On the Moduli Space of Parahoric Higgs Bundles

Karim Réga

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Karim Réga

Abstract

We study parahoric Higgs bundles over a smooth, projective curve. We start out by reviewing the notion of a parahoric group over a local field and associated group schemes constructed by Bruhat and Tits. We continue by reviewing recent criteria to prove the existence of good moduli spaces for an algebraic stack by Alper, Halpern-Leistner and Heinloth. The first result is an application of these criteria to anti-invariant Higgs bundles, which provide a specific example of parahoric Higgs bundles. Lastly, we review work on the moduli of gauged maps by Halpern-Leistner and Herrero and show how this can be used to prove the existence of a moduli space for parahoric Higgs bundles.

Lay Summary

One way of studying complex curves is by studying objects attached to them. An example of this is vector bundles on a curve. This assigns to every point x of the curve a vector space V_x in such a way that these spaces vary smoothly as the point x varies along the curve. We then want to study the moduli of vector bundles, how vector bundles on a fixed curve C can vary.

Another instance of this is Higgs bundles, introduced by Hitchin, which consist of a vector bundle on a curve C along with a Higgs field φ. This Higgs field assigns for each point x of the curve a linear transformation φ_x of the corresponding vector space V_x, again in such a way that this data varies smoothly as the point x varies along the curve. These objects arose out of studying the Yang-Mills equations. Besides this link to mathematical physics, there are also links to topology, representation theory and differential equations. In particular, there is a map from the moduli space of Higgs bundles, a space classifying Higgs bundles, to a space B which forms an algebraically completely integrable system. This is a dynamical system with many good properties.

The objects described above yield desired results for compact curves. Non-compact curves can be thought of as compact curves with a finite amount of distinguished points or punctures. Given a compact curve C with a collection of punctures, we should consider parabolic vector bundles. These are vector bundles on C such that for a puncture x, the vector space V_x has a filtration by vector subspaces, determined by the growth rate as we approach the puncture x. Similarly, we can consider parabolic Higgs bundles by requiring that the linear transformation φ_x respects the filtration of V_x.

There is an analogous notion of principal G-bundles and Higgs bundles for a group G such that the corrsponding results still hold. When trying to extend this to non-compact curves, parabolic bundles are not enough. This leads to the notion of parahoric bundles and Higgs bundles, which are our central objects of study.

In this thesis, we show the existence of a moduli space of parahoric Higgs bundles on a compact curve C, i.e. a space that classifies these bundles, for semisimple groups.

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Contents

1 Introduction

1.1 Higgs Bundles

Higgs Bundles were introduced by Hitchin in [26] by studying dimensional reduction of the 4d Yang-Mills equations to $\mathbb{R}^2$. Given a smooth projective curve C and a reductive group G over $\mathbb{C}$, a Higgs bundle consists of a pair of

- a G-bundle E and
- a section $\varphi \in H^0(C, \operatorname{End}(E) \otimes K)$,

where K is the canonical bundle of C. In [26] these were introduced for $SU(2)$ and this was extended to other semisimple groups in [25]. In the case of SU(2) a moduli space $\mathcal{M}$ for stable Higgs bundles was constructed using gauge-theoretic methods in [26]. By Serre duality, the dual of a first-order deformation of a G-bundle E is exactly a Higgs field $\varphi \in H^0(C, \operatorname{ad}(E) \otimes K)$ and it follows that this moduli space contains the cotangent bundle of the moduli space of stable bundles. This is an open subset and the natural symplectic structure on the cotangent bundle extends to the whole moduli space.

This symplectic structure forms part of an algebraically completely integrable system. Given generators c_i of degree d_i for G-invariant polynomials on the Lie algebra $\mathfrak{g}$, we can apply the corresponding polynomial to the Higgs field. This gives a morphism

$$h\colon \mathcal{M} \to B = \bigoplus_i H^0(C, K^{d_i}),$$

often called the Hitchin fibration. This morphism is an algebraically completely integrable system, and if we compactify the moduli space by including semistable bundles, then the morphism h is projective.

A GIT construction of this moduli space for vector bundles was done by Nitsure in [32]. Here, the Higgs field φ is valued in an arbitrary line bundle L, i.e. $\varphi \in H^0(C, \operatorname{ad}(E) \otimes L)$. It is also shown that the Hitchin fibration is proper in this case.

An independent construction was done by Simpson in [38]. This work also provides a correspondence between semistable Higgs bundles and representations of the fundamental group of the curve in the group G. This is known as non-abelian Hodge theory and extends results for bundles by Narasimhan and Seshadri in [31], where they exhibit a correspondence between semistable bundles and representations in a maximal compact subgroup $K \subset G$.

1.2 Parahoric Groups

The correspondence of Narasimhan and Seshadri was extended to noncompact curves by Mehta and Seshadri in [30]. Denote by C the completion of the curve and by D the divisor of points at infinity. The extra generators in the fundamental group of the noncompact curve correspond to loops around the punctures. After fixing the conjugacy classes of these loops in K, we get a correspondence between representations and semistable parabolic bundles in type A. A parabolic bundle is a triple consisting of

- a vector bundle E,
- a filtration
$$\{0\} \subset E_{x,1} \subset E_{x,2} \subset \dots E_{x,n_x-1} \subset E_{x,n_x} = E_x$$
of the fiber E_x for all $x \in D$ and
- a collection of decreasing numbers $\alpha_x = (\alpha_{x,i})_{i=1}^{n_x}$ between 0 and 1 for each x.

There is a natural generalisation to other groups, where the filtration at points $x \in D$ is replaced by a reduction to a parabolic subgroup. However, in the correspondence this does not account for all possible conjugacy classes of the loops at $x \in D$. Trying to account for the missing conjugacy classes naturally leads to parahoric subgroups.

These are certain subgroups of $G(\mathbb{C}((t)))$ that have similar properties to parabolic subgroups studied by Bruhat and Tits in [12] and [13]. In particular the preimage of a parabolic group under the reduction morphism $G(\mathbb{C}[[t]]) \to G(\mathbb{C})$ is a parahoric subgroup. In [13], given a parahoric group P, a group scheme $\mathcal{G}_P$ over $\operatorname{Spec} \mathbb{C}[[t]]$ is constructed such that the group of $\mathbb{C}[[t]]$-points is P. Globally, we say a group scheme $\mathcal{G}$ over a smooth projective curve C is a *parahoric Bruhat-Tits group scheme* if

- for each $x \in C$ the restriction $\mathcal{G}_{\mathcal{D}_x}$, where $\mathcal{D}_x$ is a formal neighbourhood of x, is a group scheme of the form $\mathcal{G}_P$ and
- $\mathcal{G}$ is generically reductive.

The points x such that $\mathcal{G}_x$ is not reductive are called *points of bad reduction* and we denote by $Ram(\mathcal{G})$ the reduced divisor consisting of these points. We say a parahoric Bruhat-Tits group scheme is *generically split* if

$$\mathcal{G}_{C \backslash Ram(\mathcal{G})} \cong G \times (C \backslash Ram(\mathcal{G})).$$

Such group schemes were studied in [34], where conjectures about their bundles were posed. Some of these conjectures were settled in [24], where it is shown that if $\mathcal{G}$ is generically semisimple, then the stack of $\mathcal{G}$-bundles has similar uniformisation properties to that of the stack of bundles for a semisimple group. The same paper also contains a description of the connected components of this stack and a description of the Picard group.

Generically split group schemes were considered in [7]. It is shown that given a generically split Bruhat Tits parahoric group scheme $\mathcal{G}$ on the curve C, there exists a Galois cover $\tilde{C}$ with Galois group Γ, so that $\mathcal{G}$-bundles are equivalent to Γ-equivariant bundles on $\tilde{C}$. Using a GIT construction, the existence of a moduli space for semistable $\mathcal{G}$-bundles is shown, alomg with a generalisation of the results of Mehta and Seshadri in [30] to this setting.

Our main objects of study are Higgs bundles for this group scheme. In analogy with parahoric Higgs bundles, first studied by Simpson in [37] to generalise non-abelian Hodge theory to this setting, there are two different notions of Higgs field.

- To get the cotangent bundle of the moduli space of parabolic bundles as an open subset of the moduli space of parabolic vector bundles, we need to consider Higgs fields that are nilpotent with respect to the flag at points $x \in D$, i.e.

  $$\varphi(E_{x,i}) \subset E_{x,i-1} \otimes K$$

  We call these *strong parabolic Higgs bundles* and a moduli space for these was constructed by Konno in [27], which carry a natural symplectic structure.

  To define a similar notion of *strong parahoric Higgs field* we construct a subbundle of $\mathrm{ad}(\mathcal{E})$ for $\mathcal{E}$ a $\mathcal{G}$-torsor whose restrictions to the Levi subalgebra at points of bad reduction vanishes. This is analogous to being nilpotent with respect to the flag, as the bundle for the Levi subgroup of the underlying parabolic group at x is the associated graded. For generically split group schemes, the moduli stack of strong parahoric $\mathcal{G}$-bundles is studied by Baraglia Kamgarpour and Varma in [8], where the existence of a symplectic structure on the stack is shown, turning the Hitchin fibration into a completely algebraically integrable system.

- Another natural choice is to define a *weak parabolic Higgs bundle* as a parabolic vector bundle together with a Higgs field φ that respects the flag at the points $x \in D$. A moduli space was constructed by Yokogawa in [41] and the existence of a Poisson structure turning the Hitchin fibration into a completely algebraically integrable system was demonstrated by Martens and Logares in [29].

  The Higgs fields that respect the flag correspond locally to elements of the Lie algebra of the underlying parabolic group, and the group schemes $\mathcal{G}$ are built so that sections locally correspond to these parabolic groups. Thus a *weak parahoric Higgs field* is a $\mathcal{G}$-bundle $\mathcal{E}$ together with a section of $\mathrm{ad}(\mathcal{E}) \otimes K$. For generically split group schemes, moduli spaces for semistable weak parahoric

Higgs bundles are constructed by Kydonakis, Sun and Zhao in [28], along with a Poisson structure on this moduli space.

We are mainly interested in weak parahoric Higgs bundles, and will mostly refer to them as just parahoric Higgs bundles.

1.3 Intrinsic GIT

A parallel motivation for our work is to apply a collection of results about the structure of moduli stacks and their good moduli spaces, often referred to as intrinsic GIT or beyond GIT.

Good moduli spaces were introduced by Alper in [1], as an approximation of a moduli stack by an algebraic space modelled on the GIT quotient. A quasicompact and quasiseparated morphism $\pi \colon \mathfrak{X} \to X$ from an algebraic stack to an algebraic space is a *good moduli space* if

- $\mathcal{O}_X \cong \pi_* \mathcal{O}_{\mathfrak{X}}$ and
- the pushforward functor $\pi_* \colon \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(X)$ is exact.

This definition implies many good properties, in particular it reproduces the orbit-closure relation for the GIT quotient and is universal for maps to algebraic spaces.

It is often hard to verify the existence of a good moduli space directly from the definition. This is remedied by the results of Alper, Halpern-Leistner and Heinloth in [4]. The main result is that in characteristic 0, an algebraic stack of finite presentation with affine diagonal admits a separated good moduli space if and only if it is Θ-complete [1] and S-complete. These are two valuative extension criteria. In loc. cit. these were applied to show the existence of a good moduli space for semistable parahoric Higgs bundles. There is a need for semistability as the whole stack does not satisfy the criteria.

This is often the case and there is a need for finding open substacks satisfying these criteria. A general framework to accomplish this is developed by Halpern-Leistner in [18], along with additional structure on the unstable locus. This is accomplished by generalising the Harder-Narasimhan stratification for the stack of vector bundles introduced in [20]. A central role in this framework is played by the stack $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$. A morphism form Θ to the stack of vector bundles on C is equivalent to a filtered vector bundle. Using this, we can reformulate the Harder-Narasimhan filtrations in terms of distinguished maps from Θ to the stack of vector bundles. This is formalised in the definition of a Θ-stratification, which is roughly a stratification of a stack $\mathfrak{X}$ along with a distinguished morphism from

[1] This was called Θ-reductivity in loc. cit. but we opt for Θ-completeness to stress that there is no link with the property of reductivity for groups.

Θ to the stack for each point x such that the image of 1 is x. The open stratum, which could be empty, is called the *semistable locus*

One way of constructing such stratifications is using a line bundle $\mathcal{L}$ on $\mathfrak{X}$ and assigning to a morphism $f\colon \Theta \to \mathfrak{X}$, the weight of the $\mathbb{G}_m$-action on the fiber of $f^*\mathcal{L}$ over 0, divided by a norm to account for multiples of the action. [2] In the case that $\mathfrak{X}$ is a quotient stack, this is analogous to the Hilbert-Mumford criterion. This assignment is called a *numerical invariant* and a stratification can be considered by assigning to each $x \in \mathfrak{X}$ the supremum of the numerical invariant for morphism such that $f(1) \cong x$. The corresponding filtration is a filtration that attains this supremum if it exists.

Given a numerical invariant, it is shown in [18] that it defines a Θ-stratification if this numerical invariant is strictly Θ-monotone and satisfies HN-Boundedness. The latter can be interpreted as a finiteness property on the morphisms $f\colon \Theta \to \mathfrak{X}$ that maximise the numerical invariant. The former is a valuative criterion analogous to Θ-completeness and this implies Θ-completeness for the semistable locus. There is an analogous criterion of S-monotonicity which implies S-completeness for the semistable locus.

It is also shown in [4] that if a stack admits a well-ordered Θ-stratification and satisfies the existence part of the valuative criterion of properness, so does the semistable locus. If this semistable locus admits a separated good moduli space, then this implies properness of that moduli space.

To summarise, to prove the existence of a proper good moduli space, our aim is to find a suitable numerical invariant, establish HN-Boundedness, Θ- and S-monotonicity, and show that the stack satisfies the existence part of the valuative criterion of properness.

1.4 Summary of Results

Our first main result concerns anti-invariant Higgs bundles. These are a special case of parahoric Higgs bundles, studied by Zelaci in [45] and [46]. Given a curve C with an involution σ, an anti-invariant Higgs bundle is a triple consisting of

- a vector bundle E with trivial determinant,
- a Higgs field φ and
- an isomorphism $\psi\colon E^* \to \sigma^*E$

satisfying some compatibility between φ and ψ.

These are equivalent to bundles for the group scheme $\mathcal{G}$ obtained by taking the Weil restriction of SL_n to the quotient C/σ and taking σ-invariants.

[2] A similar definition was also considered by Heinloth in [23] to define a semistable locus

There is a forgetful morphism from the stack of anti-invariant Higgs bundles to the stack of Higgs bundles, so we can consider the Hitchin fibration for anti-invariant Higgs bundles.

Theorem 1.1. *The moduli stack of anti-invariant Higgs bundles admits a Θ-stratification such that the semistable locus admits a separated good moduli space such that the Hitchin fibration is proper.*

The existence of a Θ-stratification is shown by defining a numerical invariant and using criteria from [19].

We show separately that the stack of semistable Higgs bundles admits a separated good moduli space (this is known but does not appear in the literature) and that the stack of semistable anti-invariant Higgs bundles maps to this stack. The existence of a separated good moduli space can be deduced by showing that Θ- and S-completeness lift along this morphism.

Finally, properness of the Hitchin fibration can be deduced from the properness for ordinary Higgs bundles.

Our second main result concerns bundles for generically semisimple parahoric Bruhat-Tits group schemes.

Theorem 1.2. *The moduli stack of parahoric Higgs bundles admits a Θ-stratification such that the semistable locus admits a separated good moduli space such that the Hitchin fibration is proper.*

The existence of the Θ-stratification and separated good moduli space are deduced by defining a numerical invariant and showing it is strictly Θ- and S-monotone and satisfies HN-Boundedness.

To prove the monotonicity we adapt arguments from [19] to our setting. Halpern-Leistner and Herrero study the moduli stack of gauged maps, parametrising maps from C to $[X/G]$ where X is a projective-over-affine scheme with a G-action for G a reductive group. The crucial element is the construction of generalised Grassmannians that are ind-projective. By tweaking the quotient stack slightly, this applies to Higgs bundles and results from [24] yield the corresponding generalised Grassmannians.

The HN-Boundedness follows by proving the existence of Harder-Narasimhan reductions in the setting of parahoric Higgs bundles. We use the formalism of complementary polyhedra developed by Behrend in [9] to deduce this, combined with work from Wissdorf in [40] and Heinloth in [23].

Finally, the properness of the Hitchin fibration is proved analogously to the proof in [4].

2 Moduli Spaces and Stratifications

In this chapter, we will review a series of definitions and results from [1], [4] and [18] that are often collectively referred to as intrinsic GIT.

First, we review the notion of a good moduli space for an algebraic stack as introduced in [1]. This is a way to approximate a stack by an algebraic space, which is modelled on quotients obtained from GIT.

To obtain such good moduli spaces, we discuss Θ- and S-completeness which are valuative criteria for a stack introduced in [4], that are equivalent to the existence of a good moduli space in characteristic 0.

For the moduli problems of interest, the moduli stack will not satisfy the criteria. This is remedied by finding an open substack that does satisfy these criteria, analogous to the need of a semistable locus in projective GIT. The framework of Θ-stratifications to find such open substacks is defined in [18]. A Θ-stratification is analogous to the Harder-Narasimhan stratification for the moduli stack of vector bundles, and we will refer to an open stratum of this stratification as the semistable locus. We first review criteria for the existence of a Θ-stratification. Next, we discuss Θ- and S-monotonicity, which are generalisations of Θ- and S-completeness. These imply the completeness criteria for the semistable locus and can thus be used to establish the existence of a good moduli space.

Lastly we sketch an idea known as infinite-dimensional GIT to prove these monotonicity properties for the moduli stacks of interest.

2.1 Good Moduli Spaces

Good moduli spaces were introduced in [1] as an approximation of an algebraic stack by an algebraic space generalising the notion of good quotient from GIT. Suppose G is a reductive group acting on an affine scheme $\operatorname{Spec} A$ and consider the morphism $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$. Two interesting properties of this morphism are

- $\Gamma([\operatorname{Spec} A/G], \mathcal{O}_{[\operatorname{Spec} A/G]}) = A^G$ and
- the pushforward functor $\pi_*\colon \mathrm{QCoh}([\operatorname{Spec} A/G]) \to \mathrm{QCoh}(\operatorname{Spec} A^G)$ is exact.

The pushforward functor sends an A-module M with a G-action to the invariants M^G. The exactness then follows because G is reductive. These properties turn out to characterise the GIT quotient and lead to the following definition.

Definition 2.1. [1, Definition 4.1] A quasicompact and quasiseparated morphism $\pi : \mathfrak{X} \to X$ from an algebraic stack $\mathfrak{X}$ to an algebraic space X is a **good moduli space** if

1. $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathfrak{X}}$ is an isomorphism, and
2. $\pi_* \colon \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(X)$ is exact.

Example 2.2.
- As discussed above, the morphism $\pi \colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space.
- If X is a projective scheme with a G-action, equipped with a G-linearisation, then $[X^{ss}/G] \to X^{ss}//G$ is a good moduli space.
- If $\mathfrak{X}$ is a Deligne-Mumford stack of characteristic 0, a coarse moduli space for this stack is a good moduli space.

A good moduli space has many good properties.

Proposition 2.3. *[1, Theorem 4.16/6.6] Let $\pi \colon \mathfrak{X} \to X$ be a good moduli space. Then*

1. *π is surjective.*
2. *π is universally closed.*
3. *For a field k and two points $x_1, x_2 \in \mathfrak{X}(k)$, $\pi(x_1) = \pi(x_2)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$.*
4. *If $\mathfrak{X}$ is locally noetherian, so is X and π_* preserves coherence.*
5. *If $\mathfrak{X}$ is finite type, so is X.*
6. *The morphism π is universal for maps to algebraic spaces.*

2.2 Existence criteria

In [4], criteria are given for when an algebraic stack admits a good moduli space.

Both criteria are filling conditions with respect to codimension 2 points of algebraic stacks over discrete valuation rings.

For the first one, Θ-completeness, we consider the stack

$$\Theta = [\operatorname{Spec} k[t]/\mathbb{G}_m],$$

where $\mathbb{G}_m$ acts with weight 1 on t. This stack has one open point corresponding to the orbit of 1, which we will often denote by 1, and likewise a closed point corresponding to the orbit of 0.

The criterion considers the base change

$$\Theta_R = \Theta \times \operatorname{Spec} R = [\operatorname{Spec} R[t]/\mathbb{G}_m]$$

where R is a DVR. This has a codimension 2 point 0 defined by the vanishing of t and a uniformizer of R.

Definition 2.4. [4, Definition 3.10] A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of locally noetherian algebraic stacks is **Θ-complete** if for every DVR R, any commutative diagram

$$\begin{array}{ccc} \Theta_R \backslash 0 & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \Theta_R & \longrightarrow & \mathfrak{Y} \end{array}$$

can be filled in uniquely with a morphism $\Theta_R \to \mathfrak{X}$.

Some useful properties of this definition are summarised in the following proposition.

Proposition 2.5. *[4, Proposition 3.17/3.21]*

1. *An affine morphism of locally noetherian algebraic stacks is Θ-complete.*
2. *If G is a reductive group acting on a locally noetherian affine scheme X, then $[X/G]$ is Θ-complete.*
3. *If $\mathfrak{X} \to \mathfrak{Y}$ has affine diagonal, it suffices to show that a filling exists after an arbitrary extension of DVRs to prove Θ-completeness.*

For the second one, S-completeness, the stack under consideration is

$$\Phi_R = [(\operatorname{Spec} R[x,y]_{1,-1}/(xy-\pi))/\mathbb{G}_m]$$

where π is a uniformiser of R and $\mathbb{G}_m$ acts with weights 1 and -1 on x and y respectively. This again has a codimension 2 point defined by the vanishing of x and y.

Definition 2.6. [4, Definition 3.38] A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of locally noetherian algebraic stacks is **S-complete** if for every DVR R, any commutative diagram

$$\begin{array}{ccc} \Phi_R \backslash 0 & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \Phi_R & \longrightarrow & \mathfrak{Y} \end{array}$$

there exists a unique map $\Phi_R \to \mathfrak{X}$ filling in the diagram.

The loci where $x \neq 0$ and $y \neq 0$ are copies of $\operatorname{Spec} R$. When both are non-zero, the result is a copy of $\operatorname{Spec} K$. We thus have that

$$\Phi_R/0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R,$$

so this can be interpreted as a weakened separatedness condition.

This definition satisfies similar properties to those of Θ-completeness.

Proposition 2.7. *[4, Proposition 3.41/3.44]*

1. *An affine morphism of locally noetherian algebraic stacks is S-complete.*

2. *If G is a reductive group acting on a locally noetherian affine scheme X, then $[X/G]$ is S-complete.*

3. *If $\mathfrak{X} \to \mathfrak{Y}$ has affine diagonal, it suffices to show that a filling exists after an arbitrary extension of DVRs to prove S-completeness.*

Interestingly, S-completeness also turns out to be a characterisation of reductivity.

Proposition 2.8. *[4, Proposition 3.47] If G is an algebraic group over a field k, G is reductive if and only if BG is S-complete. In particular, a closed point of an S-complete locally noetherian algebraic stack with affine stabilisers has a reductive stabiliser.*

These two criteria suffice to establish the existence of a good moduli space.

Theorem 2.9. *[4, Theorem A] Let $\mathfrak{X}$ be an algebraic stack of finite type with affine diagonal. Then $\mathfrak{X}$ admits a separated good moduli space if and only if it is Θ-complete and S-complete.*

The idea for the construction of this moduli space is that if $\mathfrak{X}$ is S-complete, any closed point has a reductive stabiliser. Around a closed point x [3, Theorem 1.1] implies there exists an etale morphism $f\colon \mathcal{W} \to \mathfrak{X}$ such that

- $\mathcal{W} \cong [\operatorname{Spec} A/\operatorname{GL}_N]$, and
- f induces an isomorphism of stabiliser groups at x.

So etale-locally good moduli spaces exist. The two criteria allow to glue these together to get a global good moduli space.

Example 2.10.
- Any separated Deligne-Mumford stack is Θ-complete and S-complete.

- By [2, Proposition 6.9.30], for any projective scheme X, the stack $\mathrm{Coh}(X)$ is Θ- and S-complete. However the existence theorem does not apply because this stack is not quasicompact.
- In [5] Θ and S-completeness are established for the stack $\mathrm{Bun}^{ss}_{r,d}(C)$ of semistable bundles of fixed rank r and degree d over a projective curve C.

2.3 Θ-stratifications

The previous chapter discussed criteria for the existence of a good moduli space. However, similar to GIT only producing a quotient of the semistable locus, often the whole moduli stack will not satisfy these criteria leading to the necessity of an open substack that does satisfy them. A framework to find such open substacks is developed in [18] and naturally leads to a generalisation of the Harder-Narasimhan stratification of the stack of vector bundles on a curve. Throughout this section $\mathfrak{X}$ is a quasiseparated algebraic stack locally of finite presentation with affine automorphism groups and separated inertia.

Morphisms from Θ to $\mathfrak{X}$ play a central role in this framework.

Definition 2.11. [18, Definition 1.1.11] The **stack of filtered objects** is

$$\mathrm{Filt}(\mathfrak{X}) = \mathrm{Map}(\Theta, \mathfrak{X}).$$

This is an algebraic stack by [18, Proposition 1.1.2] and is equipped with a natural evaluation morphism

$$\mathrm{ev}_1 \colon \mathrm{Filt}(\mathfrak{X}) \to \mathfrak{X} \colon f \mapsto f(1).$$

Example 2.12. 1. Let $\mathfrak{X} = [X/G]$ be the quotient stack of a projective scheme X by a reductive group G. Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, denote by $X^{\lambda,+}$ the scheme of $\mathbb{G}_m$-equivariant maps from $\mathbb{A}^1$ to X and P_λ the associated parabolic subgroup. It is shown in [18, Theorem 1.4.7] that

$$\mathrm{Filt}(\mathfrak{X}) \cong \coprod_{\lambda \in \mathrm{Hom}(\mathbb{G}_m, T)/W} X^{\lambda,+}/P_\lambda \tag{2.13}$$

where T is a maximal torus of G and W is the Weyl group.

2. Let $\mathfrak{X} = \mathrm{Bun}_G(C)$ be the stack of G-bundles on a projective curve C for a reductive group G. By [23, Lemma 1.13], a morphism $f \colon \Theta \to \mathrm{Bun}_G(C)$ corresponds to a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ up to conjugation and a reduction of structure group of the bundle $f(1)$ to the associated parabolic subgroup P_λ. The bundle $f(0)$ is the bundle obtained by extension of structure group to the associated Levi subgroup L_λ.

In the case of vector bundles, the previous example demonstrates that a morphism from Θ corresponds to a filtered vector bundle. The Harder-Narasimhan stratification is based on filtrations that are maximally destabilising, the Harder-Narasimhan filtrations. The next definition captures this idea.

Definition 2.14. [18, Definition 2.1.1/2.1.2] Let $\mathcal{X}$ be an algebraic stack over a locally Noetherian algebraic space S.

1. A **Θ-stratum** in $\mathcal{X}$ consists of a union of connected components $\mathcal{Z}^+ \subset \mathrm{Filt}(\mathcal{X})$ such that $\mathrm{ev}_1 \colon \mathcal{Z}^+ \to \mathcal{X}$ is a closed immersion.

2. A **Θ-stratification** indexed by a totally ordered set Γ is a cover of $\mathcal{X}$ by open substacks $\mathcal{X}_{\leq c}$ for $c \in \Gamma$ such that $\mathcal{X}_{\leq c} \subset \mathcal{X}_{\leq c'}$ for $c \leq c'$, along with a Θ-stratum $\mathcal{Z}_c^+$ in $\mathcal{X}$ such that the complement of the image in $\mathcal{X}_{\leq c}$ is
$$\mathcal{X}_{<c} := \bigcup_{c'<c} \mathcal{X}_{\leq c'}.$$
We require that for all $x \in \mathcal{X}$, the set $\{c \in \Gamma \,|\, x \in \mathcal{X}_{\leq c}\}$ has a minimal element. We assume that Γ has a minimal element $0 \in \Gamma$.

3. We say that a Θ-stratification is **well-ordered** if for any $x \in \mathcal{X}$ the totally ordered set $\{c \in \Gamma \,|\, \mathrm{ev}_1(\mathcal{Z}_c^+) \cap \overline{\{x\}} \neq \emptyset\}$ is well-ordered.

The open substack $\mathcal{X}^{ss} := \mathcal{X}_{\leq 0}$ is called the **semistable locus**. For any $x \in \mathcal{X}/\mathcal{X}^{ss}$, it follows from the definition that there is a unique c such that $x \in \mathrm{ev}_1(\mathcal{Z}_c^+)$. This gives rise to a canonical map $f : \Theta \to \mathcal{X}$ with $f(1) \cong x$, which we call an **HN filtration**.

One way of constructing Θ-stratifications is through numerical invariants. To define these we need a norm on graded points of $\mathfrak{X}$. In practice, these will always be pulled back along a map $\mathfrak{X} \to BG$ for a reductive group G, so we do not define these in general. If G admits a split maximal torus T, a norm on graded points of BG is the same as a Weyl invariant positive definite quadratic norm b on $X_*(T)_{\mathbb{R}}$. Filtrations give rise to cocharacters $\lambda \colon \mathbb{G}_m \to T \subset G$ and the norm on graded points assigns the number $b(\lambda)$ to a filtration.

If $\mathcal{L}$ is a $\mathbb{G}_m$-equivariant line bundle on $\mathbb{A}^1$, then its global sections are given by $H^0(\mathbb{A}^1, \mathcal{L}) = k[x] \cdot e$, where e is a section unique up to a non-zero scalar and $\mathbb{G}_m$ acts on it by $t \cdot e = t^d e$. We call the integer d the **weight** of $\mathcal{L}$ and denote it by $\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L})$.

Given a norm on graded points and a line bundle on $\mathfrak{X}$, we can use this weight to define functions μ following [18, Definition 4.1.14] and [23, Definition 1.2].

Definition 2.15. Let $\mathcal{L}$ be a line bundle on $\mathfrak{X}$ and b a norm on graded points of $\mathfrak{X}$. The **numerical invariant** associated to $\mathcal{L}$ and b is

$$\mu\colon |Filt(\mathfrak{X})| \to \mathbb{R}\colon f \mapsto \frac{\mathrm{wt}_{\mathbb{G}_m}(f^*\mathcal{L})}{b(f)}.$$

Given a numerical invariant, we can define a *stability function* on $\mathfrak{X}$ through

$$M^\mu(p) = \sup\{\mu(f) \mid f \in |\mathrm{Filt}(\mathfrak{X})| \text{ such that } f(1) = p\}.$$

This leads to the candidate substacks

$$|\mathfrak{X}|_{\leq c} = \{p \in \mathfrak{X} \mid M^\mu(p) = c\}$$

and Θ-strata

$$|\mathrm{Filt}(\mathfrak{X})|_{\leq c} = \{f \in |\mathrm{Filt}(\mathfrak{X}) \mid \mu(f) = M^\mu(f(1)) \leq c\}.$$

One criterion for when this defines a Θ-stratification is the following.

Theorem 2.16. *[18, Theorem 4.5.1] The numerical invariant μ defines a Θ-stratification if and only if it satisfies the following two properties:*

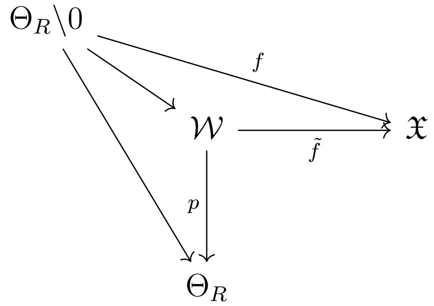
1. ***Simplified HN-Specialisation Property**: For any essentially of finite type DVR R with fraction field K and residue field κ, and any map $\xi\colon \operatorname{Spec} R \to \mathfrak{X}$ whose generic point is unstable and a HN filtration f_K, one has*
$$\mu(f_K) \leq M^\mu(\xi_{\operatorname{Spec}\kappa}),$$
and when equality holds there is an extension of DVRs $R \subset R'$ with fraction fields $K \subset K'$ such that $f_K|_{K'}$ extends to $\operatorname{Spec} R'$.

2. ***HN-Boundedness**: For any map $\xi\colon T \to \mathfrak{X}$ from a finite type affine scheme, there exists a quasicompact substack $\mathcal{X}' \subset \mathfrak{X}$ such that for all finite type points $p \in T(k)$, and filtrations f of p, there is another filtration f' with $\mu(f) \leq \mu(f')$ and $f'(0) \in \mathcal{X}'$.*

Remark 2.17. The theorem as stated in [18] contains some additional hypotheses on μ. However, our numerical invariant is of the form described in Definition 4.1.14 of loc. cit. and by Example 4.5.2 of loc. cit. it follows that these additional hypotheses are automatically satisfied.

The simplified HN-specialisation property can be replaced with another property that is similar to Θ-completeness.

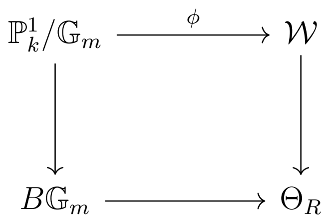
Definition 2.18. [18, Definition 5.2.1] We say μ is Θ-monotone if for any DVR R with fraction field K and residue field k and any map $f\colon \Theta_R\backslash 0 \to \mathfrak{X}$ such that $\mu(f_K) \geq 0$, there exists a commutative diagram



such that

1. $\mathcal{W}$ is a reduced and irreducible algebraic stack, $\tilde{f}$ has quasi-finite relative inertia, p is proper and representable by Deligne-Mumford stacks and is an isomorphism over $\Theta_R\backslash 0$.

2. For any commutative diagram of the form



   where the morphism from $B\mathbb{G}_m$ is a positive multiple of $0/\mathbb{G}_m \to \Theta_R$ and $\mathbb{P}^1_k/\mathbb{G}_m \to B\mathbb{G}_m \times_{\Theta_R} \mathcal{W}$ is finite, we have

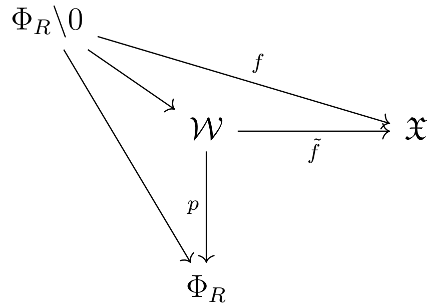
$$\mu(\tilde{f} \circ \phi_0) \leq \mu(\tilde{f} \circ \phi_\infty). \tag{2.19}$$

It is **strictly monotone** if the inequality is strict.

Proposition 2.20. *[18, Theorem 5.2.3] If μ is strictly Θ-monotone, it satisfies the simplified HN-Specialisation property. In particular, it defines a Θ-stratification if and only if it satisfies HN-Boundedness.*

It is not a coincidence that Θ-montonicity is similar to Θ-completeness and together with another property, it can also be used to prove the existence of a good moduli space for the semistable locus.

Definition 2.21. We say μ is S-monotone if for any DVR R with fraction field K and residue field k and any map $f\colon \Phi_R\backslash 0 \to \mathfrak{X}$ such that $\mu(f_K) \geq 0$, there exists a commutative diagram



satisfying the properties of Definition 2.18.

Remark 2.22. A similar criterion was defined in [23, Definition 2.3] to prove separatedness of the stable locus of a moduli stack.

Proposition 2.23. *[18, Theorem 5.5.8] Suppose μ defines a Θ-stratification and is strictly Θ-monotone (resp. S-monotone). Then the semistable locus $\mathfrak{X}^{ss}$ is Θ-complete (resp. S-complete).*

Finally, there is also a way to deduce properness of the resulting moduli space.

Proposition 2.24. *[4, Corollary 6.12] Suppose $\mathfrak{X}$ has a well-ordered Θ-stratification. If $\mathfrak{X}$ satisfies the existence part of the valuative criterion of properness, so does $\mathfrak{X}_{\leq c}$ for every $c \in \Gamma$.*

The results of this section are summarised in the following theorem.

Theorem 2.25. *[18] Suppose μ is strictly Θ-monotone and S-monotone. Then*

1. *μ defines a Θ-stratification of $\mathfrak{X}$ if and only if it satisfies HN-Boundedness.*
2. *If $\mathfrak{X}^{ss}$ is quasicompact, $\mathfrak{X}$ satisfies the existence part of the valuative criterion of properness and the Θ-stratification of $\mathfrak{X}$ is well-ordered, then it admits a good moduli space that is separated and locally of finite presentation, and the connected components of this moduli space are proper.*

2.4 Infinite-dimensional GIT

Using the results of the previous section, finding a good moduli space is reduced in part to establishing strict Θ- and S-monotonicity for a numerical invariant μ. One approach to do this, introduced in [19] is called infinite-dimensional GIT.

It is inspired by proving monotonicity in the case of projective GIT. Let X be a projective scheme with an action of a reductive group G, and $\mathcal{L}$ a G-linearised ample line bundle on X. Consider a morphism $\Theta_R \backslash 0 \to [X/G]$ and a commutative diagram.

$$\begin{array}{ccc} & & \mathcal{L} \\ & & \big| \\ \Theta_R\backslash 0 & \longrightarrow & [X/G] \\ \big\downarrow & & \big\downarrow \\ \Theta_R & \longrightarrow & BG \end{array}$$

The morphism $\Theta_R \to BG$ exists because BG is Θ-complete. We want to find a stack $\mathcal{W}$ satisfying the properties of Definition 2.18. Let $\mathcal{W}$ be the scheme-theoretic image of $\Theta_R\backslash 0 \to \Theta_R \times_{BG} [X/G]$, and denote by $\tilde{f}$ the resulting morphism to $[X/G]$. Because $\mathcal{L}$ is ample for the morphism $[X/G] \to BG$ it follows that $\tilde{f}^*\mathcal{L}$ is ample for $\mathcal{W} \to \Theta_R$.

Given a morphism $\phi\colon \mathbb{P}^1_k/\mathbb{G}_m \to \mathcal{W}$ it follows that $\phi^*\tilde{f}^*\mathcal{L}$ is ample on $\mathbb{P}^1_k$, and up to twisting by a character, there is a unique $\mathbb{G}_m$-equivariant structure on an ample line bundle, which satisfies

$$\mathrm{wt}_{\mathbb{G}_m}(\phi^*\tilde{f}^*\mathcal{L}|_{\mathbb{A}^1_0}) < \mathrm{wt}_{\mathbb{G}_m}(\phi^*\tilde{f}^*\mathcal{L}|_{\mathbb{A}^1_\infty}),$$

where $\mathbb{A}^1_0$ and $\mathbb{A}^1_\infty$ denote the two copies of $\mathbb{A}^1$ in $\mathbb{P}^1$ containing 0 and ∞ respectively. If we define a numerical invariant using the line bundle $\mathcal{L}$ and a norm b pulled back from BG, this is precisely the inequality we need to establish strict Θ-monotonicity.

Given a moduli stack $\mathfrak{X}$ and a line bundle $\mathcal{L}$ on $\mathfrak{X}$ defining a numerical invariant, a strategy for proving monotonicity inspired by the preceding discussion is then to find another stack $\mathfrak{X}'$ and a morphism $\mathfrak{X} \to \mathfrak{X}'$ such that

1. the fibers of $\mathfrak{X} \to \mathfrak{X}'$ are ind-projective ind-schemes with ample line bundle $\mathcal{L}$.

2. Given a morphism $\Theta_R\backslash 0 \to \mathfrak{X}$, there exists a morphism $\Theta_R \to \mathfrak{X}'$ such that the following diagram commutes.

$$\begin{array}{ccc} \Theta_R\backslash 0 & \longrightarrow & \mathfrak{X} \\ \big\downarrow & & \big\downarrow \\ \Theta_R & \longrightarrow & \mathfrak{X}' \end{array}$$

and similarly for Φ_R.

Given these two properties, we can construct a stack $\mathcal{W}$ in the same way done above for projective GIT, and the same proof applies to show strict monotonicity. This will be used in the final chapter to show the existence of a good moduli space for parahoric Higgs bundles.

3 Parahoric Group schemes and (Higgs) Bundles

In this chapter, we introduce the notion of parahoric groups and corresponding geometric objects.

Parahoric groups are generalisations of parabolic groups for a reductive group G, studied in [12] and [13]. Given a reductive group G over $\mathbb{C}$, these are subgroups of $G(\mathbb{C}((t)))$. We first show how this notion naturally arises when generalising the correspondence between parabolic Higgs bundles in type A and representations of the fundamental group of a punctured curve to other groups. After giving a first definition of parahoric groups, we review a choice-independent definition using the Bruhat-Tits building of G.

After this, we discuss certian group schemes over $\operatorname{Spec} \mathbb{C}[[t]]$, constructed in loc. cit., whose groups of $\mathbb{C}[[t]]$-points are parahoric groups. Given a smooth projective curve C we then consider global versions of these group schemes, and bundles for these group schemes will be a generalisation of (quasi-)parabolic bundles. We then review some results about the moduli stacks of bundles for these group schemes.

Finally, we look at Higgs bundles for such group schemes and review known results about the moduli stacks of Higgs bundles.

3.1 Parahoric Groups

3.1.1 Motivation

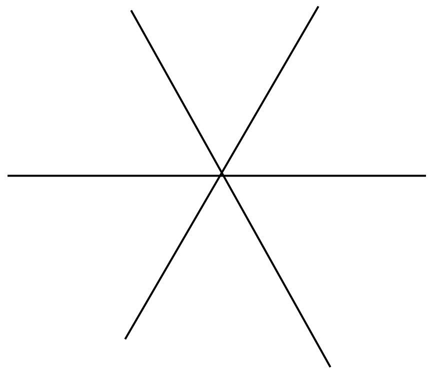
To motivate the need for parahoric bundles, we turn to the work of Mehta and Seshadri in [30], extending the results of Narasimhan and Seshadri in [31] for vector bundles on projective curves to the quasiprojective case. In the correspondence between representations of the fundamental group and parabolic vector bundles, we have to fix the conjugacy class of the loop around a marked point. This can be represented as a diagonal matrix with entries in $\mathbb{C}^*$ and the structure of this matrix gives the parabolic structure at the marked point.

A natural question that then arises is how to parametrise the conjugacy classes in a given maximal compact subgroup K of a reductive algebraic group G over $\mathbb{C}$. This can roughly be done as follows:

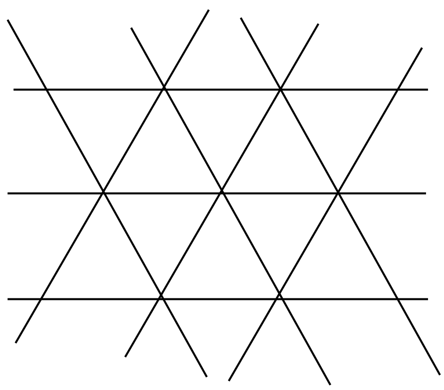
1. Fix a maximal torus T inside K. Classical results imply that every element is conjugate to one in T. We can thus restrict our attention to the conjugacy classes in T, under the remaining conjugation action of its normaliser $N(T)$.

2. Now focus on the Lie algebra $\mathfrak{t}$ of the maximal torus. We know that the action of $N(T)$ is the same as the action of the Weyl group, and a fundamental domain of this action is given by the closure of a Weyl chamber. The walls

of such a chamber correspond to the hyperplanes where a simple root [3] vanishes. This is pictured below for $\mathrm{SL}(3,\mathbb{C})$.



3. The equivalence classes for the adjoint action on the Lie algebra $\mathfrak{t}$ are classified by elements of the closure of the Weyl chamber. We can take the exponential map to T and in this way get the conjugacy classes there. We only need to account for which elements of the Lie algebra differ by an element of the kernel of this exponential map. This gives a new collection of walls which correspond to the loci where a root equals an integer. We call the closure of a connected component of the complement of these walls a *Weyl alcove*. It can be shown that one choice for this is the part of the Weyl chamber bounded by the hyperplane where the longest root equals 1. This is pictured below in the case of $\mathrm{SL}(3,\mathbb{C})$, and a Weyl alcove is given by the choice of one of the equilateral triangles. The group of symmetries of this tiling is called the *affine Weyl group*, and it is given by the semidirect product of the Weyl group and the (integer) root lattice.



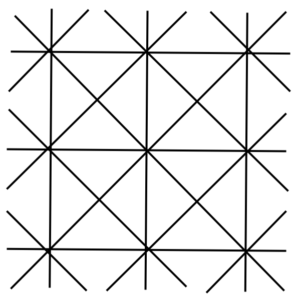
[3] The choice of a system of simple roots is implied by the choice of a Weyl chamber.

We find that the conjugacy classes are parametrised by the Weyl alcove. On the parts that correspond to the Weyl chamber, so not lying on the hyperplane defined by the longest root, we can then recover the parabolic structure as follows.

1. The Weyl chamber defines a Borel subgroup B containing the maximal torus T. A point in the Weyl chamber lies in one of its facets. These facets are in bijection with subsets of the collection of simple roots. Such subsets are in turn known to be in bijection with parabolic groups P containing B. So, given a conjugacy class for the marked point p, we get an associated parabolic group.

2. The parabolic weights can be thought of as the coordinates of the point in the Weyl alcove.

However, on the hyperplane corresponding to the longest root, we cannot in general assign a parabolic structure. For $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$ the Weyl alcove has a rotational symmetry of order n which allows us to identify points on the hyperplane corresponding to the longest root with points that do correspond to parabolic subgroups, bypassing this problem. But for other groups, this problem cannot be bypassed in general. For $\mathrm{Sp}(4,\mathbb{C})$ pictured below we see that the Weyl alcove is an isosceles triangle. Thus, this has a reflection symmetry, and while we can translate most of the points on the extra wall to one of the original Weyl chamber using this symmetry, we cannot do this for the vertex of the triangle lying on the axis of symmetry. Here, we should really have something that is more general than a parabolic structure.



There is thus a need for something more general. The answer will turn out to be a notion called parahoric groups. These are certain subgroups of $G(\mathbb{C}((t)))$ which up to conjugation correspond to the facets of the Weyl alcove. If we are in a facet of the Weyl alcove that comes from the Weyl chamber, then these will be conjugate to subgroups of $G(\mathbb{C}[[t]])$. These subgroups are the preimages of parabolic subgroups under the reduction morphism $G(\mathbb{C}[[t]]) \to G$. So this recovers the notion of parabolic subgroups, but we also get more than strictly the parabolic subgroups

If we are given a representation of the fundamental group of a punctured curve, then the images of the loops around one of the punctures determine a conjugacy class of elements of K. Through the above discussion, we get an associated parahoric group. Representations with certain fixed conjugacy classes will then correspond to bundles for parahoric group schemes through non-abelian Hodge theory. The parahoric group schemes are special group schemes over the curve C such that away from the marked points, the scheme is isomorphic to the constant group scheme associated to G, but over a formal disc around a marked point, they are isomorphic to a group scheme over $\operatorname{Spec}\mathbb{C}[[t]]$ associated to the parahoric group at that marked point.

The rest of this chapter will be devoted to studying these parahoric groups, parahoric group schemes and bundles for these group schemes.

3.1.2 Parahoric Groups

Most of this section is based on the work by Bruhat and Tits in [12] and [13], and the review paper [39]. Throughout this section, we fix a reductive group G over $\mathbb{C}$, a maximal torus T and the normaliser of this torus N. The associated root system of G is denoted by R, and we denote the corresponding root subgroups by U_a for $a \in R$. We denote by $X_*(T) = \operatorname{Hom}(\mathbb{G}_m, T)$ the group of one-parameter subgroups of T, and by V the vector space $X_*(T) \otimes \mathbb{R}$, which is canonically isomorphic to what we denoted in the previous section by $\mathfrak{t}$. We denote by A an affine space under V, which through the arguments of the previous chapter has a system of walls or hyperplanes on it. We choose a connected component of the complement of these hyperplanes C.

The main idea To construct parahoric groups, we first analyse the structure of parabolic vector bundles a bit more. Suppose E is a rank 3 parabolic vector bundle with a complete flag at the point p,

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 = E|_p.$$

We can locally choose a basis $\{e_1, e_2, e_3\}$ such that this flag looks like

$$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset E|_p.$$

Passing to the associated frame bundle, we get a $\mathrm{GL}(n, \mathbb{C})$-bundle over C with a reduction to the standard Borel subgroup at p. We want to consider the local sections of this bundle that respect this reduction. If t is a local coordinate vanishing at p, then such sections locally look like

$$\begin{pmatrix} * & * & * \\ t* & * & * \\ t* & t* & * \end{pmatrix},$$

with respect to the basis $\langle e_1, e_2, e_3 \rangle$ where $*$ denotes an arbitrary local function. We thus see that certain components of the matrix have to have a precise vanishing degree. In terms of the root subgroups, we see that the components corresponding to negative roots have to vanish of order 1, whereas in the other components anything is allowed. For other standard parabolic groups, we will have a similar pattern, but here it will only be a proper subset of the negative roots where we require the component to vanish.

In the previous section, we found a system of walls in A. We fix the closure of one of the chambers, and call it the **Weyl alcove**. Another way of making the correspondence between the parts of the Weyl alcove corresponding to the Weyl chamber and parabolic subgroups precise presents itself. If we are given such a point of the Weyl alcove, then we need to produce vanishing orders for the different root subgroups. We need a function prescribing vanishing orders for the root subgroups for each θ in this alcove,

$$m_\theta : R \to \mathbb{Z}.$$

One formula that reproduces this behaviour for parabolic groups, which can be found easily by playing around with these ingredients is

$$m_\theta(a) = -\lfloor \langle a, \theta \rangle \rfloor.$$

Here $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the Lie algebra and its dual, and $\lfloor \cdot \rfloor$ is the operation of rounding down to an integer. For a point in the interior of the Weyl alcove, which corresponds to a Borel subgroup, we get exactly the behaviour we described earlier. Similarly, for the other facets corresponding to parabolic subgroups, this gives the expected behaviour. However, there is nothing stopping us from applying this formula to the other points of this Weyl alcove, or indeed arbitrary elements of A. For instance, applying this to the point defined by the top corner of the Weyl alcove of $\mathrm{SL}(3, \mathbb{C})$, we get group elements that look like

$$\begin{pmatrix} * & t^{-1}* & t^{-1}* \\ t* & * & * \\ t* & * & * \end{pmatrix}$$

So, we end up with matrices that are allowed to have a pole at the point p. We can make sense of this as a subgroup of $G(\mathbb{C}((t)))$. We see here already some justification appearing for why we can interpret this still as a parabolic structure, since upon conjugating this matrix with $\mathrm{diag}(t, 1, 1)$ the poles disappear and we end up with ordinary matrices. A similar phenomenon will occur for the other points.

So, for every point of A, we get a subgroup of $G(\mathbb{C}((t)))$. We make this precise in the following definition.

Definition 3.1. Let $\theta \in A$, the **parahoric subgroup** P_θ of $G((t))$ associated to θ is given by

$$P_\theta = \langle T(\mathbb{C}[[t]]), U_a(t^{m_a(\theta)}\mathbb{C}[[t]]) \,|\, a \in R\rangle.$$

The group $U_a(t^{m_a(\theta)}\mathbb{C}[[t]])$ denotes the subgroup of U_a whose entries have only powers of t of order $m_a(\theta)$ or higher.

As we will see in the next chapter, for points of A not in the Weyl alcove, we get subgroups that are conjugate to those obtained from the Weyl alcove. Also, it is easily seen from the formula above that for points in the same facet, we get the same subgroup.

To describe an appropriate notion of parahoric Higgs fields later, it will also be important to us to know what the Lie algebra of a parahoric group is. This is given by the free $\mathbb{C}[[t]]$-module

$$\mathfrak{p}_\theta = \mathfrak{t}(\mathbb{C}[[t]]) \oplus \bigoplus_{a \in R} \mathfrak{g}_a(t^{m_r(\theta)}\mathbb{C}[[t]]). \tag{3.2}$$

The Lie bracket is the natural generalisation of that of the ordinary Lie algebra $\mathfrak{g}$.

3.1.3 Standard Apartments and the Building

The way we obtained the parahoric subgroups of $G(\mathbb{C}((t)))$ was dependent on a lot of different choices. In this section, we consider a more canonical way of constructing parahoric subgroups. The goal is to construct a certain combinatorial object, called the *Bruhat-Tits building*, with a canonical action of $G(\mathbb{C}((t)))$, such that parahoric subgroups are precisely the stabilisers of points in the building.

We have already motivated that an important role is played by the valuation of the elements of the different root subgroups of $G(\mathbb{C}((t)))$. In our case, all of these root subgroups are isomorphic to the additive group $\mathbb{G}_a$. We fix such isomorphisms

$$\chi_a : \mathbb{G}_a \to U_a$$

for $a \in R$, called a *pinning* of G. Over $\mathbb{C}((t))$, these additive groups have a natural filtration corresponding to the valuations of the element. For $k \in \mathbb{Z}$, the corresponding term of the filtration of U_a is

$$X_{a,k} := \chi_a(\omega^{-1}[k, \infty])^4,$$

where ω denotes the valuation on $\mathbb{C}((t))$. In terms of the affine space A, we can express this as follows. Denote by R_{af} the following subset of the affine functions on A,

$$R_{af} = \{a + k \,|\, a \in R, k \in \mathbb{Z}\},$$

[4] This is the same subgroup as $U_a(t^k\mathbb{C}[[t]])$.

which we call *affine roots*.

For each such function α, we have the associated subgroup of U_a, denoted U_α where a is the vectorial part of α. If we conjugate U_a by an element n of the normaliser of the maximal torus $\mathbb{N}(\mathbb{C}((t)))$, we land in another root subgroup. The element n specifies an element w of the Weyl group under the projection $N \to N/T$, and the root subgroup we end up in is $U_{w(a)}$. Under this natural action, we can compute that the different subgroups U_α also get permuted. In this way, we get an action of $\mathbb{N}(\mathbb{C}((t)))$ on R_{af}. This action comes from an action of $\mathbb{N}(\mathbb{C}((t)))$ by affine transformations on A by precomposing the affine function with the affine transformation.

Given an element $n \in N(\mathbb{C}((t))$, we can find the affine transformation $\nu(n)$ as follows. We can always write $n = n't$ for $n \in N(\mathbb{C})$ and $t \in T(\mathbb{C}((t)))$. We thus need to find out how these elements act separately. For n' we know that it will act through the action of the Weyl group. A simple calculation shows that

$$tU_\alpha t^{-1} = U_{\alpha+\omega(a(t))}.$$

It follows that t acts as the translation $\nu(t)$ defined by

$$a(\nu(t)) = -\omega(a(t))$$

for $a \in R$.

Example 3.3. We illustrate these notions for the case of $G = \mathrm{SL}_2$. In this case, a torus T is given by $\mathrm{diag}(f, f^{-1})$ for $f \in \mathbb{C}((t))$. The two roots with respect to this torus are

$$\pm a : T \to \mathbb{G}_m : \mathrm{diag}(f, f^{-1}) \mapsto f^{\pm 2}.$$

The root subgroups are given by

$$U_a = \{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{C}((t)),$$

and

$$U_{-a} = \{\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \mid u \in \mathbb{C}((t)).$$

The normaliser of the maximal torus is given by

$$N = T \cup \{\begin{pmatrix} 0 & n \\ -n^{-1} & 0 \end{pmatrix} \; n \in \mathbb{C}((t))\}.$$

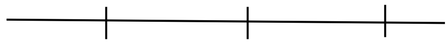
The affine space A is in this case just a one-dimensional line. The pinning is given by

$$\chi_a : \mathbb{C}((t)) \to U_a : u \mapsto \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

with a similar formula for U_{-a}. The action of the normaliser on this affine space is given as follows. For an element in the maximal torus with parameter $f \in \mathbb{C}((t))$, the affine transformation is a translation by $2\omega(f)$, which agrees with our earlier formula. An element of the other type with parameter $n \in \mathbb{C}((t))$ corresponds to the affine transformation $x \mapsto -x - 2\omega(n)$.

Given the system of affine roots R_{af}, we define the *walls* of A as $\alpha^{-1}(0)$ for $\alpha \in R_{af}$. The *chambers* are defined as the connected components of the complement of all walls. The facets of (the closure of) the chambers are also called the *facets* of A. We denote by W the group generated by the reflection through all the walls, and call it the *affine Weyl group* of G. It is easily seen that the chambers are permuted transitively by the action of the Weyl group. The facets of these chambers of the same type [5] are also permuted transitively. This system of walls recovers that constructed in the first section of this chapter. In particular, the closure of a chamber is what we called the *Weyl alcove* and this is a fundamental domain for the action of the affine Weyl group.

For the case of SL_2 considered earlier, we get a decomposition of the affine line into line segments.



In this case, the affine Weyl group consists of translations by even integers and reflections through the walls, or

$$W = \{\pm 1\} \rtimes \mathbb{Z}$$

For $\mathrm{SL}(3, \mathbb{C})$ and $\mathrm{Sp}(4, \mathbb{C})$, we get the pictures shown earlier.

In the case of groups over $\mathbb{C}$, we could associate to a Weyl chamber the Dynkin diagram, which we already know to have a relationship with respect to the subsets of parabolic subgroups up to conjugacy. There is a similar construction in this case, leading to the *affine Dynkin diagram.* We fix a Weyl alcove C and denote its walls by C_i for $i = 0, \ldots, n$. These are all of the form $\alpha_i^{-1}(0)$ for some affine roots $\alpha_i \in \Phi_{af}$. The roots α_i for $i = 0, \cdots, n$ form what is called a set of *simple affine roots.* One can show that any other affine root can be written as a $\mathbb{Z}$-linear combination of these simple ones. Suppose there is an inner product $(\cdot, \cdot)$ on the dual of V invariant by the action of the ordinary Weyl group, then we construct the affine Dynkin diagram as follows.

1. The elements α_i are the vertices of the diagram.

[5]We will make precise what is meant by type below.

2. Two distinct vertices α_i and α_j are joined by $\frac{4(a_i,a_j)^2}{(a_i,a_i)(a_j,a_j)}$ lines, where a_i and a_j are the vector parts of the corresponding affine roots. If there is more than one line, then we add an arrow pointing to the shortest root.

In general, when starting from a group only defined over $\mathbb{C}((t))$, [6] there may be some extra steps in the construction. This affine Dynkin diagram plays a crucial role in the classification of reductive groups over local fields, see Chapter 4 of [39]. The main reason we will be interested in it, is for its relationship to facets of the Weyl alcove. The different facets are all given by intersections of a collection of walls, which corresponds to a subset of the set of simple affine roots, or subsets of the affine Dynkin diagram. We call the corresponding subset of the affine Dynkin diagram the *type* of the facet. Moreover, the symmetries of the affine Dynkin diagram correspond to the symmetries of the Weyl alcove, and if G is a reductive group over $\mathbb{C}$, then the affine Dynkin diagram of $G(\mathbb{C}((t)))$ will contain the usual Dynkin diagram of G, and the set of vertices of the affine Dynkin diagram is given by the vertices of the usual Dynkin diagram plus one extra vertex.

This provides a relatively easy description of the facets in higher-dimensional situations. As already mentioned before, there is a choice of Weyl alcove which is very easy to describe. Suppose $a_1, \dots, a_n$ is a collection of simple ordinary roots. Denote by $\tilde{a}$ the longest root. Then $a_1, \dots, a_n, \tilde{a} - 1$ is a collection of simple affine roots. The Weyl alcove is given by the inequalities

$$\begin{cases} 0 \leq a_i \text{ for } i = 1, \dots, n, \\ \tilde{a} \leq 1 \end{cases}$$

If we form the affine Dynkin diagram with respect to this set of simple affine roots, $a_1, \dots, a_n$ corresponds to the usual Dynkin diagram and $\tilde{a} - 1$ corresponds to the extra vertex. Given a proper subset of the affine Dynkin diagram, the corresponding facet is the locus where the inequalities corresponding to the simple affine roots in the subset are equalities. This provides a simple way of describing points in this facet when doing calculations.

We now have a rigourous description of the affine space we associated parahoric groups to in the previous chapter. But to associate these parahoric groups, we made a choice of a maximal torus. To get a description that is independent of the torus, the basic idea is to glue all the different apartments we get for each torus together in an appropriate way. Let A be as before.

Theorem 3.4. *There exists a unique up to unique isomorphism set $\mathcal{B}(G)$ containing A with an action of $G((t))$ having the following properties:*

[6] Or more generally a local field.

1. $\mathcal{B}(G) = \bigcup_{g \in G((t))} gA$,

2. *the group* $N(\mathbb{C}((t)))$ *stabilises* A *and operates on it through* ν,

3. *for every affine root* α*, the group* U_α *fixes* $\alpha^{-1}([0,\infty))$ *pointwise.*

This set is known as the *Bruhat-Tits building* of G, and is constructed as the quotient of $G \times A$ by a suitable equivalence relation. For the details of its construction, we refer to Chapter 7.3 of [12]. The sets gA are called the *apartments* of $\mathcal{B}$, and gA can be thought of as the apartment with respect to the maximal torus gTg^{-1}. Each apartment naturally has the structure of an affine space with a decomposition into facets, as we have for A. The action of G respects this structure.

The building now allows for a canonical description of parahoric subgroups.

Definition 3.5. A **parahoric subgroup** of G is the connected component of the identity of the stabiliser of a point in the Bruhat-Tits building.

If we have fixed one apartment, and a Weyl alcove in it, then using the third point in Theorem 3.4, it can be shown that this agrees with Definition 3.1.

The assertion that all parahoric subgroups are conjugate to those obtained from a fixed Weyl alcove can now also be seen as follows. Suppose A is our fixed apartment, with C the fixed Weyl alcove. If we take a point in an apartment gA, then acting with g^{-1}, we can take it to a point in A. Because the action of $N(\mathbb{C}((t)))$ on the alcoves is transitive, we can then act with an element of this group to take it to a point in the closure of C. The stabiliser subgroup of the original point is then clearly conjugate to that of this final point.

In fact, this argument shows more. Because the action of $G(\mathbb{C}((t)))$ on $\mathcal{B}$ respects the decomposition of the building into facets, the stabiliser subgroup will be constant along facets. So up to conjugacy, the parahoric subgroup depends only on the facet the point lies in. It follows that parahoric subgroups up to conjugacy are classified by proper subsets of the affine Dynkin diagram. It is also clear that if a facet F lies in the closure of another one F', then the parahoric subgroup associated to F contains that associated to F'. It follows from this fact that the correspondence with proper subsets of the affine Dynkin diagram respects the inclusion relations.

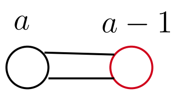
The parahoric subgroups corresponding to the interior of the alcoves are known as *Iwahori subgroups*. Because of our calculations in the previous section, we now know that these are always conjugate to the inverse image of a Borel subgroup under the reduction morphism $G(\mathbb{C}[[t]]) \to G$. Moreover, if we denote by P_0 the parahoric subgroup given by the subset which is just the usual Dynkin diagram, we see that one way of computing (a conjugate of) this subgroup is given by the formula of Definition 3.1 applied to 0. This gives just $G(\mathbb{C}[[t]])$. If we compute parahoric subgroups corresponding to subsets included in the usual Dynkin diagram, then it follows from this fact that they are conjugate to groups included in $G(\mathbb{C}[[t]])$. Moreover, they will be exactly the preimages of parabolic subgroups under the reduction morphism. This reproduces our earlier assertion that the parts of the Weyl alcove corresponding to the usual Weyl chamber give rise to parabolic subgroups.

Another important fact is that any bounded [7] subgroup of $G((t))$ which contains an Iwahori subgroup is a parahoric subgroup. This is a generalisation of the classical fact that any subgroup of G containing a Borel subgroup is a parabolic subgroup.

3.1.4 Examples

$\mathrm{SL}(2, \mathbb{C})$: For this group, the Weyl alcove looks like

and the affine Dynkin diagram is given by



The black vertex corresponds to the usual Dynkin diagram, while the red vertex is the new affine root. Reusing our earlier notation for $\mathrm{SL}(2, \mathbb{C})$, the Weyl alcove is described by the inequalities

$$0 \leq a \leq 1.$$

the black vertex corresponds to the simple affine root a, and the red vertex corresponds to $a - 1$. There are three different proper subsets of the affine Dynkin diagram, so three different facets of the Weyl alcove and up to conjugacy three different parahoric subgroups.

1. The first subset is the empty subset. This would correspond to a point in the interior of the Weyl alcove and is described by the inequality

$$0 < a < 1.$$

This corresponds to a Borel subgroup. This is a general fact, the empty subset of the affine Dynkin diagram always gives you an Iwahori subgroup. This

[7] A subset M of $G(\mathbb{C}((t)))$ is called *bounded* if for any regular function $f : G(\mathbb{C}((t))) \to \mathbb{C}((t))$ the set $\{\omega(g) \mid g \in M\}$ is bounded below.

can be seen by observing that on positive roots, the inequalities describing the point imply that we get a vanishing order of 0, whereas for negative roots, they will give 1. This describes a subgroup of $G(\mathbb{C}[[t]])$ which on reduction to G gives a Borel subgroup.

2. Next, we can take the subset consisting of the black vertex. This corresponds to the usual Dynkin diagram, and is described by

$$a = 0.$$

It is immediate that in this case, we get as a parahoric group $\mathrm{SL}(2, \mathbb{C}[[t]])$.

3. Finally, the remaining proper subset is the red vertex. The corresponding facet is given by

$$a = 1.$$

We will end up with matrices that look like

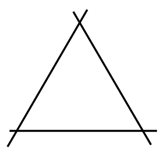
$$\begin{pmatrix} * & t^{-1}* \\ t* & * \end{pmatrix},$$

where $*$ denotes some elements of $\mathbb{C}[[t]]$. A priori, this is something new that we cannot describe by parabolic subgroups. However, if we conjugate this subgroup by $\mathrm{diag}(t, 1)$, we end up with $\mathrm{SL}(2, \mathbb{C}[[t]])$. This matrix is not an element of $\mathrm{SL}(2, \mathbb{C}[[t]])$, so this conjugation corresponds to an outer automorphism of $\mathrm{SL}(2, \mathbb{C}[[t]])$ and this realises the $\mathbb{Z}_2$ symmetry of the Dynkin diagram. This suggest that for $\mathrm{SL}(n, \mathbb{C})$ as with $\mathrm{GL}(n, \mathbb{C})$ we also do not end up with any new subgroups. While this is true, we will see later that the group schemes associated to these subgroups do give rise to new behaviour.

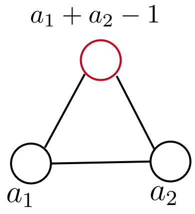
$\mathrm{SL}(3, \mathbb{C})$: A maximal torus T for this group is given by matrices of the form $\mathrm{diag}(t_1, t_2, t_1^{-1}t_2^{-1})$ for $t_1, t_2 \in \mathbb{G}_m$. The roots are given by $\pm a_1, \pm a_2, \pm(a_1 + a_2)$ for

$$\begin{cases} a_1 : T \to \mathbb{G}_m : t_1 t_2^{-1} \\ a_2 : T \to \mathbb{G}_m : t_2(t_1^{-1}t_2^{-1})^{-1} \end{cases}$$

Here, the Weyl alcoves are



and the affine Dynkin diagram is



We now focus on the proper subsets of this affine Dynkin diagram that are not subsets of the usual Dynkin diagram. There are three such subsets.

1. The first one corresponds to only the red vertex. The inequalities describing this subset of the Weyl alcove are

$$\begin{cases} 0 < a_1, a_2 \\ a_1 + a_2 = 1. \end{cases}$$

This will give matrices that look like

$$\begin{pmatrix} * & * & t^{-1}* \\ t* & * & * \\ t* & t* & * \end{pmatrix}$$

2. The second one is the subset corresponding to the red vertex and the vertex corresponding to a_1. The inequalities here are

$$\begin{cases} a_1 = 0, \\ 0 < a_2, \\ a_1 + a_2 = 1. \end{cases}$$

This gives rise to matrices looking like

$$\begin{pmatrix} * & * & t^{-1}* \\ * & * & t^{-1}* \\ t* & t* & * \end{pmatrix}$$

Conjugating by $\mathrm{diag}(t, t, 1)$, this will be again isomorphic to $\mathrm{SL}(3, \mathbb{C}[[t]])$. The previous example was a subgroup of this subgroup containing an Iwahori subgroup, It follows that once we conjugate this one, we get a subgroup of $\mathbb{G}[[t]]$, so this also corresponds to a parabolic subgroup.

3. The final subset is dealt with completely analogously and gives

$$\begin{pmatrix} * & t^{-1}* & t^{-1}* \\ t* & * & * \\ t* & * & * \end{pmatrix}$$

Here, we conjugate by $\mathrm{diag}(t, 1, 1)$.

$\mathrm{Sp}(4,\mathbb{C})$: For this group, a maximal torus T is given by $\mathrm{diag}(t_1,t_2,t_1^{-1},t_2^{-1})$ for $t_i \in \mathbb{G}_m$. Denoting by e_i the morphism

$$e_i : T \to \mathbb{G}_m : \mathrm{diag}(t_1,t_2,t_1^{-1},t_2^{-1}) \to t_i,$$

the roots are

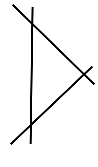
$$\pm 2e_1, \pm 2e_2, \pm e_1 + e_2, \pm e_1 - e_2.$$

The root subgroups for positive signs above are given by

$$\begin{cases} U_{2e_1} = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{G}_a\} \\ U_{2e_2} = \{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 1 \end{pmatrix} \mid x \in \mathbb{G}_a\} \\ U_{e_1+e_2} = \{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{G}_a\} \\ U_{e_1-e_2} = \{ \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \mid x \in \mathbb{G}_a\}, \end{cases}$$

and for the negative signs we take the transposes of these matrices.

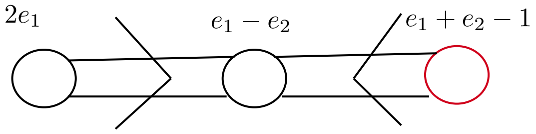
The Weyl alcoves look like



One such alcove can be described by the inequalities

$$\begin{cases} 0 \leq 2e_2, e_1 - e_2 \\ e_1 + e_2 \leq 1. \end{cases}$$

The affine Dynkin diagram looks like



We immediately see that this has a $\mathbb{Z}_2$ reflection symmetry, which corresponds to the reflection symmetry of the isosceles triangle. Using this symmetry, two of the three proper subsets of the affine Dynkin diagram can be mapped to subsets of the usual Dynkin diagram. The corresponding parahoric subgroups will then be isomorphic to those obtained from parabolic subgroups. The remaining subset is the one where we leave out the middle vertex.

This is described by the inequalities

$$\begin{cases} 2e_2 = 0 \\ 0 < e_1 - e_2 \\ e_1 + e_2 = 1. \end{cases}$$

These specify a unique point θ of the Weyl alcove determined by $2e_2 = 0, e_1 + e_2 = e_1 - e_2 = 1$. The values of the function $m_\theta(a)$ are recorded in the following table.

a	$m_\theta(a)$
$2e_1$	0
$2e_2$	0
$e_1 + e_2$	-1
$e_1 - e_2$	-1
$-2e_1$	1
$-2e_2$	0
$-e_1 - e_2$	1
$-e_1 + e_2$	1

These matrices schematically look like

$$\begin{pmatrix} * & t^{-1}* & t* & t^{-1}* \\ t* & * & t^{-1}* & * \\ * & t* & * & t* \\ t* & * & t^{-1}* & * \end{pmatrix}$$

and this subgroup is not isomorphic to $\mathrm{Sp}(4, \mathbb{C})$.

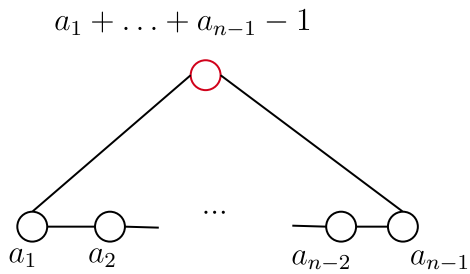
$\mathrm{SL}(n, \mathbb{C})$: Here, we work out what the maximal parahoric subgroups of $\mathrm{SL}(n, \mathbb{C})$ look like. These will be used in the next section. This has a maximal torus given by

$$T = \{\mathrm{diag}(t_1, \dots, t_n) \,|t_i \in C^* \text{ such that } \prod_{i=1}^n t_i = 1.\}$$

Denoting by e_i the character

$$e_i : T \to \mathbb{C}^* : \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i,$$

the roots are given by $e_i - e_j$ for $i \neq j$. A collection of simple roots is given by $a_i = e_{i+1} - e_i$ for $i = 1, \ldots n-1$. The affine Dynkin diagram is given by



We are interested in the parahoric group P_i which corresponds to the proper subset of this diagram obtained by deleting the vertex corresponding to a_i. This corresponds to the point in the Weyl alcove specified by $a_i = 1$ and $a_j = 0$ for $j \neq i$. Similar computations to those performed earlier, show that this subgroup consists of matrices of the form

$$\begin{pmatrix} A & t^{-1}B \\ tC & D \end{pmatrix},$$

where A is an $i \times i$ matrix, B is an $i \times (n-1)$ matrix, C $(n-i) \times i$ and $D (n-i) \times (n-i)$, all of them with entries in $\mathbb{C}[[t]]$. Conjugating by

$$\begin{pmatrix} t1_{i \times i} & 0_{i \times (n-1)} \\ 0_{(n-i) \times i} & 1_{(n-i) \times (n-i)} \end{pmatrix},$$

we see that this group is isomorphic to $\mathrm{SL}(n, \mathbb{C})$.

3.1.5 Group Schemes over $\mathrm{Spec}\, \mathbb{C}[[t]]$

Now that we have defined parahoric groups, we would like to construct group schemes over $\mathrm{Spec}\, \mathbb{C}[[t]]$ such that the $\mathbb{C}[[t]]$-points of these group schemes are exactly the parahoric groups. We can always fix an apartment A, corresponding to a torus $T \subset G((t))$ and thus an associated root system R and root subgroups U_a, containing the point of which this parahoric group is the stabiliser. In this case, we have shown that the parahoric groups derive from a function

$$f : R \to \mathbb{Z}.$$

Given this data, Bruhat and Tits constructed such group schemes as follows:

1. The first goal is to extend the torus T to a group scheme over $\operatorname{Spec} \mathbb{C}[[t]]$. This torus will always be split, which means it is isomorphic to some copies of $\mathbb{C}((t))^*$, and this can be extended as the smooth affine group scheme $\mathbb{G}_m^n$ over $\operatorname{Spec} \mathbb{C}[[t]]$. We denote this extension by $\mathcal{T}$.

2. Extending the root subgroups U_a requires more work. We want to extend this in such a way that its group of $\mathbb{C}[[t]]$-points is $U_a(t^{f(a)}\mathbb{C}[[t]])$. Because in the situation that we are in, we have an isomorphism

$$\chi_a : \mathbb{G}_a \to U_a,$$

we can define this as an extension of the additive group $\mathbb{C}((t))$. By transport of structure, we then also get the desired extension, which we will denote by $\mathcal{U}_{a,f}$.

3. Next, we need to verify that all these different extensions $\mathcal{U}_{a,f}$ are compatible. For example, if we want to consider the commutator subgroup $[\mathcal{U}_{a,f}, \mathcal{U}_{b,f}]$, where $a + b \in R$, in the group scheme we end up constructing, then this must be a subgroup of $\mathcal{U}_{a+b,f}$. Similarly, we want the commutator of $\mathcal{U}_{a,f}$ and $\mathcal{U}_{-a,f}$ to land in $\mathcal{T}$. Let us analyse this first condition on the level of $\mathbb{C}[[t]]$-points. Let $t^{f(a)}x, t^{f(b)}y$ be $\mathbb{C}[[t]]$-points of $\mathcal{U}_{a,f}$ and $\mathcal{U}_{b,f}$ respectively. Their commutator will be $t^{f(a)+f(b)}(xy-yx)$. To ensure that this is an element of $\mathcal{U}_{a+b,f}$, we need that the function f is *concave*, which means that

$$f(a) + f(b) \geq f(a + b)$$

whenever $a + b \in R$. This is indeed the case for the functions used in 3.1. This condition turns out to be enough to ensure the compatibilities we desire. [8]

4. If the function f is concave, then we now want to construct the full group scheme. The idea for this is to find a group scheme in which the schemes $\mathcal{T}, \mathcal{U}_{a,f}$ are natural subgroup schemes, in a way compatible with the extension from the group over $\mathbb{C}((t))$, and then define the group scheme $\mathcal{G}_f$ as the subgroup scheme generated by $\mathcal{T}, \mathcal{U}_{a,f}$ for $a \in R$. If the group G is adjoint, then Bruhat and Tits did this by finding a vector bundle M (or equivalently a finitely generated free $\mathbb{C}[[t]]$-module) over $\operatorname{Spec} \mathbb{C}[[t]]$ and a representation of $G(\mathbb{C}((t)))$ in the fiber V over $\mathbb{C}((t))$ such that this representation extends

[8] There is a more general construction which does not depend on the choice of such a function f, depending on what Bruhat and Tits call a schematic root datum. This is essentially the data of these compatible extensions. Given such a root datum they show that the group scheme we are looking for always exists. A concave function always gives rise to a schematic root datum. For the details, we refer to chapter 3 of [13].

to faithful actions of $\mathcal{T}$ and $\mathcal{U}_{a,f}$ on M. For this, they consider the Lie algebra $\mathfrak{p}_f$ associated to this data, given by

$$\mathfrak{p}_f = \mathfrak{t}(\mathbb{C}[[t]]) \oplus \bigoplus_{a \in R} \mathfrak{g}_a(t^{f(a)}\mathbb{C}[[t]]).$$

This is naturally a free $\mathbb{C}[[t]]$-module and the associated vector space over $\mathbb{C}((t))$ is just $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$. We can then consider the adjoint action of $G(\mathbb{C}((t)))$ on this vector space. It is clear how this extends to actions of $\mathcal{T}$ and $\mathcal{U}_{a,f}$ on $\mathfrak{p}_f$. Now, we define $\mathcal{G}_f$ as the group scheme generated by $\mathcal{T}$ and $\mathcal{U}_{a,f}$ inside $\mathrm{Aut}(M)$. The general case, is then deduced from this adjoint case.

To summarize, we have the following theorem due to Bruhat and Tits [13, Theorem 3.8.1/3.8.3/4.5.4].

Theorem 3.6. *Let $f : \Phi \to \mathbb{Z}$ be a concave function. Then there exists a unique up to unique isomorphism smooth, affine group scheme $\mathcal{G}_f$ extending $G((t))$ such that*

1. *The injection $T \to G((t))$, resp. $U_a \to G((t))$ extends to an isomorphism of $\mathcal{T}$, resp. $\mathcal{U}_{a,f}$, with a closed subgroup scheme of $\mathcal{G}_f$,*
2. *For all systems of positive roots R^+ (resp. negative roots R^-), we have an isomorphism of $\prod_{a \in R^+} \mathcal{U}_{a,f}$ (resp. $\prod_{a \in R^-} \mathcal{U}_{a,f}$) onto a closed subgroup scheme $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) given by taking the product of all factors.*
3. *Taking the product gives an isomorphism of $\mathcal{U}^- \times \mathcal{T} \times \mathcal{U}^+$ with an open subscheme of $\mathcal{G}_f$. This subscheme is often called the **big cell**.*

*We call group schemes of this type **parahoric group schemes**.*

We denote the special fiber of this group scheme by $\overline{\mathcal{G}_f}$. Because we will later want to be able to distinguish between weak and strong parahoric endomorphisms, it is important to know what Levi subgroups of this special fiber will look like. We have the following result by Bruhat and Tits [13, Proposition 4.6.10/Corollary 4.6.12].

Theorem 3.7. *The root system R_f of the Levi subgroup of $\overline{\mathcal{G}_f}$ is given by those $a \in R$ for which $f(a) + f(-a) = 0$.*

A maximal torus of this group, is of course just a maximal torus of G. An intuitive argument for this can be seen as follows. Let $t^{f(a)}x$ and $t^{f(-a)}y$ be $\mathbb{C}[[t]]$-points of $\mathcal{U}_{a,f}$ and $\mathcal{U}_{-a,f}$ respectively. The commutator of these two elements is given by $t^{f(a)+f(-a)}(xy-yx)$ where $(xy-yx)$ is a $\mathbb{C}[[t]]$-point of $\mathcal{T}$. If f is concave, it can be easily shown from Equation 3 that

$$0 \leq f(a) + f(-a)$$

for all $a \in \Phi$. If this is a strict inequality, then we see that

$$[\mathcal{U}_{a,f}(\mathbb{C}[[t]]), \mathcal{U}_{-a,f}](\mathbb{C}[[t]]) \subset \mathcal{T}(t\mathbb{C}((t))).$$

Under the surjective reduction morphism $\mathcal{G}_f(\mathbb{C}[[t]]) \to \overline{\mathcal{G}_f}(\mathbb{C})$, we see that the image of this commutator is then zero. Intuitively, this tells us that the image of one of the two root subgroups is zero. Then the image of the other one is either also zero, or a part of the unipotent radical. So, the roots we are left with to form the Levi subgroup are those for which equality holds. For a rigourous argument, we refer to chapter 4.6 of [13].

In the case that f is the function we associated to a point θ of the apartment, we see that this equality will hold if and only if $\langle \theta, a \rangle$ was already an integer. This happens if and only if θ lies in a wall associated to an affine root whose vector part is a. Otherwise, we have $f(a) + f(-a) = 1$. A system of simple roots for the Levi subgroup associated to the parahoric subgroup of a point in the Weyl alcove is thus given by exactly the vector parts of the simple affine roots whose walls the point lies on. It follows that under the correspondence between facets of the Weyl alcove or conjugacy types of parahoric subgroups and proper subsets of the affine Dynkin diagram, the subset of the affine Dynkin diagram is exactly the Dynkin diagram of the Levi subgroup

We will mostly be interested not in the actual Levi subgroup, but in the projection to the Levi subalgebra of the parahoric Lie algebra

$$\mathfrak{p}_f = \mathfrak{t}(\mathbb{C}[[t]]) \oplus \bigoplus_{a \in R} \mathfrak{g}_a(t^{f(a)}\mathbb{C}[[t]]).$$

If we consider this as an infinite-dimensional Lie algebra over $\mathbb{C}$, then

$$\mathfrak{l}_f = \mathfrak{t}(\mathbb{C}) \oplus \bigoplus_{a \in R_f} \mathfrak{g}_a(t^{f(a)}\mathbb{C})$$

is exactly the Levi subalgebra, and the reduction is given by projecting onto this subspace. Our local model for strong parahoric endomorphisms will later be exactly those elements of $\mathfrak{p}_f$ whose reduction to $\mathfrak{l}_f$ is zero.

3.2 Parahoric Bundles

3.2.1 Parahoric Group Schemes

Now that we have defined suitable group schemes locally, we are ready to look at group schemes over the full projective curve C which will generalise the parabolic structure. The idea is to consider group schemes that are genericaly reductive but at fixed points look like parahoric groups. We fix a reduced divisor D.

These group schemes were first considered by Pappas and Rapoport in [35] and [34]. The definition we give here is following that of Heinloth in [24].

Definition 3.8. A **parahoric Bruhat Tits group scheme** over C is a smooth affine group scheme $\mathcal{G}$ over C such that

1. All the fibers of $\mathcal{G}$ are connected.
2. The generic fiber of $\mathcal{G}$ is G.
3. For $x \in D$, and $\mathcal{D}_x$ a formal neighborhood of x, $\mathcal{G}|_{\mathcal{D}_x}$ is a parahoric group scheme.

If moreover

$$\mathcal{G}|_{C\backslash D} \cong G \times (C\backslash D),$$

we call the group scheme *generically split*.

The divisor D is sometimes denoted as $Ram(\mathcal{G})$ and points of this divisor are often called *points of bad reduction*.

To construct generically split group schemes, we can use the following lemma.

Lemma 3.9. *[15, Lemma 3.18] Let C be an algebraic curve over a field k, with function field $k(C)$. Let $\hat{K}$ be the completion of this function field. Let $x \in C$ be a closed point, and denote by $\mathcal{D}_x$ a formal neighborhood. Assume that we have a triple $(\mathcal{G}_1, \mathcal{G}_2, f)$ such that*

1. *$\mathcal{G}_1$ is an affine group scheme of finite type over $X\backslash\{x\}$,*
2. *$\mathcal{G}_2$ is an affine group scheme of finite presentation over $\mathcal{D}_x$,*
3. *and $f : \mathcal{G}_1 \times_{X\backslash\{x\}} \operatorname{Spec}\hat{K} \to F \times_{\mathcal{D}_x} \operatorname{Spec}\hat{K}$ is an isomorphism of group schemes over $\operatorname{Spec}\hat{K}$.*

Then there exists a unique [9] affine group scheme $\mathcal{G}$ of finite type over X such that

$$\begin{cases} \mathcal{G} \times_X X\backslash\{x\} \cong \mathcal{G}_1 \\ \mathcal{G} \times_X \mathcal{D}_x \cong \mathcal{G}_2, \end{cases}$$

and these isomorphisms are compatible with f. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are smooth, then so is $\mathcal{G}$.

[9]Unique in the sense that under an appropriate notion of isomorphism of the given data, we get an isomorphism of the corresponding group schemes.

If we have fixed a parahoric subgroup P_i for each $x_i \in D$ with associated group scheme $\mathcal{G}_i$, then we can construct a parahoric Bruhat Tits group scheme $\mathcal{G}$ such that for each $x_i \in D$,

$$\mathcal{G}|_{\mathcal{D}_{x_i}} \cong \mathcal{G}_i,$$

by repeatedly applying this lemma.

3.2.2 Parahoric Bundles

We now consider bundles for these group schemes.

Definition 3.10. Let $\mathcal{G}$ be a group scheme over C. A **$\mathcal{G}$-bundle** is given by a flat morphism $\pi : \mathcal{P} \to X$ locally of finite presentation with an action of $\mathcal{G}$, $\sigma : \mathcal{G} \times_C \mathcal{P} \to \mathcal{P}$, such that

$$(\sigma, p_2) : \mathcal{G} \times_C \mathcal{P} \to \mathcal{P} \times_C \mathcal{P}$$

is an isomorphism, where p_2 denotes the projection onto the second factor. We say that a $\mathcal{G}$-bundle is **trivial** on $U \subset C$ if there is an isomorphism $\mathcal{P}|_U \cong \mathcal{G}|_U$.

A **parahoric bundle** is a bundle for a parahoric Bruhat-Tits group scheme $\mathcal{G}$. We denote the stack of these bundles by $\mathrm{Bun}_{\mathcal{G}}$. This is a smooth algebraic stack locally of finite type.

In the case that $\mathcal{G} = G \times C$ is a constant group scheme, this reduces to the definition of a G-principal bundle.

Example 3.11.

1. Let $\mathcal{G}$ be a generically split parahoric Bruhat Tits group scheme and suppose that for all $x \in D$, $\mathcal{G}_{\mathcal{D}_x}$ corresponds to a parabolic subgroup P_x of G, or a subgroup of $G[[t]]$. In this case, we can view $\mathcal{G} \subset G \times C$ as a subgroup scheme of the constant group scheme. By extension of structure group, we then get a principal G-bundle E over C. Moreover, for each $x \in D$, we have a canonical reduction of structure group of E_x from G to P_x. So, we see that for these parahoric group schemes, we recover quasi-parabolic bundles, i.e a bundle with flags at the marked points, but without a choice of parabolic weights. The parabolic weights can be added by giving for each x a point of the standard Weyl alcove lying in the facet corresponding to P_x.

2. Suppose for simplicity that $D = \{x\}$, a single point. We consider here $G = \mathrm{SL}(n, \mathbb{C})$ and denote by $\mathcal{H}_i$ the group scheme over $\mathbb{C}[[t]]$ which corresponds to the proper subset of the affine Dynkin diagram where the vertex corresponding to a_i is left out. We let $\mathcal{G}_i$ denote the generically split parahoric Bruhat Tits group scheme with

$$\mathcal{G}_i|_{\mathcal{D}_x} = \mathcal{H}_i,$$

where $\mathcal{D}_x$ is a formal neighborhood of x.

Let $\mathcal{P}$ be a bundle for $\mathcal{G}_i$. On $U = C\backslash\{x\}$, the group scheme $\mathcal{G}_i$ is isomorphic to $\mathrm{SL}(n,\mathbb{C})\times U$, and the theorem of Harder [21] implies that $\mathcal{P}|_U \cong \mathrm{SL}_n \times U$. Similarly, we know from our computations for SL_n that

$$\mathcal{H}_i \cong \mathrm{SL}(n, \mathbb{C}[[t]]),$$

where the isomorphism comes from conjugating by

$$\begin{pmatrix} t1_{i\times i} & 0_{i\times(n-1)} \\ 0_{(n-i)\times i} & 1_{(n-i)\times(n-i)} \end{pmatrix}$$

inside $\mathrm{GL}(n, \mathbb{C}((t)))$. Globally, we get a group morphism

$$\mathcal{G}_i \to \mathrm{GL}(n,\mathbb{C}) \times C.$$

Under extension of structure group and passing to the associated vector bundle, we get a vector bundle E over C. A transition function (with respect to these trivialisations on $C\backslash\{x\}$ and a formal neighborhood around x) of this vector bundle over a punctured formal neighborhood of x is given by

$$\begin{pmatrix} t1_{i\times i} & 0_{i\times(n-1)} \\ 0_{(n-i)\times i} & 1_{(n-i)\times(n-i)} \end{pmatrix} A,$$

where $A \in \mathrm{SL}(n, \mathbb{C}((t)))$ is a transition function defining the bundle $\mathcal{P}$. The determinant of this transition function is t^i. It follows that

$$\det E \cong \mathcal{O}(-i \cdot x).$$

Conversely, because we can always write the transition function of such a vector bundle in the form given by Equation 2, we obtain all vector bundles of this type by this construction. It is then easily seen that this construction sets up a bijection between $\mathcal{G}_i$-bundles and vector bundles E of rank n such that $\det E \cong \mathcal{O}(-i{\cdot}x)$. By considering this example for arbitrary reduced divisors, we can realise all vector bundles E such that $\det E \cong L$ for L some line bundle on C in terms of bundles under some parahoric Bruhat Tits group scheme $\mathcal{G}$ such that for all $x \in C$, $\mathcal{G}|_x$ corresponds to a maximal parahoric subgroup of $\mathrm{SL}(n, \mathbb{C}((t))$. Higgs bundles where the underlying vector bundle is of this type were considered by Hausel and Thaddeus in [22], where they exhibit a beautiful connection between the Hitchin system, mirror symmetry and Langlands duality. If we allow for possible non-maximal parahoric subgroups, we get vector bundles of fixed determinant with a parabolic structure at the corresponding points.

3.2.3 Uniformization

In [34], Pappas and Rapoport conjectured that a form of the uniformization theorem should hold for the stack of $\mathcal{G}$-bundles if the generic fiber of $\mathcal{G}$ is semisimple. The main result needed to establish this was proved by Heinloth in [24].

Theorem 3.12. *Let $\mathcal{G}$ be a parahoric Bruhat Tits group scheme on C such that the generic fiber of $\mathcal{G}$ is semisimple. Then for every $x \in C$, every noetherian scheme S and every $\mathcal{P} \in Bun_{\mathcal{G}}(S)$, there exists an fppf cover $S' \to S$ such that $\mathcal{P}|_{(C\setminus x)\times S'}$ is trivial.*

A crucial ingredient for this theorem is the twisted affine Grassmannian, defined to be the functor $\mathrm{GR}_{\mathcal{G},x}$ that given $x \in C$ assigns a scheme T to $\mathcal{G}$-bundles together with a trivialisation on $C\setminus\{x\}$. We will also need the following result in the final chapter.

Proposition 3.13. *[24, Proposition 2/Lemma 18] The functor $GR_{\mathcal{G},x}$ is represented by an ind-projective ind-scheme.*

To define numerical invariants on $\mathrm{Bun}_{\mathcal{G}}$ we will also want to know what line bundles on this stack look like. This is described by the following theorem

Theorem 3.14. *[24, Theorem 7] Suppose that under the conditions of the previous theorem, the generic fiber of $\mathcal{G}$ is absolutely simple. Then there is an exact sequence*

$$0 \to \prod_{x\in D} X^*(\mathcal{G}_x) \to Pic(Bun_{\mathcal{G}}) \to \mathbb{Z} \to 0,$$

where $X^*(\mathcal{G}_x)$ denotes the cocharacter lattice of $\mathcal{G}_x$.

The morphism to $\mathbb{Z}$ is known as the central charge. For the first morphism, note that restricting to x defines a morphism

$$\mathrm{Bun}_{\mathcal{G}} \to B\mathcal{G}_x.$$

A cocharacter $\chi \in X^*(\mathcal{G}_x)$ defines a $\mathbb{G}_m$-bundle on $B\mathcal{G}_x$, or equivalently a line bundle L_χ. The pullback of this line bundle to $\mathrm{Bun}_{\mathcal{G}}$ is precisely the image of χ under this morphism.

3.2.4 Relation with (Γ, G)-bundles and representations

The motivating question of extending the results of Mehta and Seshadri to the case of general semisimple groups was considered by Balaji and Seshadri in [7] for semisimple G. They do this by showing that parahoric bundles are equivalent to what they call (Γ, G)-bundles, which are equivariant bundles on a cover $\tilde{C} \to C$ with Galois group Γ. We denote the ramification divisor of this cover by D.

Definition 3.15. [7, Definition 2.2.3] A (Γ, G)-bundle on $\tilde{C}$ is a principal G-bundle E with a lift of the action of Γ as bundle automorphisms preserving the action of G.

At a point $y \in Y$ the stabiliser Γ_y is a finite cyclic group, denoted by $\; If \mathrm{N}_y$ is a Γ_y-stable formal neighborhood, then the bundle is locally isomorphic to the (Γ_y, G)-bundle $N_y \times G$ with the Γ_y action given by a representation $\rho_y : \Gamma_y \to G$ as follows:

$$\gamma \cdot (u, g) = (\gamma u, \rho_y(\gamma) g).$$

We denote by τ_y the equivalence class of ρ_y under conjugation by G, and call it the *local type* of E at y. We define

$$\boldsymbol{\tau} := \{\tau_y \,|\, y \in p^{-1}(D)\},$$

and denote by $\mathrm{Bun}^{\boldsymbol{\tau}}_{\tilde{C}}(\Gamma, G)$ the stack of (Γ, G)-bundles with these local types. We now want to associate to this local type the data θ_x of a point in the standard Weyl alcove for each $x \in D$, to produce a parahoric Bruhat Tits group scheme $\mathcal{G}_\theta$ on C. For each $y \in p^{-1}(D)$, Γ_y is a cyclic group of finite order. It thus follows that τ_y is just given by the conjugacy class of the image of a generator, which defines a point θ_y in the standard Weyl alcove. Moreover, if $p(y) = p(y')$, then there exists $\gamma \in \Gamma$ such that $y = \gamma y'$, so they define the same conjugacy class in the standard Weyl alcove. Thus, we get associated to $\boldsymbol{\tau}$, points θ_x in the standard Weyl alcove for each $x \in D$ and these define a parahoric Bruhat Tits group scheme $\mathcal{G}_\theta$. We get the following result dure to Balaji and Seshadri.

Theorem 3.16. *[7, Definition 4.1.5] There is an isomorphism*

$$Bun^{\boldsymbol{\tau}}_{\tilde{C}}(\Gamma, G) \cong Bun_{\mathcal{G}_\theta}.$$

If E is a (Γ, G)-bundle, then the invariant pushforward $p_*(E)^\Gamma$ is a $\mathcal{G}_\theta$-bundle.

To get a moduli space, we need a notion of semistability. Balaji and Seshadri define a *parabolic line bundle* on (C, D) as a pair $(\mathcal{L}, \alpha_x \,|\, x \in D)$ where $\mathcal{L}$ is a line bundle on C and $0 \leq \alpha_x \leq 1$. The *parabolic degree* is then

$$\operatorname{pardeg} \mathcal{L} = \deg(\mathcal{L}) + \sum_{x \in D} \alpha_x.$$

Now let $\mathcal{P}_{k(C)} \subset \mathcal{G}_{k(C)}$ be a maximal parabolic subgroup of the generic fiber of $\mathcal{G}$. For a dominant character χ of $P_{k(C)}$, we get a line bundle L_χ on $C \backslash D$. Balaji and Seshadri show that this line bundle has a canonical extension to a line bundle L^θ_χ, depending only on the local data of θ_x.

Definition 3.17. [7, Definition 6.3.4] A parahoric bundle $\mathcal{P}$ with parahoric weights θ_x for $x \in D$ is **semistable** if for every maximal parabolic subgroup $\mathcal{P}_{k(C)} \subset \mathcal{G}_{k(C)}$, every reduction to $\mathcal{P}_{k(C)}$ and every dominant character χ, we have

$$\operatorname{pardeg}(L_\chi^\theta) \leq 0.$$

It is **stable** if the inequality is always strict.

Under the isomorphism of Theorem 3.16, this agrees with the notion of (semi)stability of (Γ, G)-bundles, which is a straightforward equivariant generalization of the notion of (semi)stability for G-bundles. Similarly, we can define a notion of S-equivalence, and Balaji and Seshadri show the existence of a moduli space for these classes using a GIT construction.

Theorem 3.18. *[7, Theorem 8.1.11] The set $\mathcal{N}_{\mathcal{G}_\theta}$ of S-equivalence classes of semistable bundles, has a natural structure of an irreducible normal projective variety containing the set of isomorphism classes of stable bundles as an open subset.*

3.2.5 Good Moduli Space

In [23] and [4] a numerical invariant is defined for the stack of $\mathcal{G}$-torsors for an arbitrary parahoric group scheme.

We let $\mathcal{L}_{\det}$ be the determinant line bundle whose fiber over a $\mathcal{G}$-bundle $\mathcal{E}$ is given by

$$\mathcal{L}_{\det}|_{\mathcal{E}} = \det(H^*(C, |Ad(\mathcal{E})))^{-1}.$$

Given a character χ_x of $\mathcal{G}_x$ for $x \in Ram(\mathcal{G})$, we get a line bundle $\mathcal{L}_{\chi_x}$ pulled back from the corresponding line bundle on $B\mathcal{G}_x$.

Given a choice of characters $\chi = (\chi_x)_{x \in Ram(\mathcal{G})}$, we denote by

$$\mathcal{L}_{\det,\chi} = \mathcal{L}_{\det} \otimes \bigotimes_{x \in Ram(\mathcal{G})} \mathcal{L}_{\chi_x}$$

this line bundle on $\mathrm{Bun}_{\mathcal{G}}$.This is the line bundle used to define a numerical invariant on $\mathrm{Bun}_{\mathcal{G}}$. To establish monotonicity, the twisted affine Grassmannian $\mathrm{GR}_{\mathcal{G},x}$ is used. To do this, we need the pullback of $\mathcal{L}_{\det,\chi}$ to be ample. This is automatic if $x \notin Ram(\mathcal{G})$ and otherwise we assume the characters are chosen so this is true.

To understand filtrations of the stack $\mathrm{Bun}_{\mathcal{G}}$ we need a notion of parabolic subgroup of a parahoric group scheme. Let η be the generic point of C. Given a generic cocharacter $\lambda\colon \mathbb{G}_{m,\eta} \to \mathcal{G}_\eta$, we get a generic parabolic subgroup $\mathcal{P}_\lambda, \eta$. Denote by $\mathcal{P}_\lambda$ the closure of this generic subgroup in $\mathcal{G}$. Similarly, we define a Levi subgroup $\mathcal{L}_\lambda$. Morphism from Θ can be described using these subgroups.

Lemma 3.19. *[23, Lemma 3.9] Let $\mathcal{G}$ be a a parahoric Bruhat-Tits group scheme. Any morphism $f\colon \Theta \to Bun_{\mathcal{G}}$ with $f(1) \cong \mathcal{E}$ corresponds to*

- *a generic cocharacter $\lambda\colon \mathbb{G}_{m.\eta} \to \mathcal{G}_\eta$ and*
- *a reduction of $\mathcal{E}$ to $\mathcal{P}_\lambda$.*

The collection of cocharacters χ is called *admissible* if

$$2\langle \chi_x, a^\vee \rangle \leq \mathrm{rk}\mathfrak{u}_a \langle a, a^\vee \rangle$$

for all roots a of $\mathcal{G}_x$. Given an admissible collection of characters, it is shown in [23, Proposition 3.16] that HN filtrations exist. We will expand on these filtrations when we discuss complementary polyhedra in the final chapter. These form a Θ-stratification on $\mathrm{Bun}_{\mathcal{G}}$.

Theorem 3.20. *[4, Theorem 8.1/Lemma 8.2] The HN filtrations define a Θ-stratification on $Bun_{\mathcal{G}}$ and the semistable locus $Bun_{\mathcal{G}}^{ss}$ admits a proper good moduli space.*

3.3 Parahoric Higgs bundles

3.3.1 Parahoric Higgs Fields

Throughout this chapter we assume for simplicity that G is semisimple. Most of the results should hold for reductive G as well, but not everything has been worked out in this case. As before a Higgs bundle will be a pair consisting of a bundle $\mathcal{P}$ under the group scheme $\mathcal{G}$ of interest and a section of $\mathrm{ad}(\mathcal{P}) \otimes K(D)$ which respects the structure. A natural question is how we generalise the notion of strong and weak parabolic Higgs fields.

The first is the case of strong parahoric Higgs fields, which are again constructed so that the moduli space of semistable parahoric Higgs bundles will contain the contangent bundle of the moduli space of semistable parahoric Higgs bundles as an open subset. We then get a natural symplectic structure on this side.

The other is the case of weak parahoric Higgs fields, which are sections that respect the parahoric structure. Here, we expect a Poisson structure to appear such that the strong Higgs bundles form a symplectic leaf.

The candidates for the weak parahoric Higgs fields are clear, the adjoint bundle of $\mathcal{P}$ is already set up so that it respects the parahoric structure. It is useful to see what these fields look like. The adjoint bundle is constructed through an associated bundle construction as

$$\mathrm{ad}(\mathcal{P}) = \mathcal{P} \times_{\mathcal{G}} \mathfrak{g},$$

where $\mathfrak{g}$ denotes the bundle of Lie algebras associated to the scheme $\mathcal{G}$ acted on by conjugation.For $x \in Ram(\mathcal{G})$, we have

$$\mathfrak{g}|_{\mathcal{D}_x} = \mathfrak{p}_{\theta_x},$$

where θ_x denotes a point in the Building to which the parahoric group at x is associated. This Lie algebra over $\mathbb{C}[[t]]$ is described in Equation 3.2.

Definition 3.21. Let $\mathcal{G}$ be a parahoric Bruhat Tits group scheme on C, and $\mathcal{P}$ a parahoric $\mathcal{G}$-bundle. A **weak parahoric Higgs field** is an element $\Phi \in H^0(C, \mathrm{ad}(\mathcal{P} \otimes K)$.

The case of strong parahoric Higgs fields is more subtle. Standard deformation theory shows that if a moduli space of parahoric bundles exists, at a bundle $\mathcal{P}$, the tangent space is given by $H^1(C, \mathrm{ad}(\mathcal{P}))$. Applying Serre duality, we see that we want strong parahoric Higgs fields to be sections of $\mathrm{ad}(\mathcal{P})^* \otimes K$. We study this dual bundle $\mathrm{ad}(\mathcal{P})^*$ locally. It is clear that on $C \backslash D$, we just get $\mathfrak{g}^* \times C \backslash D$, which we can identify with $\mathfrak{g} \times C \backslash D$ using the Killing form. In general, the Killing form gives a pairing

$$\kappa : \mathrm{ad}(\mathcal{P}) \otimes \mathrm{ad}(\mathcal{P}) \to \mathcal{O}_C$$

Restricting this to $\mathcal{D}_x$ for $x \in D$, we get a pairing

$$\kappa : \mathfrak{p}_{\theta_x} \otimes_{\mathbb{C}[[t]]} \mathfrak{p}_{\theta_x} \to \mathbb{C}[[t]].$$

This pairing will not be non-degenerate at x, as the fiber $\mathcal{G}_x$ at $x \in D$ will not in general be semisimple. In fact, it can only be semisimple if we consider a maximal parahoric subgroup. We will instead study the annihilator of $\mathfrak{p}_{\theta_x}$ under κ. We recall that

$$\mathfrak{g}_x = \mathfrak{t}(\mathbb{C}[[t]]) \oplus \bigoplus_{a \in \Phi} \mathfrak{g}_a(t^{m_a(\theta_x)}\mathbb{C}[[t]]).$$

For the torus part, we see that we have to take $\mathfrak{t}(t\mathbb{C}[[t]])$ as we can find a basis such that $\kappa|_{\mathfrak{t}}$ is non-degenerate and diagonal. An element of $\mathfrak{g}_a$ will be paired with one in $\mathfrak{g}_{-a}$. This gives zero at x if and only if

$$m_a(\theta_x) + m_{-a}(\theta_x) > 0.$$

In the other case, we need an extra factor of t in $\mathfrak{g}_{-a}$ to annihilate $\mathfrak{g}_a$. Putting this all together, we get that the annihilator is given by

$$\mathfrak{sp}_{\theta_x} = \mathfrak{t}(t\mathbb{C}[[t]]) \oplus \bigoplus_{a \in \Phi_{\theta_x}} \mathfrak{g}_a(t^{m_{-a}(\theta_x)+1}\mathbb{C}[[t]]) \oplus \bigoplus_{a \in \Phi \backslash \Phi_{\theta_x}} \mathfrak{g}_a(t^{m_{-a}(\theta_x)}\mathbb{C}[[t]]). \quad (3.22)$$

Using the description of the Levi subalgebra $\mathfrak{l}_{\theta_x}$ in Equation 3.1.5, these are exactly the elements of $\mathfrak{sp}_{\theta_x}$ which project to zero in $\mathfrak{l}_{\theta_x}$.

It follows that we get a bilinear pairing

$$\mathfrak{p}_{\theta_x} \otimes_{\mathbb{C}[[t]]} \mathfrak{sp}_{\theta_x} \to t\mathbb{C}[[t]]. \tag{3.23}$$

and tensoring with $t^{-1}\mathbb{C}[[t]]$, we get a pairing

$$\mathfrak{p}_{\theta_x} \otimes_{\mathbb{C}[[t]]} t^{-1}\mathfrak{sp}_{\theta_x} \to \mathbb{C}[[t]]. \tag{3.24}$$

Using our explicit description of $\mathfrak{sp}_{\theta_x}$, it follows that this pairing is non-degenerate, i.e for every element that does not reduce to zero at x, there is an element such that their pairing is nonzero at x. Locally we have that

$$\mathfrak{p}^*_{\theta_x} \cong t^{-1}\mathfrak{sp}_{\theta_x} \tag{3.25}$$

as $\mathbb{C}[[t]]$-modules.

To have a description of the dual globally, it would thus suffice to find a vector bundle and a pairing such that locally, we have these descriptions. For this, we analyse the reduction to the Levi subalgebra a bit more. We constructed a map

$$\mathfrak{p}_{\theta_x} \to \mathfrak{l}_{\theta_x}$$

for all $x \in D$. In this way we get a map

$$p : \mathrm{ad}(\mathcal{P}) \to \bigoplus_{x \in D} \mathfrak{l}_{\theta_x} \otimes O_x,$$

where O_x is the skyscraper sheaf at x.

Definition 3.26. We define the bundle of **strong parahoric endomorphisms** as

$$\mathrm{Sad}(\mathcal{P}) = \ker(p).$$

From the above construction, we see that globally the dual is given by

$$\mathrm{ad}(\mathcal{P})^* \cong \mathrm{Sad}(\mathcal{P}) \otimes \mathcal{O}(D).$$

Note that the map p we constructed depends on a choice of trivialisation at all $\mathcal{D}_x$ for $x \in D$, and also a choice of uniformiser for the completion of the local ring at x, to get an isomorphism with $\mathbb{C}[[t]]$. Changing a choice of trivialisation amounts to conjugation by an element of $G[[t]]$, whereas changing the choice of uniformiser involves multiplication by an element of $\mathbb{C}[[t]]^*$. Taking this into account, we should quotient $\mathfrak{l}_{\theta_x}$ by the conjugation action of the Levi subgroup L_{θ_x} and also the action of the rotation torus $\mathbb{C}^*$. However, this does not change the preimage of 0, so we skip these technical steps. For a construction of this evaluation map in full, (but of a different flavour) see Chapter 3 of [43].

3.3.2 Parahoric Higgs bundles

Weak Higgs Fields Weak parahoric Higgs bundles for generically split parahoric Bruhat-Tits group schemes are considered by Kydonakis, Sun and Zhao in [28].

Definition 3.27. A **weak parahoric $\mathcal{G}$-Higgs bundle** over C is a pair $(\mathcal{P}, \phi)$ consisting of

1. A $\mathcal{G}$-bundle $\mathcal{P}$ over C and
2. a weak parahoric Higgs field, $\Phi \in H^0(C, \mathrm{ad}(\mathcal{P}) \otimes K(D))$.

We denote the stack of these bundles by $\mathrm{HBun}_{\mathcal{G}}$. This is an algebraic stack locally of finite type.

Because we are mainly interested in this case, as opposed to the strong case, we will often refer to these just as parahoric Higgs bundles.

A fundamental tool to deal with $\mathcal{G}$-bundles in the work of Balaji and Seshadri is the relation between these bundles over C and certain equivariant bundles on a cover of C, or (Γ, G)-bundles. This is also used by Kydonakis, Sun and Zhao in the case of parahoric Higgs bundles. We know from the work of Balaji and Seshadri that to the parahoric type of $\mathcal{G}$, there corresponds a Galois cover $\tilde{C} \to C$ with Galois group Γ, and for each point $y \in \tilde{C}$ such that the stabiliser Γ_y is non-trivial, a representation $\rho_y : \Gamma_y \to G$. We denote again the collection of the equivalence classes of these representations by $\boldsymbol{\tau}$. We denote the preimage of D under this cover by R.

Definition 3.28. [28, Definition 3.5] Given the above data, a (Γ, G)-Higgs bundle of type $\boldsymbol{\tau}$ over $\tilde{C}$ is a pair consisting of

1. a (Γ, G)-bundle E of type $\boldsymbol{\tau}$ over $\tilde{C}$,
2. A Γ-equivariant section $\Phi \in H^0(\mathrm{ad}(E) \otimes K_{\tilde{C}}(R))$.

We denote the stack of (Γ, G)-Higgs bundles by $\mathrm{HBun}_{\tilde{C}}(\Gamma, G)$.

The following result is a natural extension of the correspondence of Balaji and Seshadri to this setting.

Theorem 3.29. *[28, Theorem 3.7] There is an isomorphism of algebraic stacks*

$$HBun_{\tilde{C}}(\Gamma, G) \cong HBun_{\mathcal{G}}.$$

To construct a moduli space, a stability condition is again introduced. This is a generalization of the condition of Balaji and Seshadri to the case of Higgs bundles. Given a parabolic subgroup $P \subset G$, we can canonically associate a subset R_P of the roots. At a point $x \in D$, we define the parahoric subgroup

$$\mathcal{P}_{\theta_x} = \langle T(\mathbb{C}[[t]]), U_a(t^{m_a(\theta_x)}) \,|\, a \in R_P \rangle.$$

Denote by $\mathcal{P}_{\boldsymbol{\theta}}$ the parahoric Bruhat Tits group scheme we can construct from this data. As usual, for a reduction $\sigma : C \to \mathcal{P}/\mathcal{P}_{\boldsymbol{\theta}}$ and a character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$, we can associate a line bundle $L^{\chi}_{\boldsymbol{\theta}}$ on C. One can show that characters of $\mathcal{P}_{\boldsymbol{\theta}}$ are in bijection with those of P.

Definition 3.30. [28, Definition 4.2] The **parahoric degree** of $\mathcal{P}$ with respect to the reduction σ and the character χ is given by

$$\operatorname{parhdeg} \mathcal{P}(\sigma, \chi) = \deg(L^{\chi}_{\boldsymbol{\theta}}) + \sum_{x \in D} \langle \theta_x, \chi \rangle.$$

We say that the reduction $\sigma : C \to \mathcal{P}/\mathcal{P}_{\boldsymbol{\theta}}$ is compatible with ϕ if there exists a lift $\phi' \in H^0(\mathrm{ad}(\sigma^*\mathcal{P}) \otimes K(D))$ such that the diagram

$$\begin{array}{ccc} & & \mathrm{ad}(\sigma^*\mathcal{P}) \otimes K(D) \\ & \nearrow & \big\downarrow \\ C & \longrightarrow & \mathrm{ad}(\mathcal{P}) \otimes K(D) \end{array}$$

commutes.

Definition 3.31. [28, Definition 4.5] A parahoric $\mathcal{G}$-Higgs bundle $(\mathcal{G}, \Phi)$ is **R-semistable** if for

- any proper parabolic subgroup $P \subset G$,
- any reduction of structure group $\sigma : C \to \mathcal{P}/\mathcal{P}_{\boldsymbol{\theta}}$ compatible with ϕ,
- and any dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$, trivial on the center of $\mathcal{P}_{\boldsymbol{\theta}}$, we have

$$\operatorname{parhdeg} \mathcal{P}(\sigma, \chi) \leq 0. \tag{3.32}$$

It is **R-stable** if the inequality is always strict.

The R in this definition stands for Ramanathan. There is a similar notion of R-(semi)stability for (Γ, G)-Higgs bundles, and these are compatible with the isomorphism of stacks. Again, we can also formulate a notion of S-equivalence. On the level of the stack, this means that the closures of the points corresponding to these Higgs bundles intersect. The main theorem of Kydonakis, Sun and Zhao is then the following.

Theorem 3.33. *[28, Theorem 6.1] There is a quasi-projective scheme $\mathcal{M}_{\mathcal{G}}$ which is a coarse moduli space for the stack of R-semistable parahoric $\mathcal{G}$-Higgs bundles. The geometric points of this scheme parametrise S-equivalence classes of R-semistable parahoric $\mathcal{G}$-Higgs bundles. Moreover, there exists a complex Poisson structure on this space.*

The complex Poisson structure is obtained in a similar way to that on the moduli space of weak parabolic Higgs bundles with a construction using Atiyah algebroids.

These parahoric $\mathcal{G}$-Higgs bundles have also been studied from a different perspective by Yun in [43]. The main difference is that in this case, Higgs field are sections of $\mathrm{ad}(\mathcal{P}) \otimes \mathcal{O}(D')$ where D' is an arbitrary divisor of degree bigger than $2g-2$, where g is the genus of the curve. Using the moduli stacks for different parahoric group schemes, Yun constructs generalisations of the Grothendieck-Springer resolution and shows that the cohomology of the fibers carries representations of affine Hecke algebras.

Strong Higgs Fields The case of strong parahoric Higgs fields has been considered by Baraglia, Kamgarpour and Varma in [8].

Definition 3.34. A **strong parahoric $\mathcal{G}$-Higgs bundle** over C is a pair $(\mathcal{P}, \phi)$ consisting of

1. A $\mathcal{G}$-bundle $\mathcal{P}$ over C and
2. a parahoric Higgs field, $\Phi \in H^0(C, \mathrm{Sad}(\mathcal{P}) \otimes K(D)$.

We denote the stack of these bundles by $\mathrm{HBun}_{\mathcal{G}}^{+}$. This is an algebraic stack locally of finite type.

Suposse $c_1, \ldots, c_n$ are a basis of invariant homogeneous polynomials of degree d_i on the Lie algebra $\mathfrak{g}$ with the adjoint action of G. For simplicity, we assume that the points corresponding to the parabolic subgroups of $\mathcal{G}$ at x lie in the standard alcove. Then it follows that $\mathfrak{sp}_{\theta_x} \subset t^{-1}\mathfrak{g}(\mathbb{C}[[t]])$. So, after choosing a local trivialisation at x, we may assume that Φ is a section of $\mathfrak{g} \otimes K(D)$. Applying c_i, we locally get a section of $(K(D))^{d_i}$. Changing the local trivialisation changes

Φ by conjugation. So the section we get locally actually does not depend on the choice of trivialisation, and we get a well-defined global section $c_i(\Phi)$ of $(K(D))^{d_i}$.

It follows from this discussion, that similar to the case of ordinary Higgs bundles, we have the **Hitchin system**

$$h_{\mathcal{G}} : \mathrm{HBun}_{\mathcal{G}}^{+} \to B_{\mathcal{G}} := \bigoplus_{i=1}^{n} H^0(C, (K(D))^{d_i}) : (\mathcal{P}, \Phi) \to (c_1(\Phi), \dots, c_n(\Phi)),$$

and we continue to call $B_{\mathcal{G}}$ the **Hitchin base**.

The main goal is to show that this is a completely integrable system. Elementary deformation theory again tells us that the tangent space to the moduli space of parahoric bundles is given by $H^1(C, \mathrm{ad}(\mathcal{P}))$. Applying Serre duality, we see that the cotangent space is given by $H^0(C, \mathrm{Sad}(\mathcal{P} \otimes K(D)))$, and it follows that

$$H_{\mathcal{G}}^{+} \cong T^*\mathrm{Bun}_{\mathcal{G}}.$$

Naively, this gives a symplectic structure on the stack $\mathrm{HBun}_{\mathcal{G}}^{+}$. However, if we are on an algebraic stack with positive-dimensional stabilisers, then more care needs to be taken when considering symplectic structures. In general, it doesn't even have to be true that

$$\dim(T^*\mathcal{Y}) = 2 \dim \mathcal{Y}$$

for a smooth algebraic stack $\mathcal{Y}$. Because of the fact that the stabilisers of points can have positive dimension, the dimension of the cotangent space can jump on different points of the stack. So, the cotangent sheaf of differentials isn't even a vector bundle in general. The way to solve this would be to take a resolution of this cotangent sheaf by vector bundles, or phrased differently, calculate the cotangent complex. This naturally leads to derived algebraic geometry, and the notion of shifted symplectic structures, see [33]. To have an honest symplectic structure, we would then look for a 0-shifted symplectic structure on the derived version of this stack.

Luckily, the naive thing actually does work out in this setting. Beilinson and Drinfeld [10] introduced the notion of a **very good stack** to make precise when this naive picture does work out. A smooth stack $\mathcal{Y}$ is **very good** if

$$\operatorname{codim}\{y \in \mathcal{Y} \mid \dim(G_y) = d\} > d \text{ for all } d > 0.$$

This implies that if $\mathcal{Y}^0$ is the largest Deligne-Mumford substack of $\mathcal{Y}$, then $T^*\mathcal{Y}^0$ is dense in $T^*\mathcal{Y}$, and also that

$$\dim(T^*\mathcal{Y}) = 2 \dim \mathcal{Y}.$$

Using properties of the nilpotent cone, $h_{\mathcal{G}}^{-1}(0)$, and the description of $\mathcal{G}$-bundles in terms of (Γ, G)-bundles, it is shown that $\mathrm{Bun}_{\mathcal{G}}$ is a very good stack, so in particular, we can naively do symplectic geometry on its cotangent bundle. They then go on to show that this map consists of Poisson commuting functions, and defines a Lagrangian fibration. To complete the proof that this defines a completely integrable system, they show that there exists an open subset $U \subset B_{\mathcal{G}}$ such that $h_{\mathcal{G}}^{-1}(U) \to U$ is a smooth, proper map and the fibers are $Z(G)$-gerbes over a disjoint union of abelian varieties.

Crucial ingredients here are the (Γ, G)-bundle description, and a version of the parahoric non-abelian Hodge correspondence, proven by Biquard, Garcia-Prada and Mundet i Riera in [11].

Filtrations We can expand the description of filtrations of $\mathrm{Bun}_{\mathcal{G}}$ for arbitrary parahoric Bruhat-Tits group schemes from [23] to the stack $\mathrm{HBun}_{\mathcal{G}}$. We will from now on consider Higgs bundles with Higgs fields valued in an arbitrary line bundle $\mathcal{L}$ on C. We consider the pullback of the line bundle $\mathcal{L}_{\mathrm{det}}$ to the stack $\mathrm{HBun}_{\mathcal{G}}$. Given a morphism $f\colon \Theta \to \mathrm{HBun}_{\mathcal{G}}$ with $f(1) \cong (\mathcal{E}, \varphi)$ we get an induced morphism $\Theta \to \mathrm{Bun}_{\mathcal{G}}$, and by Lemma 3.19 we get

- a generic cocharacter $\lambda\colon \mathbb{G}_{m,\eta} \to \mathcal{G}_\eta$ and
- a reduction of $f(1) \cong \mathcal{E}$ to $\mathcal{P}_\lambda$, denoted by $\mathcal{E}_\lambda$

The morphism will factor through $\mathrm{HBun}_{\mathcal{G}}$ if we can descend the Higgs field to the $\mathcal{P}_\lambda$-bundle $\mathcal{E}_\lambda$. The reduction induces an inclusion

$$\mathrm{ad}(\mathcal{E}_\lambda) \otimes \mathcal{L} \subset \mathrm{ad}(\mathcal{E}) \otimes \mathcal{L}.$$

A filtration of $\mathcal{E}$ will be a filtration of $(\mathcal{E}, \varphi)$ if $\varphi \in H^0(C, \mathrm{ad}(\mathcal{E}_\lambda) \otimes \mathcal{L})$. This leads to the following description of filtrations.

Lemma 3.35. *Let G be a a parahoric Bruhat-Tits group scheme. Any morphism $f\colon \Theta \to HBun_{\mathcal{G}}$ with $f(1) \cong (\mathcal{E}, \varphi)$ corresponds to*

- *a generic cocharacter $\lambda\colon \mathbb{G}_{m.\eta} \to \mathcal{G}_\eta$ and*
- *a reduction of $\mathcal{E}$ to $\mathcal{P}_\lambda$ denoted by $\mathcal{E}_\lambda$ such that*

$$\varphi \in H^0(C, \mathrm{ad}(\mathcal{E}_\lambda \otimes \mathcal{L})).$$

3.3.3 The Weak Parahoric Hitchin Fibration

The Hitchin system as constructed above for strong parahoric Higgs bundles also makes sense for weak parahoric Higgs bundles in the generically split case. When $\mathcal{G}$ is not generically split the construction above cannot be done identically. However, it still holds if we consider sections away from the points of bad reduction. So we get a map

$$h_{\mathcal{G}} : \mathrm{HBun}_{\mathcal{G}} \to B_{\mathcal{G}} := \bigoplus_{i=1}^{n} H^0(C \backslash Ram(\mathcal{G}), (K(D))^{d_i}) : (\mathcal{P}, \phi) \to (c_1(\phi), \ldots, c_n(\phi)).$$

In [43, Lemma 2.5.9] it is shown that this map lands in the subspace $H^0(C \backslash Ram(\mathcal{G}), (K(D))^{d_i}) : (\mathcal{P}, \phi) \to (c_1(\phi), \ldots, c_n(\phi))$. So for an arbitrary parahoric group scheme, we get a Hitchin fibration

$$h_{\mathcal{G}} : \mathrm{HBun}_{\mathcal{G}} \to B_{\mathcal{G}} := \bigoplus_{i=1}^{n} H^0(C, (K(D))^{d_i}) : (\mathcal{P}, \phi) \to (c_1(\phi), \ldots, c_n(\phi)), \quad (3.36)$$

Our goal is to study this map and prove that all the interesting properties present in the ordinary and parabolic case persist. More precisely, we would like to show that as soon as a good moduli space for the semistable locus of this stack with respect to a suitable stability condition exists, then the Hitchin system descends to a proper (and hopefully even projective) morphism from this moduli space to the Hitchin base.

4 Anti-invariant Higgs bundles

In this chapter, we prove the existence of a good moduli space for semistable anti-invariant Higgs bundles which is proper over the Hitchin base. Given a curve with an involution σ, these are triples consisting of a vector bundle with trivial determinant E, an isomorphism $E^* \to \sigma^* E$ and a Higgs field compatible with this isomorphism. These are equivalent to parahoric Higgs bundles on the quotient curve C/σ. They were studied in [45] and [46].

We start off by reviewing some basic properties of the stacks of Higgs bundles and introduce the definition of anti-invariant (Higgs) bundles.

To define semistability and also construct a Θ-stratification, we analyse filtrations of anti-invariant Higgs bundles. These are very similar to those of orthogonal bundles. The existence of a Θ-stratification is proven.

Finally, we apply the criteria of [4] to prove the existence of a good moduli space and the properness over the Hitchin base is proven using the same property for ordinary Higgs bundles.

In this and the next chapter, we consider Higgs bundles valued in an arbitrary line bundle $\mathcal{L}$ on C.

4.1 Ordinary Higgs bundles

We briefly summarise some definitions and results about ordinary Higgs bundles that will be used later in this chapter. We denote by HBun the stack of Higgs bundles on C valued in $\mathcal{L}$, by the stack of Higgs bundles with rank r and degree d and by $\mathrm{HBun}_{\mathrm{SL}_n}$ the stack of Higgs bundles with trivial determinant and trace-free Higgs field.

These are all algebraic stacks, locally of finite type, see for example [14, Theorem 7.18].

Proposition 4.1. *The stacks HBun and $HBun_{SL_n}$ have affine diagonal.*

Proof. We need to show that for a morphism $(a, b)\colon S \to \mathrm{HBun} \times \mathrm{HBun}$ the algebraic space $\underline{\mathrm{Isom}}_{\mathrm{HBun}(S)}(a, b)$ is a scheme affine over S. The morphisms a, b correspond to Higgs bundles $(E_a, \varphi_a), (E_b, \varphi_b)$ on $C \times S$ and elements of $\underline{\mathrm{Isom}}_{\mathrm{HBun}(S)}(a, b)$ correspond to vector bundle isomorphisms $\psi\colon E_a \to E_b$ that commute with the Higgs fields, i.e. the following diagram is commutative

$$\begin{array}{ccc} E_a & \xrightarrow{\varphi_a} & E_a \otimes \pi^*\mathcal{L} \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \psi \otimes \mathrm{id}_{\pi^* L}} \\ E_b & \xrightarrow{\varphi_b} & E_b \otimes \pi^*\mathcal{L}. \end{array} \tag{4.2}$$

There is a natural forgetful morphism HBun $\to$ Bun from the stack of Higgs bundles on C to the stack of vector bundles on C forgetting the Higgs field. This gives rise to a morphism $(a', b') : S \to \mathrm{Bun} \times \mathrm{Bun}$, which corresponds to the vector bundles E_a, E_b. Elements of $\underline{\mathrm{Isom}}_{\mathrm{Bun}(S)}(a', b')$ are vector bundle isomorphisms $f : E_a \to E_b$.

Because Bun has affine diagonal, we know that $\underline{\mathrm{Isom}}_{\mathrm{Bun}(S)}(a', b')$ is a scheme affine over S, and it is easy to see from the above interpretation of its elements that $\underline{\mathrm{Isom}}_{\mathrm{HBun}(S)}(a, b)$ is a subscheme. Commuting with the Higgs fields in Diagram 4.2 is a closed condition, so it follows that $\underline{\mathrm{Isom}}_{\mathrm{HBun}(S)}(a, b)$ is an affine scheme over S and HBun has affine diagonal.

For HBun_{SL_n} the result follows using a similar description of the isomorphisms as vector bundle isomorphisms that commute with both the Higgs fields and the respective trivialisations of the determinant. □

We will make use of the following enlargement of the stack of Higgs bundles.

Definition 4.3. For a scheme S, we define HCoh(S) as the groupoid of flat, coherent sheaves $\mathcal{E}$ on $C \times S$ together with a morphism $\varphi \colon \mathcal{E} \to \mathcal{E} \otimes \pi^*\mathcal{L}$. This defines the **stack of Higgs sheaves on** C, denoted by HCoh.

The stack HCoh is algebraic, locally of finite type. There is a natural forgetful morphism HCoh $\to$ Coh to the stack of coherent sheaves on C forgetting the Higgs field. Because HBun is the preimage of Bun under this morphism, it follows that HCoh contains HBun as an open substack.

Proposition 4.4. *HCoh is affine over Coh.*

Proof. If $S \to \mathrm{Coh}$ is a morphism from a scheme S corresponding to a family of coherent sheaves $\mathcal{E}$, the fiber product of this morphism with HCoh is given by $H^0(\mathrm{End}(\mathcal{E}) \otimes \pi^* L)$. It follows that the morphism HCoh $\to$ Coh is affine. □

4.2 Anti-invariant Higgs Bundles

From now on we assume C is equipped with an involution σ. Anti-invariant vector bundles on such a curve were studied by Zelaci in [46], [45] and [44].

Definition 4.5. An **anti-invariant vector bundle** (E, ψ, δ) is a vector bundle E on C together with a trivialisation of the determinant δ and an isomorphism

$$\psi \colon \sigma^* E \to E^*.$$

The isomorphism ψ is said to be

1. **σ-symmetric** if ${}^t\sigma^*\psi = \psi$.

2. **σ-alternating** if ${}^t\sigma^*\psi = -\psi$. The **type** τ of a σ-alternating vector bundle is defined as
$$\tau = (\tau_p)_{p\in R} \mod \pm 1$$
where R is the ramification divisor of σ and $\tau_p \in \{\pm 1\}$ is the Pfaffian of the isomorphism $\psi\colon (\sigma^*E)_p \to (E^*)_p$.

We denote the **stack of σ-symmetric anti-invariant bundles** by $\mathrm{Bun}_{\mathrm{SL}_n}^{\sigma,+}$ (resp. **σ-alternating anti-invariant bundles** by $\mathrm{Bun}_{\mathrm{SL}_n}^{\sigma,\tau}$)

Remark 4.6.

1. As outlined in [44, Section 3.2.1], if E is a stable vector bundle, then the isomorphism ψ is unique up to scalar multiplication. This implies that ${}^t\sigma^*\psi = \pm\psi$, so such bundles are always either σ-symmetric or σ-alternating.

2. We are mainly interested in the σ-symmetric and σ-alternating cases, but the results discussed throughout the rest of this section hold without any restrictions on the isomorphism ψ.

3. From now on, we will not always write out the trivialisation δ, but it will always be implicitly assumed.

Let $\mathcal{G}$ be a group scheme on C with a lift of the involution σ. A $(\sigma, \mathcal{G})$-bundle as considered in [7, Definition 2.2.3] is given by a $\mathcal{G}$-torsor E on C and a lift of the action of σ to E such that for $p \in E$ and $g \in \mathcal{G}$ we have $\sigma(g \cdot p) = \sigma(g) \cdot \sigma(p)$. It is shown in [45, Proposition 2.4] that anti-invariant vector bundles on C correspond to $(\sigma, \mathcal{G})$-bundles for various group schemes $\mathcal{G}$. For σ-symmetric bundles the associated group scheme is the constant group scheme $\mathrm{SL}_r \times C$ with action $\sigma(g, x) = ({}^tg^{-1}, x)$. For σ-alternating vector bundles, there is a very analogous construction with a slightly more complicated group scheme (that is still étale locally isomorphic to SL_r away from R).

Furthermore, in [7, Theorem 4.1.5] it is shown that $(\sigma, \mathcal{G})$-bundles correspond to torsors for the group scheme $\mathcal{H} = \mathrm{Res}_{C/C'}(\mathcal{G})^\sigma$ of σ-invariants in the Weil restriction of the group scheme to the quotient $C' = C/\sigma$. In the case of anti-invariant bundles, it is also shown in [45, Proposition 2.4] that this Weil restriction is a parahoric group scheme as defined in [34]. It subsequently follows from [24, Proposition 2.1] that $\mathrm{Bun}_{\mathrm{SL}_n}^{\sigma,+}$ and $\mathrm{Bun}_{\mathrm{SL}_n}^{\sigma,\tau}$ are smooth algebraic stacks, locally of finite type.

Remark 4.7. Anti-invariant vector bundles exhibit some analogies with symplectic and orthogonal vector bundles. If p is a fixed point of σ, it is shown in [46, Section 2]that the fiber of the parahoric group schemes over p is an extension of SO_r in the σ-symmetric case, and Sp_r for σ-alternating.

From now on, we also assume that the line bundle is invariant with a fixed isomorphism, i.e. $\sigma^*\mathcal{L} \cong \mathcal{L}$. This assumption can be made if $\mathcal{L}$ is the canonical bundle, for example.

Definition 4.8. An **anti-invariant Higgs bundle** $(E, \psi, \delta, \varphi)$ on C is an anti-invariant vector bundle (E, ψ) together with a traceless vector bundle morphism $\varphi\colon E \to E \otimes \mathcal{L}$ such that

$$\begin{array}{ccc} \sigma^*E & \xrightarrow{\psi} & E^* \\ {\scriptstyle \sigma^*\varphi}\downarrow & & \downarrow{\scriptstyle -{}^t\varphi} \\ \sigma^*E \otimes \mathcal{L} & \xrightarrow[\psi\otimes id_{\mathcal{L}}]{} & E^* \otimes \mathcal{L} \end{array} \tag{4.9}$$

We call an anti-invariant Higgs bundle **σ-symmetric** (resp. **σ-alternating**) when the underlying anti-invariant bundle is.

We denote the **stack of σ-symmetric anti-invariant Higgs bundles** by$\mathrm{HBun}_{\mathrm{SL}_n}^{\sigma,+}$ (resp. **σ-alternating anti-invariant bundles** by $\mathrm{HBun}_{\mathrm{SL}_n}^{\sigma,\tau}$)

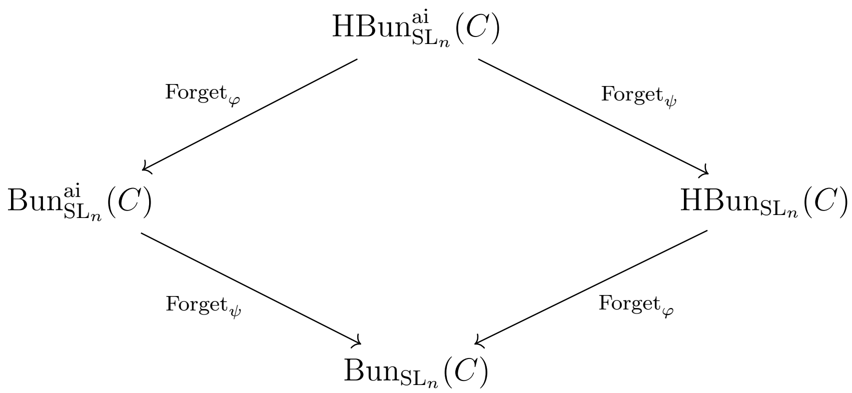
Remark 4.10.

- If we consider σ^*E and E^* as Higgs bundles with respective Higgs fields $\sigma^*\varphi$ and $-{}^t\varphi$, then this definition is the statement that ψ is an isomorphism of Higgs bundles. The reason for the minus sign is that the involution on the group scheme acts infinitesimally by a minus sign on the Lie algebra. This ensures that anti-invariant Higgs bundles correspond to the Higgs analog of $(\sigma, \mathcal{G})$-bundles and hence also parahoric Higgs bundles.
- If we assume that the underlying bundle is stable, it is shown in [44, Theorem 3.12] that an anti-invariant Higgs bundle valued in K corresponds to a cotangent vector at the underlying anti-invariant bundle. Thus, the above stacks are cotangent stacks to $\mathrm{Bun}_{\mathrm{SL}_n}^{\sigma,+}$ and $\mathrm{Bun}_{\mathrm{SL}_n}^{\sigma,\tau}$. This shows that these are algebraic stacks locally of finite type. A similar argument applies for arbitrary line bundles $\mathcal{L}$.

Proposition 4.11. *The stacks $HBun_{SL_n}^{\sigma,+}$ and $HBun_{SL_n}^{\sigma,\tau}$ have affine diagonal.*

Proof. This is proven analogously to Proposition 4.1. □

The stack of anti-invariant Higgs bundles admits a number of useful forgetful morphisms, collected in the following commutative diagram



where the superscript ai denotes either σ-symmetric or -alternating. In this diagram, the arrows pointing to the left are affine.

4.3 Filtrations and Θ-stratification

4.3.1 Filtrations

We want to show the existence of a good moduli space for anti-invariant Higgs bundles and need a notion of semistability to do this. This is done by showing the existence of a Θ-stratification. To do this, we have to understand what morphism from Θ to $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n}$ or $\mathrm{HBun}^{\sigma,\tau}_{\mathrm{SL}_n}$ looks like. Translating these maps into $\mathbb{G}_m$-equivariant information on $\mathbb{A}^1 \times C$ is the purpose of the following lemma.

Lemma 4.12. *A morphism from Θ to $HBun^{\sigma,+}_{SL_n}$ (resp. $HBun^{\sigma,\tau}_{SL_n}$) is equivalent to giving a $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}$ over $\mathbb{A}^1 \times C$ together with a $\mathbb{G}_m$-equivariant σ-symmetric (resp. σ-alternating of type τ) isomorphism $\psi\colon \sigma^*\mathcal{E} \to \mathcal{E}^*$[10] and a $\mathbb{G}_m$-equivariant morphism $\varphi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$ compatible with ψ.*

Proof. From the quotient morphism $\mathbb{A}^1 \to \Theta$, we get a family of anti-invariant Higgs bundles on C over $\mathbb{A}^1$. Factoring through the quotient stack means precisely that the bundle and corresponding sections are $\mathbb{G}_m$-equivariant. □

It is well-known that $\mathbb{G}_m$-equivariant vector bundles on $\mathbb{A}^1 \times C$ correspond to filtered vector bundles on C. We will show that the two equivariant maps serve to put restrictions on the corresponding filtration of an anti-invariant Higgs bundle on C. For this purpose we recall one way of constructing this equivalence, following [6].

[10] Here σ denotes the induced involution on $\mathbb{A}^1 \times C$, and $\sigma^*\mathcal{E}$ and $\mathcal{E}^*$ are endowed with the pullback and the dual of the action on $\mathcal{E}$ respectively.

Given a vector bundle E with an increasing filtration $(F^p(E))_{p\in\mathbb{Z}}$, we can produce a graded $\mathcal{O}_C[t]$-module given by

$$R(E, F^\bullet) = \bigoplus_{p\in\mathbb{Z}} x^p F^p(E). \tag{4.13}$$

This is known as the Rees construction. Because $\mathbb{G}_m$-equivariant morphisms of $\mathbb{G}_m$-equivariant vector bundles have to respect the grading induced by the $\mathbb{G}_m$-action, it follows that $\mathbb{G}_m$-equivariant morphisms on $\mathbb{A}^1$ correspond to endomorphisms between the corresponding vector spaces that respect the filtrations. For φ, this implies that we have to consider filtrations by Higgs subbundles, i.e subbundles F such that $\varphi(F) \subset F \otimes \mathcal{L}$. To deduce a similar conclusion for ψ, we have to figure out what the filtration $\mathcal{E}^*$ induced by the dual action looks like given a filtration of E. This is standard, but we could not find a reference.

In [6, Section 3] the equivalence between filtered vector spaces and $\mathbb{G}_m$-equivariant bundles on $\mathbb{A}^1$ is outlined. To recover the filtration given a $\mathbb{G}_m$-equivariant bundle $\mathcal{E}$ on $\mathbb{A}^1$, for $U = \mathbb{A}^1\backslash 0$ define the space

$$E = \Gamma(U, \mathcal{V}|_U)^{\mathbb{G}}_m \tag{4.14}$$

of $\mathbb{G}_m$-invariant sections away from the origin. This recovers the underlying vector space and is isomorphic to the fiber away from 0. We can define an increasing filtration on this vector space through

$$F^p(E) = \{s \in E \,|\, x^p s \text{ extends to a global section.}\} \tag{4.15}$$

where x is a coordinate on $\mathbb{A}^1$. A useful fact is that $\Gamma(\mathbb{A}^1, \mathcal{E})$ is generated by the set of all $x^p s$ for $p \in \mathbb{Z}$ and $s \in F^p(E)$.

To recover the filtration on the dual vector bundle, the following lemma will be helpful.

Lemma 4.16. *A section $\alpha \in \Gamma(U, \mathcal{E}^*)$ extends to $\mathbb{A}^1$ if and only if for every global section $s \in \Gamma(\mathbb{A}^1, \mathcal{E}) \subset \Gamma(U, \mathcal{E}\mid_U)$, the function $\alpha(s) \in \Gamma(U)$ extends to a global function.*

Proof. We will prove a slightly more general result. Suppose that M is an $R[t]$-module for some commutative ring R. The setup of the lemma corresponds to a morphism $\alpha\colon N = M \otimes_{R[t]} R[t, t^{-1}] \to R[t, t^{-1}]$ in the case $R = \mathbb{C}$. We want to show that there exists a morphism $\beta\colon M \to R[t]$ such that $\alpha = \beta \otimes 1_{R[t,t^{-1}]}$. Furthermore, we can interpret M as a submodule of N. We are moreover given that $\alpha(M) \subset R[t]$. This corresponds to the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & R[t, t^{-1}] \\ \uparrow & & \uparrow \\ M & \xrightarrow{\alpha|_M} & R[t] \end{array}$$

It thus follows that $\alpha|_M$ is the desired extension. □

Given this lemma, the filtration of the dual bundle is characterised by the following proposition.

Proposition 4.17. *Let $\mathcal{E}$ be a $\mathbb{G}_m$-equivariant bundle on $\mathbb{A}^1$ corresponding to a filtered vector space $(E, F^p(E))$. Then the $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}^*$ corresponds to the filtered vector space $(E^*, Ann(F^{-p+1}(E)))$.*

Proof. Let α be a $\mathbb{G}_m$-invariant section of $\mathcal{E}^*$ over U. Using Lemma 4.16, we see that for $p \in \mathbb{Z}$ $x^p(\alpha)$ extends to a global section if and only if for all $k \in Z$ and $v \in F^k(E)$

$$x^p \alpha(x^k v) = x^{p+k} \alpha(v)$$

extends to a global function. Because α and v are both $\mathbb{G}_m$-invariant, it follows that $\alpha(v)$ is a $\mathbb{G}_m$-invariant function, hence a constant function. When $\alpha(v) \neq 0$ this extends if and only if

$$-k \leq p.$$

It follows that $x^p\alpha$ extends if and only if for all $k \in \mathbb{Z}$ such that the $-k > p$, we have that

$$\alpha|_{F^k(V)} = 0.$$

Because the filtration on V is increasing, this is equivalent to

$$\alpha|_{F^{-p+1}(V)} = 0.$$

This proves that

$$F^p(V^*) = \text{Ann}(F^{-p+1}(V)).$$

□

To apply this construction to $\mathbb{G}_m$-equivariant vector bundles on $\mathbb{A}^1 \times C$, we define a sheaf E on C through

$$E(V) = \Gamma(U \times V, \mathcal{E})^{\mathbb{G}_m},$$

for an open $V \subset C$. Because a $\mathbb{G}_m$-invariant section is completely determined by its restriction to the fiber $\{1\} \times V$, this sheaf is isomorphic to $\mathcal{E}|_{\{1\}\times C}$. This gives us a vector bundle E on C.

We similarly define locally free subsheaves $F^p(E)$ of E through

$$F^p(E)(U \times V) = \{s \in E(V) \mid x^p s \text{ extends to } \mathbb{A}^1 \times V\}. \tag{4.18}$$

The following lemma ensures that these are vector subbundles.

Lemma 4.19. *The quotient $E/F^p(E)$ is torsion-free for all $p \in \mathbb{Z}$.*

Proof. Let $\alpha \in E/F^p(E)(V)$ for some open V and suppose that there is a function f on C such that $f\alpha = 0$. If we pick a preimage $\tilde{\alpha} \in E(V)$, then this implies that $x^p f\tilde{\alpha} \in \mathcal{E}(U \times V)$ extends to a section on $\mathbb{A}^1 \times V$, or stated otherwise, the order of vanishing in x of $x^p f\tilde{\alpha}$ is non-negative. But since f is a function on C, multiplying by f cannot change the order of vanishing, and $x^p\tilde{\alpha}$ must also extend to a global section. This implies that $\alpha = 0 \in E/F^p(E)(V)$, and thus the quotient is torsion-free. □

The statement that

$$\Gamma(\mathbb{A}^1 \times V, \mathcal{E}) = \langle x^p s \mid p \in \mathbb{Z}, s \in F^p(E)\rangle \tag{4.20}$$

likewise remains true.

The proof of Lemma 4.16 also applies to a similar extension result for $\alpha \in \Gamma(U \times V, \mathcal{E}^*)$ to $A^1 \times V$.

Putting all of this together, the proof of Proposition 4.17 goes through, which gives the following result.

Proposition 4.21. *Let $\mathcal{E}$ be a $\mathbb{G}_m$-equivariant bundle on $\mathbb{A}^1 \times C$ corresponding to a filtered vector bundle $(E, F^p(E))$ on C. Then the $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}^*$ corresponds to the filtered vector bundle $(E^*, Ann(F^{-p+1}(E)))$.*

To see how this analysis imposes conditions on the filtrations, we define an orthogonal complement with respect to ψ. This notion is due to [46, Proof of Proposition 4.2].

Definition 4.22. Let F be a subbundle of an anti-invariant bundle (E, ψ). We define $F^{\perp_\sigma}$ as the kernel of

$$E \cong \sigma^* E^* \twoheadrightarrow \sigma^* F^*,$$

where the isomorphism is given by $\sigma^*\psi$.

The isomorphism ψ thus identifies $\sigma^* F^{\perp_\sigma}$ with $\mathrm{Ann}(F)$. Putting this together with Lemma 4.12 and Proposition 4.17, we obtain the following proposition.

Proposition 4.23. *Morphisms from Θ into $HBun_{SL_n}^{\sigma,+}$ (resp. $HBun_{SL_n}^{\sigma,\tau}$) correspond to a σ-symmetric (resp. σ-alternating of type τ) anti-invariant Higgs bundle E on C together with a filtration by Higgs subbundles of the form*

$$0 = F^0 \subset F^1 \subset \ldots \subset F^k \subseteq F^{k^{\perp_\sigma}} \subset \ldots F^{1^{\perp_\sigma}} \subset F^{0^{\perp_\sigma}} = E. \tag{4.24}$$

Proof. We have already seen that this corresponds to a filtration of E by Higgs subbundles $(F^p_{p\in\mathbb{Z}}$ together with an isomorphism ψ that respects the filtrations induced on $\sigma^* E$ and E^*. This identifies $\sigma^*(F^p)$ with $\mathrm{Ann}(F^{-p+1})$. By definition, this implies that

$$F^p = F^{-p+1^{\perp_\sigma}}, \tag{4.25}$$

and this allows us to rewrite the filtration in the desired form. □

Remark 4.26. Note that the isomorphism ψ induces an isomorphism of $\sigma^*(F^{k^{\perp_\sigma}}/F^k)$ with $(F^{k^{\perp_\sigma}}/F^k)^*$ and $\sigma^*(F^j/F^{j-1})$ with $(F^{j-1^{\perp_\sigma}}/F^{j^{\perp_\sigma}})^*$. This ensures that the associated graded of the above filtration is again an anti-invariant Higgs bundle. This should be the case as the image of $0 \in \Theta$ corresponds to the associated graded of the filtration.

4.3.2 Semistability

To define the Θ-stratification we will need a line bundle on $\mathrm{HBun}_{\mathrm{SL}_n}^{\sigma,+}$ or $\mathrm{HBun}_{\mathrm{SL}_n}^{\sigma,\tau}$. We will first analyse what semistability looks like.

The stack Bun has a natural line bundle $\mathcal{L}_{\det}$ whose fiber over a vector bundle E is the determinant of its cohomology

$$\mathcal{L}_{\det,E} = \det(H^*(C, \mathrm{End}(E)))^{-1}.$$

A morphism $f\colon \Theta \to \mathrm{Bun}$ corresponds to a filtered vector bundle $(E, (E^i)_{i\in\mathbb{Z}})$. The weight $\mathrm{wt}_{\mathbb{G}_m}(f^*\mathcal{L}_{\det})$ can be calculated in terms of the ranks and degrees of the subbundles E^i. In [23, Section 1.E.c] it is shown that

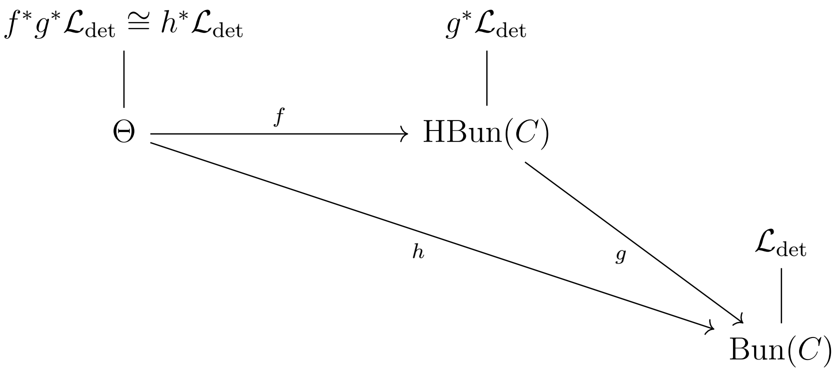
$$\mathrm{wt}_{\mathbb{G}_m}(f^*\mathcal{L}_{\det}) = 2\sum_{i\in\mathbb{Z}} \mathrm{rk}(E^i)(\mathrm{rk}(E) - \mathrm{rk}(E^i))(\mu(E^i) - \mu(E/E^i)), \tag{4.27}$$

where $\mu(E^i) = \frac{\deg(E^i)}{\mathrm{rk}(E^i)}$ is the slope of E^i. Analyzing this formula shows that this number is non-positive if and only if it is so for all 1-step filtrations, i.e. a single non-trivial subbundle. This gives the usual criterion for semistability, namely a vector bundle E is semistable if and only if $\mu(F) \leq \mu(E)$ for all non-trivial subbundles F. In the case where $\deg(E) = 0$, this reduces to the condition that $\deg(F) \leq 0$.

By pulling back along the forgetful morphism $g : \mathrm{HBun} \to \mathrm{Bun}$, we get a line bundle $g^*\mathcal{L}_{\det}$ on the stack of Higgs bundles. A morphism $f : \Theta \to \mathrm{HBun}$ corresponds to a filtration of the bundle by Higgs subbundles. Composing with the forgetful morphism, we get a morphism $h : \Theta \to \mathrm{Bun}$. The weight can be computed as

$$\mathrm{wt}_{\mathbb{G}_m}(f^*g^*\mathcal{L}_{\det}) = \mathrm{wt}_{\mathbb{G}_m}(h^*\mathcal{L}_{\det}). \tag{4.28}$$

We have



So, for semistability of Higgs bundles we get the same slope condition as for vector bundles, but only for filtrations by Higgs subbundles. This recovers the usual definition that a Higgs bundle (E, φ) is semistable if and only if for every Higgs subbundle F, we have that $\mu(F) \leq \mu(E)$.

The line bundle $\mathcal{L}_{\det}$ extends to the stack HCoh, but now to test for semistability, we also allow filtrations by Higgs subsheaves. For any Higgs sheaf, the maximal torsion subsheaf is always a Higgs subsheaf and its slope is always $+\infty$. This implies that a semistable Higgs sheaf is torsion-free and hence a Higgs bundle.

Similarly, for anti-invariant Higgs bundles, we get a slope condition on the filtrations described by Proposition 4.23. Suppose F is a subbundle of degree d of an anti-invariant Higgs bundle E. Because $F^{\perp_\sigma}$ is isomorphic to $\mathrm{Ann}(F)$, it follows that the degree of $F^{\perp_\sigma}$ is also d, and $\mathrm{rk}(F) = \mathrm{rk}(E) - \mathrm{rk}(F)$. Because of this, if

$$0 = F^0 \subset F^1 \subset \ldots \subset F^k \subseteq F^{k\perp_\sigma} \subset \ldots F^{1\perp_\sigma} \subset F^{0\perp_\sigma} = E.$$

is a filtration of E, then Equation 4.27 reduces to

$$\mathrm{wt}_{\mathbb{G}_m}(f^*\mathcal{L}_{\det}) = 4\sum_{i=1}^{k} \mathrm{rk}(F^i)(\mathrm{rk}(E) - \mathrm{rk}(F^i))(\mu(F^i) - \mu(E/F^i)). \tag{4.29}$$

From this formula, we can again reduce to the case of 1-step filtrations. In this case, a 1-step filtration looks like

$$0 \subset F \subset F^{\perp_\sigma} \subset E.$$

combining this with the fact that $\deg(E) = 0$ leads to the following definition from [46, Definition 4.1].

Definition 4.30. Let (E, φ, ψ) be an anti-invariant Higgs bundle. A subbundle F is **σ-isotropic** if $F \subset F^{\perp_\sigma}$ [11]. An anti-invariant Higgs bundle (E, φ, ψ) is **semistable** if and only if for every σ-Higgs subbundle F, we have that

$$\deg(F) \leq 0.$$

In [46, Proposition 4.2] it is shown that an anti-invariant bundle (E, ψ) is semistable if and only if it is stable as a vector bundle.

Lemma 4.31. *Let F be a subbundle of an anti-invariant bundle (E, φ, ψ) such that $\mu(F)$ is maximal among subbundles of E. Then F is a σ-isotropic subbundle.*

Proof. Consider the composition

$$\sigma^* F \to \sigma^* E \to E^* \to F^*,$$

where the middle arrow is the isomorphism ψ. Because $\mu(F^*) = -\mu(F) < 0$, the composition is a morphism from $\sigma^* F$ to one of smaller slope. Because F was chosen so that the slope is maximal among subbundles, it follows that $\sigma^* F$ and F^* are semistable, and the composition is necessarily zero. This implies that F is σ-isotropic. □

Proposition 4.32. *An anti-invariant Higgs bundle (E, φ, ψ) is semistable if and only if the underlying Higgs bundle (E, φ) is semistable.*

Proof. The if direction is clear. The other direction is a consequence of Lemma 4.31. □

4.3.3 Stratification

In this section we show the existence of a Θ-stratification for σ-symmetric bundles, the case of σ-alternating bundles is completely analogous. A crucial ingredient will be the existence of HN-filtrations.

[11] In [46], this is defined as the induced morphism $\psi : \sigma^* F \to F^*$ being zero, but this is equivalent to the one given here.

Lemma 4.33. *Let (E, φ, ψ) be an unstable σ-symmetric anti-invariant Higgs bundle and let*

$$0 = F^0 \subset F^1 \subset \ldots \subset F^k \subseteq F^{k^{\perp_\sigma}} \subset \ldots F^{1^{\perp_\sigma}} \subset F^{0^{\perp_\sigma}} = E.$$

be a filtration by Higgs subbundles denoted by f. Then this filtration maximises $\mu(f) = M^\mu((E, \varphi, \psi))$ if and only if

- $\mu(E/F^k) < \mu(F^k/F^{k-1}) < \ldots < \mu(F_2/F_1) < \mu(F_1)$,
- *and* $F^j/F^{j-1} \oplus F^{j-1^{\perp_\sigma}}/F^{j^{\perp_\sigma}}$ *for* $j = 1, \ldots k$, *and* $F^{k^{\perp_\sigma}}/F^k$ *are semistable σ-symmetric anti-invariant Higgs bundles.*

Using this proposition, we can prove completely analogously to the case of ordinary vector bundles [20, Proposition 1.3.9] and orthogonal/symplectic bundles [16, Proposition 3.2] that HN filtrations exist and are unique.

Proposition 4.34. *Let (E, φ, ψ) be an unstable σ-symmetric anti-invariant Higgs bundle. Then there exists a unique HN filtration of (E, φ, ψ).*

Proof. Lemma 4.31 implies that a maximally destabilisingHiggs subbundle of the underlying Higgs bundle E is also a maximally destabilising σ-isotropic subbundle of the anti-invariant bundle (E, φ, ψ). It follows from this that the HN filtration of the underlying Higgs bundle is also the HN filtration of the anti-invariant Higgs bundle. □

Once we know that HN filtrations exist and are unique, one of the two criteria for the existence of a Θ-stratification is immediate.

Proposition 4.35. *The stack $HBun^{\sigma,+}_{SL_n}$ satisfies HN Boundedness.*

Proof. Suppose $\xi : T \to \mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n}$ is a map from a finite type affine scheme. We know that only finitely many HN types of Higgs bundles occur in this family. Moreover, for a fixed HN type τ the stack parametrising graded Higgs bundles of this HN type is a product of stacks of semistable Higgs bundles, which are known to be quasicompact, and thus its preimage in $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n}$ is as well. Let $\mathcal{X}'$ be the union of these finitely many quasicompact stacks. For a finite type point $p \in T(k)$, if f' is the HN filtration of $\xi(p)$, then it follows by definition that for any other filtration f of $\xi(p)$, we have $\mu(f) \leq \mu(f')$, and $f'(0) \in \mathcal{X}'$ by construction. □

The HN-specialization property can be deduced from that of ordinary bundles.

Proposition 4.36. *The stack $HBun^{\sigma,+}_{SL_n}$ satisfies the simplified HN-Specialisation property.*

Proof. Let $\xi : \operatorname{Spec} R \to \mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n}$ be a morphism from the spectrum of a DVR. In [36, Theorem 2], it is shown that if E is a family of vector bundles on C over $\operatorname{Spec} R$, then any filtration over the generic point extends uniquely to the whole of $\operatorname{Spec} R$ as a filtration by locally free subsheaves. Because the conditions of being a Higgs subbundle and being σ-isotropic are closed, it follows that if the filtration over the generic point was of the type in Proposition 4.23, then the filtration over $\operatorname{Spec} R$ is as well. It follows that the HN filtration f_K extends to a filtration of $\xi_{\operatorname{Spec}\kappa}$ and this implies that

$$\mu(f_K) \leq M^{\mu}(\xi_{\operatorname{Spec}\kappa}).$$

□

Putting this all together, we obtain the following theorem.

Theorem 4.37. *The stack $HBun^{\sigma,+}_{SL_n}$ admits a well-ordered Θ-stratification such that the semistable locus is quasicompact.*

Proof. Because of Proposition 4.35 and Proposition 4.36, Theorem 2.16 now implies that the stack admits a Θ-stratification. The well-ordering is analogous to the proof of well-ordering in [4, Lemma 8.2]. Finally suppose (E, φ, ψ) is a semistable anti-invariant bundle. By analogy with [32, Corollary 3.3] it follows that there are only finitely many Harder-Narasimhan types for the underlying anti-invariant bundle (E, ψ). Denote $\mathcal{X}' \subset \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ for the substack parametrising those Harder-Narasimhan types. Because $\mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ is isomorphic to a stack of parahoric torsors, it follows by the proof of [23, Proposition 3.18] that $\mathcal{X}'$ is quaiscompact. Thus the preimage under the affine morphism $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \to \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ is as well and this contains the semistable locus by construction. □

4.4 Moduli Space of Semistable Higgs Bundles

In this section we show that the stack of semistable Higgs bundles is Θ- and S-complete and thus admits a good moduli space. Because there is already a GIT construction of this moduli space due to Nitsure in [32] and Simpson in [38], it is expected that this is true, but we could not find a proof in the literature and this is a necessary component of the proof for the anti-invariant case.

In both the criteria, we already know that there exists a unique extension to the bigger stack HCoh because of the following result.

Proposition 4.38. *The stack HCoh is Θ- and S-complete.*

Proof. The morphism HCoh $\to$ Coh is affine, so this follows from Propositions 2.5, 2.7 and Example 2.10. □

The only thing left to verify then is that the image of 0 is in the substack of semistable Higgs bundles. The arguments are analogous to those for the stack of semistable vector bundles on a curve as presented in [5].

Proposition 4.39. *$HBun^{ss}_{r,d}$ is Θ-complete*

Proof. Given a morphism $f\colon \Theta_R/0 \to \mathrm{HBun}^{ss}_{\mathrm{r,d}}$, we need to show that it extends uniquely to a morphism from Θ_R. When we compose with the inclusion $\mathrm{HBun}^{ss}_{\mathrm{r,d}} \to \mathrm{HCoh}$, then a unique extension $g\colon \Theta_R \to \mathrm{HCoh}$ exists because by Proposition 4.38, HCoh is Θ-complete.

$$\begin{array}{ccc} \Theta_R/0 & \xrightarrow[f]{} & \mathrm{HBun}^{ss}_{\mathrm{r,d}} \\ \big\downarrow & & \big\downarrow \\ \Theta_R & \xrightarrow[g]{} & \mathrm{HCoh} \end{array}$$

To get a unique lift in the above diagram, it then suffices to show that if $g : \Theta_R \to HCoh$ is a morphism such that the restriction to $\Theta_R/0$ lands in $\mathrm{HBun}^{ss}_{\mathrm{r,d}}$, then $g(0) \in \mathrm{HBun}^{ss}_{\mathrm{r,d}}$.

A morphism $\Theta_R/0$ corresponds to a semistable Higgs bundle E on C_R of rank r and degree d and a Higgs filtration

$$\{0\} = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = E_K \tag{4.40}$$

of the restriction to the generic point of the DVR such that $gr(E_K)$ is semistable of rank r and degree d.

The fact that this morphism extends in HCoh corresponds to the fact that the filtration of E_K extends globally as a filtration by Higgs subsheaves

$$\{0\} = E_0 \subset E_1 \subset \ldots \subset E_{n-1} \subset E_n = E. \tag{4.41}$$

We first show that each E_i for $i \in \{1, \ldots, n\}$ is a semistable Higgs bundle of slope $\mu := \frac{r}{d}$. Because we know that $gr(E_K)$ is semistable of slope μ, it follows that for each i, F_i/F_{i-1} is semistable of slope μ. Because F_n/F_{n-1} and F_n both have slope μ and in an extension $0 \to V_1 \to V_2 \to V_3 \to 0$, the slope of V_2 is a (non-trivial) convex combination of those of V_1 and V_3, it must hold that F_{n-1} has slope μ. By induction we can similarly show that this holds for the other non-trivial subbundles in the filtration of E_K. Because the slope is constant in flat families, it follows that the slope of the non-trivial E_i is also μ.

Because all of the E_i are now subsheaves of E of the same slope, they must also be semistable Higgs bundles. Over the residue field κ of the DVR, we now have a filtration

$$\{0\} = E_{0,\kappa} \subset E_{1,\kappa} \subset \ldots \subset E_{n-1,\kappa} \subset E_{n,\kappa} = E \tag{4.42}$$

where each nontrivial $E_{i,\kappa}$ is a semistable Higgs bundle of slope μ. To conclude that $E_{i,\kappa}/E_{i-1,\kappa}$ is semistable of slope μ, we can apply Lemma 4.43.

This shows that $gr(E_\kappa)$ is semistable of slope μ, which is precisely the statement that $g(0) \in \mathrm{HBun}^{ss}_{\mathrm{r,d}}$. □

Lemma 4.43. *Suppose E is a semistable Higgs bundle of slope μ and F is a non-trivial semistable Higgs subbundle of the same slope. Then E/F is also semistable of slope μ.*

Proof. Because E is an extension of F and E/F, it follows that the slope of E is a (non-trivial) convex combination of the slopes of F and E/F. This forces the slope of E/F to be μ. Now suppose H is another Higgs subbundle containing F, so that H/F is a Higgs subbundle of E/F. As E is semistable, the slope of H is less than or equal to μ. Because it is again a non-trivial convex combination of the slopes of F and H/F, this implies that the slope of H/F has to be less than or equal to μ. This proves that E/F is a semistable Higgs sheaf and thus a semistable Higgs bundle. □

We will prove that this stack also satisfies S-completeness by analysing opposite filtrations of semistable Higgs bundles.

Definition 4.44. Two $\mathbb{Z}$-graded filtrations

$$E_\bullet \colon \{0\} \subset \dots E_{i-1} \subset E_i \subset E_{i+1} \subset \dots E$$

$$F_\bullet \colon F \supset \dots F_{i-1} \supset F_i \supset F_{i+1} \supset \dots \{0\}$$

are **opposite** if

$$E_i/E_{i-1} \cong F_i/F_{i+1} \tag{4.45}$$

for all $i \in \mathbb{Z}$.

In [2, Proposition 6.8.32] it is shown that if $\mathcal{U} \subset \mathrm{Coh}$ is an open substack and for all opposite filtrations $E_\bullet, F_\bullet$ of $E, F \in \mathcal{U}(k)$ we have that $gr(E_\bullet) \in \mathcal{U}(k)$, then $\mathcal{U}$ is S-complete. The argument is that any morphism $\phi_R \backslash 0 \to \mathcal{U}$ extends to a morphism $g \colon \phi_R \to \mathrm{Coh}$. Restricting this to the locus where $\pi = 0$, for π some uniformiser of R, we get a morphism $g' \colon [\operatorname{Spec} k[x,y]/(xy)/\mathbb{G}_m] \to \mathrm{Coh}$. To check that $g(0) \in \mathcal{U}$, it suffices to check that the $g'(0) \in \mathcal{U}$. The morphism g' corresponds precisely to two opposite filtrations and $g'(0)$ is the associated graded of these filtrations.

If we want a similar criterion for open substacks of HCoh, the only thing that changes is that we should consider opposite Higgs filtrations. This proves the following lemma.

Lemma 4.46. *Suppose $\mathcal{U} \subset HCoh$ is an open substack and for all opposite Higgs filtrations $E_\bullet, F_\bullet$ of $E, F \in \mathcal{U}(k)$ we have that $gr(E_\bullet) \in \mathcal{U}(k)$, then $\mathcal{U}$ is S-complete.*

Opposite filtrations of a semistable Higgs bundle imply a lot of structure for the filtrations. This is captured in the following lemma.

Lemma 4.47. *Let $E_\bullet$, $F_\bullet$ be opposite Higgs filtrations of semistable Higgs bundles E, F of rank r and degree d. Then all of the non-trivial E_i(and E_i/E_{i-1}), F_i are semistable Higgs subbundles of E, F of slope $\mu := r/d$.*

Proof. Suppose that $E_i = \{0\}$ for $i \leq 0$ (and thus $F_i = F$ for $i \leq 1$) and $E_i = E$ for $i \geq n$ (and thus $F_i = \{0\}$ for $i \geq n+1$). We show that E_1 and F_2 are semistable of slope μ. Because E_1 is a Higgs subbundle of E, we have that

$$\mu(E_1) \leq \mu.$$

But because the filtrations are opposite, we also know that

$$E_1 \cong F/F_2,$$

so because it is isomorphic to a Higgs quotient of F, we have that

$$\mu(E_1) \geq \mu.$$

We conclude that $\mu(E_1) = \mu$. As $E_1 \cong F/F_2$, this also implies that $\mu(F_2) = \mu$. Because they are both Higgs subbundles of semistable Higgs bundles of slope μ, it follows that they are semistable.

The two filtrations

$$\{0\} \subset E_2/E_1 \subset \ldots \subset E_{n-1}/E_1 \subset E/E_1,$$

$$F_2 \supset F_3 \supset \ldots \supset F_n \supset \{0\} \tag{4.48}$$

are now opposite filtrations of the semistable Higgs bundles E/E_1 and F_2. We can argue by induction that F_i is a semistable Higgs bundle of slope μ for each i and similarly that E_i/E_{i-1} is. This will in turn imply that E_i is a semistable Higgs bundle for each i. □

Combining the previous lemmas, we can show that the stack of Higgs bundles is S-complete.

Proposition 4.49. *$HBun^{ss}_{r,d}$ is S-complete.*

Proof. By Lemma 4.46 it suffices to show that given two opposite filtrations of semistable Higgs bundles of rank r and degree d, the associated graded is also semistable of rank r and degree d.

By Lemma 4.47, each of the summands of the associated graded is semistable of slope $\mu := r/d$. This implies that the whole associated graded is semistable of rank r and degree d. □

Together with the Θ-completeness, this establishes the following theorem.

Theorem 4.50. *The stack $HBun^{ss}_{r,d}$ admits a good moduli space.*

Analogously, we can show that the stack of semistable SL_n-bundles $\mathrm{HBun}^{\mathrm{ss}}_{\mathrm{SL}_n}$ admits a good moduli space. The only added difficulty here consists of proving that the trivialisation of the determinant also extends over Θ_R and Φ_R in both criteria. This is done analogously to the extension of the isomorphism in the proof of Theorem 4.52

4.5 Moduli Space of anti-invariant Higgs bundles

Using the established results for the stack of Higgs bundles, we can show that the stacks of anti-invariant Higgs bundles admit good moduli spaces. Because of Lemma 4.32, we have forgetful morphisms.

$$\mathrm{HBun}^{\sigma,+;\mathrm{ss}}_{\mathrm{SL}_n} \to \mathrm{HBun}^{\mathrm{ss}}_{\mathrm{SL}_n},$$

and

$$\mathrm{HBun}^{\sigma,\tau;\mathrm{ss}}_{\mathrm{SL}_n} \to \mathrm{HBun}^{\mathrm{ss}}_{\mathrm{SL}_n}. \tag{4.51}$$

We already know the target stack admits a good moduli space and is thus Θ- and S-complete. We will show that these properties lift along the forgetful morphism.

Theorem 4.52. *The stacks $HBun^{\sigma,+;ss}_{SL_n}$ and $HBun^{\sigma,\tau;ss}_{SL_n}$ admit good moduli spaces, which we will denote by $M^{\sigma,+}_n$ and $M^{\sigma,\tau}_n$.*

Proof. We argue that the stack of σ-symmetric anti-invariant Higgs bundles is Θ-complete. The property of S-completeness and the σ-alternating case are proven completely analogously.

So suppose we are given a morphism

$$\Theta_R/0 \to \mathrm{HBun}^{\sigma,+;\mathrm{ss}}_{\mathrm{SL}_n}.$$

We can interpret this as a family of σ-symmetric anti-invariant Higgs bundles $(\mathcal{E}', \varphi', \psi')$ on $(\operatorname{Spec} R[t]/0) \times C$ equivariant with respect to the $\mathbb{G}_m$-action on $\operatorname{Spec} R[t]$.

Because $\mathrm{HBun}^{\mathrm{ss}}_{\mathrm{SL}_n}$ is Θ-complete, it follows that we can extend the underlying semistable Higgs bundle $(\mathcal{E}', \varphi')$ to a $\mathbb{G}_m$-equivariant semistable Higgs bundle $(\mathcal{E}, \varphi)$ on $\operatorname{Spec} R[t] \times C$ which admits an isomorphism ψ' away from $\{0\} \times C$. Such an isomorphism is a section of the vector bundle $\mathcal{E}^* \otimes \sigma^*\mathcal{E}^*$. Because $\{0\} \times C$ is codimension 2, it follows from algebraic Hartogs' lemma that this extends to an endomorphism

$$\psi : \sigma^*\mathcal{E} \to \mathcal{E}^*. \tag{4.53}$$

Locally, the locus where ψ is not an isomorphism is given by the vanishing of a determinant. Thus, it follows that this locus is either codimension 1 or empty. But we already know that ψ is an isomorphism outside a subset of codimension 2. It follows that ψ is an isomorphism everywhere. Because it is compatible with φ on the open subset $(\operatorname{Spec} R[t]/0) \times C$, it is compatible everywhere. Lastly, it is $\mathbb{G}_m$-equivariant because its restriction to $(\operatorname{Spec} R[t]/0) \times C$ is $\mathbb{G}_m$-equivariant and 0 is a fixed point for the action.

The anti-invariant bundle $(\mathcal{E}, \varphi, \psi)$ on $\Theta_R \times C$ then corresponds to the unique extension

$$\Theta_R \to \mathrm{HBun}^{\sigma,+;\mathrm{ss}}_{\mathrm{SL}_n}.$$

□

Remark 4.54. Completely analogously to [5, Theorem 3.12], it can be shown that these moduli spaces classify S-equivalence classes of anti-invariant Higgs bundles, for the natural generalisation of S-equivalence to this setting.

4.6 Hitchin fibration

Composing the Hitchin fibration $\mathrm{HBun}_{\mathrm{SL}_n} \to B$ with the forgetful morphism $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \to \mathrm{HBun}_{\mathrm{SL}_n}$ (respectively $\mathrm{HBun}^{\sigma,\tau}_{\mathrm{SL}_n} \to \mathrm{HBun}_{\mathrm{SL}_n}$), we get the following morphisms:

$$\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \to B \text{ and } \mathrm{HBun}^{\sigma,\tau}_{\mathrm{SL}_n} \to B.$$

Because good moduli spaces are universal for maps to algebraic spaces by [1, Theorem 6.6] we get morphisms

$$M^{\sigma,+}_n \to B \text{ and } M^{\sigma,\tau}_n \to B.$$

Theorem 4.55. *The morphism $M^{\sigma,+}_n \to B$ (resp. $M^{\sigma,\tau}_n \to B$) is proper.*

Proof. We show the theorem for the σ-symmetric case. The σ-alternating case is completely analogous. By Proposition 2.24 it suffices to show that $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \to B$

satisfies the existence part of the valuative criterion of properness. Let R be a DVR with fraction field K. We are looking for a lift $\operatorname{Spec} R \to \mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n}$ in the following diagram

$$\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & \mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} R & \longrightarrow & B
\end{array}$$

Using our various forgetful morphisms, we get a diagram

$$\begin{array}{ccccc}
\operatorname{Spec} K & \longrightarrow & \mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} & \longrightarrow & \mathrm{HBun}_{\mathrm{SL}_n} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n} & & \\
\big\downarrow & & & & \big\downarrow \\
\operatorname{Spec} R & & \longrightarrow & & B
\end{array}$$

In [23, Proposition 3.3] it is shown that if $\mathcal{G}$ is a parahoric group scheme on a curve C' and $\mathrm{Bun}_{\mathcal{G}}$ is the stack of torsors for $\mathcal{G}$ on C', then this stack satisfies the existence part of the valuative criterion. The stack $\mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ is of this form for a certain group scheme on C/σ, so it also satisfies the existence part of the valuative criterion. So we get a lift $\operatorname{Spec} R \to \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ in the above diagram.

Similarly, it is shown in [4, Lemma 6.20] that the morphism $\mathrm{HBun}_{\mathrm{SL}_n} \to B$ satisfies the existence part of the valuative criterion. Hence, there is also a lift $\operatorname{Spec} R \to \mathrm{HBun}_{\mathrm{SL}_n}$.

Combining these, we get a lift $\operatorname{Spec} R \to \mathrm{HBun}_{\mathrm{SL}_n} \times_{\mathrm{Bun}_{\mathrm{SL}_n}} \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$. This gives us the following diagram

$$\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & \mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \\
\big\downarrow & & \big\downarrow \\
 & & \mathrm{HBun}_{\mathrm{SL}_n} \times_{\mathrm{Bun}_{\mathrm{SL}_n}} \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n} \\
 & \nearrow & \big\downarrow \\
\operatorname{Spec} R & \longrightarrow & B.
\end{array}$$

The stack $\mathrm{HBun}_{\mathrm{SL}_n} \times_{\mathrm{Bun}_{\mathrm{SL}_n}} \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ parametrises triples (E, φ, ψ) consisting of a Higgs bundle (E, φ) and an isomorphism $\psi\colon \sigma^* E \to E^*$, but they need not satisfy the compatibility condition of Equation 4.9. This compatibility condition cuts out a closed substack of $\mathrm{HBun}_{\mathrm{SL}_n} \times_{\mathrm{Bun}_{\mathrm{SL}_n}} \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$, and the morphism $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n} \to \mathrm{HBun}_{\mathrm{SL}_n} \times_{\mathrm{Bun}_{\mathrm{SL}_n}} \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ is the corresponding closed immersion. It follows that the map $\operatorname{Spec} R \to \mathrm{HBun}_{\mathrm{SL}_n} \times_{\mathrm{Bun}_{\mathrm{SL}_n}} \mathrm{Bun}^{\sigma,+}_{\mathrm{SL}_n}$ lands in $\mathrm{HBun}^{\sigma,+}_{\mathrm{SL}_n}$ as it does so generically, and this is the desired lift of the map $\operatorname{Spec} R \to B$. □

Remark 4.56. A description of the image of these Hitchin systems can be found in [44, Section 5].

5 Moduli Space of Parahoric Higgs Bundles

In this chapter, we prove the existence of a good moduli space for the stack of parahoric Higgs bundles, which is proper over the Hitchin base. To establish Θ- and S-monotonicity, we follow [19], where this is shown for the moduli stack of gauged maps, i.e maps from a curve C to a $[X/G]$ where X is a projective-over-affine scheme and G is a reductive group acting on X.

For the proof of HN-boundedness, we first prove the existence of Harder-Narasimhan reductions using the theory of complementary polyhedra from [9] and its extension to Higgs bundles in [40]. The complementary polyhedron we use is constructed by combining ones defined in [40] and [23].

Finally, to prove properness, we show that the morphism from the stack of parahoric Higgs bundles to the Hitchin base satisfies the existence part of the valuative criterion of properness. This is done analogously to the proof for Higgs bundles in [4].

5.1 Monotonicity

5.1.1 Setup

Most of this section follows the arguments of Section 3 of [19]. We will be interested in the following simplified version of the setup. Let C be a smooth projective curve over an algebraically closed field k, G a reductive group over k, and X an affine scheme over k. The *moduli stack of gauged maps* $\mathcal{M}([X/G])$ [12] parametrises maps from C to $[X/G]$. The data of a map $f\colon C \to [X/G]$ is the same as a pair of

- A G-bundle E coming from the map $C \to BG$ induced by the projection $[X/G] \to BG$, and
- a section of the fibre bundle $P(X)$ associated to X.

If we let X be the Lie algebra $\mathfrak{g}$, then the sections above will be sections of the adjoint bundle. However for the purposes of Higgs bundles, we will want not a section of the adjoint bundle, but the adjoint bundle tensored with a line bundle. This can be solved by modifying the quotient stack.

Let $\mathcal{L}$ be a line bundle on C and $\rho_{\mathcal{L}}$ the total space of the corresponding $\mathbb{G}_m$-torsor. Following [42], define

$$[\mathfrak{g}/G]_{\mathcal{L}} := \rho_{\mathcal{L}} \times^{\mathbb{G}_m} [\mathfrak{g}/G],$$

[12]In [19], X is a projective-over-affine scheme, and the maps are Kontsevich stable morphims with n marked points.

for the natural $\mathbb{G}_m$-action on $\rho_{\mathcal{L}}$ and the scaling action on $\mathfrak{g}$. This is a stack over the curve C. A map from C to $[g/G]_{\mathcal{L}}$ relative to C will induce a pair consisting of

- A G-bundle E on C, and
- a section $\varphi \in H^0(\mathrm{ad}(E) \otimes \mathcal{L})$.

Now, let $\mathcal{G}$ be a semisimple parahoric Bruhat-Tits scheme on the curve C, and $\mathfrak{g}$ its Lie algebra. A morphism from C to $[\mathfrak{g}/\mathcal{G}]_{\mathcal{L}}$ will consist of a pair of

- a $\mathcal{G}$-bundle $\mathcal{E}$, and
- a Higgs field $\varphi \in H^0(\mathrm{ad}(\mathcal{E}) \otimes \mathcal{L})$.

Thus, the moduli stack of gauged maps $\mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$ is isomorphic to the stack of parahoric Higgs bundles, and for this section we will stick to this notation. In this way, Higgs bundles fit into the definiton of gauged maps. The natural projection map $\mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}}) \to \mathcal{M}(B\mathcal{G}) \cong \mathrm{Bun}_{\mathcal{G}}$ coincides with the map forgetting the Higgs field.

We will show that the proofs of monotonicity from [19] hold in this slightly more general setting. We will use the numerical invariant μ defined on the stack of parahoric bundles earlier, as

$$\mu = \frac{\mathrm{wt}_{\mathbb{G}_m}(\mathcal{L}_{\det,\chi})}{b},$$

where b is a norm on graded points of $B\mathcal{G}$, and pull it back to $\mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$.

5.1.2 Generalised Grassmannian

The main idea for proving monotonicity is infinite-dimensional GIT. For this, we need an auxiliary stack that the stack of gauged maps maps to, such that

- the fibers are (ind-)projective, and
- the auxiliary stack satisfies some filling conditions analogous to Θ- and S-completeness.

The following definition is analogous to [19, Definition 3.1].

Definition 5.1. The **stack of rational parahoric Higgs bundles** $\mathcal{M}^{\mathrm{rat}}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$ is the stack that parametrises triples consisting of

- a $\mathcal{G}$-bundle $\mathcal{E}$ on C,

- a divisor $D \subset C$ and
- a Higgs field $\varphi \in H^0(C \backslash D, \mathrm{ad}(\mathcal{E}) \otimes \mathcal{L})$ defined away from D.

An isomorphism between two triples $(\mathcal{E}_1, D_1, \varphi_1)$ and $(\mathcal{E}_2, D_2, \varphi_2)$ consists of

- equality $D_1 = D_2$ and
- an isomorphism $\phi\colon \mathcal{E}_1|_{C\backslash D_1} \to \mathcal{E}_2|_{C\backslash D_2}$ compatible with the sections φ_1 and φ_2.

There is a natural projection morphism

$$q\colon \mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}}) \times \mathrm{Div}(C) \to \mathcal{M}^{\mathrm{rat}}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})\colon ((\mathcal{E}, \varphi), D) \mapsto (\mathcal{E}, D, \varphi_{C\backslash D}).$$

This morphism will allow us to use the ideas of infinite-dimensional GIT.

A crucial part of this approach is understanding fibers of this morphism. This is described by the following adaptation of [19, Definition 3.2].

Definition 5.2. Let $T \to \mathcal{M}^{\mathrm{rat}}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$ be a morphism represented by a triple $(\mathcal{E}, D, \varphi)$ from an affine scheme T. The fiber of q over this morphism is the **generalised Grassmanian** $\mathrm{Gr}^{(\mathcal{E},D,\varphi)}$, parametrising triples consisting of

- a $\mathcal{G}$-bundle $\mathcal{E}'$,
- an isomorphism $\phi\colon \mathcal{E}|_{C\backslash D} \to \mathcal{E}'|_{C\backslash D}$ and
- an extension
$$\tilde{\varphi} \in H^0(\mathrm{ad}(\mathcal{E}') \otimes \mathcal{L})$$
of φ interpreted as a section of $\mathrm{ad}(\mathcal{E}') \otimes \mathcal{L}$ using ϕ.

Again, there is a natural morphism

$$\mathrm{Gr}^{(\mathcal{E},D,\varphi)} \to \mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})\colon (\mathcal{E}', \phi, \tilde{\varphi}) \mapsto (\mathcal{E}', \varphi).$$

We will also need a version of this generalised Grassmannian without the Higgs field, which we will denote by $\mathrm{Gr}^{(\mathcal{E},D,)}$. This comes with a natural morphism

$$\mathrm{Gr}^{(\mathcal{E},D,\varphi)} \to \mathrm{Gr}^{(\mathcal{E},D)})\colon (\mathcal{E}', \phi, \tilde{\varphi}) \mapsto (\mathcal{E}', \phi).$$

To establish monotonicity, we need to know that these are ind-projective ind-schemes. This follows from the following lemmas analogous to [19, Lemma 3.4/3.5].

Lemma 5.3. *Let D be a reduced divisor on C. The generalised Grassmannian $Gr^{(\mathcal{E},D)}$ is an ind-projective ind-scheme.*

Proof. This is completely analogous to the case when D consists of a single reduced point. In that case, this is Proposition 3.13. □

Remark 5.4.
- In [19], this is proved for arbitrary divisor and we expect this holds here as well. For the proof of monotonicity, it suffices to establish this result for reduced divisors.
- As remarked earlier, in general, the pullback of the line bundle $\mathcal{L}_{\det}$ is ample on the generalised Grassmannian only if the $\mathcal{L}_\chi$ is when pulled back to the generalised Grassmannians where the divisor consists of a single point of bad reduction. We will assume from now on that this is the case. This holds automatically if the divisor D does not contain any points of bad reduction.

Lemma 5.5. *The generalised Grassmannian $Gr^{(\mathcal{E},D,\varphi)}$ is a closed subscheme of $Gr^{(\mathcal{E},D)}$. Consequentially if D is a reduced divisor, it is an ind-projective ind-scheme.*

Proof. Because $[\mathfrak{g}/\mathcal{G}]_{\mathcal{L}}$ is affine over C, this follows identically to the proof of [19, Lemma 3.5]. □

5.1.3 Proof of monotonicity

In this section, we fix a DVR R and let Y denote either the affine scheme Spec $R[t]$ equipped with the $\mathbb{G}_m$-action of weight 1 on t, or Spec $R[x,y]/(xy-\pi)$ for a choice of uniformiser π equipped with the $\mathbb{G}_m$-action of weight 1 on x and -1 on y. We denote by 0 the point cut out by (t,π) and (x,y) respectively.

A crucial element for establishing monotonicity is proving that the morphism q satisfies certain extension properties. This is done by proving trivialisation results for $\mathcal{G}$-bundles.

Lemma 5.6. *Let R be a DVR, and $\mathcal{E}$ a $\mathcal{G}$-bundle on C_R. Up to a finite extension of R, there exists an open $U \subset C_R$ where $\mathcal{E}$ is trivialisable.*

Proof. Corollary [24, Corollary 26] shows that after a smooth cover of Spec R there exist reductions to Borel subgroups of $\mathcal{G}$. Analogous to [17, Theorem 2], it follows that after an etale cover, $\mathcal{E}$ is Zariski-locally trivialisable. As etale covers of Spec R are equivalent to finite extensions, the result follows. □

Lemma 5.7. *Let $\mathcal{E}$ be a $\mathbb{G}_m$-equivariant $\mathcal{G}$-bundle on $C\times(Y\backslash 0)$. Then there exists a Y-flat $\mathbb{G}_m$-equivariant reduced divisor $D \subset C \times Y$, a $\mathbb{G}_m$-equivariant bundle $\mathcal{E}'$ on $C \times Y$ and an isomorphism $\phi\colon \mathcal{E}|_U \to \mathcal{E}'|_U$ for $U = ((C\times Y)\backslash D)|_{Y\backslash 0}$.*

Proof. The idea is that on $Y\backslash 0$, we can trivialise the bundle $\mathbb{G}_m$-equivariantly away from a $\mathbb{G}_m$-equivariant divisor. So on $Y\backslash 0$ we can choose $\mathcal{E}'$ the trivial bundle. The trivial bundle is semistable as a parahoric bundle and because semistable parahoric bundles satisfy Θ- and S-completeness, a $\mathbb{G}_m$-equivariant extension to all of Y exists.

The only thing left to prove is that we can trivialise $\mathcal{E}$ on $C\times(Y\backslash 0)$ equivariantly away from an equivariant divisor. This can be done analogously to the proof of [23, Proposition 2.10] □

This result gives us all the ingredients to establish monotonicty for $\mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$.

Proposition 5.8. *The stack* $\mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$ *satisfies* Θ- *and S-monotonicity.*

Proof. Lemma 5.7 replaces the rational filling criterion in the proof of [19, Proposition 3.6]. The rest of the proof proceeds analogously, and is similar to the infinite-dimensional GIT picture from Section 2.4.

We briefly sketch the idea. Consider a morphism $(\mathcal{E},\varphi) : [Y\backslash 0/\mathbb{G}_m] \to \mathcal{M}([\mathfrak{g}/\mathcal{G}]_{\mathcal{L}})$. Lemma 5.7 allows us to lift this to a morphism to $\mathrm{Gr}^{(\mathcal{E},D,\varphi)}$. Taking the closure of the image of $Y\backslash 0$ gives a stack $\mathcal{W}$ which will verify the monotonicity. This stack satisfies the conditions there precisely because the ind-scheme $\mathrm{Gr}^{(\mathcal{E},D,\varphi)}$ is ind-projective with ample line bundle $\mathcal{L}_{\det,\chi}$. □

Remark 5.9. We assumed earlier that the characters chosen for the line bundle $\mathcal{L}_{\det}$ are such that when we pull back this line bundle to any generalised Grassmannian, it is ample. If the divisor does not contain any points of bad reduction this is automatic. So, if we can prove that we can always trivialise over an open containing no points of bad reduction, then this follows for any choice of characters.

5.2 HN-Boundedness

5.2.1 Complementary polyhedra

In this subsection, we summarise the framework of complementary polyhedra introduced in [9] to prove the existence of canonical parabolic subgroups of a reductive group scheme G on C. This can then be used to prove the existence of canonical reductions to parabolic subgroups for bundles for reductive group schemes. In [40], this is extended to also show the existence of canonical reductions for Higgs bundles. Much of the exposition follows [40].

We let V be a finite-dimensional real vector space of dimension n equipped with an inner product $\langle\cdot,\cdot\rangle$, and fix a root system Φ in V. In the case of bundles for a reductive group scheme G, this will be the root system with respect to a maximal torus $T\subset G_\eta$ in the generic fiber.

We call a subset $R\subset\Phi$ *parabolic* if

- for all $\alpha \in \Phi$ either $\alpha \in R$ or $-\alpha \in R$ and
- If $\alpha, \beta \in R$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in R$.

In the case of a reductive group scheme G, these are subsets of the roots that correspond to parabolic subgroups containing T and $\alpha \in R$ if the corresponding parabolic subgroup contains the root subgroup U_α.

We have a decomposition of V into walls induced by $L_\alpha = \{v \in V \mid \langle v, \alpha \rangle = 0\}$. The facets of this decomposition correspond to parabolic subsets, where a facet P corresponds to the subset

$$R(P) = \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \lambda \in P\}.$$

If P is a facet of Φ, we set $V_P = (\mathrm{span} P)^{|} \subset V$. Then $\Phi_P = \Phi \cap V_P$ is a root system in V_P. These are the roots corresponding to the Levi subgroup of P.

After fixing a basis, we call fundamental dominant weights contained in the closure of a facet P, the *vertices* of P.

Definition 5.10. [9, Definition 2.1/2.3] Let $\mathfrak{C}$ be the set of Weyl chambers of V. A family of vectors $d = (d(c))_{c \in \mathfrak{C}}$ in $V^\vee$ is a **complementary polyhedron** if the following two conditions hold:

1. If λ is a vertex of two chambers $\mathfrak{c}$ and $\mathfrak{d}$, then
$$d(\mathfrak{c})(\lambda) = d(\mathfrak{d})(\lambda).$$
2. If $\mathfrak{c}$ and $\mathfrak{d}$ have $n-1$ vertices in common and α is a root that is positive with respect to $\mathfrak{c}$ and negative with respect to $\mathfrak{d}$, then
$$d(\mathfrak{c})(\alpha) \leq d(\mathfrak{d})(\alpha).$$

If P is a facet of V and $i \colon V_P \to V$ is the corresponding inclusion, then $d_P(\mathfrak{c}) = d \circ i(\mathfrak{c})$ is a complementary polyhedron.

The Weyl chambers $\mathfrak{c}$ correspond to Borel subgroups of G_η that contain T. In the case of reductive group schemes, the vector $d(\mathfrak{c})$ is defined by assigning to a cocharacter λ the degree of the associated line bundle.

Given a complementary polyhedron, we can define a notion of semistability.

Definition 5.11. [9, Definition 3.1/3.3] For any facet P, we define the **degree** of P as

$$\deg(P) = \sum_{\alpha \in R(P)} d(\mathfrak{c})(\alpha)$$

where $\mathfrak{c}$ is any Weyl chamber containing P. The complementary polyhedron d is **semistable** if for every facet P, $\deg(P) \leq 0$.

The degree of a facet corresponds to the degree of the Lie algebra of the associated parabolic subgroup.

To define the notion of a special point we need to define numerical invariants first.

Definition 5.12. Let P be a facet of Φ and λ a vertex of P, and define

$$\Psi(P,\lambda) = \{\alpha \in \Phi \mid \langle \alpha, \lambda \rangle = 1, \langle \alpha, \mu \rangle = 0 \text{ for all other vertices } \mu\}.$$

The **numerical invariant** of P with respect to λ is

$$n(P,\lambda) = \sum_{\alpha \in \Psi(P,\lambda)} d(\mathfrak{c})(\alpha).$$

The numerical invariant will be the degree of the line bundle obtained by extension of structure group using λ.

The following definition is analogous to the definition of a Harder-Narasimhan filtration.

Definition 5.13. [9, Definition 3.5/3.7] A facet P is **special** if

1. for all vertices λ of P we have $n(P,\lambda) > 0$ and
2. the induced complementary polyhedron d_P is semistable

The first condition corresponds to the descending slopes condition for Harder-Narasimhan filtrations and the second is the semistability of the associated graded. The special facet is the largest facet P such that $\deg P$ is maximal by [9, Corollary 3.16]

The following is the crucial result to establish existence and uniqueness of canonical reductions.

Proposition 5.14. *[9, Corollary 3.14] A root system Φ with a complementary polyhedron d has a unique special facet.*

To incorporate Higgs fields into this framework, we need to incorporate a section $\varphi \in H^0(C, \mathfrak{g} \otimes \mathcal{L})$ for $\mathfrak{g}$ the Lie algebra of G. Generically, we have

$$\mathfrak{g}_\eta = (\mathcal{O}_C^{\oplus n} \bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha)_\eta$$

and we denote $\varphi_\alpha \colon \mathcal{O}_{C,\eta} \to (\mathcal{L}_\alpha \otimes \mathcal{L})_\eta$ the projection of the section φ to the corresponding root subalgebra. We define for each root α an integer $\epsilon(\alpha)$ as

- 0 if for all decompositions $\alpha = \alpha_1 + \ldots \alpha_s$ there exists an i with $\varphi_{\alpha_i} = 0$ and

- 1 if there exists a decomposition $\alpha = \alpha_1 + \ldots \alpha_s$ with $\varphi_{\alpha_i} \neq 0$ for all i.

Given a complementary polyhedron for the root system of a torus $T \subset G_\eta$, these numbers can be used to define a distortion of this complementary polyhedron that incorporates the Higgs field.

Definition 5.15. [40, Definition 3.5.4] Let d be the complementary polyhedron defined by Behrend in [9], $T \subset B \subset G_\eta$ a Borel subgroup and $R(B) \subset \Phi$ the corresponding parabolic subset. The **Higgs polyhedron** is

$$d^H_c(B) = d(B) + c \sum_{\alpha \notin R(B)} \epsilon(\alpha)\alpha^\vee.$$

Remark 5.16. This will define a complementary polyhedron for any value of C but the proof of [40, Lemma 3.5.13] shows that for any parabolic subgroup P respecting the Higgs field, i.e $\varphi \in \mathfrak{p} \otimes \mathcal{L}$ for $\mathfrak{p}$ the Lie algebra, the relevant numerical invariants are the same as those of d.

Proposition 5.17. *[40, Lemma 3.5.9] The Higgs polyhedron defines a complementary polyhedron.*

This implies that there is a canonical parabolic subgroup $P \subset G_\eta$, but it is a priori not guaranteed that φ will lie in the corresponding Lie algebra. The following result implies that this is the case.

Proposition 5.18. *[40, Lemma 3.5.16] Let P be the parabolic subgroup corresponding to the special point for d^H_c. If $c \geq |\deg(\mathcal{L})|$, then $\varphi \in \mathfrak{p} \otimes L$ where $\mathfrak{p}$ is the Lie algebra of P.*

Remark 5.19. The idea behind this proof is that for Behrends complementary polyhedron we have $n(B, \beta) = \deg \mathcal{L}_\beta$ for any root β. If α is a root such that $\varphi_\alpha \neq 0$, then $\deg(\mathcal{L}) \geq \deg(\mathcal{L}_{-\alpha})$. In [40, Lemma 3.5.15] it is shown that if $\varphi_\alpha \neq 0$ the numerical invariant $n^H(B, -\alpha)$ for $\alpha \notin R(B)$ equals $\deg(\mathcal{L}_{-\alpha}) - c$. This implies that if a Borel subgroup does not respect the Higgs field, it is not special as one of the corresponding numerical invariants is non-positive.

5.2.2 Existence of Canonical Reductions

In [9] the existence and uniqueness of canonical reductions for bundles for a reductive group scheme is reduced to the existence of canonical parabolic subgroups for reductive group schemes. If E is a G-bundle, then $\mathrm{Aut}(E)$ is a reductive group scheme, and parabolic reductions of E correspond to parabolic subgroups of $\mathrm{Aut}(E)$. Fix G a reductive group scheme over C. A *canonical parabolic subgroup* of G is a maximal parabolic subgroup such that $\deg \mathfrak{p}$ is maximal. By [9, Lemma

4.3] there is an $M > 0$ such that $\deg \mathfrak{p} \leq M$ for any parabolic subgroup P. This immediately implies that a canonical parabolic subgroup exists.

The idea for uniqueness is roughly as follows. Given two canonical parabolic subgroups P and Q, they will contain a common maximal torus T. After forming the complementary polyhedron with respect to this torus, the two parabolic subgroups should correspond to the special facet. Because this is unique, we must have that $P = Q$.

Now fix $\mathcal{G}$ a parahoric Bruhat-Tits group scheme and consider the line bundle $\mathcal{L}_{\det,\chi}$ on $\mathrm{Bun}_{\mathcal{G}}$. Given a Borel subgroup $\mathcal{B} \subset \mathcal{G}$ and a point of bad reduction x, denote by $\chi_x^{\mathcal{B}}$ the character of $\mathcal{B}$ obtained by composing χ_x with the inclusion $\mathcal{B}_x \to \mathcal{G}_x$.

A filtration f of a $\mathcal{G}$-bundle $\mathcal{E}$ corresponds to reduction to a parabolic subgroup $\mathcal{P}$ and a generic 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathcal{G}_\eta$. For a reduction to a Borel subgroup $\mathcal{B}$, it is shown in [23, Equation 2.1] that the corresponding weight is

$$\mathrm{wt}_{\mathcal{B}}(\lambda) = \sum_{\alpha\in\Phi} \deg(\mathfrak{u}_\alpha^{\mathcal{E}})\langle \alpha, \lambda\rangle + \sum_{x\in Ram(\mathcal{G})} \langle \chi_x^{\mathcal{B}}, \lambda\rangle. \tag{5.20}$$

The assignment $\mathrm{wt}_{\mathcal{B}}$ forms a complementary polyhedron if the characters χ are admissible by [23, Lemma 3.15]. We can modify this in the same way as in Definition 5.15 and denote the corresponding complementary polyhedron by $\mathrm{wt}_{\mathcal{B}}^{H,C}$.

We need to ensure that the special facet we obtain from this complementary polyhedron will correspond to a Higgs reduction. For this, we need an analogous bound to that of Remark 5.19. If $\varphi_\alpha \neq 0$ and $\alpha \notin R(B)$, then the numerical invariant $n^H(B, -\alpha)$ for this polyhedron will equal

$$n^H(B, -\alpha) = \deg(\mathfrak{u}_{-\alpha}) + \sum_{x\in Ram(\mathcal{G})} \langle \chi_x^{\mathcal{B}}, -\alpha\rangle - c.$$

Since there are finitely many roots, it follows that if we choose c large enough, then this will be non-positive. This again implies that any reduction that is not compatible with the Higgs field cannot correspond to a special facet.

If we fix a norm $\|\cdot\|$ on cocharacters of T, then [23, Lemma 3.14] implies that there exists an $M > 0$ such that $\frac{\mathrm{wt}(\lambda)}{\|\lambda\|} \leq M$, and the same is true for reductions compatible with the Higgs field.

Completely analogous to the case of group schemes, we get the following result.

Proposition 5.21. *Let $\mathcal{G}$ be a parahoric Bruhat-Tits group scheme and $(\mathcal{E}, \varphi)$ a parahoric Higgs bundle. Then there exists a unique Harder-Narasimhan reduction induced by a generic 1-paramater subgroup λ, i.e a reduction such that $\frac{wt(\lambda)}{\|\lambda\|}$ is maximal.*

5.2.3 Proof of HN-Boundedness

To prove HN-Boundedness, we first need to know that the locus of semistable parahoric Higgs bundles is quasicompact.

Proposition 5.22. *Let $\mathcal{G}$ be a parahoric Bruhat-Tits group scheme. The stack of semistable parahoric Higgs bundles $HBun_{\mathcal{G}}^{ss}$ is quasicompact.*

Proof. This can be proved analogously to the case for parahoric bundles in [23, Proposition 3.18]. □

A filtration f of a parahoric Higgs bundle corresponds to a generic 1-parameter subgroup λ and a reduction to the corresponding parabolic subgroup $\mathcal{P}_\lambda$. If this filtration corresponds to a Harder-Narasimhan reduction, the extension to the Levi subgroup $\mathcal{L}_\lambda$ is semistable. We will call the group $\mathcal{L}_\lambda$ the *type* of the Harder-Narasimhan filtration.

Lemma 5.23. *Let T be a finite type affine scheme and $g : T \to HBun_{\mathcal{G}}$. Only finitely many types appear in the Harder-Narasimhan filtrations of this family.*

Proof. The complementary polyhedron in Equation 5.20 is defined by cohomological data. By upper-semicontinuity it follows that only finitely many different complementary polyhedra appear in any finite type family. As a consequence only finitely many different special facets appear, and the type of a Harder-Narasimhan filtration is determined by this facet. □

The previous two results are the main ingredients to prove that the stack $\mathrm{HBun}_{\mathcal{G}}$ satisfies HN-Boundedness.

Proposition 5.24. *Let $\mathcal{G}$ be a parahoric Bruhat-Tits group scheme. The stack $HBun_{\mathcal{G}}$ satisfies HN-Boundedness with respect to the numerical invariant μ.*

Proof. Let T be an affine Noetherian scheme of finite type and fix a morphism $g\colon T \to \mathrm{HBun}_{\mathcal{G}}$. Denote by $\mathcal{L}_i$ for $i = 1, \dots, m$ the finitely many types of Harder-Narasimhan filtrations appearing in this family by Lemma 5.23. For a fixed $\mathcal{L}_i$ it follows that $\mathrm{HBun}_{\mathcal{L}_i}^{ss}$ is quasicompact by Proposition 5.22. Denote by $\mathcal{U}_i$ the image of the morphism $\mathrm{HBun}_{\mathcal{L}_i}^{ss} \to \mathrm{HBun}_{\mathcal{G}}$ obtained by extension of structure group using the inclusion $\mathcal{L}_i \to \mathcal{G}$. Let $\mathcal{U}$ be the union of these $\mathcal{U}_i$. It is a quasicompact substack of $\mathrm{HBun}_{\mathcal{G}}$

Let $t\colon \operatorname{Spec} k \to T$ be a geometric point and f a filtration of $g(t)$. denote by f' the Harder-Narasimhan filtration of $g(t)$. It immediately follows that

$$\mu(f) \leq \mu(f').$$

The point $f'(0)$ corresponds to the bundle for the associated Levi subgroup and it is semistable. So by construction we have $f'(0) \in \mathcal{U}$. □

5.3 Good Moduli Space and Properness of the Hitchin fibration

The only thing that is missing to prove our desired theorem is the following lemma.

Lemma 5.25. *Let $\mathcal{G}$ be a parahoric Bruhat-Tits group scheme. The morphism $HBun_{\mathcal{G}} \to B_{\mathcal{G}}$ to the Hitchin base satisfies the existence part of the valuative criterion of properness.*

Proof. This is proven for reductive group schemes G in [4, Lemma 6.20]. The idea is as follows. Let R be a DVR and $(\mathcal{E}_K, \varphi_K)$ be a Higgs bundle with image in $B(R)$. After a possible finite extension of K, we may assume $\mathcal{E}_K$ is trivial at the generic point $K(\eta)$ of C_K. We can identify the Higgs field at the generic point with an element $X_K \in \mathfrak{g}(K(\eta))$. In the case that the derived group of G is simply connected, it is shown in loc. cit. that after a finite extension, we can conjugate X_K to an element of $\mathfrak{g}(R_\eta)$. This allows to extend the Higgs field over the generic point $R(\eta)$ of C_R. Because sections of affine bundles extend in codimension 2, the Higgs field will extend to any extension of $\mathcal{E}_K$ that is trivial over $R(\eta)$. Finally, it is proven that the general result derives from the one where the derived group of G is simply connected.

For a parahoric Bruhat Tits-group scheme, the Lie algebras $\mathfrak{g}(K(\eta))$ and $\mathfrak{g}(R(\eta))$ are still reductive, so the proof still applies. □

Theorem 5.26. *Let $\mathcal{G}$ be a parahoric Bruhat-Tits group scheme. The stack $HBun_{\mathcal{G}}$ admits a Θ-stratification induced by the numerical invariant μ. The semistable locus $HBun_{\mathcal{G}}$ of this stratification admits a separated good moduli space that is proper over the Hitchin base $B_{\mathcal{G}}$.*

Proof. Because of Propositions 5.8 and 5.22 it follows from Theorem 2.25 that the stack $\mathrm{HBun}_{\mathcal{G}}$ admits a Θ-stratification. Analogously to [4, Lemma 8.2], it can be shown that this stratification is well-ordered. Combining Proposition 5.8 and Lemma 5.25 with Theorem 2.25 we deduce that $\mathrm{HBun}_{\mathcal{G}}$ admits a separated good moduli space that is proper over $B_{\mathcal{G}}$ □

Remark 5.27.
- The points of this good moduli space can be identified with S-equivalence classes of parahoric Higgs bundles, where two parahoric Higgs bundles are identified if there exist filtrations of both of them such that the associated graded is isomorphic.
- All of the results in this chapter still hold if we replace the adjoint bundle $\mathrm{ad}(\mathcal{E})$ with the bundle $\mathrm{Sad}(\mathcal{E})$. So, our results also hold for strong parahoric Higgs bundles.

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