} + \|a\|_{H^{r^*}} < \epsilon$$

then there exists a solution $(\psi, A) \in C\big([-1,1], H^s \times H^r\big)$ to (CSD) with $(\psi, A)(0) = (f, a)$. Moreover solution depends continuously on the initial data and if we let $\psi = (u_+, u_-)^T$ and $A_\pm = A_0 \mp A_1$ then

$$u_\pm \in Z^{s,b}_\pm(S) \cap Y^{s,0}_\pm(S), \qquad A_\pm \in Z^{r,b}_\pm(S) \cap Y^{s,0}_\pm(S)$$

for any $b > \frac{1}{2}$ with $s \leqslant b \leqslant r^* + 1$ and $S = [-1,1] \times \mathbb{R}$.

Assume for the moment that Theorem 3.3.1 holds, we deduce Theorem 3.1.1 as follows. Let $(f, a) \in H^s \times H^r$ with $\frac{-1}{2} < r \leqslant s \leqslant r+1$. Theorem 3.3.1 together with a scaling argument then gives a solution $(\psi, A) \in C([-T,T], H^s \times H^r)$ that depends continuously on the initial data, where T only depends on some negative power of $\|f\|_{H^r} + \|a\|_{H^{r^*}}$ with $r^* > \frac{-1}{2}$ and $s - 1 \leqslant r^* \leqslant r$, see Figure 3.2. The uniqueness we leave until the next section. Hence, to complete the proof of Theorem 3.1.1 it only remains to check that any additional regularity persists in time.

Suppose the initial data has additional smoothness $(f, a) \in H^{r_0} \times H^{s_0}$ with $s_0 > s$, $r_0 > r$, and $r_0 \leqslant s_0 \leqslant r_0 + 1$. Applying the local existence result we have $(\psi, A) \in C\big((-T_0, T_0), H^{s_0} \times H^{r_0}\big)$ for some $T_0 > 0$. Persistence of regularity will follow if we can obtain $T_0 \geqslant T$. To this end, we note that it is enough to show that if $T_0 < T$ and

$$\limsup_{t\to T_0} \big(\|\psi(t)\|_{H^{s_0}} + \|A(t)\|_{H^{r_0}}\big) = \infty \tag{3.10}$$

then we also have

$$\limsup_{t\to T_0} \big(\|\psi(t)\|_{H^{s}} + \|A(t)\|_{H^{r}}\big) = \infty. \tag{3.11}$$

This is done in steps as follows. We first deduce by the rescaled version of Theorem 3.3.1 that

$$\limsup_{t\to T_0} \big(\|\psi(t)\|_{H^{r_0}} + \|A(t)\|_{H^{\max\{s_0-1,\frac{-1}{2}+\epsilon\}}}\big) = \infty \tag{3.12}$$

for any sufficiently small $\epsilon > 0$. Since if not, then we can choose some sequence of points $t_n \to T_0$ with $\sup_n \|\psi(t_n)\|_{H^{r_0}} + \|A(t_n)\|_{H^{\max\{s_0-1,\frac{-1}{2}+\epsilon\}}} < \infty$. Taking t_n sufficiently close to T and applying a rescaled version of Theorem 3.3.1 with initial data $(\psi(t_n), A(t_n))$, we can extend our solution beyond T_0, contradicting (3.10). Therefore, provided $T_0 < \infty$ and (3.10) holds, we must have (3.12).

Repeating this argument with (3.10) replaced with (3.12) we obtain

$$\limsup_{t\to T_0} \big(\|\psi(t)\|_{H^{\max\{s_0-1,\frac{-1}{2}+\epsilon\}}} + \|A(t)\|_{H^{\max\{r_0-1,\frac{-1}{2}+\epsilon\}}}\big) = \infty.$$

We continue in this manner and observe that after k iterations, the $H^{\max\{s_0-k,\frac{-1}{2}+\epsilon\}} \times H^{\max\{r_0-k,\frac{-1}{2}+\epsilon\}}$ norm must blow up as we approach T_0. Taking k such that $s_0 - k \leqslant s$ and $r_0 - k \leqslant r$ we obtain (3.11) as required.

We now come to the proof of small data local well-posedness for (CSD).

Proof of Theorem 3.3.1. Let $\frac{-1}{2} < r \leqslant s \leqslant r+1$ and choose $b > \frac{1}{2}$ with $s \leqslant b \leqslant r^*+1$. Note that this is possible since $r^* \geqslant s-1$ and $r^* > \frac{-1}{2}$. Let $r \leqslant s' \leqslant s$. We claim that Lemma 3.2.2 and Lemma 3.2.3 imply the estimates

$$\|uv\|_{Y^{r,-1}_{\pm}} \leqslant \|uv\|_{Y^{s',-1}_{\pm}} \lesssim \|u\|_{Z^{s',b}_{\pm}} \|v\|_{Z^{r^*,b}_{\mp}} \tag{3.13}$$

and

$$\|uv\|_{Z^{r,b-1}_{\pm}} \leqslant \|uv\|_{Z^{s',b-1}_{\pm}} \lesssim \|u\|_{Z^{s',b}_{\pm}} \|v\|_{Z^{r^*,b}_{\mp}}. \tag{3.14}$$

To obtain the estimate (3.13), an application of Lemma 3.2.2 reduces the problem to showing that there exists $a_0, b_0 \in \mathbb{R}$ such that

$$a_0 \prec \{s', b\}, \qquad b_0 \prec \{r^*, b\}, \qquad s' \prec \{a_0, b_0+1\}$$
$$s' + b > \frac{-1}{2}, \qquad r^* + b > \frac{-1}{2}.$$

Since $r^* \leqslant r \leqslant s' \leqslant s \leqslant b$, we let $a_0 = s'$, $b_0 = r^*$. It is clear that $s' \prec \{s', b\}$ and $r^* \prec \{r^*, b\}$. Thus the only remaining conditions are

$$s' + r^* + 1 \geqslant 0, \qquad s' \leqslant r^* + 1, \qquad s' < s' + r^* + 1 - \frac{1}{2}.$$

But these also hold provided $r^*, s' > \frac{-1}{2}$ and $s' \leqslant r^* + 1$, which follows since $s' \leqslant s \leqslant r^* + 1$.

Consequently (3.13) holds.

The remaining estimate, (3.14), follows from Lemma 3.2.3 provided that

$$s' \prec \{s', b\}, \qquad b-1 \prec \{r^*, b\}.$$

Using the assumptions $s', r^* > \frac{-1}{2}$ and $b > \frac{1}{2}$ this reduces to

$$\begin{aligned} s' &\leqslant b, & s' &< s' + b - \frac{1}{2} \\ b-1 &\leqslant r^*, & b-1 &< r^* + b - \frac{1}{2}. \end{aligned}$$

These inequalities also hold in view of the assumptions $\frac{-1}{2} < s' \leqslant b$ and $\frac{1}{2} < b \leqslant r^* + 1$. Therefore (3.13) and (3.14) both hold.

It suffices to consider the system (3.8) and (3.9) with the assumption

$$\sum_{\pm} \|f_\pm\|_{H^r} + \|a_\pm\|_{H^{r^*}} < \epsilon.$$

Let $S = (-1,1) \times \mathbb{R}$ and define the Banach space $E^s = \big\{ v = (v_+, v_-) \,\big|\, v_\pm \in Z^{s,b}_\pm(S) \cap Y^{s,0}_\pm(S) \big\}$ with norm

$$\|v\|_{E^s} = \sum_{\pm} \|v_\pm\|_{Y^{s,0}_\pm(S)} + \|v_\pm\|_{Z^{s,b}_\pm(S)}$$

Note that since $Y^{s,0}_\pm(S) \subset L^\infty_t H^s_x(S)$ we have $\|v\|_{L^\infty_t H^s_x(S)} \lesssim \|v\|_{E^s}$ and so $v \in E^s$ implies $v \in C([-1,1], H^s)$. Let $\Gamma = \sum_\pm \|f_\pm\|_{H^s} + \|a_\pm\|_{H^r}$ and define the closed subset $\mathcal{X}_\epsilon \subset E^s \times E^r$ by

$$\mathcal{X}_\epsilon = \{ \|u\|_{E^r} + \|A\|_{E^{r^*}} \leqslant 2C\epsilon \} \cap \{ \|u\|_{E^s} + \|A\|_{E^r} \leqslant 2C\Gamma \}.$$

Define the map $\mathcal{S} : \mathcal{X}_\epsilon \longrightarrow \mathcal{X}_\epsilon$ by letting $\mathcal{S}(u, A) = (v, B)$ be the solution to

$$\begin{aligned} i(\partial_t \pm \partial_x) v_\pm &= m u_\mp + A_\mp u_\pm \\ (\partial_t \pm \partial_x) B_\pm &= \pm \Re(u_+ \overline{u}_-) \\ v_\pm(0) = f_\pm, &\qquad B_\pm(0) = a_\pm. \end{aligned} \tag{3.15}$$

Then using Lemma 3.2.1 together with (3.13) and (3.14) we obtain

$$\|v\|_{E^s} + \|B\|_{E^r} \lesssim \sum_{\pm} \big(\|f_\pm\|_{H^s} + \|a_\pm\|_{H^r} \big) + |m| \big(\|u\|_{E^s} + \|A\|_{E^r} \big) + \big(\|u\|_{E^r} + \|A\|_{E^{r^*}} \big) \big(\|u\|_{E^s} + \|A\|_{E^r} \big)$$

and

$$\|v\|_{E^r} + \|B\|_{E^{r^*}} \lesssim \sum_{\pm} \big(\|f_\pm\|_{H^r} + \|a_\pm\|_{H^{r^*}} \big) + |m| \big(\|u\|_{E^r} + \|A\|_{E^{r^*}} \big) + \big(\|u\|_{E^r} + \|A\|_{E^{r^*}} \big)^2.$$

The assumption $(u, A) \in \mathcal{X}_\epsilon$ then gives the inequalities

$$\begin{aligned}\|v\|_{E^s} + \|B\|_{E^r} &\leqslant C\Gamma + C\epsilon\Gamma + C^2\epsilon\Gamma\\ \|v\|_{E^r} + \|B\|_{E^{r*}} &\leqslant C\epsilon + C\epsilon^2 + C^2\epsilon^2\end{aligned}$$

Therefore, provided ϵ is sufficiently small, depending only on the constants in (3.13), (3.14), and (3.2), we see that $\mathcal{S}$ is well defined. A similar argument shows that $\mathcal{S}$ is a contraction mapping. Consequently we have existence, uniqueness in $\mathcal{X}_\epsilon$, and continuous dependence on the initial data. □

3.4 Uniqueness

In this section we will complete the proof of Theorem 3.1.1 and show that the solution obtained in Section 3.3 is unique. More precisely, we will prove the following.

Proposition 3.4.1. *Let $\frac{-1}{2} < r \leqslant s \leqslant r+1$, $T > 0$, and $b > \frac{1}{2}$. Define $S_T = (-T, T) \times \mathbb{R}$. Assume (u, A) and (v, B) are solutions to (3.8) and (3.9) with $u_\pm, v_\pm \in Z^{s,b}_\pm(S_T)$ and $A_\pm, B_\pm \in Z^{r,b}_\pm(S_T)$. If $(u, A)(0) = (v, B)(0)$ then $(u, A) = (v, B)$ on S_T.*

The proof of Proposition 3.4.1 is slightly involved as we need to understand the behaviour of the energy inequality Lemma 3.2.1 on the domain S_T for small T. For the $Y^{s,b}$ component this is reasonably straightforward.

Lemma 3.4.2. *Let $s \in \mathbb{R}$, $0 < T < 1$, and $0 < \epsilon < 1$. Suppose ψ is a solution to*

$$\begin{aligned}\partial_t\psi \pm \partial_x\psi &= F\\ \psi(0) &= f.\end{aligned}$$

Let $\rho \in C^\infty_0$ and define $\rho_T(t) = \rho\big(\frac{t}{T}\big)$. Then

$$\begin{aligned}\|\rho_T(t)\psi\|_{Y^{s,0}_\pm} &\lesssim_\rho \|f\|_{H^s} + \big\|\langle\xi\rangle^s \min\{T, |\tau \pm \xi|^{-1}\}\widetilde{F}\big\|_{L^2_\xi L^1_\tau} &(3.16)\\ &\lesssim \|f\|_{H^s} + T^\epsilon\|F\|_{Y^{s,\epsilon-1}_\pm} &(3.17)\end{aligned}$$

with constant independent of T.

Proof. It is easy to see that (3.16) follows from the estimate

$$\Big\|\mathcal{F}_t\Big[\rho_T(t)\int_0^t e^{\mp i(t-s)\xi}\widehat{F}(s)ds\Big](\tau,\xi)\Big\|_{L^2_\xi L^1_\tau} \lesssim \|\min\{T, |\tau \pm \xi|^{-1}\}\widetilde{F}\|_{L^2_\xi L^1_\tau}. \tag{3.18}$$

Note that by scaling it is sufficient to consider the case $T = 1$. Consequently $\min\{1, |\tau \pm \xi|^{-1}\} \approx \langle\tau \pm \xi\rangle^{-1}$ and so (3.18) follows from Lemma 3.2 in [63]. The remaining inequality (3.17) then follows by observing that since $0 < T < 1$,

$$\min\{T, |\tau \pm \xi|^{-1}\} \lesssim T^\epsilon\langle\tau \pm \xi\rangle^{\epsilon-1}.$$

□

It remains to control the $Z^{s,b}_\pm$ component of the energy inequality. This is significantly more difficult as both multipliers $\langle\tau + \xi\rangle$ and $\langle\tau - \xi\rangle$ involve the time variable. This observation, together with the fact that $Y^{s,0}$ has a different scaling to $Z^{s,b}$, is the main difficulty in the following proposition.

Proposition 3.4.3. *Let* $\frac{-1}{2} < s \leqslant 0$ *and* $0 < T < 1$. *Choose* $0 < \epsilon < \frac{1}{2}$ *and let* $\frac{1}{2} < b < \min\{1+s, 1-\epsilon\}$. *Assume* $\rho, \sigma \in C_0^\infty$ *with* $\rho(t) = 1$ *for* $t \in [-1,1]$, $\sigma(t) = 1$ *for* $t \in$ *supp* ρ, *and*

$$\text{supp } \rho \subset \text{supp } \sigma \subset [-2,2].$$

Define $\rho_T(t) = \rho\left(\frac{t}{T}\right)$ *and* $\sigma_T(t) = \sigma\left(\frac{t}{T}\right)$. *Let* ψ *be a solution to*

$$\partial_t \psi \pm \partial_x \psi = F.$$

Then

$$\|\rho_T(t)\psi\|_{Z^{s,b}_{\pm}} \lesssim T^{\frac{1}{2}-b}\|\sigma_T(t)\psi\|_{Y^{s,0}_{\pm}} + T^{\epsilon}\|F\|_{Z^{s,b-1+\epsilon}_{\pm}} \tag{3.19}$$

with the implied constant independent of T.

Proof. We only prove the $+$ case as the $-$ case is similar. Note that since $\sigma_T(t) = 1$ on supp ρ_T we may simply write $\psi = \sigma_T\psi$. Let $\Omega \subset \mathbb{R}^2$ and define

$$I(\Omega) = \left\| \langle \tau+\xi\rangle^b \langle \tau-\xi\rangle^s \int_{\mathbb{R}} \widehat{\rho_T}(\tau-\lambda)\widetilde{\psi}(\lambda,\xi)d\lambda \right\|_{L^2_{\tau,\xi}(\Omega)}.$$

We break $\mathbb{R}^2$ into different regions and estimate each region separately. We first consider the set

$$\Omega_1 = \{|\tau+\xi| \leqslant T^{-1}\}$$

and split this into the regions $2|\tau-\xi| \geqslant |\xi|$ and $2|\tau-\xi| \leqslant |\xi|$. In the former region, since $s \leqslant 0$ and $\langle \tau+\xi\rangle^b \leqslant T^{-b}$,

$$\begin{aligned} I(\Omega_1 \cap \{2|\tau-\xi| \geqslant |\xi|\}) &\lesssim T^{-b}\left\| \int_{\mathbb{R}} \widehat{\rho_T}(\tau-\lambda)\langle\xi\rangle^s \widetilde{\psi}(\lambda,\xi)d\lambda \right\|_{L^2_{\tau,\xi}} \\ &\lesssim T^{-b}\|\widehat{\rho_T}(\tau)\|_{L^2_\tau}\|\psi\|_{Y^{s,0}_+} \\ &\lesssim_\rho T^{\frac{1}{2}-b}\|\psi\|_{Y^{s,0}_+}. \end{aligned}$$

On the other hand if $2|\tau-\xi| \leqslant |\xi|$ then $|\tau| \approx |\xi| \approx |\tau+\xi| \lesssim T^{-1}$. Hence

$$\begin{aligned} I(\Omega_1 \cap \{2|\tau-\xi| \leqslant |\xi|\}) &\lesssim \left\| \langle\tau+\xi\rangle^{b-s}\langle\tau-\xi\rangle^s \int \widehat{\rho_T}(\tau-\lambda)\langle\xi\rangle^s\widehat{\psi}(\lambda,\xi)d\lambda \right\|_{L^2_{\tau,\xi}(|\tau-\xi|,|\tau+\xi|\lesssim T^{-1})} \\ &\lesssim T^{s-b}\|\widehat{\rho_T}\|_{L^\infty_\rho}\|\langle\tau\rangle^s\|_{L^2_\tau(|\tau|\leqslant T^{-1})}\|\psi\|_{Y^{s,0}_+} \\ &\lesssim_\rho T^{s-b} \times T \times T^{-\frac{1}{2}-s}\|\psi\|_{Y^{s,0}_+} \\ &= T^{\frac{1}{2}-b}\|\psi\|_{Y^{s,0}_+}. \end{aligned}$$

Therefore

$$I(\Omega_1) \lesssim T^{\frac{1}{2}-b}\|\psi\|_{Y^{s,0}_+}.$$

We now consider the region $\Omega_2 = \{|\tau+\xi| \geqslant T^{-1}\}$. Note that

$$\begin{aligned}
\big(\rho_T(t)\psi\big)\widetilde{\ }(\tau,\xi) &= \frac{1}{i(\tau+\xi)}\int i\big((\tau-\lambda)+(\lambda+\xi)\big)\widehat{\rho_T}(\tau-\lambda)\widetilde{\psi}(\lambda,\xi)d\lambda \\
&= \frac{1}{i(\tau+\xi)}\Big[T^{-1}\big((\partial_t\rho)_T\psi\big)\widetilde{\ }(\tau,\xi) + \big(\rho_T F\big)\widetilde{\ }(\tau,\xi)\Big]
\end{aligned}$$

and so, using the fact that $|\tau+\xi| \geqslant T^{-1} \gg 1$ implies $|\tau+\xi| \approx \langle\tau+\xi\rangle$, we have

$$\begin{aligned}
I(\Omega_2) \leqslant T^{-1}&\Big\|\langle\tau+\xi\rangle^{b-1}\langle\tau-\xi\rangle^s\big((\partial_t\rho)_T\psi\big)\widetilde{\ }\Big\|_{L^2_{\tau,\xi}(\Omega_2)} \\
&+ \Big\|\langle\tau+\xi\rangle^{b-1}\langle\tau-\xi\rangle^s\big(\rho_T F\big)\widetilde{\ }\Big\|_{L^2_{\tau,\xi}(\Omega_2)}.
\end{aligned} \tag{3.20}$$

We estimate each of these terms separately. For the first term we follow the Ω_1 case and decompose Ω_2 into $2|\tau-\xi| \geqslant |\xi|$ and $2|\tau-\xi| \leqslant |\xi|$. In the former region we use the fact that $\langle\tau+\xi\rangle^{b-1} \leqslant T^{1-b}$ to deduce that

$$\begin{aligned}
T^{-1}\Big\|\langle\tau+\xi\rangle^{b-1}\langle\tau-\xi\rangle^s &\int \widehat{(\partial_t\rho)_T}(\tau-\lambda)\widetilde{\psi}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2\cap\{2|\tau-\xi|\geqslant|\xi|\})} \\
&\lesssim T^{-b}\Big\|\int \widehat{(\partial_t\rho)_T}(\tau-\lambda)\langle\xi\rangle^s\widetilde{\psi}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}} \\
&\lesssim T^{-b}\|\widehat{(\partial_t\rho)_T}\|_{L^2}\|\psi\|_{Y^{s,0}_+} \\
&\lesssim_\rho T^{\frac{1}{2}-b}\|\psi\|_{Y^{s,0}_+}.
\end{aligned}$$

On the other hand for $2|\tau-\xi| \leqslant |\xi|$ we have $|\tau+\xi| \approx |\xi|$ and so

$$\begin{aligned}
T^{-1}\Big\|\langle\tau+\xi\rangle^{b-1}\langle\tau-\xi\rangle^s &\int \widehat{(\partial_t\rho)_T}(\tau-\lambda)\widetilde{\psi}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2\cap\{2|\tau-\xi|\leqslant|\xi|\})} \\
&\lesssim T^{-1}\Big\|\langle\tau+\xi\rangle^{b-1-s}\langle\tau-\xi\rangle^s\int \widehat{(\partial_t\rho)_T}(\tau-\lambda)\langle\xi\rangle^s\widetilde{\psi}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2)} \\
&\lesssim T^{s-b}\|\psi\|_{Y^{s,0}_+}\sup_{\xi,\lambda}\big\|\langle\tau-\xi\rangle^s\widehat{(\partial_t\rho)_T}(\tau-\lambda)\big\|_{L^2_\tau}.
\end{aligned}$$

To control the $\partial_t\rho$ term we use

$$\begin{aligned}
\big\|\langle\tau-\xi\rangle^s\widehat{(\partial_t\rho)_T}(\tau-\lambda)\big\|_{L^2_\tau} &\lesssim \big\|\langle\tau-\xi\rangle^s\widehat{(\partial_t\rho)_T}(\tau-\lambda)\big\|_{L^2_\tau(|\tau-\xi|\leqslant T^{-1})} \\
&\qquad + \big\|\langle\tau-\xi\rangle^s\widehat{(\partial_t\rho)_T}(\tau-\lambda)\big\|_{L^2_\tau(|\tau-\xi|\geqslant T^{-1})} \\
&\lesssim \|\langle\tau\rangle^s\|_{L^2_\tau(|\tau|\leqslant T^{-1})}\big\|\widehat{(\partial_t\rho)_T}\big\|_{L^\infty} + T^{-s}\|\partial_t\rho_T\|_{L^2} \\
&\lesssim_\rho T^{\frac{1}{2}-s}
\end{aligned}$$

and so we can estimate the first term in (3.20).

Finally, to estimate the remaining term in (3.20), we write

$$\begin{aligned}\Big\|\langle\tau+\xi\rangle^{b-1}\langle\tau-\xi\rangle^{s}\int\widehat{\rho_T}(\lambda-\tau)\widetilde{F}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2)}\\ \lesssim T^{\epsilon}\Big\|\langle\tau+\xi\rangle^{b-1+\epsilon}\langle\tau-\xi\rangle^{s}\int_{2|\tau+\xi|\leqslant|\lambda+\xi|}\widehat{\rho_T}(\lambda-\tau)\widetilde{F}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2)}\\ +T^{\epsilon}\Big\|\langle\tau+\xi\rangle^{b-1+\epsilon}\langle\tau-\xi\rangle^{s}\int_{2|\tau+\xi|\geqslant|\lambda+\xi|}\widehat{\rho_T}(\lambda-\tau)\widetilde{F}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2)}.\end{aligned}$$

In the region $2|\tau+\xi|\leqslant|\lambda+\xi|$ we have $|\lambda+\xi|\approx|\tau-\lambda|$ and so, using the fact that $|\tau+\xi|\geqslant T^{-1}$,

$$\begin{aligned}\Big\|\langle\tau+\xi\rangle^{b-1+\epsilon}\langle\tau-\xi\rangle^{s}\int_{2|\tau+\xi|\leqslant|\lambda+\xi|}\widehat{\rho_T}(\lambda-\tau)\widetilde{F}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2)}\\ \lesssim\Big\|\langle\tau-\xi\rangle^{s}\int(T|\tau-\lambda|)^{1-b-\epsilon}\big|\widehat{\rho_T}(\tau-\lambda)\big|\langle\lambda+\xi\rangle^{b-1+\epsilon}\big|\widetilde{F}(\lambda,\xi)\big|d\lambda\Big\|_{L^2_{\tau,\xi}}\\ \lesssim_{\rho}\|F\|_{Z^{s,b-1+\epsilon}_{+}}\end{aligned}$$

where the last line follows from an application of Lemma 3.2.4. On the other hand, if $2|\tau+\xi|\geqslant|\lambda+\xi|$, we can simply use the estimate $\langle\tau+\xi\rangle^{b-1+\epsilon}\lesssim\langle\lambda+\xi\rangle^{b-1+\epsilon}$ followed by another application of Lemma 3.2.4. Therefore we have

$$\Big\|\langle\tau+\xi\rangle^{b-1}\langle\tau-\xi\rangle^{s}\int\widehat{\rho_T}(\lambda-\tau)\widetilde{F}(\lambda,\xi)d\lambda\Big\|_{L^2_{\tau,\xi}(\Omega_2)}\lesssim T^{\epsilon}\|F\|_{Z^{s,b-1+\epsilon}_{+}}$$

and consequently the result follows.

□

We remark that the factor $T^{\frac{1}{2}-b}$ in front of the term $\|\psi\|_{Y^{s,0}_{\pm}}$ in (3.19) is not ideal as for T small, this will blow up since $b>\frac{1}{2}$. This cannot be avoided, as a simple scaling argument shows that this is in fact the best possible exponent on T. Essentially the problem arises because the spaces $Y^{s,0}$ and $Z^{s,b}$ scale differently. More precisely the $Y^{s,0}$ space scales like $Z^{s,b}$ at the endpoint $b=\frac{1}{2}$. However, the term $T^{\frac{1}{2}-b}$ is not a huge problem, because if we take b sufficiently close to $\frac{1}{2}$, then we can safely absorb this into the inhomogeneous term $T^{\epsilon}\|F\|_{Y^{s,\epsilon-1}_{\pm}}$ in Lemma 3.4.2.

Corollary 3.4.4. *Let $\frac{-1}{2}<s<0$, $0<\epsilon<\frac{1}{6}$, and $\frac{1}{2}<b<\min\{1+\epsilon,1+s\}$. Assume $0<T<1$ and define $S_T=(-T,T)\times\mathbb{R}$. Let ψ be the solution to*

$$\partial_t\psi\pm\partial_x\psi=F$$

with $\psi(0)=f$. Then

$$\|\psi\|_{Z^{s,b}_{\pm}(S_T)}\lesssim T^{\frac{1}{2}-b}\|f\|_{H^s}+T^{\epsilon}\inf_{F'=F\ on\ S_T}\Big(\|F'\|_{Y^{s,-1+2\epsilon}_{\pm}}+\|F'\|_{Z^{s,b-1+2\epsilon}_{\pm}}\Big).$$

Proof. Follows from Lemma 3.4.2 and Proposition 3.4.3. □

We now come to the proof of Proposition 3.4.1.

Proof of Proposition 3.4.1. It is enough to consider the case $\frac{-1}{2}<r\leqslant s<0$. Choose $\epsilon>0$ sufficiently small such that

$$r>\frac{-1}{2}+4\epsilon \tag{3.21}$$

and

$$\frac{1}{2} < b < \frac{1}{2} + \epsilon. \tag{3.22}$$

A standard argument using Corollary 3.4.4 reduces the problem to obtaining the estimates

$$\|\psi\phi\|_{Z_{\pm}^{s,b-1+2\epsilon}} \lesssim \|\psi\|_{Z_{\pm}^{s,b}} \|\phi\|_{Z_{\mp}^{r,b}}, \tag{3.23}$$

$$\|\psi\phi\|_{Y_{\pm}^{s,2\epsilon-1}} \lesssim \|\psi\|_{Z_{\pm}^{s,b}} \|\phi\|_{Z_{\mp}^{r,b}}, \tag{3.24}$$

$$\|\psi\|_{Y_{\pm}^{s,2\epsilon-1}} \lesssim \|\psi\|_{Z_{\mp}^{s,b}}. \tag{3.25}$$

We start with (3.23). By Lemma 3.2.3 we need

$$s \prec \{s, b\}, \qquad b - 1 + 2\epsilon \prec \{b, r\}.$$

The first condition is straightforward since $s > \frac{-1}{2}$ and $b > \frac{1}{2}$. For the second we need

$$b + r \geqslant 0, \qquad b - 1 + 2\epsilon \leqslant \min\{b, r\}, \qquad b - 1 + 2\epsilon < b + r - \frac{1}{2}$$

which all hold in view of the assumptions (3.21) and (3.22).

To prove (3.24), we observe that by an application of the triangle inequality on the Fourier transform side, it suffices to show that

$$\|\psi\phi\|_{Y_{\pm}^{s,-1}} \lesssim \|\psi\|_{Z_{\pm}^{s,b-2\epsilon}} \|\phi\|_{Z_{\mp}^{r-2\epsilon,b}}.$$

By letting $a_0 = s$ and $b_0 = r - 4\epsilon$ in Lemma 3.2.2, we can reduce this to showing

$$\begin{aligned} s \prec \{s, b\}, \qquad r - 4\epsilon \prec \{b - 2\epsilon, r - 2\epsilon\}, \qquad s \prec \{s, r + 1 - 4\epsilon\} \\ s + b - 2\epsilon > \frac{-1}{2}, \qquad r + b - 2\epsilon > \frac{-1}{2}. \end{aligned} \tag{3.26}$$

The first condition is obvious. For the second condition we need

$$b - 2\epsilon + r - 2\epsilon \geqslant 0, \qquad r - 4\epsilon \leqslant \min\{b - 2\epsilon, r - 2\epsilon\}, \qquad r - 4\epsilon < b - 2\epsilon + r - 2\epsilon - \frac{1}{2}$$

which all follow from (3.21) and (3.22). The third condition in (3.26) can be written as

$$s + r + 1 - 4\epsilon \geqslant 0, \qquad s \leqslant \min\{s, r + 1 - 4\epsilon\}, \qquad s < s + r + 1 - 4\epsilon - \frac{1}{2}$$

and again each of these inequalities follows from (3.21), (3.22) and $r \leqslant s < 0$. The remaining conditions in (3.26) are also easily seen to be satisfied and so (3.24) follows.

Finally to prove (3.25) we use Holder's inequality to obtain

$$\begin{aligned} \|\psi\|_{Y_{\pm}^{s,2\epsilon-1}} = \|\langle\xi\rangle^s \int_{\mathbb{R}} \langle\tau \pm \xi\rangle^{2\epsilon-1} |\widehat{\psi}| d\tau\|_{L^2_\xi} &\lesssim \|\langle\xi\rangle^s \widehat{\psi}\|_{L^2_{\tau,\xi}} \\ &\lesssim \|\langle\tau \mp \xi\rangle^s \langle\tau \pm \xi\rangle^{|s|} \widehat{\psi}\|_{L^2_{\tau,\xi}} \\ &\leqslant \|\psi\|_{Z_{\mp}^{s,b}}. \end{aligned}$$

□

3.5 Global Existence

Here we prove Corollary 3.1.5.

Proof of Corollary 3.1.5. The persistence of regularity in Theorem 3.1.1 shows that it suffices to prove global existence in the case $s = 0$ and $\frac{-1}{2} < r \leqslant 0$. Let $(u_\pm, A_\pm)$ be the solution to (3.8) and (3.9) given by Theorem 3.1.1 with initial data $(f_\pm, a_\pm) \in L^2 \times H^r$. We extend $(u_\pm, A_\pm)$ to some maximal interval of existence $(-T, T)$. To show the solution is global in time, it is enough to show that if $T < \infty$ then we have the bound

$$\sup_{t\in(-T,T)} \|A_\pm(t)\|_{H^r} < \infty. \tag{3.27}$$

Since supposing (3.27) holds, we can extend the solution past $(-T, T)$ by using the L^2 conservation of $u_\pm$, together with the local well-posedness of Theorem 3.1.1, contradicting the fact that $(-T, T)$ was the maximal time of existence. Consequently we must have $T = \infty$.

To obtain the bound (3.27) we make use of the decomposition introduced in Chapter 2. We split the Dirac component of our solution $u_\pm$ into a mass free part $u^L_\pm$ satisfying

$$i(\partial_t u^L_\pm \pm \partial_x u^L_\pm) = -A_\mp u^L_\pm$$
$$u^L_\pm(0) = f_\pm$$

and a term $u^N_\pm$ with vanishing initial data

$$i(\partial_t u^N_\pm \pm \partial_x u^N_\pm) = m u_\mp - A_\mp u^N_\pm$$
$$u^N_\pm(0) = 0.$$

Observe that $u_\pm = u^L_\pm + u^N_\pm$. Since $A_\pm$ is real valued, a computation shows that

$$\partial_t \big|u^L_\pm\big|^2 \pm \partial_x \big|u^L_\pm\big|^2 = 0$$

and

$$\partial_t \big|u^N_\pm\big|^2 \pm \partial_x \big|u^N_\pm\big|^2 = 2m\Im(u_\mp \overline{u}^N_\pm).$$

Hence

$$|u^L_\pm(t,x)| = |f_\pm(x \mp t)| \tag{3.28}$$

and, arguing as in the proof of Proposition 2.4.1,

$$\sup_{|t|<T} \big(\|u^N_+(t)\|_{L^\infty_x} + \|u^N_-(t)\|_{L^\infty_x}\big) \lesssim_{T,m} \|f_+\|_{L^2} + \|f_-\|_{L^2}. \tag{3.29}$$

To obtain the bound (3.27), we note that the equation for $A_\pm$ easily leads to

$$\|A_+(t)\|_{H^r_x} + \|A_-(t)\|_{H^r_x} \lesssim \|a_+\|_{H^r} + \|a_-\|_{H^r} + \int_0^t \|u_+ \overline{u}_-\|_{L^2_x} ds \tag{3.30}$$

and so it suffices to bound $\int_{|s|<T} \|u_+(s)\overline{u}_-(s)\|_{L^2_x} ds$ in terms of the initial data $f_\pm$. If we now

use the decomposition $u_\pm = u_\pm^L + u_\pm^N$ we have

$$u_+\overline{u}_- = u_+\overline{u}_-^L + u_+\overline{u}_-^N = u_+^L\overline{u}_-^L + u_+^N\overline{u}_-^L + u_+\overline{u}_-^N. \tag{3.31}$$

The terms involving $u_\pm^N$ are straightforward by (3.29), while for the remaining term Hölder's inequality followed by a change of variables gives

$$\int_{|s|<T} \|u_+^L(s)\overline{u}_-^L(s)\|_{L^2_x(\mathbb{R})} ds \lesssim_T \|f_+(x-s)\overline{f}_-(x+s)\|_{L^2_{s,x}(\mathbb{R}^2)} \lesssim \|f_+\|_{L^2}\|f_-\|_{L^2}.$$

Therefore the required bound (3.27) follows. □

Chapter 4

Bilinear Estimates and Applications to Global Well-Posedness for the Dirac-Klein-Gordon Equation on $\mathbb{R}^{1+1}$

We prove new bilinear estimates for the $X^{s,b}_{\pm}(\mathbb{R}^2)$ spaces which are optimal up to endpoints. These estimates are often used in the theory of nonlinear Dirac equations on $\mathbb{R}^{1+1}$. The proof of the bilinear estimates follows from a dyadic decomposition together with some simplifications due to Tao. As an application, by using the I-method of Colliander, Keel, Staffilani, Takaoka, and Tao, we extend the work of Tesfahun [82] on global existence below the charge class for the Dirac-Klein-Gordon equation on $\mathbb{R}^{1+1}$.

4.1 Introduction

We consider the problem of proving bilinear estimates in the Bourgain-Klainerman-Machedon type spaces $X^{s,b}_{\pm}$ on $\mathbb{R}^2$, where we define the spaces $X^{s,b}_{\pm}$ via the norm

$$\|\psi\|_{X^{s,b}_{\pm}} = \big\|\langle \tau \pm \xi\rangle^b \langle \xi\rangle^s \widetilde{\psi}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}$$

with $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$. These spaces have been used in the low regularity theory of various nonlinear Dirac equations in one space dimension, [60, 75], as well as the Dirac-Klein-Gordon (DKG) system [67, 73]. Although, as we have seen in the first half of this thesis, product Sobolev spaces based on the null coordinates $x \pm t$ have also proved useful. In applications of the $X^{s,b}_{\pm}$ spaces to low regularity well-posedness, we often require product estimates of the form

$$\|uv\|_{X^{-s_1,-b_1}_{\pm_1}} \lesssim \|u\|_{X^{s_2,b_2}_{\pm_2}} \|v\|_{X^{s_3,b_3}_{\pm_3}} \tag{4.1}$$

where $s_j, b_j \in \mathbb{R}$ and $\pm_j$ are independent choices of $\pm$. A number of estimates of this form, for specific values of s_j and b_j have appeared previously in the literature [60, 73, 75]. The case where $\pm_1 = \pm_2 = \pm_3$ is not particularly interesting, as a simple change of variables reduces (4.1) to two applications of the 1-dimensional Sobolev product estimate

$$\|fg\|_{H^{-s_1}(\mathbb{R})} \lesssim \|f\|_{H^{s_2}(\mathbb{R})}\|g\|_{H^{s_3}(\mathbb{R})}.$$

Thus leading to the conditions[1]

$$b_j + b_k > 0, \qquad b_1 + b_2 + b_3 > \frac{1}{2} \tag{4.2}$$

and

$$s_j + s_k > 0, \qquad s_1 + s_2 + s_3 > \frac{1}{2} \tag{4.3}$$

where $j \neq k$. On the other hand, if we have $\pm_1 = \pm_2 = \pm$ and $\pm_3 = \mp$, then we can make significant improvements over (4.3). This observation allows one to exploit the null structure that is often found in nonlinear hyperbolic systems in one dimension, see for instance the discussion leading to (1.6) in the introduction.

To state our first result we use the following conventions. For a set of real numbers $\{a_1, a_2, a_3\}$, we let $a_{max} = \max_i a_i$, $a_{min} = \min_i a_i$, and use a_{med} to denote the median. If $a \in \mathbb{R}$ then we define

$$a_+ = \begin{cases} a & a > 0 \\ 0 & a \leqslant 0. \end{cases}$$

We state our product estimate in the dual form.

Theorem 4.1.1. *Let s_j, $b_j \in \mathbb{R}$, $j = 1, 2, 3$ satisfy*

$$b_1 + b_2 + b_3 > \frac{1}{2}, \qquad b_j + b_k > 0, \qquad (j \neq k) \tag{4.4}$$

and for $k \in \{1, 2\}$

$$\begin{aligned} s_1 + s_2 &\geqslant 0, \\ s_k + s_3 &> -b_{min}, \\ s_k + s_3 &> \frac{1}{2} - b_1 - b_2 - b_3, \\ s_1 + s_2 + s_3 &> \frac{1}{2} - b_3, \\ s_1 + s_2 + s_3 &> \Big(\frac{1}{2} - b_{max}\Big)_+ + \Big(\frac{1}{2} - b_{med}\Big)_+ - b_{min}. \end{aligned} \tag{4.5}$$

Then

$$\Big|\int_{\mathbb{R}^2} \Pi_{j=1}^3 \psi_j(t,x)dxdt\Big| \lesssim \|\psi_1\|_{X_\pm^{s_1,b_1}}\|\psi_2\|_{X_\pm^{s_2,b_2}}\|\psi_3\|_{X_\mp^{s_3,b_3}}. \tag{4.6}$$

Moreover the conditions (4.4) and (4.5) are sharp up to equality.

Remark 4.1.2. There are cases where we can allow equality in (4.4) or (4.5), for instance the

[1] For the sake of exposition, we are ignoring the endpoint cases. The sharp result allows one of the inequalities in (4.2) to replaced with an equality, a similar comment applies to the condition (4.3).

case

$$s_1 = s_2 = s_3 = 0, \qquad b_1 = 0, \qquad b_2 = b_3 = \frac{1}{2} + \epsilon$$

holds [73, Corollary 1]. We have not attempted to list or prove the endpoint cases here, as this would significantly complicate the statement of Theorem 4.1.1. Additionally, Theorem 4.1.1 is sufficient for our intended application to global well-posedness for the Dirac-Klein-Gordon equation.

Define the Wave-Sobolev spaces $H^{s,b}$ by using the norm

$$\|\psi\|_{H^{s,b}} = \big\| \langle |\tau| - |\xi| \rangle^b \langle \xi \rangle^s \widetilde{\psi}(\tau, \xi) \|_{L^2_{\tau,\xi}(\mathbb{R}^2)}.$$

Then as a simple corollary to Theorem 4.1.1 we can replace one of the $X^{s,b}_\pm$ norms on the righthand side of (4.6) with a $H^{s,b}$ norm.

Corollary 4.1.3. *Let $r, s_1, s_2, b_j \in \mathbb{R}$, $j = 1, 2, 3$ satisfy*

$$b_1 + b_2 + b_3 > \frac{1}{2}, \qquad b_j + b_k > 0, \qquad (j \neq k)$$

and for $k \in \{1, 2\}$

$$\begin{aligned} s_k + r &\geqslant 0, \\ s_k + r &> -b_{min} \\ s_1 + s_2 &> -b_{min}, \\ s_1 + s_2 &> \frac{1}{2} - b_1 - b_2 - b_3, \\ s_1 + s_2 + r &> \frac{1}{2} - b_k, \\ s_1 + s_2 + r &> \Big(\frac{1}{2} - b_{max}\Big)_+ + \Big(\frac{1}{2} - b_{med}\Big)_+ - b_{min}. \end{aligned}$$

Then

$$\Big| \int_{\mathbb{R}^2} \Pi_{j=1}^3 \psi_j(t, x) dx dt \Big| \lesssim \|\psi_1\|_{X^{s_1,b_1}_+} \|\psi_2\|_{X^{s_2,b_2}_-} \|\psi_3\|_{H^{r,b_3}}.$$

Proof. We decompose ψ_3 into the regions $\{(\tau, \xi) \in \mathbb{R}^{1+1} \mid \pm \tau\xi \geqslant 0\}$ and observe that on the first region $\langle |\tau| - |\xi| \rangle = \langle \tau - \xi \rangle$ while in the second region $\langle |\tau| - |\xi| \rangle = \langle \tau + \xi \rangle$. The corollary now follows from two applications of Theorem 4.1.1. □

Remark 4.1.4. This result should be compared to the similar estimates contained in [73] and [82]. We also note that the decomposition used in the proof of Corollary 4.1.3 can be used to give bilinear estimates in the Wave-Sobolev spaces $H^{r,b}$, giving an alternative (though closely related) proof of Theorem 7.1 in [33] (up to endpoints).

The second main result contained in this chapter concerns the global existence problem for the DKG equation on $\mathbb{R}^{1+1}$. The DKG equation can be written as

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi \\ \big(-\Box + m^2 \big)\phi &= \overline{\psi}\psi \end{aligned} \tag{4.7}$$

	$\psi(0) \in H^s$	$\phi[0] \in H^r \times H^{r-1}$
Chadam [20], 1973	$s = 1$	$r = 1$
Bournaveas [11], 2000	$s = 0$	$r = 1$
Fang [39], 2004	$s = 0$	$\frac{1}{2} < r \leqslant 1$
Bournaveas-Gibbson [16, 17], 2006	$s = 0$	$\frac{1}{4} < r \leqslant 1$
Machihara [61], Pecher [67], 2006	$s = 0$	$0 < r \leqslant 1$
Machihara-Nakanishi-Tsugawa [63], 2010	$s = 0$	$r = 0$
Selberg [72], 2007	$\frac{-1}{8} < s < 0$	$s + \sqrt{s^2 + s} < r \leqslant 1 + s$
Tesfahun [82], 2009	$\frac{-1}{8} < s < 0$	$s + \sqrt{s^2 - s} < r \leqslant 1 + s$

Figure 4.1: Summary of previous GWP results for the DKG equation

with initial data

$$\psi(0) = \psi_0 \in H^s, \qquad \phi(0) = \phi_0 \in H^r, \qquad \partial_t \phi(0) = \phi_1 \in H^{r-1} \tag{4.8}$$

for some values of $s, r \in \mathbb{R}$. The Dirac spinor $\psi \in \mathbb{C}^2$, and the real-valued scalar field $\phi \in \mathbb{R}$, are functions of $(t, x) \in \mathbb{R}^{1+1}$. The matrices γ^μ are defined in (1.3) and $m, M \in \mathbb{R}$ are constants.

There are two main features of the DKG equation (4.7) which we wish to highlight here. The first feature concerns the conservation of charge which can be stated as follows: if (ψ, ϕ) is a smooth solution to (4.7) with sufficient decay at infinity, then for all times $t \in \mathbb{R}$ we have

$$\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2}. \tag{4.9}$$

The conservation of charge is crucial in controlling the global behaviour of the solution (ψ, ϕ). The second feature we would like to note is that the nonlinearity in the DKG equation has null structure. Roughly speaking, this refers to the fact that the nonlinear terms in (4.7) behave significantly better than generic products. The null structure is a crucial component in the low regularity existence theory for the DKG equation and has been used by a number of authors [17, 40, 61, 67, 73]. The observation that null structure can be used to improve local existence results for nonlinear wave equations is due to Klainerman and Machedon in [49].

The question of local well-posedness (LWP) for the DKG equation was first considered by Chadam [20]. Subsequently, much progress has been made by numerous authors [17, 40, 61, 67, 73]. The best result to date is due to Machihara, Nakanishi, and Tsugawa [63] where it was shown that (4.7) with initial data (4.8) is locally well-posed provided

$$s > -\frac{1}{2}, \qquad |s| \leqslant r \leqslant s + 1.$$

Moreover, this region is essentially sharp, except possibly at the endpoint $s = -\frac{1}{2}$. More precisely, outside this region the solution map is either ill-posed, or fails to be twice differentiable; see [63] for a more precise statement.

In the current article we are interested in the minimum regularity required on the initial data (4.8) to ensure that the corresponding local in time solution (ψ, ϕ) to (4.7) can be extended globally in time. Global well-posedness (GWP) in the high regularity case $s = r = 1$ was first proven by Chadam [20], this was then progressively lowered to $s \geqslant 0$ by a number of authors [11, 16, 40, 67] by exploiting the conservation of charge (4.9) together with the local well-posedness theory, see Figure 4.1. The first result below the charge class was due to Selberg [72] where it was shown that the DKG equation is GWP in the region[2]

$$-\frac{1}{8} < s < 0, \qquad -s + \sqrt{s^2 - s} < r \leqslant s + 1.$$

Note that when $s < 0$, the conservation of charge cannot be used directly since $\psi \notin L^2$, thus the problem of global existence is significantly more difficult. Instead Selberg made use of the Fourier truncation method of Bourgain [10], which allows one to take initial data just below a conserved quantity. There is a difficulty in directly applying this method to the DKG equation however, as there is no conservation law for the scalar ϕ. Instead, one needs to exploit the fact the nonlinearity for ϕ depends only on the spinor ψ. Thus, as we have control over ψ via the conservation of charge, we should be able to estimate the growth of ϕ. This strategy was implemented by Selberg via an induction argument involving the cascade of free waves [72]. We should note that a related idea was used by Colliander-Holmer-Tzirakis to prove GWP for the Zakharov and Klein-Gordon-Schrdinger systems [26]. We remark that in the $s \geqslant 0$ case, the growth of the H^r norm of ϕ could be controlled by the standard energy inequality together with the one dimensional product estimate, see for instance [16]. If we implemented the same argument in this setting we would end up with the restriction $s > \frac{-1}{10}$. To improve this we need to use the more involved induction on free waves argument introduced by Selberg.

Currently, the best result for GWP for the DKG equation is due to Tesfahun [82] where the GWP region of Selberg was extended to

$$-\frac{1}{8} < s < 0, \qquad s + \sqrt{s^2 - s} < r \leqslant s + 1.$$

The improvement comes from applying the I-method of Colliander, Keel, Staffilani, Takaoka, and Tao, see for instance [23] for an introduction to the I-method. In the current article, we prove the following.

Theorem 4.1.5. *The DKG equation (4.7) is globally well-posed for initial data $\psi_0 \in H^s$, $(\phi_0, \phi_1) \in H^r \times H^{r-1}$ provided*

$$-\frac{1}{6} < s < 0, \qquad s - \frac{1}{4} + \sqrt{\left(s - \frac{1}{4}\right)^2 - s} < r \leqslant s + 1.$$

The proof of Theorem 4.1.5 follows the argument used in [82] together with the bilinear estimates in Theorem 4.1.1. More precisely, we use the I-method together with the induction on free waves approach of Selberg. The main idea, following the usual I-method, is to define a mild smoothing operator I such that, firstly, for some large constant N, we have the estimate

$$\|If\|_{L^2(\mathbb{R})} \lesssim N^{-s} \|f\|_{H^s(\mathbb{R})} \lesssim N^{-s} \|If\|_{L^2}. \tag{4.10}$$

Secondly, we require I to be the identity on low frequencies. We then try to estimate the growth of $\|I\psi(t)\|_{L^2}$ in terms of t. It turns out that despite the fact that $I\psi$ no longer solves the DKG equation, there is sufficient cancellation of frequencies to ensure that the charge $\|I\psi(t)\|_{L^2_x}$

[2] Note that this also gives GWP in the region $s > 0$, $|s| \leqslant r \leqslant s + 1$ by persistence of regularity, see for instance [73].

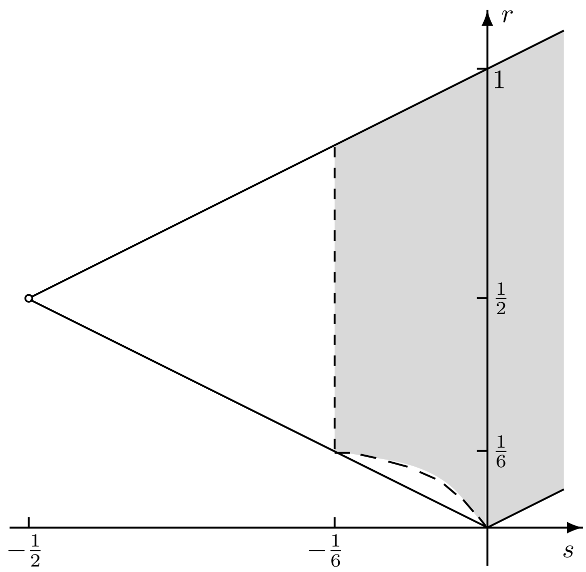

Figure 4.2: Global well-posedness holds in the shaded region by Theorem 4.1.5. Local well-posedness holds inside the the lines $r = |s|$ and $r = s + 1$ for $s > \frac{-1}{2}$ by [63].

is almost conserved. This almost conservation property follows from the usual proof of the conservation of charge, together with a number of applications of Theorem 4.1.1. Thus we can estimate the growth of $\|\psi(t)\|_{H^s}$ from (4.10). The induction on free waves approach of Selberg then allows us to control the scalar field ϕ and completes the proof of Theorem 4.1.5, thus we obtain GWP in the region in Figure 4.2.

We now give a brief outline of this chapter. In Section 4.2, we recall some properties of the $X^{s,b}$ and $H^{r,b}$ spaces which we require in the proof of Theorem 4.1.5. The proof of Theorem 4.1.5 is contained in Section 4.3. In Section 4.4 we prove that the conditions in Theorem 4.1.1 are sufficient for the estimate (4.6). Finally, the counter examples showing that Theorem 4.1.1 is sharp up to equality are contained in Section 4.5.

4.2 Linear Estimates

Here we briefly recall some of the important properties of the $X^{s,b}_{\pm}$ and $\mathcal{H}^{r,b}$ spaces which we make use of in the proof of Theorem 4.1.5, for more details we refer the reader to [31] and [79]. We start by defining

$$S_{\Delta T} = (0, \Delta T) \times \mathbb{R}.$$

This notation will be used throughout this chapter. We need a number of properties of the localised spaces $X^{s,b}_{\pm}(S_{\Delta T})$. Firstly, we note that if $b > \frac{1}{2}$, then $u \in X^{s,b}(S_{\Delta T})$ implies $u \in C([0, \Delta T], H^s)$. Secondly, we have the following lemma.

Lemma 4.2.1. *Let $s \in \mathbb{R}$, $0 < \Delta T < 1$, and $\nu \in C^\infty_0(\mathbb{R})$. If $-\frac{1}{2} < b_1 \leqslant b_2 < \frac{1}{2}$ then*

$$\Big\| \nu\Big(\frac{t}{\Delta T}\Big) u(t,x) \Big\|_{X^{s,b_1}_{\pm}} \lesssim \Delta T^{b_2 - b_1} \|u\|_{X^{s,b_2}_{\pm}}.$$

Consequently, we have $\|u\|_{X^{s,b_1}_\pm(S_{\Delta T})} \lesssim \Delta T^{b_2-b_1}\|u\|_{X^{s,b_2}_\pm(S_{\Delta T})}$. *Moreover if* $-\frac{1}{2} < b < \frac{1}{2}$ *then*

$$\|\mathbb{1}_{[0,\Delta T]}(t)u\|_{X^{s,b}_\pm} \lesssim \|u\|_{X^{s,b}_\pm(S_{\Delta T})}$$

with constant independent of ΔT.

Proof. The first conclusion is well known and can be found in, for instance, [79]. The second conclusion is perhaps not as well known and for the convenience of the reader we include the proof here. The definition of $X^{s,b}_\pm(S_{\Delta T})$ together with a change of variables on the frequency side shows that is suffices to prove

$$\|\mathbb{1}_{[0,\Delta T]}(t)f\|_{H^b} \lesssim \|f\|_{H^b}. \tag{4.11}$$

By duality we may assume that $0 < b < \frac{1}{2}$. Then by a well-known characterisation of the Sobolev spaces H^s, (see Theorem 1.2.2) we have

$$\begin{aligned}\|\mathbb{1}_{[0,\Delta T]}f\|^2_{H^b} &\approx \|\mathbb{1}_{[0,\Delta T]}f\|^2_{L^2} + \int_{\mathbb{R}^2} \frac{|\mathbb{1}_{[0,\Delta T]}(t)f(t) - \mathbb{1}_{[0,\Delta T]}(t')f(t')|^2}{|t-t'|^{1+2b}} dtdt' \\ &\lesssim \|f\|^2_{L^2} + \int_0^{\Delta T}\int_0^{\Delta T} \frac{|f(t)-f(t')|^2}{|t-t'|^{1+2b}} dtdt' + 2\int_0^{\Delta T}\int_{t'\notin[0,\Delta T]} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt'dt \\ &\lesssim \|f\|^2_{H^b} + 2\int_0^{\Delta T}\int_{t'\notin[0,\Delta T]} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt'dt.\end{aligned}$$

To complete the proof we use Hardy's inequality (see Theorem 1.2.6, or alternatively [79, Lemma A.2]) together with the assumption $0 < b < \frac{1}{2}$ to deduce that

$$\begin{aligned}\int_0^{\Delta T}\int_{t'\notin[0,\Delta T]} \frac{|f(t)|^2}{|t-t'|^{1+2b}} dt'dt &\lesssim \int_0^{\Delta T} |f(t)|^2\Big(\frac{1}{|t|^{2b}} + \frac{1}{|t-\Delta T|^{2b}}\Big)dt \\ &\lesssim \Big\|\frac{f(t)}{|t|^b}\Big\|^2_{L^2} + \Big\|\frac{f(t)}{|t-\Delta T|^b}\Big\|^2_{L^2} \\ &\lesssim \|f\|^2_{H^b}.\end{aligned}$$

□

Remark 4.2.2. An obvious way to try to prove (4.11) would be to follow the proof of Lemma 3.2.4 in Chapter 3 and use the estimate

$$\|fg\|_{H^b} \lesssim \|f\|_{B^{\frac{1}{2}}_{2,1}} \|g\|_{H^b}$$

(see Theorem 1.2.13). However, the sharp cutoff $\mathbb{1}_{[0,T]}$ just barely fails to belong to $B^{\frac{1}{2}}_{2,1}$, and so this approach does not work. Thus, we need to use the more direct approach given above.

To control the solution to the Dirac equation we make use of the energy estimate for the $X^{s,b}_\pm$ spaces, see [31, Lemma 5] for a proof.

Lemma 4.2.3. *Let* $s \in \mathbb{R}$, $b > \frac{1}{2}$, *and* $0 < \Delta T < 1$. *Suppose* $f \in H^s$, $F \in X^{s,b-1}_\pm(S_{\Delta T})$, *and let* u *be the solution to*

$$\begin{aligned}\partial_t u \pm \partial_x u &= F \\ u(0) &= f.\end{aligned}$$

Then $u \in X^{s,b}_{\pm}(S_{\Delta T})$ and we have the estimate

$$\|u\|_{X^{s,b}_{\pm}(S_{\Delta T})} \lesssim \|f\|_{H^s} + \|F\|_{X^{s,b-1}_{\pm}(S_{\Delta T})}.$$

We also require the $H^{r,b}$ versions of the above results.

Lemma 4.2.4. *Let $r \in \mathbb{R}$, $0 < \Delta T < 1$, and $\nu \in C_0^\infty(\mathbb{R})$. Then if $-\frac{1}{2} < b_1 \leqslant b_2 < \frac{1}{2}$ we have*

$$\left\| \nu\Big(\frac{t}{\Delta T}\Big) u(t,x) \right\|_{H^{r,b_1}} \lesssim \Delta T^{b_2 - b_1} \|u\|_{H^{r,b_2}}.$$

Consequently, we have $\|u\|_{H^{r,b_1}(S_{\Delta T})} \lesssim \Delta T^{b_2-b_1} \|u\|_{H^{r,b_2}(S_{\Delta T})}$.

Define $\mathcal{H}^{r,b}$ as the completion of $\mathcal{S}(\mathbb{R}^2)$ using the norm

$$\|u\|_{\mathcal{H}^{r,b}} = \|u\|_{H^{r,b}} + \|\partial_t u\|_{H^{r-1,b}}.$$

Then, provided $b > \frac{1}{2}$, we have $\mathcal{H}^{r,b}(S_{\Delta T}) \subset C([0,\Delta T], H^s) \cap C^1([0,\Delta T], H^{s-1})$ where the embedding is continuous. This can also be written as

$$\|u[t]\|_{L_t^\infty H_x^s} \lesssim \|u\|_{\mathcal{H}^{r,b}}$$

where we use the shorthand $\|u[t]\|_{H_x^s} = \|u(t)\|_{H_x^s} + \|\partial_t u(t)\|_{H_x^s}$. We require the following $\mathcal{H}^{r,b}$ counterpart to Lemma 4.2.3.

Lemma 4.2.5. *Let $r \in \mathbb{R}$, $b > \frac{1}{2}$, $0 < \Delta T < 1$, and $m \in \mathbb{R}$. Suppose $f \in H^r$, $g \in H^{r-1}$, and $F \in H^{r-1,b-1}(S_{\Delta T})$ and let u be the solution to*

$$\begin{aligned} \Box u &= m^2 u + F \\ u(0) = f, &\qquad \partial_t u(0) = g. \end{aligned}$$

Then $u \in \mathcal{H}^{r,b}(S_{\Delta T})$ and we have the estimate

$$\|u\|_{\mathcal{H}^{r,b}(S_{\Delta T})} \lesssim \|f\|_{H^r} + \|g\|_{H^{r-1}} + \|F\|_{H^{r-1,b-1}(S_{\Delta T})}.$$

Proof. See [82]. □

4.3 Global Well-Posedness for the Dirac-Klein-Gordon Equation

We are now ready to consider the proof of global well-posedness for the DKG equation. To uncover the null structure for the DKG equation, we let $\psi = (\psi_+, \psi_-)^T$. Then the DKG equation (4.7) can be written as

$$\begin{aligned} \partial_t \psi_\pm \pm \partial_x \psi_\pm &= -iM\psi_\mp + i\phi\psi_\mp \\ \Box\phi &= m^2\phi - 2\Re\big(\psi_+ \overline{\psi}_-\big) \end{aligned} \tag{4.12}$$

with initial data

$$\psi_\pm(0) = f_\pm \in H^s, \qquad \phi(0) = \phi_0 \in H^r, \qquad \partial_t\phi(0) = \phi_1 \in H^{r-1}. \tag{4.13}$$

Note that the right hand side of (4.12) has the bilinear product $\psi_+\overline{\psi}_-$, which, as we have seen in Theorem 4.1.1, behaves significantly better than the corresponding product with $++$. The $+-$ structure can also be seen in the term $\phi\psi_\pm$ via a duality argument [73]. These are the key observations used in the local well-posedness theory for the DKG equation.

To prove the global well-posedness result of Theorem 4.1.5, by the local well-posedness result in [73], it suffices to prove that the data norms $\|\psi_\pm(T)\|_{H^s}$, $\|u[T]\|_{H^r}$ remain finite for all large times $0 < T < \infty$. To this end, we make use of the I-method together with ideas from [72] and [82]. Let $\rho_0 \in C^\infty$ be even, decreasing, and satisfy

$$\rho_0(\xi) = \begin{cases} 1 & |\xi| < 1 \\ |\xi|^s & |\xi| > 2. \end{cases}$$

Let $\rho(\xi) = \rho_0\Big(\frac{|\xi|}{N}\Big)$ and define the I operator by $\widehat{I\psi}(\xi) = \rho(\xi)\widehat{\psi}(\xi)$. We have the following straightforward estimates. Firstly, since $s < 0$, we have for any $\sigma \in \mathbb{R}$,

$$\|f\|_{H^\sigma} \lesssim \|If\|_{H^{\sigma-s}} \lesssim N^{-s}\|f\|_{H^\sigma}. \tag{4.14}$$

In particular, by taking $\sigma = s$, we observe that to obtain control over $\|\psi(t)\|_{H^s_x}$, it suffices to estimate $\|I\psi(t)\|_{L^2_x}$. Secondly, if supp $\widehat{g} \subset \{|\xi| \gtrsim N\}$, $s < 0$, and $s_1 < s_2$, then we can trade regularity for decay in terms of N,

$$\|g\|_{H^{s_1}} \lesssim N^{s_1-s_2}\|g\|_{H^{s_2}} \approx N^{s_1-s_2+s}\|Ig\|_{H^{s_2-s}}. \tag{4.15}$$

Thirdly, we note that the I operator is the identity on low frequencies, so if supp $\widehat{f} \subset \{|\xi| < N\}$ then $If = f$. Finally, if f is real-valued, then If is also real-valued since ρ was assumed to be even.

The I-method proceeds as follows. Assume we have a local solution

$$\psi_\pm \in C\big([0, \Delta T], H^s\big), \qquad \phi \in C\big([0, \Delta T], H^r\big) \cap C^1\big([0, \Delta T], H^{r-1}\big)$$

to (4.12), (4.13). Note that from (4.14) we have $I\psi(t) \in L^2_x$. We would like to use the conservation of charge to control $\|I\psi(t)\|_{L^2_x}$. However $I\psi$ is no longer a solution to (4.12) and so we can not expect $\|I\psi(t)\|_{L^2_x}$ to be conserved. Despite this, if we follow the proof of conservation of charge, then

$$\begin{aligned}
\partial_t \int_{\mathbb{R}} |I\psi_+(t)|^2 + |I\psi_-(t)|^2 dx &= 2\Re\Big(\int_{\mathbb{R}} \overline{I\psi}_+ \partial_t I\psi_+ + \overline{I\psi}_- \partial_t I\psi_- dx \Big) \\
&= 2\Re\Big(\int_{\mathbb{R}} \overline{I\psi}_+ \big(-\partial_x I\psi_+ - iMI\psi_- + iI(\phi\psi_-)\big) \\
&\qquad\qquad + \overline{I\psi}_- \big(\partial_x I\psi_- - iMI\psi_+ + iI(\phi\psi_+)\big) dx \Big) \\
&= 2\Re\Big(i \int_{\mathbb{R}} \overline{I\psi}_+ I(\phi\psi_-) + \overline{I\psi}_- I(\phi\psi_+) dx \Big).
\end{aligned} \tag{4.16}$$

Now as ϕ is real-valued, $I^2\phi$ is also real-valued and hence

$$2\Re\Big(iI^2\phi\big(\overline{I\psi}_+ I\psi_- + \overline{I\psi}_- I\psi_+\big)\Big) = 0.$$

Subtracting this term from (4.16) and using the fundamental theorem of Calculus then gives

$$\sup_{t'\in[0,\Delta T]}\big(\|I\psi_+(t')\|_{L^2_x}^2+\|I\psi_-(t')\|_{L^2_x}^2\big)$$
$$\leqslant \|f_+\|_{L^2}^2+\|f_-\|_{L^2}^2+2\sum_{\pm}\sup_{t'\in[0,\Delta T]}\Big|\int_0^{t'}\int_{\mathbb{R}}\big(I(\phi\psi_\pm)-I^2\phi I\psi_\pm\big)\overline{I\psi_\mp}dxdt\Big|. \quad (4.17)$$

Thus provided we can show the last term in (4.17) is small, we can deduce that over a small time $[0,\Delta T]$, $\|I\psi_\pm(t)\|_{L^2}$ does not grow to large. The first step in this direction is the following.

Lemma 4.3.1. *Let $\frac{-1}{4}<s<0$ and $-s<r\leqslant 1+2s$. Assume $b=\frac{1}{2}+\epsilon$ with $\epsilon>0$ sufficiently small. Then for any $\Delta T\ll 1$, $N\gg 1$ we have*

$$\sup_{t'\in[0,\Delta T]}\Big|\int_0^{t'}\int_{\mathbb{R}}\big(I(\phi u)-I^2\phi Iu\big)\overline{Iv}\,dxdt\Big|$$
$$\lesssim \Delta T^{\frac{1}{2}-2\epsilon}N^{2s-r+2\epsilon}\|I^2\phi\|_{H^{r-2s,b}(S_{\Delta T})}\|Iu\|_{X^{0,b}_{\pm}(S_{\Delta T})}\|Iv\|_{X^{0,b}_{\mp}(S_{\Delta T})} \quad (4.18)$$

where $S_{\Delta T}=(0,\Delta T)\times\mathbb{R}$.

Proof. See Subsection 4.3.1 below. ☐

Remark 4.3.2. The use of $I^2\phi$ instead of just ϕ or $I\phi$ on the right hand side of (4.18) may require some explanation. Roughly speaking, the larger the negative exponent on N in (4.18), the better the eventual GWP result will be. Moreover, an examination of the proof of Lemma 4.3.1 shows that the exponent on N depends entirely on the number of derivatives on ϕ. In other words, we could replace the term $N^{2s-r}\|I^2\phi\|_{H^{r-2s,b}}$ with $N^{ks-r}\|I^k\phi\|_{H^{r-ks,b}}$ for any $k\in\mathbb{N}$ (provided $r-ks\leqslant 1$). However, the size of ϕ with respect to N ends up being of the order N^{-2s}. This follows by observing that schematically ϕ is a solution to $\Box\phi=\psi^2$, and by (4.14), the low frequency component of ψ^2 is essentially of size N^{-2s}. Thus it is natural to take $I^2\phi$, which via (4.14), also has size roughly N^{-2s}.

Remark 4.3.3. The powers of ΔT and N on the right hand side of (4.18) are essentially sharp if we are working in the spaces $X^{s,b}_{\pm}$, $H^{s,b}$. This follows from the counter examples in Section 4.5 together with a scaling argument.

Lemma 4.3.1 allows us to estimate the growth of $\|I\psi_\pm(t)\|_{L^2}$ on $[0,\Delta T]$, provided that we can control the size of the norms $\|I\psi_\pm\|_{X^{0,b}_{\pm}(S_{\Delta T})}$ and $\|I^2\phi\|_{H^{r-2s,b}(S_{\Delta T})}$. This control is provided by a modification of the usual local well-posedness theory.

Lemma 4.3.4. *Let $\frac{-1}{6}<s<0$, $-s<r\leqslant\frac{1}{2}+2s$, and $b=\frac{1}{2}+\epsilon$ with $\epsilon>0$ sufficiently small. Assume $f_\pm\in H^s$ and $\phi[0]\in H^r\times H^{r-1}$. Choose $\Delta T\ll 1$ and $N\gg 1$ such that*

$$\Big(\Delta T^{\frac{1}{2}+r-2s-3\epsilon}+N^{-r+2s+2\epsilon}\Big)\|I^2\phi[0]\|_{H^{r-2s}}\ll 1 \quad (4.19)$$

and

$$\Big(\Delta T^{1-\epsilon}+N^{-\frac{1}{2}+2\epsilon}\Big)\Big(\|If_+\|_{L^2}+\|If_-\|_{L^2}\Big)^2\ll 1. \quad (4.20)$$

Then the Dirac-Klein-Gordon equation (4.12) with initial data (4.13) is locally well-posed on the domain $[0,\Delta T]\times\mathbb{R}$. Moreover, the solution (ψ,ϕ) satisfies

$$\|I\psi_+\|_{X^{0,b}_{+}(S_{\Delta T})}+\|I\psi_-\|_{X^{0,b}_{-}(S_{\Delta T})}\lesssim\|If_+\|_{L^2}+\|If_-\|_{L^2}$$

and

$$\|I^2\phi\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} \lesssim \|I^2\phi[0]\|_{H^{r-2s}} + \big(\|If_+\|_{L^2} + \|If_-\|_{L^2}\big)^2.$$

Proof. See Subsection 4.3.2 below. □

Remark 4.3.5. Note that since $\|I^2\phi[0]\|_{H^{r-2s}} \lesssim \|\phi[0]\|_{H^r} dN^{-2s}$, by choosing N sufficiently large and ΔT sufficiently small, we can ensure that the inequality (4.19) is satisfied. A similar comment applies to (4.20).

Remark 4.3.6. The reason that we can extend the work of Tesfahun [82] is due to the conclusions in Lemma 4.3.1 and Lemma 4.3.4. In more detail, Lemma 4.3.1 improves [82, Lemma 8] by adding a power of ΔT on the right hand side of (4.18). Since ΔT will be taken small, this is a significant gain. Similarly, Lemma 4.3.4 extends [82, Theorem 8] by having a larger exponent on ΔT in (4.19). As a consequence, we can take ΔT larger, which improves the eventual GWP result. The point here is that the larger ΔT becomes, the fewer time steps of length ΔT are required to reach a large time T.

We now follow the argument used in [82] and sketch the proof of Theorem 4.1.5. The persistence of regularity result in [73] shows that it suffices to prove GWP in the case

$$-\frac{1}{6} < s < 0, \qquad s - \frac{1}{4} + \sqrt{\left(s - \frac{1}{4}\right)^2 - s} < r < \frac{1}{2} + 2s. \tag{4.21}$$

Note that this region is non-empty as the intersection of the curves $s - \frac{1}{4} + \sqrt{\left(s - \frac{1}{4}\right)^2 - s}$ and $\frac{1}{2} + 2s$ occurs at $s = -\frac{1}{6}$.

Choose some large time $T > 0$ and assume $\epsilon > 0$ is small. Let N be some large fixed constant to be chosen later depending on the initial data $\|\psi(0)\|_{H^s}$ and $\|\phi[0]\|_{H^r}$, as well as the various constants appearing in Lemma 4.3.1 and Lemma 4.3.4. Take $\Delta T = N^{\frac{4s-2\epsilon}{1+2r-4s-6\epsilon}}$. If N is sufficiently large then from (4.14)

$$\begin{aligned}\Big(\Delta T^{\frac{1}{2}+r-2s-3\epsilon} + N^{-r+2s+2\epsilon}\Big)\|I^2\phi[0]\|_{H^{r-2s}} &\ll 1\\ \Big(\Delta T^{1-\epsilon} + N^{-\frac{1}{2}+2\epsilon}\Big)\Big(\|If_+\|_{L^2} + \|If_-\|_{L^2}\Big)^2 &\ll 1.\end{aligned}$$

Therefore by Lemma 4.3.4 we get a solution (ψ, ϕ) to (4.12) on $[0, \Delta T]$. We would now like to repeat this argument $\frac{T}{\Delta T}$ times to advance to the time T. The only obstruction is the possible growth of the norms $\|I\psi_\pm(t)\|_{L^2}$ and $\|I^2\phi[t]\|_{H^{r-2s}}$. Our aim is to use Lemma (4.3.1) to show that $\|I\psi_\pm(t)\|_{L^2}$ is "almost conserved" and consequently obtain large time control over the norm $\|I\psi_\pm(t)\|_{L^2}$. This is accomplished by using an induction argument as follows.

Assume $n \lesssim \frac{T}{\Delta T}$ and suppose we have a solution (ψ, ϕ) on $[0, n\Delta T]$ with the bounds

$$\sup_{t\in[0,n\Delta T]} \Big(\|I\psi_+(t)\|^2_{L^2_x} + \|I\psi_-(t)\|^2_{L^2_x}\Big) \leqslant 2\|If_+\|^2_{L^2_x} + 2\|If_-\|^2_{L^2_x} \tag{4.22}$$

and

$$\sup_{t\in[0,n\Delta T]} \|I^2\phi[t]\|_{H^{r-2s}_x} \leqslant C^*\Big(\|I^2\phi[0]\|_{H^{r-2s}_x} + \big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\big)^2\Big) \tag{4.23}$$

where the constant C^* is some large constant independent of N, ΔT, and n. If N is sufficiently large, depending on C^* and the initial data $\|f_\pm\|_{H^s}$, $\|\phi[0]\|_{H^r}$, then we can apply Lemma 4.3.4 with initial data $\psi(n\Delta T)$, $\big(\phi(n\Delta T), \partial_t\phi(n\Delta T)\big)$, and extend the solution to $[0,(n+1)\Delta T]$.

Suppose we could show that the bounds (4.22) and (4.23) on $[0, n\Delta T]$ implied that they also hold on the larger interval $[0,(n+1)\Delta T]$ with the same constant C^*. Then by induction we would have (4.22) and (4.23) on $[0,T]$. Since T was arbitrary, Theorem 4.1.5 would follow. Thus it suffices to verify the estimates (4.22) and (4.23) on the interval $[0,(n+1)\Delta T]$. We break this into two parts, proving the bound on $\|I\psi_\pm(t)\|_{L^2}$, and then estimating $\|I^2\phi[t]\|_{H^{r-2s}}$.

Bound on the Spinor $\psi_\pm$. Let

$$\Gamma(z) = \sup_{t\in[0,z]} \Big(\|I\psi_+(t)\|^2_{L^2_x} + \|I\psi_-(t)\|^2_{L^2_x} \Big).$$

Note that the bounds (4.22) and (4.23) imply that

$$\begin{aligned} \Gamma(n\Delta T) \leqslant 2\Gamma(0) \leqslant AN^{-2s} \\ \sup_{t\in[0,n\Delta T]} \|I^2\phi[t]\|_{H^{r-2s}_x} \leqslant BN^{-2s} \end{aligned} \tag{4.24}$$

where $A = A(C^*, T, \|\psi(0)\|_{H^s})$ and $B = B(C^*, T, \|\psi(0)\|_{H^s}, \|\phi[0]\|_{H^r})$ depend on the initial data, the constant C^*, and T, but are independent of n, N, and ΔT. If we now combine Lemma 4.3.1, Lemma 4.3.4 together with (4.17) we obtain the following control on the growth of $\Gamma(t)$.

Corollary 4.3.7 (Almost conservation law). *Let $\frac{-1}{6} < s < 0$ and $-s < r \leqslant \frac{1}{2} + 2s$ and $b = \frac{1}{2}+\epsilon$ with $\epsilon > 0$ sufficiently small. Suppose*

$$\Delta T = N^{\frac{4s-2\epsilon}{1+2r-4s-6\epsilon}}$$

and we have the bounds (4.24). Then provided N is sufficiently large,

$$\Gamma(\Delta T) \leqslant \Gamma(0) + C\Delta T^{\frac{1}{2}-2\epsilon} N^{-r+2\epsilon}\big(A+B\big)\Gamma(0).$$

Proof. By Lemma 4.3.1, Lemma 4.3.4, and (4.17) it suffices to show that

$$\Delta T^{\frac{1}{2}+r-2s-3\epsilon} N^{-2s} B + N^{-r+2\epsilon} B \ll 1$$

and

$$\Delta T^{1-\epsilon} N^{-2s} A + N^{2\epsilon-\frac{1}{2}-2s} B \ll 1.$$

However these inequalities follow provided $\Delta T = N^{\frac{4s-2\epsilon}{1+2r-4s-6\epsilon}}$ and we choose N sufficiently large. □

The previous corollary, together with (4.24) shows that

$$\begin{aligned} \Gamma\big((n+1)\Delta T\big) &\leqslant \Gamma(n\Delta T) + C\Delta T^{\frac{1}{2}-2\epsilon} N^{-r+2\epsilon}(A+B)\Gamma(n\Delta T) \\ &\leqslant \Gamma(n\Delta T) + 2C\Delta T^{\frac{1}{2}-2\epsilon} N^{-r+2\epsilon}(A+B)\Gamma(0). \end{aligned}$$

Repeating this argument n times we obtain control over $\Gamma(t)$ at time $(n+1)\Delta T$

$$\Gamma\big((n+1)\Delta T\big) \leqslant \Gamma(0) + 2Cn\Delta T^{\frac{1}{2}-2\epsilon} N^{-r+2\epsilon}(A+B)\Gamma(0).$$

Since the number of steps $n \lesssim \frac{T}{\Delta T}$ we get

$$\Gamma\big((n+1)\Delta T\big) \leqslant \Gamma(0) + 2CT\Delta T^{-\frac{1}{2}-2\epsilon} N^{-r+2\epsilon}(A+B)\Gamma(0).$$

We want to make the coefficient of the second term small. Thus we need to ensure that, using the requirement on ΔT in Corollary 4.3.7,

$$2CT\Delta T^{-\frac{1}{2}-2\epsilon} N^{-r+2\epsilon}(A+B) \approx N^{\frac{-(1+4\epsilon)(2s-\epsilon)}{1+2r-4s-6\epsilon}-r+2\epsilon} \ll 1. \tag{4.25}$$

By choosing N large, and $\epsilon > 0$ sufficiently small, we see that (4.25) will follow provided $-2s - r\big(1+2r-4s\big) < 0$. Rearranging, we get the quadratic polynomial $2r^2 + (1-4s)r + 2s > 0$ and so we need

$$s - \frac{1}{4} + \sqrt{\Big(s - \frac{1}{4}\Big)^2 - s} < r.$$

Therefore, provided we choose N large enough, depending on T, A, and B, we get

$$\Gamma\big((n+1)\Delta T\big) \leqslant 2\Gamma(0)$$

as required.

Bound on ϕ. Recall that our goal was to show that, if the bounds (4.22) and (4.23) hold for $t \in [0, n\Delta T]$, then in fact they also held on the larger domain $[0, (n+1)\Delta T]$ (with the same constants). The bound for $\|I\psi_\pm\|_{L^2}$ was obtained above. Thus it remains to bound $\|I^2\phi[t]\|_{H^{r-2s}}$ on the interval $[0, (n+1)\Delta T]$. The argument that gives the required bound makes use of an idea due to Selberg in [72] on induction of free waves. The idea is to break ϕ into a sum of homogeneous waves, together with an inhomogeneous term and then use an induction argument to estimate the contribution that each of these homogeneous waves makes to the size of $\|I^2\phi[t]\|_{H^{r-2s}}$. We note that this idea was also used in [82].

We begin by observing that the induction assumptions (4.22) and (4.23) together with Lemma 4.3.4 give for every $0 \leqslant j \leqslant n$

$$\|I\psi_+\|_{X_+^{0,b}(S_j)} + \|I\psi_-\|_{X_-^{0,b}(S_j)} \leqslant C_1\Big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\Big) \tag{4.26}$$

where $S_j = \big(j\Delta T, (j+1)\Delta T\big) \times \mathbb{R}$ and the constant C_1 is independent of C^*, j, n, N, and ΔT. Suppose we could show that (4.26) implies that

$$\sup_{t \in [n\Delta T, (n+1)\Delta T]} \|I^2\phi[t]\|_{H^{r-2s}} \leqslant C_2\Big(\|I^2\phi[0]\|_{H^{r-2s}_x} + \Big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\Big)^2\Big). \tag{4.27}$$

Then by taking $C^* = C_2$ we see that the bound (4.23) holds for $t \in [0, (n+1)\Delta T]$. Thus by induction, together with the fact that the constants in (4.22) and (4.23) are independent of n, we would obtain control over the solution on $[0, T]$ and Theorem 4.1.5 would follow.

We now show that (4.26) implies (4.27). We make use of the following result which is a variant of a corresponding result in [82].

Lemma 4.3.8. *Let $m \in \mathbb{R}$, $0 < \Delta T < 1$, $\frac{-1}{4} < s < 0$, $0 < r < \frac{1}{2} + 2s$, and $b > \frac{1}{2}$. Assume*

$u \in X_+^{s,b}(S_{\Delta T})$ and $v \in X_-^{s,b}(S_{\Delta T})$. Then there exists a unique solution $\Phi \in \mathcal{H}^{r,b}(S_{\Delta T})$ to

$$\Box\Phi = \Re(uv) + m^2\Phi$$
$$\Phi(0) = \partial_t\Phi(0) = 0.$$

Moreover we have

$$\sup_{t\in[0,\Delta T]} \|I^2\Phi[t]\|_{H_x^{r-2s}} \lesssim \big(\Delta T + N^{-\frac{1}{2}+2\epsilon}\big)\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}. \tag{4.28}$$

Proof. The existence/uniqueness claim follows from Lemma 4.2.5 together with an application of Theorem 4.1.1. To prove (4.28) we write $\Phi = \Phi_1 + \Phi_2$ where

$$\Box\Phi_1 = \Re(u_{low}v_{low}) + m^2\Phi_1$$
$$\Phi_1(0) = 0, \qquad \partial_t\Phi_1(0) = 0.$$

and $\widehat{u_{low}} = \mathbb{1}_{|\xi|<\frac{N}{2}}\widehat{u}$, $\widehat{v_{low}} = \mathbb{1}_{|\xi|<\frac{N}{2}}\widehat{v}$. The standard representation of solutions to the Klein-Gordon equation, together with the Sobolev product law and the observation that $I^2(u_{low}v_{low}) = u_{low}v_{low}$, gives

$$\begin{aligned}\sup_{t\in[0,\Delta T]} \|I^2\Phi_1[t]\|_{H_x^{r-2s}} &\lesssim \int_0^{\Delta T} \|u_{low}(t)v_{low}(t)\|_{H_x^{r-2s-1}}dt \\ &\lesssim \int_0^{\Delta T} \|u_{low}(t)\|_{L_x^2}\|v_{low}(t)\|_{L_x^2}dt \\ &\lesssim \Delta T\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}.\end{aligned}$$

To bound the remaining term, Φ_2, we note that by the energy estimate for $H^{s,b}$ spaces in Lemma 4.2.5,

$$\sup_{t\in[0,\Delta T]} \|I^2\Phi_2[t]\|_{H_x^{r-2s}} \tag{4.29}$$
$$\begin{aligned}&\lesssim \|I^2\Phi_2\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} \\ &\lesssim \|I^2(uv - u_{low}v_{low})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \\ &\lesssim \|u_{low}v_{hi}\|_{H^{-\frac{1}{2},b-1}(S_{\Delta T})} + \|u_{hi}v_{low}\|_{H^{-\frac{1}{2},b-1}(S_{\Delta T})} + \|u_{hi}v_{hi}\|_{H^{-\frac{1}{2},b-1}(S_{\Delta T})}\end{aligned} \tag{4.30}$$

where $u_{hi} = u - u_{low}$ is the high frequency component of u, v_{hi} is defined similarly, and we used the assumption $r < \frac{1}{2} + 2s$. By Corollary 4.1.3 we have the estimate

$$\|\psi_1\psi_2\|_{H^{-\frac{1}{2},b-1}} \lesssim \|\psi_1\|_{X_+^{-\frac{1}{2}-s_1+2\epsilon,b}}\|\psi_2\|_{X_-^{s_1,b}} \tag{4.31}$$

for $\frac{-1}{2} < s_1 \leqslant 0$. To control the first term in (4.30) we use (4.31) with $s_1 = -\frac{1}{2} + 2\epsilon$ together with (4.15) to obtain

$$\begin{aligned}\|u_{low}v_{hi}\|_{H^{-\frac{1}{2},b-1}(S_{\Delta T})} &\lesssim \|u_{low}\|_{X_+^{0,b}(S_{\Delta T})}\|v_{hi}\|_{X_-^{-\frac{1}{2}+2\epsilon,b}(S_{\Delta T})} \\ &\lesssim N^{-\frac{1}{2}+2\epsilon}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}\end{aligned}$$

A similar application of (4.31) allows us to estimate the second term in (4.30). Finally, for the last term in (4.30) we use (4.15) and (4.31) with $s_1 = s$ to deduce that

$$\begin{aligned}\|u_{hi}v_{hi}\|_{H^{-\frac{1}{2},b}(S_{\Delta T})} &\lesssim \|u_{hi}\|_{X_+^{-\frac{1}{2}-s+2\epsilon,b}(S_{\Delta T})}\|v_{hi}\|_{X_-^{s,b}(S_{\Delta T})}\\ &\lesssim N^{-\frac{1}{2}+2\epsilon}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}\end{aligned}$$

where we needed $-\frac{1}{2} - s + 2\epsilon \leqslant s$ which holds provided $s > -\frac{1}{4}$ and ϵ sufficiently small.

□

Remark 4.3.9. The lack of complex conjugation in the previous lemma causes no difficulties as the $X^{s,b}_\pm$ spaces we consider in this chapter are invariant with respect to complex conjugation, i.e. $u \in X^{s,b}_\pm \iff \overline{u} \in X^{s,b}_\pm$.

Remark 4.3.10. If we tried to apply the I-method to the CSD equation considered in Chapter 3, the main obstruction is the estimate for the gauge potential A that corresponds to Lemma 4.3.8. Essentially since the equation for the gauge A is first order in time, we only gain $\Delta T^{\frac{1}{2}}$ instead of the full factor of ΔT in Lemma 4.3.8. This leads to substantial difficulties later in the proof and prevents a simple application of the method used here to the CSD equation.

We now have the necessary results to control the growth of $\|I^2\phi[t]\|_{H^{r-2s}}$. Let $0 \leqslant j \leqslant n$ and define $\phi_j^{(0)}$ to be the solution to

$$\begin{aligned}\Box\phi_j^{(0)} &= m^2\phi_j^{(0)}\\ \phi_j^{(0)}(j\Delta T) = \phi(j\Delta T), &\qquad \partial_t\phi_j^{(0)}(j\Delta T) = \partial_t\phi(j\Delta T).\end{aligned} \tag{4.32}$$

Let $\Phi_j = \phi - \phi_j^{(0)}$ be the inhomogeneous component of ϕ. The inequality (4.26) together with Lemma 4.3.8 and the assumption $\Delta T = N^{\frac{4s-2\epsilon}{1+2r-4s-6\epsilon}}$, shows that for every $0 \leqslant j \leqslant n$

$$\sup_{t\in[j\Delta T,(j+1)\Delta T]}\|I^2\Phi_j[t]\|_{H_x^{r-2s}} \lesssim \Delta T\big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\big)^2. \tag{4.33}$$

We now claim that for $1 \leqslant j \leqslant n$ we have the estimate

$$\begin{aligned}&\sup_{t\in[0,(n+1)\Delta T]}\|I^2\phi_j^{(0)}[t]\|_{H_x^{r-2s}}\\ &\qquad\leqslant \sup_{t\in[0,(n+1)\Delta T]}\|I^2\phi_{j-1}^{(0)}[t]\|_{H_x^{r-2s}} + C\Delta T\big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\big)^2.\end{aligned} \tag{4.34}$$

Assume for the moment that (4.34) holds. Then after n applications of (4.34), together with the standard energy inequality for the homogeneous wave equation, we obtain

$$\begin{aligned}\sup_{t\in[0,(n+1)\Delta T]}\|I^2\phi_n^{(0)}[t]\|_{H_x^{r-2s}} &\leqslant \sup_{t\in[0,(n+1)\Delta T]}\|I^2\phi_0^{(0)}[t]\|_{H_x^{r-2s}} + Cn\Delta T\big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\big)^2\\ &\lesssim \|I^2\phi[0]\|_{H_x^{r-2s}} + Cn\Delta T\big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\big)^2.\end{aligned} \tag{4.35}$$

If we now combine (4.33) and (4.35) we see that since $n \lesssim \frac{T}{\Delta T}$

$$\begin{aligned}\sup_{t\in[n\Delta T,(n+1)\Delta T]}\|I^2\phi[t]\|_{H_x^{r-2s}} &\leqslant \sup_{t\in[n\Delta T,(n+1)\Delta T]}\|I^2\phi_n^{(0)}[t]\|_{H_x^{r-2s}} + \sup_{t\in[n\Delta T,(n+1)\Delta T]}\|I^2\Phi_n[t]\|_{H_x^{r-2s}}\\ &\lesssim \|I^2\phi[0]\|_{H_x^{r-2s}} + (n+1)\Delta T\Big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\Big)^2\\ &\lesssim \|I^2\phi[0]\|_{H_x^{r-2s}} + \Big(\|If_+\|_{L^2_x} + \|If_-\|_{L^2_x}\Big)^2\end{aligned}$$

where the implied constant is independent of N, C^*, and ΔT. Thus we obtain (4.27) as required.

It only remains to prove (4.34). We begin by observing that

$$\big(\phi_j^{(0)} - \phi_{j-1}^{(0)}\big)(j\Delta T) = \phi(j\Delta T) - \phi_{j-1}^{(0)}(j\Delta T) = \Phi_{j-1}(j\Delta T).$$

Hence the difference $\phi_j^{(0)} - \phi_{j-1}^{(0)}$ satisfies the equation

$$\begin{aligned} \Box(\phi_j^{(0)} - \phi_{j-1}^{(0)}) &= m^2(\phi_j^{(0)} - \phi_{j-1}^{(0)}) \\ \big(\phi_j^{(0)} - \phi_{j-1}^{(0)}\big)(j\Delta T) &= \Phi_{j-1}(j\Delta T), \\ \partial_t\big(\phi_j^{(0)} - \phi_{j-1}^{(0)}\big)(j\Delta T) &= \partial_t\Phi_{j-1}(j\Delta T). \end{aligned}$$

Therefore

$$\begin{aligned} &\sup_{t\in[0,(n+1)\Delta T]} \|I^2\phi_j^{(0)}[t]\|_{H_x^{r-2s}} \\ &\qquad \leqslant \sup_{t\in[0,(n+1)\Delta T]} \|I^2\phi_{j-1}^{(0)}[t]\|_{H_x^{r-2s}} + \sup_{t\in[0,(n+1)\Delta T]} \|I^2\big(\phi_j^{(0)} - \phi_{j-1}^{(0)}\big)[t]\|_{H_x^{r-2s}} \\ &\qquad \leqslant \sup_{t\in[0,(n+1)\Delta T]} \|I^2\phi_{j-1}^{(0)}[t]\|_{H_x^{r-2s}} + C\|\Phi_{j-1}[j\Delta T]\|_{H_x^{r-2s}} \end{aligned}$$

and so (4.34) follows from (4.33). Consequently, we deduce that the induction assumptions (4.22) and (4.23) hold on the larger interval $[0,(n+1)\Delta T]$ and hence Theorem 4.1.5 follows.

4.3.1 Proof of Lemma 4.3.1

Let $Q(f,g) = I(fg) - I^2 f I g$. Note that

$$\widehat{Q(f,g)}(\xi) = \int_{\mathbb{R}} \big(\rho(\xi) - \rho(\xi-\eta)^2\rho(\eta)\big)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta.$$

An application of Cauchy-Schwarz together with Lemma 4.2.1 gives

$$\begin{aligned} \Big|\int_0^{t'}\int_{\mathbb{R}} \big(I(\phi u) - I^2\phi I u\big)\overline{Iv}\,dxdt\Big| &\lesssim \|\mathbb{1}_{[0,t']}Q(\phi,u)\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}} \|Iv\|_{X_{\mp}^{0,\frac{1}{2}-\epsilon}([0,t']\times\mathbb{R})} \\ &\lesssim \|Q(\phi,u)\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}([0,t']\times\mathbb{R})} \|Iv\|_{X_{\mp}^{0,\frac{1}{2}-\epsilon}([0,t']\times\mathbb{R})} \\ &\lesssim \|Q(\phi,u)\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}(S_{\Delta T})} \|Iv\|_{X_{\mp}^{0,b}(S_{\Delta T})}. \end{aligned}$$

Thus, by the definition of $X_{\pm}^{0,b}(S_{\Delta T})$, it suffices to prove that

$$\|Q(\phi,u)\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}(S_{\Delta T})} \lesssim \Delta T^{\frac{1}{2}-2\epsilon} N^{2s-r+2\epsilon} \|I^2\phi\|_{H^{r-2s,b}} \|Iu\|_{X_{\pm}^{0,b}}. \tag{4.36}$$

where we may assume that ϕ and u are supported in $[-\Delta T, 2\Delta T]\times\mathbb{R}$. Note that since the I operator only acts on the spatial variable x, $I^2\phi$ and Iu are also supported in $[-\Delta T, 2\Delta T]\times\mathbb{R}$. Write $\phi = \phi_{low} + \phi_{hi}$ and $u = u_{low} + u_{hi}$ where, as in the proof of Lemma 4.3.8, we define $\widetilde{\phi}_{low} = \mathbb{1}_{|\xi|\leqslant \frac{N}{2}}\widetilde{\phi}$, and u_{low} is defined similarly. We consider each of the possible interactions separately.

- **Case 1** (*low-low*). In this case we simply note that $Q(\phi,u) = 0$ and hence (4.36) holds trivially.

- **Case 2** (*low-high*). We need to use the smoothing property of the bilinear form $Q(\phi, u)$ to transfer a derivative from ϕ_{low} to u_{hi}. More precisely, suppose $|\xi - \eta| < \frac{N}{2}$ and $|\eta| > \frac{N}{2}$. Then since $\rho'(z) \lesssim N^{-s}|z|^{s-1}$ for $|z| \geqslant \frac{N}{2}$ we have

$$\begin{aligned}|\rho(\xi) - \rho(\xi-\eta)^2\rho(\eta)| &= |\rho(\xi) - \rho(\eta)| \\ &\lesssim N^{-s}|\eta|^{s-1}|\xi-\eta| \\ &\approx \rho(\eta)\frac{|\xi-\eta|}{|\eta|} \lesssim \rho(\eta)\frac{|\xi-\eta|^{r-2s}}{|\eta|^{r-2s}}\end{aligned}$$

provided $r - 2s < 1$. Hence

$$|\widetilde{Q(\phi_{low}, u_{hi})}(\tau,\xi)| \lesssim \int_{\mathbb{R}^2} |\xi-\eta|^{r-2s}|\widetilde{\phi}_{low}(\tau-\lambda, \xi-\eta)||\eta|^{-r+2s}\rho(\eta)|\widetilde{u}_{hi}(\lambda,\eta)|d\lambda d\eta.$$

Thus we can move the derivative $|\nabla|^{r-2s}$ from u_{hi} to ϕ_{low}, where we let $\widehat{(|\nabla|^s f)}(\xi) = |\xi|^s\widehat{f}(\xi)$. This is the essential step which allows us to prove (4.36) in the *low-hi* case. We now apply (4.15) and Theorem 4.1.1 with $s_1 = s_2 = 0$, $s_3 = 2\epsilon$, $b_1 = \frac{1}{2} - \epsilon$, $b_2 = 0$, and $b_3 = b$ to obtain

$$\begin{aligned}\|Q(\phi_{low}, u_{hi})\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}(S_{\Delta T})} &\lesssim \left\||\nabla|^{r-2s}\phi_{low}|\nabla|^{-r+2s}Iu_{hi}\right\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}} \\ &\lesssim \left\||\nabla|^{r-2s}\phi_{low}\right\|_{L^2_{t,x}}\left\||\nabla|^{-r+2s}Iu_{hi}\right\|_{X_{\pm}^{2\epsilon,b}} \\ &\lesssim \Delta T^{\frac{1}{2}}N^{-r+2s+2\epsilon}\|I^2\phi\|_{L^\infty_t H^{r-2s}_x}\|Iu\|_{X_{\pm}^{0,b}} \\ &\lesssim \Delta T^{\frac{1}{2}}N^{-r+2s+2\epsilon}\|I^2\phi\|_{H^{r-2s,b}}\|Iu\|_{X_{\pm}^{0,b}}\end{aligned}$$

where we used the assumption supp $\phi \subset \{[-\Delta T, 2\Delta T] \times \mathbb{R}\}$.

- **Case 3** (*high-low*). In this case we do not have to transfer any regularity and we simply use the estimate $\rho(\xi) - \rho(\xi-\eta)^2\rho(\eta) \lesssim 1$. Then (4.15) together with an identical application of Theorem 4.1.1 to the *low-hi* case gives

$$\begin{aligned}\|Q(\phi_{hi}, u_{low})\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}(S_{\Delta T})} &\lesssim \left\|\phi_{hi}u_{low}\right\|_{X_{\mp}^{0,-\frac{1}{2}+\epsilon}} \\ &\lesssim \left\|\phi_{hi}\right\|_{L^2_{t,x}}\left\|u_{low}\right\|_{X_{\pm}^{2\epsilon,b}} \\ &\lesssim \Delta T^{\frac{1}{2}}N^{2s-r+2\epsilon}\|I^2\phi\|_{L^\infty_t H^{r-2s}_x}\|Iu\|_{X_{\pm}^{0,b}} \\ &\lesssim \Delta T^{\frac{1}{2}}N^{2s-r+2\epsilon}\|I^2\phi\|_{H^{r-2s,b}}\|Iu\|_{X_{\pm}^{0,b}}\end{aligned}$$

where as before, we used the assumption supp $\phi \subset \{[-\Delta T, 2\Delta T] \times \mathbb{R}\}$.

- **Case 4** (*high-high*). This is the most difficult case and we need to make full use of the generality of Theorem 4.1.1 to obtain the term $\Delta T^{\frac{1}{2}-\epsilon}$. We decompose $\phi_{hi} = \phi^+_{hi} + \phi^-_{hi}$ where

$$\widetilde{\phi}^+_{hi} = \mathbb{1}_{\{\tau\xi<0\}}\widetilde{\phi}_{hi}$$

is the restriction of $\widetilde{\phi}_{hi}$ to the second and fourth quadrants of $\mathbb{R}^{1+1}$. Note that $\|\phi^\pm\|_{X^{s,b}_\pm} \lesssim \|\phi\|_{H^{s,b}}$. Assume that we have $\pm = +$, $\mp = -$ in (4.36), it will be clear that the proof will also apply to the $\pm = -$, $\mp = +$ case.

- **Case 4a** (*high-high* $+$). As in the *high-low* case we start by discarding the smoothing multiplier Q. We now apply Theorem 4.1.1 with $s_1 = -s + 2\epsilon$, $s_2 = s$, $s_3 = 0$, $b_1 = b_2 = \frac{1}{4}$, and $b_3 = \frac{1}{2} - \epsilon$ to obtain

$$\begin{aligned}
\|Q(\phi^+_{hi}, u_{hi})\|_{X_-^{0,-\frac{1}{2}+\epsilon}(S_{\Delta T})} &\lesssim \|\phi^+_{hi} u_{hi}\|_{X_-^{0,-\frac{1}{2}+\epsilon}} \\
&\lesssim \|\phi^+_{hi}\|_{X_+^{-s+2\epsilon,\frac{1}{4}}} \|u_{hi}\|_{X_+^{s,\frac{1}{4}}} \\
&\lesssim N^{2s-r+2\epsilon} \|I^2\phi\|_{H^{r-2s,\frac{1}{4}}} \|Iu\|_{X_+^{0,\frac{1}{4}}} \\
&\lesssim \Delta T^{\frac{1}{2}-\epsilon} N^{2s-r+2\epsilon} \|I^2\phi\|_{H^{r-2s,b}} \|Iu\|_{X_+^{0,b}}
\end{aligned}$$

where we needed $-s < r$, $\epsilon > 0$ sufficiently small, and in the final line we used the assumption that ϕ, u, are compactly supported in the interval $[-\Delta T, 2\Delta T]$ together with Lemma 4.2.1 and Lemma 4.2.4.

- **Case 4b** (*high-high* $-$). Here we first apply Lemma 4.2.1, discard the multiplier Q, and then apply Theorem 4.1.1 with $s_1 = 0$, $s_2 = -s + \epsilon$, $s_3 = s$, $b_1 = b_2 = \frac{1}{4}$, and $b_3 = \frac{1}{2} + \epsilon$ to obtain

$$\begin{aligned}
\|Q(\phi^-_{hi}, u_{hi})\|_{X_-^{0,-\frac{1}{2}+\epsilon}(S_{\Delta T})} &\lesssim \Delta T^{\frac{1}{4}-\epsilon} \|\phi^-_{hi} u_{hi}\|_{X_-^{0,-\frac{1}{4}}} \\
&\lesssim \Delta T^{\frac{1}{4}-\epsilon} \|\phi^-_{hi}\|_{X_-^{-s+\epsilon,\frac{1}{4}}} \|u_{hi}\|_{X_+^{s,b}} \\
&\lesssim \Delta T^{\frac{1}{4}-\epsilon} N^{2s-r+\epsilon} \|I^2\phi\|_{H^{r-2s,\frac{1}{4}}} \|Iu\|_{X_+^{0,b}} \\
&\lesssim \Delta T^{\frac{1}{2}-2\epsilon} N^{2s-r+\epsilon} \|I^2\phi\|_{H^{r-2s,b}} \|Iu\|_{X_+^{0,b}}
\end{aligned}$$

where, as previously, we used the assumption on the support of ϕ in the last line.

4.3.2 Proof of Lemma 4.3.4

Lemma 4.3.4 follows by a standard fixed point argument using Lemma 4.2.3, Lemma 4.2.5, and the estimates

$$\|I(uv)\|_{X_\pm^{0,b-1}(S_{\Delta T})} \lesssim \Big(\Delta T^{\frac{1}{2}+r-2s-3\epsilon} + N^{-r+2s+2\epsilon}\Big) \|I^2 u\|_{H^{r-2s,b}(S_{\Delta T})} \|Iv\|_{X_\mp^{0,b}(S_{\Delta T})} \tag{4.37}$$

and

$$\|I^2(uv)\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \lesssim \Big(\Delta T^{1-\epsilon} + N^{-\frac{1}{2}+2\epsilon}\Big) \|Iu\|_{X_+^{0,b}(S_{\Delta T})} \|Iv\|_{X_-^{0,b}(S_{\Delta T})}. \tag{4.38}$$

See for instance [82].

We start by proving (4.37). As in the proof of Lemma 4.3.1, we decompose $u = u_{low} + u_{hi}$ and $v = v_{low} + v_{hi}$.

- **Case 1** (*low-low*). We split $u_{low} = u^+_{low} + u^-_{low}$ where we use the same notation as in

Subsection 4.3.1, Case 4. Observe that an application of Theorem 4.1.1 gives

$$\int_{\mathbb{R}^2} \Pi_{j=1}^3 \psi_j dxdt \lesssim \|\psi_1\|_{X_\pm^{0,\epsilon}} \|\psi_2\|_{X_\pm^{r-2s,\frac{1}{2}-r+2s+\frac{\epsilon}{2}}} \|\psi_3\|_{X_\mp^{0,\frac{1}{2}-\epsilon}} \tag{4.39}$$

provided that $0 < r-2s < \frac{1}{2}$ and $\epsilon > 0$ is sufficiently small. Hence, using Lemma 4.2.1 together with two applications of (4.39) we see that

$$\begin{aligned}\|I(u_{low}v_{low})\|_{X_\pm^{0,b-1}(S_{\Delta T})} &\lesssim \Delta T^{\frac{1}{2}-2\epsilon} \|u_{low}^{\pm} v_{low}\|_{X_\pm^{0,-\epsilon}(S_{\Delta T})} + \|u_{low}^{\mp} v_{low}\|_{X_\pm^{0,b-1}(S_{\Delta T})} \\ &\lesssim \Delta T^{\frac{1}{2}-2\epsilon} \|u_{low}^{\pm}\|_{X_\pm^{r-2s,\frac{1}{2}-r+2s+\frac{\epsilon}{2}}(S_{\Delta T})} \|v_{low}\|_{X_\mp^{0,\frac{1}{2}+\epsilon}(S_{\Delta T})} \\ &\qquad + \|u_{low}^{\mp}\|_{X_\mp^{r-2s,\frac{1}{2}-r+2s+\frac{\epsilon}{2}}(S_{\Delta T})} \|v_{low}\|_{X_\mp^{0,\epsilon}(S_{\Delta T})} \\ &\lesssim \Delta T^{\frac{1}{2}+r-2s-3\epsilon} \|I^2 u\|_{H^{r-2s,b}(S_{\Delta T})} \|Iv\|_{X_\pm^{0,b}(S_{\Delta T})}.\end{aligned}$$

- **Case 2** (*low-high*). Note that Corollary 4.1.3 implies that

$$\|\psi\varphi\|_{X_\pm^{0,b-1}} \lesssim \|\psi\|_{H^{s_1,b}} \|\psi\|_{X_\mp^{s_2,b}} \tag{4.40}$$

provided

$$s_1 > 0, \qquad s_2 > -\frac{1}{2} + \epsilon, \qquad s_1 + s_2 > \epsilon.$$

We now apply (4.40) with $s_1 = r-2s$, $s_2 = 2s-r+2\epsilon$ to get

$$\begin{aligned}\|I(u_{low}v_{hi})\|_{X_\pm^{0,b-1}(S_{\Delta T})} &\lesssim \|u_{low}\|_{H^{r-2s,b}(S_{\Delta T})} \|v_{hi}\|_{X_\mp^{2s-r+2\epsilon,b}(S_{\Delta T})} \\ &\lesssim N^{2s-r+2\epsilon} \|I^2 u\|_{H^{r-2s,b}(S_{\Delta T})} \|Iv\|_{X_\mp^{0,b}(S_{\Delta T})}.\end{aligned}$$

- **Case 3** (*high-low*). An application of (4.40) with $s_1 = 2\epsilon$, $s_2 = 0$ gives

$$\begin{aligned}\|I(u_{hi}v_{low})\|_{X_\pm^{0,b-1}(S_{\Delta T})} &\lesssim \|u_{hi}\|_{H^{2\epsilon,b}(S_{\Delta T})} \|v_{low}\|_{X_\mp^{0,b}(S_{\Delta T})} \\ &\lesssim N^{2s-r+2\epsilon} \|I^2 u\|_{H^{r-2s,b}(S_{\Delta T})} \|Iv\|_{X_\mp^{0,b}(S_{\Delta T})}.\end{aligned}$$

- **Case 4** (*high-high*). We apply (4.40) with $s_1 = r$, $s_2 = -r+2\epsilon$ and observe that

$$\begin{aligned}\|I(u_{hi}v_{hi})\|_{X_\pm^{0,b-1}(S_{\Delta T})} &\lesssim \|u_{hi}\|_{H^{r,b}(S_{\Delta T})} \|v_{hi}\|_{X_\mp^{-r+2\epsilon,b}(S_{\Delta T})} \\ &\lesssim N^{2s-r+2\epsilon} \|I^2 u\|_{H^{r-2s,b}(S_{\Delta T})} \|Iv\|_{X_\mp^{0,b}(S_{\Delta T})}\end{aligned}$$

where we used the assumption $r > -s$ together with (4.15).

We now prove prove (4.38). We again break $u = u_{low} + u_{hi}$ and $v = v_{low} + v_{hi}$ and consider each of the possible interactions separately.

- **Case 1** (*low-low*). Corollary 4.1.3 together with the assumption $r - 2s < \frac{1}{2}$ gives

$$\begin{aligned} \|I^2(u_{low}v_{low})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} &\lesssim \|u_{low}v_{low}\|_{H^{-\frac{1}{2},b-1}(S_{\Delta T})} \\ &\lesssim \|u_{low}\|_{X_+^{0,\epsilon}(S_{\Delta T})}\|v_{low}\|_{X_-^{0,\epsilon}(S_{\Delta T})} \\ &\lesssim \Delta T^{1-2\epsilon}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}. \end{aligned}$$

- **Case 2** (*low-high*). The remaining cases use the estimate

$$\|\psi\varphi\|_{H^{-\frac{1}{2},b-1}} \lesssim \|\psi\|_{X_+^{s_1,b}}\|\varphi\|_{X_-^{s_2,b}} \tag{4.41}$$

which, by Corollary 4.1.3, holds provided

$$s_1 > -\frac{1}{2}, \qquad s_2 > -\frac{1}{2}, \qquad s_1 + s_2 > -\frac{1}{2} + \epsilon.$$

The *low-high* case now follows by taking $s_1 = 0$, $s_2 = -\frac{1}{2} + 2\epsilon$ and observing that

$$\begin{aligned} \|I^2(u_{low}v_{hi})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} &\lesssim \|u_{low}v_{hi}\|_{H^{-\frac{1}{2},b-1}(S_{\Delta T})} \\ &\lesssim \|u_{low}\|_{X_+^{0,b}(S_{\Delta T})}\|v_{hi}\|_{X_-^{-\frac{1}{2}+2\epsilon,b}(S_{\Delta T})} \\ &\lesssim N^{-\frac{1}{2}+2\epsilon}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}. \end{aligned}$$

- **Case 3** (*high-low*). We let $s_1 = -\frac{1}{2} + 2\epsilon$, $s_2 = 0$ in (4.41) and use an identical argument to the previous case.

- **Case 4** (*high-high*). As before, we use (4.41) with $s_1 = -\frac{1}{2} + 2\epsilon - s$ and $s_2 = s$ and apply a similar argument to the above cases.

4.4 Bilinear Estimates

In this section we prove Theorem 4.1.1. To help simplify the proof, we start by introducing some notation. Let $m : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$ and consider the inequality

$$\Big| \int_\Gamma m(\tau,\xi)\Pi_{j=1}^3 f_j(\tau_j,\xi_j)d\sigma(\tau,\xi)\Big| \lesssim \Pi_{j=1}^3\|f_j\|_{L^2_{\tau,\xi}} \tag{4.42}$$

where $\tau, \xi \in \mathbb{R}^3$, $\Gamma = \{\xi_1 + \xi_2 + \xi_3 = 0, \quad \tau_1 + \tau_2 + \tau_3 = 0\}$, and $d\sigma$ is the surface measure on the hypersurface Γ. Without loss of generality, we may assume $f_j \geqslant 0$ as we are using L^2 norms on the right hand side of (4.42). Note that the $X^{s,b}$ estimate contained in Theorem 4.1.1 can be written in the form (4.42) after applying Plancherel and relabeling.

Following Tao in [78], for a multiplier m, we use the notation $\|m\|_{[3,\mathbb{R}\times\mathbb{R}]}$ to denote the optimal constant in (4.42). This norm $\|\cdot\|_{[3,\mathbb{R}\times\mathbb{R}]}$ was studied in detail in [78]. We recall the following elementary properties. Firstly, if $m_1 \leqslant m_2$ then it is easy to see that $\|m_1\|_{[3,\mathbb{R}\times\mathbb{R}]} \leqslant \|m_2\|_{[3,\mathbb{R}\times\mathbb{R}]}$. Secondly, via Cauchy-Schwarz, for $j,k \in \{1,2,3\}$, $j \neq k$, we have the characteristic function estimate

$$\|\mathbb{1}_A(\tau_j,\xi_j)\mathbb{1}_B(\tau_k,\xi_k)\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \sup_{(\tau,\xi)\in\mathbb{R}^2} \big|\{(\lambda,\eta) \in A \ : \ (\tau-\lambda,\xi-\eta) \in B\}\big|^{\frac{1}{2}} \tag{4.43}$$

where $A, B \subset \mathbb{R}^2$, and $|\Omega|$ denotes the measure of the set $\Omega \subset \mathbb{R}^2$. We refer the reader to [78] for a proof as well a number of other properties of the norm $\| \cdot \|_{[3,\mathbb{R}\times\mathbb{R}]}$.

Let

$$\lambda_1 = \tau_1 \pm \xi_1, \qquad \lambda_2 = \tau_2 \pm \xi_2, \qquad \lambda_3 = \tau_3 \mp \xi_3.$$

Note that if $(\tau, \xi) \in \Gamma$, then

$$\lambda_1 + \lambda_2 + \lambda_3 = \pm 2\xi_3. \tag{4.44}$$

Let $N_j, L_j \in 2^{\mathbb{N}}$, $j = 1, 2, 3$, be dyadic numbers. Our aim is to decompose the ξ_j and λ_j variables dyadically, and reduce the problem of estimating $\|m\|_{[3,\mathbb{R}\times\mathbb{R}]}$ to trying to bound the frequency localised version

$$\Big\| m(\tau,\xi) \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\, |\lambda_j| \approx L_j\}} \Big\|_{[3,\mathbb{R}\times\mathbb{R}]}$$

together with computing a dyadic summation. Note that if we restrict $|\xi_j| \approx N_j$, then since $\xi_1 + \xi_2 + \xi_3 = 0$ we must have $N_{max} \approx N_{med}$ where $N_{max} = \max\{N_1, N_2, N_3\}$, we define N_{med} and N_{min} similarly. If we also restrict $|\lambda_j| \approx L_j$, then (4.44) implies that $L_{max} \approx \max\{L_{med}, N_3\}$. Hence

$$1 \approx \sum_{N_{max}\approx N_{med}} \sum_{L_{max}\approx \max\{N_3, L_{med}\}} \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\, |\lambda_j|\approx L_j\}}.$$

Combining these observations with results from [78] leads to the following.

Lemma 4.4.1.

$$\|m\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \sup_N \sum_{N_{max}\approx N_{med}\approx N} \sum_{L_{max}\approx \max\{N_3, L_{med}\}} \Big\| m(\tau,\xi) \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\, |\lambda_j|\approx L_j\}} \Big\|_{[3,\mathbb{R}\times\mathbb{R}]}.$$

Proof. The inequality follows from the triangle inequality together with [78, Lemma 3.11]. Alternatively, we can just compute by hand. For ease of notation, let $a_{N_1} = \|f_1 \mathbb{1}_{|\xi_1|\approx N_1}\|_{L^2}$, $b_{N_2} = \|f_2 \mathbb{1}_{|\xi_2|\approx N_2}\|_{L^2}$, $c_{N_3} = \|f_3 \mathbb{1}_{|\xi_3|\approx N_3}\|_{L^2}$, and $A_{N_1,N_2,N_3} = \big\| m(\tau,\xi) \Pi_{j=1}^3 \mathbb{1}_{|\xi_j|\approx N_j} \big\|_{[3,\mathbb{R}\times\mathbb{R}]}$. Then since ξ_j lie on the surface Γ, we have $\xi_1 + \xi_2 + \xi_3 = 0$ and so

$$\begin{aligned}
\int_\Gamma m(\tau,\xi) \Pi_{j=1}^3 f_j(\tau_j,\xi_j) d\sigma(\tau,\xi) &= \sum_{N_{max}\approx N_{med}} \sum_{N_{min}\leqslant N_{med}} \int_\Gamma m(\tau,\xi) \Pi_{j=1}^3 f_j(\tau_j,\xi_j) \mathbb{1}_{|\xi_j|\approx N_j} d\sigma(\tau,\xi) \\
&\leqslant \sum_{N_{max}\approx N_{med}} \sum_{N_{min}\leqslant N_{med}} a_{N_1} b_{N_2} c_{N_3} A_{N_1,N_2,N_3}.
\end{aligned}$$

Without loss of generality we may assume that $N_1 \geqslant N_2 \geqslant N_3$ and so $N_1 \approx N_2$. For simplicity we also assume that $N_1 = N_2$ as the general case $N_1 \approx N_2$ is essentially the same. Then

$$\begin{aligned}
\int_\Gamma m(\tau,\xi) \Pi_{j=1}^3 f_j(\tau_j,\xi_j) d\sigma(\tau,\xi) &\leqslant \sum_{N_1} a_{N_1} b_{N_1} \sum_{N_3\leqslant N_1} c_{N_3} A_{N_1,N_1,N_3} \\
&\lesssim \Big(\sup_{N_3} c_{N_3}\Big) \Big(\sup_{N_1} \sum_{N_3\leqslant N_1} A_{N_1,N_1,N_3}\Big) \sum_{N_1} a_{N_1} b_{N_1} \\
&\lesssim \Big(\sup_{N_1} \sum_{N_3\leqslant N_1} A_{N_1,N_1,N_3}\Big) \Pi_{j=1}^3 \|f_j\|_{L^2}.
\end{aligned}$$

Thus we have

$$\|m\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \sup_N \sum_{N_{max}\approx N_{med}\approx N} \sum_{N_{min}\leqslant N_{med}} \left\| m(\tau,\xi)\Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}.$$

To decompose the λ_j variables follows an similar argument. We omit the details.

□

We now come to the proof of Theorem 4.1.1. To begin with, by taking the Fourier transform and relabeling, the required estimate (4.6) is equivalent to showing

$$\Big| \int_\Gamma \mathfrak{m}(\tau,\xi)\Pi_{j=1}^3 f_j(\tau_j,\xi_j)d\sigma(\tau,\xi)\Big| \lesssim \Pi_{j=1}^3 \|f_j\|_{L^2_{\tau,\xi}} \tag{4.45}$$

where

$$\mathfrak{m}(\tau,\xi) = \frac{\langle\xi_1\rangle^{-s_1}\langle\xi_2\rangle^{-s_2}\langle\xi_3\rangle^{-s_3}}{\langle\tau_1\pm\xi_1\rangle^{b_1}\langle\tau_2\pm\xi_2\rangle^{b_2}\langle\tau_3\mp\xi_3\rangle^{b_3}}.$$

Note that Theorem 4.1.1 follows from the estimate $\|\mathfrak{m}\|_{[3,\mathbb{R}\times\mathbb{R}]} < \infty$. Now since

$$\left\|\mathfrak{m}\,\Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\,|\lambda_j|\approx L_j\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} \approx \left\|\Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\,|\lambda_j|\approx L_j\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\Pi_{j=1}^3 N_j^{-s_j}L_j^{-b_j},$$

an application of Lemma 4.4.1 shows that is suffices to estimate, for every $N\in 2^{\mathbb{N}}$,

$$\sum_{N_{max}\approx N_{med}\approx N} \sum_{L_{max}\approx \max\{L_{med},N_3\}} \left(\Pi_{j=1}^3 N_j^{-s_j}L_j^{-b_j}\right)\left\|\Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\,|\lambda_j|\approx L_j\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}. \tag{4.46}$$

The first step to estimate this sum is the following estimate on the size of the frequency localised multiplier.

Lemma 4.4.2.

$$\left\|\Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\,|\lambda_j|\approx L_j\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim \min\left\{N_{min}^{\frac12}L_{min}^{\frac12},\ L_1^{\frac12}L_3^{\frac12},\ L_2^{\frac12}L_3^{\frac12}\right\}$$

Proof. Let $I = \left\|\Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\,|\lambda_j|\approx L_j\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}$. If we let $A = \mathbb{1}_{\{|\lambda_j|\approx L_j,\,|\xi_j|\approx N_j\}}$ and $B = \mathbb{1}_{\{|\lambda_k|\approx L_k,\,|\xi_k|\approx N_k\}}$ in (4.43), then an application of Fubini gives

$$\begin{aligned}
I &\lesssim \left\|\mathbb{1}_{\{|\xi_j|\approx N_j,\,|\lambda_j|\approx L_j\}}\mathbb{1}_{\{|\xi_k|\approx N_k,\,|\lambda_k|\approx L_k\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]}\\
&\lesssim \sup_{\lambda,\xi\in\mathbb{R}} \left|\left\{|\lambda_j|\approx L_j \;:\; |\lambda-\lambda_j|\approx L_k\right\}\right|^{\frac12}\left|\left\{|\xi_j|\approx N_j \;:\; |\xi-\xi_j|\approx N_k\right\}\right|^{\frac12}\\
&\lesssim \min\{L_j^{\frac12},L_k^{\frac12}\}\min\{N_j^{\frac12},N_k^{\frac12}\}
\end{aligned}$$

and hence $I\lesssim L_{min}^{\frac12}N_{min}^{\frac12}$. On the other hand, another application of (4.43) together with a change of variables gives

$$\begin{aligned}
I \lesssim \left\|\mathbb{1}_{\{|\lambda_1|\approx L_1\}}\mathbb{1}_{\{|\lambda_3|\approx L_3\}}\right\|_{[3,\mathbb{R}\times\mathbb{R}]} &\lesssim \sup_{\tau,\xi\in\mathbb{R}}\left|\left\{|\tau_1\pm\xi_1|\approx L_1 \;:\; |\tau\mp\xi-(\tau_1\mp\xi_1)|\approx L_3\right\}\right|^{\frac12}\\
&\lesssim L_1^{\frac12}L_3^{\frac12}.
\end{aligned}$$

A similar argument gives $I\lesssim L_2^{\frac12}L_3^{\frac12}$ and hence lemma follows. □

We are now ready to preform the computations needed to estimate the dyadic summation

(4.46). We split this into two parts, by computing the inner summation and then the outer summation. We note the following estimate

$$\sum_{a\leqslant N\leqslant b} N^{\delta} \approx \begin{cases} a^{\delta} & \delta<0 \\ \log(b) & \delta=0 \\ b^{\delta} & \delta>0 \end{cases}$$

which we use repeatedly. Moreover, we have $\log(r) \lesssim r^{\epsilon}$ for any $\epsilon > 0$ and $r \geqslant 1$.

Lemma 4.4.3. *Let $b_j + b_k > 0$ and $b_1 + b_2 + b_3 > \frac{1}{2}$. Then for any sufficiently small $\epsilon > 0$*

$$\sum_{L_{max}\approx \max\{L_{med},N_3\}} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \big\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\, |\lambda_j|\approx L_j\}} \big\|_{[3,\mathbb{R}\times\mathbb{R}]}$$
$$\lesssim N_3^{\epsilon}\Big(N_3^{\frac{1}{2}-b_1-b_2-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_{min}} N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+}\Big).$$

Proof. We split into the cases $L_{med} \leqslant N_3$ and $L_{med} \geqslant N_3$.

- **Case 1 ($L_{med} \leqslant N_3$).** Since the the righthand side of Lemma 4.4.2 does not behave symmetrically with respect to the sizes of the L_j, we need to decompose further into $L_{max} = L_3$ and $L_{max} \neq L_3$.

- **Case 1a ($L_{med} \leqslant N_3$ and $L_{max} \neq L_3$).** We have by Lemma 4.4.2

$$\big\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\, |\lambda_j|\approx L_j\}} \big\|_{[3,\mathbb{R}\times\mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} \min\{N_{min}^{\frac{1}{2}}, L_{med}^{\frac{1}{2}}\}.$$

Since the righthand side is symmetric under permutations of $\{1,2,3\}$, we may assume $L_1 \geqslant L_2 \geqslant L_3$. Then for any $\epsilon > 0$

$$\begin{aligned}\sum_{L_{max}\approx N_3 \gtrsim L_{med}} & L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \big\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j|\approx N_j,\, |\lambda_j|\approx L_j\}} \big\|_{[3,\mathbb{R}\times\mathbb{R}]} \\ &\lesssim N_3^{-b_1} \sum_{L_2\leqslant N_3} L_2^{-b_2} \min\{N_{min}^{\frac{1}{2}}, L_2^{\frac{1}{2}}\} \sum_{L_3\leqslant L_2} L_3^{\frac{1}{2}-b_3} \\ &\lesssim N_3^{-b_1} \sum_{L_2\leqslant N_3} L_2^{(\frac{1}{2}-b_3)_+ - b_2} \log(L_2) \min\{N_{min}^{\frac{1}{2}}, L_2^{\frac{1}{2}}\} \\ &\lesssim N_3^{-b_1+\frac{\epsilon}{2}} \sum_{L_2\leqslant N_{min}} L_2^{(\frac{1}{2}-b_3)_+ + \frac{1}{2}-b_2} \\ &\quad + N_{min}^{\frac{1}{2}} N_3^{-b_1+\frac{\epsilon}{2}} \sum_{N_{min}\leqslant L_2\leqslant N_3} L_2^{(\frac{1}{2}-b_3)_+ - b_2} \end{aligned} \tag{4.47}$$

Now for the first sum in (4.47) we have

$$\begin{aligned} N_3^{-b_1} \sum_{L_2\leqslant N_{min}} L_2^{(\frac{1}{2}-b_3)_+ + \frac{1}{2}-b_2} &\lesssim N_{min}^{\left((\frac{1}{2}-b_3)_+ + \frac{1}{2}-b_2\right)_+} N_3^{-b_1} \log(N_{min}) \\ &\lesssim N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} N_3^{-b_{min}+\frac{\epsilon}{2}}. \end{aligned}$$

For the second sum we first consider the case $(\frac{1}{2} - b_3)_+ - b_2 > 0$. Then

$$N_{min}^{\frac{1}{2}} N_3^{-b_1} \sum_{N_{min} \leqslant L_2 \leqslant N_3} L_2^{(\frac{1}{2}-b_3)_+ - b_2} \lesssim N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_3)_+ - b_1 - b_2}$$
$$\lesssim N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_{max})_+ - b_{med} - b_{min}}$$

On the other hand if $(\frac{1}{2} - b_3)_+ - b_2 \leqslant 0$ we get

$$N_{min}^{\frac{1}{2}} N_3^{-b_1} \sum_{N_{min} \leqslant L_2 \leqslant N_3} L_2^{(\frac{1}{2}-b_3)_+ - b_2} \lesssim N_{min}^{\frac{1}{2} - b_2 + (\frac{1}{2}-b_3)_+} N_3^{-b_1} \log(N_3)$$
$$\lesssim N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} N_3^{-b_{min} + \frac{\epsilon}{2}}.$$

Together with (4.47) this then gives

$$\sum_{L_{max} \approx N_3 \gtrsim L_{med}} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \left\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j| \approx N_j, \, |\lambda_j| \approx L_j\}} \right\|_{[3, \mathbb{R}\times\mathbb{R}]}$$
$$\lesssim N_3^{\epsilon} \Big(N_3^{(\frac{1}{2}-b_{max})_+ - b_{med} - b_{min}} N_{min}^{\frac{1}{2}} + N_3^{-b_{min}} N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} \Big)$$
$$\lesssim N_3^{\epsilon} \Big(N_3^{\frac{1}{2}-b_1-b_2-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_{min}} N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} \Big)$$

where we used the inequality

$$N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_{max})_+ - b_{med} - b_{min}} \leqslant N_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}-b_1-b_2-b_3} + N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} N_3^{-b_{min}}. \qquad (4.48)$$

which is trivial if $b_{max} < \frac{1}{2}$. On the other hand, if $b_{max} \geqslant \frac{1}{2}$, then (4.48) follows by noting that since $b_j + b_k > 0$ we have $b_{med} > 0$ and so

$$N_{min}^{\frac{1}{2}} N_3^{-b_{med}-b_{min}} \leqslant N_{min}^{\frac{1}{2}-b_{med}} N_3^{-b_{min}} \leqslant N_{min}^{(\frac{1}{2}-b_{med})_+} N_3^{-b_{min}}$$

as required.

- **Case 1b ($L_{med} \leqslant N_3$ and $L_{max} = L_3$).** Lemma 4.4.2 together with the assumption $L_{max} = L_3$ gives

$$\left\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j| \approx N_j, \, |\lambda_j| \approx L_j\}} \right\|_{[3, \mathbb{R}\times\mathbb{R}]} \lesssim L_{min}^{\frac{1}{2}} N_{min}^{\frac{1}{2}}.$$

Suppose $L_1 \leqslant L_2$. Then

$$\sum_{L_{max} \approx N_3 \gtrsim L_{med}} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \left\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j| \approx N_j, \, |\lambda_j| \approx L_j\}} \right\|_{[3, \mathbb{R}\times\mathbb{R}]} \qquad (4.49)$$
$$\lesssim N_{min}^{\frac{1}{2}} N_3^{-b_3} \sum_{L_2 \leqslant N_3} L_2^{-b_2} \sum_{L_1 \leqslant L_2} L_1^{\frac{1}{2}-b_1}$$
$$\lesssim N_{min}^{\frac{1}{2}} N_3^{-b_3} \sum_{L_2 \leqslant N_3} L_2^{(\frac{1}{2}-b_1)_+ - b_2} \log(L_2)$$
$$\lesssim N_{min}^{\frac{1}{2}} N_3^{((\frac{1}{2}-b_1)_+ - b_2)_+ - b_3 + \epsilon}$$

for any $\epsilon > 0$. If we have

$$N_{min}^{\frac{1}{2}} N_3^{((\frac{1}{2}-b_1)_+ - b_2)_+ - b_3} \leqslant N_3^{\frac{1}{2}-b_1-b_2-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_{min}} N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} \tag{4.50}$$

then we get

$$(4.49) \lesssim N_3^{\epsilon}\Big(N_3^{\frac{1}{2}-b_1-b_2-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_{min}} N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+}\Big)$$

as required. The case $L_1 \geqslant L_2$ follows an identical argument and so it remains to show (4.50). To this end note that if $(\frac{1}{2} - b_1)_+ - b_2 < 0$ then we simply have

$$N_{min}^{\frac{1}{2}} N_3^{((\frac{1}{2}-b_1)_+ - b_2)_+ - b_3} = N_{min}^{\frac{1}{2}} N_3^{-b_3}.$$

On the other hand, if $(\frac{1}{2} - b_1)_+ - b_2 \geqslant 0$, then by using (4.48) we have

$$\begin{aligned} N_{min}^{\frac{1}{2}} N_3^{((\frac{1}{2}-b_1)_+ - b_2)_+ - b_3} &= N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_1)_+ - b_2 - b_3} \\ &\leqslant N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_{max})_+ - b_{med} - b_{min}} \\ &\leqslant N_{min}^{\frac{1}{2}} N_3^{\frac{1}{2}-b_1-b_2-b_3} + N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+} N_3^{-b_{min}} \end{aligned}$$

and so we obtain (4.50).

- **Case 2 ($L_{med} \geqslant N_3$).** In this case we have $L_{max} \approx L_{med}$ and by Lemma 4.4.2

$$\big\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j| \approx N_j,\, |\lambda_j| \approx L_j\}} \big\|_{[3, \mathbb{R}\times\mathbb{R}]} \lesssim N_{min}^{\frac{1}{2}} L_{min}^{\frac{1}{2}}.$$

Suppose $L_1 \geqslant L_2 \geqslant L_3$. Then

$$\begin{aligned} \sum_{L_{max} \approx L_{med} \gtrsim N_3} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} N_{min}^{\frac{1}{2}} L_{min}^{\frac{1}{2}} &\approx N_{min}^{\frac{1}{2}} \sum_{L_2 \gtrsim N_3} L_2^{-b_1-b_2} \sum_{L_3 \leqslant L_2} L_3^{\frac{1}{2}-b_3} \\ &\lesssim N_{min}^{\frac{1}{2}} \sum_{L_2 \gtrsim N_3} L_2^{(\frac{1}{2}-b_3)_+ - b_1 - b_2} \log(L_2) \\ &\lesssim N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_3)_+ - b_1 - b_2 + \epsilon} \\ &\lesssim N_{min}^{\frac{1}{2}} N_3^{(\frac{1}{2}-b_{max})_+ - b_{med} - b_{min} + \epsilon} \end{aligned}$$

provided $b_1 + b_2 + b_3 > \frac{1}{2}$, $b_j + b_k > 0$, and we choose $\epsilon > 0$ sufficiently small. Since this argument also holds for all other size combinations of the L_j, we get from (4.48)

$$\begin{aligned} &\sum_{L_{max} \approx L_{med} \gtrsim N_3} L_1^{-b_1} L_2^{-b_2} L_3^{-b_3} \big\| \Pi_{j=1}^3 \mathbb{1}_{\{|\xi_j| \approx N_j,\, |\lambda_j| \approx L_j\}} \big\|_{[3, \mathbb{R}\times\mathbb{R}]} \\ &\qquad \lesssim N_3^{\epsilon}\Big(N_3^{\frac{1}{2}-b_1-b_2-b_3} N_{min}^{\frac{1}{2}} + N_3^{-b_{min}} N_{min}^{(\frac{1}{2}-b_{max})_+ + (\frac{1}{2}-b_{med})_+}\Big) \end{aligned}$$

and so lemma follows.

□

We now come to the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. By Lemma 4.4.1 and Lemma 4.4.3 it suffices to estimate the sum

$$\sup_N \sum_{N_{max} \approx N_{med} \approx N} \left(\Pi_{j=1}^3 N_j^{-s_j} \right) N_{min}^{\alpha} N_3^{-\beta}$$

for the pairs

$$(\alpha, \beta) \in \left\{ \left(\frac{1}{2},\ b_1 + b_2 + b_3 - \frac{1}{2} - \epsilon \right),\ \left(\frac{1}{2},\ b_3 - \epsilon \right),\ \left(\left(\frac{1}{2} - b_{max} \right)_+ + \left(\frac{1}{2} - b_{med} \right)_+,\ b_{min} - \epsilon \right) \right\}$$

where $\epsilon > 0$ may be taken arbitrarily small. Let $s_1' = s_1$, $s_2' = s_2$, and $s_3' = s_3 + \beta$. Then we have to show

$$\sup_N \sum_{N_{max} \approx N_{med} \approx N} \left(\Pi_{j=1}^3 N_j^{-s_j'} \right) N_{min}^{\alpha} < \infty.$$

Since this summation is symmetric with respect to the N_j, we may assume $N_1 \leqslant N_2 \leqslant N_3$. Then

$$\sum_{N_{max} \approx N_{med} \approx N} \left(\Pi_{j=1}^3 N_j^{-s_j'} \right) N_{min}^{\alpha} \lesssim N^{-s_2' - s_3'} \sum_{N_1 \leqslant N} N_1^{-s_1' + \alpha} < \infty$$

provided $s_j' + s_k' \geqslant 0$ and $s_1' + s_2' + s_3' > \alpha$. These conditions hold by the assumptions in Theorem 4.1.1 provided we choose ϵ sufficiently small.

□

4.5 Counter Examples

Here we prove that the conditions in Theorem 4.1.1 are sharp up to equality.

Proposition 4.5.1. *Assume the estimate (4.6) holds. Then we must have*

$$b_j + b_k \geqslant 0, \qquad b_1 + b_2 + b_3 \geqslant \frac{1}{2} \tag{4.51}$$

and for $k \in \{1, 2\}$

$$s_1 + s_2 \geqslant 0, \tag{4.52}$$

$$s_k + s_3 \geqslant -b_{min}, \tag{4.53}$$

$$s_k + s_3 \geqslant \frac{1}{2} - b_1 - b_2 - b_3, \tag{4.54}$$

$$s_1 + s_2 + s_3 \geqslant \frac{1}{2} - b_3, \tag{4.55}$$

$$s_1 + s_2 + s_3 \geqslant \left(\frac{1}{2} - b_{max} \right)_+ + \left(\frac{1}{2} - b_{med} \right)_+ - b_{min}. \tag{4.56}$$

Remark 4.5.2. We note that in some regions the ± structure in (4.1.1) is redundant and so the counter examples for the Wave-Sobolev spaces used in [34] and [73] would apply. In fact, the counterexamples in [34] already essentially show that we must have (4.51), (4.52), and (4.56). On the other hand, the conditions (4.53 - 4.55) reflect the ± structure and thus cannot be deduced from [34].

Proof. It suffices to find necessary conditions for the estimate (4.45). Moreover we may assume

$\pm = +$ since the case $\pm = -$ follows by a reflection in the τ_j variables. Let $\lambda \gg 1$ be some large parameter. The main idea is as follows. Assume we have sets $A, B, C \subset \mathbb{R}^{1+1}$ with

$$|A| \approx \lambda^{d_1}, \qquad |B| \approx \lambda^{d_2}, \qquad |C| \approx \lambda^{d_3}. \tag{4.57}$$

Moreover, suppose that if $(\tau_2, \xi_2) \in B$ and $(\tau_3, \xi_3) \in C$, then

$$-(\tau_2 + \tau_3, \xi_2 + \xi_3) \in A \tag{4.58}$$

and

$$\frac{\langle \xi_2 + \xi_3 \rangle^{-s_1} \langle \xi_2 \rangle^{-s_2} \langle \xi_3 \rangle^{-s_3}}{\langle \tau_2 + \tau_3 + \xi_2 + \xi_3 \rangle^{b_1} \langle \tau_2 + \xi_2 \rangle^{b_2} \langle \tau_3 - \xi_3 \rangle^{b_3}} \approx \lambda^{-\delta}. \tag{4.59}$$

Let $f_1 = \mathbb{1}_A$, $f_2 = \mathbb{1}_B$, $f_3 = \mathbb{1}_C$. Then using the conditions (4.57 - 4.59) we have

$$\begin{aligned} \int_\Gamma \mathfrak{m}(\tau, \xi) \Pi_{j=1}^3 f_j(\tau_j, \xi_j) d\sigma(\tau, \xi) &\gtrsim \lambda^{-\delta} \int_B \int_C d\tau_3 d\xi_3 d\tau_2 d\xi_2 \\ &\approx \lambda^{d_2 + d_3 - \delta}. \end{aligned}$$

Therefore, assuming that the inequality (4.45) holds, we must have

$$\lambda^{d_2 + d_3 - \delta} \lesssim |A|^{\frac{1}{2}} |B|^{\frac{1}{2}} |C|^{\frac{1}{2}} \approx \lambda^{\frac{d_1 + d_2 + d_3}{2}}.$$

By choosing λ large, we then derive the necessary condition

$$\delta + \frac{d_1 - d_2 - d_3}{2} \geqslant 0. \tag{4.60}$$

Thus it will suffice to find sets A, B, and C satisfying the conditions (4.57 - 4.59) with particular values of δ, d_1, d_2, and d_3.

- **Necessity of (4.51).** We first show that $b_j + b_k \geqslant 0$. Since the estimate (4.45) is symmetric in b_1, b_2, it suffices to consider the pairs $(j, k) \in \{(1, 2), (1, 3)\}$. For the first pair, we choose

$$B = \{|\tau + \lambda| \leqslant 1, |\xi| \leqslant 1\}, \quad C = \{|\tau| \leqslant 1, |\xi| \leqslant 1\}, \quad A = \{|\tau - \lambda| \leqslant 2, |\xi| \leqslant 2\}.$$

Then the conditions (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = b_1 + b_2$ and so from (4.60) we obtain the necessary condition $b_1 + b_2 \geqslant 0$.

On the other hand, for the pair $(1, 3)$ we choose

$$B = \{|\tau| \leqslant 1, |\xi| \leqslant 1\}, \; C = \{|\tau + \lambda| \leqslant 1, |\xi| \leqslant 1\}, \; A = \{|\tau - \lambda| \leqslant 2, |\xi| \leqslant 2\}.$$

Then as in the previous case, the conditions (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = b_1 + b_3$ and so from (4.60) we obtain the necessary condition $b_1 + b_3 \geqslant 0$.

To show the second condition in (4.51) is also necessary, we take

$$B = \{|\tau - 2\lambda| \leqslant \lambda, |\xi| \leqslant 1\}, \; C = \{|\tau - 2\lambda| \leqslant \lambda, |\xi| \leqslant 1\}, \; A = \{|\tau + 4\lambda| \leqslant 2\lambda, |\xi| \leqslant 2\}.$$

Then (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 1$ and $\delta = b_1 + b_2 + b_3$ which leads to the condition

$b_1 + b_2 + b_3 \geqslant \frac{1}{2}$.

• **Necessity of (4.52).** Let

$$B = \{|\tau - \lambda| \leqslant 1,\ |\xi + \lambda| \leqslant 1\},\ C = \{|\tau| \leqslant 1,\ |\xi| \leqslant 1\},\ A = \{|\tau + \lambda| \leqslant 2,\ |\xi - \lambda| \leqslant 2\}.$$

Then (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_2$ and so we must have (4.52).

• **Necessity of (4.53).** By symmetry we may assume $k = 1$. Suppose $b_{min} = b_1$ and choose

$$B = \{|\tau| \leqslant 1,\ |\xi| \leqslant 1\}, \qquad C = \{|\tau - \lambda| \leqslant 1,\ \ |\xi - \lambda| \leqslant 1\},$$

and

$$A = \{|\tau + \lambda| \leqslant 2,\ \ |\xi + \lambda| \leqslant 2\}.$$

Then (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_3 + b_1$ and so we must have $s_1 + s_3 + b_1 \geqslant 0$.

On the other hand, if $b_{min} = b_2$ we let

$$B = \{|\tau + 2\lambda| \leqslant 1,\ \ |\xi| \leqslant 1\}, \qquad C = \{|\tau - \lambda| \leqslant 1,\ \ |\xi - \lambda| \leqslant 1\},$$

and

$$A = \{|\tau - \lambda| \leqslant 2,\ \ |\xi + \lambda| \leqslant 2\}.$$

Then (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_3 + b_2$ and so we obtain the condition $s_1 + s_3 + b_2 \geqslant 0$.

The final case, $b_{min} = b_3$, follows by taking

$$B = \{|\tau| \leqslant 1,\ \ |\xi| \leqslant 1\}, \qquad C = \{|\tau - \lambda| \leqslant 1, |\xi + \lambda| \leqslant 1\},$$

and

$$A = \{|\tau + \lambda| \leqslant 2,\ \ |\xi - \lambda| \leqslant 2\}.$$

Again the conditions (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 0$ and $\delta = s_1 + s_3 + b_3$. Hence (4.53) is necessary.

• **Necessity of (4.54).** As in the previous case, by symmetry, we may assume $k = 1$. Let

$$B = \Big\{|\tau - \lambda| \leqslant \frac{\lambda}{4},\ \ |\xi| \leqslant 1\Big\}, \qquad C = \Big\{|\tau| \leqslant \frac{\lambda}{4},\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4}\Big\},$$

and

$$A = \Big\{|\tau + \lambda| \leqslant \frac{\lambda}{2},\ \ |\xi + \lambda| \leqslant \frac{\lambda}{2}\Big\}.$$

Then (4.57 - 4.59) hold with $d_1 = d_3 = 2$, $d_2 = 1$, and $\delta = s_1 + s_3 + b_1 + b_2 + b_3$. Thus we obtain the necessary condition (4.54).

• **Necessity of (4.55).** In this case we choose

$$B = \Big\{|\tau + \xi| \leqslant 1,\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4}\Big\}, \qquad C = \Big\{|\tau + \xi| \leqslant 1,\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4}\Big\},$$

and

$$A = \Big\{ |\tau + \xi| \leqslant 2,\ \ |\xi + 2\lambda| \leqslant \frac{\lambda}{2} \Big\}.$$

Then a simple computation shows that (4.57 - 4.59) hold with $d_1 = d_2 = d_3 = 1$, and $\delta = s_1 + s_2 + s_3 + b_3$. So we see that (4.55) is necessary.

- **Necessity of (4.56).** We break this into the 3 conditions

$$s_1 + s_2 + s_3 \geqslant 1 - b_1 - b_2 - b_3, \qquad s_1 + s_2 + s_3 \geqslant \frac{1}{2} - b_j - b_k, \qquad s_1 + s_2 + s_3 \geqslant -b_{min}. \quad (4.61)$$

For the first inequality, we take

$$B = \Big\{ |\tau| \leqslant \frac{\lambda}{4},\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4} \Big\}, \qquad C = \Big\{ |\tau| \leqslant \frac{\lambda}{4},\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4} \Big\},$$

and

$$A = \Big\{ |\tau| \leqslant \frac{\lambda}{2},\ \ |\xi + 2\lambda| \leqslant \frac{\lambda}{2} \Big\}.$$

Then we have (4.57 - 4.59) with $d_1 = d_2 = d_3 = 2$, and $\delta = s_1 + s_2 + s_3 + b_1 + b_2 + b_3$. Therefore we must have $s_1 + s_2 + s_3 \geqslant 1 - b_1 - b_2 - b_3$.

We now consider the second inequality in (4.61). By symmetry, it suffices to consider $(j,k) \in \{(1,2),(1,3)\}$. Let

$$B = \Big\{ |\tau + \xi - \lambda| \leqslant \frac{\lambda}{4},\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4} \Big\}, C = \Big\{ |\tau - \xi| \leqslant 1,\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4} \Big\},$$

and

$$A = \Big\{ |\tau + \xi + 3\lambda| \leqslant \lambda,\ \ |\xi + 2\lambda| \leqslant \frac{\lambda}{2} \Big\}.$$

Then (4.57 - 4.59) hold with $d_1 = d_2 = 2$, $d_3 = 1$, and $\delta = s_1 + s_2 + s_3 + b_1 + b_2$. Therefore we must have $s_1 + s_2 + s_3 > \frac{1}{2} - b_1 - b_2$. On the other hand, for the case $(j,k) = (1,3)$, we take

$$B = \Big\{ |\tau + \xi| \leqslant 1,\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4} \Big\}, \qquad C = \Big\{ |\tau| \leqslant \frac{\lambda}{4},\ \ |\xi - \lambda| \leqslant \frac{\lambda}{4} \Big\},$$

and

$$A = \Big\{ |\tau + \xi + \lambda| \leqslant \frac{3\lambda}{4},\ \ |\xi + 2\lambda| \leqslant \frac{\lambda}{2} \Big\}.$$

A simple computation shows that (4.57 - 4.59) are satisfied with $d_1 = d_3 = 2$, $d_2 = 1$, and $\delta = s_1 + s_2 + s_3 + b_1 + b_3$.

Finally, the third condition in (4.61) follows from the conditions (4.52) and (4.53).

□

Chapter 5

Local Well-posedness for the Space-Time Monopole Equation in Lorenz Gauge

It is known from the work of Czubak [28] that the space-time Monopole equation is locally well-posed in the Coulomb gauge for small initial data in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{4}$. Here we prove local well-posedness for arbitrary initial data in $H^s(\mathbb{R}^2)$ with $s > \frac{1}{4}$ in the Lorenz gauge.

5.1 Introduction

The space-time Monopole equation is

$$F_A = *D_A\phi \tag{5.1}$$

where F_A is the curvature of a one-form connection $A = A_\alpha dx^\alpha$, D_A is a covariant derivative of the Higgs field ϕ, and $*$ is the Hodge star operator with respect to the Minkowski metric diag(-1, 1, 1) on $\mathbb{R}^{1+2}$. The components of the connection $A = A_\alpha dx^\alpha$, and the Higgs field ϕ, are maps from $\mathbb{R}^{1+2}$ into $\mathfrak{g}$

$$A_\alpha : \mathbb{R}^{1+2} \to \mathfrak{g}, \qquad \phi : \mathbb{R}^{1+2} \to \mathfrak{g},$$

where $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\cdot, \cdot]$. For simplicity we will always assume $\mathfrak{g}$ is the Lie algebra of a matrix Lie group such as $SO(d)$ or $SU(d)$. The curvature F_A of the connection A, and the covariant derivative $D_A\phi$ of the Higgs field ϕ, are given by

$$F_A = \frac{1}{2}\big(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]\big) dx^\alpha \wedge dx^\beta, \qquad D_A\phi = \big(\partial_\alpha\phi + [A_\alpha, \phi]\big)dx^\alpha.$$

The space-time Monopole equation is an example of a non-abelian gauge field theory and can be derived by dimensional reduction from the anti-selfdual Yang-Mills equations, see for instance [29] or [65]. It was first introduced by Ward in [86] as a hyperbolic analog of the Bogomolny equations, or magnetic monopole equations, which describe a point source of magnetic charge. The space-time Monopole equation is an example of a completely integrable system and has an equivalent formulation as a Lax pair. The Lax pair formulation of (5.1), together with the inverse scattering transform, was used by Dai-Terng-Uhlenbeck in [29] to prove global existence and uniqueness up to a gauge transform from small initial data in $W^{2,1}(\mathbb{R}^2)$. The survey [29] also contained a number of other interesting results related to the space-time Monopole equation.

In the current article we study the local well-posedness of the initial value problem for the space-time Monopole equation from rough initial data in $H^s(\mathbb{R}^2)$. We can think of the equation (5.1) as a system which is roughly of the form[1]

$$\Box u = |\nabla|^{-1} B(\partial u, \partial u) \tag{5.2}$$

where B is some bilinear form. It is well known since the seminal paper of Klainerman-Machedon [50], that to prove well-posedness results close to scaling for nonlinear wave equations of the form (5.2), the bilinear form B must satisfy certain cancelation properties known as null structure (at least in low dimensions $d \leqslant 4$). Consequently, the local behavior of the space-time Monopole equation depends crucially on the presence of null structure.

The space-time Monopole equation (5.1) is gauge invariant. More precisely if (A, ϕ) is a solution to (5.1) then so is $(A_g, \phi_g) = (gAg^{-1} + gdg^{-1}, g\phi g^{-1})$ where the gauge transform $g : \mathbb{R}^{1+2} \to G$ is smooth map into the Lie group G. Note that if we choose $g(0)$ to be the identity in G, then we have the existence of two different solutions (A, ϕ) and (A_g, ϕ_g) with the same initial data. Thus to obtain a wellposed problem we need to specify a choice of gauge. Traditionally, for nonlinear hyperbolic systems with a gauge freedom such as Maxwell-Klein-Gordon or Maxwell-Dirac, the gauge was chosen to satisfy the Coulomb condition $\partial^j A_j = 0$, but more recently null structure has been discovered in the Lorenz gauge as well [32, 74]. In the Coulomb gauge, the system (5.1) can be written as a nonlinear system of wave equations for (A_1, A_2, ϕ) coupled with a nonlinear elliptic equation for A_0. The advantage of this gauge is that usually the estimates for the elliptic component A_0 are quite favorable. Recently[2] Czubak [28], showed that the space-time Monopole equations in the Coulomb gauge are locally well-posed for small initial data in H^s with $s > \frac{1}{4}$. The small data assumption is an artifact of the choice of the Coulomb gauge, as the existence of a global Coulomb gauge requires a smallness condition.

In the current chapter we instead consider the Lorenz gauge condition

$$\partial_\alpha A^\alpha = 0.$$

With this choice of gauge the space-time Monopole equations can be written as a purely hyperbolic system and the small data assumption is not needed. Additionally our proof is substantially shorter as we do not have to combine elliptic estimates with hyperbolic estimates, which can often be technically very inconvenient. Our main result is the following.

Theorem 5.1.1. *Assume $s > \frac{1}{4}$ and $\phi_0, a \in H^s(\mathbb{R}^2)$. Then there exists*

$$T = T(\|\phi_0\|_{H^s(\mathbb{R}^2)}, \|a\|_{H^s(\mathbb{R}^2)}) > 0$$

[1] The exact formulation depends on the choice of gauge, see below.
[2] Though the result was obtain earlier in Czubak's PhD thesis [27].

such that the space-time Monopole equation (5.1) coupled with the Lorenz gauge condition

$$\partial^\alpha A_\alpha = 0$$

has a solution $(\phi, A) \in C([-T,T], H^s(\mathbb{R}^2))$ *with* $(\phi(0), A(0)) = (\phi_0, a)$. *Moreover the solution is unique in some subspace of* $C([-T,T], H^s(\mathbb{R}^2))$*, the solution map depends continuously on the initial data, and any additional regularity persists in time*[3].

Remark 5.1.2. The space-time Monopole equation is invariant under the scaling $\lambda A(\lambda t, \lambda x)$, $\lambda\phi(\lambda t, \lambda x)$. Thus (5.1) is L^2 critical and so ideally we would like to prove local well-posedness for $s \geqslant 0$. However the space-time Monopole equation is essentially a system of nonlinear wave equations, and the fact that we are working in $\mathbb{R}^{1+2}$ means that there is a gap between what scaling predicts, and the regularity possible via standard null form estimates. More precisely, consider the equation

$$\Box u = Q$$

where Q is a combination of the null forms

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v.$$

Then the scale invariant space is $H^1 \times L^2$, but standard null form estimates only give well-posedness for $(u(0), \partial_t u(0)) \in H^s \times H^{s-1}$ for $s > \frac{5}{4}$. Below $\frac{5}{4}$, it can be shown that the first iterate leaves the data space H^s, see [88]. Thus in some sense the regularity $H^{\frac{1}{4}}$ in Theorem 5.1.1 and the work of Czubak [28], is the limit for iterative methods. On the other hand the space-time Monopole has additional structure which is not used in the proof of Theorem 5.1.1. Hence it may be possible to remove the restriction $s > \frac{1}{4}$ by exploiting the structure in a different way.

5.2 Preliminaries

Recall that the Hodge star operator, $*$, is defined for $\omega \in \bigwedge^p(M)$ by

$$(*\omega)_{\lambda_{p+1}\dots\lambda_d} = \frac{1}{p!}\eta_{\lambda_1\dots\lambda_d}\omega^{\lambda_1\dots\lambda_p}$$

where (M, g) is a pseudo Riemannian manifold, η is the volume form with respect to the metric g, and the previous formula is given in some local coordinate system. If we couple the space-time Monopole equation (5.1) with the Lorenz gauge condition

$$\partial^\mu A_\mu = 0$$

[3]More precisely if $\phi_0, a \in H^r(\mathbb{R}^2)$ for some $r \geqslant s$, then we also have $(\phi, A) \in C([-T,T], H^r(\mathbb{R}^2))$ with T only depending on $\|\phi_0\|_{H^s(\mathbb{R}^2)}$ and $\|a\|_{H^s(\mathbb{R}^2)}$.

and write out the resulting system in terms of ϕ and the components A_α we obtain

$$\begin{aligned}
\partial_t\phi + \partial_1 A_2 - \partial_2 A_1 &= [A_2, A_1] + [\phi, A_0]\\
\partial_t A_0 - \partial_1 A_1 - \partial_2 A_2 &= 0\\
\partial_t A_1 - \partial_1 A_0 - \partial_2 \phi &= [A_2, \phi] + [A_1, A_0]\\
\partial_t A_2 + \partial_1 \phi - \partial_2 A_0 &= [\phi, A_1] + [A_2, A_0].
\end{aligned}$$

Define $u, v : \mathbb{R}^{1+2} \to \mathfrak{g} \times \mathfrak{g}$ by

$$u = \begin{pmatrix} A_0 + A_1 \\ \phi + A_2 \end{pmatrix} \qquad v = \begin{pmatrix} A_0 - A_1 \\ \phi - A_2 \end{pmatrix}.$$

Then since

$$[A_2, A_1] + [\phi, A_0] \pm \big([\phi, A_1] + [A_2, A_0]\big) = [\phi \pm A_2, A_0 \pm A_1]$$

and

$$[A_2, \phi] + [A_1, A_0] = \frac{1}{2}\big([A_2 - \phi, A_2 + \phi] + [A_1 - A_0, A_1 + A_0]\big)$$

we can write the Monopole equation as

$$\begin{aligned}
\partial_t u_1 - \partial_1 u_1 - \partial_2 u_2 &= \frac{1}{2}\big(u \cdot v - v \cdot u\big)\\
\partial_t u_2 + \partial_1 u_2 - \partial_2 u_1 &= [u_2, u_1]\\
\partial_t v_1 + \partial_1 v_1 + \partial_2 v_2 &= \frac{1}{2}\big(v \cdot u - u \cdot v\big)\\
\partial_t v_2 - \partial_1 v_2 + \partial_2 v_1 &= [v_2, v_1].
\end{aligned}$$

Define the matrices

$$\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and let $\alpha = (\alpha_1, \alpha_2)$. Then we can rewrite the previous equations in the more concise form

$$\begin{cases} \partial_t u - \big(\alpha \cdot \nabla\big) u = N(u, v) \\ \partial_t v + \big(\alpha \cdot \nabla\big) v = N(v, u) \end{cases} \tag{5.3}$$

where

$$N(a, b) = \begin{pmatrix} \frac{1}{2}\big(a \cdot b - b \cdot a\big) \\ \beta a \cdot a \end{pmatrix}.$$

We can now restate Theorem 5.1.1 as follows.

Theorem 5.2.1. *Assume $s > \frac{1}{4}$ and $f, g \in H^s$. Then there exists $T = T(\|f\|_{H^s}, \|g\|_{H^s}) > 0$ such that (5.3) has a solution $(u, v) \in C([-T, T], H^s)$ with $(u(0), v(0)) = (f, g)$. Moreover, the solution is unique in some subspace of $C([-T, T], H^s)$, the solution map depends continuously on the initial data, and any additional regularity persists in time*[4]*.*

Note that Theorem 5.1.1 follows immediately from Theorem 5.2.1. To prove Theorem 5.2.1

[4] More precisely if $f, g \in H^r$ for some $r \geqslant s$, then we also have $(u, v) \in C([-T, T], H^r)$ with T only depending on $\|f\|_{H^s}$ and $\|g\|_{H^s}$.

we will first diagonalise the left hand side of (5.3). Define the projections $\mathcal{P}_\pm$ to be the operator with Fourier multiplier $p_\pm(\xi) = \frac{1}{2}\Big(I \pm \frac{1}{|\xi|}\alpha\cdot\xi\Big)$ so

$$\widehat{\mathcal{P}_\pm f}(\xi) = p_\pm(\xi)\widehat{f}(\xi).$$

It is easy to see that

$$f = \mathcal{P}_+ f + \mathcal{P}_- f, \qquad \alpha\cdot\nabla = i|\nabla|\big(\mathcal{P}_+ - \mathcal{P}_-\big)$$

and $\mathcal{P}_\pm^2 = \mathcal{P}_\pm$, $\mathcal{P}_\pm\mathcal{P}_\mp = 0$. Therefore we can rewrite the above as

$$\partial_t u_\pm \mp i|\nabla|u_\pm = \mathcal{P}_\pm N(u,v)$$
$$\partial_t v_\pm \mp i|\nabla|v_\pm = \mathcal{P}_\mp N(v,u)$$

where $u_\pm = \mathcal{P}_\pm u$ and $v_\pm = \mathcal{P}_\mp v$. With this formulation we see that, for short times at least, u_+ and v_+ should have Fourier support concentrated on the forwards light cone $\{\tau - |\xi| = 0\}$, while u_- and v_- should have Fourier support concentrated on the backwards light cone $\{\tau + |\xi| = 0\}$. Thus the natural spaces to iterate in are the spaces $X^{s,b}_\pm$ defined by using the norm

$$\|\psi\|_{X^{s,b}_\pm} = \big\|\langle \tau \mp |\xi|\rangle^b \langle\xi\rangle^s \widehat{\psi}(\tau,\xi)\big\|_{L^2_{\tau,\xi}}.$$

We also let $H^{s,b}$ be the closely related Wave-Sobolev space defined by

$$\|\psi\|_{H^{s,b}} = \big\|\langle |\tau| - |\xi|\rangle^b \langle\xi\rangle^s \widehat{\psi}(\tau,\xi)\big\|_{L^2_\xi}.$$

We will iterate in the spaces $u_+, v_+ \in X^{s,b}_+$ and $u_-, v_- \in X^{s,b}_-$ for some $\frac{1}{2} < b < 1$ to be chosen later. It is well known that the proof of Theorem 5.2.1 reduces to proving the estimates

$$\|\mathcal{P}_\pm N(u,v)\|_{X^{s,b-1+\epsilon}_\pm} \lesssim \big(\|u_+\|_{X^{s,b}_+} + \|u_-\|_{X^{s,b}_-} + \|v_+\|_{X^{s,b}_+} + \|v_-\|_{X^{s,b}_-}\big)^2 \tag{5.4}$$

and

$$\|\mathcal{P}_\mp N(v,u)\|_{X^{s,b-1+\epsilon}_\pm} \lesssim \big(\|u_+\|_{X^{s,b}_+} + \|u_-\|_{X^{s,b}_-} + \|v_+\|_{X^{s,b}_+} + \|v_-\|_{X^{s,b}_-}\big)^2 \tag{5.5}$$

where $\epsilon > 0$ is some small constant depending on s and $b > \frac{1}{2}$, see for instance [71] or Section 3 in [5]. Since $\mathcal{P}_\pm$ is a bounded operator on H^s, and $\big||\tau| - |\xi|\big| \leqslant \big|\tau \pm |\xi|\big|$, we see that provided $b + \epsilon < 1$, the estimates (5.4) and (5.5) follow from

$$\|N(u,v)\|_{H^{s,b-1+\epsilon}} \lesssim \big(\|u_+\|_{X^{s,b}_+} + \|u_-\|_{X^{s,b}_-} + \|v_+\|_{X^{s,b}_+} + \|v_-\|_{X^{s,b}_-}\big)^2$$

and

$$\|N(v,u)\|_{H^{s,b-1+\epsilon}} \lesssim \big(\|u_+\|_{X^{s,b}_+} + \|u_-\|_{X^{s,b}_-} + \|v_+\|_{X^{s,b}_+} + \|v_-\|_{X^{s,b}_-}\big)^2.$$

Now recalling that $u = \mathcal{P}_+ u_+ + \mathcal{P}_- u_-$, $v = \mathcal{P}_- v_+ + \mathcal{P}_+ v_-$, and

$$N(a,b) = \begin{pmatrix} \frac{1}{2}\big(a\cdot b - b\cdot a\big) \\ \beta a\cdot a \end{pmatrix},$$

we can reduce this further to just proving the estimates

$$\|\mathcal{P}_{\pm_1}\Psi\cdot\mathcal{P}_{\pm_2}\Phi\|_{H^{s,b-1+\epsilon}}\lesssim\|\Psi\|_{X^{s,b}_{\pm_1}}\|\Phi\|_{X^{s,b}_{\mp_2}},$$

$$\|\beta\mathcal{P}_{\pm_1}\Psi\cdot\mathcal{P}_{\pm_2}\Phi\|_{H^{s,b-1+\epsilon}}\lesssim\|\Psi\|_{X^{s,b}_{\pm_1}}\|\Phi\|_{X^{s,b}_{\pm_2}},$$

and

$$\|\beta\mathcal{P}_{\pm_1}\Psi\cdot\mathcal{P}_{\pm_2}\Phi\|_{H^{s,b-1+\epsilon}}\lesssim\|\Psi\|_{X^{s,b}_{\mp_1}}\|\Phi\|_{X^{s,b}_{\mp_2}},$$

where $\pm_1$ and $\pm_2$ are independent choices of $+$ and $-$, and Ψ and Φ are functions taking values in $\mathfrak{g}\times\mathfrak{g}$. Observe that

$$\|\psi(-t,x)\|_{X^{s,b}_{\pm}}=\|\psi(t,x)\|_{X^{s,b}_{\mp}},\qquad\|\psi(t,-x)\|_{X^{s,b}_{\pm}}=\|\psi(t,x)\|_{X^{s,b}_{\pm}}.$$

Similarly

$$\|\psi(-t,x)\|_{H^{s,b}}=\|\psi(t,x)\|_{H^{s,b}},\qquad\|\psi(t,-x)\|_{H^{s,b}}=\|\psi(t,x)\|_{H^{s,b}}$$

and $\mathcal{P}_{\pm}\big(f(-\cdot)\big)(x)=\mathcal{P}_{\mp}f(-x)$. Furthermore a computation shows that $\beta\mathcal{P}_{\pm}=\mathcal{P}_{\mp}\beta$. Therefore, combining these observations, it suffices to prove

$$\|\mathcal{P}_{+}\Psi\cdot\mathcal{P}_{\pm}\Phi\|_{H^{s,b-1+\epsilon}}\lesssim\|\Psi\|_{X^{s,b}_{+}}\|\Phi\|_{X^{s,b}_{\mp}}\tag{5.6}$$

It is well known that nonlinear wave equations are only well behaved at low regularities if the nonlinear terms satisfy a null condition. The thesis of Czubak showed that the Monopole equation in the Coulomb gauge has null structure. Here we will show that the nonlinear term $\mathcal{P}_+\Psi\cdot\mathcal{P}_\pm\Phi$ also has null structure in the sense that the worst interaction for parallel waves vanishes. An easy computation shows that $p_\pm(\xi)^T=p_\pm(\xi)$ and so

$$\mathcal{P}_{+}\widehat{\Psi\cdot\mathcal{P}}_{\pm}\Phi(\xi)=\int_{\mathbb{R}^2}p_{\pm}(\eta)p_{+}(\xi-\eta)\widehat{\Psi}(\xi-\eta)\cdot\widehat{\Phi}(\eta)d\eta.$$

Thus the symbol of $\mathcal{P}_+\Psi\cdot\mathcal{P}_\pm\Phi$ is given by $p_\pm(\eta)p_+(\xi)$. The null structure is then contained in the following lemma.

Lemma 5.2.2. *We have the estimate*

$$|p_{+}(\eta)p_{\pm}(\xi)|\lesssim\theta(\xi,-\eta)$$

where $\theta(\xi,\eta)$ denotes the (positive) angle between ξ and η.

Proof. The $(+,+)$ case follows from the computation

$$\begin{aligned}4p_{+}(\eta)p_{+}(\xi)&=\left(I+\frac{1}{|\eta|}\alpha\cdot\eta\right)\left(I+\frac{1}{|\xi|}\alpha\cdot\xi\right)\\&=I+\frac{1}{|\eta||\xi|}\begin{pmatrix}\eta_1&\eta_2\\\eta_2&-\eta_1\end{pmatrix}\begin{pmatrix}\xi_1&\xi_2\\\xi_2&-\xi_1\end{pmatrix}+\Big(\frac{\eta}{|\eta|}+\frac{\xi}{|\xi|}\Big)\cdot\alpha\\&=\Big(1+\frac{\xi\cdot\eta}{|\xi||\eta|}\Big)I+\Big(\frac{\xi_2\eta_1}{|\xi||\eta|}-\frac{\xi_1\eta_2}{|\xi||\eta|}\Big)\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\Big(\frac{\xi}{|\xi|}+\frac{\eta}{|\eta|}\Big)\cdot\alpha\end{aligned}$$

together with the easy estimates $\Big(1+\frac{\xi\cdot\eta}{|\xi||\eta|}\Big)\lesssim\theta(\xi,-\eta)$, $\Big(\frac{\xi_2\eta_1}{|\xi||\eta|}-\frac{\xi_1\eta_2}{|\xi||\eta|}\Big)\lesssim\theta(\xi,-\eta)$, and

$\left(\frac{\xi}{|\xi|} + \frac{\eta}{|\eta|}\right) \lesssim \theta(\xi, -\eta)$. If we now note that $p_-(\eta) = p_+(-\eta)$ we obtain the $(+,-)$ case by replacing η with $-\eta$ in the previous computation.

□

Define $Q_\pm(\psi, \phi)$ by

$$\widehat{Q_\pm(\psi,\phi)}(\xi) = \int_{\mathbb{R}^2} \theta(\xi - \eta, \pm\eta)\widehat{\psi}(\xi - \eta)\widehat{\phi}(\eta)d\eta.$$

Then by Lemma 5.2.2 we have reduced the proof of Theorem 5.2.1 to proving

$$\|Q_\pm(\psi,\phi)\|_{H^{s,b-1+\epsilon}} \lesssim \|\psi\|_{X_+^{s,b}} \|\phi\|_{X_\pm^{s,b}}.$$

This estimate is essentially well known and follows from the work of Klainerman-Selberg [53], Foschi-Klainerman [41], using ideas from [30]. However as we could not find this inequality explicitly stated in the literature, we will include a proof in the next section. We note that the standard null form estimates for the wave equation in $\mathbb{R}^{1+2}$ were proven by Zhou [88]. The origin of these types of estimates is the seminal paper of Klainerman-Machedon [49].

5.3 Null-Form Estimates

Here we prove the following estimate.

Theorem 5.3.1. *Let $s > \frac{1}{4}$. Then there exists $b > \frac{3}{4}$ and $\epsilon > 0$ with $b + \epsilon < 1$ such that*

$$\|Q_\pm(\psi,\phi)\|_{H^{s,b-1+\epsilon}} \lesssim \|\psi\|_{X_+^{s,b}} \|\phi\|_{X_\pm^{s,b}}. \tag{5.7}$$

Note that this completes the proof of Theorem 5.2.1. To prove Theorem 5.3.1 we need to introduce some notation. Let

$$r_+ = |\xi - \eta| + |\eta| - |\xi|, \qquad r_- = |\xi| - \big||\xi - \eta| - |\eta|\big|,$$

and define the bilinear operator $S^\alpha_\pm(\psi,\phi)$ by

$$\widehat{S^\alpha_\pm(\psi,\phi)}(\xi) = \int_{\mathbb{R}^2} r^\alpha_\pm \widehat{\psi}(\xi - \eta)\widehat{\phi}(\eta)d\eta.$$

Moreover define the Fourier multipliers $|\nabla|^s$, Λ^s, and $\Omega^b_\pm$ by

$$\widehat{|\nabla|^s\psi}(\xi) = |\xi|^s\widehat{\psi}(\xi), \qquad \widehat{\Lambda^s\psi}(\xi) = \langle\xi\rangle^s\widehat{\psi}(\xi), \qquad \widehat{\Omega^b_\pm\psi}(\tau,\xi) = \langle\tau \mp |\xi|\rangle^b\widehat{\psi}(\xi).$$

Then we have the following estimate, which follows from [41] and is the analogue of Theorem 3.5 in [53] for the $X^{s,b}_\pm$ spaces.

Theorem 5.3.2. *Let $s, \alpha, s_1, s_2 \in \mathbb{R}$ and $b' > \frac{1}{2}$. Then the estimate*

$$\||\nabla|^s S^\alpha_\pm(\psi,\phi)\|_{L^2_{t,x}} \lesssim \||\nabla|^{s_1}\psi\|_{X_+^{0,b'}} \||\nabla|^{s_2}\phi\|_{X_\pm^{0,b'}} \tag{5.8}$$

holds provided

$$\begin{aligned} s+\alpha &= s_1+s_2-\frac{1}{2} \\ \alpha &\geqslant \frac{1}{4} \\ s_i &\leqslant \alpha+\frac{1}{2} \\ s_1+s_2 &\geqslant \frac{1}{2} \\ s &> \frac{-1}{2} \end{aligned}$$

and $(s_i,\alpha)\neq(\frac{3}{4},\frac{1}{4})$, $(s_1+s_2,\alpha)\neq(\frac{1}{2},\frac{1}{4})$.

Proof. The hard work is contained in the result of Foschi-Klainerman [41] where the following estimate is proven

$$\big\||\nabla|^s D_-^\alpha(e^{it|\nabla|}f\, e^{\pm it|\nabla|}g)\big\|_{L^2_{t,x}(\mathbb{R}^{1+2})} \lesssim \||\nabla|^{s_1}f\|_{L^2(\mathbb{R}^2)}\||\nabla|^{s_2}g\|_{L^2(\mathbb{R}^2)}$$

under the above conditions on the exponents s, s_1, s_2, α where $\widetilde{D_-^\alpha\psi}(\tau,\xi)=\big||\tau|-|\xi|\big|^\alpha\widetilde{\psi}$. It is easy to see that

$$D_-^\alpha(e^{it|\nabla|}f\, e^{\pm it|\nabla|}g)=S_\pm^\alpha(e^{it|\nabla|}f, e^{\pm it|\nabla|}g).$$

Now since the operator $S_\pm^\alpha$ only acts on the ξ variable, the expression on the lefthand side of (5.8) is invariant under multiplication by the modulations $e^{it\tau_0}$. Therefore an application of the Transference principle[5] completes the proof. □

Theorem 5.3.1 will now follow by using an argument from [30].

Proof of Theorem 5.3.1. We begin by noting that since the left and righthand sides of (5.7) only depend on the size of the Fourier transform of ψ and ϕ, we can use the triangle inequality to write

$$(1+|\xi|^2)^{\frac{a}{2}} \lesssim (1+|\xi-\eta|^2)^{\frac{a}{2}}(1+|\eta|^2)^{\frac{a}{2}}$$

and hence reduce to the case $\frac{1}{4}<s<\frac{1}{2}$. Choose $\epsilon>0$ and $b>\frac{3}{4}$ so that $s=b-\frac{1}{2}+\epsilon$. Note that $b+\epsilon<1$.

We now deal with the low frequency case. Assume the product $\psi\phi$ has Fourier support contained in the set $\{|\xi|<1\}$. Let $\rho\in C_0^\infty(\mathbb{R}^2)$ with $\widehat{\rho}=1$ for $|\xi|<1$. Then

$$\psi\phi=\rho*(\psi\phi) \tag{5.9}$$

where the convolution is with respect to the x variable. By discarding the smoothing multiplier $\langle|\tau|-|\xi|\rangle^{b-1+\epsilon}$ and the null form $Q_\pm$, and using the assumption $\langle\xi\rangle\lesssim 1$ together with (5.9),

[5]See for instance Lemma 2.9 in [79].

we have

$$\begin{aligned}\|Q_\pm(\psi,\phi)\|_{H^{s,b-1+\epsilon}} &\lesssim \|\rho*(\psi\phi)\|_{L^2_{t,x}}\\ &\lesssim \|\psi\phi\|_{L^2_tL^1_x}\\ &\lesssim \|\psi\|_{L^\infty_tL^2_x}\|\phi\|_{L^2_{t,x}}\\ &\lesssim \|\psi\|_{X^{s,b}_+}\|\phi\|_{X^{s,b}_\pm}.\end{aligned}$$

Therefore the low frequency case follows.

Since we may now assume $|\xi|>1$, it suffices to prove

$$\||\nabla|^sQ_\pm(\psi,\phi)\|_{H^{0,b-1+\epsilon}} \lesssim \|\psi\|_{X^{s,b}_+}\|\phi\|_{X^{s,b}_\pm}. \tag{5.10}$$

To this end we will need the following estimate on the symbol of $Q_\pm$,

$$\theta^2(\xi-\eta,\eta)\approx\frac{|\xi-\eta|+|\eta|}{|\xi-\eta||\eta|}r_+, \qquad \theta^2(\xi-\eta,-\eta)\approx\frac{|\xi|}{|\xi-\eta||\eta|}r_-. \tag{5.11}$$

Note that these estimates gives us a smoothing derivative $|\nabla|^{-1}$ at the cost of a hyperbolic derivative $r_\pm$. To prove (5.11) note that

$$\begin{aligned}(|\eta|+|\xi-\eta|-|\xi|)(|\eta|+|\xi-\eta|+|\xi|) &= 2\big(|\eta||\xi-\eta|-\eta\cdot(\xi-\eta)\big)\\ &= 2|\eta||\xi-\eta|\big(1-\cos(\theta(\xi-\eta,\eta))\big)\end{aligned}$$

which proves the first estimate. For the second we have

$$\begin{aligned}\big(|\xi|+\big||\xi-\eta|-|\eta|\big|\big)\big(|\xi|-\big||\xi-\eta|-|\eta|\big|\big) &= 2\big(|\xi-\eta||\eta|+\eta\cdot(\xi-\eta)\big)\\ &= 2|\xi-\eta||\eta|\big(1-\cos(\theta(\xi-\eta,-\eta))\big)\end{aligned}$$

and since $|\xi|\geqslant\big||\xi-\eta|-|\eta|\big|$ we have $|\xi|\approx|\xi|+\big||\xi-\eta|-|\eta|\big|$ which gives the second estimate. We also need the following estimate[6]

$$r_\pm\leqslant\big||\tau|-|\xi|\big|+\big||\tau-\lambda-|\xi-\eta|\big|+\big|\lambda\mp|\eta|\big|$$

which leads to

$$r_\pm\lesssim\langle|\tau|-|\xi|\rangle\langle\tau-\lambda-|\xi-\eta|\rangle\langle\lambda\mp|\eta|\rangle. \tag{5.12}$$

We are now ready to prove the + case. Combining the estimates for θ and r_+ and assuming $|\eta|>|\xi-\eta|$ (as we may be symmetry) we have

$$\theta(\xi-\eta,\eta)\lesssim\frac{r_+^{\frac12}}{|\xi-\eta|^{\frac12}}\lesssim\frac{r_+^{b-\frac12+\epsilon}}{|\xi-\eta|^{\frac12}}\langle|\tau|-|\xi|\rangle^{1-b-\epsilon}\langle\tau-\lambda-|\xi-\eta|\rangle^{1-b-\epsilon}\langle\lambda-|\eta|\rangle^{1-b-\epsilon}.$$

[6]The + case follows by writing

$$r_+=(\tau-|\xi|)-(\tau-\lambda-|\xi-\eta|)-(\lambda-|\eta|).$$

If $\tau>0$ the triangle inequality gives inequality while if $\tau<0$ then the term $(\tau-|\xi|)$ is less than zero and so can be discarded. The − case follows from a similar computation after we note that

$$r_-\leqslant\begin{cases}|\xi|+|\xi-\eta|-|\eta|\\ |\xi|-|\xi-\eta|+|\eta|.\end{cases}$$

and so

$$\||\nabla|^s Q_+(\psi,\phi)\|_{H^{0,b-1+\epsilon}} \lesssim \left\||\nabla|^s S_+^{b-\frac{1}{2}+\epsilon}\big(\Omega_+^{1-b-\epsilon}\psi, |\nabla|^{-\frac{1}{2}}\Omega_+^{1-b-\epsilon}\phi\big)\right\|_{L^2_{t,x}}.$$

Therefore the + case follows from Theorem 5.3.2 by taking[7] $b' = 2b-1+\epsilon$, $s_1 = s$, $s_2 = s+\frac{1}{2}$, and $\alpha = b - \frac{1}{2} + \epsilon$. It is easy to check that the required conditions on α, s_1, s_2, s, and b' are satisfied. To obtain the $-$ case we note that (5.11) and (5.12) give the estimate

$$\theta(\xi-\eta,-\eta) \lesssim \frac{|\xi|^{\frac{1}{2}} r_-^{\frac{1}{2}}}{|\xi-\eta|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}} \lesssim \frac{|\xi|^{\frac{1}{2}} r_-^{b-\frac{1}{2}+\epsilon}}{|\xi-\eta|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}} \langle|\tau|-|\xi|\rangle^{1-b-\epsilon}\langle\tau-\lambda-|\xi-\eta|\rangle^{1-b-\epsilon}\langle\lambda+|\eta|\rangle^{1-b-\epsilon}.$$

Thus

$$\||\nabla|^s Q_-(\psi,\phi)\|_{H^{0,b-1+\epsilon}} \lesssim \left\||\nabla|^{s+\frac{1}{2}} S_-^{b-\frac{1}{2}+\epsilon}\big(|\nabla|^{-\frac{1}{2}}\Omega_+^{1-b-\epsilon}\psi, |\nabla|^{-\frac{1}{2}}\Omega_-^{1-b-\epsilon}\phi\big)\right\|_{L^2_{t,x}}$$

and so the required estimate follows from Theorem 5.3.2 by taking $b' = 2b-1+\epsilon$, $s_1 = s+\frac{1}{2}$, $s_2 = s+\frac{1}{2}$, and $\alpha = b-\frac{1}{2}+\epsilon$. Again it is easy to check that the required conditions are satisfied.

□

[7] This is where we require the assumption $s > \frac{1}{4}$. As to apply Theorem 5.3.2 we need $\alpha > \frac{1}{4}$ and $s+\alpha = s_1+s_2-\frac{1}{2}$ which implies $s = \alpha > \frac{1}{4}$. Note that if we could take $\alpha = 0$ then we would have local well-posedness for all $s > 0$. However, heuristically speaking, since we have to assume $\alpha > \frac{1}{4}$ we can only use the null form $Q_\pm$ to cancel half the hyperbolic derivative $\langle|\tau|-|\xi|\rangle^{-\frac{1}{2}}$. See the related discussion after Theorem 3.3 in [30].

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