Aspects of Scattering Amplitudes: On Strong Backgrounds and in Twistor Space

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. This work has not been submitted for any other degree or processional qualification.

(*Sonja Klisch*)

To those who taught me how to love the universe:
my parents, my teachers, and my friends.

My chief occupation, despite appearances, has always been love.

– Albert Camus, *Notebooks 1942-1951*

Abstract

In this thesis we study the properties of scattering amplitudes in two contexts: in the presence of strong background fields; and their twistorial representations.

In the first part, we consider scattering amplitudes in strong background fields: classical solutions of the equations of motion which are treated non-perturbatively. Our focus is on highly symmetric, but physically relevant, plane wave and spherically symmetric backgrounds. In the case of plane waves, we review the calculation and properties of these amplitudes. For spherically symmetric backgrounds we propose a semi-classical approximation to the solutions of the equations of motion that is well-suited to amplitude calculations. Thus, we obtain the first expression for semi-classical 3-point amplitudes in Coulomb and Schwarzschild backgrounds.

Amplitudes on backgrounds are of particular interest due to their applications in computing classical observables such as gravitational waveforms arising from astrophysical phenomena. We conclude the first part by using amplitudes on plane wave backgrounds to calculate the classical radiation waveform of scattering events on these backgrounds.

In the second part, we explore aspects of tree-level amplitudes formulated as localised integrals over the moduli space of maps to twistor space. These are representations closely related to twistor string theory that provide all-multiplicity closed-form expressions for scattering amplitudes in gauge theory and gravity, graded by the helicity of the external particles.

A remarkable aspect of amplitudes in gauge theory and gravity is that they are related via the double copy. In the final section of this thesis, we study how these twistor space representations of amplitudes manifest the double copy via applications of graph theory. In doing so, we find new twistorial representations of biadjoint scalar amplitudes. Connecting to the first part of this thesis, amplitudes on certain self-dual backgrounds in gauge theory and gravity also enjoy all-multiplicity representations as moduli space integrals of maps to twistor space. Generalising these arguments, we make the first step towards an all-multiplicity statement of the double copy for amplitudes on self-dual backgrounds.

Lay summary

The fundamental particles of our universe are best described using quantum field theory (QFT). This is a theory where each particle type correponds to a field, and particles are represented by fluctuations in this field. Different fields may interact with each other, governed by the action of the theory, allowing particles to turn into other particles. Scattering amplitudes are objects calculated in QFT that encode the probability of a certain initial particle state becoming some final particle state. These are the essential objects that enable high-precision measurements of the masses and coupling constants describing the matter of our universe.

In this thesis, we study aspects of scattering amplitudes in strong backgrounds, and in a certain algebro-geometric space called twistor space. Amplitudes are usually calculated in vacuum flat space, with no sources or fields except for the particles being scattered — for example, this is a good description for the interior of a collider. However, there are cases of interest where we have a strong background (e.g. a strong field or source) and would like to calculate observables of QFT. For example, this may be particle scattering in the presence of a strong laser (here modelled by a plane wave), or astrophysically in the presence of a black hole. In the first part of this thesis, we consider and calculate scattering amplitudes on these backgrounds. In addition, we demonstrate how amplitudes on backgrounds can be used to calculate the gravitational waveform (fluctuations in the gravitational field) of astrophysical scattering events.

Scattering amplitudes also have remarkable mathematical properties (most well-studied back in flat space). One of these is known as the double copy: a powerful relations between scattering amplitudes of gluons and those of gravitons (the quanta of gravity). Another is the surprising simplicity of the tree-level scattering amplitudes of these theories in four dimensions. It turns out that the leading contributions to amplitudes are best captured using twistor space: a non-local description of spacetime. Here, the scattering amplitude of any number of gluons or gravitons is given by a single closed expression (when the usual spacetime calculation may contain 1000+ terms). In the second part of this thesis we manifest the connection between these two ideas, and derive a new twistorial version of the double copy. We also explore what this may tell us about the double copy for amplitudes on strong backgrounds.

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List of publications

- [1] T. Adamo, A. Cristofoli, A. Ilderton, and **S. Klisch**, "Scattering amplitudes for self-force," *Class. Quant. Grav.* **41** no. 6, (2024) 065006, arXiv:2307.00431 [hep-th]

  Chapter 3 is based on this publication.

- [2] T. Adamo, A. Cristofoli, A. Ilderton, and **S. Klisch**, "All Order Gravitational Waveforms from Scattering Amplitudes," *Phys. Rev. Lett.* **131** no. 1, (2023) 011601, arXiv:2210.04696 [hep-th]

  Chapter 4 is based on this publication.

- [3] T. Adamo and **S. Klisch**, "The KLT kernel in twistor space," arXiv:2406.04539 [hep-th]

  Chapters 5 and 6 are based on this preprint.

Contents

Chapter 1

Introduction

This thesis concerns the study of *scattering amplitudes*. Scattering amplitudes are fundamental objects in theoretical particle physics, but also feature in more diverse fields such as condensed matter and astrophysics. They are the key calculational tool for constraining the values of coupling constants in the standard model, testing new theories, and finding constraints on yet undiscovered matter. The usual method of calculating scattering amplitudes in quantum field theory is using the action of the theory and a path integral formalism built from this action. This naturally assumes a perturbative regime — a small coupling constant that represents the coupling of fields to each other and to themselves. With this assumption, scattering amplitudes are expanded into a series of Feynman diagrams, organised by the number of loops present, which are summed to arrive at the final expression.

However, it is well known that this perturbative expansion can break down for some theories (notably gravity) at high energies due to the running of the couplings under renormalisation flow. Additionally, the loop expansion is known to not converge in general due to a radius of convergence of zero. Despite this, we have an asymptotic series which provides reasonable approximations in appropriate kinematic regimes. Moreover, statements that are all-orders in the coupling constants provide a glimpse into elusive non-perturbative physics.

But even sticking to a finite order in the Feynman diagram expansion, there exist remarkable resummation properties. At tree-level (leading order in the coupling), certain external helicity configurations of either gluons or gravitons resum the many Feynman diagrams into a single term: the Parke-Taylor [4] and Hodges formulae [5] respectively. This property can be extended to generic helicity configurations, giving closed-form expressions (coming from either the twistor string [6–8] or the Cachazo-He-Yuan (CHY) formalism [9–11]) that are completely disconnected from any Feynman diagram origins.

This suggests that scattering amplitudes may arise from something more fundamental than Feynman diagrams. This is already the case in string theory, where each order in the perturbative expansion is given by counting the genus of the string worldsheet. At tree-level, massless theories are beautifully described using the Cachazo-He-Yuan formalism, with foundations coming from (ambi-)twistor string theories [12–14] — string theories with certain projective spaces as target space. Beyond this are even more geometric approaches, such as Grassmannians, the amplituhedron, or tropical geometry. Overall, the story of the past decades has been that perhaps Feynman integrals are not the best way to describe scattering amplitudes, and the true fundamentals of particle physics lie in a geometric construction.

All of the above approaches have been studied primarily for quantum field theory in flat space vacuum — where all fields are expanded around a trivial solution. Expanding scattering amplitudes instead around a *non-trivial background* corresponds to an infinite resummation of field excitations that represent the background. These scattering amplitudes can be represented as a series in the coupling constants of the small perturbations. Here the perturbation is a small fluctuation on the background that does not affect the background geometry itself[1]. Scattering amplitudes on backgrounds therefore probe some of the non-perturbative properties of amplitudes - as well as those that would only appear at much higher orders in conventional perturbation theory. These are for example geodesic motion, the memory effect, the structure of the Green's function (including tail effects), the presence of horizons, and particle pair-production.

A big question is whether the same structures as in flat space amplitudes can exist for non-trivial backgrounds. However, in contrast to flat space, amplitudes calculated on backgrounds have not been studied as extensively. As an example, the Parke-Taylor formula for Yang-Mills on flat space was first conjectured after a tree-level 6-point gluon calculation using Feynman diagrams. On physically relevant backgrounds, the state of the art [15] is currently 4-points, fully non-linear in the background fields. There is thus a huge calculational gap, but recent progress has been made to fill this gap with modern amplitude methods such as twistor sigma models [16] and the worldline formalism [17–19].

This thesis explores the progress made in two aspects of scattering amplitudes. One is progress in the study of scattering amplitudes on backgrounds. Whilst interesting in their own right for their mathematical properties, they also have applications in the context of classical observables built from scattering amplitudes and the self-force expansion [20]. We will study these applications here.

Scattering amplitudes are usually associated with quantum effects. However, they can also capture the classical physics associated with processes such as the gravitational wave emissions from certain events. Observables such as the classical waveform of a process can be computed using scattering amplitudes by taking a careful $\hbar \to 0$ limit in the Kosower-Maybee-O'Connell (KMOC) framework [21]. Explicitly, the scattering of two black holes, and the emission of gravitational radition, can be modelled in the large distance limit as two massive scalar particles interacting and emitting a graviton[2]. Advances in the calculation of these waveforms are useful for understanding the possible matching waveforms for gravitational wave observations.

In this thesis we will take a different perspective: we treat one of the black holes as a background field, and consider the scattering of a single probe scalar particle on this

[1]With large enough coupling, or running of this coupling such as near high energy regions like the horizon of a black hole, the perturbation will back-react on the geometry and give rise to many non-trivial quantum effects. In this thesis we will not focus on these.

[2]Whilst this captures only *scattering* rather than the more physically relevant mergers of black holes, there is progress towards analytically continuing scattering results to closed orbits.

background, emitting radiation. In classical general relativity, this is closely related to an expansion known as the self-force expansion. The expansion parameter in this scheme is the mass ratio m/M of the mass of the big black hole (background) M and the small black hole (probe) m. The first order in this expansion captures geodesic motion, while higher orders take account of backreaction effects on the motion of the particle and the spacetime.

From the amplitudes perspective, the calculation of the waveform involves a certain three-point amplitude (a massive scalar emitting radiation) on this background. For general backgrounds, this is a very non-trivial calculation. Here we will consider two different simplifications: a semiclassical (WKB) approximation to scattering amplitudes on a Schwarzschild (and Coulomb) background, and approximating the background as a plane wave. The former is motivated by seeking *classical* observables as output. The latter is motivated by the Penrose limit [22], which states that any generic spacetime can be viewed as a plane wave spacetime in a certain limit. On plane wave backgrounds, the form of the amplitudes will allow us to calculate explicitly the classical waveform arising from such a scattering event, the first all-orders in the coupling result of this form.

The other aspect studied in this thesis is that of scattering amplitudes represented as moduli space integrals over maps to twistor space [6, 23, 24]. These are particular representations of tree-level amplitudes at all multiplicities of external legs, derived from certain twistor string theories in Yang-Mills and gravity. In particular, these formulae are graded in complexity by their external helicity configurations. Whilst they only capture the tree-level dynamics of scattering processes, there are still questions about them that have remained open.

Twistor theory, first developed by Roger Penrose in the 60s, has revealed itself in the last decades to be a natural place to consider quantum field theories, starting with the development of a twistor string theory by Witten [6] to describe the scattering of gluons. Whilst many of these twistor models still fail to capture loop effects (they have anomalies in all but very specially chosen theories), they are the natural place to consider the tree-level scattering of gluons and gravitons in four dimensions. Intuitively, this is due to the integrability of the self-dual sectors of both theories, which is naturally captured on twistor space.

For example, a self-dual gauge field in four dimensions satisfying $*F = iF$ corresponds in twistor space to a holomorphic vector bundle over $\mathbb{PT}$ valued on the Lie algebra [25]. A self-dual gravitational field corresponds in twistor space to a holomorphic deformation of the complex structure [26]. The up-shot of this is that a highly non-linear problem becomes linearised on twistor space. Exploiting this, the amplitude of two negative helicity, and $n-2$ positive helicity particles can be viewed as anti-self-dual perturbations on a self-dual background, where the self-dual background is linearised in twistor space. We can then derive the amplitudes from two-point correlation functions on this background, giving precisely the Parke-Taylor and Hodges formulae mentioned earlier in this introduction.

Twistor string theory was first developed in the context of $\mathcal{N} = 4$ super-Yang-Mills (sYM) [6, 7, 27]. It was noticed that scattering amplitudes of this theory are supported on certain holomorphic curves on twistor space - labelled by helicity (or in the supersymmetric case R-charge sector). The embedding of these curves is described by the twistor string, and correlation functions of this string (with appropriate vertex operators) are the tree-level scattering amplitudes of sYM, the expression for which is the Roiban-Spradlin-Volovich-Witten (RSVW) formula [6, 23]. A similar story in $\mathcal{N} = 8$ supergravity [8] gives rise to the Cachazo-Skinner formula [24, 28].

These formulae were the basis for a revolution in the study of tree-level scattering amplitudes known as the CHY formalism [9–11]. The RSVW and Cachazo-Skinner formulae are both localised to a sum of solutions of the polarised scattering equations. Remarkably, these equations can be generalised to arbitrary dimensions, giving rise to the scattering equations. The scattering equations describe the scattering of massless particles in a whole zoo of theories, much beyond those that can be described using twistor strings. However, they lose the aspects of 'helicity grading' that was so powerful in providing simple expressions for MHV scattering. On the other hand, they have a natural manifestation of the *double copy*.

The double copy (see [29] for references) is a relation between scattering amplitudes in gauge theory and in gravity. Heuristically, it is a way of finding gravity amplitudes from the 'square' of gauge theory amplitudes. In practise, this is manifested using colour-kinematics (CK) duality or the Kawai-Lewellen-Tye (KLT) double copy for tree-level amplitudes. The main advantage of this method to calculate gravitational scattering amplitudes is that the relative complexity of interactions in gauge theory — in comparison to gravity — is much lower. CK duality is one of the main tools that allows for the calculation of many-loop scattering amplitudes in supergravity, with additional applications in the study of gravitational waves.

The second part of this thesis is a careful study of the double copy relation between the RSVW and Cachazo-Skinner formulae. This turns out to reveal a novel formulation of the double copy, adapted to helicity-graded formulae. In addition to scattering amplitudes in flat space, there exist formulae for scattering on certain self-dual backgrounds and anti-de-Sitter (AdS), making use of integrable structures mentioned previously in this introduction. This program of study therefore reveals an avenue of research towards finding a double copy structure in the presence of background fields.

Outline of this thesis. The main body of this thesis is divided into two parts. Part I concerns scattering amplitudes on backgrounds, whilst Part II focuses on the double copy properties of amplitudes coming from twistor space.

The first part starts with chapter 2, which introduces the basic properties of amplitudes calculations on backgrounds, and the main properties of some of the background fields we will be considering. Chapter 3 concerns the calculation of scattering amplitudes on Schwarzschild and Coulomb, whilst in chapter 4 we calculate the radiation waveforms for scattering of a massive particle in plane wave backgrounds in gravity and electromagnetism.

The second part consists of two chapters. Chapter 5 is an introduction to twistor space and amplitudes calculated using twistor theory. Following this, chapter 6 features a derivation and proof of various properties of the double copy for helicity-graded amplitude formulae.

Finally, chapter 7 concludes the thesis. Additional material related to previous chapters can be found in the appendices in part III.

Part I

On Strong Backgrounds

Chapter 2

Technical Introduction

In this chapter we will introduce the basic concepts needed for calculating scattering amplitudes in the presence of strong background fields. We will start in Section 2.1 by introducing plane wave backgrounds in electromagnetism and gravity. In Section 2.2 we review the basic tools for calculating scattering amplitudes on generic backgrounds. We will then review the three-point amplitude in plane wave backgrounds that will play a central role in chapter 4.

2.1 Plane wave backgrounds in electromagnetism and gravity

Plane wave backgrounds are certain highly symmetric solutions to the fully non-linear classical equations of motion in electromagnetism or general relativity (GR) [30, 31]. In both theories, they represent a field of pure radiation propagating from past to future null infinity, along a specified null direction. The remarkable feature of these backgrounds is that they admit exact, closed solutions to the background-coupled equations of motion associated with momentum eigenstates. This is in stark contrast to many other backgrounds where solving the equations of motion in an appropriate form is already a huge undertaking.

In this section, we use notation for plane waves, and their related quantities, as in [32, 33]. We note that this notation differs in general to other contemporary works on generic plane waves, such as [34, 35].

Electromagnetic plane waves The natural coordinates on which to define plane waves are lightfront coordinates. Here the flat metric is written as

$$ds^2 = 2dx^+ \, dx^- - \delta_{ab} dx^a dx^b, \tag{2.1.1}$$

where $a, b = 1, \dots, d-2$. The transverse coordinates x^a will often be denoted collectively by $x^\perp$. The transformation to the Cartesian coordinates is trivial for $x^\perp$, and $x^\pm = x^0 \pm x^{d-1}$. Throughout this introduction we will use the vector $n = \partial_+$ (or $n_\mu = \delta^-_\mu$) associated to these coordinates.

Electromagnetic plane waves are described by the potential

$$A_\mu(x)\,dx^\mu = A_a(x^-)\,dx^a \tag{2.1.2}$$

or equivalently (after a gauge transformation $A \to A - d\phi$ with $\phi(x) = x^a A_x(x^-)$)

$$\tilde{A}_\mu(x)\,dx^\mu = -x^a\,\dot{A}_a(x^-)\,dx^-. \tag{2.1.3}$$

The solutions to the vacuum Maxwell equations have field strength

$$F_{\mu\nu}(x^-)\,dx^\mu \wedge dx^\nu = \dot{A}_a(x^-)dx^- \wedge dx^a. \tag{2.1.4}$$

We will consider in particular *sandwich* electromagnetic plane waves. Here there exists some initial and final light-front times x_i^- and x_f^- so that $\dot{A}_a(x^- < x_i^-) = 0$ and $\dot{A}_a(x^- > x_f^-) = 0$. This gives us a notion of flatness in these asymptotic regions, and in particular allows for the definition of an S-matrix on this background. We can always shift the potential A_a by a constant value, however the difference

$$a_{\infty\perp} = A_\perp(+\infty) - A_\perp(-\infty) \tag{2.1.5}$$

is invariant. This encodes the *electromagnetic memory effect* associated to this background [36]. The memory effect describes the permanent change in momentum of charged matter after the wave has passed through it. It will be particularly useful for us to define in-going and out-going gauges satisfying

$$A_\perp^{\text{in}}(x^- < x_i^-) = 0, \qquad A_\perp^{\text{out}}(x^- > x_f^-) = 0. \tag{2.1.6}$$

These are simply related by a gauge transformation with $\phi(x) = x^a a_{\infty\perp}$.

Free fields on electromagnetic plane waves The scalar free field solutions with charge e, mass m, and associated momentum $p_\mu = (p_+, p_\perp, p_-)$ on this background are [37]

$$\Phi_p(x) = \exp[i\,\phi_p(x)], \tag{2.1.7}$$

$$\phi_p(x) = p_+x^+ + (p_\perp + eA_\perp(x^-))x^\perp + \frac{1}{2p_+}\int^{x_-} ds\Big[(p_\perp + eA_\perp(s))^2 + m^2\Big]. \tag{2.1.8}$$

By adding labels 'in' or 'out' to $A_\perp$ appearing in this expression, one can specify that the scalar solutions looks like the flat $e^{ip\cdot x}$ in the in-going ($x^- < x_i^-$) or out-going ($x^- > x_f^-$) regions.

It's possible to define a *dressed momentum* for these fields, via

$$P_\mu\,dX^\mu := -ie^{-i\phi_p}D_\mu\,e^{i\phi_p}dX^\mu \tag{2.1.9}$$

$$P_\mu(x) = p_\mu - eA_\mu(x) + n_\mu\frac{2eA(x)\cdot p - e^2A^2(x)}{2\,p_+} \tag{2.1.10}$$

where the covariant derivative is defined with respect to the potential and the charge of the scalar field $D_\mu = \partial_\mu - ieA_\mu$. Notably this momentum remains on-shell even when x^- takes values in the sandwich region.

As photons do not interact with electromagnetic backgrounds, their free field solutions on these backgrounds remain unchanged compared to the flat case. For completeness they take the form

$$a_\mu(x) = \epsilon_\mu \, \exp[ik \cdot x], \qquad k \cdot x = k_+ x^+ + k_\perp x^\perp + \frac{k_\perp k^\perp}{2\,k_+} x^- \tag{2.1.11}$$

where we've chosen lightfront and Lorenz gauge so that $\epsilon \cdot n = 0$ and $k \cdot \epsilon = 0$.

Gravitational plane waves A gravitational plane wave is described in Brinkmann coordinates [30] by

$$ds^2 = 2\,dx^+\,dx^- - \delta_{ab}\,dx^a\,dx^b - \kappa\,H_{ab}(x^-)\,x^a x^b\,(dx^-)^2 \tag{2.1.12}$$

where $a, b = 1, \ldots d-2$ and $x^\pm = x^0 \pm x^{d-1}$. This is a solution to the vacuum Einstein equations provided that the $(d-2) \times (d-2)$ matrix $H_{ab}(x^-)$ is trace-free: $H^a_a(x^-) = 0$. From hereon we will absorb the gravitation coupling into the background $\kappa\,H_{ab} \to H_{ab}$ — this means that expressions containing H or its dependents are automatically a resummation of κ terms. We will additonally impose that the wave profile is *sandwich* [38]: there exist initial and final light-front times, x_i^- and x_f^- so that $H_{ab}(x^- < x_i^-) = 0$ and $H_{ab}(x^- > x_f^-) = 0$.

One can see that the null vector $n = \partial_+$, as defined earlier, is a covariantly constant Killing vector of any metric of this form. Other geometric quantities related to this metric will also play an important role in the rest of this thesis. Firstly the Riemann tensor has non-vanishing components

$$R^a_{\ -b-}(x^-) = -H^a{}_b(x^-), \tag{2.1.13}$$

as well as index permutations of this. Given the traceless condition on H_{ab} the Ricci tensor and scalar are both zero. The geodesic deviation equation for the transverse coordinates takes the form

$$\ddot{e}_a(x^-) = H_{ab}(x^-)\,e^b(x^-), \qquad e^a(x^-) = \Delta x^a. \tag{2.1.14}$$

We choose a basis of $(d-2)$ linearly independent vectors $E^i{}_a(x^-)$ spanning this set of solutions, defining a set of transverse vielbeins for this metric. The index $i = 1, \ldots, d-2$ spans the set. By definition the vielbeins obey

$$\ddot{E}_{i\,a} = H_{ab}\,E_i{}^b, \qquad \dot{E}_{[i}{}^a\,E_{j]}{}^b = 0, \tag{2.1.15}$$

where the Brinkmann indices $a, b, \ldots$ are raised by the flat transverse metric δ^{ab}. The inverse $E^i{}_a(x^-)$ satisfies

$$\sum_{i=1}^{d-2} E^i{}_a(x^-)\,E_{i\,b}(x^-) = \delta_{ab}, \tag{2.1.16}$$

and we can define a metric of the geodesic distances through

$$\gamma_{ij}(x^-) := E_{(i}{}^a(x^-)\,E_{j)\,a}(x^-). \tag{2.1.17}$$

The inverse of this metric is $\gamma^{ij}(x^-)$, satisfying $\gamma_{ij}\gamma^{jk} = \delta_i{}^k$. From this, we can build a physically equivalent metric describing gravitational plane waves, known as the *Einstein-Rosen metric* [31]

$$ds^2 = dX^- \, dX^+ - \gamma_{ij}(X^-)\, dX^i \, dX^j. \tag{2.1.18}$$

The Einstein-Rosen coordinates can be obtained from Brinkmann coordinates using the coordinate transformation

$$X^- = x^-, \tag{2.1.19a}$$

$$X^+ = x^+ + \frac{1}{2}\dot{E}^i{}_a E_{i\,b} x^a x^b, \tag{2.1.19b}$$

$$X^i = E^i{}_a x^a. \tag{2.1.19c}$$

A further object useful for describing the geometric optics of these backgrounds is the deformation tensor

$$\sigma_{ab}(x^-) \coloneqq \dot{E}^i{}_a(x^-)\, E_{i\,b}(x^-). \tag{2.1.20}$$

It encodes the expansion and shear of the null geodesic congruence through its trace and trace-free parts respectively.

It will be useful to understand the general properties of these objects. The function $H_{ab}(x^-)$ uniquely defines the plane wave, however all other quantities defined above feature some coordinate ambiguity. More precisely, (2.1.15) gives a set of unique solutions provided we have appropriate initial conditions. Natural ones to consider are the *in-going* and *out-going* initial conditions

$$E^{\text{in}}_{i\,a}(x^- < x^-_i) = \delta_{i\,a}, \qquad E^{\text{out}}_{i\,a}(x^- > x^-_f) = \delta_{i\,a}. \tag{2.1.21}$$

These have the general opposite asymptotics

$$E^{\text{in}}_{i\,a}(x^- > x^-_f) = b^{\text{in}}_{i\,a} + x^- \, c^{\text{in}}_{i\,a}, \qquad E^{\text{out}}_{i\,a}(x^- < x^-_i) = b^{\text{out}}_{i\ \ a} + x^- \, c^{\text{out}}_{i\ \ a} \tag{2.1.22}$$

which satisfy

$$b^{a\,\text{in}}_{[i} \, c^{\text{in}}_{j]\,a} = b^{a\,\text{out}}_{[i} \, c^{\text{out}}_{j]\,a} = 0, \qquad b_i{}^{a\,\text{out}} = \delta_{ib}\, \delta^{bj}\, b_j{}^{b\,\text{in}}. \tag{2.1.23}$$

This is imposed from the second equation in (2.1.15) and the conservation of the Wronskian between the two solutions. Additionally, rotating the coordinates, $c_{i\,a}$ can be set to be a symmetric matrix.

At this point it should be noted that a big motivator for considering plane waves is that they can describe important properties of more generic space-times such as the memory and tail effects. In fact, applying a certain limit, called the Penrose limit [39, 40] to any space-time, will result in a plane wave space time. The Penrose limit takes as input a space-time and a null geodesic γ with affine parameter z in this space time. As a succession of observers approach γ more and more closely (i.e. travelling with greater and greater speeds), they set their clocks to run faster and faster, until their times match z along γ. The spacetime they observe in this limit is a gravitational plane wave.

However, despite this generality, plane waves have a key issue in that they are not globally hyperbolic [41]. This is due to a phenomenon known as null geodesic focusing, or caustics, and stems from the fact that the Einstein-Rosen form of the metric (2.1.18) is in general singular at some values of x^-. However, the S-matrix is still well-defined on these backgrounds [42].

Free fields on gravitational plane waves In Brinkmann coordinates the scalar wave equation is

$$\left(2\partial_-\partial_+ + H_{ab}(x^-)x^a x^b\,\partial_+^2 - \partial_a\partial^a\right)\Phi = 0. \tag{2.1.24}$$

This is solved by [43]

$$\Phi(x) = \Omega(x^-)\,e^{i\,\phi_k}, \qquad \Omega(x^-) \coloneqq |\gamma^{-1}(x^-)|^{1/4} = |E(x^-)|^{-1/2} \tag{2.1.25}$$

associated to a null momentum $(k_\perp^2/k_+, k_+, k_\perp)$ where

$$\phi_k \coloneqq \frac{k_+}{2}\sigma_{ab}x^a x^b + k_i E^i_a\,x^a + k_+\,x^+ + \frac{k_i k_j}{2\,k_+}F^{ij}, \qquad F^{ij}(x^-) = \int^{x^-} ds\,\gamma^{ij}(s). \tag{2.1.26}$$

This solution depends inherently on what initial conditions we specify to solve the differential equation (2.1.15) for the vielbeins. Choosing $E^{\text{in}}_{i\,a}$ this is a solution that is flat ($e^{ik\cdot x}$) in the asymptotic past, whilst choosing $E^{\text{out}}_{i\,a}$ is flat in the asymptotic future

$$\Phi^{\text{in}}(x) = e^{ik\cdot x} \quad \text{when} \quad x^- < x_i^- \tag{2.1.27}$$

$$\Phi^{\text{out}}(x) = e^{ik\cdot x} \quad \text{when} \quad x^- > x_f^-. \tag{2.1.28}$$

These solutions have a naturally associated 'dressed' momentum, which will be used throughout the calculations. This is

$$\begin{aligned} K_\mu(x)dx^\mu &\coloneqq d\phi_k \\ &= k_+ dx^+ + \left(\frac{k_+}{2}\dot\sigma_{bc}x^b x^c + k_i \dot E^i_a x^b + \frac{k_i k_j}{2k_+}\gamma^{ij}\right)dx^- + \left(k_i E^i_a + k_+\,\sigma_{ab}x^b\right)dx^a. \end{aligned} \tag{2.1.29}$$

Gravitational perturbations can be constructed out of the scalar solution using the spin-raising operators introduced in [44], generalised beyond four dimensions:

$$\mathcal{R}^a \coloneqq dx^-\delta^{ab}\frac{\partial}{\partial x^b} + dx^a\frac{\partial}{\partial x^+}. \tag{2.1.30}$$

Applying this twice to the scalar solution, with some associated polarisation vector $\epsilon_{ab} = \epsilon_a\epsilon_b$, which constrains our solution to lightfront and Lorenz gauge (apropriately dressed on these backgrounds), we have

$$h_{\mu\nu}(x)dx^\mu dx^\nu = \frac{1}{k_+^2}\epsilon_a\mathcal{R}^a\left(\epsilon_b\mathcal{R}^b\Phi\right) =: \mathbb{P}^{ab}{}_{\mu\nu}(x)\,\epsilon_a\epsilon_b\,dx^\mu dx^\nu\,\Phi(x); \tag{2.1.31}$$

$$\mathbb{P}^{ab}{}_{\mu\nu} = \mathbb{P}^a{}_\mu(x)\mathbb{P}^b{}_\nu(x) - in_\mu n_\nu\frac{\sigma^{ab}(x^-)}{k_+}, \qquad \mathbb{P}^a{}_\mu := \frac{K^a(x)\,n_\mu}{k_+} + \delta^a_\mu \tag{2.1.32}$$

in terms of the dressed momentum (2.1.29). The $n_\mu n_\nu$ term in $\mathbb{P}^{ab}{}_{\mu\nu}$ is often interpreted as the tail [34] of the gravitational radiation, which comes from solutions on this space violating Huygen's principle [45]. Huygen's principle states that the propagation of information for massless fields is supported on the null cone. In gravitational plane waves, the fields also scatter off the background geometry, which means that the Green's functions have support also within the null cone. It can be shown [32] that the terms proportional to the shear σ_{ab} appearing in the polarisation tensor generate these tail effects.

2.2 Scattering amplitudes on backgrounds

In the next two chapters we will use the perturbiner method to calculate tree-level scattering amplitudes on strong backgrounds. The perturbiner method is a functional method for generating elements of the tree-level S-matrix. It provides a way of calculating amplitudes, or their corresponding objects in spacetimes where the S-matrix is not well-defined, with arbitrary external data. We encourage the reader to refer to [46–59] for the original and recent work on this topic. The review here is primarily based on Appendix A in [32].

In the perturbiner definition of scattering amplitudes, we consider an action of the form

$$S[\Phi] = \int d^dX\,(\mathcal{L}_{\text{kin}}[\Phi] + \mathcal{L}_{\text{int}}[\Phi]) \tag{2.2.33}$$

where Φ are scalar fields defined on the background. Whilst in this introduction, we will only consider scalar fields, it is straightforward to generalise these consideration to higher spin fields including gauge fixing. Here $\mathcal{L}_{\text{kin}}$ is quadratic in the field, and captures the free part of the theory. In particular, it defines a differential operator $\mathcal{D}^2$ via the Euler-Lagrange equations, which annihilates the free solutions $\phi(x)$. The interaction part $\mathcal{L}_{\text{int}}$ is higher order in the couplings of the theory, and contains the interaction terms.

The classical solutions to the non-linear equations of motion of this theory are generated recursively using the integral formula

$$\Phi^{[n]}(X) := \sum_{i=1}^{n}\epsilon_i\phi_i(X) + \int d^dY\,\Delta(X,Y)\,\frac{\delta\mathcal{L}_{\text{int}}}{\delta\Phi(Y)}\bigg|_{\Phi(Y)=\Phi^{[n]}(Y)}. \tag{2.2.34}$$

Here the fields $\{\phi_i\}$ are n solutions to the free equations of motion with respect to $\mathcal{L}_{\text{kin}}$, with appropriate (but currently arbitrary) boundary condtions. The object $\Delta(X,Y)$ is a Green's function corresponding to the differential operator $\mathcal{D}^2$.

Neglecting boundary terms (for example using an appropriate $\pm i\epsilon$ prescription for the external states), the n-point tree-level scattering amplitudes for the states $\{\phi_i\}$ is the multi-linear piece of the classical action with respect to the fully non-linear classical solution

$$\mathcal{M}_n^{(0)}(\phi_1, \dots, \phi_n) = \left.\frac{\partial^n S[\Phi^{[n]}]}{\partial\epsilon_1 \cdots \partial\epsilon_n}\right|_{\epsilon_1=\dots=\epsilon_n=0} . \tag{2.2.35}$$

It can be shown that this corresponds to the usual definition of the connected part of the S-matrix for flat backgrounds. On non-trivial backgrounds where the S-matrix does not exist (such as black hole spacetimes) this object will still encode important dynamics, as will be discussed in chapter 3.

The next chapters will mainly concern tree-level *three-point amplitudes*. Assuming that we can neglect boundary terms, these just involve the free parts of (2.2.34), and are given by an integral over spacetime

$$\mathcal{M}_3^{(0)}(\phi_1, \phi_2, \phi_3) = \int d^d X \left.\frac{\partial^3 \mathcal{L}_{\text{int}}[\epsilon_1\phi_1 + \epsilon_2\phi_2 + \epsilon_3\phi_3]}{\partial\epsilon_1\partial\epsilon_2\partial\epsilon_3}\right|_{\epsilon_1=\epsilon_2=\epsilon_3=0} \tag{2.2.36}$$

of the interaction part of the Lagrangian.

The generalisation to actions with multiple types of fields is straightforward, by considering the iterated solution (2.2.34) sourced in addition by the other fields.

2.2.1 Scattering amplitudes on plane wave backgrounds

Here we will consider two instructive examples of three-point amplitudes calculated using the method illustrated above. Both of these will play a key role in chapter 4.

Electromagnetism We consider the action of a massive, charged scalar particle in an electromagnetic plane wave background, coupled to the electromagnetic field. Seeking to calculate the three-point scattering amplitude of two scalars and a photon on this background, the interaction part of the action has the cubic component

$$S[\Phi, a] \supset 2e \int d^d x \, a_\mu \, (D^\mu \Phi) \, \Phi. \tag{2.2.37}$$

If we consider the scattering of an in-going scalar, going to an out-going scalar and an emitted out-going photon, the three-point amplitude is, see e.g. [15],

$$\mathcal{M}_3^{(0)}(a_k^{\text{out}}, \Phi_{p'}^{\text{out}}, \Phi_p^{\text{in}}) = \\ ie(2\pi)^{d-1}\delta^{d-1}_{+,\perp}(p' + k - p - ea_\perp) \int_{-\infty}^{+\infty} dx^- \, \epsilon_\mu \, (P^\mu_{\text{in}} + P'^\mu_{\text{out}}) \, \exp\left[i\mathcal{V}[p, k]\right]. \tag{2.2.38}$$

The *Volkov exponent* $\mathcal{V}[p,k]$ in this scattering scenario is given by the x^- part of the exponential and equal to

$$\mathcal{V}[p,k] = \int^{x^-} \mathrm{d}s \left[-\frac{(p+eA^{\mathrm{in}}(s))\cdot(p+eA^{\mathrm{in}}(s))}{2\,p_+} + \frac{(p'+eA^{\mathrm{out}}(s))\cdot(p'+eA^{\mathrm{out}}(s))}{2\,p'_+} \right] + k_-\,x^- \tag{2.2.39}$$

in terms of the in- and out-going gauge fields (2.1.6). Additionally, we recall the definition of the electromagnetic memory $a_\perp$ from (2.1.5). For all in-going asymptotics (or equivalently, no memory $a_\perp = 0$) the formula simplifies to

$$\begin{aligned}\mathcal{M}_3^{(0)}(a_k, \Phi_{p'}, \Phi_p) = \mathrm{i}e(2\pi)^{d-1}\delta^{d-1}_{+,\perp}(p'+k-p)\int_{-\infty}^{+\infty} \mathrm{d}x^- \epsilon_\mu (P^\mu + P'^\mu)&\\ \times \exp\left[\mathrm{i}\int_{-\infty}^{x^-} \mathrm{d}s \frac{k\cdot P(s)}{p_+ - k_+}\right]&\end{aligned} \tag{2.2.40}$$

on the support of the momentum-conserving δ-functions. This is the form we will be using in chapter 4.

Gravity Consider the action describing a massive scalar coupled to gravity in a plane wave background. The interaction part of the action has cubic component

$$S[\Phi, h] \supset \kappa \int \mathrm{d}^d x \sqrt{|g|} h^{\mu\nu} \left(2\partial_\mu \Phi\, \partial_\nu \Phi - g_{\mu\nu} \left(\partial_\rho \Phi\, \partial^\rho \Phi - \frac{m^2}{2}\Phi^2 \right) \right) \tag{2.2.41}$$

where $g_{\mu\nu}$ is the background plane wave metric. Using the solutions to the free equations of motion on this background for the scalar field and the (trace-free) gravitational perturbation, and evaluating (2.2.36) we have [33]

$$\begin{aligned}\mathcal{M}_3^{(0)}(h_k^{\mathrm{out}}, \phi_{p'}^{\mathrm{out}}, \phi_p^{\mathrm{in}}) = -4\,\kappa \int \mathrm{d}^d x\, \Omega^2_{\mathrm{out}}(x^-)\Omega_{\mathrm{in}}(x^-)\, \mathcal{E}_{\mathrm{out}\,\mu\nu}\, P^\mu_{\mathrm{in}} P'^\nu_{\mathrm{out}}&\\ \times \exp[-\mathrm{i}\phi_p^{\mathrm{in}} + \mathrm{i}\phi_{p'}^{\mathrm{out}} + \mathrm{i}\phi_k^{\mathrm{out}}]&\end{aligned} \tag{2.2.42}$$

describing the scattering of an in-going scalar with momentum p, an out-going scalar with momentum p' and an outgoing graviton with momentum k, with geometric objects defined previously in this section. Due to the symmetries of plane wave backgrounds, it's possible to do all but the x^- integral in these kinds of amplitudes. For generic boundary conditions, the transverse $\mathrm{d}^{d-2}x^\perp$ integrals involve Gaussian integrals. This is much simplified if we assume that all fields have the same in/out-going asymptotics, or (equivalently) that the plane wave has *no memory* which will be discussed further in chapter 4.

Using all in-going asymptotics in (2.2.42) instead, and doing all possible integrals, we find

$$\mathcal{M}_3^{(0)}(h_k, \phi_{p'}, \phi_p) = -4\,\kappa(2\pi)^{d-1}\delta^{d-1}_{+,\perp}(p'+k-p)\int_{-\infty}^{+\infty} \frac{\mathrm{d}x^-}{\sqrt{|E|}} \mathcal{E}_{\mu\nu} P^\mu P'^\nu(x^-) e^{\mathrm{i}\mathcal{V}[p,k]} \tag{2.2.43}$$

where $\mathcal{V}[p,k]$ is the gravitation Volkov exponent

$$\begin{aligned}\mathcal{V}[p,k] &\coloneqq \int^{x^-} \mathrm{d}s \frac{P_\mu K_\nu g^{\mu\nu}(s)}{(p-k)_+} \\ &= \frac{1}{(p-k)_+}\int^{x^-} \mathrm{d}s\, \gamma^{ij}(s)\left(\frac{p_+}{k_+}k_i k_j + \frac{k_+}{p_+}p_i p_j - p_i k_j\right). \end{aligned} \tag{2.2.44}$$

Chapter 3

Scattering Amplitudes on Spherically Symmetric Backgrounds

3.1 Introduction

Our main motivation for studying amplitudes on backgrounds in this and the next chapter is to study the two-body problem, or more generally, the classical observables arising from astrophysical events.

Perturbation theory in general relativity has been developed extensively over the past 100 years using the two-body problem as a natural laboratory (cf., [60–63]). In this case, the presence of multiple scales allows for several perturbative schemes which may be investigated individually, as in the effective field theory approach to the two-body problem first pioneered by Goldberger and Rothstein [64]; see [65–67] for reviews of subsequent developments. Among these schemes, the post-Minkowskian ('PM') expansion [68–76], valid for weak gravitational fields but with no restrictions on velocity, has received renewed attention. This interest follows a remarkable calculation, building upon [77, 78], for the PM expansion of the conservative potential of a compact binary system, using on-shell scattering amplitudes [79, 80]. This calculation and subsequent works [81–90] have demonstrated the possibility of systematically organizing the PM expansion solely in terms of on-shell amplitudes in Minkowski spacetime and their classical limits, bypassing ordinary perturbative methods in classical relativity and providing an alternative way to understand the two-body problem (cf., [21, 91, 92] and reviews [93–95]).

In particular, it has been demonstrated – for instance, using the formalism developed by [21, 96–99] – that the PM expansion of classical observables such as the scattering angle, waveform and power emitted are determined by corresponding on-shell, perturbative scattering amplitudes. In essence, the classical observable, to a given PM precision, is determined by an on-shell phase space integral over the classical limit of on-shell scattering amplitudes (or products thereof) computed in the ordinary perturbative expansion of a quantum field theory. For example, the classical waveform for gravitational radiation emitted by the scattering of two Schwarzschild black holes is determined to leading PM order by the tree-level 5-point amplitude for two massive scalars to scatter and emit a single graviton in the field theory of massive scalars minimally coupled to general relativity [98].

Consequently, it is natural to consider whether the same tools can be applied to other perturbative expansions relevant to the two-body problem in general relativity. One example is the self-force expansion [20, 100–104]. This expansion assumes the existence of an exact solution to the Einstein field equation, referred to as the *background* and it is defined by studying deviations from geodesic motion in powers of a dimensionless parameter given by the mass of the particle and a natural mass scale associated with the background.

The self-force expansion on a Schwarzschild (or Kerr) background is perhaps the most notable example, due to its significance in the effective-one-body description which accurately describes the two-body problem in general relativity for a (non-spinning) binary system of masses m and $M \gg m$ [105–108]. At zeroth-order in the mass ratio m/M, the motion of the probe particle follows a geodesic, and no radiation is emitted. At next-to-leading order, the emission of gravitational waves due to the particle's acceleration and its consequent backreaction is taken into account by solving Einstein's field equations to the same order [109–113], or equivalently, considering EFT in curved spacetime [114, 115]. Interestingly, most of the literature on this topic has focused on the study of self-force for initially *bound* orbits, while the investigation of the *unbound* case has only gained attention quite recently [116–123]. This setting seems highly amenable to scattering amplitude methods, where initial data is expressed in terms of on-shell, unbound states.

An intriguing question arises: what are the necessary amplitude building blocks to compute self-force corrections to scattering observables on a Schwarzschild background? In this chapter, we provide an answer: the required building blocks are represented by semiclassical on-shell amplitudes on the corresponding Schwarzschild background, constructed within the perturbiner approach (cf., [46–58]) as reviewed in chapter 2.2. Here tree-level scattering amplitudes are computed from solutions of the classical equations of motion, determined by the multiplicity of the scattering process of interest, and by the quantum numbers of the scattered states. While this is also the natural input for standard (i.e., Feynman diagram) calculations of scattering amplitudes, the power of the perturbiner approach is that it can be used even when the S-matrix does not exist, as on a Schwarzschild background [42, 124–128]. We show that the building blocks of self-force corrections constructed in this way are controlled by Hamilton's principal function on the background. Furthermore, we demonstrate that they may also be interpreted in terms of resummed perturbative amplitudes. We thus argue that our approach provides a conceptual pathway to the self-force approximation for unbound orbits solely in terms of on-shell amplitudes in vacuum[1].

Our focus is on the scattering of scalar particles and the leading order emission of radiation in static spherically symmetric backgrounds. We find it useful to consider electromagnetism alongside the more complicated gravitational setting (see [134, 135] for studies of the two-body probem in scalar QED using the electromagnetic analogy of the PM expansion), so our investigation concerns scattering in Coulomb and Schwarzschild backgrounds, respectively.

[1]At the geodesic level, the possibility of exploring all-order results from resummed scattering amplitudes in vacuum has been investigated in [129], using an algebraic relation between scattering amplitudes and the Hamiltonian in an isotropic gauge [80, 130–132], valid to all orders only in 4-dimensions [133].

This chapter is organized as follows: In Section 3.2 we introduce the notion of *semiclassical scattering amplitudes*, the calculation of which will be the focus of much of this chapter. Section 3.3 applies this formalism to elastic scattering of massive scalars on Coulomb and Schwarzschild backgrounds, where we introduce a novel method to define the semiclassical wavefunctions in terms of Hamilton's principal function (HPF) for the background and certain 'matching coefficients' to ensure proper asymptotic behaviour. We see that this reproduces the well-known expressions for elastic scattering and geodesic motion in terms of the radial action of the background.

We then proceed to the computation of the semiclassical photon emission amplitude in Section 3.4. We see that this is controlled by the HPF (rather than the radial action), and demonstrate that in the classical, weak field limit the semiclassical amplitude on the Coulomb background gives the probe limit of the classical 5-point photon emission amplitude from two scalars. This implies that the classical, probe limit of the 5-point amplitude is in fact a linear function of the HPF itself, highlighting that it is the HPF, rather than radial action, which controls radiation. Section 3.5 deals with the semiclassical graviton emission amplitude, where the definition of the emitted graviton state presents new complications. While we are only able to define the amplitude schematically on the full Schwarzschild metric, linearising the background enables a more explicit but still rich computation. The semiclassical amplitude is again controlled by the HPF, with its classical weak field limit reproducing the classical probe limit of the 5-point graviton emission amplitude from massive scalars in Minkowski spacetime. Section 3.6 concludes with a discussion of future directions and how classical, self-force observables can be constructed from our results.

3.2 Semiclassical scattering amplitudes

In standard perturbation theory around a trivial vacuum, tree-level scattering amplitudes can be given a purely variational definition, as multi-linear pieces of the classical action evaluated on recursively constructed solutions to the equations of motion. The order to which one constructs the solution perturbatively in the coupling and its boundary conditions are dictated by the multiplicity of the scattering process and the asymptotic quantum numbers of the scattered states, respectively. This is sometimes called the *perturbiner approach* to scattering amplitudes [46–58], which trades the combinatorial computations of traditional Feynman rules for computations in differential equations and variational calculus.

The perturbiner approach can easily be extended to scattering amplitudes in background (gauge and gravitational) fields by extracting multilinear pieces of the classical background field action (cf., [32, 33, 136–138])[2], as reviewed in chapter 2. When the background fields admit an S-matrix (e.g., ultrarelativistic beams, shockwaves and sandwich plane waves in gauge theory and gravity), the amplitudes obtained from the perturbiner approach agree with those obtained using background-coupled Feynman rules. However, even when the S-matrix does *not* exist – as in black hole spacetimes [42, 124–128] – the perturbiner approach remains well-defined. The resulting amplitudes are gauge-invariant quantities which contain all of

[2]An equivalent definition of these quantities as 'on-shell' correlators has been given in [139–141]

the dynamical information expected from tree-level background field scattering amplitudes which is needed to compute observable quantities; see [142] for explicit examples in the case of electromagnetic horizons. As such we continue to use the word 'amplitudes' for the output of perturbiner calculations. Other potential ambiguities associated with a particular background (e.g., lack of a unique choice of vacuum) will manifest themselves in the choices of admissible boundary conditions for the background-coupled equations of motion.

The recursive construction of solutions to the equations of motion in the perturbiner approach is seeded with solutions to the free equations of motion with boundary conditions corresponding to asymptotic scattering states; this is simply the perturbiner version of LSZ truncation. The external states corresponding to the full tree-level S-matrix (which will include quantum information when there are massive particles involved) are thus exact solutions to the 'free' equations of motion on the background. However, on many backgrounds including Coulomb potentials in QED and black holes in general relativity (linearised or fully non-linear), the required solutions are so complicated that explicit calculations of scattering amplitudes – particularly in the presence of emitted radiation – have simply not been possible.

To address this, we introduce here a tractable approximation of tree-level scattering amplitudes in background fields which we refer to as *semiclassical scattering amplitudes*. These semiclassical amplitudes are defined by taking the semiclassical WKB approximation for the external states in the scattering process – that is, by approximating solutions to the free equations of motion in the background – and using these as the input for the perturbiner approach. To be precise:

Definition 1 (Semiclassical scattering amplitude)**.** A tree-level scattering amplitude (in the sense of the perturbiner approach) with all external legs defined by solving the free equations of motion using the WKB expansion to leading order in the $\hbar \to 0$ limit.

For external massless fields, where $\hbar$ does not enter the equations of motion, this semiclassical prescription is not an approximation: the free equation of motion is classically exact. Thus, massless legs in a semiclassical scattering amplitude are represented by solutions to their full equation of motion, without approximation.

As we will see, semiclassical amplitudes defined in this way are controlled by Hamilton's principal function (HPF): the solution to the Hamilton-Jacobi equations for a particle on the background. Even when there are massless external legs which interact with the background – as in the emission of gravitational radiation on a curved spacetime – this remains the case, as the graviton wavefunction can be written in terms of the HPF and a background-dressed polarization tensor, which is itself related to the HPF through the linearised Einstein equation.

The massive free field equations on the asymptotically flat, static and spherically symmetric backgrounds (i.e., Coulomb and Schwarzschild) that we consider in this chapter are *not* WKB exact, so these semiclassical amplitudes contain less information than the full tree-level S-matrix, but – as we will see – they encode the underlying classical probe dynamics in the background and its weak field limit.

3.3 Semiclassical scalar wavefunctions & elastic scattering

The external wavefunctions for any scattering process in a background are defined by solving the free, background-coupled equations of motion for the asymptotic incoming and outgoing states involved in the process. For massive scalar particles coupled to electromagnetism or gravity, these are solutions to the Klein-Gordon equation on the given background with appropriate boundary conditions. The standard approach to solving these equations for spherically symmetric backgrounds like a Coulomb field or the Schwarzschild metric is to separate variables, reducing the problem to a second-order radial ODE for the coefficient functions of a spherical harmonic expansion (cf., [143–151]). The 2-point (or, more properly, $1 \to 1$) amplitude for elastic scattering is then read off – at least, in principle – from the asymptotic expansion of these radial wavefunctions (cf., [152–155]). Simplifications arise under assumptions such as small momentum exchange, ultrarelativistic limits or a perturbative description of the background (e.g., [138, 156–160]).

Here, we show that semiclassical scalar wavefunctions in static, spherically symmetric backgrounds can be determined (at all-orders in the coupling, for generic scattering angle) by using a WKB ansatz combined with an asymptotic matching condition to the usual radial wavefunctions. This procedure is inspired by a similar 'patching' approach to solving the Klein-Gordon equation in the WKB-exact, ultrarelativistic setting where the background is localized on a lightfront [41, 156, 160, 161], suitably generalized to non-WKB-exact backgrounds like Coulomb and Schwarzschild. States constructed in this way have the substantial advantage of being highly amenable to calculation in the relativistic, covariant framework of background QFT. As a warm-up, we show how the 2-point, elastic scattering amplitudes for semiclassical scalars are obtained in a straightforward way using these states.

3.3.1 Semiclassical scalar states on static, spherically symmetric backgrounds

On-shell complex scalar fields coupled to background electromagnetic or gravitational fields are defined by solutions to the Klein-Gordon equation in the background. Let $A_\mu(x)$ denote a background electromagnetic field solving Maxwell's equations in Minkowski spacetime and $g_{\mu\nu}(x)$ denote a background metric solving the Einstein equations. Working in Lorenz gauge for the electromagnetic background ($\partial^\mu A_\mu = 0$) and de Donder gauge for the gravitational background ($g^{\mu\nu}\,\Gamma^\alpha_{\mu\nu} = 0$ for $\Gamma^\alpha_{\mu\nu}$ the Christoffel symbols of $g_{\mu\nu}$), the Klein-Gordon equations become

$$\left(\Box - \frac{2i}{\hbar}\, e\, A^\mu \partial_\mu - \frac{e^2}{\hbar^2}\, A^2 + \frac{m^2}{\hbar^2}\right)\phi(x) = 0\,, \qquad \left(g^{\mu\nu}\partial_\mu\partial_\nu - \frac{m^2}{\hbar^2}\right)\phi(x) = 0\,. \tag{3.3.1}$$

In the first equation, $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the Minkowski spacetime d'Alembertian, e is the charge of the complex scalar and all indices are contracted via the Minkowski metric.

To define semiclassical states from solutions to these equations, we employ a WKB approximation

$$\phi(x) = e^{i\,\frac{S(x)}{\hbar}}\,. \tag{3.3.2}$$

In the $\hbar \to 0$ semiclassical limit, the Klein-Gordon equations (3.3.1) become the Hamilton-

Jacobi equations for the background:

$$\eta^{\mu\nu}\,(\partial_\mu S - e\,A_\mu)\,(\partial_\nu S - e\,A_\nu) = m^2\,, \qquad g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S = m^2\,. \tag{3.3.3}$$

In other words, in the semiclassical limit the field equations impose that the WKB phase $S(x)$ becomes Hamilton's principal function (HPF) for the background. From now on, we will set $\hbar = 1$ in most expressions, with the implicit understanding that we work in the semiclassical limit described by (3.3.3); the dropping of quantum contributions will be highlighted where necessary.

Now, to obtain scattering states parametrized by an on-shell asymptotic momentum p_μ obeying $\eta^{\mu\nu}p_\mu p_\nu = m^2$, we follow [138] and make a weak-field expansion of the WKB phase

$$S(x) = \sum_{n=0}^{\infty} S^{(n)}(x)\,, \qquad S^{(n)}(x) \propto e^n,\; G^n\,, \tag{3.3.4}$$

where e is the elementary charge in the electromagnetic case and G is Newton's constant in the gravitationally-coupled case. We also assume the existence of a similar weak-field expansion of the background fields themselves; for the inherently linear electromagnetic background this is trivial, while for gravity it implies

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + \sum_{n=2}^{\infty} H^{(n)}_{\mu\nu} \tag{3.3.5}$$

with $H_{\mu\nu} \propto G$ the leading, linear correction to Minkowski spacetime and $H^{(n)}_{\mu\nu} \propto G^n$ encoding the higher, non-linear terms.

This leads to a system of coupled differential equations at each order in the expansion:

$$\partial_\mu S^{(0)}\,\partial^\mu S^{(0)} = m^2\,, \tag{3.3.6}$$

$$\text{EM:}\quad \partial^\mu S^{(0)}\,\partial_\mu S^{(1)} = e\,A^\mu\,\partial_\mu S^{(0)}\,, \quad \text{GR:}\quad 2\,\partial^\mu S^{(0)}\,\partial_\mu S^{(1)} = H^{\mu\nu}\,\partial_\mu S^{(0)}\,\partial_\nu S^{(0)}, \tag{3.3.7}$$

to subleading order in the weak field expansion, with all indices in all equations now contracted using the Minkowski metric. The (theory-independent) leading equation for $S^{(0)}$ can then be solved in terms of an on-shell momentum p_μ:

$$S^{(0)}(x) = p\cdot x\,, \qquad p^2 = \eta^{\mu\nu}p_\mu\,p_\nu = m^2\,, \tag{3.3.8}$$

in Minkowski spacetime.

We now make an additional simplifying assumption, which in effect restricts us to the cases of interest: we assume that the background field is spherically symmetric. By Birkhoff's theorems in both electromagnetism [162] and general relativity [163–165] this implies that the backgrounds are static and asymptotically flat. In particular, if we assume that scattering occurs in a vacuum region of spacetime (i.e., outside of any sources), the electromagnetic and gravitational background fields are the Coulomb gauge potential and Schwarzschild metric, respectively. In terms of the fields entering the PDEs defining $S^{(1)}$ in (3.3.7), we have, in spherical polar coordinates (t, r, θ, ϕ)

$$A_\mu = \frac{Q\, U_\mu}{4\,\pi\, r}\,, \qquad H_{\mu\nu} = \frac{2\, G\, \mathcal{P}_{\mu\nu}}{r}\,, \tag{3.3.9}$$

where Q is the charge of the Coulomb potential, $U_\mu = (1, 0, 0, 0)$ and $\mathcal{P}_{\mu\nu}$ is the constant tensor

$$\mathcal{P}_{\mu\nu} := M\left(\eta_{\mu\alpha}\,\eta_{\nu\beta} + \eta_{\mu\beta}\,\eta_{\nu\alpha} - \eta_{\mu\nu}\,\eta_{\alpha\beta}\right) U^\alpha\, U^\beta\,, \tag{3.3.10}$$

for M the Schwarzschild mass. The equations (3.3.7) for $S^{(1)}$ are then easily integrated to give

$$S^{(1)}(x) = \frac{eQ\, p\cdot U}{4\,\pi\,|\vec{p}|}\, \log(|\vec{p}|r + \vec{p}\cdot\vec{r})\,, \tag{3.3.11}$$

for electromagnetism and

$$S^{(1)}(x) = \frac{G\, \mathcal{P}^{\mu\nu}\, p_\mu\, p_\nu}{|\vec{p}|}\, \log(|\vec{p}|r + \vec{p}\cdot\vec{r}) \tag{3.3.12}$$

for gravity. In both expressions, $\vec{p}$ denotes the spatial components on the on-shell momentum p_μ with (Euclidean) norm $|\vec{p}|$.

It should be noted that in the context of large-distance, the phase $S^{(1)}$ is equal to an *eikonal phase*, which is a function of the radial distance r alone (cf., [138, 151, 158]). However, if one wishes to consider situations beyond elastic 2-point scattering, the non-trivial angular dependence of $S^{(1)}$ is crucial, as we will show later.

More generally, one can proceed recursively to solve for the HPF (3.3.4) order-by-order in the weak-field expansion. Let $S_p(x)$ denote the resulting *all-order* HPF corresponding to on-shell asymptotic momentum p_μ; a general solution to the free field equations (3.3.1) can then be determined by taking an on-shell linear combination of such particular solutions:

$$\phi(x) = \int d\Phi(p)\, \Lambda(p)\, e^{i\, S_p(x)}\,, \tag{3.3.13}$$

where (defining $\hat{d}^4p \equiv d^4p/(2\pi)^4$ and $\hat{\delta}(\cdot) = 2\pi\delta(\cdot)$),

$$d\Phi(p) := \hat{d}^4p\ \Theta(p^0)\, \hat{\delta}(p^2 - m^2)\,, \tag{3.3.14}$$

is the Lorentz-invariant on-shell measure and $\Lambda(p)$ are as-yet-undetermined coefficients.

At this point, we take inspiration from a similar procedure for constructing general solutions to the Klein-Gordon equation on plane- or pp-wave backgrounds which are localized on a lightfront [41, 156, 160, 161]. In that context, the coefficients in the on-shell superposition (3.3.13) are determined by matching conditions at this lightfront, namely that the equation of motion is solved on the lightfront itself. Such backgrounds which include impulsive plane waves, ultrarelativistic shockwaves and beams are, of course, very different from Coulomb or Schwarzschild: they are WKB exact, so the procedure determines fully quantum mechanical scattering states. The lesson we wish to apply to the context of (3.3.13) in a Coulomb or Schwarzschild background is that the coefficients in the sum can be fixed by demanding that the solution has desired properties in a certain region of spacetime.

In Coulomb or Schwarzschild, the natural matching region is at spatial infinity[3], $r \to \infty$. Here, our solution should agree with solutions to the Klein-Gordon equation obtained in the 'standard' way, by separating variables and expanding in spherical harmonic modes. The radial wavefunctions obtained in this way encode the elastic scattering amplitude in their asymptotic behaviour as $r \to \infty$, so clearly the solutions (3.3.13) must agree with them in this asymptotic region. This is enough to fix the coefficients $\Lambda(p)$.

To begin, recall that general solutions to the Klein-Gordon equations (3.3.1) in a static, spherically symmetric background can be written in terms of a spherical harmonic expansion

$$\phi_p(x) = \frac{4\pi\, e^{i\,E\,t}}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{x})\, \overline{Y_\ell^m}(\hat{p})\, R_{\ell m}(r)\,, \tag{3.3.15}$$

where $E \equiv p^0$, $Y_\ell^m(\hat{x})$ are the usual spherical harmonics evaluated at (θ, φ) on the unit sphere $\hat{x} = (\sin\theta\, \cos\varphi,\, \sin\theta\, \sin\varphi,\, \cos\varphi)$ and $\hat{p} = \vec{p}/|\vec{p}|$ is the unit vector associated to the spatial momentum. The non-trivial part comes from solving for the radial wavefunction modes $R_{\ell m}(r)$, which are determined by a second-order ODE of Schrödinger type. The functional form of these radial wavefunctions can be quite complicated: in Coulomb or linearized Schwarzschild backgrounds they are Whittaker/confluent hypergeometric functions, while in a fully non-linear Schwarzschild metric they are confluent Heun functions. Luckily, we will only require the asymptotic form of the radial wavefunctions as $r \to \infty$, and their asymptotic expansions are well-known (cf., [168] Sections 33 and 31, respectively).

First, we consider the asymptotic expansion of the semiclassical WKB ansatz. For $S^{(0)} = p \cdot x$ one invokes the plane wave expansion and spherical harmonic addition theorem to find

$$e^{i\,p\cdot x} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^\ell\, j_\ell(|\vec{p}|r)\, Y_\ell^m(-\hat{x})\, \overline{Y_\ell^m}(\hat{p})\, e^{i\,E\,t}\,, \tag{3.3.16}$$

where j_ℓ are the spherical Bessel functions. As $r \to \infty$ these obey

$$j_\ell(|\vec{p}|r) = \frac{e^{i\,(|\vec{p}|r - \ell\,\pi/2)}}{2\,i\,|\vec{p}|\,r} - \frac{e^{-i\,(|\vec{p}|r - \ell\,\pi/2)}}{2\,i\,|\vec{p}|\,r} + O(r^{-2})\,. \tag{3.3.17}$$

The two leading terms correspond to left or right-moving particles in the semiclassical state, respectively; only the second term is relevant for the desired state in the on-shell sum (3.3.13), so the first term is discarded by hand[4].

[3]In spherical polar coordinates, $r \to \infty$ is the asymptotic boundary region associated to the Coulomb potential and Schwarzschild metric. However, one could instead write the background fields in some other coordinate system with a different boundary; for instance, retarded Bondi coordinates would give future null infinity $\mathscr{I}^+$ as the natural boundary. The matching conditions and wavefunctions in such alternative coordinates will certainly look very different to those in spherical coordinates, but diffeomorphism invariance ensures that the corresponding scatting amplitudes themselves will agree; see [166, 167] for some 2-point examples obtained from $\mathscr{I}$.

[4]If we proceeded naïvely, keeping the first term leads to an apparent un-physical divergence in the wave-

From (3.3.11) – (3.3.12) it is clear that the next correction to the HPF, $S^{(1)}(x)$ grows like $\log r$ as $r \to \infty$, while all terms in the HPF $S^{(n)}$ for $n \geqslant 2$ scale as $O(r^{-1})$ (see [169–172] for explicit expressions at $n = 2$ in the eikonal regime); this follows simply by inspection of the structure of the Hamilton-Jacobi equations at higher-orders in the weak-field expansion. Thus, it follows that the semiclassical WKB wavefunction, to all orders in the coupling, behaves at large r as

$$e^{i\,S_p(x)} = \frac{2\pi\,i}{|\vec{p}|\,r}\left(\sum_{\ell=0}^{\infty} Y_\ell^m(\hat{x})\,\overline{Y_\ell^m}(\hat{p})\right) e^{i\left(E\,t-|\vec{p}|\,r+S^{(1)}(x)\right)} + O(r^{-2})\,, \tag{3.3.18}$$

for an outgoing state. Using the spherical harmonic completeness relations, this is further simplified to

$$e^{i\,S_p(x)} = \frac{2\pi\,i}{|\vec{p}|\,r}\,\delta^2_{\Omega_x}(\hat{x}-\hat{p})\,e^{i\left(E\,t-|\vec{p}|\,r+S^{(1)}(x)\right)} + O(r^{-2})\,, \tag{3.3.19}$$

where

$$\delta^2_{\Omega_x}(\hat{x}-\hat{p}) := \frac{1}{\sin\theta}\,\delta^2(\hat{x}-\hat{p})\,, \tag{3.3.20}$$

covariantly localizes the angular dependence to that of the on-shell momentum. On the support of these delta functions

$$S^{(1)}(x) \longrightarrow C_p\,\log(2|\vec{p}|\,r)\,, \tag{3.3.21}$$

for C_p the theory-dependent constant pre-factor determined by (3.3.11) and (3.3.12) in electromagnetism and gravity, respectively.

Feeding (3.3.19) and (3.3.21) into (3.3.13), the asymptotic behaviour of the general solution is

$$\phi(x) = \frac{2\pi i}{E\,r}\int_0^\infty |\vec{p}|\,d|\vec{p}|\,\Lambda(|\vec{p}|,\hat{x})\,e^{i(E\,t-|\vec{p}|\,r+C_p\log(2|\vec{p}|\,r))} + O(r^{-2})\,, \tag{3.3.22}$$

where three of the four on-shell phase space integrals have been done trivially against delta functions.

At this point, the asymptotic matching condition between (3.3.22) and an exact solution of the form (3.3.15) with momentum p'_μ and energy E reads:

$$\begin{aligned}\frac{i}{E}\int_0^\infty |\vec{p}|\,d|\vec{p}|\,\Lambda^{p'}(|\vec{p}|,\hat{x})\,\lim_{r\to\infty} e^{i(E\,t-|\vec{p}|\,r+C_p\log(2|\vec{p}|\,r))}&\\ = 2\,e^{i\,E\,t}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{x})\,\overline{Y_\ell^m}(\hat{p}\,')\,\lim_{r\to\infty} R_{\ell m}(r)\,,&\end{aligned} \tag{3.3.23}$$

where the superscript on $\Lambda^{p'}$ denotes the fact that these coefficients are being fixed by

function. To see that the first term corresponds to a *finite* contribution which simply has the wrong scattering behaviour requires a more careful treatment of all harmonic and asymptotic expansions, which is described in Appendix A.1.

matching with a solution of momentum p'. This condition can be solved by taking

$$\Lambda^{p'}(|\vec{p}|,\hat{p}) = -\frac{2i\,E}{|\vec{p}|}\,\delta(|\vec{p}|-|\vec{p}\,'|) \\ \times \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{p})\,\overline{Y_\ell^m}(\hat{p}\,')\,\lim_{r\to\infty} R_{\ell m}(r)\,e^{i(|\vec{p}|\,r - C_p\log(2|\vec{p}|\,r))}\,. \tag{3.3.24}$$

Now, as $r\to\infty$ the radial wavefunctions behave at leading order as (cf., [151]):

$$R_{\ell m}(r) \xrightarrow{r\to\infty} e^{-i(|\vec{p}|\,r - C_p\log(2|\vec{p}|\,r)) + i\,B_\ell}\,, \tag{3.3.25}$$

where B_ℓ depend on the kinematics and mode number ℓ but are otherwise constant. In other words, the r-dependence appearing in the large-r limit part of the matching condition (3.3.24) precisely cancels, leaving a finite result.

The combination of spherical harmonics and large-r limits appearing in (3.3.24) can then be repackaged into a (finite) partial wave sum known as the elastic scattering amplitude [151]:

$$\begin{aligned} f^{p'}(|\vec{p}|,\hat{p}) &:= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\hat{p})\,\overline{Y_\ell^m}(\hat{p}\,')\,\lim_{r\to\infty} R_{\ell m}(r)\,e^{i(|\vec{p}|\,r - C_p\log(2|\vec{p}|\,r))} \\ &= \sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\hat{p}\cdot\hat{p}\,')\,e^{2i\,I_\ell(r=\infty)-i\pi\ell}\,, \end{aligned} \tag{3.3.26}$$

where P_ℓ are the Legendre polynomials and $I_\ell(r=\infty)$ is the *radial action* [173] at infinity, regularized by the subtraction of a divergent accumulation phase. The radial action takes the explicit forms (cf., [151])

$$I_\ell^{\text{Coulomb}}(r) = \int_{r_{\text{turn}}}^{r}\sqrt{\frac{\vec{p}^{\,2}\,s^2 + \frac{E\,eQ}{2\pi}\,s - \nu_\ell^2}{s^2}}\,ds\,, \tag{3.3.27}$$

for $\nu_\ell = \sqrt{\ell^2 - \frac{eQ}{4\pi}} - 1/2$ in Coulomb and

$$I_\ell^{\text{Sch}}(r) = \int_{r_{\text{turn}}}^{r}\sqrt{\frac{\vec{p}^{\,2}\,s^2 + 2GMm\,s - \frac{s-2GM}{s}\ell^2}{(s-2GM)^2}}\,ds\,. \tag{3.3.28}$$

in Schwarzschild, and the accumulation phase takes the universal form $|\vec{p}|r + \eta\,\log(2|\vec{p}|r)$ for

$$\eta^{\text{Coulomb}} := \frac{E\,e\,Q}{2\pi\,|\vec{p}|}\,,\qquad \eta^{\text{Sch}} := \frac{G\,M}{|\vec{p}|}\left(E^2+\vec{p}^{\,2}\right)\,. \tag{3.3.29}$$

In both cases, r_{turn} is the value of s for which the integrand – the radial momentum of a freely falling probe – vanishes.

It is worth mentioning that in the limit where the angle between $\hat{p}$ and $\hat{p}\,'$ is small, it is well-known that the partial wave sum in (3.3.26) can be expressed as an eikonal integral (cf., [149, 174–179]). In this eikonal limit, the coefficients $\Lambda(p)$ are still given by (3.3.32), but the amplitude f is now given by

$$f^{p'}(|\vec{p}|,\,\hat{p}) = i\,|\vec{p}| \int d^2x^\perp\, e^{i\,x^\perp\cdot(\hat{p}-\hat{p}\,')}\, e^{i(2I(|x^\perp|)-\pi\,|\vec{p}|\,|x^\perp|)}\,, \tag{3.3.30}$$

where

$$I(|x^\perp|) := I_{|\vec{p}|\,|x^\perp|-\frac{1}{2}}(r=\infty)\,, \tag{3.3.31}$$

in terms of the radial action. This is due to the eikonal limit of the partial wave sum being dominated by large ℓ contributions, and the eikonal integral (3.3.30) is similarly dominated by a large $|x^\perp|$ saddle point [180–183].

To summarize, for arbitrary scattering angle the matching condition fixes

$$\Lambda^{p'}(|\vec{p}|,\hat{p}) = -\frac{2i\,E}{|\vec{p}|}\,\delta(|\vec{p}|-|\vec{p}\,'|)\, f^{p'}(|\vec{p}|,\,\hat{p})\,, \tag{3.3.32}$$

with $f^{p'}(|\vec{p}|,\,\hat{p})$ determined by the radial action of the background. Feeding this back into the initial form of the general solution (3.3.13) gives the final expression for an outgoing state associated with momentum p^μ:

$$\begin{aligned}\phi_p(x) &= \int d\Phi(l)\,\Lambda^p(l)\,e^{i\,S_l(x)}\Big|_{l^0=p^0}\\ &= -i\,|\vec{p}| \int d^2\Omega_l\, f^p(|\vec{p}|,\,\hat{l})\,e^{i\,S_l(x)}\Big|_{l^0=p^0}\,,\end{aligned} \tag{3.3.33}$$

upon performing two of the on-shell phase space integrals. The remaining integral is over the celestial sphere with measure

$$d^2\Omega_l := \sin\theta_{\hat{\ell}}\, d\theta_{\hat{l}}\, d\varphi_{\hat{l}}\,, \tag{3.3.34}$$

corresponding to the angles defined by $\hat{l}$.

3.3.2 Scattering without emission from the radial action

Usually, the elastic $1\to 1$ scattering amplitude in a static, spherically symmetric background is read off from the asymptotic expansion of the radial wavefunction, with the results for Coulomb and Schwarzschild backgrounds being well-known [143, 144, 146, 149–151, 154]. Here, we show how these results can be obtained directly from the perturbiner approach described in chapter 2.2.1. This is important for two reasons: firstly, it extends the results of [138] connecting $1\to 1$ scattering in linearised Schwarzschild with the eikonal amplitude to $1\to 1$ scattering in the *exact, non-linear* Schwarzschild metric, and secondly, it serves as a useful warm-up for the case of $1\to 2$ scattering with emission that we will consider in subsequent sections.

The 2-point amplitude in this framework is given by the quadratic part of the background coupled classical action, evaluated on the sum of an incoming and outgoing state, taking only the contribution linear in each state. The fact that these states solve the free equation of motion means that only a boundary term contributes[5], and for background Coulomb of Schwarzschild fields (expressed in spherical coordinates) it is straightforward to show that this is given by

$$\langle p'|\,\mathcal{S}\,|p\rangle := \lim_{r\to\infty}\int_{\mathbb{R}\times S^2} dt\, d^2\Omega_x\, r^2\, \bar{\phi}_p^{\text{in}}\, \partial_r \phi_{p'}^{\text{out}}\,, \tag{3.3.35}$$

for $1\to 1$ scattering of a charged or gravitationally-coupled complex scalar with initial momentum p and final momentum p'. As the amplitude is totally localised at spatial infinity, the scattering conditions at this boundary are that the incoming and outgoing states appearing in (3.3.35) are given by

$$\phi_p^{\text{in}}(x) = e^{i\,S_p(x)}\,, \qquad \phi_{p'}^{\text{out}}(x) = -i\,|\vec{p}\,'|\int d^2\Omega_l\, f^{p'}(|\vec{p}\,'|,\,\hat{l})\, e^{i\,S_l(x)}\Big|_{l^0=(p')^0}\,, \tag{3.3.36}$$

in terms of the HPF and background dressing (3.3.33).

Plugging these into (3.3.35) and exploiting the asymptotic expansion (3.3.19) gives

$$\begin{aligned}\langle p'|\,\mathcal{S}\,|p\rangle = \frac{|\vec{p}\,'|}{|\vec{p}|}\lim_{r\to\infty}\int\limits_{\mathbb{R}\times(S^2)^2} dt\, d^2\Omega_x\, d^2\Omega_l\, \delta^2_{\Omega_x}(\hat{x}-\hat{p})\,\delta^2_{\Omega_x}(\hat{x}-\hat{l})\, f^{p'}(|\vec{p}\,'|,\,\hat{l})\\ \times \exp\!\big[i\left((E'-E)\,t + (|\vec{p}|-|\vec{p}\,'|)\,r + C_{p'}\,\log(2|\vec{p}\,'|r) - C_p\,\log(2|\vec{p}|r)\right)\big]\,,\end{aligned} \tag{3.3.37}$$

dropping all terms which vanish in the $r\to\infty$ limit. The time integral is now performed to give a delta function setting $E'=E$, and thus $|\vec{p}|=|\vec{p}\,'|$, as the masses of the incoming and outgoing states are equal. This immediately removes all remaining r-dependence from the integrand, rendering the $r\to\infty$ limit trivial and leaving

$$\begin{aligned}\langle p'|\,\mathcal{S}\,|p\rangle &= \hat{\delta}(E'-E)\int\limits_{S^2\times S^2} d^2\Omega_x\, d^2\Omega_l\, \delta^2_{\Omega_x}(\hat{x}-\hat{p})\,\delta^2_{\Omega_x}(\hat{x}-\hat{l})\, f^{p'}(|\vec{p}|,\,\hat{l})\\ &= \hat{\delta}(E'-E)\, f^{p'}(|\vec{p}|,\,\hat{p})\,.\end{aligned} \tag{3.3.38}$$

In particular, we find that the well-known result that the $1\to 1$ elastic scattering amplitude is given by $f^{p'}(|\vec{p}|,\,\hat{p})$. By (3.3.26), this also confirms that the $1\to 1$ amplitude on any spherically symmetric background is captured exactly, without any small angle or leading-order eikonal approximation, by the full outgoing amplitude, which is itself controlled by the radial action associated with the background [151].

The scattering angle: It is interesting to see how classical physics, such as the scattering angle, is encoded in the 2-point amplitude (3.3.38). The Legendre polynomials appearing

[5]For the Schwarzschild metric, there is another boundary at $r = 2GM$, the event horizon. By ignoring this boundary's contributions to the 2-point amplitude, we are implicitly considering elastic scattering with sufficiently large impact parameter that the probe's interaction with the horizon is vanishingly small. This assumption is implicit in most considerations of $1\to 1$ scattering in black hole spacetimes, although there are several interesting studies that consider horizon effects in different frameworks (e.g., [184–191]).

in (3.3.26) describe the angular dependence of the wavefunction for the outgoing state and contain both classical and quantum information; in keeping with the correspondence principle [192], the classical information is encoded only by certain values of ℓ in the partial wave sum, namely (cf., [155], Chapter VII §49 and XVII §127):

$$\theta\,\ell \gg 1\,, \qquad (\pi-\theta)\,\ell \gg 1\,, \tag{3.3.39}$$

for $\cos\theta = \hat{p}\cdot\hat{p}\,'$ (not to be confused with the spherical polar coordinate θ). The condition of classicality for the angular part of the wavefunction can thus be expressed as: for a given value of $\theta \neq 0, \pi$, the classical limit occurs for large values of ℓ, which corresponds to a small variation in the De Broglie wavelength. In this limit, the Legendre polynomials can be expanded as

$$P_\ell(\cos\theta) = \sqrt{\frac{\theta}{\sin\theta}}\, J_0\!\left(\left(\ell+\frac{1}{2}\right)\theta\right) + O((\theta\ell)^{-3/2})\,, \qquad \ell \gg \theta^{-1}\,, \tag{3.3.40}$$

where J_0 is the Bessel function of the first kind. Substituting this expansion into (3.3.26) does not result in any loss of information so long as the resulting expression is understood in the classical (i.e., large ℓ) limit.

In particular, this eliminates the need to use a small-angle approximation to arrive at the expansion (3.3.40), as was done in [179]. One then defines

$$b := \frac{(\ell+1/2)}{|\vec{p}\,'|}\,, \tag{3.3.41}$$

so that the high-energy limit is now equivalent to assuming small variations of b and the sum over ℓ can be replaced with an integral over b. The 2-point amplitude then becomes

$$\begin{aligned}\langle p'|\,\mathcal{S}\,|p\rangle &= 2i\,\hat{\delta}(E'-E)\,|\vec{p}|\sqrt{\frac{\theta}{\sin\theta}}\int_0^\infty b\,db\,J_0(|\vec{p}|\,\theta\,b)\,e^{i(2\,I(b)-\pi\,|\vec{p}|\,b)}\\ &= \frac{i}{\pi}\,\hat{\delta}(E'-E)\,|\vec{p}|\sqrt{\frac{\theta}{\sin\theta}}\int_0^\infty b\,db\int_0^{2\pi} d\varphi\, e^{i\,|\vec{p}|\,\theta\,b\,\cos\varphi}\,e^{i(2\,I(b)-\pi\,|\vec{p}|\,b)}\,,\end{aligned} \tag{3.3.42}$$

where $I(b)$ is defined in (3.3.31) and the last line follows from the integral representation of the Bessel function. Performing all integrals via saddle point approximation, one finds that the integral is sharply peaked at (b^*, φ^*) such that

$$\theta + \frac{2}{|\vec{p}|}\frac{dI}{db}(b^*) - \pi = 0\,, \qquad \varphi^* = 0\,. \tag{3.3.43}$$

Finally, since the scattering angle is $\chi := \pi - \theta$, we can use (3.3.41) to obtain an expression for this classical observable as a function of the radial action to all orders:

$$\chi = 2\,\frac{dI_{\ell^*}(r=\infty)}{d\ell}\,, \tag{3.3.44}$$

which is still valid for large angles[6].

3.4 Photon emission

At 2-points, we have seen that the WKB approximation for the external states captures the full elastic scattering amplitude through the leading-order correction to the HPF in the weak field expansion. In particular, the fact that the 2-point amplitude is localized as an asymptotic boundary term means that all contributions come from the purely radial part of $S^{(1)}$, which is controlled by the radial action of the background. Beyond elastic scattering, when the emission of radiation plays a role, this is no longer the case.

In this section, we compute the semiclassical amplitude for photon emission from a charged scalar scattering on the Coulomb background. After showing that this amplitude is controlled by the HPF, we consider the classical weak-field limit of the amplitude in the Coulomb background, recovering the classical part of the probe limit of 5-point scattering between two charged scalars with single photon emission in a trivial vacuum. This confirms that our semiclassical amplitude contains the expected physical information at leading order in perturbation theory, but also demonstrates that the radial action is not sufficient to describe classical two-body physics in the presence of emitted radiation at infinity. Indeed, the full angular dependence of the HPF is required to obtain the correct perturbative result.

3.4.1 Semiclassical photon emission amplitude

Following the perturbiner description of tree-level scattering amplitudes (cf., [32, 33, 46–53, 138]), the 3-point amplitude for photon emission from a complex, charged scalar in a background gauge field is given by (i times) the tri-linear terms of the scalar QED action, evaluated on solutions of the free, background-coupled equations of motion:

$$\langle p', k| \mathcal{S} |p\rangle = -e \int d^4x \left(a^{\text{out}}_{k\,\mu}\, \bar{\phi}^{\text{in}}_p\, D^\mu \phi^{\text{out}}_{p'} - a^{\text{out}}_{k\,\mu}\, (\overline{D}^\mu \bar{\phi}^{\text{in}}_p)\, \phi^{\text{out}}_{p'} \right) , \qquad (3.4.45)$$

where ϕ^{in}_p, $\phi^{\text{out}}_{p'}$ are the incoming and outgoing scalar fields with momenta p and p', respectively $a^{\text{out}}_{k\,\mu}$ is the outgoing photon with (massless) momentum k, and $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative defined by the background gauge field. To obtain the *semiclassical* scattering amplitude, we simply evaluate this expression using wavefunctions defined by the WKB approximation – and hence the HPF of the Coulomb background – in accordance with Definition 1.

Since the photon is massless and does not interact with the background gauge field, it can be described exactly by an ordinary plane wave momentum eigenstate, while the scalar

[6]Of course, this relation can also be derived using standard Hamilton-Jacobi analysis [173]; what is relevant for us is that it can be related to scattering amplitudes.

wavefunctions are given by the general solutions (3.3.33)[7]. Explicitly, we have:

$$
\begin{aligned}
\bar{\phi}^{\text{in}}_{p} &= \int \mathrm{d}\Phi(l)\,\overline{\Lambda^{p}(l)}\,e^{-i\,S_l}\Big|_{l^0=p^0}\,,\\
\phi^{\text{out}}_{p'} &= \int \mathrm{d}\Phi(l')\,\Lambda^{p'}(l')\,e^{i\,S_{l'}}\Big|_{(l')^0=(p')^0}\,,\\
a^{\text{out}}_{k\,\mu} &= \varepsilon_\mu\,e^{i\,k\cdot x}\,,
\end{aligned}
\tag{3.4.46}
$$

where ε_μ is the photon polarization vector, on-shell with respect to the photon momentum k^μ in Lorenz gauge (i.e., $k^2 = 0 = k\cdot\varepsilon$). the massless photon momentum. With these external wavefunctions, the semiclassical photon emission amplitude is

$$
\begin{aligned}
\langle p', k|\,\mathcal{S}\,|p\rangle = i\,e \int \mathrm{d}^4x\,\mathrm{d}\Phi(l)\,\mathrm{d}\Phi(l')\,\overline{\Lambda^{p}(l)}\,\Lambda^{p'}(l')\\
\times\,\varepsilon\cdot(\partial S_l + \partial S_{l'} + 2\,e\,A)\;e^{i\,(k\cdot x + S_{l'} - S_l)}\Big|^{l^0=p^0}_{(l')^0=(p')^0}\,,
\end{aligned}
\tag{3.4.47}
$$

with A_μ given explicitly by (3.3.9). Aside from performing the trivial time integral (which results in an energy-conserving delta function), this is as far as the semiclassical scattering amplitude can be evaluated analytically (at least, without making further approximations or simplifications).

This complexity is simply an example of the more general fact that amplitudes on strong backgrounds (even with WKB-exact wavefunctions) are generically highly non-trivial functions of the scattering data (cf., [15, 193, 194]). Unlike in a trivial vacuum, it is not usually possible to perform all spacetime vertex integrals, even at low numbers of points.

Given this complexity, it is natural to ask if there are any checks that we can perform on our results. Clearly, the semiclassical amplitude (3.4.47) contains *some* of the information in the full tree-level 3-point amplitude for photon emission on the Coulomb background, but it is not at all obvious that this corresponds to the probe limit of leading classical, perturbative contributions to photon emission. To check this, one must re-expand (3.4.47) in powers of the coupling to the background to see if an appropriate, purely perturbative result is obtained.

3.4.2 The classical and weak field limits

If the semiclassical 3-point amplitude is truly capturing classical perturbative physics, then the leading classical contribution to $\langle p', k|\,\mathcal{S}\,|p\rangle$ in an expansion in the coupling to the Coulomb background Q should recover part of the 5-point tree-level perturbative amplitude for two massive scalar charges to scatter and emit a photon. The part of this amplitude we should recover is its leading classical behaviour (since we work in a WKB expansion) in the *probe limit*, in which the recoil of one charge (which is essentially a background, since it is not affected by the scattering) is neglected.

[7]As amplitudes at 3-points and higher are not localized to a spacetime boundary, both incoming and outgoing wavefunctions must be fully dressed by the background.

The expansion of (3.4.47) to first order in Q contains two qualitatively distinct contributions. One set of terms will arise by taking the order Q contributions from each factor of the matching coefficients $\overline{\Lambda^{p}(l)}$ and $\Lambda^{p'}(l')$, with all occurrences of the HPF being restricted to $S^{(0)}$. In other words, taking factors of Q from the first line of (3.4.47) only. The other set of terms arises from taking order Q contributions only from the second line of (3.4.47), with only order Q^0 contributions from the matching coefficients.

Let us begin by considering the first set of contributions to the weak-field limit:

$$\begin{aligned} &e \int d^4x\, d^2\Omega_l\, d^2\Omega_{l'} \left(\overline{f^p(|\vec{p}|,\, \hat{l})}\, f^{p'}(|\vec{p}\,'|,\, \hat{l}\,') \right) \Big|_{O(Q)} \varepsilon \cdot (l + l')\, e^{i\,(k+l'-l)\cdot x} \Big|^{l^0=p^0}_{(l')^0=(p')^0} \\ &= 2e \int \hat{\delta}^4(k + l' - l)\, d^2\Omega_l\, d^2\Omega_{l'} \left(\overline{f^p(|\vec{p}|,\, \hat{l})}\, f^{p'}(|\vec{p}\,'|,\, \hat{l}\,') \right) \Big|_{O(Q)} \varepsilon \cdot l \Big|^{l^0=p^0}_{(l')^0=(p')^0} , \end{aligned} \tag{3.4.48}$$

ignoring irrelevant overall factors. As this is proportional to on-shell 3-point momentum conserving delta functions, it is immediately vanishing; however, it is interesting to see that these contributions also vanish if one first takes the classical limit rather than using the overall momentum conserving delta functions.

As we saw in Section 3.3, in the classical limit the elastic amplitude behaves as an eikonal-like integral, meaning the weak field expansion has the form

$$f^p(|\vec{p}|,\, \hat{l}) = \delta^2_{\Omega_l}(\hat{p} - \hat{l}) + i\, \mathcal{A}_4(\hat{l} - \hat{p}) + O(Q^2)\,, \tag{3.4.49}$$

where $\mathcal{A}_4$ is the tree-level, single photon exchange amplitude between two charged scalars with exchanged momentum $\hat{l} - \hat{p}$. This arises through the eikonal phase, which is precisely the inverse Fourier transform of $\mathcal{A}_4$. Furthermore, at leading order in the classical limit, the massless photon momentum k scales as $\hbar$ times the classical wavenumber (cf., [21]). Thus, in the classical, weak field limit the contribution (3.4.48) is given by

$$\begin{aligned} &i\, e^2\, Q \int \hat{\delta}^4(l' - l)\, d^2\Omega_l\, d^2\Omega_{l'}\, \varepsilon \cdot l \left(\delta^2_{\Omega_l}(\hat{p} - \hat{l})\, \mathcal{A}_4(\hat{l}\,' - \hat{p}\,') - \delta^2_{\Omega_{l'}}(\hat{p}\,' - \hat{l}\,')\, \overline{\mathcal{A}_4}(\hat{p} - \hat{l}) \right) \Big|^{l^0=p^0}_{(l')^0=(p')^0} \\ &\qquad = i\, e^2\, Q\, \varepsilon \cdot p \left(\mathcal{A}_4(\hat{p} - \hat{p}\,') - \overline{\mathcal{A}_4}(\hat{p} - \hat{p}\,') \right) \\ &\qquad = -2\, e^2\, Q\, \varepsilon \cdot p\, \mathrm{Im}\, \mathcal{A}_4(\hat{p} - \hat{p}\,') = 0\,. \end{aligned} \tag{3.4.50}$$

In the second and third lines, irrelevant energy conserving delta functions have been omitted, and the imaginary part of $\mathcal{A}_4$ vanishes as it is a real function of the Mandelstam invariants (i.e., s/t), or equivalently as a consequence of the optical theorem.

In other words, the contributions to the classical, weak field limit of $\langle p', k|\, \mathcal{S}\, |p\rangle$ which arise from perturbatively expanding the matching coefficients vanish (which should be contrasted with the case of the 2-point amplitude), and the *only* non-trivial contributions arise from the perturbative expansion of the HPF or the explicit background insertion in (3.4.47). Collecting these terms requires expanding the WKB exponents, so we now replace $S_p(x) \to p \cdot x + S^{(1)}_p$ as in (3.3.11), with an additional subscript to keep track of the scattering momenta, and keep terms linear in Q. With this, (3.4.47) simplifies to

$$\langle p', k|\,\mathcal{S}\,|p\rangle\,\big|_Q = ie\int d^4x\, e^{i(k+p'-p)\cdot x}\Big[\varepsilon\cdot(p+p')\big(i\,S^{(1)}_{p'}(x) - i\,S^{(1)}_p(x)\big) \\ + \varepsilon\cdot\partial\big(S^{(1)}_{p'}(x)+S^{(1)}_p(x)\big) + 2e\,\varepsilon\cdot A(x)\Big]\,, \tag{3.4.51}$$

in which the dressing factors have been replaced by their zeroth-order expressions, given by the first term in (3.4.49), allowing the integrals over l and l' to be performed.

The position integrals can now be performed as a simple Fourier transform (as taking the classical and weak field limits removes all non-trivial x–dependence from the exponent). The Fourier transform of $S^{(1)}_p$ is

$$S^{(1)}_p(x) := -i\int \hat{d}^4\ell\, e^{-i\ell\cdot x}\,\tilde{S}(p;\ell)\,, \qquad \tilde{S}(p;\ell) = e\,Q\,\frac{\hat{\delta}(U\cdot\ell)}{|\vec{\ell}|^2}\,\frac{U\cdot p}{\ell\cdot p}\,, \tag{3.4.52}$$

where the factor of $-i$ is included for convenience, while the Fourier transform of the background Coulomb field is

$$\tilde{A}_\mu(\ell) = U_\mu\,\frac{Q\,\hat{\delta}(U\cdot\ell)}{|\vec{\ell}|^2}\,. \tag{3.4.53}$$

With this, (3.4.51) becomes

$$\langle p', k|\,\mathcal{S}\,|p\rangle\,\big|_Q = ie\Big[\varepsilon\cdot(p+p')(\tilde{S}(p';\ell)-\tilde{S}(p;\ell))-\varepsilon\cdot\ell\,(\tilde{S}(p';\ell)+\tilde{S}(p;\ell))+2e\,\varepsilon\cdot\tilde{A}(\ell)\Big]\,, \tag{3.4.54}$$

in which $\ell = k + p' - p$ is the total momentum transfer. It is now apparent that the leading classical and weak field limit of our 3-point amplitude is fully determined by the Fourier transform of the leading WKB phase $S^{(1)}$.

At this point, to obtain the *classical*, weak field limit of $\langle p', k|\,\mathcal{S}\,|p\rangle$ we need to work consistently only to leading order in the classical limit. Recall that the massless momentum k scales as $\hbar$ times its classical wavenumber, and we also expect the recoil of the massive scalar probe to be small compared to its own rest mass in the classical limit. We thus write $p'_\mu = p_\mu + q_\mu$ in which q scales as $\hbar$ to leading order [21]. In terms of these variables

$$\langle p', k|\,\mathcal{S}\,|p\rangle\,\big|_Q = 2i\,e\Big[\varepsilon\cdot p\,\big(\tilde{S}(p+q;\ell)-\tilde{S}(p;\ell)\big) - \varepsilon\cdot q\,\tilde{S}(p;\ell) + e\,\varepsilon\cdot\tilde{A}(\ell)\Big]\,. \tag{3.4.55}$$

Taking the classical limit of (3.4.55) is then equivalent to extracting the leading term in a Taylor expansion in which both k and q are of the same small order (i.e., $k, q \sim \hbar$). When performing this expansion it is useful to note that, by definition, $q^2 + 2p\cdot q = 0$ and so $p\cdot q$ is actually of order $\hbar^2$.

Let us consider this expansion for each of the various terms in (3.4.55) explicitly. The entire expression is proportional to an overall factor of $eQ\hat{\delta}(U\cdot\ell)/|\vec{\ell}|^2$, so we begin by stripping this off from each term and then expanding. For instance,

$$\varepsilon \cdot q\, \tilde{S}(p;\ell) \propto \varepsilon \cdot q\, \frac{U \cdot p}{(k+q)\cdot p} \xrightarrow{\hbar \to 0} \varepsilon \cdot q\, \frac{U\cdot p}{k \cdot p} + O(\hbar)\,, \tag{3.4.56}$$

where the leading term is $\hbar$-independent. The expansion of the combination $\tilde{S}(p+q;\ell) - \tilde{S}(p;\ell)$ is slightly more subtle:

$$\begin{aligned}\tilde{S}(p+q;\ell) - \tilde{S}(p;\ell) &\propto \frac{U\cdot(p+q)}{(k+q)\cdot(p+q)} - \frac{U\cdot p}{(k+q)\cdot p} \\ &= U \cdot p \left[\frac{1}{(k+q)\cdot(p+q)} - \frac{1}{(k+q)\cdot p}\right] + \frac{U\cdot q}{(k+q)\cdot(p+q)}\,.\end{aligned} \tag{3.4.57}$$

Each of the terms in the large brackets here appears to have super-classical ($\sim \hbar^{-1}$) behaviour, but this cancels between them leaving $\hbar$-independent leading behaviour:

$$\frac{1}{(k+q)\cdot(p+q)} - \frac{1}{(k+q)\cdot p} = \cancel{\frac{1}{k\cdot p} - \frac{1}{k \cdot p}} - \frac{k\cdot q + q^2}{(k\cdot p)^2} \xrightarrow{\hbar\to 0} \frac{k \cdot q}{(k\cdot p)^2} + O(\hbar)\,, \tag{3.4.58}$$

so that

$$\tilde{S}(p+q;\ell) - \tilde{S}(p;\ell) \propto U\cdot p\, \frac{k\cdot q}{(k\cdot p)^2} + \frac{U \cdot q}{k\cdot p} + O(\hbar)\,, \tag{3.4.59}$$

as desired. Here, we imposed that the total recoil $k+q$ is small (to stay within the regime of validity for the background field approach), which means we only keep the leading term in $1/(k+q)^2$ and, in effect, allows us to use the identity $q^2 + 2k\cdot q = 0$ in the numerator. We are thus working to leading order in $\rho := |q+k|/M$, which is the recoil of the heavy particle in units of its mass; we denote this by 'LO(ρ)'.

Assembling (3.4.56)–(3.4.59), the final result for the leading classical and weak field limit of the 3-point semiclassical amplitude on a Coulomb field is thus

$$\begin{aligned}\lim_{\hbar\to 0} \langle p', k|\, \mathcal{S}\, |p\rangle \big|^{\mathrm{LO}(\rho)}_{e^2 Q} =& \hat{\delta}\big(U\cdot(q+k)\big)\frac{2ie^2 Q}{(k+q)^2}\times \\ & \left[-\varepsilon\cdot U + \frac{\varepsilon\cdot q}{k\cdot p} U\cdot p - \frac{\varepsilon \cdot p}{k\cdot p} U \cdot q - \frac{\varepsilon\cdot p\, U\cdot p\, k\cdot q}{(k\cdot p)^2}\right].\end{aligned} \tag{3.4.60}$$

This result can now be compared to direct calculations in perturbative scalar QED; the necessary results are provided in Appendix A.2. Let $\mathcal{A}_5$ be the perturbative 5-point amplitude for photon emission (momentum k) in the scattering of two massive scalar charges ($p \to p'$ and $P = MU \to P'$), stripped of its momentum-conserving delta functions. We find that our 3-point semiclassical photon emission amplitude on the Coulomb background is directly related to $\mathcal{A}_5$ via

$$\lim_{\hbar\to 0} \langle p', k|\, \mathcal{S}\, |p\rangle \big|^{\mathrm{LO}(\rho)}_{e^2 Q} = \lim_{\hbar \to 0} \frac{\hat{\delta}(U\cdot(q+k))}{2M}\, \mathcal{A}_5^{\mathrm{LO}(\rho)}(p, P \to p+q, P-q-k, k)\,. \tag{3.4.61}$$

This relationship is sufficient to guarantee that our semiclassical amplitude will reproduce known classical observables, such as the waveform [98], in the weak field and probe limits. A further check is provided by the literature: (3.4.60) recovers the extreme mass limit ratio of e.g. (5.48) in [21], which is itself the leading classical limit of the full five-point amplitude. As an aside, this demonstrates the commutativity of the probe and classical limits.

3.4.3 The 5-point amplitude and Hamilton's principal function

If the Hamilton-Jacobi equation is separable, the solution of the radial part is the radial action of the system, $I(r)$. It is often assumed that the radial action alone is sufficient to describe a classical two-body system (at least in the probe limit). However, as discussed in Section 3.3.2, the results here show that it will *not* hold when, for instance, considering the classical waveform at infinity.

Working to leading order in the self-force expansion, radiative observables are controlled by the 3-point amplitude (3.4.47). Investigating even the simplest weak-field limit of this amplitude, we have now seen in (3.4.54) and (3.4.61) that it is determined by the Fourier transform of the leading order HPF $S^{(1)}(x)$, which comprises both the radial action and non-trivial angular dependence, as is evident from the explicit position-space representation (3.3.11). Had we not used the full HPF, but only the radial action, we would not have recovered the correct 5-point amplitude.

This is perhaps the simplest counterexample to the claim that the radial action controls all dynamics of a point particle on a (spherically symmetric) background.

We conclude this section by rewriting the result (3.4.61) in a manner which echoes the known relation between the radial action and the perturbative 4-point amplitude. Inspecting (3.4.60), we observe that it can be written as

$$\lim_{\hbar\to 0} \langle p', k|\,\mathcal{S}\,|p\rangle\,\Big|_{e^2Q}^{\mathrm{LO}(\rho)} = \hat{\delta}(U\cdot\ell)\,\frac{2ie^2Q}{\ell^2}\left[\frac{\varepsilon\cdot\ell}{\ell\cdot p}\,U\cdot p + \frac{\varepsilon\cdot p}{\ell\cdot p}\,U\cdot k - \frac{\varepsilon\cdot p\,U\cdot p\,k\cdot\ell}{(\ell\cdot p)^2}\right], \quad (3.4.62)$$

in which $\ell \equiv q + k$, we have used $q\cdot p = 0$ and, for simplicity, chosen the gauge such that $U\cdot\varepsilon = 0$. We have also used the delta function to flip the sign of the second term, because doing so makes it clear that the entire result can be re-packaged in terms of the Fourier transform of the HPF (3.4.52):

$$\begin{aligned}\lim_{\hbar\to 0} \langle p', k|\,\mathcal{S}\,|p\rangle\,\Big|_{e^2Q}^{\mathrm{LO}(\rho)} &= \hat{\delta}(U\cdot\ell)\,\frac{2ie^2Q}{\ell^2}\left[\frac{\varepsilon\cdot\ell}{\ell\cdot p}\,U\cdot p + \varepsilon\cdot p\,k\cdot\partial_p\left(\frac{U\cdot p}{\ell\cdot p}\right)\right] \\ &= -2e\left[\varepsilon\cdot\ell\,\tilde{S}(p;\ell) + \varepsilon\cdot p\,k\cdot\partial_p\,\tilde{S}(p;\ell)\right].\end{aligned} \quad (3.4.63)$$

This Fourier transform can be undone by integrating over ℓ, keeping in mind that the polarisation vectors depend on $k = \ell - q \to -i\partial_x - q$. We could either express the inverse Fourier transform of (3.4.63) in terms of $\hat{\varepsilon} := \varepsilon(-i\partial_x - q)$ or, since we have fixed the gauge, simply strip the polarisation vectors from both sides of (3.4.63) before performing the transform. In this case we find, expressing the classical limit of $\langle p', k|\,\mathcal{S}\,|p\rangle\,\big|_{e^2Q}^{\mathrm{LO}(\rho)}$ in terms of $\mathcal{A}_5 \equiv \varepsilon^\mu \mathcal{A}_{5\mu}$

from (3.4.61),

$$
\begin{aligned}
\lim_{\hbar\to 0}\int d^4\ell\, e^{-i\ell\cdot x}\,\frac{\hat{\delta}(U\cdot\ell)}{2M}\,\mathcal{A}^{\mathrm{LO}(\rho)}_{5\mu}(p,P\to p+q,P-\ell,\ell-q)& \\
=2e\left[\frac{\partial S^{(1)}_p}{\partial x^\mu}(x)-ip_\mu\,k^\nu\,\frac{\partial S^{(1)}_p}{\partial p^\nu}(x)\right]&\,.
\end{aligned}
\tag{3.4.64}
$$

Thus, the Fourier transform of the 5-point amplitude is a linear function of the HPF and its derivatives (with respect to both position and asymptotic momentum). This is highly reminiscent of the result that the Fourier transform of the 4-point amplitude is the leading order radial action, but again emphasises that it is the HPF that controls radiation.

3.5 Graviton emission

We now turn to the semiclassical graviton emission amplitude on a Schwarzschild spacetime. This computation is significantly more complicated than its scalar QED counterpart for two reasons: firstly, the emitted graviton ‘sees’ the background field (unlike the photon in QED) so defining the semiclassical graviton wavefunction is non-trivial; and secondly, because effects due to the event horizon must be accounted for in the fully non-linear black hole background. At 2-points, the issue of the event horizon could be ignored with the physically reasonable assumption that elastic scattering occurs at sufficiently large impact parameter, or equivalently that the 2-point amplitude only receives contributions from the asymptotic boundary. However, at 3-points and beyond, interactions must be integrated over the whole spacetime manifold and the event horizon simply cannot be ignored.

In this section, we outline how the semiclassical graviton wavefunction on fully non-linear Schwarzschild could be determined within our framework, but to circumvent dealing with horizon effects – and to present a more concrete calculation – we then simplify to a *linearised* Schwarzschild background, where the event horizon does not play a role. Even with this simplification, determining the graviton wavefunction and computing the 3-point semiclassical emission amplitude is non-trivial, and we confirm that this is controlled by the HPF of the background. As in the QED case, we show that the classical weak-field limit of the amplitude on linearised Schwarzschild recovers the classical part of the probe limit of 5-point scattering between massive scalars with single graviton emission in Minkowski spacetime.

3.5.1 Semiclassical graviton states

We begin by considering a generic vacuum spacetime background with metric $g_{\mu\nu}$, and a linearised metric perturbation $h_{\mu\nu}$ on this background. Define

$$
\bar{h}_{\mu\nu}:=h_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,h^\sigma{}_\sigma\,,
\tag{3.5.65}
$$

and impose the covariant Lorenz gauge

$$\nabla^{\mu}\bar{h}_{\mu\nu} = 0\,, \tag{3.5.66}$$

where all indices are raised and lowered with background metric g and ∇_{μ} is the Levi-Civita connection of the background. In this gauge, the linearised Einstein equations governing the gravitational perturbation become:

$$\nabla^2\bar{h}_{\mu\nu} + 2\,R_{\mu\rho\nu\sigma}\,\bar{h}^{\rho\sigma} = 0\,, \tag{3.5.67}$$

where $\nabla^2 := g^{\alpha\beta}\,\nabla_{\alpha}\nabla_{\beta}$, $R_{\mu\rho\nu\sigma}$ is the Riemann curvature tensor of the background and we have used the assumption that the background is vacuum, and hence that its Ricci tensor vanishes.

There is no $\hbar$ entering this equation: the graviton is a massless wave and so the linearised Einstein equation is classically exact. However, it is nevertheless useful to make an ansatz for the graviton wavefunction which is motivated by the desire to have a Fourier basis:

$$\bar{h}_{\mu\nu}(x) = \mathcal{E}_{\mu\nu}(x)\,e^{i\,S(x)}\,, \tag{3.5.68}$$

where S is the HPF solving the *massless* Hamilton-Jacobi equation on the background

$$S(x) = k\cdot x + \frac{G\,\mathcal{P}^{\mu\nu}\,k_{\mu}\,k_{\nu}}{|\vec{k}|}\,\log(|\vec{k}|r + \vec{k}\cdot\vec{r}) + O(G^2)\,. \tag{3.5.69}$$

Note that the graviton polarization tensor, $\mathcal{E}_{\mu\nu}(x)$, acquires non-trivial spacetime dependence due to its interaction with the background. The choice to factor out the HPF phase is motivated by the form of known exact graviton wavefunctions in other background fields, for example plane waves, see [32]. As we will see, the choice is natural: the asymptotic behavior at spatial infinity of our graviton state contains an accumulated phase similar to that in the scalar wavefunction. This is completely consistent, since it matches the asymptotic behavior of a gravitational perturbation around Schwarzschild, and it is equivalent to requiring, as a boundary condition, a free phase in tortoise coordinates (see, for example, Section 12.2.5 of [63]). In particular, the polarisation tensor becomes equal to the free-field polarisation at large distance, simplifying our matching conditions.

A straightforward calculation with this ansatz shows that the linearised Einstein equations and Lorenz gauge condition become

$$\nabla^2\mathcal{E}_{\mu\nu} + i\,\mathcal{E}_{\mu\nu}\,\nabla^2 S + 2i\,\partial_{\alpha}S\,\nabla^{\alpha}\mathcal{E}_{\mu\nu} + 2\,R_{\mu\rho\nu\sigma}\,\mathcal{E}^{\rho\sigma} = 0\,, \tag{3.5.70}$$

and

$$\nabla^{\mu}\mathcal{E}_{\mu\nu} + i\,\mathcal{E}_{\mu\nu}\,\partial^{\mu}S = 0\,, \tag{3.5.71}$$

respectively. In particular, this means that with the ansatz (3.5.68), the background-dressed semiclassical graviton polarization $\mathcal{E}_{\mu\nu}$ is determined by the HPF (and the background geometry) through (3.5.70) and gauge consistency condition (3.5.71)[8]. Note that if we additionally impose traceless gauge (i.e., $h^{\mu}_{\mu} = 0$), then $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and these are the equations for the graviton polarization itself.

We make the additional assumption that the dressed polarization admits a weak field expansion

$$\mathcal{E}_{\mu\nu}(x) = \varepsilon_{\mu\nu} + \sum_{n=1}^{\infty} \mathcal{E}^{(n)}_{\mu\nu}(x)\,, \tag{3.5.72}$$

with $\varepsilon_{\mu\nu}$ the on-shell graviton polarization in Minkowski spacetime in transverse traceless gauge ($\eta^{\mu\nu}\varepsilon_{\mu\nu} = 0 = \eta^{\sigma\mu}\,k_\sigma\,\varepsilon_{\mu\nu}$) and $\mathcal{E}^{(n)}_{\mu\nu}$ of order G^n. Thus, the dressed polarization tensor can be determined order-by-order by solving the linearised Einstein equation with the weak field expansion of the HPF itself. For instance, using the weak field expansion (3.3.5) of the background spacetime metric, (3.5.70) at linear order in G becomes:

$$(\Box + 2i\,k\cdot\partial)\,\mathcal{E}^{(1)}_{\mu\nu} + i\,\varepsilon_{\mu\nu}\left(\Box S^{(1)} - k_\sigma\,\eta^{\alpha\beta}\,\Gamma^{(1)\,\sigma}_{\alpha\beta}\right) - 4i\,k^\alpha\,\varepsilon_{\sigma(\mu}\,\Gamma^{(1)\,\sigma}_{\nu)\alpha} + 2\,R^{(1)}_{\mu\rho\nu\sigma}\,\varepsilon^{\rho\sigma} = 0\,, \tag{3.5.73}$$

where all indices are now raised and lowered with the *Minkowski* metric, and $\Gamma^{(1)\,\alpha}_{\beta\gamma}$ and $R^{(1)}_{\mu\rho\nu\sigma}$ are the linearised Christoffel symbols and Riemann curvature tensor, both constructed from $H_{\mu\nu}$.

With the specification that we are working with the Schwarzschild metric, for which $H_{\mu\nu}$ is given by (3.3.9), it follows that we can solve for $\mathcal{E}^{(1)}_{\mu\nu}$ using a Green's function:

$$\begin{aligned}\mathcal{E}^{(1)}_{\mu\nu}(x) = -2\,\varepsilon_{\beta(\mu}\,k^\alpha \int \hat{d}^4\ell\,\frac{e^{-i\,\ell\cdot x}}{\ell^2 - 2\ell\cdot k + i\,\epsilon}\left(\tilde{H}^{\beta}_{\nu)}\,\ell_\alpha + \ell_{\nu)}\,\tilde{H}^{\beta}_{\alpha} - \tilde{H}_{\nu)\alpha}\,\ell^\beta\right)\\ - i\,\varepsilon_{\mu\nu}\int \hat{d}^4\ell\,\frac{|\vec{\ell}|^2\,e^{-i\,\ell\cdot x}}{\ell^2 - 2\ell\cdot k + i\,\epsilon}\,\tilde{S}(k;\ell)\\ - \varepsilon^{\rho\sigma}\int \hat{d}^4\ell\,\frac{e^{-i\,\ell\cdot x}}{\ell^2 - 2\ell\cdot k + i\,\epsilon}\left(-\ell_\mu\ell_\nu\,\tilde{H}_{\rho\sigma} + \ell_\nu\ell_\rho\,\tilde{H}_{\mu\sigma} + \ell_\sigma\ell_\mu\,\tilde{H}_{\rho\nu} - \ell_\sigma\ell_\rho\,\tilde{H}_{\mu\nu}\right)\,,\end{aligned} \tag{3.5.74}$$

where $i\,\epsilon$ denotes a choice of contour prescription for inverting the differential operator acting on $\mathcal{E}^{(1)}_{\mu\nu}$ in (3.5.73) in momentum space. In this expression

$$\tilde{H}_{\mu\nu} = 4\pi\,G\,\frac{\mathcal{P}_{\mu\nu}\,\hat{\delta}(U\cdot\ell)}{|\vec{\ell}|^2}\,, \qquad \tilde{S}(k;\ell) = 2\pi i\,G\,\frac{\hat{\delta}(U\cdot\ell)\,\mathcal{P}^{\alpha\beta}\,k_\alpha\,k_\beta}{|\vec{\ell}|^2\,\ell\cdot k}\,, \tag{3.5.75}$$

so it follows that all of the ℓ_μ appearing in (3.5.74) are effectively $\vec{\ell}_\mu$ and purely spatial. Proceeding in this fashion, one can recursively solve for the dressed polarization order-by-order in the weak field expansion, similarly to how one constructs the HPF itself. It can be verified that $\mathcal{E}^{(1)}_{\mu\nu}$ obeys the Lorenz gauge condition (3.5.71) to linear order in the coupling at leading order in r^{-1}, which is the regime of validity for our approximation, to this order.

[8]Observe that, as the graviton wavefunction is meant to describe a classical wave, its background-dressed polarization does not obey the transport equation and the gauge consistency condition of the strict geometric optics limit (for more on this limit, see [60, 195–197]).

One can also verify that it satisfies the traceless gauge condition at this order. Additionally, it is easily seen that $\lim_{r\to\infty} \mathcal{E}^{(1)}_{\mu\nu}(x) = 0$, reducing the graviton polarization to the free field polarization at large r.

Just as we did with scalar states in Section 3.3, we construct a general graviton wavefunction by taking an on-shell linear combination of the solutions to the linearised Einstein equations in the form of our ansatz:

$$h_{\mu\nu}(x) = \int d\Phi(k)\, \Lambda_{\mu\nu}{}^{\rho\sigma}(k)\, \mathcal{E}_{\rho\sigma}(x)\, e^{i\, S_k(x)}\,, \tag{3.5.76}$$

where $d\Phi(k)$ is the massless Lorentz-invariant on-shell measure and $\Lambda_{\mu\nu}{}^{\rho\sigma}(k)$ are the tensorial coefficients of the on-shell combination. Although our exposition above of the linearised Einstein equations (3.5.70) was in Lorenz gauge, this expression for a general graviton is schematically true in *any* gauge: the difference with Lorenz gauge will be in the structure of the PDE determining $\mathcal{E}_{\mu\nu}$ from the HPF.

In the fully non-linear Schwarzschild spacetime, the coefficients in (3.5.76) must be consistent with boundary values of exact solutions to the linearised Einstein equations both asymptotically ($r \to \infty$) *and* at the event horizon ($r \to 2GM$). The mechanism for doing this is to consider the linearised Einstein equations with separation of variables; this was initially done long ago in the Regge-Wheeler gauge, where gravitational perturbations are governed by simple 1-dimensional radial Schrödinger equations with different potentials depending on whether they are of axial or polar type [144, 146, 148, 198]. This is still the case in Lorenz gauge [199], where the radial part of the perturbation is controlled by a generalized Regge-Wheeler-Zerilli equation. The presence of dissipative horizon dynamics (such as quasinormal modes) would then correspond to allowing the HPF to become *complex*, with corresponding damped, non-oscillatory behaviour at the event horizon. Matching at the event horizon must also be taken into account for the scalar wavefunctions – defined in Section 3.3 only through a matching at infinity – governed by the behaviour of the radial scalar wavefunctions of the Klein-Gordon equation in the Schwarzschild metric near the horizon (cf., [200–202]).

While it would be extremely interesting to understand how the matching at infinity and event horizon is implemented in the fully non-linear setting, doing so explicitly is beyond the scope of this chapter. To give a precise example of the general semiclassical graviton wavefunction (3.5.76), we focus on the case of the *linearised* Schwarzschild metric. In this case, the expansion of the dressed polarization tensor truncates with $\mathcal{E}^{(1)}_{\mu\nu}$ given by (3.5.74). In the linearised metric, there is no event horizon and the only matching condition is at infinity, as $r \to \infty$. However, we have already seen that the dressed polarisation reduces to the free field polarization $\varepsilon_{\mu\nu}$ as $r \to \infty$. Combined with the asymptotic behaviour of the pure HPF phase part in (3.5.76), the matching coefficients automatically reduce to those of a massless scalar:

$$\Lambda^{k'}_{\mu\nu}{}^{\rho\sigma}(k)\Big|_{\text{lin. Schw.}} = -2i\, \delta^{\rho}_{(\mu}\, \delta^{\sigma}_{\nu)}\, \delta(|\vec{k}| - |\vec{k}\,'|)\, f^{k'}(|\vec{k}|,\, \hat{k})\,, \tag{3.5.77}$$

where $f^{k'}(|\vec{k}|,\, \hat{k})$ is the *massless* (i.e., $m = 0$) scalar elastic scattering amplitude on Schwarzschild.

In other words, the general wavefunction for a graviton of momentum k in the linearised Schwarzschild metric is given by

$$h_{k\,\mu\nu}(x) = -i\,|\vec{k}| \int d^2\Omega_l\, f^k(|\vec{k}|,\, \hat{l})\, \mathcal{E}_{\mu\nu}(x)\, e^{i\,S_l(x)}\Big|_{l^0=k^0}\,, \tag{3.5.78}$$

where all 4-momenta appearing in this expression are null and $\mathcal{E}_{\mu\nu}$ is implicitly on-shell with respect to l.

3.5.2 Semiclassical graviton emission amplitude

Once again following the perturbiner prescription for tree-level scattering amplitudes, the 3-point amplitude for graviton emission from a massive complex scalar in a curved background spacetime is given by the tri-linear terms in the gravitationally coupled scalar action[9]:

$$\kappa \int d^4x\, \sqrt{-|g|}\, h^{\text{out}}_{k\,\mu\nu} \left[2\,\partial^\mu \phi^{\text{out}}_{p'}\, \partial^\nu \bar{\phi}^{\text{in}}_{p} - g^{\mu\nu}\left(\partial_\alpha \phi^{\text{out}}_{p'}\, \partial^\alpha \bar{\phi}^{\text{in}}_{p} - \frac{m^2}{2}\, \phi^{\text{out}}_{p'}\, \bar{\phi}^{\text{in}}_{p}\right)\right]\,, \tag{3.5.79}$$

where $\kappa = \sqrt{8\pi G}$ is the gravitational coupling constant, all indices are raised and lowered with the background metric $g_{\mu\nu}$ and $|g|$ is its determinant. For the fully non-linear Schwarzschild black hole, we can only give a schematic refinement of this expression, with semiclassical wavefunctions

$$\begin{aligned}
\bar{\phi}^{\text{in}}_{p} &= \int d\Phi(l)\, \overline{\Lambda^p(l)}\, e^{-i\,S_l}\Big|_{l^0=p^0}\,,\\
\phi^{\text{out}}_{p'} &= \int d\Phi(l')\, \Lambda^{p'}(l')\, e^{i\,S_{l'}}\Big|_{(l')^0=(p')^0}\,,\\
h^{\text{out}}_{k\,\mu\nu} &= \int d\Phi(k')\, \Lambda^{k}_{\mu\nu}{}^{\rho\sigma}(k')\, \mathcal{E}_{\rho\sigma}\, e^{i\,S_{k'}}\Big|_{(k')^0=k^0}\,,
\end{aligned} \tag{3.5.80}$$

where the graviton wavefunction and dressed polarization are defined in Regge-Wheeler gauge and the scalar and tensorial matching coefficients are determined by agreement with exact solutions to the Klein-Gordon and Regge-Wheeler-Zerilli equations, respectively, at the event horizon and asymptotically. We re-emphasize that we have certainly not described this matching at any sort of technical level. With this in mind, the semiclassical graviton

[9]We work throughout with a minimally coupled scalar field. Relaxing this assumption would amount to modeling finite size effects for the object moving in the background, as is customary in an EFT language. While this does not play a role at the level of the discussion in this chapter, we expect it to be necessary at higher orders to ensure finite self-force corrections to scattering observables. This is because the notion of a point particle is not assumed but derived when considering the self-force approximation. For an example of where divergences might arise, see [122].

emission amplitude is given schematically by:

$$\langle p', k|\,\mathcal{S}\,|p\rangle = -\kappa \int\limits_{\mathbb{R}^{1,3}\setminus\overline{B(r_S)}} d^4x\,\sqrt{-|g|}\int d\Phi(l)\,d\Phi(l')\,d\Phi(k')\,\overline{\Lambda^p(l)}\,\Lambda^{p'}(l')\,\Lambda^{k}_{\mu\nu}{}^{\rho\sigma}(k')\,\mathcal{E}_{\rho\sigma}$$
$$\times\left[2\partial^\mu S_{l'}\,\partial^\nu S_l - g^{\mu\nu}\left(\partial S_{l'}\cdot\partial S_l+\frac{m^2}{2}\right)\right] e^{i(S_{k'}+S_{l'}-S_l)}\Big|^{l^0=p^0}_{(l')^0=(p')^0,\;(k')^0=k^0}\,, \tag{3.5.81}$$

where the region of integration over spacetime is the exterior ($r > r_S = 2GM$) of the black hole.

For the case of the *linearised* Schwarzschild background, we can be significantly more explicit. By working in the traceless Lorenz gauge, only the first term in the integrand of (3.5.79) survives, there is a well-defined S-matrix and the only matching conditions are at infinity, simplifying the structure of the outgoing graviton wavefunction (3.5.78). This leads to the semiclassical graviton emission amplitude

$$\langle p', k|\,\mathcal{S}\,|p\rangle = -2\,\kappa\int d^4x\,d\Phi(l)\,d\Phi(l')\,d\Phi(k')\,\sqrt{-|g|}\,\overline{\Lambda^p(l)}\,\Lambda^{p'}(l')\,\Lambda^k(k')$$
$$\times\;\mathcal{E}_{\mu\nu}\,\partial^\mu S_{l'}\,\partial^\nu S_l\,e^{i(S_{k'}+S_{l'}-S_l)}\Big|^{l^0=p^0}_{(l')^0=(p')^0,\;(k')^0=k^0} \tag{3.5.82}$$

where $g_{\mu\nu}$ is now the linearised Schwarzschild metric in the spherical coordinates of (3.3.9) and the HPFs are defined with respect to this metric.

3.5.3 The classical and weak field limits

To check the classical, weak field limit of the semiclassical graviton emission amplitude, it suffices to start with the answer (3.5.82) on linearised Schwarzschild. As in the photon emission calculation, it is convenient to divide contributions to this limit into those coming from the matching coefficients – or equivalently, elastic amplitudes – and everything else. In (3.5.82), there are three such matching coefficients: one for each incoming/outgoing scalar and one for the emitted graviton. It is easy to see that the contributions coming from perturbatively expanding each of these in turn to linear order in G will be proportional to on-shell 3-point momentum conservation, so all such contributions vanish for exactly the same reason as in the QED calculation

Thus, the only contributions to the classical, weak field limit of $\langle p', k|\,\mathcal{S}\,|p\rangle$ come by taking powers of G from the HPF, the dressed polarization, or explicit insertions of the background metric. A straightforward calculation shows that

$$\begin{aligned}\langle p', k|\,\mathcal{S}\,|p\rangle\,|_{\kappa^3} = -2\kappa \Bigg[&\varepsilon_{\mu\nu}\, p^\mu\, p^\nu\, \left(\tilde{S}(k; k+q) + \tilde{S}(p+q; k+q) - \tilde{S}(p; k+q)\right) \\ &- \varepsilon_{\mu\nu}\, p^\mu\, q^\nu\, \left(\tilde{S}(p; k+q) + \tilde{S}(p+q; k+q)\right) + \tilde{\mathcal{E}}^{(1)}_{\mu\nu}(k+q)\, p^\mu\, (p+q)^\nu \\ &+ \frac{1}{2}\, \tilde{H}^\sigma_\sigma(k+q)\, \varepsilon_{\mu\nu}\, p^\mu\, p^\nu - \tilde{H}^{\mu\sigma}(k+q)\, \varepsilon_{\mu\nu}\, p^\nu\, (p+q)_\sigma - \tilde{H}^{\nu\sigma}(k+q)\, \varepsilon_{\mu\nu}\, p^\mu\, p_\sigma \Bigg]\,,\end{aligned} \tag{3.5.83}$$

with $p'_\mu = p_\mu + q_\mu$ and the Fourier transformed quantities are defined by (3.5.75) and (3.5.74). To take the classical limit of this expression, we expand all quantities to leading order in the $\hbar \to 0$ limit, keeping in mind that massless momenta k_μ, q_μ scale linearly with $\hbar$ in this limit. Furthermore, as in Section 3.4, we impose that the total recoil $k + q$ is small, and exploit identities such as $q^2 + 2p \cdot q = 0$, which tell us that $p \cdot q$ scales like $\hbar^2$ in the classical limit. The result is

$$\begin{aligned}\lim_{\hbar\to 0} \langle p', k|\,\mathcal{S}\,|p\rangle\,|^{\mathrm{LO}(\rho)}_{\kappa^3} = \hat{\delta}(U\cdot(q+k))\, \frac{64\, M^2\, \kappa^3}{(q+k)^2}\, \varepsilon_{\mu\nu} \Bigg[&U^\mu\, U^\nu\, \frac{(k\cdot p)^2}{q^2} - 2\, U^\mu\, q^\nu\, \frac{k\cdot p\, p\cdot U}{q^2} \\ &+ \frac{q^\mu\, q^\nu}{q^2}\left((p\cdot U)^2 - \frac{m^2}{2}\right) + \frac{2\, p^\mu\, U^\nu}{q^2}\left(k\cdot p\, q\cdot U - \frac{q^2}{2}\,(p\cdot U)\right) \\ &- \frac{p^\mu\, q^\nu}{q^2}\left(\frac{q^2\, m^2}{2\, k\cdot p} + 2\, p\cdot U\, q\cdot U - \frac{q^2\,(p\cdot U)^2}{k\cdot p}\right) \\ &- p^\mu\, p^\nu\left(\frac{m^2\, q^2}{8\,(k\cdot p)^2} - \frac{(p\cdot U)^2\, q^2}{4\,(k\cdot p)^2} + \frac{p\cdot U\, q\cdot U}{k\cdot p} - \frac{(q\cdot U)^2}{q^2}\right)\Bigg]\,,\end{aligned} \tag{3.5.84}$$

for the classical weak field limit of the semiclassical graviton emission amplitude.

This can now be compared against the perturbative 5-point amplitude for two scalars of masses $M \gg m$ to scatter and emit a single graviton in Minkowski spacetime [203]. Let $\mathcal{A}_5$ denote this tree-level amplitude, stripped of its overall momentum conserving delta functions. By comparing with the analysis of this amplitude in Appendix A.2, we find the relationship

$$\lim_{\hbar\to 0}\, \langle p', k|\,\mathcal{S}\,|p\rangle\,|^{\mathrm{LO}(\rho)}_{\kappa^3} = \lim_{\hbar\to 0}\, \frac{\hat{\delta}(U\cdot(q+k))}{2M}\, \mathcal{A}^{\mathrm{LO}(\rho)}_5(p, P\to p+q, P-q-k, k)\,, \tag{3.5.85}$$

between the classical weak field limit of our semiclassical 3-point amplitude in Schwarzschild and the classical, probe limit of the perturbative 5-point amplitude in Minkowski spacetime. Once again, this demonstrates that the semiclassical amplitude will recover known classical observables in the weak field and probe limits.

As the Coulomb electromagnetic field and the Schwarzschild metric are related by classical double copy [204], one may be tempted to look for a double copy relationship between our graviton (3.5.82) and photon (3.4.47) emission amplitudes on those respective backgrounds. However, at this stage we have only superficial comments to make in this direction. Firstly, there is a fairly obvious 'double copy' relationship between the first-order HPFs, replacing electromagnetic charge with a second copy of probe momentum and tensorial structure corresponding to subtracting the dilaton. Secondly, as the perturbative limits of both emission amplitudes correspond to the probe limits of the corresponding 5-point perturbative amplitudes, the known double copy relationship [203] between those amplitudes is similarly recovered in the perturbative limit.

It would, of course, be interesting to explore a more enlightening notion of double copy between the emission amplitudes that fully manifests the non-perturbative nature of the backgrounds.

3.6 Conclusions

Scattering amplitudes serve as the natural building blocks for studies of the two-body problem in general relativity. When defined in a Minkowski vacuum, they provide the integrands necessary to extract classical observables within the perturbative Post-Minkowskian (PM) approximation [21]. On a generic background, they define the on-shell integrands for perturbative self-force corrections to observables of scattering orbits [2, 205], as will be shown in the next chapter. The relevance of this perturbative scheme for the two-body problem is twofold.

First, and practically, while ground-based observatories continue to generate waveform templates by combining information from post-Newtonian calculations and numerical relativity in the strong field regime, this approach will no longer be suitable for extreme mass ratio inspiral waveforms, such as those accessible to eLISA[10]. This motivates efforts to explore the self-force expansion using modern tools such as those coming from quantum field theory (QFT), where on-shell data defines the scattering problem. Secondly, on a conceptual level, this approach offers an intriguing way to inform other perturbative schemes, such as PM calculations. For instance, Damour demonstrated that the calculation of the gravitational scattering angle at the first self-force order determines the complete two-body potential through 4PM (i.e., order G^4) to all orders in the mass ratio [208], providing a concrete motivation for investigating these calculations using scattering amplitudes. This perspective opens up, in particular, possibilities to understand perturbation schemes that do not rely on weak field assumptions, using only perturbative amplitudes in vacuum and their resummation. This, in turn, would offer potential applications of powerful methods like the double copy, generalized unitarity, and BCFW recursion relations in a strong field regime.

In this chapter, we have already seen a few examples of this. The main results concerning the semiclassical 2-point (3.3.38) and 3-point amplitudes on Coulomb (3.4.47) and Schwarzschild (3.5.82) demonstrate that these on-shell quantities encode the expected weak field probe limit dynamics for the radiative sector of classical scattering. Understanding these quantities at the perturbative level solely in terms of on-shell data has allowed us to revisit the entire scattering amplitude defined on the background as a resummation of amplitudes in

[10] For more details, see [206], Chapter II.2 of [207] and section 1.2 of [120]

vacuum. This perspective reveals known relations, such as the connection between the radial action and the 4-point amplitude in vacuum [83, 151]. However, for the 5-point amplitude in vacuum, the structure is more intricate. We have found that it receives contributions not only from the radial action but also from the angular part of the Hamilton's principal function. This implies that even the simplest amplitude with emission on a background carries significant information that is not available in the radial action alone.

Looking to the future, it would be interesting to explore the extraction of classical observables from the background field amplitudes that we have considered. Their use can be made systematic following the methods in the following chapter, and by a proper counting of the matrix elements on the background in powers of the mass ratio. For example, the first deviation from geodesic motion will appear in the form of radiation emitted by a particle moving on the background. This would result in a formula for the total power emitted, which would in turn generate a correction to geodesic motion due to momentum balance. This correction will depend on the mass of the particle. At leading order, the formula for the radiated momentum can be expressed as on-shell integral over a wavepacket $\phi(p)$ - sharply localized to ensure a well defined classical limit - and the impact parameter b as:

$$K^{\mu} = \sum_{\eta} \int d\Phi(p, p', l, k)\, \phi(p)\, \overline{\phi}(p')\, e^{i\, b\cdot(p-p')/\hbar}\, \langle p'|\, \mathcal{S}^{\dagger}\, |l, k^{\eta}\rangle\, \langle l, k^{\eta}|\, \mathcal{S}\, |p\rangle\, k^{\mu}\,. \tag{3.6.86}$$

Here, the sum is over the helicity of the emitted photon or graviton and the $\hbar \to 0$ limit of the entire expression of the right-hand-side is implicit.

To perform all of the integrals in this expression using our results, a combination of six eikonal-type integrals would be necessary, along with additional contributions. Although we have not considered classical observables in this chapter, our approach offers an alternative method to address self-force corrections to geodesic motion, solely based on on-shell data. By contrast, the standard approach to first-order in self-force corrections relies on the 'MiSaTaQuWa equations' [209, 210][11]. Results for radiative observables obtained in this way (such as the radiated momentum or waveform) could be cross-checked against results found by numerically solving the Regge-Wheeler-Zerilli equations [212, 213]. It would also be interesting to contrast the derivation of similar classical observables, which incorporate contributions of all orders in the coupling, with a conjecture presented in [99] regarding the final semiclassical state in a two-body scattering scenario involving radiation. When one of the objects has a significantly larger mass than the other, it would effectively act as a background as in our calculation. Finally, it would be interesting to extend our results with radiation to a Kerr background, motivated by the possibility of describing exact geodesic motion with perturbative methods [214, 215]. We hope to explore this in future work.

[11] For a perturbative treatment of the latter, see [211]. In this case, real and imaginary parts of loop amplitudes in vacuum play relevant but different roles in the radiative physics.

Chapter 4

Waveforms from Scattering Amplitudes on Strong Backgrounds

4.1 Introduction

The observation of gravitational waves has brought renewed importance to the study of general relativity and its observables. Surprisingly, scattering amplitudes – one of the key outputs of *quantum* field theory – are providing a new way to study *classical* general relativity; for reviews see [93–95]. A key tool in this program has been the development of a formalism to systematise the extraction of classical physical observables from scattering amplitudes [21]. So far, all observables computed with this approach are valid for weak fields only: they are obtained from amplitudes at finite PM order, so truncated at a corresponding fixed order in the coupling [96–99, 216–223]. This is in sharp contrast with other approaches to gravitational dynamics such as the self-force paradigm [20, 100, 102, 104, 121, 224], where perturbation theory is implemented around a curved background and the weak field limit is not considered.

To address this gap, the amplitudes-based approach can be generalised to curved backgrounds by means of strong field scattering amplitudes and their classical limits [205]. This provides an alternative route to the computation of classical observables, as strong field amplitudes encode a substantial amount of information about higher-order processes [138, 156–159, 225–227] and finite size effects [228–230] in trivial backgrounds, and can also admit remarkably compact formulae [231–233]. A key aspect is that even first order perturbation theory around a curved background – which we refer to as 'first post-background', or 1PB, order – encodes *infinitely* many orders of the PM expansion. This is analogous to the relation between the PM and post-Newtonian (PN) expansions for bound orbits, where a fixed contribution of the former encodes infinitely many orders of the latter due to the virial theorem.

Here we show how classical observables encoding all-order results can be extracted from scattering amplitudes. We derive expressions for the gravitational waveform emitted by a point particle scattering on a gravitational plane wave background (an exact solution to the nonlinear Einstein equations), encoding all-order contributions in the PM expansion when the flat spacetime limit is taken, as well as tail effects which usually enter at high order in the PM approximation. These plane waves are not just good models of gravitational waves, but also provide a local description of *any* spacetime in the neighbourhood of a null geodesic [22]. We also perform an analogous calculation of, and compare with, electromagnetic waveforms for charged particles scattering on electromagnetic plane waves (which does not seem to appear in the literature, despite a long history of related studies [15, 193, 194, 234]), which sheds new light on the interface between QED and gravitational observables [134, 235–237].

4.1.1 Waveforms from amplitudes on curved backgrounds

Asymptotic waveforms. Let $|\Psi\rangle$ be a superposition of massive particle states describing a free particle of mass m as in [205]:

$$|\Psi\rangle = \int d\Phi(p)\, \phi(p)\, e^{ip\cdot b/\hbar}\, |p\rangle \;, \tag{4.1.1}$$

such that $\langle\Psi|\Psi\rangle = 1$, where $d\Phi(p) := \hat{d}^4p\, \Theta(p_0)\, \hat{\delta}(p^2 - m^2)$ is the Lorentz-invariant on-shell measure, while $\hat{\delta}(x) := 2\pi\delta(x)$ and $\hat{d}x := \frac{dx}{2\pi}$ throughout. The impact parameter b specifies the origin of the frame of reference of the initial state. The wavepacket $\phi(p)$ is a square-integrable function with a well-defined classical limit on these backgrounds (cf., [21])[1]. This state is evolved on an electromagnetic or gravitational plane wave background, in the latter case a solution to the fully non-linear Einstein equations. In terms of the S-matrix $\mathcal{S}$ on that background, the time-evolved state is

$$\mathcal{S}\,|\Psi\rangle = \int d\Phi(p)\phi(p)e^{ip\cdot b/\hbar}\mathcal{S}\,|p\rangle \,. \tag{4.1.2}$$

We may introduce a completeness relation into (4.1.2) to see that it is fixed in terms of n-point scattering amplitudes on the curved background

$$\mathcal{S}\,|\Psi\rangle = \sum_{\text{states}} \int d\Phi(p, p', k_1, \ldots, k_n)\, \phi(p)e^{ip\cdot b/\hbar} \\ \times \sum_{\eta_1,\ldots,\eta_n} |p', k_1^{\eta_1}, \ldots, k_n^{\eta_n}\rangle\, \langle p', k_1^{\eta_1}, \ldots, k_n^{\eta_n}|\,\mathcal{S}\,|p\rangle \,. \tag{4.1.3}$$

To get to this expression, we assume that the process is not energetic enough to create new massive particles. The first sum runs over all possible massless states that can be emitted in this progress, and the second over the quantum numbers (in our case, polarisations) of the emitted states. Additionally, we use shorthand notation $d\Phi(p_1, \ldots, p_n) := \prod_{j=1}^{n} d\Phi(p_j)$.

Our interest is in the classical gravitational or electromagnetic radiation emitted by a scalar particle as it scatters on these backgrounds, as measured by an asymptotic ob-

[1] In flat space, this means that its Compton wavelength l_c, the wavepacket spread l_w and the scattering length d_μ obey the 'Goldilocks' relations $l_c \ll l_w \ll \sqrt{-d^2}$. However, in plane wave backgrounds it makes more sense to use the approach from [98], where the Goldilock's relation is with respect to a characteristic wavelength λ of the background $l_c \ll l_w \ll \lambda$.

server at future null infinity. This observable is encoded in the classical limit of $\langle O_{\vec{\mu}}(x)\rangle := \langle\Psi|\, \mathcal{S}^\dagger \mathbb{O}_{\vec{\mu}}(x)\mathcal{S}\,|\Psi\rangle$ in which $\mathbb{O}_{\vec{\mu}}(x)$ represents the field strength $F_{\mu\nu}(x)$ or curvature tensor $R_{\mu\nu\sigma\rho}(x)$ at future null infinity, and where $\vec{\mu}$ denotes a number of indices appropriate to the radiated field. In terms of creation and annihilation operators $a^\dagger_\eta(k)$ and $a_\eta(k)$ for a state of helicity η, these operators have the *schematic* form

$$\mathbb{O}_{\vec{\mu}}(x) = \int d\Phi(k)\, e^{-ik\cdot x/\hbar}\, C^\eta_{\vec{\mu}}(k)\, a_\eta(k) + \text{c.c.}\,, \tag{4.1.4}$$

where $C^\eta_{\vec{\mu}}(k)$ is a placeholder for the polarisation content of the field, and a sum over helicities η is implied. Now, with coordinates $x^\mu = (t, \mathbf{x})$, future null infinity corresponds to $r \equiv |\mathbf{x}| \to \infty$ while $u = t - r$ is held constant. The mode expansion then becomes, to leading order in $1/r$ [98],

$$\mathbb{O}_{\vec{\mu}}(x) = -\frac{i\hbar^2}{4\pi r}\int_0^\infty \hat{d}\omega\, e^{-i\omega u}\, C^\eta_{\vec{\mu}}(k)\, a_\eta(k)\Big|_{k=\hbar\omega\hat{x}} + \text{c.c.} \tag{4.1.5}$$

in which we define the null vector $\hat{x}^\mu = (1, \hat{\mathbf{x}})$ and ω is a classical ($\hbar$-independent) frequency; this parametrisation will be useful later when we take the classical limit.

Following [98], the waveform $W_{\vec{\mu}}(u, \hat{x})$ is defined simply as the (expectation value of) the coefficient of this leading $1/r$ term. It is a function of u and the two angular degrees of freedom encoded in $\hat{x}^\mu$. So for the Maxwell and Riemann tensors, respectively, we have

$$\langle F_{\mu\nu}(x)\rangle = \frac{1}{r}W_{\mu\nu}(u,\hat{x}) + O(r^{-2})\,, \qquad \langle R_{\mu\nu\sigma\rho}(x)\rangle = \frac{1}{r}W_{\mu\nu\sigma\rho}(u,\hat{x}) + O(r^{-2})\,. \tag{4.1.6}$$

The amplitudes contributing to the waveform are easily identified from inserting (4.1.3) into the expectation value; the leading contribution is at 1PB, meaning order e (κ) in QED (gravity) but all orders in the background fields, and comes from interference between tree-level 2-point and 3-point amplitudes since

$$\langle\Psi|\, \mathcal{S}^\dagger a_\eta(k)\mathcal{S}\,|\Psi\rangle = \int d\Phi(p')\langle\Psi|\, \mathcal{S}^\dagger\,|p'\rangle_{\text{tree}}\,\langle p', k^\eta|\,\mathcal{S}\,|\Psi\rangle_{\text{tree}} + \cdots\,. \tag{4.1.7}$$

It is easy to show that this combination of amplitudes reproduces the radiation emitted due to geodesic motion, i.e. the first contribution of self-force effects [104].

We stress that, unlike in vacuum, two-point amplitudes on backgrounds are not trivial even at tree-level, encoding e.g. memory effects [205]. Suppressing the 'tree' subscript from here on we arrive at, in QED

$$W_{\mu\nu}(u,\hat{x}) = -\frac{\hbar^{\frac{1}{2}}}{2\pi}\int_0^\infty \hat{d}\omega\, e^{-i\omega u}\, k_{[\mu}\,\varepsilon^{-\eta}_{\nu]}\int d\Phi(p')\,\langle\Psi|\,\mathcal{S}^\dagger\,|p'\rangle\,\langle p', k^\eta|\,\mathcal{S}\,|\Psi\rangle\Big|_{k=\hbar\omega\hat{x}} + \text{c.c.} \tag{4.1.8}$$

where $\varepsilon^{\eta}_{\mu} \equiv \varepsilon^{\eta}_{\mu}(k)$ is the photon polarisation, while in gravity

$$W_{\mu\nu\sigma\rho}(u,\hat{x}) = \frac{i\kappa}{2\pi\hbar^{\frac{1}{2}}} \int_0^\infty \hat{d}\omega\, e^{-i\omega u}\, k_{[\mu}\varepsilon^{-\eta}_{\nu]} k_{[\sigma}\varepsilon^{-\eta}_{\rho]} \int d\Phi(p')\, \langle\Psi|\, \mathcal{S}^\dagger\, |p'\rangle\, \langle p', k^\eta|\, \mathcal{S}\, |\Psi\rangle \Big|_{k=\hbar\omega\hat{x}} + \text{c.c.}\,. \tag{4.1.9}$$

Weak memory. Plane wave backgrounds, as introduced in chapter 2, naturally carry a notion of ' velocity memory'. In electromagnetism this is encoded by $a_{\infty\perp}$ as defined in (2.1.5), whilst in gravity it's given by the functional form of the vielbeine $E^a_i(x^-) = b^i_a + c^i_a x^-$, as defined in (2.1.22). By inspecting the form of the dressed momenta, (2.1.10) and (2.1.29), on both of these backgrounds, it's possible to see explicitly how these terms shift the asymptotic momenta of the particles after crossing the background.

This memory effect is encoded in the tree-level 2-point amplitude, which is a boundary term of the classical action (see [138] for a recent review). In electromagnetism, this is given by

$$\langle p'|\, \mathcal{S}\, |p\rangle = 2p_+ \hat{\delta}^3_{+,\perp}(p' - p + ea_\infty) e^{i\hat{s}_p/\hbar}, \tag{4.1.10}$$

$$\hat{s}_p \coloneqq \int_{x_i^-}^{x_f^-} ds\, s \frac{d}{ds} \frac{e^2 A_\perp^2(s) - 2ep^\perp A_\perp(s)}{2p_+}. \tag{4.1.11}$$

From this we can see that taking $a_\infty \to 0$, the 2-point amplitude is

$$\langle p'|\, \mathcal{S}\, |p\rangle \to 2p_+\, \hat{\delta}^3_{+,\perp}(p' - p)\, e^{i\hat{s}_p/\hbar}. \tag{4.1.12}$$

Note that this remains different to the flat-space amplitude, as there is still no localisation in the x^- component of the momentum. Importantly, this encodes the position memory effect felt by the particle after passing through the plane wave.

A similar calculation of the two-point in gravity gives the result

$$\langle p'|\, \mathcal{S}\, |p\rangle = 2p_+\, \hat{\delta}(p'_+ - p_+)\, F(p'_\perp, p_\perp, p_+), \tag{4.1.13}$$

$$\begin{aligned} F(p'_\perp, p_\perp, p_+) \coloneqq \frac{2\pi}{p_+\sqrt{\det(c)}\,\hbar} &\exp\left[-\frac{i\, p_i p_j}{2\, p_+\, \hbar} \int_{x_i^-}^{x_f^-} ds\, s \frac{d}{ds}\gamma^{ij}(s)\right] \\ &\times \exp\left[\frac{i}{2\, p_+\, \hbar}(p'_a - p_i b^i_a) b^{ak} (c^{-1})^b_k (p'_b - p_j b^j_b)\right]. \end{aligned} \tag{4.1.14}$$

Scaling the velocity memory matric $c^i_a \sim \lambda\, c^i_a$, taking $\lambda \to 0$ and $b^i_a \to \delta^i_a$, we see that using the stationary phase approximation

$$\langle p'|\, \mathcal{S}\, |p\rangle \to 2p_+ \hat{\delta}^3(p' - p) e^{i\hat{r}_p/\hbar} \tag{4.1.15}$$

where $\hat{r}_p$ is the exponent on the first line of (4.1.14). Overall these results mean that the tree-level 2-point amplitude relevant for our calculations always collapses to

$$\langle p'|\,\mathcal{S}\,|\Psi\rangle \to \int \! d\Phi(p)\phi(p)\, 2p_+\, \hat{\delta}^3_{+,\perp}(p'-p)\, e^{i\theta(p)/\hbar} = e^{i\theta(p')/\hbar}\phi(p')\,, \tag{4.1.16}$$

in the zero memory limit, where θ is theory-dependent; for our purposes it can be absorbed into a redefinition of u, but in general it will encode position memory effects on the scattered scalar [238].

4.2 Electromagnetism

In this section we construct the classical limit of the electromagnetic waveform $W_{\mu\nu}(u,\hat{x})$ from (4.1.8). Given our assumption of no memory, the only ingredient required is the 3-point amplitude on an electromagnetic plane wave background, to which we now turn.

Tree-level 3-point amplitude. We require the amplitude for an incoming charged scalar to emit a photon. Let the incoming momentum be p_μ (with $p^2=m^2$), the outgoing be p'_μ (also with $p'^2=m^2$) and the emitted photon have null momentum k_μ and helicity η. The amplitude can be calculated by evaluating the cubic part of the action on the appropriate scattering states in a plane wave background, see e.g. [15] and as shown in Section 2.2.1, in equation (2.2.40). The result is

$$\langle p', k^\eta|\mathcal{S}|\Psi\rangle = \int \! d\Phi(p)\phi(p)\, e^{ip\cdot b/\hbar}\, \hat{\delta}^3_{+,\perp}(p'+k-p)\mathcal{A}_3(p\to p'+k^\eta)\,, \tag{4.2.17}$$

$$\mathcal{A}_3(p\to p'+k^\eta) = -\frac{2ie}{\hbar^{3/2}}\int_y \varepsilon^\eta(k)\cdot P(y)\exp\left[\frac{i}{\hbar}\int_{-\infty}^{y} dz\, \frac{k\cdot P(z)}{p_+-k_+}\right], \tag{4.2.18}$$

where $\int_{-\infty}^{+\infty}dy := \int_y$, while y and z are integration variables denoting lightfront times. The 'dressed' momentum $P_\mu(y)$ is the classical momentum of the particle in the background, given in (2.1.10), and recalled here,

$$P_\mu(y) = p_\mu - eA_\mu(y) + n_\mu \frac{2eA(y)\cdot p - e^2A^2(y)}{2p_+}\,, \tag{4.2.19}$$

where $a_\mu(y)=\delta^\perp_\mu a_\perp(y)$, obeying $P^2(y)=m^2$. Note that only three components of overall momentum are conserved in (4.2.18), as the background breaks translation symmetry in y^-.

Calculation of the waveform. We assemble the waveform (4.1.8) from (4.2.18) and (4.1.16), which gives the expression

$$W_{\mu\nu}(u,\hat{x}) = \frac{ie}{\pi\hbar}\sum_{\eta}\int_{\omega} e^{-i\omega u}k_{[\mu}\epsilon^{-\eta}_{\nu]}\int d\Phi(p',p)\overline{\phi(p')}\phi(p)e^{-i(p'-p)\cdot b/\hbar}$$
$$\times\,\hat{\delta}^3_{+,\perp}(p'+k-p)\int_{y}\epsilon^{\eta}(k)\cdot P(y)\exp\left[\frac{i}{\hbar}\int_{-\infty}^{y}\frac{k\cdot P(z)}{p_+-k_+}\right]\Bigg|_{k=\hbar\omega\hat{x}}. \tag{4.2.20}$$

The first step is to perform the sum over photon helicity using the completeness relation in lightfront gauge

$$\sum_{\eta}\epsilon^{\eta}_{\mu}\epsilon^{-\eta}_{\nu} = -\eta_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k_+}. \tag{4.2.21}$$

Upon inserting this into the waveform all gauge-dependent pieces vanish by anti-symmetry or generate boundary terms which can be ignored [36], leaving only a contribution from $-\eta_{\mu\nu}$. An immediate simplification in the classical limit is that the delta function sets $p'=p$, and thus the wavepacket appears as $|\phi(p)|^2$. This means that under the usual assumption that ϕ is sharply peaked around some classical momentum, we can integrate over p, localising the integrand at the on-shell momentum of the incoming particle, which we continue to write as p for simplicity. This gives

$$W_{\mu\nu}(u,\hat{x}) = -\frac{ie}{4\pi^2 p_+}\int_{y,\omega}\omega\, e^{-i\omega(u-\hat{x}\cdot X(y))}\,\hat{x}_{[\mu}P_{\nu]}(y)\,, \tag{4.2.22}$$

in which $X^\mu(y)$ is the classical particle orbit, obeying $X'_\mu(y) = P_\mu(y)/p_+$. The frequency integral can be performed by writing it as a derivative with respect to y and integrating by parts. This yields a very compact final expression for the classical waveform:

$$W_{\mu\nu}(u,\hat{x}) = \frac{e}{2\pi}\int_{y}\delta(u-\hat{x}\cdot X(y))\,\frac{d}{dy}\frac{\hat{x}_{[\mu}P_{\nu]}(y)}{\hat{x}\cdot P(y)} = \frac{e}{2\pi}\sum_{\text{sols}}\frac{p_+}{\hat{x}\cdot P}\frac{d}{dx}\frac{\hat{x}_{[\mu}P_{\nu]}}{\hat{x}\cdot P} \tag{4.2.23}$$

where the sum runs over all solutions of the delta-function constraint. It can be checked that this matches the result obtained directly from classical electrodynamics; see Appendix B.1. We now highlight several properties of the waveform.

4.2.1 Properties of the waveform

First observe that, due to the derivative, the waveform is vanishing in the absence of acceleration. Indeed the final integration by parts, performed as part of the evaluation of the frequency interval, corresponds to removing Coulomb field contributions from the asymptotic waveform, i.e. restricting to the radiation field which is of interest [239].

Next, recall from (4.2.19) that the dressed momentum P, hence the orbit X, is quadratic in the background ea: it follows immediately that the waveform contains terms of *all* orders in the background, and hence the coupling e. This is both explicit, due to the presence of P

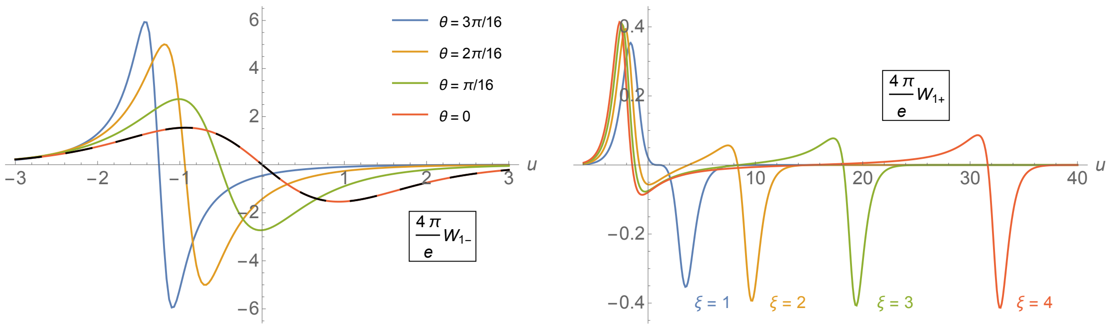


Figure 4.1: Two examples of the waveform $W_{\mu\nu}(u, \hat{x})$ for a particle at rest struck by the wave $ea_1 = m\xi\text{sech}^2(\nu x^-)$ and $a_2 = 0$, for strength ξ and frequency ν. We work in units where $\nu = 1$. *Left*: W_{1-} as a function of u for various θ. We have fixed $\xi = 2$ and $\phi = 0$. At $\theta = 0$ (red/black dashed curve), the waveform is a multiple of the driving field $F_{\mu\nu}$ as in (4.2.24), but is very different for larger angles. *Right*: W_{1+} for various ξ at fixed scattering angles $\theta = \pi$, $\phi = 0$, showing a nonlinear change in the waveform as the strength ξ of the background increases.

in the denominator, and implicit, in that one must solve the delta-function constraint. This requires inverting $\hat{x} \cdot X(y)$ which will introduce arbitrary non-polynomial dependence on the coupling. (Even for the simple field choice of an unphysical 'box' electric field, solving the constraint means solving a cubic equation.) In general, there will be *multiple* solutions to the constraint, meaning that the waveform at any given u and $\hat{x}$ is sourced at several points on the particle orbit.

We examine $W_{\mu\nu}$ by choosing a specific plane wave profile and other kinematic data; Fig. 4.1 illustrates the rich structure found in the classical waveform for a 'Sauter pulse' defined by $ea_1 = m\xi\text{sech}^2(\nu y^-)$ and $a_2 = 0$ for strength ξ and frequency ν. Furthermore, the all-orders property of the waveform can be made explicit in the case of an impulsive plane wave, for which all integrals can be performed explicitly; see Section 4.2.2.

For any plane wave profile, we can consider the waveform aligned with the direction of the background: $\hat{x}_\mu = \sqrt{2}n_\mu$ (the factor results from conventions). Parameterising $\hat{x}^\mu$ by azimuthal and polar angles ϕ and θ, respectively, alignment with the background corresponds to $\theta = 0$. At this collinear point the argument of the delta function is simply $u - \sqrt{2}y$, and thus has a single point of support. Most of the structure in the waveform vanishes due to contraction or commutation with n_μ, and one finds

$$W_{\mu\nu}\big|_{\theta=0} = -\frac{e^2}{4\pi}\frac{F_{\mu\nu}\left(\frac{u}{\sqrt{2}}\right)}{p_+\sqrt{2}}\ , \tag{4.2.24}$$

a result we will contrast with gravity in Sec. 4.3.1. If we consider any other point on the celestial sphere, the waveform has a far richer structure, though – see again Fig. 4.1.

4.2.2 The impulsive case

Here we calculate the waveform for a particularly simple example of an electromagnetic plane wave, capturing explicitly the dependence on the velocity memory effect. An impulsive plane wave in electromagnetism has electric fields $E_\perp(x^-) = E_\perp \delta(x^-)$, for $E_\perp$ constant. We write $a_\mu = \delta^\perp_\mu E_\perp \Theta(x^-) \equiv E_\mu \Theta(x^-)$. An incoming particle with momentum p_μ for $x^- < 0$ is kicked by the wave to momentum P_μ at $x^- > 0$ where

$$P_\mu = p_\mu - eE_\mu + n_\mu \frac{2eE\cdot p - e^2 E\cdot E}{2n\cdot p} \,. \tag{4.2.25}$$

This is a velocity memory effect on the scalar after passing through the wave. Whilst we neglected this in the previous, more general discussion, in an impulsive plane wave we cannot (as the wave turns off without memory). However, the addition of memory does not impact the final expression for the electrodynamics waveform, which holds for *any* plane wave. The waveform is most easily evaluated using (4.2.22) by splitting the dx^- integral into two parts: $x^- \gtrless 0$. The remainder of the calculation is trivial, and one finds

$$W_{\mu\nu}(u,\hat{x}) = \frac{e}{2\pi}\,\delta(u)\left[\frac{\hat{x}_{[\mu}P_{\nu]}}{\hat{x}\cdot P} - \frac{\hat{x}_{[\mu}\,p_{\nu]}}{\hat{x}\cdot p}\right]. \tag{4.2.26}$$

The waveform manifestly contains terms all orders in the coupling e. It is supported on the same singularity structure in u as the driving electric field is in x^-. The tensor structure clearly derives directly from standard soft factors for momentum transfer $p \to P$. Neglecting memory in this case amounts to assuming E_μ is parametrically small, in which case one replaces $P \to p$ and the waveform vanishes. For an impulsive background, the waveform is thus 'pure memory.'

4.3 Gravity

3-point amplitude in gravity We now require the tree-level 3-point amplitude for a massive scalar emitting a graviton, on the gravitational plane wave background. Let the on-shell incoming/outgoing momentum for the scalar be p_μ/p'_μ, but let k_μ now label the emitted graviton momentum. In contrast to QED, all particles are 'dressed' in gravity: in scattering calculations, any particle of asymptotic momentum l_μ and mass m has the dressed momentum [32,33] (2.1.29)

$$\begin{aligned} L_\mu(y)dy^\mu = {} & l_+ dy^+ + \big(l_i E^i_a(y^-) + l_+\sigma_{ab}(y^-)y^b\big)dy^a \\ & + \left(\frac{m^2}{2l_+} + \gamma^{ij}(y^-)\frac{l_i l_j}{2l_+} + \frac{l_+}{2}\dot{\sigma}_{bc}(y^-)y^b y^c + l_i \dot{E}^i_b(y^-)y^b\right)dy^- \,, \end{aligned} \tag{4.3.27}$$

which obeys $g^{\mu\nu}L_\mu L_\nu = m^2$. Note that, in contrast to the dressed momentum (4.2.19) in QED, the gravitational dressing depends on the perpendicular coordinates y^a. The outgoing graviton polarisation also becomes 'dressed' by the background, see (2.1.31); it is conveniently expressed in terms of a projector acting on the free polarisation:

$$\mathcal{E}^{\eta}_{\mu\nu}(k;y) = \mathbb{P}_{\mu\nu\sigma\rho}(k;y)\varepsilon^{\sigma\rho}_{\eta} := \left[\mathbb{P}_{\mu\rho}(k;y)\mathbb{P}_{\nu\sigma}(k;y) - \frac{i\hbar}{k_+}n_\mu n_\nu \delta^a_\rho \delta^b_\sigma \sigma_{ab}(y)\right]\varepsilon^{\sigma\rho}_{\eta}\ , \quad (4.3.28)$$

where $\mathbb{P}_{\mu\nu}(k;y) = g_{\mu\nu}(y) - 2K_{(\mu}(y)n_{\nu)}/k_+$ contains the dressed momentum $K_\mu(y)$ of the graviton. With these ingredients and the simplification of negligible memory, we can write down the required amplitude [33] (2.2.43):

$$\mathcal{A}_3(p \to p' + k^\eta) = -\frac{2i\kappa}{\hbar^{3/2}}\int_y \frac{\exp\left[i\mathcal{V}(y)\right]}{\sqrt{|E(y)|}}\,\mathcal{E}^{\eta}_{\mu\nu}(k;y)P^\mu(y)P'^\nu(y)\,, \quad (4.3.29)$$

where (4.2.17) still holds, the exponent is

$$\mathcal{V}(y) := \frac{1}{\hbar}\int_{-\infty}^{y^-} dz\,\frac{P_\mu(z)K_\nu(z)g^{\mu\nu}(z)}{p_+ - k_+}\,, \quad (4.3.30)$$

and $|E(y)|$ is the zweibein determinant. It can be checked that all contractions between dressed momenta and polarisations appearing are independent of the transverse coordinates, even though their constituents are not. Hence the integrand in (4.3.29) is a function of only y^-, and is (trivially) evaluated on the classical particle orbit parametrized by y^-.

Calculation of the waveform The calculation proceeds as in QED: assemble the waveform (4.1.9) from the three-point amplitude (4.3.29) and the two-point amplitude (4.1.16). Similarly to the electromagnetic case, we can restrict the sum over graviton polarisations to physical degrees of freedom using the gauge-invariant portion of the completeness relation. We only need the leading classical behaviour of the 3-point amplitude (4.3.29). Inspecting the powers $\hbar$ in the amplitude and in the definition of the waveform (4.1.9), it is again clear that all pre-factors of $\hbar$ are absent, and the classical limit is obtained by setting $\hbar = 0$ everywhere else. This again allows the wavepacket to be integrated out, arriving at

$$\begin{aligned} W_{\mu\nu\sigma\rho}(u,\hat{x}) = \frac{\kappa^2}{\pi}\int_0^\infty \hat{d}\omega\,\omega^2\,e^{-i\omega u}\,\hat{x}_{[\mu}\hat{x}_{[\sigma}\int_y \frac{e^{i\omega\bar{\mathcal{V}}(y)}}{\sqrt{|E(y)|}} \\ \times\left[\eta_{\nu]\gamma}\eta_{\rho]\delta} - \frac{1}{2}\eta_{\nu]\rho]}\eta_{\gamma\delta}\right]\bar{\mathbb{P}}^{\alpha\beta\gamma\delta}(\hat{x},y)P_\alpha(y)P_\beta(y) + \text{c.c.}\ , \end{aligned} \quad (4.3.31)$$

in which the reduced exponent $\bar{\mathcal{V}}$ is

$$\bar{\mathcal{V}}(y) = \frac{1}{2p_+}\int_{-\infty}^{y} dz\,\frac{m^2}{p_+}\hat{x}^+ + \gamma^{ij}(z)\left(\frac{p_+}{\hat{x}_+}\hat{x}_i\hat{x}_j + \frac{\hat{x}_+}{p_+}p_ip_j - 2p_i\hat{x}_j\right) = \hat{x}\cdot X(y)\,, \quad (4.3.32)$$

for $X(y)$ the classical particle orbit and $\bar{\mathbb{P}}_{\mu\nu\sigma\rho}(\hat{x},y) := \mathbb{P}_{\mu\nu\sigma\rho}(k,y)|_{\bar{k}=\omega\hat{x}}$ evaluated on the classical orbit. Note that $\bar{\mathcal{V}}$ is independent of ω. Now, the trace-like term in (4.3.31) arising from the polarisation sum can be simplified by observing that

$$\eta_{\gamma\delta}\,\bar{\mathbb{P}}^{\alpha\beta\gamma\delta}(\hat{x},y)P_\alpha(y)P_\beta(y) = m^2 + \frac{2ip_+^2}{\omega\hat{x}_+}\left[i\partial_-\bar{\mathcal{V}}(y) - \frac{1}{2}\sigma^a{}_a(y)\right]. \tag{4.3.33}$$

Using the Jacobi identity for determinant derivatives, it can be checked that the term in brackets is exactly the y^- derivative of the entire integrand in (4.3.31), and hence gives a boundary term which can be dropped, leaving only the mass term.

It remains to perform the ω integral. However, in contrast to QED, the projector $\bar{\mathbb{P}}^{\alpha\beta\gamma\delta}(\hat{x},y)$ contains terms with different scaling in ω. We highlight this by defining

$$T^0_{\nu\rho}(\hat{x},y) := \frac{\mathbb{P}_{\nu\alpha}(\hat{x},y)\mathbb{P}_{\rho\beta}(\hat{x},y)P^\alpha(y)P^\beta(y) - \frac{1}{2}\eta_{\nu\rho}m^2}{\sqrt{|E(y)|}}\,, \qquad T^1_{\nu\rho}(\hat{x},y) := \frac{\delta^a_\nu\delta^b_\rho\sigma_{ab}(y)}{\hat{x}_+\sqrt{|E(y)|}}p_+^2\,, \tag{4.3.34}$$

such that the integrand scales in the frequency as $\sim \omega^2 T^0 - i\omega T^1$. Combining the presented term in (4.3.31) with its complex conjugate and trading explicit ω factors for y^--derivatives gives our final result for the waveform:

$$W_{\mu\nu\sigma\rho}(u,\hat{x}) = -\frac{\kappa^2}{\pi}\hat{x}_{[\mu}\hat{x}_{[\sigma}\int_y \delta(u-\bar{\mathcal{V}}(y))\,\left[\mathcal{D}^2\,T^0_{\rho]\nu]}(\hat{x},y) - \mathcal{D}\,T^1_{\rho]\nu]}(\hat{x},y)\right], \tag{4.3.35}$$

in which the derivative $\mathcal{D}$ acts as

$$\mathcal{D}f(y) := \frac{d}{dy}\left(\frac{f(y)}{\partial_-\bar{\mathcal{V}}(y)}\right). \tag{4.3.36}$$

Once again, this result can be confirmed by comparison with the calculation in classical general relativity; see Appendix B.1.

4.3.1 Properties of the waveform

Some insight into the gravitational waveform is provided by observing from (4.3.32) that $\bar{\mathcal{V}}$ is determined by the 0PB classical orbit $X^\mu(y)$ of a particle crossing the plane wave spacetime. The orbit itself goes like the integral of the transverse metric $\gamma^{ij} = E^{(i|a|}E^{j)}_a$. Reinstating explicit dependence on the gravitational coupling by taking $H_{ab} \to \kappa H_{ab}$, it is clear that the integral of γ^{ij} will contain terms which are at least linear in κ. Since (4.3.35) contains terms which go like $\bar{\mathcal{V}}^{-1}$, as well as an integral localised in terms of $\bar{\mathcal{V}}$, it follows that the waveform will contain terms of *all orders* in the background and hence in κ.

While the non-linearity of general relativity makes it harder to evaluate the waveform analytically for test plane wave profiles, progress can be made in the impulsive case where $\kappa H_{ab}(x^-) = \delta(x^-)\,\kappa\,\mathrm{diag}(\lambda,-\lambda)$. This is demonstrated in Section 4.3.2, along with the resulting waveform which is explicitly all-orders in $\kappa\lambda$.

The structure of (4.3.35) indicates the presence of *tail* effects in the gravitational waveform. This follows from the fact that the two terms in the waveform descend directly from those in the polarization tensor (4.3.28). The background dressing of this polarization is directly related to the failure of Huygens' principle for gravitational perturbations in plane wave spacetimes: initial data localized on a lightcone spreads outside of the lightcone as it evolves [32,34,45]. These effects are present in both the T^0 and T^1 terms of the 1PB waveform, with the T^1 contribution being pure tail; by comparison, in the PM expansion of the two-body problem tail effects only emerge at fourth-order (e.g., [88]).

These tail effects are a consequence of the inherent non-linearity of gravity compared to electromagnetism, and this leads to another interesting feature of the gravitational waveform which is not present in QED. Consider the case, as in (4.2.24), where the direction of observation $\hat{x}^\mu$ aligns with the wave direction n^μ, corresponding to azimuthal angle $\theta = 0$. The background metric is not asymptotically flat in this direction, so we approach it with caution. For any $\theta \neq 0$ the gravitational waveform is well-defined, but in the limit $\theta \to 0$, it is divergent. To see this, one uses the small-θ expansion of $\hat{x}_\mu$:

$$\hat{x}_j = \sin\theta\{\cos\phi, \sin\phi\} \sim \theta \ , \quad \hat{x}_+ = \frac{1-\cos\theta}{\sqrt{2}} \sim \theta^2 \ , \quad \hat{x}_- = \frac{1+\cos\theta}{\sqrt{2}} \sim 1 \ . \tag{4.3.37}$$

With this, it is simplest to pick components of W, and also to focus on the pure tail term which contains the deformation tensor σ. The contribution of this term to W_{-a-b} is

$$W_{-a-b} = \frac{\kappa^2 p_+^2 \hat{x}_- \hat{x}_-}{\pi \hat{x}_+} \int_y \delta(u - \bar{\mathcal{V}}(y)) \ \mathcal{D} \, \frac{\sigma_{ab}(y)}{\sqrt{|E(y)|}} + \dots \sim \frac{1}{\theta^2} \ . \tag{4.3.38}$$

in which the $1/\hat{x}_+$ term generates the divergence (it is easily seen that $\bar{\mathcal{V}}$ and $\partial_- \bar{\mathcal{V}}$ remain finite in the limit $\theta \to 0$). The divergence reflects the fact that it is not possible to 'scatter' gravitons in the n_μ direction, in which the background is not asymptotically flat; the interaction between the emitted radiation and the background never switches off.

4.3.2 The impulsive case

We again consider the explicit example of an impulsive plane wave, this time in gravity. The situation in gravity is rather more intricate than for electromagnetism in Section 4.3.2, even for an impulsive background, and the structures provide an interesting contrast with electrodynamics. The impulsive metric is given by taking, in (2.1.12), $\kappa H_{ab}(x^-) \to \kappa\delta(x^-)H_{ab}$ with H_{ab} now constant (though still symmetric and traceless). We can, without loss of generality, choose coordinates to diagonalise $H_{ab} = \mathrm{diag}(\lambda, -\lambda)$. In contrast to QED, the memory effects present in this metric cannot, in general, be directly treated with the expressions in the text. In order to present the full impulsive result, we compute the relevant amplitudes directly, using the wavefunctions in [32].

The momentum kick is given by replacing $eE_\mu \to \kappa p_+ \delta^a_\mu H_{ac} b^c$ in (4.2.25), in which b is the transverse impact parameter (a dependence not present in electrodynamics). From here there are, as in electrodynamics, two contributions: one from before the impulse ($x^- < 0$) and one from after the impulse ($x^- > 0$). That from $x^- > 0$ yields a term similar to (4.2.26), while that from $x^- < 0$ is more complicated. One finds

$$
\begin{aligned}
W_{\mu\nu\sigma\rho}(u,\hat{x}) = \frac{\kappa^2}{4\pi}\,\delta'(u)\,\hat{x}_{[\mu}\hat{x}_{[\sigma}\varepsilon^{-\eta}_{\rho]}\varepsilon^{-\eta}_{\nu]}\varepsilon^{\eta}_{\alpha}\varepsilon^{\eta}_{\beta}\left[\frac{P^\alpha P^\beta}{\hat{x}\cdot P} - \frac{p^\alpha p^\beta}{\hat{x}\cdot p}\right] \\
+ \frac{i\kappa^2}{4\pi}\hat{x}_{[\mu}\hat{x}_{[\sigma}\varepsilon^{-\eta}_{\rho]}\varepsilon^{-\eta}_{\nu]}\int_0^\infty \hat{d}\omega\,\omega^2 e^{-i\omega u}\int \hat{d}^2\ell_\perp\,(\varepsilon^\eta(\ell)\cdot p)^2\frac{1}{\ell\cdot p}\mathcal{F}(\ell-k) + \text{ c.c.},
\end{aligned}
\tag{4.3.39}
$$

in which the sum of polarisations η is implied, $k_\mu \equiv \omega\hat{x}_\mu$, ε_μ *without* argument is $\varepsilon_\mu(\hat{x})$, ℓ_μ is a null vector with fixed longitudinal component $\ell_+ = k_+$, and

$$
\mathcal{F}(\ell) := \frac{e^{ib^a\ell_a - i\ell^a H^{-1}_{ab}\ell^b/(2k_+)}}{\kappa\lambda k_+} - \hat{\delta}_\perp(\ell)\ . \tag{4.3.40}
$$

Consider the first line in (4.3.39); the tensor structure is a double copy of the soft factor structure in electrodynamics, but the singularity is now $\delta'(u)$ rather than $\delta(u)$. The second line of (4.3.39) is, in a sense, 'pure tail' and we have not yet found a very compact expression for the remaining integrals for general $\hat{x}^\mu$ and u. Nevertheless, the result as presented is clearly of all orders in κ.

Moreover, we can consider a special case which allows a direct, if tedious, calculation of *all* terms in the impulsive waveform. First, we choose $p_\perp = 0$, so that the wave-particle collision is 'head on'. Second, we choose the impact parameter as $b^\perp = 0$; this has the effect of turning off memory, since the kicked momentum P collapses back to incoming p. Finally, we choose a particular point of observation on the celestial sphere, $\theta = \pi$ (antipodal to the direction in which the background is not asymptotically flat), which sets $\hat{x}_\mu \to \sqrt{2}\delta^+_\mu$. In this case, the first line of (4.3.39) vanishes – this is clearly due to the assumption of no memory, as in QED. The second line remains and simplifies considerably. By performing the angular integration in $\ell_\perp$, one finds that the remaining integrals are independent of helicity and the helicity sum can then be performed directly.

From here one writes the ω factors as derivatives with respect to u and combines the presented term in (4.3.39) with its complex conjugate. This gives, writing $q \equiv |\ell_\perp|$,

$$
W_{\mu\nu\sigma\rho} = \kappa^2\frac{p_+^3}{\kappa\lambda}\,\delta^+_{[\mu}\delta^+_{[\sigma}(-1)^{(a)}\delta^a_{\rho]}\delta^a_{\nu]}\partial_u^2\int_0^\infty \hat{d}q\,\frac{\hat{d}q\,q^3}{2m^2+p_+^2q^2}\int_0^\infty \hat{d}\omega\,J_1\!\left(\frac{\omega\,q^2}{\kappa\lambda\sqrt{8}}\right)\cos(\omega u)\ , \tag{4.3.41}
$$

in which a sum over $a \in (1,2)$ is implied and J_1 is a Bessel function of the first kind. The remaining integrals may be evaluated using [240], resulting in

$$
W_{\mu\nu\sigma\rho} = -\frac{\kappa^2 p_+}{\pi^2\sqrt{8}}\,\delta^+_{[\mu}\delta^+_{[\sigma}(-1)^{(a)}\delta^a_{\rho]}\delta^a_{\nu]}\ \frac{\partial^2}{\partial u^2}\left(\frac{\nu\log(\nu+\sqrt{\nu^2-1})}{\sqrt{\nu^2-1}}\right)\ , \tag{4.3.42}
$$

$$
\text{for}\qquad \nu := \kappa\lambda\sqrt{2}\frac{p_+^2}{m^2}\,|u|\ .
$$

Thus, unlike the 'soft' terms in the first line of (4.3.39), the 'pure tail' terms are not localised.

Taking the derivatives in (4.3.42), one finds that they do include a localised piece at the origin, proportional to $\delta(u)$ like in electrodynamics, rather than $\delta'(u)$ as in the first line of (4.3.39). This is not unexpected if the waveform is supported entirely on T^1: recalling the discussion around (4.3.34), in Fourier space the contribution from T^1 carries the same frequency dependence as electrodynamics.

4.4 Conclusions

In this chapter we have derived the gravitational waveform emitted by a massive particle when it scatters off a gravitational plane wave background, a solution to the fully non-linear Einstein equation. Analogous formulae have been presented for the electromagnetic case. These waveforms are manifestly *all-orders* in the coupling, and exhibit a rich structure including tail effects that usually enters to higher order in the PM expansion. Our results underline the power of using strong field amplitudes to study classical physics [205]. It would be interesting to go to higher orders in the PB expansion, including higher points and loops. There is no conceptual obstacle in pushing higher order calculations, and we expect this to provide access to observables of interest in classical gravity. In this chapter we neglected the velocity memory effect for generic plane waves: one may wonder how including these effects in general would change our results. It would also be interesting to consider other physically interesting strong backgrounds, like black holes using the amplitudes expressions from the previous chapter, or beams of gravitational radiation.

Part II

In Twistor Space

Chapter 5

Technical Introduction

In this chapter we introduce formulae for all-multiplicity tree-level scattering amplitudes in gauge theory and gravity, taking their origins in twistor theory. These formulae will play an essential role in chapter 6. We start with a gentle introduction of MHV (maximally helicity violating) scattering in section 5.1, and then introduce twistor space and formulae for scattering amplitudes with generic helicity configurations in section 5.2.

5.1 MHV scattering amplitudes

In four spacetime dimensions, gluons and gravitons have only two on-shell polarizations which can be labeled by *helicity*. This is a sign corresponding to whether the linearised field strength of the gluon or graviton is self-dual (positive helicity) or anti-self-dual (negative helicity). With this, the tree-level S-matrices of Yang-Mills and general relativity in 4-dimensions can be graded by the number of negative helicity external particles in the scattering process (where all external particles are assumed to be outgoing). Consistency and classical integrability of the self-dual sectors in these theories ensure that the tree-level amplitudes with less than two negative helicity particles vanish, so the first non-trivial amplitude in this helicity grading, known as the *maximal helicity violating* (MHV) configuration, involves two negative helicity and arbitrarily many positive helicity external particles.

In general, the n-point, tree-level scattering amplitude with $k+2$ negative helicity external particles is called the tree-level N^{k}MHV amplitude, denoted by $\mathcal{M}_{n,k+1}$. For gluon scattering, it is often useful to further decompose the tree-level scattering amplitude into colour-ordered partial amplitudes. All tree-amplitudes in Yang-Mills theory (with gauge group SU(N), for instance) can be decomposed as

$$\mathcal{M}_{n,k+1} = \sum_{\rho \in S_n \backslash \mathbb{Z}_n} \operatorname{tr}\left(\mathsf{T}^{a_{\rho(1)}} \cdots \mathsf{T}^{a_{\rho(n)}}\right) \mathcal{A}_{n,k+1}[\rho]\,, \tag{5.1.1}$$

where the sum is over non-cyclic permutations on n labels and T^a are generators of the adjoint representation of the gauge group. All kinematic information is packaged in the *colour-ordered partial amplitude* $\mathcal{A}_{n,k+1}[\rho]$; knowing this partial amplitude, with the explicit colour-dependence stripped off, is equivalent to knowing the full tree-level amplitude.

Remarkably, MHV amplitudes in gauge theory and gravity have simple, compact expressions for arbitrary multiplicity when written in the appropriate variables. In 4-dimensions, a (complex) momentum k^μ can be written as a 2×2 matrix

$$k^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} k^0 + k^3 & k^1 - ik^2 \\ k^1 + ik^2 & k^0 - k^3 \end{pmatrix} . \tag{5.1.2}$$

Null momenta ($k^2 = 0$) have the aditional property that this matrix can be factorised $k^{\alpha\dot{\alpha}} = \kappa^\alpha \, \tilde{\kappa}^{\dot{\alpha}}$, where $\alpha = 0, 1$ and $\dot{\alpha} = \dot{0}, \dot{1}$ are negative and positive chirality $\mathrm{SL}(2, \mathbb{C})$ Weyl spinor indices, respectively. This representation has a redundancy in the rescaling $\kappa \to r\,\kappa$, $\tilde{\kappa} \to r^{-1}\,\tilde{\kappa}$ for r any non-vanishing complex number, known as the action of the little group. External polarisations scale under the action of the little group and this helps constrain the form of certain amplitudes.

Spinor indices are raised and lowered using the $\mathrm{SL}(2, \mathbb{C})$-invariant Levi-Civita symbols, and we adopt the conventions:

$$\kappa^\alpha = \epsilon^{\alpha\beta}\, \kappa_\beta \,, \qquad \kappa_\alpha = \kappa^\beta\, \epsilon_{\beta\alpha} \,, \tag{5.1.3}$$

and similarly for dotted indices. These are easily seen to be consistent with the normalisation $\epsilon^{\alpha\beta}\, \epsilon_{\gamma\beta} = \delta^\alpha_\gamma$. A tree-level scattering amplitude involving n gluons or gravitons will be a rational function of the Lorentz-invariant kinematical quantities

$$\langle i\, j\rangle := \epsilon_{\beta\alpha}\, \kappa_i^\alpha\, \kappa_j^\beta \,, \qquad [i\, j] := \epsilon_{\dot{\beta}\dot{\alpha}}\, \tilde{\kappa}_i^{\dot{\alpha}}\, \tilde{\kappa}_j^{\dot{\beta}} \,, \tag{5.1.4}$$

for $i, j = 1, \ldots, n$. For a review of how these methods are useful for amplitude calculations, see [241].

In these variables, tree-level MHV gluon and graviton scattering amplitudes take a very simple form. The famous Parke-Taylor formula [4] for MHV gluon scattering is given by

$$\mathcal{A}_{n,1}[12\cdots n] = \delta^4\left(\sum_{i=1}^{n} k_i\right) \, \frac{\langle a\, b\rangle^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \,, \tag{5.1.5}$$

where gluons a, b have negative helicity. This formula manifests the cyclicity associated with the colour-ordering (broken only by the choice of the two negative helicity gluons), and scales under the little group for each gluon with homogeneity appropriate to that gluon's helicity.

The Hodges formulae [5] for tree-level MHV graviton scattering is the corresponding expression[1] in general relativity:

$$\mathcal{M}_{n,1} = \delta^4\left(\sum_{i=1}^{n} k_i\right) \langle 1\,2\rangle^8 \det{}'(H)\,, \tag{5.1.6}$$

where gravitons 1,2 have negative helicity. Here, H is a $(n-2)\times(n-2)$ matrix whose entries are

$$H_{ij} = \begin{cases} \frac{[i\,j]}{\langle i\,j\rangle} & \text{if } i \neq j \\ -\sum_{j\neq i} \frac{[i\,j]}{\langle i\,j\rangle}\frac{\langle 1\,j\rangle\,\langle 2\,j\rangle}{\langle 1\,i\rangle\,\langle 2\,i\rangle} & \text{if } i = j \end{cases}, \qquad i,j \in \{3,\ldots,n\}\,, \tag{5.1.7}$$

while

$$\det{}'(H) := \frac{|H^i_i|}{\langle 1\,2\rangle^2\,\langle 1\,i\rangle^2\,\langle 2\,i\rangle^2}\,, \tag{5.1.8}$$

is a reduced determinant which is independent of the choice of positive helicity graviton i on the support of momentum conservation.

5.2 N^{k}MHV scattering and twistor space

Similar compact expressions for gluon and graviton scattering exist beyond the MHV sector. These are found by writing the S-matrix in terms of a localized integral over the moduli space of rational, holomorphic maps from the Riemann sphere to twistor space, which we will introduce below. The degree d of the underlying map is related to the N^{k}MHV degree of the corresponding scattering amplitude by $d = k+1$.

The twistor space of (complexified) Minkowski space is given by an open subset of 3-dimensional complex projective space (we follow the notation and conventions of [245]). Let $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ be holomorphic, homogeneous coordinates on $\mathbb{P}^3$. Twistor space is then the open subset

$$\mathbb{PT} = \left\{Z^A \in \mathbb{P}^3 \,|\, I_{AB}Z^A \neq 0\right\} \tag{5.2.9}$$

where the (Minkowski) infinity twistor is

$$I_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}. \tag{5.2.10}$$

The condition $I_{AB}Z^A \neq 0$ then amounts to $\lambda_\alpha \neq 0$.

The relationship between twistor space $\mathbb{PT}$ and spacetime is known as the *twistor correspondence*. This takes the form of a non-local relation

$$\mu^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_\alpha \tag{5.2.11}$$

known as the incidence relations. Here $x^{\alpha\dot\alpha}$ is a point on (complexified) Minkowski space

[1]There are several earlier explicit formulae for the gravitational MHV amplitude [137, 242–244], but none of them explicitly manifests the permutation invariance of the external gravitons or is as compact as the Hodges formula.

identified with the usual coordinates x^μ via the Pauli matrices as before

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{5.2.12}$$

In effect this means that points in spacetime correspond to complex lines (of degree $d = 1$) in $\mathbb{PT}$ via the equation (5.2.11). Projectively, this should be considered as a linear embedding of a Riemann sphere into twistor space. In fact *any* holomorphic linear embedding of a Riemann sphere can be put into the form of the incidence relations, and therefore uniquely related to a spacetime point.

Conversely, fixing a point $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$ in twistor space, (5.2.11) defines a complex plane $\mathbb{C}^2 \subset \mathbb{C}^4$ of complexified Minkowski space. This 2-plane is totally null: every tangent vector to it is null and of the form $\lambda^\alpha \tilde{\lambda}^{\dot{\alpha}}$ where $\tilde{\lambda}^{\dot{\alpha}}$ is free whilst λ^α is defined by the point Z^A.

One of the powers of this non-local construction on twistor space is that fields satisfying zero-rest-mass (z.r.m.) equations on four-dimensional Minkowski space are in one-to-one correspondence with cohomology classes on $\mathbb{PT}$. In gauge theory and gravity we can represent the field strength and curvature in terms of so-called self-dual (SD) and anti-self-dual (ASD) components corresponding to positive and negative helicity polarisations. The equations that these components satisfy ($|h| = 0$ corresponds to scalar fields, $|h| = 1$ to gauge theory, $|h| = 2$ to gravity) are

$$h > 0, \qquad \partial^{\beta\dot{\alpha}_1} \tilde{\phi}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2|h|}} = 0, \tag{5.2.13}$$

$$h = 0, \qquad \Box\phi = 0. \tag{5.2.14}$$

$$h < 0, \qquad \partial^{\alpha_1\dot{\beta}} \phi_{\alpha_1 \cdots \alpha_{2|h|}} = 0. \tag{5.2.15}$$

These z.r.m. fields are related to cohomology classes on twistor space via the Penrose transform [246, 247]. Explicitly, there is an isomorphism

$$\{\text{helicity } h \text{ z.r.m. fields on } \mathbb{M}_{\mathbb{C}}\} \cong H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2)). \tag{5.2.16}$$

The right-hand-side means that we consider $(0, 1)$ forms on $\mathbb{PT}$ with projective weight $2h-2$. The cohomology group is then given by such forms that obey $\bar{\partial} f = 0$ and $f \neq \bar{\partial} g$ for any function g.

For example, to represent fields of helicity h with a definite momentum $k_{\alpha\dot{\alpha}} = \kappa_\alpha \tilde{\kappa}_{\dot{\alpha}}$ a representatative of the cohomology class we can use is

$$\phi(Z) = \int_{\mathbb{C}^*} \frac{dt}{t^{2h-1}} \bar{\delta}^2(\kappa - t_i \lambda) e^{it\,[\mu\,\kappa]}. \tag{5.2.17}$$

Here the holomorphic delta function in N-dimensions is defined, for some holomorphic quantity $z \in \mathbb{C}^N$, by

$$\bar{\delta}^N(z) := \frac{1}{(2\pi i)^N} \bigwedge_{a=1}^{N} \bar{\partial} \left(\frac{1}{z_a} \right), \tag{5.2.18}$$

and the integral over the scaling parameter t ensures that each representative has the appropriate homogeneity on twistor space.

Returning now from this brief excursion, we remind that N^kMHV scattering amplitudes in Yang-Mills and gravity requires us to consider degree $d = k + 1$ curves on twistor space. Whilst the incidence relations describe the moduli space of maps of degree 1 from $\mathbb{P}^1$ to PT, we will now describe these more generic degree curves. Let $\sigma^{\mathbf{a}} = (\sigma^{\mathbf{0}}, \sigma^{\mathbf{1}})$ be holomorphic homogeneous coordinates on $\mathbb{P}^1$, the Riemann sphere. A holomorphic map of degree d from $\mathbb{P}^1$ to $\mathbb{PT}$ can be parameterized as

$$Z^A(\sigma) = U^A_{\mathbf{a}_1\cdots\mathbf{a}_d}\, \sigma^{\mathbf{a}_1}\cdots\sigma^{\mathbf{a}_d} \equiv U^A_{\mathbf{a}(d)}\, \sigma^{\mathbf{a}(d)}\,, \tag{5.2.19}$$

where the $4(d+1)$ components of $U^A_{\alpha(d)}$ are the moduli of the map. This is a redundant parameterization, due to the $\mathrm{SL}(2,\mathbb{C})$ automorphisms of $\mathbb{P}^1$ and $\mathbb{C}^*$ projective rescalings of the homogeneous coordinates, so the integration measure on the moduli space of such maps is given by

$$d\mu_d := \frac{d^{4(d+1)}U}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\,. \tag{5.2.20}$$

Here, the quotient by the (infinite) volume of $\mathrm{GL}(2,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^*$ is understood in the Faddeev-Popov sense.

The extension of the Parke-Taylor formula to the full tree-level S-matrix of Yang-Mills theory is known as the Roiban-Spradlin-Volovich-Witten (RSVW) formula [6, 23]. Let $\tilde{\mathbf{g}} \subset \{1,\ldots,n\}$ denote the set of $d+1$ negative helicity gluons in the N^{d-1}MHV amplitude, and $\mathbf{g} = \{1,\ldots,n\} \setminus \tilde{\mathbf{g}}$ be its complement. Denote the Möbius-invariant inner product between homogeneous coordinates on $\mathbb{P}^1$ as

$$(i\,j) := \epsilon_{\mathbf{ba}}\, \sigma_i^{\mathbf{a}}\, \sigma_j^{\mathbf{b}}\,. \tag{5.2.21}$$

Define the Vandermonde determinant of the set $\tilde{\mathbf{g}}$ on $\mathbb{P}^1$ by

$$|\tilde{\mathbf{g}}| := \prod_{\substack{i,j\in\tilde{\mathbf{g}}\\ i<j}} (i\,j)\,. \tag{5.2.22}$$

This object is homogeneous of degree d in each σ_i for $i \in \tilde{\mathbf{g}}$. Finally, define the quantity

$$\mathrm{PT}_n[\rho] := \prod_{i=1}^{n} \frac{D\sigma_{\rho(i)}}{(\rho(i)\,\rho(i+1))}\,, \tag{5.2.23}$$

where $\rho \in S_n \setminus \mathbb{Z}_n$,

$$D\sigma := (\sigma\, d\sigma)\,, \tag{5.2.24}$$

is the holomorphic 1-form trivializing the canonical bundle of $\mathbb{P}^1$ and the product in (5.2.23) is understood cyclically.

With these ingredients, the RSVW formula for the tree-level N^{d-1}MHV gluon amplitude is [6, 23]

$$\mathcal{A}_{n,d}[\rho] = \int d\mu_d \, |\tilde{\mathbf{g}}|^4 \, \mathrm{PT}_n[\rho] \prod_{i \in \mathrm{g}} a_i(Z(\sigma_i)) \prod_{j \in \tilde{\mathrm{g}}} b_j(Z(\sigma_j)) \,. \tag{5.2.25}$$

Here the $\{a_i(Z)\}$ and $\{b_j(Z)\}$ are, respectively, the positive and negative helicity gluon wavefunctions on twistor space as described in (5.2.16).

By choosing the twistor wavefunctions to represent momentum eigenstates as in (5.2.17), one can show that in fact *all* of the integrals in (5.2.25) are localized in terms of the kinematics, leaving a rational function of those kinematics with support dictated by overall 4-momentum conservation. Here we recall that for positive and negative helicity gluons respectively these momentum eigenstates take the form [23, 248, 249]:

$$\begin{aligned} a_i(Z) &= \int_{\mathbb{C}^*} \frac{dt_i}{t_i} \, \bar{\delta}^2(\kappa_i - t_i \, \lambda) \, e^{i t_i \, [\mu \, i]} \,, \\ b_j(Z) &= \int_{\mathbb{C}^*} t_j^3 \, dt_j \, \bar{\delta}^2(\kappa_j - t_j \, \lambda) \, e^{i t_j \, [\mu \, j]} \,. \end{aligned} \tag{5.2.26}$$

Inserting these wavefunctions into (5.2.25), all moduli integrals for the $\mu^{\dot{\alpha}}(\sigma)$ part of the holomorphic map $Z : \mathbb{P}^1 \to \mathbb{PT}$ can be done against the exponentials to give delta functions. Combined with the holomorphic delta functions in each wavefunction, this gives $2(d+1)+2n$ total constraints, with $2(d+1)+2n-4$ remaining integrals (accounting for the $\mathrm{GL}(2,\mathbb{C})$ redundancy in the moduli measure). Thus, each remaining integral is localized by a constraint, leaving 4 additional delta functions; it is straightforward to show that these correspond precisely to momentum conservation. Furthermore, the integrand of (5.2.25) after performing the $\mu^{\dot{\alpha}}(\sigma)$-moduli integrals is a rational function of the remaining moduli as well as the kinematic data. The constraints are similarly rational functions, implying that the result, after all integrals have been localized against the constraints, is itself a rational function of the kinematic data, as required for tree-level scattering amplitudes. The moduli integrals get localised to a set of solutions known as the polarised scattering equations. At N^{d-1}MHV, these equations have $E(n-3, d-1)$ solutions, where $E(a, b)$ denotes the Eulerian numbers.

More generally, it can be proved that the RSVW formula is a correct representation of the tree-level S-matrix of Yang-Mills theory using tree-level unitarity arguments based on the factorization properties of the formula [250–253]. When $d = 1$, the localization of the moduli integrals can be done explicitly and the Parke-Taylor amplitude (5.1.5) is recovered for the MHV sector.

The gravitational version of this story, the Cachazo-Skinner formula, follows similarly, though it requires introducing several more involved ingredients. External gravitons in the scattering process are represented on twistor space via the Penrose transform as twistor wavefunctions

$$h(Z) \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2)) \,, \qquad \tilde{h}(Z) \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-6)) \,, \tag{5.2.27}$$

for positive and negative helicity, respectively. The momentum eigenstate representatives for these wavefunctions are

$$h_i(Z) = \int_{\mathbb{C}^*} \frac{dt_i}{t_i^3}\,\bar{\delta}^2(\kappa_i - t_i\,\lambda)\,e^{it_i\,[\mu\,i]}\,,$$
$$\tilde{h}_j(Z) = \int_{\mathbb{C}^*} t_j^5\,dt_j\,\bar{\delta}^2(\kappa_j - t_j\,\lambda)\,e^{it_j\,[\mu\,j]}\,. \tag{5.2.28}$$

Let $\tilde{\mathbf{h}} \subset \{1, \ldots, n\}$ be the set of $d+1$ negative helicity gravitons in a N^{d-1}MHV scattering process, and let $\mathbf{h}$ be the complementary set of positive helicity gravitons.

Define a $(n-d-1) \times (n-d-1)$ matrix $\mathbb{H}$ with entries[2]

$$\mathbb{H}_{ij} = -t_i\,t_j\,\frac{[i\,j]}{(i\,j)}\sqrt{D\sigma_i\,D\sigma_j}\,, \qquad i,j \in \mathbf{h}\,, i \neq j\,,$$
$$\mathbb{H}_{ii} = t_i\,D\sigma_i \sum_{\substack{j\in\mathbf{h}\\ j\neq i}} t_j\,\frac{[i\,j]}{(i\,j)} \prod_{l\in\tilde{\mathbf{h}}} \frac{(j\,l)}{(i\,l)}\,, \quad i \in \mathbf{h}\,. \tag{5.2.29}$$

It can be shown that this matrix has corank 1, so it has a natural (generally non-vanishing) associated reduced determinant

$$\det{}'(\mathbb{H}) := \frac{|\mathbb{H}^i_i|}{\left|\tilde{\mathbf{h}} \cup \{i\}\right|^2} \prod_{j\in\tilde{\mathbf{h}}\cup\{i\}} D\sigma_i\,, \tag{5.2.30}$$

for any $i \in \mathbf{h}$. It can be shown that $\det{}'(\mathbb{H})$ is independent of the choice of this i.

Next, define the 'dual' $(d+1) \times (d+1)$ matrix $\mathbb{H}^\vee$

$$\mathbb{H}^\vee_{ij} = \frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(i\,j)}\,, \qquad i,j \in \tilde{\mathbf{h}}\,, i \neq j$$
$$\mathbb{H}^\vee_{ii} = -\frac{\langle\lambda(\sigma_i)\,d\lambda(\sigma_i)\rangle}{D\sigma_i}\,, \qquad i \in \tilde{\mathbf{h}}\,. \tag{5.2.31}$$

The matrix $\mathbb{H}^\vee$ also has corank 1, with the natural (generally non-vanishing) reduced determinant

$$\det{}'\left(\mathbb{H}^\vee\right) := \frac{|\mathbb{H}^{\vee\,i}_i|}{\left|\tilde{\mathbf{h}} \setminus \{i\}\right|^2}\,, \tag{5.2.32}$$

for any $i \in \tilde{\mathbf{h}}$. It can be shown that not only is $\det{}'(\mathbb{H}^\vee)$ independent of this choice of $i \in \tilde{\mathbf{h}}$, but it is actually independent of *all* the marked points $\{\sigma_1, \ldots, \sigma_n\}$ [8]. In particular, this reduced determinant is equal to the *resultant* (cf., [255]) of the $\lambda_\alpha(\sigma)$ components of the holomorphic map $Z : \mathbb{P}^1 \to \mathbb{PT}$ [256]. This gives $\det{}'(\mathbb{H}^\vee)$ an algebro-geometric meaning in its own right; among other properties, it ensures that $\det{}'(\mathbb{H}^\vee)$ vanishes whenever $\lambda_\alpha(\sigma) = 0$

[2] In general, the diagonal entries of $\mathbb{H}$ are defined in terms of an arbitrary choice of section $\mathfrak{s}(\sigma)$ of $\mathcal{O}(d+1) \to \mathbb{P}^1$ [24,28]. Here, we make the simple choice $\mathfrak{s}(\sigma) = \prod_{l\in\tilde{\mathbf{h}}}(\sigma\,l)$; while the amplitude itself is invariant for different choices, our later calculations in chapter 6 of the integral kernel will change. This is not surprising, since double copy representations (such as colour-kinematic numerators) are generically gauge-dependent [254].

– the case when the holomorphic map Z lands outside of twistor space.

With these ingredients in place, the Cachazo-Skinner formula for the tree-level scattering amplitudes of general relativity is:

$$\mathcal{M}_{n,d} = \int d\mu_d \, |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}) \det{}'\left(\mathbb{H}^\vee\right) \prod_{i\in\mathbf{h}} h_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j(Z(\sigma_j)) \,. \tag{5.2.33}$$

As with the RSVW formula, when evaluated on the momentum eigenstates (5.2.28) all of the integrals in this expression are localized against delta function constraints, with four remaining delta functions imposing momentum conservation. The formula is a rational function of the kinematic data and can be shown to exhibit the correct factorization properties [28, 253], thereby establishing that it is a representation of the tree-level S-matrix of general relativity. Furthermore, when $d = 1$ it is simple to show that the Cachazo-Skinner formula reduces to the Hodges formula (5.1.6) for the graviton MHV amplitude.

It should be noted that both the RSVW and Cachazo-Skinner formulae can be obtained as correlation functions in *twistor string theories*: chiral 2d CFTs governing holomorphic maps from a closed Riemann surface to twistor space. This connection was first suggested long ago by Nair [27], who observed that the Parke-Taylor formula could arise in this way, and generalized to the full tree-level S-matrix of Yang-Mills theory by Witten [6] and others [7, 257, 258]. A twistor string theory for gravity was later discovered by Skinner [8]. For the considerations in the remainder of this thesis, we will however not need to exploit any underlying twistor string description of the amplitudes.

Chapter 6

The Double Copy in Twistor Space

6.1 Introduction

The double copy is a relation between scattering amplitudes in gravitational and non-gravitational quantum field theories (QFTs), which has been the subject of more than 38 years of active study (see [29, 259–262] for reviews). The original incarnation, due to Kawai-Lewellen-Tye (KLT) [263], expresses closed string amplitudes as a product of two open string amplitudes, multiplied via a kinematic object now known as the *KLT* or *momentum kernel*. The field theory limit of this statement gives a powerful relation between tree amplitudes in general relativity and Yang-Mills:

$$\mathcal{M}_n^{\mathrm{GR}} = \sum_{\alpha,\beta\in S_{n-3}} \mathcal{A}_n^{\mathrm{YM}}[12\alpha n]\, S_n[\alpha|\beta]\, \mathcal{A}_n^{\mathrm{YM}}[2\beta 1 n]\,, \tag{6.1.1}$$

where $\mathcal{M}_n^{\mathrm{GR}}$ is the n-point tree-level graviton amplitude and $\mathcal{A}_n^{\mathrm{YM}}[12\alpha n]$ is the n-point tree-level, colour-ordered partial amplitude for gluon scattering in colour ordering $12\alpha n$. The sum is over the set of orderings $\alpha, \beta \in S_{n-3}$, each of size $(n-3)!$, and $S_n[\alpha|\beta]$ is the KLT kernel, a polynomial of degree $n-3$ in 2-particle Mandelstam invariants whose form is known explicitly for arbitrary n [264–267]. KLT kernels linking a remarkable array of other QFTs and string theories at tree-level have also been constructed [268, 269].

The double copy has been extended to loop-level via the notion of *colour-kinematics duality* [254, 270, 271], where the complexity of gravity amplitudes is traded for the simplicity of gluon amplitudes whose kinematic numerators are in a suitable representation. At tree-level, the existence of this colour-kinematic representation is in one-to-one correspondence with the KLT kernel [264, 272–275], and it has been shown to exist (or with some modifications) for 4-point scattering with maximal supersymmetry through 5-loops [276, 277]. The existence of a corresponding KLT kernel beyond tree-level is less clear, although there has been substantial progress in the study of closely related string monodromy relations at higher genus [278–286]. Moreover, the double copy toolkit has recently been applied to precision-frontier calculations in the gravitational 2-body problem; the most recent example at writing is the calculation of the scattering angle between two supersymmetric black holes to *fifth-order* in the Post-Minkowski expansion and first-order in the self-force expansion of $\mathcal{N} = 8$ supergravity [287].

Yet despite these myriad advances, there are still some surprising gaps in our understanding of double copy. For instance, it is not known how to explicitly double copy some of the most famous scattering amplitude formulae, which are written in a helicity graded representation for the external states. In this chapter, we establish a new double copy description of tree-level scattering using helicity graded representations of gluon and graviton amplitudes.

Indeed, the study of scattering amplitudes in four-dimensions has enjoyed a parallel set of advances inspired by the natural organisation of amplitudes into helicity sectors (also known as R-charge sectors in supersymmetric theories). The simplest non-vanishing tree-level amplitudes in gauge theory and gravity are those that are *maximally helicity violating* (MHV), with two negative helicity and arbitrarily many positive helicity particles[1]. It is a remarkable fact that these amplitudes in gauge theory and gravity admit compact, closed-form expressions at arbitrary multiplicity in the external positive helicity particles, known as the Parke-Taylor [4] and Hodges formulae [5][2], for MHV gluon and graviton scattering, respectively.

The incredible simplicity of scattering amplitudes in the MHV sector can be explained by the fact that both Yang-Mills and general relativity admit perturbative expansions around the self-dual sector in asymptotically flat spacetimes [288–291]. The MHV sectors represent the first non-trivial term in this perturbative expansion around a classically integrable subsector. One can then ask if a similar level of simplicity is lurking at all-orders in the expansion around self-duality at tree-level.

Answering this question involves using *twistor theory* [292], an algebro-geometric formalism which manifests the integrability of the self-dual sector in gauge theory and gravity [25, 26, 293]. Indeed, it was noted long ago that the Parke-Taylor formula can be written in terms of a certain worldsheet correlator in twistor space [27], and Witten extended this idea to the full tree-level S-matrix of Yang-Mills theory [6]. Here, the tree-level N^{k}MHV amplitude, with $\mathrm{k}+2$ negative helicity and arbitrarily many positive helicity external gluons, is given by an integral over the moduli space of holomorphic, degree $\mathrm{k}+1$ maps from the Riemann sphere to twistor space, which is localised by kinematic constraints. This can be written in a compact form for arbitrary multiplicity, and is known as the Roiban-Spradlin-Volovich-Witten (RSVW) formula [23] for the helicity-graded tree-level S-matrix of Yang-Mills theory. There is also a helicity-graded formula for the tree-level S-matrix of general relativity, again written as a localised moduli integral over holomorphic rational maps to twistor space, known as the Cachazo-Skinner formula [24].

While studies of helicity graded all-multiplicity amplitudes and the double copy were initially closely entangled[3], the two paths have since diverged significantly. In particular, there

[1]The conjugate configuration with two positive and arbitrarily many negative helicity particles is known as $\overline{\text{MHV}}$ configuration, and is equal to the complex conjugate of the MHV amplitude for Lorentzian-real kinematics.

[2]Other expressions for the graviton MHV amplitude were known before [137, 242–244], but these lack the manifest permutation invariance in the external gravitons which is manifest in Hodges' formula.

[3]Indeed, the first investigations of gravitational MHV scattering were closely tied to a double copy of the Parke-Taylor formula [242].

is no obvious imprint of double copy visible in the Hodges formula for gravitational MHV scattering, or, more generally, in the Cachazo-Skinner formula for the helicity graded tree-level graviton S-matrix. Of course, on a basic level we know that a double copy relationship between the RSVW formula and the Cachazo-Skinner formula must exist: the KLT double copy (6.1.1) preserves the helicity grading. Yet an explicit double copy construction of the Cachazo-Skinner representation of the graviton amplitudes has remained elusive[4]. Some recent progress was made in determining colour-kinematic numerators for the MHV sector directly from the Hodges formula [295].

In some sense, this opacity is compensated by the clarity of the double copy in the Cachazo-He-Yuan (CHY) formalism [9–11], where tree-level amplitudes are calculated by localising integrands on the $(n-3)!$ solution of the scattering equations, a set of contraints linking kinematics with the moduli of a punctured Riemann sphere. Here, double copy is beautifully apparent at the level of the integrands; this representation of double copy led [11] to the discovery that the matrix inverse of the KLT momentum kernel is equal to the amplitudes of another quantum field theory - biadjoint scalar (BAS) theory. Mathematically, this is an extremely non-trivial fact, as *a priori* this requires the calculation of the inverse of an $(n-3)! \times (n-3)!$ matrix. However, in the CHY formalism, the momentum kernel is a 1×1 matrix, with the remaining complexity absorbed by the scattering equations. Since then, the relation between the momentum kernel and BAS theory has been studied independently of CHY [268, 269, 296–301], and proven in [302].

The scattering equations and the degree d maps that underlie N^{d-1}MHV amplitudes are closely connected. In fact, the moduli integrals in the RSVW or Cachazo-Skinner formulae are localised onto a subset of size $E(n-3, d-1)$ (the Eulerian number) of the $(n-3)!$ solutions to the scattering equations [303–305]. It is thus possible to 'grade' the CHY formulae by helicity to obtain the corresponding N^{d-1}MHV amplitudes in gauge theory and gravity. However, this involves a rather non-trivial basis change at the level of the moduli integrals [306–310] which obscures the initially obvious double copy structure as well as any link with BAS amplitudes[5].

In this chapter we provide the link that connects these two organising principles of amplitudes: the double copy and helicity grading. Our main results can be summarised as follows.

Theorem 1: The integrand of the Cachazo-Skinner formula $\mathcal{J}^{\mathrm{GR}}_{n,d}$ is the product of two colour-ordered RSVW integrands $\mathcal{J}^{\mathrm{YM}}_{n,d}[\alpha]$ multiplied via a *helicity-graded integral kernel* $S_{n,d}[\alpha|\beta]$:

$$\mathcal{J}^{\mathrm{GR}}_{n,d} = \sum_{\substack{\rho,\omega \in \Omega_d \\ \bar{\rho},\bar{\omega} \in \Omega_{n-d-2}}} \mathcal{J}^{\mathrm{YM}}_{n,d}[\rho\bar{\rho}]\, S_{n,d}[\rho\bar{\rho}|\omega^T\bar{\omega}]\, \mathcal{J}^{\mathrm{YM}}_{n,d}[\omega^T\bar{\omega}]\,, \tag{6.1.2}$$

where Ω_d, Ω_{n-d-2} are certain subsets of colour orderings with $d!$ and $(n-d-2)!$

[4]A formula for N^kMHV graviton scattering was developed by Cachazo-Geyer [294] building on a sort of double copy from the RSVW formula, but this formula remains a conjecture and has not been shown to be equal to the Cachazo-Skinner formula.

[5]In [311], a formula for BAS scattering based on rational maps to twistor space was given, but it involves a sum over different degrees, meaning that it cannot be the inverse of a KLT kernel which preserves the helicity grading.

elements, respectively. This kernel $S_{n,d}$, for which we give an explicit formula, is an object at the level of *integrands* on twistor space and admits a chiral splitting according to the helicity configurations of the external particles, as does the basis of colour orderings.

Theorem 2: The inverse of the helicity-graded integral kernel (viewed as a linear map on the bases of colour orderings) defines a new formula for all tree-level, colour-ordered partial amplitudes of BAS theory. Schematically,

$$m_n[\rho\bar{\rho}|\omega^T\bar{\omega}^T] = \int d\mu_d\, S^{-1}_{n,d}[\rho\bar{\rho}|\omega^T\bar{\omega}] \prod_{i=1}^{n} \varphi_i\,, \tag{6.1.3}$$

where the integral is over the moduli space of maps of degree d from the Riemann sphere to twistor space, and external kinematics are encoded in the twistor wavefunctions $\{\varphi_i\}$. Note that this expression for the partial amplitude $m_n[\rho\bar{\rho}|\omega^T\bar{\omega}]$ depends on the degree of the map only through the form of the colour-orderings.

The derivation of the integral kernel underpinning Theorem 2 rests on a non-trivial re-writing of the integrand of the Cachazo-Skinner formula using various tools from graph theory. The integral kernel $S_{n,d}$ should be viewed as the KLT kernel in twistor space; indeed, in the MHV sector ($d = 1$) where all moduli integrals can be performed explicitly, the resulting momentum space expression is the well-known KLT momentum kernel. The proof of the new formula for BAS tree-amplitudes in Theorem 2 is by iteration, using BCFW recursion adapted to twistor space. The result is remarkable, as the final expression is a scalar amplitude, independent of helicity.

The graph-theoretic methods utilized here can naturally be adapted to any twistor space formula which has a similar mathematical structure to the original Cachazo-Skinner formula. This allows us to find generalizations of the integral kernel on certain *curved* four-dimensional spacetimes; namely, anti-de Sitter space [312] and self-dual radiative spacetimes [233]. In both cases, this leads to tantalizing hints at the possibility of an explicit manifestation of double copy for scattering in curved spacetimes, although we observe several issues that prevent us from drawing any definitive conclusions in this regard.

The chapter is structured as follows. Section 6.2 is a review of results in graph theory that will be used throughout this chapter. Section 6.3 contains the derivation of the helicity-graded integral kernel. The derivation rests on the graph theoretic properties of the gravity integrand, and results in a chirally split momentum kernel. The special case of the MHV configuration is highlighted and reproduces the usual field theory KLT momentum kernel. In Section 6.4 the twistor space BAS formula is presented and justified. It is then proven using BCFW recursion techniques. Section 6.5 applies the methods of the previous two sections to amplitude formulae on non-trivial backgrounds: AdS and self-dual radiative gauge fields and spacetimes. Finally, Section 6.6 concludes, while Appendices C.1 and C.2 verify expected physical properties of the BAS formula.

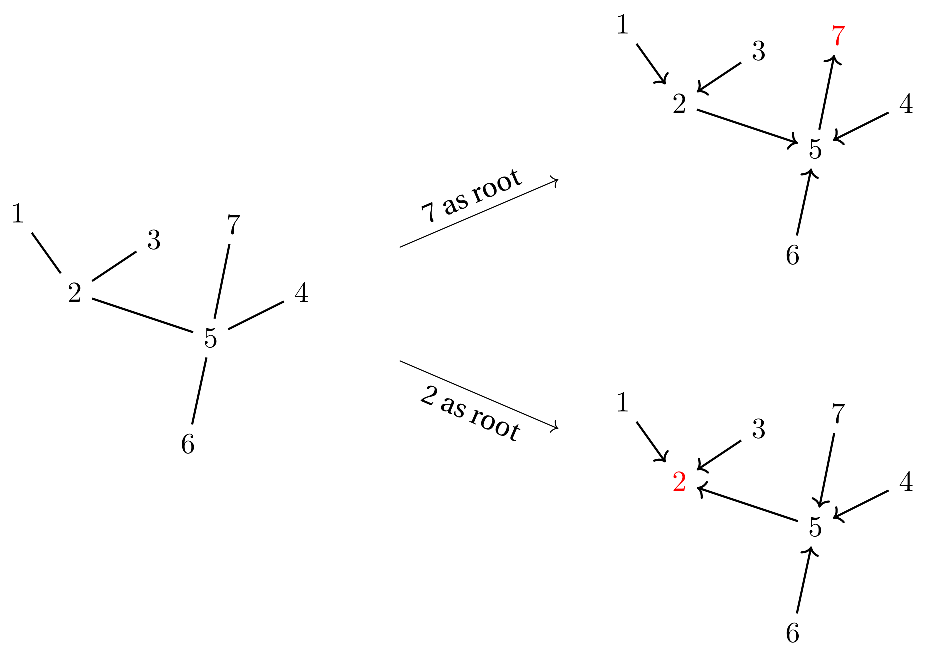


Figure 6.1: Labelling a point as a 'root' gives a unique orientation to an undirected tree graph.

6.2 Review: relevant concepts in graph theory

Many of the key arguments in this chapter make use of results in graph theory. Most of these are standard results in algebraic combinatorics, but for the sake of completeness (and those readers unfamiliar with graph theory), we include a review of the necessary concepts here. Standard textbook references are [313–315], with concise introductions to some of the relevant theorems in the amplitudes literature to be found in [295, 316, 317].

A *graph* $G = (V, E)$ is a set V of vertices, and a set E of edges. Each edge $e \in E$ is a pair of two vertices $(v - w)$ if the graph is *undirected*, or $(v \to w)$ if the graph is *directed*. The graph is *simple* if there is no more than one edge connecting any two vertices, and no edge connects the same vertex. A *tree graph* is a connected simple graph without cycles. A *spanning tree* of a connected graph G is a connected tree on the same vertices and with the set of edges a subset of the set of edges of G.

Given an undirected tree graph G on the vertex set $V = \{1, 2, \ldots, n\}$, we can fix some $b \in V$ to be the 'root' (also sometimes called a 'sink') of G. Then one can uniquely assign a direction to the edges of G so that each vertex $i \neq b$ has exactly one outgoing edge, and b only has incoming edges. Thus for a given vertex $i \neq k$ in G, we may define $o(i) \in V$ to be the vertex connected to i via the outgoing edge. Given an undirected tree graph G and a choice of root $b \in V$, we denote the directed tree graph arising in this way as G^b.

From now on, we will consider graphs inscribed on the Riemann sphere $\mathbb{P}^1$, so that to each vertex $i \in V$ there is an associated point on $\mathbb{P}^1$, written in homogeneous coordinates as σ_i^a.

$$
\begin{tikzcd}[column sep=small, row sep=small]
3 \arrow[dr] & & 4 \arrow[dl] & \\
& 2 \arrow[dr] & & 1 \arrow[dl] \\
& & b &
\end{tikzcd}
\qquad\qquad
\begin{aligned}
\Omega_{G^b} = \{&b1234,\ b1243,\\
&b2134,\ b2143,\ b2314,\\
&b2413,\ b2341,\ b2431\}
\end{aligned}
$$

Figure 6.2: A tree graph G^b and its set of compatible orderings Ω_{G^b}.

Now, directed tree graphs can be related to orderings on the set of vertices. An *ordering* of V is a word that uses each letter $v \in V$ exactly once. For an ordering ρ on V we define the *broken Parke-Taylor* factor $\mathrm{pt}_n[\rho]$ in the ring of rational functions on $(\mathbb{P}^1)^n$ as the product

$$
\mathrm{pt}_n[\rho] = \prod_{i \in V \setminus \{\rho(n)\}} \frac{1}{(\rho(i)\, \rho(i+1))}\,, \tag{6.2.4}
$$

where $\rho(i+1)$ is the letter following $\rho(i)$ in the word, and $\rho(n)$ is the last element of ρ. For example,

$$
\mathrm{pt}_n[12\cdots n] = \frac{1}{(1\,2)\,(2\,3)\cdots(n-1\,n)}\,, \tag{6.2.5}
$$

and in general these broken Parke-Taylor factors are related to their 'full' counterparts (5.2.23) by $\mathrm{pt}_n[\rho] = \mathrm{PT}_n[\rho](\rho(n)\,\rho(1))$.

It is a simple corollary of a result in [295][6] that:

Proposition 6.2.1. *Let G^b be a directed tree graph on $\mathbb{P}^1$ with vertices $\{1 \ldots n\}$ and root b, and let $x \in \mathbb{P}^1$ be a non-intersecting point. Then:*

$$
\prod_{\substack{\text{edges}\\ i \to j}} \frac{(x\,j)}{(i\,j)(i\,x)} = \sum_{\substack{\rho \\ o(i) <_{b\rho} i}} \mathrm{pt}_n[b\rho]\, \frac{(b\,x)}{(\rho(n)\,x)}\,. \tag{6.2.6}
$$

where the sum is over all orderings ρ on $\{1, \ldots, n\} \setminus \{b\}$ such that $o(i)$ – the vertex connected to i by its unique outgoing edge – precedes i in the word $b\rho$ for all $i \neq b$.

In an intuitive sense, the sum appearing on the right-hand-side of the identity (6.2.6) is over all orderings on the vertices which are compatible with the directed tree graph G^b.

More precisely, define an ordering ρ on $\{1, \ldots, n\} \setminus \{b\}$ to be *compatible* with the directed tree G^b if $o(i) <_{b\rho} i$ for all i in the tree. The set of all orderings compatible with G^b will be denoted by Ω_{G^b}. See Figure 6.2 for an example.

Conversely, we say that a directed tree graph G^b is *compatible* with an ordering ρ if $o(i) <_{b\rho} i$ for all vertices $i \neq b$ of G^b. Therefore, to each ordering ρ, we can associate a set of compatible directed trees $\mathcal{T}^b_\rho$ that satisfy this relation. See Figure 6.3 for an example. Further, it will be useful to consider the intersection $\mathcal{T}^b_{\rho,\omega} = \mathcal{T}^b_\rho \cap \mathcal{T}^b_\omega$ of all directed trees that are simultaneously compatible with the orderings ρ and ω. See Figure 6.4 for an example.

[6] The proposition in [295] is the case where we fix $x = (1, 0)$, and all other points are of the form $\sigma^a_i = (z_i, 1)$. Similar relations also arise in the study of arrangements of hyperplanes in complex affine spaces [318].

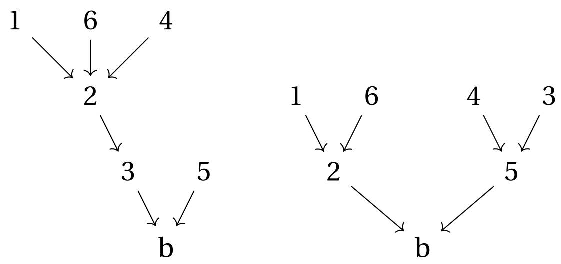


Figure 6.3: The ordering $(b532461)$ is compatible with both of these directed tree graphs rooted at b. (Note that this is a non-exhaustive list.)

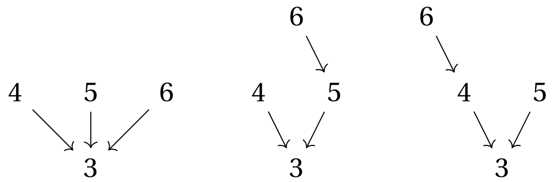


Figure 6.4: All tree graphs rooted at 3 which are simultaneously compatible with the orderings $(3456|3546)$.

We now consider *weighted graphs*, which are graphs where each edge $(i-j)$ is assigned a 'weight' $w_{ij} \in \mathbb{C}$.

Proposition 6.2.2. *The following summations are equivalent:*

$$\sum_{T \in \mathcal{T}^b_{\rho,\omega}} \prod_{(i \to j) \in T} w_{ij} = \prod_{j \neq b} \sum_{\substack{i <_{b\rho} j \\ i <_{b\omega} j}} w_{ij}, \tag{6.2.7}$$

for symmetric weights $w_{ij} = w_{ji}$.

Proof: We proceed by induction. As the base case, consider two vertices b and c, so the only possible ordering for ρ and ω is the singlet c. In this case, (6.2.7) is simply $w_{bc} = w_{bc}$, which is true for symmetric weights.

Now, suppose the equality (6.2.7) holds when $b\rho,\ b\omega$ are some orderings on $n-1$ letters. We can now create a new orderings $b\rho^+$, $b\omega^+$ on n letters by inserting the new letter n anywhere after b into either ordering. Without loss of generality, we may assume that n is the last letter in $b\rho^+$ (upon relabelling our alphabet appropriately). All compatible directed trees in $\mathcal{T}^b_{\rho^+,\omega^+}$ can now be constructed by joining the n vertex appropriately to trees from $\mathcal{T}^b_{\rho,\omega}$. Consider some $T \in \mathcal{T}^b_{\rho,\omega}$. It's possible to generate a tree with n vertices from T by either putting n in the middle of an existing edge, or connecting it to a vertex with a single line. Since n is the *last* letter in ρ^+, for a compatible tree there can be no $(k \to n)$ edge. Therefore the only possibility is that n has valence one, with one $(n \to k)$ edge joining it to T. The new vertex n can only join onto vertices k such that $k <_{b\omega} n$, and so the compatible trees formed out of T can be labelled by this joining vertex i.

Each tree in $\mathcal{T}^{b}_{\rho^+,\omega^+}$ is described uniquely in this way. In particular, given any $T^+ \in \mathcal{T}^{b}_{\rho^+,\omega^+}$, the corresponding 'base tree' $T \in \mathcal{T}^{b}_{\rho,\omega}$ is easily identified by removing the n vertex (which, by the preceding argument, must have valence one), which is uniquely connected to some other vertex, k. Therefore.

$$\sum_{T\in\mathcal{T}^{b}_{\rho^+,\omega^+}} \prod_{(i\to j)\in T} w_{ij} = \sum_{T\in\mathcal{T}^{b}_{\rho,\omega}} \left(\sum_{k<_{b\omega}n} w_{kn} \prod_{(i\to j)\in T} w_{ij} \right) \tag{6.2.8}$$

$$= \sum_{k<_{b\omega}n} w_{kn} \sum_{T\in\mathcal{T}^{b}_{\rho,\omega}} \prod_{(i\to j)\in T} w_{ij} \tag{6.2.9}$$

$$= \sum_{k<_{b\omega}n} w_{kn} \prod_{j\neq b,n} \sum_{\substack{i<_{b\rho}j \\ i<_{b\omega}j}} w_{ij} \tag{6.2.10}$$

$$= \prod_{j\neq b} \sum_{\substack{i<_{b\rho^+}j \\ i<_{b\omega^+}j}} w_{ij}\,, \tag{6.2.11}$$

as desired. □

A further key result from graph theory tells us how to sum all of the weights of spanning tree graphs of G. Suppose G has n vertices, and let $w_{ij} \in \mathbb{C}$ again denote the weight assigned to each edge $(i-j)$, with $w_{ij} = 0$ if $(i-j) \notin E$. One then constructs the $n \times n$ *weighted Laplacian matrix of* G, $W(G)$, with the entries

$$W(G)_{ij} = \begin{cases} \sum_{(k-i)\in E} w_{ik} & \text{if } i = j, \\ -w_{ij} & \text{if } i \neq j, \end{cases} \tag{6.2.12}$$

This matrix is degenerate, as the sum of elements in each column and row vanishes, so its determinant is zero. However, the minor corresponding to removing the row and column corresponding to vertex b, $|W(G)^{b}_{b}|$, is generically non-vanishing.

The following is a classical result in graph theory:

Theorem (Weighted Matrix-Tree Theorem). *The minor* $|W(G)^{b}_{b}|$*, obtained by deleting the* b^{th} *row and column of the weighted Laplacian matrix of* G *is independent of* b*, and equal to*

$$\begin{aligned} \left|W(G)^{i}_{i}\right| &= \sum_{\substack{T \\ \text{spanning } G}} \left(\prod_{(i-j)\in E(T)} w_{ij} \right) \\ &= \sum_{\substack{T^b \\ \text{spanning } G}} \left(\prod_{(i\to j)\in E(T^b)} w_{ij} \right)\,, \end{aligned} \tag{6.2.13}$$

where the first sum is over all spanning trees of G*, and the second sum is over all directed spanning trees of* G *rooted at* $b \in V(G)$*.*

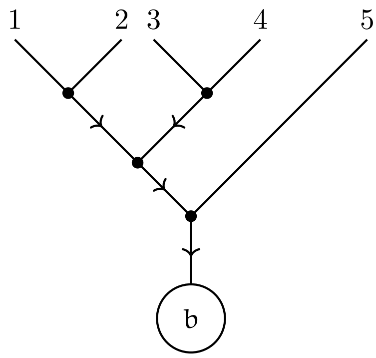


Figure 6.5: The binary tree corresponding to the complete bracketing $[[[1,2],[3,4]],5]$ of the leaves $\mathsf{L} = \{1,2,3,4,5\}$, with root b.

A further refinement of directed tree graphs is the concept of a *rooted binary tree*. These are tree graphs, with a choice of root inducing a direction, where each vertex (other than the root) in the graph has exactly two incoming edges (hence the 'binary') and one outgoing edge; the root itself has only a single incoming edge. As such, rooted binary trees are naturally labelled by the (semi-ordered) set of leaves L, which encode the (unique) external edge structure of the graph. By labelling the external edges of the binary tree, the label L can be encoded using the nested bracket notation, as illustrated in Figure 6.5.

There is a natural notion of weighting which can be associated to a rooted binary tree. Suppose a rooted binary tree has $n-1$ external edges, which we label $\{1,\dots,n-1\}$, and let E denote the set of *internal* edges in the rooted binary tree. To any internal edge $e \in \mathsf{E}$, we can assign a weight $w_e \in \mathbb{C}$. Now, any $e \in \mathsf{E}$ is uniquely identified with the set $\mathsf{I}(e) \subset \mathsf{L}$ of leaves in the tree which are 'upstream' (with respect to the direction of the graph) from e. Equivalently, $\mathsf{I}(e)$ labels the (smaller) rooted binary tree rooted at the endpoint of the edge e. We can then assign

$$w_e = \sum_{\{i,j\}\subset \mathsf{I}(e)} w_{ij} \tag{6.2.14}$$

in terms of some symmetric $w_{ij} \in \mathbb{C}$. We denote the product of weights for a rooted binary tree T by

$$w_{\mathsf{T}} = \prod_{e\in\mathsf{E}} w_e\,, \tag{6.2.15}$$

and the sum of all weights by

$$w_{\text{total}} = \sum_{e\in\mathsf{E}} w_e\,. \tag{6.2.16}$$

The sum w_{total} can be viewed as the total weight of the rooted binary tree 'flowing' into the root.

Just like tree graphs, binary trees can also be *compatible* with pairs of orderings on L. A binary tree T is compatible with the two orderings ρ, ω on L if it is planar with respect to both. In practice, it is possible to algorithmically construct all binary trees compatible with two given orderings (cf., [11]). We denote the set of binary trees, rooted at the vertex b, which are compatible with both ρ and ω by $\mathcal{BT}^b_{\rho,\omega}$. There is also a *sign* associated with each pair of orderings $\rho,\ \omega$, given by the number of times the ordering ω winds around ρ (see [296] for further discussion).

The following result is an easy generalization of one obtained in [302] for a specific class of weightings associated with cubic Feynman diagrams:

Proposition 6.2.3. *Let ρ, ω be orderings on the letters $\{2, \ldots, n-1\}$ and $w_{ij} = w_{ji} \in \mathbb{C}$. Then the two maps*

$$S[1\rho|1\omega] = \sum_{T \in \mathcal{T}^n_{1\rho,1\omega}} \prod_{(i\to j)\in E(T)} w_{ij}\,, \tag{6.2.17}$$

$$T[1\rho|1\omega] = \pm\frac{1}{w_{\text{total}}} \sum_{T \in \mathcal{BT}^n_{1\rho,1\omega}} \frac{1}{w_T}\,, \tag{6.2.18}$$

on the space of orderings are inverses of each other.

The genesis of this result lies in the close relationship between weighted, rooted binary trees and the tree-level Feynman diagrams of *biadjoint scalar* (BAS) *theory*, a theory of massless scalars valued in the tensor product of two Lie algebras, interacting cubically via the structure constants [11, 319, 320]. The tree-level scattering amplitudes of BAS theory can be decomposed into doubly-colour-ordered partial amplitudes $m_n[\rho|\omega]$, and the Feynman diagrams contributing to such a partial amplitude correspond to the rooted binary trees in $\mathcal{BT}^n_{\rho,\omega}$, where the root is chosen (without loss of generality) to correspond to the n^{th} external scalar.

The weights naturally assigned to these trees are simply the Mandelstam invariants corresponding to the momenta flowing through each internal edge of the Feynman diagram. In particular, if $\{k_i^\mu\}_{i=1,\ldots,n}$ are the massless external momenta for the n-point BAS scattering process, let

$$k_I^\mu := \sum_{i\in I} k_i^\mu\,. \tag{6.2.19}$$

Then (as discussed above), the edge e in the binary tree is equivalently labeled by some $I(e) \subset L$ corresponding to the 'leaves' contributing to the edge, and one can make the assignment

$$w_I \equiv s_I = \sum_{\{i,j\}\subset I(e)} s_{ij}\,, \qquad s_{ij} := 2\,k_i \cdot k_j\,, \tag{6.2.20}$$

so that

$$2\,k_I \cdot k_J = \sum_{i\in I,\, j\in J} s_{ij}\,. \tag{6.2.21}$$

In this language, Proposition 6.2.3 is the statement of the relation between the tree-level KLT kernel, encoded by the map S in (6.2.17), and the tree-level partial amplitudes, encoded by

$$m_n[1\rho n|1\omega n] = w_{\text{total}}\, T[1\rho|1\omega]\,, \tag{6.2.22}$$

with T defined by (6.2.18).

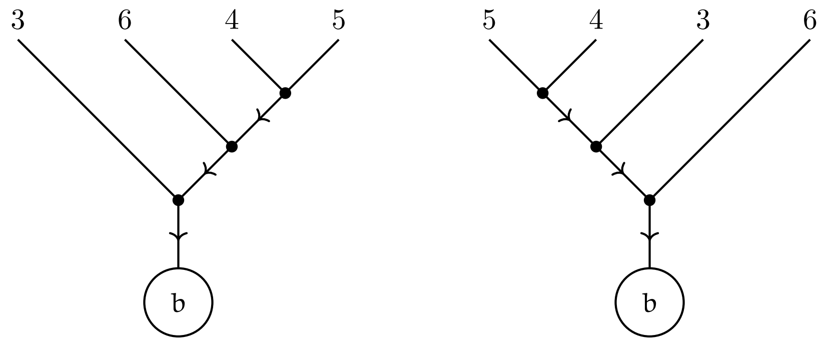


Figure 6.6: The two binary trees compatible with the orderings $(3456|3546)$, rooted at b. Legs with arrows are those for which weights are naturally assigned taking contributions from the total momentum of the subtree it connects. Both of these come with a coefficient -1 from the winding number.

6.3 Helicity-graded integral kernel

We now use the machinery of graph theory, reviewed in Section 6.2, to re-write the Cachazo-Skinner formula (5.2.33) introduced in chapter 5 in a fashion that manifests a KLT-like double copy structure. In particular, the result is expressed in terms of an integral kernel, which is graded by the helicity configuration and defined as an integrand factor to be integrated over the moduli space of rational maps from the Riemann sphere to twistor space of the appropriate degree (fixed by the helicity configuration). The result builds on the colour-kinematics dual numerators derived for the MHV sector in [295], but extended to identify a double copy kernel and generalised to all helicity configurations.

6.3.1 Reduced determinants from orderings and directed trees

The key idea is to rewrite the reduced determinants appearing in the Cachazo-Skinner formula, namely $\det{}'(\mathbb{H})$ and $\det{}'(\mathbb{H}^\vee)$, into a form resembling a minor of a weighted Laplacian matrix[7]. Then the weighted matrix-tree theorem will enable us to translate the formula into the language of sums over weighted graphs, where we can apply further results from Section 6.2.

To begin, consider the matrix $\mathbb{H}$, whose entries are given by (5.2.29). Using the definition of the reduced determinant (5.2.30) and elementary properties of determinants, it is straightforward to show that $\det{}'(\mathbb{H})$ can be written as

$$\det{}'(\mathbb{H}) = \frac{|\mathrm{B}^{\mathfrak{b}}_{\mathfrak{b}}|}{|\tilde{\mathbf{h}}|^2} \prod_{\substack{\mathrm{j}\in\mathbf{h}\\ \mathrm{l}\in\tilde{\mathbf{h}}}} \frac{1}{(\mathrm{j}\,\mathrm{l})^2} \prod_{\mathrm{k}=1}^{n} \mathrm{D}\sigma_{\mathrm{k}}\,, \tag{6.3.23}$$

[7]The connection with weighted Laplacian matrices has been noted many times before, particularly for the reduced determinant of $\mathbb{H}$ (cf., [8, 16, 316, 317]).

where $b \in \mathbf{h}$ is arbitrary and B is a $(n-d-1) \times (n-d-1)$ matrix with entries

$$\begin{aligned} B_{ij} &= \mathbb{H}_{ij} \prod_{l \in \tilde{\mathbf{h}}} (i\,l)\,(j\,l)\,, \qquad i, j \in \mathbf{h}\,,\ i \neq j\,, \\ B_{ii} &= -\sum_{\substack{j \in \mathbf{h} \\ j \neq i}} B_{ij}\,, \qquad i \in \mathbf{h}\,. \end{aligned} \tag{6.3.24}$$

In particular, B has the form of a weighted Laplacian matrix for the simple graph on the set of vertices $\mathbf{h}$, where the weight corresponding to the edge $(i - j)$, for $i, j \in \mathbf{h}$, is

$$B_{ij} := -t_i\, t_j \frac{[i\,j]}{(i\,j)} \prod_{l \in \tilde{\mathbf{h}}} (i\,l)\,(j\,l)\,. \tag{6.3.25}$$

Note that, as required, this is symmetric (i.e., $B_{ij} = B_{ji}$).

By the weighted matrix tree theorem, this means that the reduced determinant of $\mathbb{H}$ is equal to

$$\det{}'(\mathbb{H}) = \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{\substack{k \in \mathbf{h} \\ l \in \tilde{\mathbf{h}}}} \frac{1}{(k\,l)^2} \sum_{\substack{T^b \\ \text{spanning } \mathbf{h}}} \prod_{(i \to j)} B_{ij}\,, \tag{6.3.26}$$

where we have stripped off an overall factor of $\prod_{i=1}^{n} D\sigma_i$ and the sum is over directed trees on the set $\mathbf{h}$, with the direction fixed by the (arbitrary) choice of $b \in \mathbf{h}$. Now, for each T^b – that is, for each directed tree rooted at b – each vertex $i \in \mathbf{h} \setminus \{b\}$ appears only once as the source of an edge $(i \to j)$, as every vertex other than b has a unique outgoing edge. Consequently, it follows that

$$\begin{aligned} \det{}'(\mathbb{H}) &= \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{\substack{k \in \mathbf{h} \\ l \in \tilde{\mathbf{h}}}} \frac{1}{(k\,l)^2} \prod_{\substack{m \in \tilde{\mathbf{h}} \\ a \in \mathbf{h} \setminus \{b\}}} t_a\,(a\,m) \sum_{\substack{T^b \\ \text{spanning } \mathbf{h}}} \prod_{(i \to j)} \left(-t_j \frac{[i\,j]}{(i\,j)} \prod_{l \in \tilde{\mathbf{h}}} (j\,l) \right) \\ &= \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{l \in \tilde{\mathbf{h}}} \frac{1}{(b\,l)^2} \prod_{\substack{m \in \tilde{\mathbf{h}} \\ a \in \mathbf{h} \setminus \{b\}}} \frac{t_a}{(a\,m)} \sum_{\substack{T^b \\ \text{spanning } \mathbf{h}}} \prod_{(i \to j)} \left(-t_j \frac{[i\,j]}{(i\,j)} \prod_{l \in \tilde{\mathbf{h}}} (j\,l) \right) \\ &= \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{l \in \tilde{\mathbf{h}}} \frac{1}{(b\,l)^2} \sum_{\substack{T^b \\ \text{spanning } \mathbf{h}}} \prod_{(i \to j)} \left(-t_i\, t_j \frac{[i\,j]}{(i\,j)} \prod_{l \in \tilde{\mathbf{h}}} \frac{(j\,l)}{(i\,l)} \right)\,. \end{aligned} \tag{6.3.27}$$

Now, pick any two distinct $x, y \in \tilde{\mathbf{h}}$; these can be used to trivially re-express the reduced determinant as

$$\det{}'(\mathbb{H}) = \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{l\in\tilde{\mathbf{h}}} \frac{1}{(b\,l)^2} \sum_{\substack{T^b\\ \text{spanning } \mathbf{h}}} \prod_{(i\to j)} \left(-t_i\, t_j\, [i\,j]\, (i\,j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{x,y\}} \frac{(j\,l)}{(i\,l)} \right)$$
$$\times \prod_{(i\to j)} \frac{(x\,j)}{(i\,j)\,(i\,x)} \prod_{(i\to j)} \frac{(y\,j)}{(i\,j)\,(i\,y)}\,, \tag{6.3.28}$$

which can now be further processed using Proposition 6.2.1.

In particular, applying (6.2.6) to each of the factors in the second line of (6.3.28), we arrive at an expression for the reduced determinant in terms of the broken Parke-Taylor factors (6.2.4):

$$\det{}'(\mathbb{H}) = \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{l\in\tilde{\mathbf{h}}} \frac{1}{(b\,l)^2} \sum_{\substack{T^b\\ \text{spanning } \mathbf{h}}} \prod_{(i\to j)} \left(-t_i\, t_j\, [i\,j]\, (i\,j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{x,y\}} \frac{(j\,l)}{(i\,l)} \right)$$
$$\times \sum_{\rho\in\Omega_{T^b(\mathbf{h})}} \mathrm{pt}_{n-d-1}[b\rho]\, \frac{(b\,x)}{(\rho^*\,x)} \sum_{\omega\in\Omega_{T^b(\mathbf{h})}} \mathrm{pt}_{n-d-1}[b\omega]\, \frac{(b\,y)}{(\omega^*\,y)}\,, \tag{6.3.29}$$

where the sums in the second line are over orderings on $\mathbf{h}\setminus\{b\}$ compatible with the directed, rooted tree T^b, and $\rho^*, \omega^* \in \mathbf{h}\setminus\{b\}$ denote the last elements of the orderings ρ and ω, respectively. The summations over graphs and compatible orderings can now be swapped to give

$$\det{}'(\mathbb{H}) = \frac{(b\,x)\,(b\,y)}{\left|\tilde{\mathbf{h}}\cup\{b\}\right|^2} \sum_{\rho,\omega\in\mathcal{S}(\mathbf{h}\setminus\{b\})} \mathrm{pt}_{n-d}[b\rho x]\, \mathrm{pt}_{n-d}[b\omega y]$$
$$\times \sum_{T\in\mathcal{T}^b_{\rho,\omega}} \prod_{(i\to j)\in E(T)} \left(-t_i\, t_j\, [i\,j]\, (i\,j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{x,y\}} \frac{(j\,l)}{(i\,l)} \right)\,, \tag{6.3.30}$$

where $\mathcal{S}(\mathbf{h}\setminus\{b\}) \cong S_{n-d-2}$ is the group of permutations on the $n-d-2$ letters of $\mathbf{h}\setminus\{b\}$. Note that the summation in the second line is now over directed trees on $\mathbf{h}$ rooted at b which are compatible with the orderings ρ and ω.

A sequence of similar manipulations can also be performed to express the reduced determinant of $\mathbb{H}^\vee$, with entries (5.2.31), in terms of a sum of orderings and compatible directed trees. The steps closely follow those we have gone through for $\det{}'(\mathbb{H})$; only the initial part of the argument – connecting the reduced determinant of $\mathbb{H}^\vee$ to the determinant of a weighted Laplacian matrix – differs significantly.

This is primarily because the form of the diagonal entries of $\mathbb{H}^\vee$ does not, as stated in (5.2.31), take the form of a sum, as needed for the general form of a weighted Laplacian matrix. However, there is a non-trivial identity[8] (first proven in a slightly more general form in [8]) for each $i \in \tilde{\mathbf{h}}$:

$$\frac{\langle \lambda(\sigma_i)\, d\lambda(\sigma_i)\rangle}{D\sigma_i} = \sum_{\substack{j\in\tilde{\mathbf{h}} \\ j\neq i}} \frac{\langle \lambda(\sigma_i)\, \lambda(\sigma_j)\rangle}{(i\,j)} \prod_{k\in\tilde{\mathbf{h}}\setminus\{i,j\}} \frac{(k\,i)}{(k\,j)}\,, \tag{6.3.31}$$

which enables us to perform the necessary manipulations. Indeed, using the basic properties of determinants and the definition (6.3.23) of $\det{}'(\mathbb{H}^\vee)$, one finds that

$$\det{}'(\mathbb{H}^\vee) = |\tilde{\mathbf{h}}|^2 \left| B^{\vee}{}_{a}^{\ a} \right|\,, \tag{6.3.32}$$

where $B^\vee$ is a $(d+1)\times(d+1)$ matrix with entries

$$\begin{aligned} B^\vee_{ij} &= -\mathbb{H}^\vee_{ij} \prod_{k\in\tilde{\mathbf{h}}\setminus\{i\}} \frac{1}{(i\,k)} \prod_{l\in\tilde{\mathbf{h}}\setminus\{j\}} \frac{1}{(l\,j)}\,, \qquad i,j\in\tilde{\mathbf{h}}\,,\ i\neq j\,, \\ B^\vee_{ii} &= -\sum_{\substack{j\in\tilde{\mathbf{h}} \\ j\neq i}} B^\vee_{ij}\,, \qquad i\in\tilde{\mathbf{h}}\,. \end{aligned} \tag{6.3.33}$$

In particular, $B^\vee$ is a symmetric matrix of corank 1, with the structure of a weighted Laplacian matrix for the simple graph on the set of vertices $\tilde{\mathbf{h}}$. Note that this matrix carries projective weight on $\mathbb{P}^1$: each entry $B^\vee_{ij}$ has homogeneity -2 each σ_i, for $i\in\tilde{\mathbf{h}}$.

At this point, one can apply the weighted matrix tree theorem, Proposition 6.2.1 and a bit of algebra to arrive at the identity

$$\begin{aligned} \det{}'(\mathbb{H}^\vee) = \frac{(a\,s)\,(a\,t)}{\left|\tilde{\mathbf{h}}\setminus\{a\}\right|^2} &\sum_{\bar{\rho},\bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}}\setminus\{a\})} \mathrm{pt}_{d+2}[a\bar{\rho}s]\,\mathrm{pt}_{d+2}[a\bar{\omega}t] \\ &\times \sum_{\bar{T}\in\mathcal{T}^a_{\bar{\rho},\bar{\omega}}} \prod_{(i\to j)\in E(\bar{T})} \left(\langle \lambda(\sigma_i)\,\lambda(\sigma_j)\rangle\,(i\,j) \prod_{k\in(\tilde{\mathbf{h}}\cup\{s,t\})\setminus\{i,j\}} \frac{(k\,i)}{(k\,j)} \right)\,, \end{aligned} \tag{6.3.34}$$

where $s,t\in\mathbb{P}^1$ are arbitrary points which do not coincide with the vertices of $\tilde{\mathbf{h}}$, and $\mathcal{S}(\tilde{\mathbf{h}}\setminus\{a\})\cong S_d$ is the group of permutations on the d letters of $\tilde{\mathbf{h}}\setminus\{a\}$.

6.3.2 The momentum kernel

At this point, the expressions (6.3.30) and (6.3.34) lead to a new representation of the Cachazo-Skinner formula for the helicity-graded, tree-level graviton S-matrix. *A priori*, it may seem that all that we have accomplished at this point is to turn the compact, manifestly permutation-invariant integrand of (5.2.33) into an unwieldy mess of sums over permutations and com-

[8]The right-hand side of this identity is actually the form of the diagonal entries for $\mathbb{H}^\vee$ originally used in the formulation of the Cachazo-Skinner formula [24, 28].

patible tree graphs. However, the utility of this description lies in the fact that the broken Parke-Taylor factors appearing in (6.3.30) and (6.3.34) can be auspiciously combined.

In particular, note that the points $x, y \in \tilde{\mathbf{h}}$ appearing in (6.3.30) and $s, t \in \mathbb{P}^1$ appearing in (6.3.34) are arbitrarily chosen. So without loss of generality, we can set $s = b \in \mathbf{h}$ and $x = a \in \tilde{\mathbf{h}}$. Then making use of the definition (6.2.4), it follows that

$$\mathrm{pt}_{n-d}[b\rho a]\,\mathrm{pt}_{n-d}[b\omega y]\,\mathrm{pt}_{d+2}[a\bar{\rho}b]\,\mathrm{pt}_{d+2}[a\bar{\omega}t] = \frac{(a\,b)\,(\omega^*\,\bar{\omega}^*)}{(y\,\omega^*)\,(\bar{\omega}^*\,t)}\,\frac{\mathrm{PT}_n[a\bar{\rho}b\rho]\,\mathrm{PT}_n[\bar{\omega}^{\mathrm{T}}ab\omega]}{\prod_{i=1}^n D\sigma_i^2}\,, \tag{6.3.35}$$

where ω^* and $\bar{\omega}^*$ denote the final entries in the orderings ω and $\bar{\omega}$, respectively.

This enables us to re-write the tree-level N^{d-1}MHV graviton amplitude as

$$\begin{aligned}\mathcal{M}_{n,d} = \sum_{\substack{b\rho,b\omega\in\mathcal{S}(\mathbf{h})\\ a\bar{\rho},a\bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d}\,|\tilde{\mathbf{h}}|^4\,\frac{(a\,b)\,(\omega^*\,\bar{\omega}^*)\,(b\,y)\,(a\,t)}{(y\,\omega^*)\,(\bar{\omega}^*\,t)} \prod_{k\in\tilde{\mathbf{h}}\setminus\{a\}} \frac{(k\,a)^2}{(k\,b)^2}\,\mathrm{PT}_n[a\bar{\rho}b\rho] \\ \mathrm{PT}_n[\bar{\omega}^{\mathrm{T}}ab\omega]\left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}}\prod_{(i\to j)\in E(T)}\left(-t_i\,t_j\,[i\,j]\,(i\,j)\prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}}\frac{(j\,l)}{(i\,l)}\right)\right] \\ \left[\sum_{\bar{T}\in\mathcal{T}^a_{\bar{\rho},\bar{\omega}}}\prod_{(i\to j)\in E(\bar{T})}\left(\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle\,(i\,j)\prod_{k\in(\tilde{\mathbf{h}}\cup\{b,t\})\setminus\{i,j\}}\frac{(k\,i)}{(k\,j)}\right)\right]\prod_{i\in\mathbf{h}}h_i\prod_{j\in\tilde{\mathbf{h}}}\tilde{h}_j\,,\end{aligned} \tag{6.3.36}$$

where

$$d\mu_{n,d} := \frac{d\mu_d}{\prod_{i=1}^n D\sigma_i}\,, \tag{6.3.37}$$

and we have abbreviated $h_i \equiv h_i(Z(\sigma_i))$, $\tilde{h}_j \equiv \tilde{h}_j(Z(\sigma_j))$. By further manipulating the summations over tree graphs, this can be equivalently written as

$$\begin{aligned}\mathcal{M}_{n,d} = \sum_{\substack{b\rho,b\omega\in\mathcal{S}(\mathbf{h})\\ a\bar{\rho},a\bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d}\,|\tilde{\mathbf{h}}|^8\,\frac{(a\,b)\,(\omega^*\,\bar{\omega}^*)\,(b\,y)\,(a\,t)\,(y\,t)^2}{(y\,\omega^*)\,(\bar{\omega}^*\,t)} \prod_{\substack{k\in\mathbf{h}\setminus\{b,t\}\\ l\in\tilde{\mathbf{h}}\setminus\{a,y\}}} \frac{1}{(k\,l)^2}\,\mathrm{PT}_n[a\bar{\rho}b\rho] \\ \times\,\mathrm{PT}_n[\bar{\omega}^{\mathrm{T}}ab\omega]\left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}}\prod_{(i\to j)\in E(T)}\left(-t_i\,t_j\,[i\,j]\,(i\,j)\prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}}(i\,l)\,(j\,l)\right)\right] \\ \left[\sum_{\bar{T}\in\mathcal{T}^a_{\bar{\rho},\bar{\omega}}}\prod_{(i\to j)\in E(\bar{T})}\left(\frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(i\,j)}\prod_{k\in(\tilde{\mathbf{h}}\cup\{b,t\})\setminus\{i,j\}}\frac{1}{(k\,i)\,(k\,j)}\right)\right]\prod_{i\in\mathbf{h}}h_i\prod_{j\in\tilde{\mathbf{h}}}\tilde{h}_j\,.\end{aligned} \tag{6.3.38}$$

At this stage, the advantage of processing the Cachazo-Skinner formula in this way becomes clear: there is now a double copy structure apparent in the formula. In particular, the integrand now contains *two* copies of the $\mathbb{P}^1$ Parke-Taylor factor PT_n, each of which defines the integrand of the RSVW formula for the tree-level gluon S-matrix, glued together with an integral 'kernel' and summed over a basis of colour-orderings.

To see this more explicitly, define

$$\phi_{ij} := -t_i\, t_j\, [i\, j]\, (i\, j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (i\, l)\, (j\, l)\,, \qquad i, j \in \mathbf{h}\,, \tag{6.3.39}$$

which is weightless in each $i, j \in \mathbf{h}$ and weight 2 in each $l \in \tilde{\mathbf{h}} \setminus \{a, y\}$,

$$\tilde{\phi}_{ij} := \frac{\langle\lambda(\sigma_i)\,\lambda(\sigma_j)\rangle}{(i\, j)} \prod_{k\in(\tilde{\mathbf{h}}\cup\{b,t\})\setminus\{i,j\}} \frac{1}{(k\, i)\,(k\, j)}\,, \qquad i, j \in \tilde{\mathbf{h}}\,, \tag{6.3.40}$$

which carries weight -2 in each $k \in \tilde{\mathbf{h}} \cup \{b, t\}$, and

$$\mathcal{D}[\omega, \bar{\omega}] := \frac{(a\, b)\,(\omega^*\, \bar{\omega}^*)\,(b\, y)\,(a\, t)\,(y\, t)^2}{(y\, \omega^*)\,(\bar{\omega}^*\, t)} \prod_{\substack{k\in\mathbf{h}\setminus\{b,t\}\\ l\in\tilde{\mathbf{h}}\setminus\{a,y\}}} \frac{1}{(k\, l)^2}\,, \tag{6.3.41}$$

which has weight $+2$ in $\{a, b, t, y\}$, weight $-2(d-1)$ in all elements of $\mathbf{h} \setminus \{b, t\}$ and weight $-2(n-d-3)$ in all elements of $\tilde{\mathbf{h}} \setminus \{a, y\}$. Let

$$S_{n,d}[\rho, \bar{\rho}|\omega, \bar{\omega}] := \mathcal{D}[\omega, \bar{\omega}] \left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}} \prod_{(i\to j)\in E(T)} \phi_{ij}\right] \left[\sum_{\bar{T}\in\mathcal{T}^a_{\bar{\rho},\bar{\omega}}} \prod_{(i\to j)\in E(\bar{T})} \tilde{\phi}_{ij}\right], \tag{6.3.42}$$

be the *integral kernel*; this can be viewed as a linear transformation $S : \mathcal{S}(\mathbf{h}\setminus\{b\}) \times \mathcal{S}(\tilde{\mathbf{h}}\setminus\{a\}) \to \mathcal{S}(\mathbf{h} \setminus \{b\}) \times \mathcal{S}(\tilde{\mathbf{h}} \setminus \{a\})$ on the space of orderings.

In terms of this integral kernel, the graviton tree amplitude becomes

$$\mathcal{M}_{n,d} = \sum_{\substack{b\rho,b\omega\in\mathcal{S}(\mathbf{h})\\ a\bar{\rho},a\bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d}\, |\tilde{\mathbf{h}}|^8\, \mathrm{PT}_n[a\bar{\rho}b\rho]\, S_{n,d}[\rho, \bar{\rho}|\omega, \bar{\omega}]\, \mathrm{PT}_n[\bar{\omega}^{\mathrm{T}} a b \omega] \prod_{i\in\mathbf{h}} h_i \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j\,. \tag{6.3.43}$$

The double copy structure here can be made even more explicit by writing the RSVW formula for the tree-level N^{d-1}MHV colour-ordered partial amplitude as

$$\mathcal{A}_{n,d}[\rho] = \int d\mu_d\, \mathcal{I}^{\tilde{\mathbf{g}}}_n[\rho] \prod_{i\in\mathbf{g}} a_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{g}}} b_j(Z(\sigma_j))\,, \tag{6.3.44}$$

for the integrand

$$\mathcal{I}^{\tilde{\mathbf{g}}}_n[\rho] := |\tilde{\mathbf{g}}|^4\, \mathrm{PT}_n[\rho]\,. \tag{6.3.45}$$

In particular, the new representation (6.3.43) of the Cachazo-Skinner formula is

$$\mathcal{M}_{n,d} = \sum_{\substack{b\rho, b\omega \in \mathcal{S}(\mathbf{h}) \\ a\bar{\rho}, a\bar{\omega} \in \mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d}\, \mathcal{J}_n^{\tilde{\mathbf{h}}}[a\bar{\rho}b\rho]\, S_{n,d}[\rho, \bar{\rho}|\omega, \bar{\omega}]\, \mathcal{J}_n^{\tilde{\mathbf{h}}}[\bar{\omega}^T ab\omega] \prod_{i\in\mathbf{h}} h_i \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j\,, \qquad (6.3.46)$$

which is clearly a double copy of the Yang-Mills integrand, glued together via the integral kernel.

Indeed, each of $\mathcal{J}_n^{\tilde{\mathbf{h}}}[a\bar{\rho}b\rho]$ and $\mathcal{J}_n^{\tilde{\mathbf{h}}}[\bar{\omega}^T ab\omega]$ can be viewed as a vector in the $(n-d-2)!\times d!$-dimensional space of orderings $\mathcal{S}(\mathbf{h}\setminus\{b\})\times\mathcal{S}(\tilde{\mathbf{h}}\setminus\{a\})$. The sum over orderings in (6.3.46) then simply corresponds to the multiplication of these two vectors via the square matrix S corresponding to the integral kernel.

It should be noted that this formula is independent of the choices of $\{a,b,y,t\}$. While this is far from obvious from the final expression (6.3.46), we saw that at each stage in the derivation where these choices were introduced, it was clear that they were entirely arbitrary.

At this point, it is worth comparing the structure of (6.3.46) with the more standard double copy structure arising from the usual KLT momentum kernel [263]. In particular, the KLT double copy representation of tree-level graviton scattering amplitudes takes the form

$$\mathcal{M}_{n,d} = \sum_{\alpha,\beta\in S_{n-3}} \mathcal{A}_{n,d}[(n-1)n\alpha 1]\, S^{\mathrm{KLT}}[\alpha|\beta]\, \mathcal{A}_{n,d}[1\beta(n-1)n]\,, \qquad (6.3.47)$$

where $S^{\mathrm{KLT}}[\alpha|\beta]$ is the KLT momentum kernel, a rational function of the kinematic invariants given by [265, 266]

$$S^{\mathrm{KLT}}[\alpha|\beta] = \prod_{i=2}^{n-2}\left(s_{1\alpha(i)} + \sum_{j>i}^{n-2} \theta_\beta(\alpha(i),\alpha(j))\, s_{\alpha(i)\alpha(j)}\right)\,, \qquad (6.3.48)$$

where $s_{ij} := (k_i+k_j)^2$ are the Mandelstam invariants and

$$\theta_\beta(\alpha(i),\alpha(j)) = \begin{cases} 0 & \text{if } \alpha(i)<\alpha(j) \text{ in } \beta \\ 1 & \text{otherwise} \end{cases}\,. \qquad (6.3.49)$$

Here, the choice of three external legs, 1, $n-1$ and n is arbitrary – any three legs can be chosen; the point is that the graviton takes the form of a sum over two copies of a basis of $(n-3)!$ colour-ordered gluon amplitudes, multiplied together via the KLT kernel. The size of this basis can be understood as the reduction of the naïve number of colour-orderings, namely $n!$, to $(n-3)!$ due to the photon decoupling [321], Kleiss-Kuijff [322, 323] and Bern-Carassco-Johansson [254] relations.

By contrast, our expression (6.3.46) is a sum over integrals of a basis of $(n-d-2)!\times d!$ colour-ordered gluon integrands multiplied together by the integral kernel. A closer comparison with momentum space formulations of KLT double copy becomes clear when we perform a chiral splitting of our kernel (6.3.42).

6.3.3 Chirally split momentum kernel

It is easy to see that the integral kernel (6.3.42) respects a chiral splitting in terms of the underlying helicity decomposition of the external states in the scattering amplitude. In particular, let $\hat{\rho} := \rho \cup \bar{\rho}$ be the ordering on $\{1, \ldots, n\} \setminus \{a, b\}$ (i.e., and element of S_{n-2}) arising from the union of the orderings ρ on $\mathbf{h} \setminus \{b\}$ and $\bar{\rho}$ on $\tilde{\mathbf{h}} \setminus \{a\}$, with $\hat{\omega} := \omega \cup \bar{\omega}$ defined similarly. Then the prefactor $\mathcal{D}(\omega, \bar{\omega}) = \mathcal{D}(\hat{\omega})$ can be viewed as a diagonal matrix D acting as a linear transformation on S_{n-2}:

$$D = \mathrm{diag}\left(\mathcal{D}(\hat{\omega})\right) , \tag{6.3.50}$$

whose inverse D^{-1} is simply given by inverting $\mathcal{D}(\hat{\omega})$ for each ordering.

Then we define the *reduced kernel* $\mathbb{S}_{n,d} := S_{n,d}\, D^{-1}$, which admits a natural chiral splitting in terms of orderings on $\tilde{\mathbf{h}}$ and $\mathbf{h}$ separately. To see this, define the negative helicity kernel as

$$\mathbb{S}_{\tilde{\mathbf{h}}}[\bar{\rho}|\bar{\omega}] := \sum_{\bar{T} \in \mathcal{T}^{a}_{\bar{\rho},\bar{\omega}}} \prod_{(i \to j) \in E(\bar{T})} \tilde{\phi}_{ij} , \tag{6.3.51}$$

and the positive helicity kernel as

$$\mathbb{S}_{\mathbf{h}}[\rho|\omega] := \sum_{T \in \mathcal{T}^{b}_{\rho,\omega}} \prod_{(i \to j) \in E(T)} \phi_{ij} , \tag{6.3.52}$$

upon which the reduced kernel splits as

$$\mathbb{S}_{n,d} = \mathbb{S}_{\tilde{\mathbf{h}}} \otimes \mathbb{S}_{\mathbf{h}} . \tag{6.3.53}$$

In particular, the reduced kernel admits a chiral splitting that respects the underlying decomposition of sums over (partial) colour-orderings of the single-copy gluon amplitudes.

Our results so far can be summarized as the following:

Theorem 1. *The tree-level S-matrix of general relativity in Minkowski spacetime admits the helicity-graded double copy representation:*

$$\mathcal{M}_{n,d} = \sum_{\substack{b\rho, b\omega \in \mathcal{S}(\mathbf{h}) \\ a\bar{\rho}, a\bar{\omega} \in \mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d}\, \mathcal{J}^{\tilde{\mathbf{h}}}_{n}[a\bar{\rho}b\rho]\, S_{n,d}[\hat{\rho}|\hat{\omega}]\, \mathcal{J}^{\tilde{\mathbf{h}}}_{n}[\bar{\omega}^{T} ab\omega] \prod_{i \in \mathbf{h}} h_i \prod_{j \in \tilde{\mathbf{h}}} \tilde{h}_j , \tag{6.3.54}$$

in terms of the integral kernel $S_{n,d}$*, which itself obeys a chiral splitting:*

$$S_{n,d}[\hat{\rho}|\hat{\omega}] = \mathcal{D}(\hat{\omega})\, \mathbb{S}_{\tilde{\mathbf{h}}}[\bar{\rho}|\bar{\omega}]\, \mathbb{S}_{\mathbf{h}}[\rho|\omega] , \tag{6.3.55}$$

in terms of $\mathbb{S}_{\tilde{\mathbf{h}}}$ *and* $\mathbb{S}_{\mathbf{h}}$ *the negative and positive helicity kernels* (6.3.51) *and* (6.3.52)*, respectively.*

The chiral splitting (6.3.55) of the integral kernel helps to make the connection between the representation (6.3.54) and other field theory KLT double copy representations in the literature more apparent. Although the standard version of the KLT double copy is as given in (6.3.47), a generalization of this formula in terms of 'shortened' KLT kernels was derived in [265–267, 324] using (the field theory limit of) string monodromy relations[9]. This generalized KLT double copy can also be stated in helicity-graded form as.

$$\mathcal{M}_{n,d} = (-1)^{n-3} \sum_{\hat{\alpha}\in S_{n-3}} \sum_{\substack{\rho\in\mathcal{S}(\mathbf{h}\setminus\{b,n\}) \\ \bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}}\setminus\{a\})}} \mathcal{A}_{n,d}[a\hat{\alpha}bn]\, \mathbb{S}^{\mathrm{KLT}}[\alpha|\rho(\alpha)] \\ \times\, \widetilde{\mathbb{S}}^{\mathrm{KLT}}[\bar{\omega}(\bar{\alpha})|\bar{\alpha}]\, \mathcal{A}_{n,d}[\bar{\omega}(\alpha)ab\rho(\alpha)n]\,, \quad (6.3.56)$$

where the first sum is over orderings $\hat{\alpha} = \alpha \cup \bar{\alpha}$ on $n-3$ letters, decomposed into $\alpha \in S_{n-d-3}$ and $\bar{\alpha} \in S_d$. The chirally-split KLT momentum kernels are

$$\mathbb{S}^{\mathrm{KLT}}[\alpha|\rho(\alpha)] := \prod_{i\in\mathbf{h}\setminus\{b,n\}} \left(s_{\alpha(i)\,b} + \sum_{j>i} \theta_{\rho(\alpha)}(\alpha(i), \alpha(j))\, s_{\alpha(i)\,\alpha(j)} \right), \quad (6.3.57)$$

and

$$\widetilde{\mathbb{S}}^{\mathrm{KLT}}[\bar{\omega}(\bar{\alpha})|\bar{\alpha}] := \prod_{i\in\tilde{\mathbf{h}}\setminus\{a\}} \left(s_{\bar{\alpha}(i)\,a} + \sum_{j<i} \theta_{\bar{\omega}(\bar{\alpha})}(\bar{\alpha}(j), \bar{\alpha}(i))\, s_{\bar{\alpha}(i)\,\bar{\alpha}(j)} \right), \quad (6.3.58)$$

with the ordering function θ defined as before in (6.3.49) and s_{ij} the Mandelstam invariants.

Clearly, this expression more closely resembles our representation (6.3.54) than the generic KLT double copy (6.3.47). The sums over chirally-split orderings $\rho, \bar{\omega}$ also appear in our formula, although the overall sum over orderings $\hat{\alpha} \in S_{n-3}$ does not appear in our expression. Instead, we have an *integral* over the moduli space of rational maps from $\mathbb{P}^1$ to $\mathbb{PT}$, as well as additional complementary sums over orderings $\bar{\rho}$ and ω.

These differences are not so stark as they may seem, and it is interesting to speculate that the two formulae (6.3.54) and (6.3.56) are in fact equivalent. The moduli integral in (6.3.54) is *completely localised* against the delta function constraints appearing in the twistor wavefunctions, just as it is in the RSVW and Cachazo-Skinner formulae. These constraints are a helicity-graded version of the *scattering equations* – rational constraints localizing n marked points on $\mathbb{P}^1$ in terms of on-shell kinematic data – so the moduli integral is really a sum over solutions to these helicity-graded scattering equations. At N^{d-1}MHV, the number of such solution is given by the Eulerian number $E(n-3, d-1)$ [303–305], so the total number of terms summed in (6.3.54) is $E(n-3, d-1) \times (d\,!)^2 \times ((n-d-2)!)^2$. This is generally higher than the number of term in the sums (6.3.56) and (6.3.47), however there may be some form of redundancy in the solutions owing to a version of KLT orthogonality [9, 294].

While these speculations suggest a very close relationship between the chirally split, integral kernel manifestation of double copy given by (6.3.54) and the more standard momentum space formulation of (6.3.56), their equivalence is, for generic d, purely conjectural. However,

[9]We will use one particular version of this generalization, a special case of which appeared much earlier in [325].

for the MHV helicity configuration (corresponding to $d = 1$) where (6.3.56) coincides with the un-graded KLT double copy (6.3.47), we can make this equivalence completely precise.

6.3.4 Example: the MHV configuration

The MHV helicity configuration is particularly straightforward to analyse, as the moduli integrals can be performed explicitly. In this case, the various ingredients of (6.3.54) simplify considerably.

Fixing $d = 1$, without loss of generality let $\tilde{\mathbf{h}} = \{1, 2\}$ and then we can choose $a = 2$, $y = 1$. Furthermore, the orderings $\bar{\rho}$, $\bar{\omega}$ on $\tilde{\mathbf{h}} \setminus \{a\}$ are now an ordering on a single letter; namely, $\bar{\rho} = \bar{\omega} = 1$, and thus $\bar{\omega}^* = 1$ trivially. This means hat for the $d = 1$ configuration,

$$\mathcal{D}(\hat{\omega}) = (b\,1)\,(b\,2)\,(t\,1)\,(t\,2)\,, \tag{6.3.59}$$

and

$$\mathbb{S}_{\tilde{\mathbf{h}}}[1|1] = \frac{\langle\lambda(\sigma_1)\,\lambda(\sigma_2)\rangle}{(1\,2)}\,\frac{1}{(b\,1)\,(b\,2)\,(t\,1)\,(t\,2)}\,. \tag{6.3.60}$$

Furthermore, the residual $GL(2,\mathbb{C})$ freedom in the parametrization of the map moduli $U^A_{\mathbf{a}}$ can be fixed by setting

$$U^A_{\mathbf{a}} = \left(x^{\beta\dot{\alpha}}\,\delta_{\beta\mathbf{a}},\,\delta_{\alpha\mathbf{a}}\right)\,, \tag{6.3.61}$$

where this choice of gauge for the moduli identifies the anti-self-dual $SL(2,\mathbb{C})$ spinor indices $\alpha, \beta, \ldots$ with the holomorphic homogeneous coordinate indices $\mathbf{a}, \mathbf{b}, \ldots$ of $\mathbb{P}^1$; the four $x^{\alpha\dot{\alpha}}$ are the remaining, gauge-fixed moduli of the degree one map.

With these choices, the $d = 1$ double copy representation (6.3.54) simplifies to

$$\begin{aligned}\mathcal{M}_{n,1} = \sum_{b\rho, b\omega\in\mathcal{S}(\mathbf{h})} \int d^4x\,(1\,2)^6\,\mathrm{pt}_n[1b\rho2]\,\mathrm{pt}_n[2b\omega1]\left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}}\prod_{(i\to j)\in E(T)} -t_i\,t_j\,[i\,j]\,(i\,j)\right]\\ \times\prod_{k=1,2} t_k^5\,dt_k\,\bar{\delta}^2(\kappa_k - t_k\,\sigma_k)\,e^{i\,t_k\,x^{\alpha\dot{\alpha}}\,\sigma_{\alpha\,k}\,\tilde{\kappa}_{\dot{\alpha}\,k}}\prod_{l=3}^{n}\frac{dt_l}{t_l^3}\,\bar{\delta}^2(\kappa_l - t_l\,\sigma_l)\,e^{i\,t_l\,x^{\alpha\dot{\alpha}}\,\sigma_{\alpha\,l}\,\tilde{\kappa}_{\dot{\alpha}\,l}}\,.\end{aligned} \tag{6.3.62}$$

At this stage, all of the integrals can be performed explicitly against holomorphic delta functions or as simple exponential integrals, leaving

$$\begin{aligned}\mathcal{M}_{n,1} &= (2\pi)^4\,\delta^4\!\left(\sum_{i=1}^n k_i\right)\sum_{\rho,\omega\in S_{n-3}}\hat{\mathcal{A}}_{n,1}[1b\rho2]\left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}}\prod_{(i\to j)\in E(T)} -[i\,j]\,\langle i\,j\rangle\right]\hat{\mathcal{A}}_{n,1}[12b\omega]\\ &= (2\pi)^4\,\delta^4\!\left(\sum_{i=1}^n k_i\right)\sum_{\rho,\omega\in S_{n-3}}\hat{\mathcal{A}}_{n,1}[1b\rho2]\left[\sum_{T\in\mathcal{T}^b_{\rho,\omega}}\prod_{(i\to j)\in E(T)} s_{ij}\right]\hat{\mathcal{A}}_{n,1}[12b\omega]\,,\end{aligned} \tag{6.3.63}$$

where $\hat{\mathcal{A}}_{n,1}$ is the colour-ordered Parke-Taylor formula (5.1.5), stripped of its overall momentum-conserving delta functions.

Now, by Proposition 6.2.2 and the results of [267] it follows that

$$\sum_{T\in\mathcal{T}^{b}_{\rho,\omega}} \prod_{(i\to j)\in E(T)} s_{ij} = \prod_{\substack{j=3\\ i\neq b}}^{n} \sum_{\substack{i<_{b\rho}j\\ i<_{b\omega}j}} s_{ij} = S^{\mathrm{KLT}}[\rho|\omega]\,. \tag{6.3.64}$$

In particular, it means that (6.3.63) is equivalent to

$$\mathcal{M}_{n,1} = (2\pi)^4\,\delta^4\left(\sum_{i=1}^{n} k_i\right) \sum_{\rho,\omega\in S_{n-3}} \hat{\mathcal{A}}_{n,1}[1b\rho 2]\,S^{\mathrm{KLT}}[\rho|\omega]\,\hat{\mathcal{A}}_{n,1}[12b\omega]\,, \tag{6.3.65}$$

which reproduces the known field theory KLT double copy for MHV amplitudes [242]. A representation similar to this in terms of colour-kinematics dual numerators was also derived directly from the Hodges formula by Frost in [295]. Here we have demonstrated that our general N^{d-1}MHV double copy representation (6.3.54) is equivalent to the Berends-Giele-Kuijf formula for graviton scattering in the MHV sector.

6.4 Biadjoint scalar amplitudes in twistor space

It is a well-known fact that the *inverse* of the field theory KLT kernel encodes the tree-level S-matrix of biadjoint scalar (BAS) theory [9, 296, 302]; these tree-level scattering amplitudes can be decomposed into doubly colour-ordered partial amplitudes $m_n[\alpha|\beta]$ for $\alpha, \beta\in S_n$. In particular, written in terms of permutations ρ, ω on $n-3$ letters,

$$m_n[12b\rho|21b\omega] = (2\pi)^4\,\delta^4\left(\sum_{i=1}^{n} k_i\right) S^{-1}_{\mathrm{KLT}}[\rho|\omega]\,, \tag{6.4.66}$$

where the field theory KLT kernel for these orderings is given by (6.3.64), and S^{-1}_{KLT} is its inverse, computed via Proposition 6.2.3:

$$S^{-1}_{\mathrm{KLT}}[\rho|\omega] = \frac{1}{s_{34\cdots n}} \sum_{T\in\mathcal{BT}_{b\rho,b\omega}} \frac{1}{s_T} = \frac{1}{s_{12}} \sum_{T\in\mathcal{BT}_{b\rho,b\omega}} \frac{1}{s_T}\,, \tag{6.4.67}$$

where the second equality follows on the support of overall momentum conservation.

In light of this connection between inverted field theory KLT kernels and BAS tree-level amplitudes, it seems natural to ask if there is some way to encode BAS scattering amplitudes in twistor space by inverting the integral kernel (6.3.55). Remarkably, this is indeed the case, resulting in a series of representations, graded by degree, all of which encode BAS tree amplitudes that see the degree only through their colour-orderings. We first present the resulting formula, and then prove that it is correct by looking at its factorization properties. The detailed proof is contained in Section 6.4.2.

6.4.1 BAS tree-amplitudes

To invert the integral kernel $S_{n,d}[\hat{\rho}|\hat{\omega}]$, we need to invert each of its constituent parts, as linear maps on the space of ordering: $\mathcal{D}(\hat{\omega})$, $\mathbb{S}_{\tilde{\mathbf{h}}}[\bar{\rho}|\bar{\omega}]$ and $\mathbb{S}_{\mathbf{h}}[\rho|\omega]$. The inversion of $\mathcal{D}(\hat{\omega})$ is trivial,

$$\mathcal{D}^{-1}(\hat{\omega}) = \frac{(y\,\omega^*)\,(\bar{\omega}^*\,t)}{(a\,b)\,(\omega^*\,\bar{\omega}^*)\,(b\,y)\,(a\,t)\,(y\,t)^2} \prod_{\substack{k\in\mathbf{h}\setminus\{b,t\}\\ l\in\tilde{\mathbf{h}}\setminus\{a,y\}}} (k\,l)^2\,, \tag{6.4.68}$$

while the inversion of the chiral factors $\mathbb{S}_{\tilde{\mathbf{h}}}$, $\mathbb{S}_{\mathbf{h}}$ is accomplished using Proposition 6.2.3.

This gives

$$\mathbb{S}_{\mathbf{h}}^{-1}[\rho|\omega] = \frac{1}{\phi_{\text{total}}} \sum_{T\in\mathcal{BT}_{b\rho,b\omega}} \frac{1}{\phi_T}\,, \tag{6.4.69}$$

and

$$\mathbb{S}_{\tilde{\mathbf{h}}}^{-1}[\bar{\rho}|\bar{\omega}] = \frac{1}{\tilde{\phi}_{\text{total}}} \sum_{\bar{T}\in\mathcal{BT}_{a\bar{\rho},a\bar{\omega}}} \frac{1}{\tilde{\phi}_{\bar{T}}}\,. \tag{6.4.70}$$

Here, recall that ϕ_{ij} and $\tilde{\phi}_{ij}$ are the weights (6.3.39) and (6.3.40). The 'total' quantities appearing as prefactors in these expressions are

$$\phi_{\text{total}} = \sum_{\substack{i,j\in\mathbf{h}\\ i\neq j}} \phi_{ij}\,, \qquad \tilde{\phi}_{\text{total}} = \sum_{\substack{i,j\in\tilde{\mathbf{h}}\\ i\neq j}} \tilde{\phi}_{ij}\,, \tag{6.4.71}$$

while ϕ_T, $\tilde{\phi}_{\bar{T}}$ denote the product over all edge weights appearing in the binary trees T, $\bar{T}$, respectively. Here each edge weight is $\phi_E = \sum_{i,j\in I(E)} \phi_{ij}$ where $I(E)$ is the set of leaves *upstream* from E, and similarly for $\tilde{\phi}_{\tilde{E}}$.

Collecting these results, one has

$$S_{n,d}^{-1}[\hat{\rho}|\hat{\omega}] = \mathcal{D}^{-1}(\hat{\omega})\,\mathbb{S}_{\mathbf{h}}^{-1}[\rho|\omega]\,\mathbb{S}_{\tilde{\mathbf{h}}}^{-1}[\bar{\rho}|\bar{\omega}]\,, \tag{6.4.72}$$

which is an inverse of the integral kernel (6.3.42) in the sense that

$$\sum_{\substack{\omega\in S_{n-d-2}\\ \bar{\omega}\in S_d}} S_{n,d}[\hat{\rho}|\hat{\omega}]\,S_{n,d}^{-1}[\hat{\omega}|\hat{\alpha}] = \sum_{\substack{\omega\in S_{n-d-2}\\ \bar{\omega}\in S_d}} S_{n,d}^{-1}[\hat{\alpha}|\hat{\omega}]\,S_{n,d}[\hat{\omega}|\hat{\rho}] = \delta_{\rho\alpha}\,\delta_{\bar{\rho}\bar{\alpha}}\,, \tag{6.4.73}$$

where the Kronecker deltas represent the identity on the space of linear maps between permutations.

Armed with the inverse integral kernel (6.4.72), we are now in a position to state the following result:

Theorem 2. *Let*

$$m_{n,d}[a\bar{\rho}b\rho|\bar{\omega}^T ab\omega] := \int d\mu_d\, S_{n,d}^{-1}[\hat{\rho}|\hat{\omega}] \prod_{i=1}^{n} \varphi_i(Z(\sigma_i))\,D\sigma_i\,, \tag{6.4.74}$$

for $\varphi_i \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-2))$. *This expression is equal to the doubly colour-ordered tree-level partial amplitudes of BAS theory, in the sense that*

$$m_{n,d}[a\bar{\rho}b\rho|\bar{\omega}^{\mathrm{T}}ab\omega] = m_n[a\bar{\rho}b\rho|\bar{\omega}^{\mathrm{T}}ab\omega]\,, \tag{6.4.75}$$

for each $d = 1, \dots, n-3$.

We will prove this theorem in the following subsection, but for now, let us make a few elementary observations about the formula (6.4.74). Firstly, it is easy to check that the formula is mathematically well-defined, in the sense that all of the integrals make sense projectively.

To see this, observe that external states in this formula are still partitioned into 'positive helicity' and 'negative helicity' sets, $\mathbf{h}$ and $\tilde{\mathbf{h}}$, respectively, despite the fact that there is no helicity associated with the massless scalars being described by the twistor wavefunctions φ_i. Now, $\mathcal{D}^{-1}(\hat{\omega})$ carries weight -2 in $\{a, b, t, y\}$, weight $2(d-1)$ in all elements of $\mathbf{h} \setminus \{b, t\}$ and weight $2(n-d-3)$ in all elements of $\tilde{\mathbf{h}} \setminus \{a, y\}$. Meanwhile, $\mathbb{S}_{\mathbf{h}}^{-1}$ carries weight $-2(n-d-2)$ in each element of $\tilde{\mathbf{h}} \setminus \{a, y\}$ and $\mathbb{S}_{\tilde{\mathbf{h}}}^{-1}$ carries weight $2d$ in each element of $\tilde{\mathbf{h}} \cup \{b, t\}$. Combined with the scaling weight $-2d$ of each twistor wavefunction and the weight 2 holomorphic forms $D\sigma_i$, it follows that the integrand of (6.4.74) is a $(1,1)$-form, homogeneous of weight zero in each insertion σ_i on the Riemann sphere.

Furthermore, when the twistor wavefunctions correspond to massless scalar momentum eigenstates

$$\varphi_i(Z) = \int_{\mathbb{C}^*} t_i\, dt_i\, \bar{\delta}^2(\kappa_i - t_i\,\lambda)\, e^{it_i\,[\mu\, i]}\,, \tag{6.4.76}$$

the same counting of integrals versus delta functions that holds for the RSVW and Cachazo-Skinner formulae shows that all of the integrals appearing in (6.4.74) are localized against delta functions, with four remaining delta functions encoding momentum conservation.

In this sense, the formula for $m_{n,d}$ passes the most basic sanity tests, and it can also be evaluated explicitly when $d = 1$. In this case, following the same steps as in our evaluation of $\mathcal{M}_{n,1}$ above, the formula evaluates to

$$\begin{aligned} m_{n,1}[12b\rho|21b\omega] &= \frac{1}{s_{3\cdots n}} \sum_{T \in \mathcal{BT}_{b\rho,b\omega}} \frac{1}{s_T} \int d^4x\, e^{i(k_1+\cdots+k_n)\cdot x} \\ &= (2\pi)^4\, \delta^4\left(\sum_{i=1}^{n} k_i\right) \frac{1}{s_{12}} \sum_{T \in \mathcal{BT}_{b\rho,b\omega}} \frac{1}{s_T}\,, \end{aligned} \tag{6.4.77}$$

which is precisely the BAS tree-amplitude $m_n[12b\rho|21b\omega]$. A sort of 'parity symmetry' inherited from the graviton amplitude (6.3.54) (shown in Appendix C.1) means that we also obtain the concrete, correct expression for $d = n-3$.

The remarkable claim is that this is true *for all degrees* d: namely, (6.4.74) is equal to the BAS tree amplitude, which is sensitive to the degree of the underlying map to twistor space only through the arrangement of its double colour-orderings. Intuitively, the counting in degree can be interpreted as follows. Due to the specific structure $[a\bar{\rho}b\rho|\bar{\omega}^{\mathrm{T}}ab\omega]$ of the

$$S^{-1}_{n,d}[\hat\rho|\hat\omega] = \mathcal{D}^{-1}(\hat\omega) \sum_{\hat T = \tilde T \cup T}$$

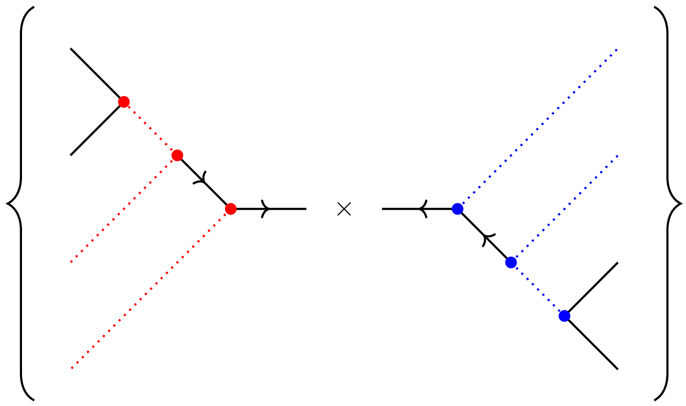


Figure 6.7: The tree decomposition of $S^{-1}_{n,d}[\hat\rho|\hat\omega]$, where the sum is over all trees compatible with the orderings on both $\tilde{\mathbf{h}}$ and $\mathbf{h}$.

colour-orderings, the corresponding BAS amplitude is divided into two halves: the binary trees on the left-hand set of $(d+1)$ particles, and the binary trees on the right-hand set of $(n-d-1)$ particles. These trees are treated independently, and so the degree of the map can be interpreted as enumerating the 'complexity' of the trees contained in the corresponding BAS formula.

6.4.2 Proof of BAS formula

The proof proceeds by using an argument based on BCFW recursion [326] in twistor space [137], applied to each tree sub-amplitude of $\mathcal{S}_{\mathbf{h}}[\rho|\omega]$, to establish that each of these sub-amplitudes is equivalent to a trivalent Feynman diagram contributing to the tree-level BAS scattering amplitude in the relevant colour-ordering. In this way, we establish that the formula (6.4.74) captures the full tree-level Feynman diagram expansion of BAS theory. Our notation and method will closely follow [28].

The relevant tree structure is apparent in (6.4.74), in which $S^{-1}_{n,d}[\hat\rho|\hat\omega]$ can be expanded as

$$S^{-1}_{n,d}[\hat\rho|\hat\omega] = \mathcal{D}^{-1}(\hat\omega) \sum_{\substack{T\in\mathcal{BT}_{b\rho,b\omega}\\ \tilde T\in\mathcal{BT}_{a\tilde\rho,a\tilde\omega}}} \frac{1}{\phi_{\text{total}}}\frac{1}{\phi_T} \times \frac{1}{\tilde\phi_{\text{total}}}\frac{1}{\tilde\phi_{\tilde T}}. \tag{6.4.78}$$

This decomposition is also illustrated in Figure 6.7. From this we define a tree sub-amplitude integrand to be, for the tree $\hat T = T \cup \tilde T$,

$$S^{-1\,\hat T}_{n,d}[\hat\rho|\hat\omega] = \mathcal{D}^{-1}(\hat\omega)\frac{1}{\phi_{\text{total}}}\frac{1}{\phi_T} \times \frac{1}{\tilde\phi_{\text{total}}}\frac{1}{\tilde\phi_{\tilde T}}. \tag{6.4.79}$$

The proof will then proceed on each of these subamplitudes, and show that each one is equal to the equivalent trivalent Feynman diagram in BAS theory. The formula (6.4.74) thus captures the full tree-level Feynman diagram expansion of the theory.

The factorization analysis is made easier by explicitly performing the moduli integrals for the $\mu^{\dot\alpha}(\sigma)$-components of the degree d holomorphic map to twistor space in (6.4.74), evaluated on the momentum eigenstates (6.4.76). This leads to the equivalent formula

$$\mathfrak{m}_{n,d}[a\bar\rho b\rho|\bar\omega^{T}ab\omega] = \int \frac{d^{2(d+1)}\lambda}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\,\mathcal{D}^{-1}(\hat\omega)\,\mathbb{S}^{-1}_{\mathbf{h}}[\rho|\omega]\,\mathbb{S}^{-1}_{\tilde{\mathbf{h}}}[\bar\rho|\bar\omega] \\ \times\,\delta^{2(d+1)}\left(\sum_{i=1}^{n} t_i\,\tilde\kappa_i^{\dot\alpha}\,\sigma_i^{\mathbf{a}(d)}\right)\prod_{i=1}^{n}\mathrm{D}\sigma_i\,t_i\,\mathrm{d}t_i\,\bar\delta^2(\kappa_i - t_i\,\lambda(\sigma_i))\,, \quad (6.4.80)$$

where an irrelevant overall factor of $(2\pi)^{2(d+1)}$ has been ignored. As the factorization analysis is also essentially local in the moduli space of holomorphic maps, it is also convenient to work in an affine patch $\sigma^{\mathbf{a}} = (1, u)$ of $\mathbb{P}^1$, under which the remaining components of the holomorphic map are written

$$\lambda_\alpha(\sigma) = \lambda_\alpha(u) = \sum_{r=0}^{d}\lambda_{\alpha\,r}\,u^r\,, \quad (6.4.81)$$

with the set of $2(d+1)$ complex parameters $\{\lambda_{\alpha\,r}\}$ the remaining moduli of the map. The formula (6.4.80) in this representation becomes

$$\mathfrak{m}_{n,d}[a\bar\rho b\rho|\bar\omega^{T}ab\omega] = \int \frac{d^{2(d+1)}\lambda}{\mathrm{vol}\,\mathrm{GL}(2,\mathbb{C})}\,\mathcal{D}^{-1}(\hat\omega)\,\mathbb{S}^{-1}_{\mathbf{h}}[\rho|\omega]\,\mathbb{S}^{-1}_{\tilde{\mathbf{h}}}[\bar\rho|\bar\omega] \\ \times\,\prod_{r=0}^{d}\delta^{2}\left(\sum_{i=1}^{n} t_i\,\tilde\kappa_i^{\dot\alpha}\,u_i^{r}\right)\prod_{i=1}^{n}\mathrm{d}u_i\,t_i\,\mathrm{d}t_i\,\bar\delta^2(\kappa_i - t_i\,\lambda(u_i))\,, \quad (6.4.82)$$

where all quantities are now evaluated on the affine patch, where $(i\,j) = u_i - u_j$ for all $i, j = 1, \ldots, n$.

The BCFW shift

Consider a BCFW-deformation of the n-particle kinematics: this is a one-parameter complex deformation which preserves the on-shell condition for each external 4-momentum as well as overall 4-momentum conservation. Without loss of generality, we can take this shift to act on the momenta of particles $1, n \in \mathbf{h}$ as[10]:

$$\kappa_1 \to \kappa_1 + z\,\kappa_n\,, \qquad \tilde\kappa_n \to \tilde\kappa_n - z\,\tilde\kappa_1\,, \quad (6.4.83)$$

where $z \in \mathbb{C}$ is the shift parameter. Under this shift, the quantity $\mathfrak{m}_{n,d}$ becomes a rational function of z, $\mathfrak{m}_{n,d}(z)$. We say that such a shift is *admissible* if $\mathfrak{m}_{n,d}(z)$, described by the formula (6.4.74) under the shift (6.4.83), is analytic as $|z| \to \infty$ – in other words, there is no pole at $z \to \infty$.

[10]The proof is qualitatively un-changed if we take the two shifted particles to be in $\tilde{\mathbf{h}}$. We expect that taking the shifted particles in different sets would yield the same result, with added complexity in the argument.

General shift behaviour: Under the shift (6.4.83), the various constituents of the formula (6.4.82) pick up dependence on the shift parameter z; for instance, the delta functions

$$\bar{\delta}^2\left(\kappa_1 + z\,\kappa_n - t_1 \sum_{r=0}^{d} \lambda_r\, u_1^r\right) \prod_{r=0}^{d} \delta^2\left(\sum_{i=1}^{n} t_i \tilde{\kappa}_i\, u_i^r - z\, t_n\, \tilde{\kappa}_1\, u_n^r\right) \tag{6.4.84}$$

now have explicit z-dependence. The arguments of these delta functions are apparently divergent as $z \to \infty$, but this can be absorbed by introducing a new scaling parameter and affine coordinate on $\mathbb{P}^1$ for the particle 1, defined by

$$\hat{t}_1 := t_1 - z\, t_n\,, \qquad \hat{t}_1\, \hat{u}_1^d := t_1\, u_1^d - z\, t_n\, u_n^d\,, \tag{6.4.85}$$

which imply that

$$u_1 = \left(\frac{\hat{t}_1\, \hat{u}_1^d + z\, t_n\, u_n^d}{\hat{t}_1 + z\, t_n}\right)^{1/d} . \tag{6.4.86}$$

On the support of the (un-shifted) delta function

$$\bar{\delta}^2\left(\kappa_n - t_n \sum_{r=0}^{d} \lambda_r\, u_n^r\right) , \tag{6.4.87}$$

in (6.4.82), it then follows that

$$\kappa_1 + z\,\kappa_n - t_1 \sum_{r=0}^{d} \lambda_r\, u_1^r = \kappa_1 - \hat{t}_1 \sum_{r=0}^{d} \lambda_r \left(\frac{d-r}{d}\, u_n^r + \frac{r}{d}\, \hat{u}_1^d\, u_n^{r-d}\right) + O(z^{-1})\,, \tag{6.4.88}$$

and

$$\sum_{i=1}^{n} t_i \tilde{\kappa}_i\, u_i^r - z\, t_n\, \tilde{\kappa}_1\, u_n^r = \sum_{i=2}^{n} t_i\, \tilde{\kappa}_i\, u_i^r + \hat{t}_1\, \tilde{\kappa}_1 \left(\frac{d-r}{d}\, u_n^r + \frac{r}{d}\, \hat{u}_1^d\, u_n^{r-d}\right) + O(z^{-1})\,. \tag{6.4.89}$$

In particular, the arguments of the delta functions (6.4.84) are finite in the $z \to \infty$ limit when expressed in terms of the new variables (6.4.85).

One must now account for the additional z-dependence in (6.4.82). With the assumption that $1, n \in \mathbf{h}$, we can also take $1, n \neq b, t$ (where $b, t \in \mathbf{h}$ are the arbitrarily-chosen points singled out in the definition of $\mathcal{D}^{-1}(\hat{\omega})$ and ϕ_{ij}), in which case only $\mathcal{D}^{-1}(\hat{\omega})$ and $\mathbb{S}_{\mathbf{h}}^{-1}$ in the first line of (6.4.82) have any z-dependence. It is easy to see that

$$\begin{aligned}\mathcal{D}^{-1}(\hat{\omega}) \xrightarrow{z\to\infty} &\frac{(u_y - u_{\omega^*})\,(u_{\bar{\omega}^*} - u_t)}{(u_a - u_b)\,(u_{\omega^*} - u_{\bar{\omega}^*})\,(u_b - u_y)\,(u_a - u_t)\,(u_y - u_t)^2} \\ &\times \prod_{j\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_n - u_j)^4 \prod_{\substack{k\in\mathbf{h}\setminus\{1,n,b,t\} \\ l\in\tilde{\mathbf{h}}\setminus\{a,y\}}} (u_k - u_l)^2 + O(z^{-1})\,,\end{aligned} \tag{6.4.90}$$

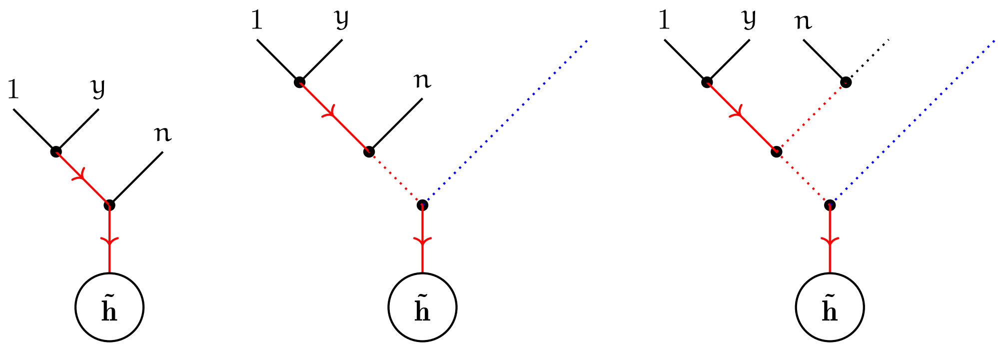


Figure 6.8: Examples of the type of shift configuration for the points 1 and n that are admissible. Note that 1 has a unique neighbour y, and the next closest point is n. The red lines indicate a decay in z^{-1} for the propagator, whilst the blue lines indicate that the lines go as z^0. Overall, the large-z falloff due to these propagators will be at least z^{-2}, rendering the large-z falloff of at least $m_{n,d}(z) \sim z^{-1}$. Note that we only consider cases with at least two points in $\mathbf{h}$, otherwise we are already in the $\overline{\text{MHV}}$ sector which we can evaluate directly from (6.4.77).

while the z-dependence of $\mathbb{S}_{\mathbf{h}}^{-1}$ is inherited from that of the weights ϕ_{ij}:

$$\phi_{ij} \xrightarrow{z\to\infty} -t_i\, t_j\, [i\, j]\, (u_i - u_j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_i - u_l)\,(u_j - u_l)\,, \qquad i,j \neq 1,n\,, \tag{6.4.91}$$

$$\phi_{1j} \xrightarrow{z\to\infty} -z\, t_n\, t_j\, [1\, j]\, (u_n - u_j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_n - u_l)\,(u_j - u_l) + O(z^0)\,, \qquad j \neq n\,, \tag{6.4.92}$$

$$\phi_{1j} \xrightarrow{z\to\infty} z\, t_n\, t_j\, [1\, j]\, (u_n - u_j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_n - u_l)\,(u_j - u_l) + O(z^0)\,, \qquad j \neq 1\,, \tag{6.4.93}$$

and

$$\phi_{1n} \xrightarrow{z\to\infty} -t_n\, \hat{t}_1\, [1\, n]\, u_n \left(\frac{\hat{u}_1^d}{u_n^d} + 1\right) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_n - u_l)^2 + O(z^{-1})\,, \qquad j \neq n\,. \tag{6.4.94}$$

In addition, we must account for potential large-z dependence elsewhere in the integration measure:

$$du_1 \xrightarrow{z\to\infty} \frac{\hat{t}_1}{z\, t_n}\, u_n^{1-d}\, \hat{u}_1^{d-1}\, d\hat{u}_1 + O(z^{-2})\,, \tag{6.4.95}$$

and

$$t_1\, dt_1 \xrightarrow{z\to\infty} z\, t_n\, d\hat{t}_1 + O(z^0)\,, \tag{6.4.96}$$

where terms which wedge to zero in (6.4.82) have been dropped in the last expression. Finally, observe that as the shifted particles $1, n \in \mathbf{h}$, there is no z-dependence coming from the factor $\mathbb{S}_{\tilde{\mathbf{h}}}^{-1}$.

Admissible shifts: Consider each binary tree over $\mathbf{h}$ in the sum appearing in (6.4.82) separately. From (6.4.90) – (6.4.96), it follows that the tree-independent constituents of $m_{n,d}(z)$ contribute large-z behaviour of the form z^0. The shift (6.4.83) is *admissible* for the given tree if the tree is such that it contains at least two factors of ϕ_{1i}^{-1} or ϕ_{ni}^{-1} for some $i \neq 1, n$. Examples of trees, where 1 has a unique nearest neighbour, are shown in Figure 6.8. If a given binary tree in the sum is such that the shift on $1, n$ is not admissible, then we simply shift two other particles in $\mathbf{h}$ such that the shift is admissible for this tree.

Overall, this means that choosing an admissible shift for each binary tree in the sum gives large-z behaviour

$$m_{n,d}(z)\Big|_{\text{admissible}}^{\hat{T}} \xrightarrow{z\to\infty} O(z^{-2})\,. \tag{6.4.97}$$

This means that, by choosing an admissible shift for each binary tree contributing to (6.4.82), there is no pole at $z \to \infty$ in the deformed quantity $m_{n,d}(z)$.

Factorisation

At this point, we restrict our focus to a single term in the sum over the set of binary trees $\mathcal{BT}_{b\rho,b\omega}$ on $\mathbf{h}$ and $\mathcal{BT}_{a\bar{\rho},a\bar{\omega}}$ on $\tilde{\mathbf{h}}$ compatible with the colour-orderings. Let this single term correspond to the binary tree $\hat{T} = T \cup \bar{T}$, and (without loss of generality) assume that the shift (6.4.83) is admissible for this binary tree. We have established that the contribution of this term to the shifted expression (6.4.82), which we will denote by $m_{n,d}^{\hat{T}}(z)$, has no pole as $z \to \infty$. Our next task is to establish that the only other poles contained in $m_{n,d}^{\hat{T}}(z)$ correspond to certain multi-particle factorization channels.

To do this, we observe that $m_{n,d}^{\hat{T}}$ is, by definition, associated to a degree d holomorphic rational curve in twistor space, and consider a degeneration of this underlying curve. If $m_{n,d}^{\hat{T}}$ develops poles in the modulus controlling this degeneration, then by standard arguments it will have a pole of the corresponding order in the shift parameter z corresponding to a multi-particle factorization compatible with the curve degeneration.

The degeneration of the curve has a standard description by modelling the underlying $\mathbb{P}^1$, with homogeneous coordinates $\sigma^{\mathbf{a}} = (1, u)$ in the chosen affine patch, as a conic in $\mathbb{P}^2$:

$$\Sigma_s = \{xy = s^2 z^2\} \subset \mathbb{P}^2, \tag{6.4.98}$$

where s is a parameter and $[x, y, z]$ are homogeneous coordinates on $\mathbb{P}^2$, related to $\sigma^{\mathbf{a}}$ by

$$(x, y, z) = \left((\sigma^{\mathbf{0}})^2, (\sigma^{\mathbf{1}})^2, \frac{\sigma^{\mathbf{0}}\,\sigma^{\mathbf{1}}}{s}\right)\,. \tag{6.4.99}$$

As $s \to 0$, $\Sigma_s \cong \mathbb{P}^1$ degenerates into two components:

$$\lim_{s\to 0} \Sigma_s = \Sigma_L \cup \Sigma_R. \tag{6.4.100}$$

These components are defined by $\Sigma_L = \{y = 0\}$ and $\Sigma_R = \{x = 0\}$, with intrinsic homogeneous coordinates

$$\sigma_L^{\mathbf{a}} = (z, x) = \sigma^{\mathbf{0}} \left(\frac{\sigma^{\mathbf{1}}}{s}, \sigma^{\mathbf{0}} \right), \qquad \sigma_R^{\mathbf{a}} = (z, y) = \sigma^{\mathbf{1}} \left(\frac{\sigma^{\mathbf{0}}}{s}, \sigma^{\mathbf{1}} \right). \tag{6.4.101}$$

The affine coordinate $u = \sigma^{\mathbf{1}}/\sigma^{\mathbf{0}}$ on Σ_s is related to the corresponding affine coordinates on $\Sigma_{L,R}$ by

$$u_L = \frac{s}{u}, \qquad u_R = s\,u. \tag{6.4.102}$$

The point at which the two components intersect in the degenerate $s \to 0$ limit (which we will call the node $\bullet$) is fixed at $u_{L,R} = 0$ in both coordinate patches.

Now, in the $s \to 0$ limit, the n marked points on Σ_s are distributed among the two resulting components Σ_L and Σ_R. Let n_L, n_R denote the number of initial marked points distributed on Σ_L and Σ_R, respectively, with $n_L + n_R = n$. To understand the dependence of the integrand of $m_{n,d}^{\hat{T}}$ on s, it is useful to observe that holomorphic separations $(u_i - u_j)$ between marked points behave differently depending on where those points wind up in the degenerate limit. It follows that

$$u_i - u_j = \begin{cases} s\,\frac{u_{jL} - u_{iL}}{u_{iL}\,u_{jL}}, & i, j \in L \\ \frac{u_{iR} - u_{jR}}{s}, & i, j \in R \\ \frac{s^2 - u_{iL} u_{jR}}{s\,u_{iL}}, & i \in L, j \in R \end{cases}, \tag{6.4.103}$$

in terms of the affine coordinates on Σ_L and Σ_R. In addition, various other ingredients in $m_{n,d}^{\hat{T}}$ have standard behaviour in the degenerate limit which were determined in [28]. The invariant measure

$$d\nu = \frac{1}{\mathrm{vol}\,SL(2,\mathbb{C})} \prod_{i=1}^{n} du_i\,, \tag{6.4.104}$$

on the marked point locations behaves as

$$d\nu = s^{n_L - n_R - 4}\,ds^2\,d\nu_L\,d\nu_R \prod_{i\in L} \frac{1}{u_{iL}^2} + O(s^{n_L - n_R - 2})\,, \tag{6.4.105}$$

where

$$d\nu_L := \frac{1}{\mathrm{vol}\,SL(2,\mathbb{C})} \prod_{i\in L\cup\bullet} du_{iL}\,, \qquad d\nu_R := \frac{1}{\mathrm{vol}\,SL(2,\mathbb{C})} \prod_{i\in R\cup\bullet} du_{iR}\,. \tag{6.4.106}$$

In particular, in the degenerate limit, the node $\bullet$ becomes a marked point, fixed by the choice of affine coordinate to be located at the origin in the affine patches on both Σ_L and Σ_R.

The behaviour of the degree d holomorphic map $Z : \Sigma_s \to \mathbb{PT}$ in the $s \to 0$ limit is also easily deduced [28]. To do this, a conveniently re-scaled set of map moduli $\{\mathcal{Z}_r\}$ can be defined by taking

$$\mathcal{Z}_r = s^r\,U_{d_L - r}\,, \qquad \mathcal{Z}_\bullet = U_{d_L}\,, \qquad \mathcal{Y}_r = s^r\,U_{d_L + r}\,, \tag{6.4.107}$$

for $d_L + d_R = d$, under which the holomorphic map admits a degeneration compatible with that of Σ_s itself:

$$\begin{aligned} Z(u;s) &= \sum_{r=0}^{d} U_r\, u^r \\ &= u^{d_L}\left(\sum_{r=1}^{d_L} \mathcal{Z}_r\, u_L^r + \mathcal{Z}_\bullet + \sum_{b=1}^{d_R} \mathcal{Y}_b\, \frac{s^{2b}}{u_L^b}\right) \quad u \in \Sigma_L \\ &= u^{d_L}\left(\sum_{b=1}^{d_L} \mathcal{Z}_b\, \frac{s^{2b}}{u_R^b} + \mathcal{Z}_\bullet + \sum_{r=1}^{d_R} \mathcal{Y}_r\, u_R^r\right) \quad u \in \Sigma_R\,. \end{aligned} \tag{6.4.108}$$

In particular, note that as $s \to 0$ the holomorphic map $Z : \Sigma_s \to \mathbb{PT}$ degenerates into a degree d_L holomorphic map on Σ_L and a degree d_R holomorphic map on Σ_R, up to the shared overall factor of u^{d_L}.

Implementing the re-scaling (6.4.107) at the level of the measure on the map moduli introduces explicit s-dependence:

$$\frac{d^{4(d+1)}U}{\mathrm{vol}\,\mathbb{C}^*} = s^{2d_L(d_L+1)}\, s^{2d_R(d_R+1)}\, \frac{d^4\mathcal{Z}_\bullet}{\mathrm{vol}\,\mathbb{C}^*} \prod_{r=1}^{d_L} d^4\mathcal{Z}_r \prod_{b=1}^{d_R} d^4\mathcal{Y}_b\,, \tag{6.4.109}$$

which nearly represents a splitting of the measure into measures for two distinct maps of degree d_L and d_R, respectively. The obstruction to this splitting is due to the fact that the node $\bullet \in \Sigma_{L,R}$ is mapped to the *same* point $\mathcal{Z}_\bullet \in \mathbb{PT}$ for both degenerate maps. The image of the node on the Σ_R side can trivially be disentangled by inserting a factor of unity into the integrand:

$$1 = \int D^3\mathcal{Y}_\bullet\, D^3Z\, \frac{dv}{v}\, r^3\, dr\, \bar{\delta}^4(Z - v\,\mathcal{Z}_\bullet)\, \bar{\delta}^4(Z - r\,\mathcal{Y}_\bullet)\,. \tag{6.4.110}$$

The holomorphic delta functions here set

$$\mathcal{Y}_\bullet = \frac{v}{r}\,\mathcal{Z}_\bullet\,, \tag{6.4.111}$$

so the point $\mathcal{Y}_\bullet \in \mathbb{PT}$ is (projectively) just another copy of the image of the node as $s \to 0$. Observing that

$$D^3\mathcal{Y}_\bullet = \frac{d^4\mathcal{Y}_\bullet}{\mathrm{vol}\,\mathbb{C}^*}\,, \tag{6.4.112}$$

including the factor of unity (6.4.110) allows us to completely factorize the measure on the map moduli into degree d_L and d_R components:

$$D^3\mathcal{Y}_\bullet\, \frac{d^{4(d+1)}U}{\mathrm{vol}\,\mathbb{C}^*} = s^{2d_L(d_L+1)}\, s^{2d_R(d_R+1)}\, \frac{d^4\mathcal{Z}_\bullet}{\mathrm{vol}\,\mathbb{C}^*} \prod_{r=1}^{d_L} d^4\mathcal{Z}_r\, \frac{d^4\mathcal{Y}_\bullet}{\mathrm{vol}\,\mathbb{C}^*} \prod_{b=1}^{d_R} d^4\mathcal{Y}_b\,, \tag{6.4.113}$$

as desired.

However, there are still some inconvenient overall factors in the degenerate maps as

$s \to 0$. In particular, at this stage

$$Z(u)|_{\Sigma_L} = u^{d_L} \left(\mathcal{Z}_\bullet + \sum_{r=1}^{d_L} \mathcal{Z}_r \, u_L^r + O(s^2) \right) , \tag{6.4.114}$$

$$Z(u)|_{\Sigma_R} = u^{d_L} \, \frac{\nu}{r} \left(\mathcal{Y}_\bullet + \sum_{b=1}^{d_R} \mathcal{Y}_b \, u_R^b + O(s^2) \right) , \tag{6.4.115}$$

where we have re-scaled $\mathcal{Y}_b \to (\nu/r)\, \mathcal{Y}_b$ to obtain an overall factor of ν/r on the Σ_R component. These overall factors can be removed from the degenerate maps by exploiting the fact that all ingredients of $m^{\hat{T}}_{n,d}$ are homogeneous functions of $Z(u)$. By the ingredients

$$\varphi_i(Z(u_i)) \to \begin{cases} u_i^{-2d_L} \, \varphi_i(Z(u_i)) & \text{if } u_i \in \Sigma_L \\ u_i^{-2d_L} \, \frac{r^2}{\nu^2} \, \varphi_i(Z(u_i)) & \text{if } u_i \in \Sigma_R \end{cases} , \tag{6.4.116}$$

as well as

$$\phi_{ij} \to \begin{cases} (u_i \, u_j)^{-d_L} \, \phi_{ij} & \text{if } u_i, u_j \in \Sigma_L \\ (u_i \, u_j)^{-d_L} \, \frac{r}{\nu} \, \phi_{ij} & \text{if } u_{i/j} \in \Sigma_L, \; u_{j/i} \in \Sigma_R \\ (u_i \, u_j)^{-d_L} \, \frac{r^2}{\nu^2} \, \phi_{ij} & \text{if } u_i, u_j \in \Sigma_R \end{cases} , \tag{6.4.117}$$

$$\tilde{\phi}_{ij} \to \begin{cases} (u_i \, u_j)^{d_L} \, \tilde{\phi}_{ij} & \text{if } u_i, u_j \in \Sigma_L \\ (u_i \, u_j)^{d_L} \, \frac{\nu}{r} \, \tilde{\phi}_{ij} & \text{if } u_{i/j} \in \Sigma_L, \; u_{j/i} \in \Sigma_R \\ (u_i \, u_j)^{d_L} \, \frac{\nu^2}{r^2} \, \tilde{\phi}_{ij} & \text{if } u_i, u_j \in \Sigma_R \end{cases} , \tag{6.4.118}$$

as well as

$$\prod_{b=1}^{d_R} d^4 \mathcal{Y}_b \to \left(\frac{\nu}{r} \right)^{4d_R} \prod_{b=1}^{d_R} d^4 \mathcal{Y}_b \, , \tag{6.4.119}$$

the overall factors can be removed from the degenerate limit of the holomorphic map. That is, after these re-scalings it follows that

$$Z(u_L) = \mathcal{Z}_\bullet + \sum_{r=1}^{d_L} \mathcal{Z}_r \, u_L^r + O(s^2) \, , \qquad Z(u_R) = \mathcal{Y}_\bullet + \sum_{b=1}^{d_R} \mathcal{Y}_b \, u_R^b + O(s^2) \, , \tag{6.4.120}$$

in the degenerate limit.

Isolating singular contributions: The goal is to determine the leading behaviour of the formula as $s \to 0$. In particular, we are interested in contributions which are singular; that is, terms which have poles in s. All contributions which are regular as $s \to 0$ are irrelevant as they do not correspond to kinematic poles and therefore will not result in poles in the BCFW shift parameter. At this point, we can tightly constrain the parameters $d_{L,R}$ and $n_{L,R}$ underlying the degeneration by demanding that we look at the *most* singular contributions.

In [28], it was established that

$$\frac{\det'(\mathbb{H})\det'(\mathbb{H}^\vee)}{\prod_{i=1}^n du_i} \sim s^{2d_L(d_L+1)}\, s^{2d_R(d_R+1)}\, s^{2d_L(n_R-n_L)}\, s^{n_R-n_L+2}\,, \tag{6.4.121}$$

as $s \to 0$. Now, we have shown that

$$\det'(\mathbb{H})\det'(\mathbb{H}^\vee) = \sum_{\substack{b\rho, b\omega\in\mathcal{S}(\mathbf{h})\\ a\bar\rho, a\bar\omega\in\mathcal{S}(\tilde{\mathbf{h}})}} \mathrm{PT}_n[a\bar\rho b\rho]\, S_{n,d}[\rho,\bar\rho|\omega,\bar\omega]\, \mathrm{PT}_n[\bar\omega^T a b\omega]\,, \tag{6.4.122}$$

and as it is straightforward to determine the behaviour of the Parke-Taylor factors as $s \to 0$, this allows us to infer the behaviour of $S_{n,d}$ and its inverse by comparison with (6.4.121).

For a generic $\mathrm{PT}_n[\alpha]$ on Σ_s, the behaviour as $s \to 0$ depends on how many times the colour-ordering α crosses the node in the degenerate limit. Clearly, it must cross an even number of times, $2m_b$, where m_b denotes the number of crossings or, equivalently, the number of colour traces which will appear on each of Σ_L and Σ_R as $s \to 0$ – see Figure 6.9. A straightforward calculation shows that

$$\frac{\mathrm{PT}_n[\alpha]}{\prod_{i=1}^n du_i} \sim s^{n_R-n_L+2m_b}\,, \tag{6.4.123}$$

as $s \to 0$. Thus, letting m_B denote the total number of colour-trace crossings for *both* Parke-Taylor factors in (6.4.122), it follows that

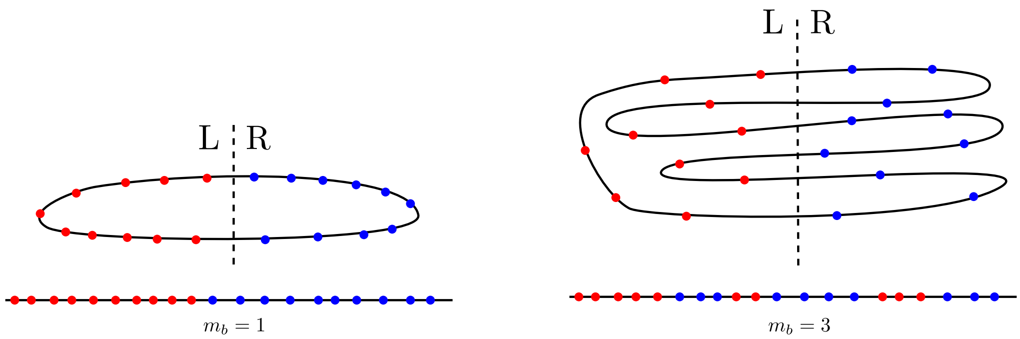
$$S^{-1}_{n,d}[\rho,\bar\rho|\omega,\bar\omega] \sim s^{2m_B}\, s^{-2d_L(d_L+1)}\, s^{-2d_R(d_R+1)}\, s^{2d_L(n_L-n_R)}\, s^{n_R-n_L-2}\,, \tag{6.4.124}$$

upon comparison with (6.4.121).

Since $m^T_{n,d}$ goes like $S^{-1}_{n,d}$, the most singular contributions will be those with the *fewest* colour-order crossings of the node; that is, with the smallest possible value of m_B. Now, configurations with $m_B = 0$ are not allowed; we have already established that (without loss of generality) $1, n \in \mathbf{h}$ must be on opposite sides of the degeneration, and this forces each Parke-Taylor factor in (6.4.122) to cross the degeneration at least once, meaning that $m_B \geqslant 2$ with the most singular contributions corresponding to $m_B = 2$. This can only happen if all points in $\tilde{\mathbf{h}}$ are on a single side of the degeneration; otherwise m_B will be greater than two. See Figure 6.10 for an example. In other words, if $\tilde{\mathbf{h}}$ is distributed across both Σ_L and Σ_R then this configuration cannot be the most singular as $s \to 0$.

So in order to study the most singular contributions to the formula, we can (again, without loss of generality) assume that all elements of $\tilde{\mathbf{h}}$ are located on Σ_R as $s \to 0$. By definition, this fixes $d_L = 0$ and $d_R = d$, and it is easy to see that this in turn fixes $n_L = 2$, $n_R = n - 2$: all other configurations will have vanishing support in terms of the various delta functions appearing in (6.4.82).

Having established that the most singular contribution to the integrand occurs for the configuration where $n_L = 2$ and $d_L = 0$, we can now determine precisely how the entire integrand of $m^{\hat{T}}_{n,d}$ behaves in the degenerate limit. Collecting (6.4.105), (6.4.113) and using



(a) The case in which the trace is broken the minimal number of times. In this case $m_b = 1$.

(b) For this colour ordering the trace is broken multiple times, and we end up with 3 trace components on each side, so $m_b = 3$.

Figure 6.9: Two examples of the possible trace configurations for a given distribution of points among the left and right sides, and their corresponding values of m_b. An easy way of counting m_b is counting how many connected traces end up (e.g.) on the right side.

$$\begin{array}{cc} \tilde{\mathbf{h}} & \mathbf{h} \\ a\ \tilde{\sigma} & b\ \sigma \\ \tilde{\rho}\ a & b\ \rho \end{array} \qquad \left.\begin{array}{l} m_b = 1 \\ \\ m_b = 2 \end{array}\right\} m_B = 3$$

Figure 6.10: When $\tilde{\mathbf{h}}$ is distributed among the two sides the values of m_B is always at least 3. Here this is illustrated in a simple configuration.

(6.4.116), (6.4.120), it follows that

$$\frac{d^{(4d+1)}U}{\mathrm{vol}\,\mathbb{C}^*}\,d\nu\prod_{i=1}^{n}\varphi_i(Z(u_i)) = s^{2d(d+1)-n}\,ds^2\left[d\nu_L\,\frac{d^4\mathcal{Z}_\bullet}{\mathrm{vol}\,\mathbb{C}^*}\prod_{i\in\mathbf{h}_L}\frac{\varphi_i(Z(u_{iL}))}{(u_\bullet-u_i)_L^2}\right]\left(\frac{\nu}{r}\right)^{4d-2n+4}$$
$$\times\left[d\nu_R\,\frac{d^4\mathcal{Y}_\bullet}{\mathrm{vol}\,\mathbb{C}^*}\prod_{b=1}^{d}d^4\mathcal{Y}_b\prod_{j\in\mathbf{h}_R\cup\tilde{\mathbf{h}}}\varphi_j(Z(u_{jR}))\right]+O(s^{2d(d+1)-n+2})\,,\quad(6.4.125)$$

where $\mathbf{h}_L\subset\mathbf{h}$ is the set comprising the two marked points which are left on Σ_L as $s\to 0$. From (6.4.124), we also have that

$$S^{-1}_{n,d}[\hat{\rho}|\hat{\omega}]\sim s^{-2d(d+1)+n-2}+O(s^{-2d(d+1)+n})\,,\qquad(6.4.126)$$

so that overall

$$m^{\hat{T}}_{n,d}=\int\frac{ds^2}{s^2}\left(1+O(s^2)\right)\,,\qquad(6.4.127)$$

in the degenerate limit.

In particular, this establishes that the maximally singular $m_B=2$ configuration, with $d_L=0$ and $n_L=2$, is the *only* singular configuration, having a simple pole in s^2. All other configurations will not contribute to the BCFW recursion; see Figure 6.11 for an illustration.

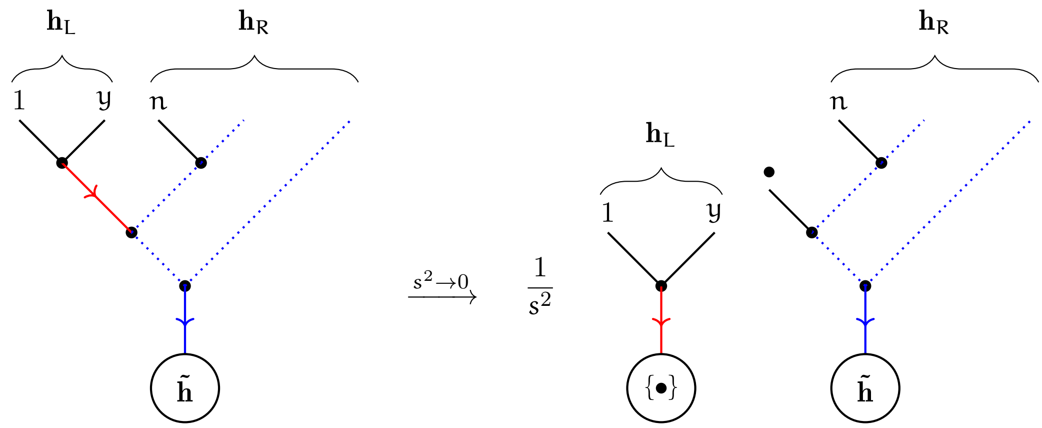


Figure 6.11: The general decomposition of the product of propagators into propagators on the two halves. On the left, the red propagator takes contributions only from $\mathbf{h}_L$, whereas the leading (in s^2) contributions to the blue propagators come only from $\mathbf{h}_R$ on the right. In effect this 'plucks' a branch from the tree.

Factorisation: We can now focus entirely on this singular configuration, and determine the precise behaviour of each ingredient in $S^{-1\,\hat{T}}_{n,d}$ as $s\to 0$. Of the three factors making up $S^{-1\,\hat{T}}_{n,d}$, the easiest to consider is $\mathbb{S}^{-1}_{\tilde{\mathbf{h}}}[\bar{\rho}|\bar{\omega}]$, since this only depends on points in $\tilde{\mathbf{h}}\cup\{b,t\}$, all of which can be taken to lie on Σ_R in the degenerate limit. Therefore, it follows from

$$\tilde{\phi}_{ij} = \frac{\nu^2}{r^2}\, s^{2d+3}\, \tilde{\phi}_{ij\,R} + O(s^{2d+5})\,, \tag{6.4.128}$$

that

$$\mathbb{S}_{\tilde{\mathbf{h}}}^{-1}[\bar{\rho}|\bar{\omega}] = \left(\frac{\nu}{r}\right)^{-2d} s^{-d(2d+3)}\, \mathbb{S}_{\tilde{\mathbf{h}}}^{-1}[\bar{\rho}|\bar{\omega}]_R\,, \tag{6.4.129}$$

where the subscript 'R' on quantities such as $\tilde{\phi}_{ij\,R}$ or $\mathbb{S}_{\tilde{\mathbf{h}}}^{-1}[\bar{\rho}|\bar{\omega}]_R$ indicates that these are now defined in the affine coordinate patch on Σ_R. Observe that the scaling (6.4.129) is independent of $\mathbf{h}_L$, the distribution of points on Σ_L.

Next, we consider the pre-factor $\mathcal{D}^{-1}(\hat{\omega})$. Without loss of generality[11], we can take all of the points $b, t, \omega^* \in \mathbf{h}$ to lie on Σ_R in the limit $s \to 0$. In this case, we have

$$\begin{aligned}
\mathcal{D}^{-1}(\hat{\omega}) &= \frac{s^{4-2(d-1)(n-d-3)}\,(u_{\omega^*}-u_y)_R\,(u_t-u_{\bar{\omega}^*})_R}{(u_a-u_b)_R\,(u_{\omega^*}-u_{\bar{\omega}^*})_R\,(u_b-u_y)_R\,(u_a-u_t)_R\,(u_y-u_t)_R^2} \\
&\quad\times \prod_{\substack{i\in\mathbf{h}_R\setminus\{b,t\}\\ j\in\tilde{\mathbf{h}}\setminus\{a,y\}}} (u_i-u_j)_R^2 \prod_{j\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_\bullet-u_j)_R^2 + \cdots \\
&= s^{4-2(d-1)(n-d-3)}\, \mathcal{D}^{-1}(\hat{\omega})_R \prod_{j\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_\bullet-u_j)_R^2 + \cdots\,,
\end{aligned} \tag{6.4.130}$$

where the '$+\cdots$' indicates terms which are higher-order in s.

Finally, we must account for the behaviour of $\mathbb{S}_{\mathbf{h}}^{-1}[\rho|\omega]$, which is the most complicated due to its dependence on $\mathbf{h}_L$. Using (6.4.103) and (6.4.117), one finds that

$$\phi_{ij} = \begin{cases} -s^{3-2d}\, t_i t_j\, [i\,j]\, \dfrac{(u_i-u_j)_L}{(u_\bullet-u_i)_L\,(u_\bullet-u_j)_L} \displaystyle\prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_\bullet-u_l)_R^2\,, & i,j\in L \\ -\dfrac{s^{1-2d}\,r^2}{\nu^2}\, t_i t_j\, [i\,j]\,(u_i-u_j)_R \displaystyle\prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_i-u_l)_R\,(u_i-u_l)_R\,, & i,j\in L \\ -\dfrac{s^{1-2d}\,r}{\nu}\, t_i t_j\, [i\,j]\,(u_\bullet-u_j)_R \displaystyle\prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_\bullet-u_l)_R\,(u_j-u_l)_R\,, & i\in L,\ j\in R \end{cases} \tag{6.4.131}$$

Recall that for a specified binary tree T

$$\mathbb{S}_{\mathbf{h}}^{-1\,T}[\rho|\omega] = \frac{1}{\phi_{\text{total}}} \prod_{E\in T} \frac{1}{\phi_E}\,, \tag{6.4.132}$$

where

$$\phi_E = \sum_{(i\to j)\in E} \phi_{ij}\,. \tag{6.4.133}$$

[11] It is a straightforward, if somewhat tedious, calculation to show that the same result is obtained with a generic distribution of these points.

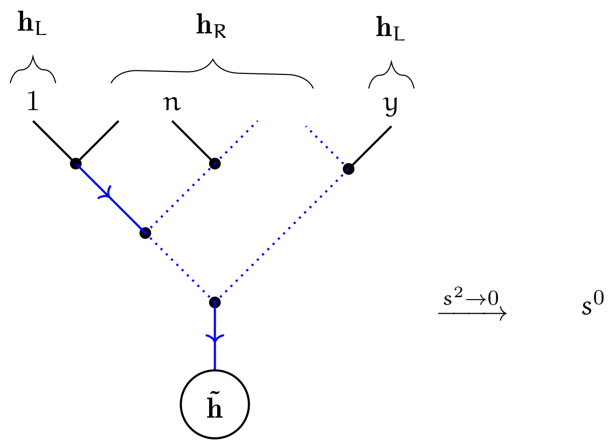


Figure 6.12: Example of a configuration where we have to split up the left and right leaves in the sub-amplitude, so that 1 and y are no longer adjacent. Note that this now has more blue propagators (scaling as s overall in the contribution from this factorisation channel, compared to s^{-1} coming from red propagators) than the unbroken configuration, and therefore will not create a pole (as all other contributions remain unchanged).

Any $\mathbf{h}_L$ which is not connected with respect to T (i.e., appearing as a leaf in the tree T) will involve more factors of ϕ_{ij}^{-1} with one index valued in L and the other valued in R and fewer factors of ϕ_{ij}^{-1} with both indices valued in L in contrast to connected contributions (i.e., where $\mathbf{h}_L$ forms a leaf of T).

Consequently, from (6.4.131), we see that such disconnected/non-leaf contributions will scale with a less singular power of s as $s \to 0$ than connected/leaf contributions; see Figure 6.12. However, we already established that the maximally-singular contribution to $m_{n,d}^{T}$ is in fact the only singular one, so to obtain the factorization behaviour of $\mathbb{S}_{\mathbf{h}}^{-1}$ corresponding to (6.4.126), we must take the terms in (6.4.132) with the most singular scaling possible, corresponding to $\mathbf{h}_L$ being a single leaf of T, as illustrated in Figure 6.11. This means that on Σ_L, the node $u_{\bullet L}$ is treated as being the only member of $\tilde{\mathbf{h}}_L$, while on Σ_R the node $u_{\bullet R}$ is treated as an additional element of $\mathbf{h}_R$.

Explicitly, this leads to

$$\mathbb{S}_{\mathbf{h}}^{-1\,T}[\rho|\omega] = s^{d(2n-2d-3)-n}\left(\frac{\nu}{r}\right)^{-2(d-n+2)} \prod_{j\in\tilde{\mathbf{h}}\setminus\{a,y\}} (u_\bullet - u_j)_R^{-2}\, \mathbb{S}_{\mathbf{h}_L}^{-1\,T_L}\, \mathbb{S}_{\mathbf{h}_R\cup\{\bullet_R\}}^{-1\,T_R} + \cdots\,, \tag{6.4.134}$$

where $\mathbb{S}_{\mathbf{h}_R\cup\{\bullet_R\}}^{-1\,T_R}$ is defined in the obvious way on Σ_R and $\mathbb{S}_{\mathbf{h}_L}^{-1\,T_L}$ is defined in terms of

$$\phi_{ij}^{L} := -t_i\, t_j\, [i\, j]\, \frac{(u_i - u_j)_L}{(u_i - u_\bullet)_L\, (u_j - u_\bullet)_L}\,, \tag{6.4.135}$$

for $\mathbf{h}_L = \{i, j\}$. The binary tree T_L is composed of a single cubic vertex with external edges

$\{i,j\} = \mathbf{h}_L$ (where one of i,j is 1 or n) and rooted at $\bullet_L$. It then immediately follows that

$$\mathbb{S}_{\mathbf{h}_L}^{-1\,T_L} = -\frac{(u_i - u_\bullet)_L\,(u_j - u_\bullet)_L}{t_i t_j\,[i\,j]\,(u_i - u_j)_L}\,, \tag{6.4.136}$$

from the extremely simple structure of T_L.

Collecting factors: At this stage, we have isolated the leading behaviour of all of the ingredients of $m_{n,d}^{\hat{T}}$ in the $s \to 0$ degeneration limit. Collecting (6.4.110), (6.4.125), (6.4.129), (6.4.130) and (6.4.134), one observes a remarkable cancellation among the various powers of s, ν and r to leave:

$$\begin{aligned} m_{n,d}^{\hat{T}} = \int \frac{ds^2}{s^2}\,D^3Z &\left[\frac{d^4\mathcal{Z}_\bullet}{\mathrm{vol}\,\mathbb{C}^*}\,d\nu_L\,\mathbb{S}_{\mathbf{h}_L}^{-1\,T_L}\,\nu\,d\nu\,\bar{\delta}^4(Z-\nu\,\mathcal{Z}_\bullet)\prod_{i\in\mathbf{h}_L}\frac{\varphi_i(Z(u_{iL}))}{(u_\bullet - u_i)_L^2}\right] \\ \times &\left[\frac{d^{4(d+1)}\mathcal{Y}}{\mathrm{vol}\,\mathbb{C}^*}\,d\nu_R\,\mathcal{D}_R^{-1}\,\mathbb{S}_{\mathbf{h}\cup\{\bullet_R\}}^{-1\,T_R}\,\mathbb{S}_{\tilde{\mathbf{h}}}^{-1\,\bar{T}}\,r\,dr\,\bar{\delta}^4(Z - r\,\mathcal{Y}_\bullet)\prod_{j\in\mathbf{h}_R\cup\tilde{\mathbf{h}}}\varphi_j(Z(u_{jR}))\right]\,, \end{aligned} \tag{6.4.137}$$

where we have dropped all regular terms as $s \to 0$, suppressed explicit dependence on the two colour-orderings and

$$d^{4(d+1)}\mathcal{Y} := d^4\mathcal{Y}_\bullet \prod_{b=1}^{d} d^4\mathcal{Y}_b\,, \tag{6.4.138}$$

denotes the measure on the map moduli corresponding to the right-hand branch of the degeneration.

Now, observe that as a distribution

$$\int_{\mathbb{C}^*} \nu\,d\nu\,\bar{\delta}^4(Z - \nu\,\mathcal{Z}_\bullet) = \bar{\delta}^3_{-2,-2}(Z, Z(u_{\bullet L}))\,, \tag{6.4.139}$$

where the right-hand-side is a holomorphic delta function, that is, a $(0,3)$-distribution on $\mathbb{P}^3$, with support where the node on Σ_L is mapped to the point $Z \in \mathbb{PT}$ (i.e., $Z(u_{\bullet L}) \propto Z$) which is homogeneous of weight -2 in both of its arguments. In (6.4.137), this distribution is integrated against both D^3Z and $du_{\bullet L}$. Consequently, viewed as a differential form

$$\bar{\delta}^3_{-2,-2}(Z, Z(u_{\bullet L})) \in \Omega^{0,2}(\mathbb{PT}_Z, \mathcal{O}(-2)) \otimes \Omega^{0,1}(\mathbb{P}^1_{\bullet_L}, \mathcal{O}(-2))\,, \tag{6.4.140}$$

where the line bundles in each factor are understood to be defined over the respective projective varieties.

Integration against $du_{\bullet L}$ reduces the distribution to a $(0,2)$-distribution with compact support on $\mathbb{PT}$. Together with the integral over D^3Z appearing in (6.4.137), this defines an exact pairing with twistor representatives of massless scalars, which, by the Penrose transform, are classes in $H^{0,1}(\mathbb{PT}, \mathcal{O}(-2))$. So given any $\varphi(Z) \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-2))$, the integral

$$\int_{\mathbb{PT}_Z \times \mathbb{P}^1_{\bullet_L}} D^3Z \, du_{\bullet L} \, \bar{\delta}^3_{-2,-2}(Z, Z(u_{\bullet L})) \, \varphi(Z) \,, \tag{6.4.141}$$

is simply a complex number. Observing that, similarly,

$$\int_{\mathbb{C}^*} r \, dr \, \bar{\delta}^4(Z - r\, \mathcal{Y}_\bullet) = \bar{\delta}^3_{-2,-2}(Z, Z(u_{\bullet R})) \,, \tag{6.4.142}$$

it follows that the integral over D^3Z is playing the role of a sum over a complete basis of massless, on-shell scalar wavefunctions inserted at the nodes on Σ_L and Σ_R, respectively.

Consequently, we can define the wavefunctions

$$\varphi_{\bullet_L}(Z(u_{\bullet L})) := \bar{\delta}^3_{-2,-2}(Z, Z(u_{\bullet L})) \,, \qquad \varphi_{\bullet_R}(Z(u_{\bullet R})) := \bar{\delta}^3_{-2,-2}(Z, Z(u_{\bullet R})) \,, \tag{6.4.143}$$

at each node. The factorisation behaviour (6.4.137) can then be written as

$$m^{\hat{T}}_{n,d} = \int D^3Z \, ds^2 \left[m^{T_L}_{3,0}(\mathbf{h}_L \cup \{\bullet_L\}) \, \frac{1}{s^2} \, m^{T_R \cup \bar{T}}_{n-1,d}(\mathbf{h}_R \cup \{\bullet_R\} \cup \tilde{\mathbf{h}}) + O(s^2) \right] \,, \tag{6.4.144}$$

where

$$m^{T_L}_{3,0}(\mathbf{h}_L \cup \{\bullet_L\}) = \int \frac{d^4 \mathcal{Z}_\bullet}{\mathrm{vol}\, \mathbb{C}^*} \, d\nu_L \prod_{i \in \mathbf{h}_L} \frac{1}{(u_i - u_\bullet)^2_L} \, \mathcal{S}^{-1\, T_L}_{\mathbf{h}_L} \prod_{j \in \mathbf{h}_L \cup \{\bullet_L\}} \varphi_j(Z(u_{jL})) \,, \tag{6.4.145}$$

and

$$m^{T_R \cup \bar{T}}_{n-1,d}(\mathbf{h}_R \cup \{\bullet_R\} \cup \tilde{\mathbf{h}}) = \int \frac{d^{4(d+1)} \mathcal{Y}}{\mathrm{vol}\, \mathbb{C}^*} \, d\nu_R \, \mathcal{D}^{-1}_R \, \mathcal{S}^{-1\, T_R}_{\mathbf{h} \cup \{\bullet_R\}} \, \mathcal{S}^{-1\, \bar{T}}_{\tilde{\mathbf{h}}} \prod_{i \in \mathbf{h}_R \cup \tilde{\mathbf{h}} \cup \{\bullet_R\}} \varphi_j(Z(u_{jR})) \,. \tag{6.4.146}$$

As the notation suggests, these two objects are simply the formula (6.4.74) for the contributions to the BAS tree amplitudes corresponding to the binary rooted trees T_L and $T_R \cup \bar{T}$, respectively. In the case of $m^{T_R \cup \bar{T}}_{n-1,d}$ given by (6.4.146), this is obvious by comparison with the ingredients of (6.4.74), but for $m^{T_L}_{3,0}$ given by (6.4.145) the correspondence is less clear, as this expression is defined for a degree-zero map, whereas (6.4.74) is defined for $d \geqslant 1$. In fact, $m^{T_L}_{3,0}$ is equal to the 3-point amplitude of BAS theory, as can be verified by evaluating on momentum eigenstate wavefunctions.

Lemma 6.4.1. *The quantity* $m^{T}_{3,0}$, *for* T *the trivial cubic graph with a single vertex, is equal to the 3-point tree amplitude of BAS theory, in the sense that*

$$m^{T}_{3,0} = \delta^4\left(\sum_{i=1}^{3} k_i \right) \,, \tag{6.4.147}$$

when evaluated on momentum eigenstates.

Proof: Without loss of generality, let T be the trivial cubic graph whose external legs are labeled by 1,2 and 3, and evaluate (6.4.145) on twistor momentum eigenstate representatives for the external wavefunctions. In this case, (6.4.145) is given by

$$m^{T}_{3,0} = \frac{1}{[1\,2]}\int \frac{d^4\mathcal{Z}_\bullet}{\mathrm{vol}\,\mathbb{C}^*}\,\frac{du_1\,du_2\,du_3}{\mathrm{vol}\,SL(2,\mathbb{C})}\,\frac{t_3\,dt_1\,dt_2\,dt_3}{(u_1-u_2)\,(u_2-u_3)\,(u_3-u_1)} \\ \times \prod_{i=1}^{3}\bar{\delta}^2(\kappa_i - t_i\,\lambda_\bullet)\,e^{i\,t_i\,[\mu_\bullet\,i]}\,. \quad (6.4.148)$$

The $SL(2,\mathbb{C})$ freedom in the integral can be used to eliminate all dependence on the marked points (u_1, u_2, u_3) with unit Jacobian, and the $\mathbb{C}^*$ freedom can be used to fix $t_3 = 1$ with unit Jacobian. The integrals over $d^4\mathcal{Z} = d^2\mu_\bullet\,d^2\lambda_\bullet$ can then be performed against the exponential and delta functions to leave

$$\frac{1}{[1\,2]}\int dt_1\,dt_2\,\delta^2(\kappa_1 - t_1\,\kappa_3)\,\delta^2(\kappa_2 - t_2\,\kappa_3)\,\delta^2(t_1\,\tilde{\kappa}_1 + t_2\,\tilde{\kappa}_2 + \tilde{\kappa}_3)\,, \quad (6.4.149)$$

ignoring an irrelevant overall factor of $(2\pi)^2$. The two scaling integrals can now be performed against the final set of delta functions to give

$$-\frac{1}{[1\,2]^2}\,\delta^2\left(\kappa_1 + \frac{[2\,3]\,\kappa_3}{[2\,1]}\right)\,\delta^2\left(\kappa_2 + \frac{[1\,3]\,\kappa_3}{[1\,2]}\right) = \delta^4\left(\sum_{i=1}^{3} k_i\right)\,, \quad (6.4.150)$$

as claimed. □

In other words, as the notation suggest, $m^{T_L}_{3,0}$ is indeed equal to the tree-level BAS 3-point amplitude, simply written in terms of a degree zero (as opposed to a degree one) holomorphic rational map to twistor space. As (6.4.144) captures all of the singular behaviour as $s \to 0$, it follows that

$$\operatorname*{Res}_{s\to 0}\, m^{\hat{T}}_{n,d}(\mathbf{h}\cup\tilde{\mathbf{h}}) = \sum_{\text{sing. trees}} \int D^3Z\, m^{T_L}_{3,0}(\mathbf{h}_L \cup \{Z\})\; m^{T_R\cup\bar{T}}_{n-1,d}(\mathbf{h}_R \cup \{Z\} \cup \tilde{\mathbf{h}})\,, \quad (6.4.151)$$

where the sum is over all decompositions of T into T_L and T_R containing the simple pole in s^2 as $s \to 0$. As showed above, the only such decompositions correspond to T_L being a cubic tree graph with a single vertex – see Figure 6.13.

This process can then be repeated on $m^{T_R\cup\bar{T}}_{n-1,d}$, splitting off further cubic tree graphs until the elements of $\mathbf{h}$ are exhausted. At this point, we have a $\overline{\text{MHV}}$ amplitude on $\bar{T}$ with two points remaining in $\mathbf{h}$, which we know is equal to the corresponding BAS amplitude via (6.4.77) and 'partity symmetry' (shown in Appendix C.1). We have therefore decomposed $m^{T}_{n,d}$ totally into single-vertex cubic graphs on $\mathbf{h}$ with each decomposition taking the form of (6.4.151), and the expected Feynman diagram contribution from the tree on $\tilde{\mathbf{h}}$, viewed as a $\overline{\text{MHV}}$ amplitude.

Now, with the starting expression $m^{\hat{T}}_{n,d}$ evaluated on momentum eigenstates, it is known that factorisation of the form (6.4.151) in twistor space is equivalent to a multi-particle

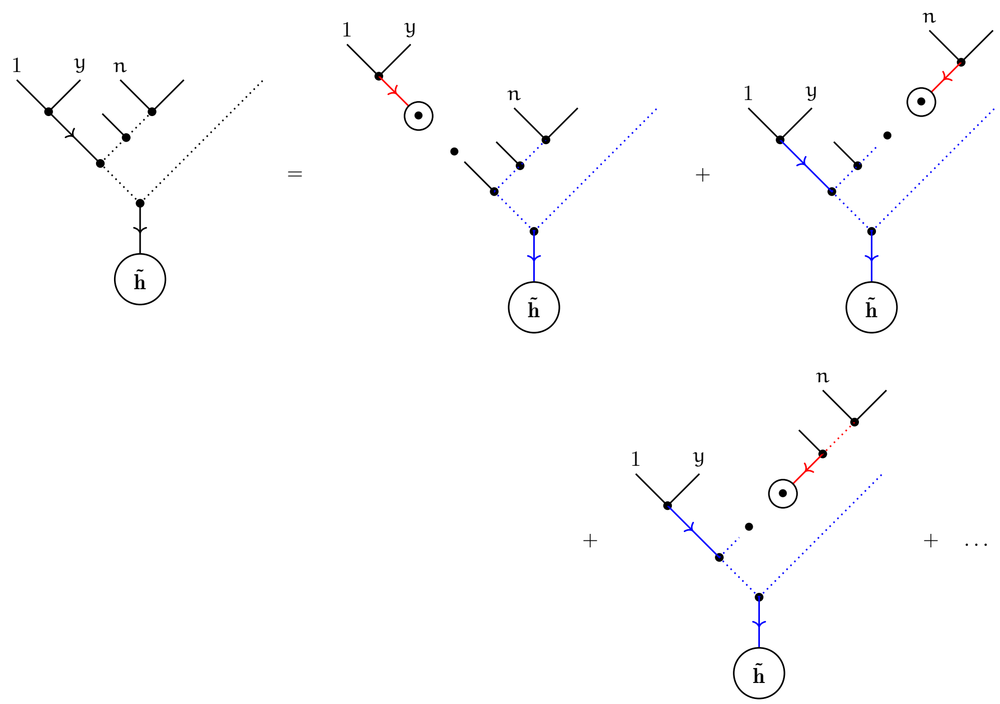


Figure 6.13: Possible poles picked up by a tree subamplitude . Note that only the first line contributes because the rest contain $d = 0$ amplitudes with more than 3 external particles, which evaluate to zero.

factorization channel in momentum space [251, 327–329]. Thus, we have established that $m^{\hat{T}}_{n,d}$ contains all of the factorization channels of the Feynman diagram $\hat{T}$ contributing to the BAS tree amplitude m_n, with the correct residues when evaluated on the singularities of those channels. The same argument works for every other term in the sum over binary trees in (6.4.78). It is easy to see that there are no other singularities in the formula – we have already established that there are no poles for large values of the BCFW shift parameter, and other spurious poles would spoil soft limits, which we check in Appendix C.2 – meaning that the formula (6.4.74) has the same singularity structure as the BAS tree-amplitude. Therefore, it is in fact equal to the BAS tree-amplitude, as claimed in (6.4.75), and obeys the same BCFW recursion relation.

6.5 Beyond flat spacetime

The astute reader will have observed that the graph theoretic arguments used in Section 6.3 to derive the integral kernel on twistor space can be applied to *any* formula based on determinants which encode a weighted counting of tree graphs via the matrix-tree theorem. Furthermore, twistor theory can be used to describe any solution to the vacuum Einstein equations with self-dual Weyl curvature and vanishing trace-free Ricci curvature [26, 330].

Consequently, one can ask if there are integral formulae for graviton scattering amplitudes in curved spacetimes which also admit an integral kernel representation indicative of double copy.

In fact, there are two known generalisations of the Cachazo-Skinner formula which also have this character. The first [312] is an adaptation of the formula to govern holomorphic maps from the Riemann sphere to the twistor space of AdS_4, aiming to describe tree-level graviton boundary correlation functions[12]. The second [231, 233] is a formula for the tree-level graviton scattering amplitudes in a class of chiral, asymptotically flat curved spacetimes known as self-dual radiative spacetimes. These can both be found reviewed in appendix C.3.

In this section, we present the integral kernels that can be derived for these formulae; this is achieved using identical methods to those explained in detail in Section 6.3, so we gloss over most of the details. The resulting integral kernel defines a representation of the formula which is suggestive of a double copy, and we comment on potential difficulties with this interpretation in each case.

6.5.1 AdS

The expression in AdS is given by

$$\mathcal{M}^{\Lambda}_{n,d} = \int d\mu_d \, |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}_\Lambda) \det{}'\left(\mathbb{H}^{\vee}_{\Lambda}\right) \prod_{i\in\mathbf{h}} h_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j(Z(\sigma_j)) \,, \tag{6.5.152}$$

with the ingredients reviewed in Appendix C.3.1.

Now, by following the exact same steps as in Section 6.3, it is possible to re-write the formula (6.5.152) as

$$\mathcal{M}^{\Lambda}_{n,d} = \sum_{\substack{b\rho, b\omega\in\mathcal{S}(\mathbf{h})\\ a\bar{\rho}, a\bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d} \, \mathcal{J}^{\tilde{\mathbf{h}}}_n[a\bar{\rho}b\rho] \, S^{\Lambda}_{n,d}[\rho,\bar{\rho}|\omega,\bar{\omega}] \, \mathcal{J}^{\tilde{\mathbf{h}}}_n[\bar{\omega}^{\mathrm{T}}ab\omega] \prod_{i\in\mathbf{h}} h_i \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j \,, \tag{6.5.153}$$

in terms of an AdS_4 integral kernel

$$S^{\Lambda}_{n,d} = \mathcal{D}(\hat{\omega}) \, \mathbb{S}^{\Lambda}_{\mathbf{h}}[\rho|\omega] \, \mathbb{S}^{\Lambda}_{\tilde{\mathbf{h}}}[\bar{\rho}|\bar{\omega}] \,. \tag{6.5.154}$$

The prefactor $\mathcal{D}(\hat{\omega})$ is the same as before (6.3.41), while the positive and negative helicity kernels have the same structure as (6.3.52) and (6.3.51), but defined in terms of

$$\phi_{ij} = (i\,j) \left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_j)}\right] \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (i\,l)\,(j\,l) \,, \quad i,j\in\mathbf{h} \,, \tag{6.5.155}$$

[12] We emphasize that although this formula is mathematically well-defined, its precise relationship to graviton boundary correlators in AdS_4, computed in positions space and for the Poincaré patch, remains unclear.

and

$$\tilde{\phi}_{ij} = \frac{\langle Z(\sigma_i),\, Z(\sigma_j)\rangle}{(i\,j)} \prod_{k\in(\tilde{\mathbf{h}}\cup\{b,t\})\setminus\{i,j\}} \frac{1}{(k\,i)\,(k\,j)}\,, \quad i,j\in\tilde{\mathbf{h}}\,, \tag{6.5.156}$$

respectively.

The integrand $\mathcal{I}_n^{\tilde{g}}[\alpha]$ for Yang-Mills theory on AdS_4 is functionally equivalent to that of (5.2.25), as can be seen in Appendix C.3.1, in equation (C.3.41). Therefore, taken at face value, (6.5.153) seems to provide an explicit twistorial double copy on AdS_4 in terms of the integral kernel (6.5.154).

While this conclusion is certainly tantalizing, a few words of caution are in order. Firstly, we reiterate that the precise connection between (6.5.152) or the RSVW formula in AdS_4 with graviton/gluon boundary correlators has not been established. While these formulae certainly know something about the tree-level boundary correlators in AdS, they could be missing pure boundary contributions such as those which contribute to correlators with all external particles of the same helicity [331]. Secondly, we do not currently have any physical interpretation of the inverse of the AdS kernel (6.5.154). If this were a true manifestation of double copy, one would expect this inverse to encode information about BAS 'scattering' in some background (possibly AdS), but this does not seem to be immediately obvious. Finally, we observe that other explorations of double copy in AdS using explicitly Witten diagram/boundary correlator computations in spacetime have found structures which seem much more complicated than (6.5.153) (cf., [141,332–343]), so an explicit matching between our formulae and position or momentum space expressions is required before drawing any conclusions.

6.5.2 Self-dual radiative spacetimes

A review of self-dual radiative spacetimes, and their gravitational scattering amplitudes can be found in Appendix C.3.3. We recall that the N^{d-1}MHV graviton tree-amplitude on the SD radiative background is given by:

$$\mathcal{M}_{n,d} = \sum_{t=0}^{n-d-3} \sum_{p_1,\ldots,p_t} \int d\mu_d\, |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}^\vee) \prod_{i\in\mathbf{h}} h_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j(Z(\sigma_j))$$
$$\times \left(\prod_{m=1}^{t} D\sigma_m \wedge D\bar{\lambda}(\sigma_m)\, \frac{N^{(p_m-2)}(\sigma_m)}{p_m!} \frac{\partial^{p_m}}{\partial \varepsilon_m^{p_m}} \right) \det{}'(\mathcal{H}) \Bigg|_{\varepsilon=0} . \tag{6.5.157}$$

The ingredients that enter into this formula are defined at length near equations (C.3.69).

The matrix $\mathbb{H}^\vee$ is unchanged from Minkowski space due to the self-duality of the background, $\mathbb{H}$ is replaced in each term of (6.5.157) by a $(n+t-d-1)\times(n+t-d-1)$ matrix $\mathcal{H}$. For our purposes, all that is required is the decomposition of $\det{}'(\mathcal{H})$ according to the matrix-tree theorem. This follows by the same methods as in flat space, to give

$$\det{}'(\mathcal{H}) = \frac{1}{|\tilde{\mathbf{h}}|^2} \prod_{\substack{k \in \mathbf{h} \cup \mathbf{t} \\ l \in \tilde{\mathbf{h}}}} \frac{1}{(k\,l)^2} \sum_{\substack{T^b \\ \text{spanning } \mathbf{h} \cup \mathbf{t}}} \prod_{(i \to j)} \frac{[K_i\, K_j]}{(i\,j)} \prod_{q \in \tilde{\mathbf{h}}} (i\,q)\,(j\,q)\,, \tag{6.5.158}$$

where $\mathbf{t}$ is the set of $m = 1, \ldots, t$ background insertions and

$$K_i^{\dot\alpha} = \begin{cases} i\, t_i\, H^{\dot\beta\dot\alpha}(U, \sigma_i)\, \tilde\kappa_{i\,\dot\beta} & i \in \mathbf{h} \\ i\, \varepsilon_i\, H^{\dot\beta\dot\alpha}(U, \sigma_i)\, \bar\lambda_{\dot\beta}(\sigma_i) & i \in \mathbf{t} \end{cases}, \tag{6.5.159}$$

for $H^{\dot\alpha\dot\beta}(U, \sigma)$ the holomorphic frame of the self-dual spinor bundle on the SD radiative spactime, pulled back to the degree d holomorphic curve in $\mathbb{P}\mathscr{T}$:

$$\bar\partial H^{\dot\alpha\dot\beta}(U, \sigma) = \bar\lambda^{\dot\alpha}(\sigma)\, \bar\lambda_{\dot\gamma}(\sigma)\, H^{\dot\gamma\dot\beta}(U, \sigma)\, N(\sigma)\, D\bar\lambda(\sigma)\,. \tag{6.5.160}$$

Using Proposition 6.2.1, (6.5.158) can be written as

$$\begin{aligned} \det{}'(\mathcal{H}) = \frac{(b\,x)\,(b\,y)}{\left|\tilde{\mathbf{h}} \cup \{b\}\right|^2} \sum_{\rho^+, \omega^+ \in \mathcal{S}(\mathbf{h} \cup \mathbf{t} \setminus \{b\})} \mathrm{pt}_{n+t-d}[b\rho^+ x]\, \mathrm{pt}_{n+t-d}[b\omega^+ y] \\ \times \sum_{T \in \mathcal{T}^b_{\rho^+, \omega^+}} \prod_{(i \to j) \in E(T)} \left([K_i\, K_j]\, (i\,j) \prod_{l \in \tilde{\mathbf{h}} \setminus \{x, y\}} \frac{(j\,l)}{(i\,l)} \right)\,, \end{aligned} \tag{6.5.161}$$

in terms of broken Parke-Taylor factors.

As it stands, this decomposition is not yet appropriate to defining a momentum kernel, as it involves orderings on the set $\mathbf{h} \cup \mathbf{t}$ which includes background insertions. To compensate for this, we can define a 'correction' factor for any $(n + t - d)$-point broken Parke-Taylor factor which breaks it down to the desired $(n - d)$-point object:

$$\mathrm{pt}_{n+t-d}[b\rho^+ x] := \mathrm{pt}_{n-d}[b\rho x]\, \mathcal{C}(b\rho^+ x; b\rho x)\,, \tag{6.5.162}$$

where $b\rho$ on the right-hand-side is now an ordering on $\mathbf{h}$ alone. By definition, the correction factor $\mathcal{C}(b\rho^+ x; b\rho x)$ will be homogeneous of weight -2 on $\mathbb{P}^1$ in each element of $\mathbf{t}$ but homogeneous of weight zero in its other arguments. For example,

$$\mathcal{C}(b123t_1x; b123x) = \frac{(3\,x)}{(3\,t_1)\,(t_1\,x)}\,, \tag{6.5.163}$$

where t_1 denotes a single background insertion.

This allows the graviton amplitude (6.5.157) to be rewritten in terms of an integral kernel:

$$\mathcal{M}_{n,d} = \sum_{\substack{b\rho, b\omega \in \mathcal{S}(\mathbf{h}) \\ a\bar\rho, a\bar\omega \in \mathcal{S}(\tilde{\mathbf{h}})}} \int d\mu_{n,d}\, \mathcal{J}^{\tilde{\mathbf{h}}}_n[a\bar\rho b\rho]\, S^N_{n,d}[\rho, \bar\rho | \omega, \bar\omega]\, \mathcal{J}^{\tilde{\mathbf{h}}}_n[\bar\omega^T a b\omega] \prod_{i \in \mathbf{h}} h_i \prod_{j \in \tilde{\mathbf{h}}} \tilde{h}_j\,, \tag{6.5.164}$$

where the background-dependent integral kernel still admits a chiral splitting

$$\mathbb{S}^{\mathrm{N}}_{n,d}[\rho,\bar{\rho}|\omega,\bar{\omega}] = \mathcal{D}(\hat{\omega})\,\mathbb{S}_{\tilde{\mathbf{h}}}[\bar{\rho}|\bar{\omega}]\,\mathbb{S}^{\mathrm{N}}_{\mathbf{h}}[\rho|\omega]\,, \tag{6.5.165}$$

with only the positive helicity factor $\mathbb{S}^{\mathrm{N}}_{\mathbf{h}}$ sensitive to the background. Explicitly, the positive helicity factor of the integral kernel is given by

$$\begin{aligned}\mathbb{S}^{\mathrm{N}}_{\mathbf{h}}[\rho|\omega] = \sum_{t=0}^{n-d-3}\sum_{p_1,\ldots,p_t}&\left(\prod_{m=1}^{t} \mathrm{D}\sigma_m\wedge \mathrm{D}\bar{\lambda}(\sigma_m)\,\frac{\mathrm{N}^{(p_m-2)}(\sigma_m)}{p_m!}\,\frac{\partial^{p_m}}{\partial\varepsilon_m^{p_m}}\right)\\ &\sum_{b\rho^+,b\omega^+\in\mathcal{S}_{\rho,\omega}(\mathbf{h}\cup\mathbf{t})}\Bigg[\mathcal{C}(b\rho^+x;b\rho x)\,\mathcal{C}(b\omega^+y;b\omega y)\\ &\qquad\left.\sum_{T\in\mathcal{T}^{b}_{\rho^+,\omega^+}}\prod_{(i\to j)\in E(T)}[K_i\,K_j]\,(i\,j)\prod_{l\in\tilde{\mathbf{h}}\setminus\{x,y\}}(i\,l)\,(j\,l)\Bigg]\right|_{\varepsilon=0}\,.\end{aligned} \tag{6.5.166}$$

The sum over $b\rho^+, b\omega^+\in\mathcal{S}_{\rho,\omega}(\mathbf{h}\cup\mathbf{t})$ denotes summing over all compatible extensions of the orderings $b\rho$, $b\omega$ to include the t background insertions.

The double copy interpretation of this formula can be made clearer by considering the scattering of gluons on a Cartan-valued SD radiative gauge field background, as reviewed in Appendix C.3.2. The amplitude is given by

$$\mathcal{A}_{n,d}[\rho] = \int \mathrm{d}\mu_d\,|\tilde{\mathbf{g}}|^4\,\mathrm{PT}_n[\rho]\prod_{i\in\mathbf{g}} a_i\,e^{e_i\,g(U,\sigma_i)}\prod_{j\in\tilde{\mathbf{g}}} b_j\,e^{e_j\,g(U,\sigma_j)}\,, \tag{6.5.167}$$

and (6.5.164) becomes a double copy formula when written as:

$$\mathcal{M}_{n,d} = \sum_{\substack{b\rho,b\omega\in\mathcal{S}(\mathbf{h})\\ a\bar{\rho},a\bar{\omega}\in\mathcal{S}(\tilde{\mathbf{h}})}}\int \mathrm{d}\mu_{n,d}\,\mathcal{I}^{\tilde{\mathbf{h}},e}_n[a\bar{\rho}b\rho]\,\mathbb{S}^{\mathrm{N}}_{n,d}[\rho,\bar{\rho}|\omega,\bar{\omega}]\,\mathcal{I}^{\tilde{\mathbf{h}},-e}_n[\bar{\omega}^{\mathrm{T}}ab\omega]\prod_{i\in\mathbf{h}}h_i\prod_{j\in\tilde{\mathbf{h}}}\tilde{h}_j\,, \tag{6.5.168}$$

where $\mathcal{I}^{\tilde{\mathbf{h}},e}_n$ denotes the integrand of (6.5.167) with Cartan-valued charges $e=\{e_1,\ldots,e_n\}$.

In particular, by taking the double copy between gluon amplitudes with sign-reversed background charges, the exponentials depending on the gauge field background in (6.5.167) cancel, leaving (6.5.164) as required. Precisely this charge-reversed prescription was found previously in constructing double copies for the only known examples of 3-point amplitudes on *non-chiral* radiative backgrounds in gauge theory and gravity [32, 33]. In this sense, (6.5.164) does indeed have some double copy interpretation, but it seems strange that such a formula would correctly capture double copy for Cartan-valued SD radiative gauge backgrounds but not fully non-abelian ones. It is not known how to replace the charge-reversed double copy prescription when the gauge field background becomes non-abelian. Furthermore, we do not know how to invert the positive helicity integral kernel (6.5.166) as a linear map on the space of colour-orderings, akin to [296], meaning that we have no plausible link with BAS theory in any form, as would be expected for a truly robust statement of double copy.

6.6 Conclusion

The question posed in the introduction of this chapter was: is there a helicity-graded representation of all-multiplicity tree-level gluon and graviton scattering amplitudes which manifests double copy? More specifically, what is the double copy relation between the RSVW formula for gauge theory, and the Cachazo-Skinner formula for gravity? In this chapter, we have definitively answered these questions.

We found a natural double copy structure of the integrands, in the form of a helicity-graded integral kernel in Section 6.3, derived through quite elementary graph theoretic and combinatorial results. We additionally proved that the inverse of this object should be viewed as the integrand for BAS theory in Section 6.4. A connection like this should be expected of a robust double copy structure. We extended the analysis further to formulae on non-trivial backgrounds – though the interpretation of these expressions remains open.

Other double copies: Whilst the derivation of this double copy structure followed a simple combinatorial path, its interpretation is a bit more mysterious. Comparing the structure and properties to other versions of the double copy – field theory KLT or CHY formulae – it seems to take a middle ground. At MHV level, our formula (6.3.54) is equivalent to the KLT double copy with a certain choice of basis. However, for generic helicity, the formula expresses gravity as a sum over a strictly *smaller* basis of colour orderings. In this way it could be viewed as being computationally 'simpler' than KLT. Of course, this relative simplicity is compensated for by the helicity-graded solutions to the scattering equations implied by the integral.

From the other perspective, we can compare our formula to the CHY double copy. Here, the double copy is naturally manifested as a multiplicative rule on a single term, evaluated on the support of the full set of solutions to the scattering equations. Since our formula only requires a helicity-graded subset of these equations, it is simpler in that sense. However, the integrand is much more complicated due to the presence of the sums over colour-orderings.

It remains to be seen whether a closer analysis of the relationship between our integral kernel, momentum space KLT and the CHY formulae might provide clues towards new relations between solutions to the scattering equations. Additionally, it seems that the structure of the basis of colour-orderings is essential to the relation – and encodes the degree of the maps, and thus the helicity configuration of the external legs. This fact – also at the level of the BAS amplitude – appears very mysterious. Why should the double copy care about helicity in this way?

In relation to other versions of the double copy, one may wonder if there is a way to rephrase our results as some kind of colour-kinematic dual statement. Indeed, the basis of this work was the calculation of the colour-kinematics dual numerators arising from the Hodges formula by Frost in [295]. There is also a question whether this structure is somehow related to a double copy at the level of twistor actions, as explored in [344], or more generally for Chern-Simons theories [345].

Finally, it would be remiss to not comment on approaches to double copy similar to ours in the literature. The first of these is the Cachazo-Geyer formula [294], giving a proposal for all-multiplicity graviton amplitudes constructed out of RSVW integrands and the spacetime KLT kernel. It would be interesting to see whether this formula can be related to our new representation of the gravity amplitude via the double copy to give a sharp equivalence. This is not at all obvious, as our kernel is chirally spit and the colour-orderings are helicity graded.

Another double copy for the RSVW and Cachazo-Skinner integrands was proposed in [311] in the form of certain 'scalar blocks'. It was noted in this paper that the objects constructed have unphysical poles, and must be summed over degree to give the biadjoint scalar amplitude. However, they do provide a *minimal* basis for the double copy in the sense that N^{d-1}MHV amplitudes in gravity are given as a sum of $E(n-3, d-1)$ distinct objects built from two gauge theory factors. The object we have found is a bit different in nature, and captures the scalar properties in a more physical way.

BAS theory: The new formula for the doubly colour-ordered BAS amplitude (6.4.74) does not provide any calculational advantages for computing these amplitudes (the expansion in trees is precisely that of the usual Feynman diagrams, and there is an additional integral over maps). That being said, it is interesting that twistor space encodes the structure of propagators in this chiral way. It would be interesting to see whether there are any connections to the recent studies of double copy and BAS theory (or $\mathrm{tr}(\phi^3)$ theory) via positive geometry [300, 346–350].

Non-trivial backgrounds: Our exploration of the double copy beyond flat space in Section 6.5 suggests that the scalar theory associated to the kernel will be deformed in some way. The examples presented in this chapter are definitely not exhaustive, and we expect that the true relations will arise from different configurations. But notably, with twistor space, we have one of the best handles on all-multiplicity data for certain backgrounds, and this may prove the playground to test the existence of double copy beyond flat space. Perhaps the deformation of the kernel in various theories can be related to the KLT bootstrap program [268], a generalisation of the Berends-Giele currents used in [302, 320], or the stringy version of [296].

In the same vein, one should compare our construction to existing versions of the double copy on non-trivial backgrounds. The objects we consider might live in a different space, but the final result should be intuitively similar to these spacetime approaches [32, 33, 140, 141, 332–343, 351, 352]. In connection to CHY, perhaps our AdS formula can be related via a version of the scattering equations in (A)dS [353–356].

Chapter 7

Conclusions

In this thesis I have demonstrated the progress made in two key aspects of scattering amplitudes: on strong backgrounds and in twistor space.

On strong backgrounds, we were motivated by the close connection that can be drawn between on-shell scattering amplitudes and classical observables — especially waveforms emerging from astrophysical scattering events. Often these events involve two black holes, and our methods focussed particularly on small mass ratios where one of the black holes can be treated as a small probe particle and the other as a background field. From the scattering amplitudes perspective this leads us to naturally consider scattering of the small probe particle on a strong background sourced by the more massive particle.

In chapter 3 we considered the scattering of a massive particle on Schwarzschild and Coulomb, emitting radiation in the process. We proposed a new basis expansion for the external states, amenable for calculating classical observables on these backgrounds. In particular, we calculated the first (semi-classical) three-point amplitude of this form in Schwarzschild. Whilst we neglected the presence of horizon and higher-order effects in these calculations, we outline how these can be included in the formalism.

Amplitudes on backgrounds of this form (a massive scalar emitting radiation) play a key role in the calculation of waveforms. In chapter 4, we calculated the waveform created by a massive particle crossing a plane wave background. In contrast to similar calculations using amplitudes on flat space, our results contain all-order contributions in the coupling constant and are non-perturbative in the background. Plane wave backgrounds can additionally be related to any generic spacetime via the Penrose limit, so we anticipate that more generic background calculations will exhibit similar features.

This work highlights the importance of amplitudes on strong backgrounds in calculations even removed from their usual domain of particle physics. It also highlights the many avenues of research available from this perspective. It would be exciting to extend the amplitudes calculation in chapter 3 to exact expressions at all-orders, and in particular inclusion of horizon effects. This would start relating our work to quasinormal modes and other aspects of black holes. Furthermore, we may be able to detect signatures of these effects in gravitational waveforms.

For classical calculations viewed through the lens of amplitudes, amplitudes on flat space provide the advantage that they are on-shell objects that we understand very well. In particular, we have a handle on loop-level calculations (which correspond to complicated back-reaction effects) and we also have a powerful relation between gravity and gauge theory (the double copy) which simplifies drastically any gravitational calculation. This wealth of knowledge gives considerable advantages compared to the usual classical approach. There are many hints that amplitudes on strong backgrounds exhibit similar properties — establishing these would impart similar advantages for classical calculations where the background is treated non-perturbatively.

Overall, I hope I have imparted on the reader in the course of Part I why studying amplitudes on strong backgrounds and their properties is so exciting. I look forward to future developments in this field. Of particular interest are advances in the the double copy on strong backgrounds [32,33,340,351,352], holographic descriptions of amplitudes on asymptotically flat backgrounds [167,357–359], and further applications of amplitudes on backgrounds to other aspects of astrophysics (for example, see [360,361]).

In the second part of this thesis, we considered aspects of helicity-graded representations of tree-level scattering amplitudes in gravity and in gauge theory. I presented the manifestation of the double copy for these two representations, taking the form of a relation between integrands. This answered a long-standing question about the precise double copy relation between the two simplest representations of generic helicity, all-multiplicity scattering in Yang-Mills and gravity in four dimensions. Additionally, we have shed light on how some ubiquitous structures present in gravitational tree-level formulae — weighted tree graphs — are closely linked to the colour-ordered structures essential in gauge theoretic formulae.

This opens the way for finding new formulae with twistorial representations, and studying new forms of the double copy. The advantage of twistorial representations of amplitudes is that they are helicity-graded, and are simpler to evaluate than conventional representations like the CHY formulae. Additionally, as illustrated in chapter 6.5, we now have access to investigating the form of the double copy on certain self-dual backgrounds. Whilst none of the representations have been illuminating so far, there are still many background configurations that are natural to consider, and for which twistorial derivations are possible.

I would like to highlight that twistor theory is proving more and more essential to understanding scattering amplitudes. For one, it is the natural place to consider scattering amplitudes on self-dual backgrounds, and is a veritable factory for all-multiplicity expressions [232,233,362,363]. Additionally, twistor theory is proving to be a natural place to look for a holographic description of asymptotically flat space times (e.g. [364–367]). Studying the objects derived using twistor theory, and for example what they imply about amplitudes on strong backgrounds in general, is therefore more promising than ever.

Part III

Appendix

Appendix A

Calculations in spherically symmetric backgrounds

A.1 Detailed asymptotic expansions

In this appendix, we provide a rigorous derivation of the matching coefficients (3.3.24), coming from the asymptotic matching of our WKB ansatz to the known exact solution of the Klein-Gordon equation on the same backgrounds (3.3.15).

We start with the ansatz

$$\phi(x) = \int d\Phi(p)\, \Lambda(p)\, e^{i\, S_p(x)}\,, \tag{A.1.1}$$

for some unknown coefficients $\Lambda(p)$ that will be matched onto the exact solution asymptotically. Expanding $e^{i\, S_p(x)}$ in spherical harmonics gives the general result

$$e^{i\, S_p(x)} = e^{i\, E t} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2}\, c_\ell(r) P_\ell(\cos\theta)\,, \tag{A.1.2}$$

where $\cos\theta = \hat{p}\cdot\hat{x}$ and the coefficients $c_\ell(r)$ are determined via

$$c_\ell(r) = \int_0^\pi d\theta\, \sin\theta\, e^{i(S_p(r,\cos\theta) - E t)} P_\ell(\cos\theta)\,. \tag{A.1.3}$$

The ansatz (A.1.1) is thus equivalent to

$$\phi(x) = 2\pi \int d\Phi(p)\, \Lambda(p)\, e^{i\, E t} \sum_{\ell,m} c_\ell(r)\, Y_\ell^m(\hat{x})\, \overline{Y_\ell^m}(\hat{p})\,, \tag{A.1.4}$$

which we want to match asymptotically onto the known solution

$$\phi_{p'}(x) = \frac{4\pi\, e^{i\, E' t}}{r} \sum_{\ell,m} Y_\ell^m(\hat{x})\, \overline{Y_\ell^m}(\hat{p}\,')\, R_{\ell m}(p';r), \tag{A.1.5}$$

where we have explicitly denoted the dependence of the radial wavefunction on the momentum p'.

Since the spherical harmonics are orthogonal, we can equate coefficients between (A.1.4) and (A.1.5) at large distances to find

$$\lim_{r\to\infty}\int d\Phi(p)\,\Lambda(p)\,e^{i\,E\,t}\sum_{\ell,m} c_\ell(r)\,Y_\ell^m(\hat{x})\,\overline{Y_\ell^m}(\hat{p})\,\frac{2\,e^{i\,E' t}}{r}\sum_{\ell,m} Y_\ell^m(\hat{x})\,\overline{Y_\ell^m}(\hat{p})\,R_{\ell m}(p';r)\,. \tag{A.1.6}$$

This equality requires $E = E'$, and this in turn implies that $|\vec{p}| = |\vec{p}\,'|$. Exploiting this and the underlying spherical symmetry, (A.1.6) implies that

$$\Lambda^{p'}(p) = \delta\big(|\vec{p}| - |\vec{p}\,'|\big)\sum_{\ell',m'} Y_{\ell'}^{m'}(\hat{p})\,\overline{Y_{\ell'}^{m'}}(\hat{p}\,')\,d_{\ell' m'}(p')\,, \tag{A.1.7}$$

in terms of some as-yet-undetermined coefficients $d_{\ell' m'}(p')$. Substituting this back into (A.1.4), the two mode sums are identified due to the orthogonality of the spherical harmonics in $\hat{p}$. This means that the asymptotic matching condition is reduced to:

$$\lim_{r\to\infty} d_{\ell m}(p')\,c_\ell(r) = \frac{2\,E}{|\vec{p}|^2}\,\lim_{r\to\infty}\frac{R_{\ell m}(p';r)}{r}\,, \tag{A.1.8}$$

with the factors of $E/|\vec{p}|^2$ arising from the on-shell integrals on the left-hand-side of (A.1.6). This in turn gives the expression

$$\Lambda^{p'}(p) = \frac{2\,E}{|\vec{p}|^2}\,\delta\big(|\vec{p}| - |\vec{p}\,'|\big)\sum_{\ell,m} Y_\ell^m(\hat{p})\,\overline{Y_\ell^m}(\hat{p}\,')\,\lim_{r\to\infty}\frac{R_{\ell m}(p';r)}{r\,c_\ell(r)}\,, \tag{A.1.9}$$

for the matching coefficients.

To further process this expression, we need to investigate the asymptotic properties of the $c_\ell(r)$ defined by (A.1.3). Observe that

$$\begin{aligned} c_\ell(r) &= \int_0^\pi d\theta\,\sin\theta\,e^{i(-\vec{p}\cdot\vec{x}+C_p\,\log(|\vec{p}|\,r)+C_p\,\log(1+\cos\theta))}\,P_\ell(\cos\theta) + O(r^{-2}) \\ &= e^{i\,C_p\,\log(|\vec{p}|\,r)}\int_{-1}^{1} dx\,e^{i(C_p\,\log(1+x)-|\vec{p}|\,r\,x)}\,P_\ell(x) + O(r^{-2})\,, \end{aligned} \tag{A.1.10}$$

where C_p is the theory-dependent constant defined by (3.3.11) and (3.3.12), and the integration variable is $x \equiv \cos\theta$ in the last line. Defining

$$\tilde{c}_\ell(r) := \int_{-1}^{1} dx\,e^{i(C_p\,\log(1+x)-|\vec{p}|\,r\,x)}\,P_\ell(x)\,, \tag{A.1.11}$$

we observe that

$$\tilde{c}_\ell'(r) = -i\,|\vec{p}|\int_{-1}^{1} dx\,e^{i(C_p\,\log(1+x)-|\vec{p}|\,r\,x)}\,x\,P_\ell(x)\,. \tag{A.1.12}$$

Now, the recurrence relation

$$(2\ell+1)\,x\,P_\ell(x) = (\ell+1)\,P_{\ell+1}(x) + \ell\,P_{\ell-1}(x)\,, \tag{A.1.13}$$

for Legendre polynomials can be combined with (A.1.11) and (A.1.12) to deduce the recurrence relation

$$\tilde{c}_{\ell+1}(r) = \frac{i\,(2\ell+1)}{|\vec{p}|\,(\ell+1)}\,\tilde{c}_\ell'(r) - \frac{\ell}{\ell+1}\,\tilde{c}_{\ell-1}(r)\,, \tag{A.1.14}$$

for the $\tilde{c}_\ell$s. By computing the first few of these coefficients explicitly, one soon arrives at

$$\tilde{c}_\ell(r) = \frac{i\,e^{i(C_p\,\log 2-|\vec{p}|\,r)}}{|\vec{p}|\,r} + (-1)^\ell\,\frac{C_p\,\Gamma(i\,C_p)\,e^{i(|\vec{p}|\,r-C_p\,\log(i|\vec{p}|\,r))}}{|\vec{p}|\,r} + O(r^{-2})\,, \tag{A.1.15}$$

which can easily be proven to solve (A.1.14) for all ℓ by induction. Note that for $C_p \to 0$, this reproduces the asymptotic expansion for spherical Bessel functions, as expected.

In (A.1.15), we only want contributions which will contribute to the wavefunction as waves travelling like $e^{i(Et-|\vec{p}|\,r)}$, so we discard the second term. Note that this un-wanted term is perfectly finite; it simply has the wrong scattering behaviour. Plugging the first term of (A.1.15) into (A.1.10) and then into (A.1.9) gives:

$$\Lambda^{p'}(p) = -\frac{2i\,E}{|\vec{p}|}\,\delta\big(|\vec{p}|-|\vec{p}\,'|\big)\sum_{\ell,m} Y_\ell^m(\hat{p})\,\overline{Y_\ell^m}(\hat{p}\,')\,\lim_{r\to\infty} R_{\ell m}(p';r)\,e^{i(|\vec{p}|\,r-C_p\,\log(2|\vec{p}|\,r))}\,, \tag{A.1.16}$$

matching (3.3.24) from the text.

A.2 The probe and classical limits of 5-point amplitudes

In this appendix we describe the probe and classical limits of the 5-point amplitudes in scalar QED and gravitationally coupled scalars which appear from the classical, weak field limits of our semiclassical 3-point amplitudes on Coulomb and Schwarzschild backgrounds. We will first consider the probe limit and show how this is equivalent to a background field calculation. We then take a further, classical limit, which provides the check on the semiclassical calculations of Sections 3.4 and 3.5.

We begin with the scalar QED calculation. In vacuum, consider the emission of a photon, momentum k and helicity η, in the scattering of two charges, one of mass m, momentum $p \to p'$, the other of mass M, momentum $P \to P'$. We can assume without loss of generality that $P_\mu = MU_\mu$ as in (3.3.9). The 5-point amplitude $\mathcal{M}_5$ is easily calculated in scalar QED, with the result

$$\begin{aligned}\mathcal{M}_5 = {}& ie^2Q\,\hat{\delta}^4(P'+p'+k-p-P)\,\times \\ & \frac{(P'+P)^\mu}{(P-P')^2}\left[-2\varepsilon_\mu - \frac{\varepsilon\cdot p'}{k\cdot p'}\,(p'+p+k)_\mu + \frac{\varepsilon\cdot p}{k\cdot p}\,(p'+p-k)_\mu\right] \\ & + (p\leftrightarrow P, p'\leftrightarrow P')\,.\end{aligned} \tag{A.2.17}$$

Consider the momentum-conserving delta functions. To see how the probe limit arises, we split these into temporal and spatial pieces:

$$\delta^3\left(\vec{P}' + \vec{\ell}\right)\,\delta\left(\sqrt{M^2+\vec{\ell}^{\,2}} + k_0 - p_0 - M\right)\,, \qquad \vec{\ell} := \vec{p}\,' + \vec{k} - \vec{p}\,. \tag{A.2.18}$$

We take M to be large, and assume the recoil of that particle is negligible, which quantitatively means assuming the *total momentum transfer* $\vec{\ell}$ obeys $\vec{\ell}^{\,2} \ll M^2$. This allows us to expand the square root in the temporal delta function. We similarly expand in powers of $\vec{\ell}$ in the full QED amplitude. The leading order terms are those shown explicitly in (A.2.17), which go as $1/\vec{\ell}^2$. The result is, writing $\ell_\mu = k_\mu + p'_\mu - p_\mu$,

$$\begin{aligned}
&\mathcal{M}_5 \to 2M\,\hat{\delta}^3\left(\vec{P}' + \vec{\ell}\right)\widetilde{\mathcal{M}}_5\,,\\
&\widetilde{\mathcal{M}}_5 := \hat{\delta}(U\cdot\ell)\,\frac{2ie^2Q}{\ell^2}\left[-\varepsilon\cdot U + \frac{\varepsilon\cdot p'}{k\cdot p'}\,U\cdot p - \frac{\varepsilon\cdot p}{k\cdot p}\,U\cdot p'\right].
\end{aligned} \tag{A.2.19}$$

It can be checked by direct calculation that $\widetilde{\mathcal{M}}_5$ is precisely the *three*-point amplitude ($p \to p'$ with emission of photon k) on a Coulomb background, calculated to leading order in the background charge Q. The differences between $\mathcal{M}_5$ and $\widetilde{\mathcal{M}}_5$ are resolved at the level of physical predictions: the additional $\hat{\delta}^3$ and factor of $2M$ in $\mathcal{M}_5$ relative to $\widetilde{\mathcal{M}}_5$ are absorbed, in *observables*, by state normalisation and final state integrals over the heavy particle momentum, see also [138]. Hence (A.2.19) is indeed the probe limit, physical predictions from which agree exactly with those obtained in a direct background field calculation.

In our semiclassical approach used in the text, we work to leading nontrivial order in $\hbar$. To compare to the perturabtive QED results here, we thus need to identify the leading classical behaviour of (A.2.19). To do so we recall that massless momenta k scale as $\hbar$ times classical wavenumber, while continuing to impose that the recoil of the mass M particle is small compared to its own rest mass. Hence, we write $p'_\mu = p_\mu + q_\mu$ in which q also scales as $\hbar$ to leading order [21]. Taking the classical limit of (A.2.19) is then equivalent to taking a Taylor expansion in which both k and q are of the same small order. This expansion yields the leading classical behaviour

$$\widetilde{\mathcal{M}}_5 \to \hat{\delta}(U\cdot(q+k))\,\frac{2ie^2Q}{(q+k)^2}\left[-\varepsilon\cdot U + \frac{U\cdot p\,q\cdot\varepsilon}{k\cdot p} - \frac{U\cdot q\,p\cdot\varepsilon}{k\cdot p} - \frac{k\cdot q\,U\cdot p\,p\cdot\varepsilon}{(k\cdot p)^2}\right]. \tag{A.2.20}$$

We confirm in the text that this is indeed what is recovered from our WKB analysis.

For gravity, the implementation of the probe and classical limits happens in exactly the same fashion; the only distinction is the functional form of the 5-point amplitude for graviton emission from the scattering of two massive scalars. Needless to say, this computation, in Feynman diagrams, is significantly more complicated than its scalar QED cousin due to the appearance of the cubic graviton vertex. In this case it is easiest to avoid Feynman diagrams by employing double copy methods [203], which also allow for the classical limit to be taken immediately. Employing the same momentum notation as before, the classical 5-point graviton emission amplitude is

$$\mathcal{M}_5 = 16\, M^2\, \kappa^3\, \hat{\delta}^4\big(P' + p' + k - p - P\big)\ \frac{\varepsilon_{\mu\nu}}{\ell^2\,(p-p')^2} \times$$
$$\bigg[4\,(k\cdot p\, U^\mu - k\cdot U\, p^\mu)\,(k\cdot p\, U^\nu - k\cdot U\, p^\nu) + 4\, p\cdot U\ \Big(k\cdot p\, U^{(\mu} - k\cdot U\, p^{(\mu}\Big)\ Q^{\nu)}$$
$$+\bigg((p\cdot U)^2 - \frac{m^2}{2}\bigg)\bigg(Q^\mu\, Q^\nu - \frac{\ell^2\,(p-p')^2}{(k\cdot p)^2\,(k\cdot U)^2}\ (k\cdot p\, U^\mu - k\cdot U\, p^\mu)\,(k\cdot p\, U^\nu - k\cdot U\, p^\nu)\bigg)\bigg]\,, \tag{A.2.21}$$

where

$$Q^\mu := -2(p'-p)^\mu - k^\mu - \frac{(p-p')^2\, p^\mu}{k\cdot p} + \frac{\ell^2\, U^\mu}{k\cdot U}\,. \tag{A.2.22}$$

Applying the probe limit to this expression, we arrive at (3.5.84) in Section 3.5.

Appendix B

Classical derivation of the waveform

B.1 Classical checks

This appendix contains a classical derivation of the waveforms in electromagnetism and gravity. Schematically, these stem from radiation fields 'A' generated by sources 'J' representing particles moving on a background, which take the form

$$A_\sigma(x) := \int d^4y\, G_{\rm ret}(x,y)\, J_\sigma(y)\ , \tag{B.1.1}$$

in which the subscript σ is a placeholder for any number of vector indices or spin labels. The retarded Green's function is the inverse of ∇^2 in a flat or curved background, and is therefore theory-dependent:

$$G^{\rm EM}_{\rm ret}(x-y) = i\Theta(x^- - y^-)\int \frac{\hat{d}\bar{k}_+\hat{d}^2\bar{k}_\perp}{2\bar{k}_+}\Theta(\bar{k}_+)\, e^{-i\bar{k}\cdot(x-y)}\ , \tag{B.1.2}$$

$$G^{\rm GR}_{\rm ret}(x,y) = \frac{i\Theta(x^- - y^-)}{\sqrt{|E(x)|}}\int \frac{\hat{d}\bar{k}_+\hat{d}^2\bar{k}_\perp}{2\bar{k}_+}\Theta(\bar{k}_+)\frac{e^{-i\mathcal{F}(x,y)}}{\sqrt{|E(y)|}}\ , \tag{B.1.3}$$

in which $\bar{k}_\mu$ is on-shell and

$$\begin{aligned}\mathcal{F}(x,y) = \bar{k}_+(x-y)^+ + \bar{k}_i(E^i_a(x)x^a - E^i_a(y)y^a)\\ + \frac{\bar{k}_+}{2}(\sigma_{ab}(x)x^ax^b - \sigma_{ab}(y)y^ay^b) + \frac{\bar{k}_i\bar{k}_j}{2\bar{k}_+}\int_{y^-}^{x^-} ds\, \gamma^{ij}(s).\end{aligned} \tag{B.1.4}$$

We measure the waveform at future null infinity, hence we write the coordinate x^μ in the coordinate system $(u, r, \hat{\mathbf{x}})$ where $r = |\mathbf{x}|$, $\mathbf{x} = r\hat{\mathbf{x}}$ and $u = t - r$; the asymptotic limit is reached by taking $r \to \infty$ at fixed u and angular coordinates $\hat{\mathbf{x}}$. As long as our measurement device is not in the beam of the wave (corresponding to $\hat{x}^3 = 1$), then we can set the initial $\sqrt{|E(x)|} = 1$ in (B.1.3) to unity. The step function can also be set to unity in the limit. With this, the 'Fourier transformed' version of (B.1.1) is

$$A_\sigma(x) = i\int \frac{\hat{d}\bar{k}_+ \hat{d}^2\bar{k}_\perp}{2\bar{k}_+} e^{-i\bar{k}\cdot x}\, \bar{J}_\sigma(\bar{k})\ , \tag{B.1.5}$$

where $\bar{J}_\sigma(\bar{k})$ theory-dependent. In the $r \to \infty$ limit the leading behaviour of this quantity is, performing a saddle point calculation of the $\bar{k}_\perp$ integrals as in the text,

$$A_\sigma(x) \sim \frac{1}{4\pi r}\int_0^\infty \hat{d}\omega\, e^{-i\omega u}\bar{J}_\sigma(\omega\hat{x}_\mu)\ +\ \text{c.c.}\,, \tag{B.1.6}$$

where $\hat{x}_\mu = (1, \hat{\mathbf{x}})$ in Cartesian coordinates. We now turn to specifics in electromagnetism and gravity.

Electromagnetism

In electromagnetism J is the vector current for a particle moving in a background field,

$$\bar{J}_\mu(\bar{k}) = -e\int_y e^{i\bar{k}\cdot X}\frac{\partial}{\partial y}\frac{X'_\mu(y)}{i\bar{k}\cdot X'(y)} \tag{B.1.7}$$

where $X_\mu(y)$ is the particle orbit, and dashes represent derivatives with respect to y^-. Note that this form of the current generates the radiation field; the Coulomb fields from outside the wave have been subtracted. Substituting into (B.1.6) we obtain the gauge potential of the radiation field:

$$A_\mu(u,\hat{x}) \sim \frac{ie}{4\pi r}\int_0^\infty \hat{d}\omega \int_y\ e^{-i\omega(u-\hat{x}\cdot X(y))}\frac{1}{\omega}\frac{\partial}{\partial y}\frac{X'_\mu(y)}{\hat{x}\cdot X'(y)}\ +\ \text{c.c.} \tag{B.1.8}$$

The radiated field strength is $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ which, up to subleading corrections in $1/r$, we can obtain by adding factors of $-i\omega\hat{x}$ to the integrand of (B.1.8). The factors of ω outside the exponential cancel in $F_{\mu\nu}$ allowing us to perform the ω-integral to find

$$F_{\mu\nu}(u,\hat{x}) \sim \frac{1}{r}W_{\mu\nu}(u,\hat{x}) = \frac{e}{2\pi r}\int_y \delta(u - \hat{x}\cdot X(y))\frac{\partial}{\partial y}\frac{\hat{x}_{[\mu}X'_{\nu]}(y)}{\hat{x}\cdot X'(y)}\ , \tag{B.1.9}$$

where '$\sim$' denotes equality up to subleading terms in r^{-1}. This exactly the waveform derived in the text from the classical limit of the quantum result.

Gravity

The gravitational radiation of a massive scalar moving in a background is sourced by the stress-energy tensor

$$T_{\mu\nu}(y) = \frac{P_\mu(y)P_\nu(y)}{p_+}\,\delta^3_{+,\perp}(y - X(y))\,, \tag{B.1.10}$$

in which $X_\mu(y)$ is once again the particle orbit and $P_\mu(y) = p_+ X'_\mu(y)$. For convenience we define $\widetilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\alpha_\alpha$ as shorthand for a 'trace-reversed' $T_{\mu\nu}$. From the Einstein field equations, one can derive that the sourced gravitational field satisfies, imposing lightfront gauge $n^\mu h_{\mu\nu} = 0$,

$$h_{\mu\nu} = -\frac{2}{\nabla^2} t_{\mu\nu} + \frac{2}{\nabla^2} n_\mu n_\nu h^{ab} H_{ab} \; . \tag{B.1.11}$$

in which the modified stress-energy tensor $t_{\mu\nu}$ is defined by

$$t_{\mu\nu} \coloneqq \kappa^2 \left[\widetilde{T}_{\mu\nu} - 2 \frac{\nabla_{(\mu} \widetilde{T}_{\nu)+}}{\partial_+} + \nabla_{(\mu} \nabla_{\nu)} \frac{\widetilde{T}_{++}}{\partial_+^2} \right] . \tag{B.1.12}$$

In the notation of (B.1.1), the source 'J' is now

$$\begin{aligned} \bar{t}_{\mu\nu}(k) = 2\kappa^2 \int_{x_i^-}^{x_f^-} & dy \, e^{i\mathcal{V}(y)} \\ \times \frac{\partial}{\partial y} & \left[\frac{1}{\sqrt{|E(y)|}} \left(\mathbb{P}_{\mu\alpha} \mathbb{P}_{\nu\beta} P^\alpha P^\beta - \frac{1}{2} \eta_{\mu\nu} m^2 - \frac{i p_+^2 \delta^a_\mu \delta^b_\nu \sigma_{ab}}{k_+} \right) \frac{1}{i\partial_- \mathcal{V}(y)} \right] , \end{aligned} \tag{B.1.13}$$

where $\mathbb{P}_{\mu\nu}(k; y^-)$ are the projectors defined in (4.3.28) and

$$\mathcal{V}(y) = \int_{-\infty}^{y} dz \, \frac{g^{\mu\nu}(z) \, \bar{K}_\mu(z) \, P_\nu(z)}{p_+} \, . \tag{B.1.14}$$

To arrive at this expression one uses integration by parts to shift the derivatives present in (B.1.12) onto the propagator. Additionally, we ignore the second term in (B.1.11) since we are only interested in radiative contributions. See [33] for details. Substituting into (B.1.6) we obtain an expression for the asymptotic metric perturbation

$$h_{\mu\nu}(u, \hat{x}) = \frac{-i\kappa^2}{2\pi r} \int_0^\infty \hat{d}\omega \int_{y_i^-}^{y_f^-} dy \, \frac{e^{-i\omega(u - \bar{\mathcal{V}}(y))}}{\omega} \frac{\partial}{\partial y} \left[\left(T^0_{\mu\nu} - \frac{i}{\omega} T^1_{\mu\nu} \right) \frac{1}{\partial_- \bar{\mathcal{V}}(y)} \right] + \text{c.c.} \, , \tag{B.1.15}$$

in which the same 'effective' energy-momentum tensors defined in (4.3.34) have appeared, along with with the reduced exponent

$$\bar{\mathcal{V}}(y) = \frac{1}{\omega} \mathcal{V}(y) \Big|_{\bar{k} = \omega \hat{x}} . \tag{B.1.16}$$

From here we form the linearised curvature $R_{\mu\nu\sigma\rho} = 2\partial_{[\mu} \partial_{[\sigma} h_{\nu]\rho]}$, with each derivative introducing a factor of $(-i)\omega\hat{x}$ into the integrand. We can then integrate in ω to obtain

$$R_{\mu\nu\sigma\rho}(u, \hat{x}) \sim -\frac{\kappa^2}{\pi r} \hat{x}_{[\mu} \hat{x}_{[\sigma} \int_y \delta(u - \bar{\mathcal{V}}(y^-)) \left[\mathcal{D}^2 T^0_{\nu]\rho]}(u, \hat{x}) - \mathcal{D} T^1_{\nu]\rho]}(u, \hat{x}) \right] , \tag{B.1.17}$$

where $\mathcal{D}$ is defined in (4.3.36). This confirms the classical limit of our QFT calculations.

Appendix C

Properties of the KLT kernel and backgrounds in twistor space

C.1 'Parity' invariance of the twistor BAS amplitude

Even though massless scalar particles do not have a helicity (i.e., all of them are helicity zero), the BAS amplitude (6.4.74) is graded by the degree of the holomorphic map to twistor space, which is traditionally associated with the helicity of the external particles[1]. A desirable property of the formula would be that it is 'parity' symmetric in the sense that changing the degree of the map to $\tilde{d} = n - d - 2$ and interchanging angle and square brackets reproduces the same quantity.

One can show this following methods similar to [23, 248, 368] by first noting that, when evaluated on the twistor momentum eigenstates (6.4.76) we have

$$\langle \lambda(\sigma_i)\, \lambda(\sigma_j) \rangle = \frac{\langle i\, j \rangle}{t_i\, t_j}\,, \tag{C.1.1}$$

where the t_i are the scaling parameters associated to each momentum eigenstate in twistor space. Now, change variables from t_i to s_i for each $i = 1, \ldots, n$, where

$$s_i\, t_i := \prod_{j \neq i} \frac{1}{(i\, j)}\,, \tag{C.1.2}$$

where the product is over *all* insertion points $j \neq i$. As shown in [23], this transforms the moduli integral over degree d curves to a moduli integral over degree $n - d - 2$ curves, up to a factor

$$\prod_{i=1}^{n} s_i^4 \prod_{j \neq i} (i\, j)^2. \tag{C.1.3}$$

[1] In the BAS formula, degree is instead related to the structure of the colour ordering.

At the same time, the measure in the wavefunctions transforms to

$$t_i \, dt_i = -\frac{ds_i}{s_i^3} \prod_{j \neq i} \frac{1}{(i\,j)^2} \,. \tag{C.1.4}$$

These two Jacobians combine to give the scaling parameter measure $s_i \, ds_i$ appropriate to scalar momentum eigenstates on twistor space. Therefore, all integral measures are invariant up to a sign, and it remains to be checked that the integrand behaves in the same way.

The prefactor $\mathcal{D}^{-1}$ will not change, and is already symmetric in the exchange of $\mathbf{h}$ and $\tilde{\mathbf{h}}$. The transformation of the quantities ϕ_{ij}, $\tilde{\phi}_{ij}$ is less trivial. Recall that

$$\phi_{ij} = -t_i t_j \, [i\,j](i\,j) \prod_{l \in \tilde{\mathbf{h}} \backslash \{a,y\}} (i\,l)(j\,l) \,, \tag{C.1.5}$$

so changing variables to s_i gives

$$\phi_{ij} = -\frac{[i\,j]}{s_i s_j}(i\,j) \prod_{l \neq i} \frac{1}{(i\,l)} \prod_{k \neq j} \frac{1}{(j\,k)} \prod_{l \in \tilde{\mathbf{h}} \backslash \{ay\}} (i\,l)(j\,l) \tag{C.1.6}$$

$$= -\frac{[i\,j]}{s_i s_j}(i\,j) \prod_{l \in \mathbf{h} \cup \{a,y\} \backslash \{i\}} \frac{1}{(i\,l)} \prod_{k \in \mathbf{h} \cup \{a,y\} \backslash \{j\}} \frac{1}{(j\,k)} \tag{C.1.7}$$

$$= -\frac{[i\,j]}{s_i s_j (i\,j)} \prod_{l \in \mathbf{h} \cup \{a,y\} \backslash \{i,j\}} \frac{1}{(i\,l)(j\,l)} \,, \tag{C.1.8}$$

which, upon also transforming $[i\,j] \to \langle i\,j \rangle$, reproduces the structure of $-\tilde{\phi}_{ij}$.

A similar calculation for $\tilde{\phi}_{ij}$ yields

$$\tilde{\phi}_{ij} = \langle i\,j \rangle \, s_i s_j (i\,j) \prod_{l \in \mathbf{h} \backslash \{b,t\}} (i\,l)(j\,l) \tag{C.1.9}$$

under the reparametrisation from t_i to s_i. Once again, transforming $\langle i\,j \rangle \to [i\,j]$ reproduces $-\phi_{ij}$ as desired. The overall minus sign cancels the minus sign in the twistor wavefunction. Thus, we have established that $\bar{m}_{n,d} = m_{n,n-d-2}$. This fact is used in the proof of Theorem 2 to evaluate the N^{n-3}MHV amplitudes at the end of the iterative argument.

C.2 Soft limits

In this section we verify that the proposed formula (6.4.74) reproduces the soft limits of biadjoint scalar amplitudes. Multi-particle soft limits also follow easily as a corrollary of this derivation. These two results combined verify that there are no unphysical poles in the proposed formula, as these would present themselves in an anomalous multi-particle soft factor. This section follows closely the soft analysis in [23] and [368].

We again analyse the amplitude via its tree sub-amplitudes (6.4.78), each of which will correspond to the equivalent Feynman diagram. We will analyse the limit of sending a leg

$n \in \mathbf{h}$ soft, which, by the 'parity' invariance shown in Appendix C.1, is the same as sending a particle in $\tilde{\mathbf{h}}$ soft. In the spinor-helicity formalism, we can describe this soft limit as $\kappa_n^\alpha \to 0$ whilst $\tilde{\kappa}_n^{\dot\alpha}$ stays constant (the holomorphic soft limit), or as $\tilde{\kappa}_n^{\dot\alpha} \to 0$ whilst κ_n^α stays constant (the anti-holomorphic soft limit). In a scalar theory, the soft limit should not depend on this choice, so we will verify both.

Holomorphic soft limit: Starting with the holomorphic soft limit, the only dependence on n is in the associated twistor wavefunctions $\varphi_n(Z(\sigma))$ and the quantities ϕ_{nj}. Therefore, we are interested in analysing the singularity structure of

$$\int dt_n\, t_n\, D\sigma_n \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (n\,l)^2 \left(\prod_{I(E)\ni n} \frac{1}{\phi_E}\right) \bar{\delta}^2(\kappa_n - t_n\lambda(\sigma_n))\, e^{it_n\,[\mu(\sigma_n)\,n]}\,, \tag{C.2.10}$$

where the second product appearing here is over all 'propagators' in the primary tree that contain contributions from particle n, given by

$$\phi_E = \sum_{\{i,j\}\subset I(E)} \phi_{ij}\,, \qquad \phi_{ij} = -t_i t_j\,[i\,j]\,(i\,j) \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (i\,l)\,(j\,l)\,. \tag{C.2.11}$$

In (C.2.10), the integral over the scaling parameter t_n can be performed against one of the holomorphic delta functions, leaving

$$\int \frac{D\sigma_n\,\langle a\,b\rangle}{\langle\lambda(\sigma_n)\,a\rangle^2} \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (n\,l)^2 \prod_{E\ni n} \frac{1}{\phi_E}\, \bar{\delta}\!\left(\frac{\langle n\,b\rangle}{\langle n\,a\rangle} - \frac{\langle\lambda(\sigma_n)\,b\rangle}{\langle\lambda(\sigma_n)\,a\rangle}\right) e^{it_n\,[\mu(\sigma_n)\,n]}, \quad t_n \to \frac{\langle n\,a\rangle}{\langle\lambda(\sigma_n)\,a\rangle}\,, \tag{C.2.12}$$

where $\{a^\alpha, b^\alpha\}$ is an arbitrary. choice of (constant) spinor dyad used to decompose the two holomorphic delta functions.

Since t_n is fixed to be of order κ_n by this integration, terms coming from the exponential in (C.2.12) can be neglected, as they will be subleading in the holomorphic soft expansion. On the support of the other holomorphic delta functions in the subamplitude (which we have not written explicitly as they do not contain singular behaviour in the soft limit), the argument of the remaining holomorphic delta function in (C.2.12) can be rewritten as

$$\bar{\delta}\left(\frac{\langle n\,j\rangle\langle a\,b\rangle}{\langle n\,a\rangle\langle j\,a\rangle} - \left[\frac{\langle\lambda(\sigma_j)\,b\rangle}{\langle\lambda(\sigma_j)\,a\rangle} - \frac{\langle\lambda(\sigma_n)\,b\rangle}{\langle\lambda(\sigma_n)\,a\rangle}\right]\right)\,, \tag{C.2.13}$$

using $\langle j\,b\rangle/\langle j\,a\rangle - \langle\lambda(\sigma_j)\,b\rangle/\langle\lambda(\sigma_j)\,a\rangle = 0$ for some $j \neq n$. The quantity in the brackets here is a rational function of σ_n and vanishes when $\sigma_n \to \sigma_j$. Therefore, we can introduce a new variable $\omega := (n\,j)$, and express the bracketed quantity as $\omega\, F(\omega, \sigma_j)$ for some function F that is regular at $\omega = 0$. In terms of this ω, the other contributions to (C.2.12) are

$$\phi_{ni}(\omega) = -\frac{t_i\,\langle n\,a\rangle\,[n\,i]\,(\omega - (i\,j))}{\langle\lambda(\sigma_n)\,a\rangle} \prod_{l\in\tilde{\mathbf{h}}\setminus\{a,y\}} (\omega - (j\,l))\,(i\,l)\,, \tag{C.2.14}$$

and the prefactor

$$\prod_{l\in\tilde{h}\backslash\{a,y\}} (n\,l)^2 = \prod_{l\in\tilde{h}\backslash\{a,y\}} (\omega - (j\,l))^2 \,. \tag{C.2.15}$$

The remaining integral in (C.2.12) thus takes the form

$$\int_{\mathbb{C}^*} d\omega\, f(\omega)\, \bar{\delta}\left(\frac{\langle n\,j\rangle\,\langle a\,b\rangle}{\langle n\,a\rangle\,\langle j\,a\rangle} - \omega\, F(\omega, \sigma_j) \right) , \tag{C.2.16}$$

where

$$f(\omega) = \frac{\langle a\,b\rangle}{\langle \lambda(\sigma_n)\,a\rangle^2} \prod_{l\in\tilde{h}\backslash\{a,y\}} (\omega - (j\,l))^2 \prod_{E\ni n} \frac{1}{\phi_E(\omega)} \tag{C.2.17}$$

is a rational function of ω.

We now investigate the roots of the argument of the holomorphic delta-function in (C.2.16) as $\langle n\,j\rangle \to 0$. One of these is when ω becomes small, of the same order as $\langle n\,j\rangle$, whilst the function $F(\omega, \sigma_j)$ remains of order unity. The other roots are when $F(\omega, \sigma_j)$ becomes small. For the latter case, performing the integral will not give us singular behaviour unless this root is also pole of $f(\omega)$. However, since $f(\omega)$ has poles determined by kinematic square brackets while $F(\omega, \sigma_j)$ only depends on kinematic angle brackets, for generic momentum configurations these will not have coinciding singular points. Additionally, due to the form of $F(\omega, \sigma_j)$ from (C.2.13), there is no singularity coming from $1/\langle \lambda(\sigma_n)\,a\rangle$ in (C.2.17).

Thus, we need only consider the case where ω becomes of order $\langle n\,j\rangle$. This leads to a singularity if $\phi_E = \phi_{nj}$ for one of the edges; that is, if the soft scalar n attaches directly to an external j line. Here

$$\phi_{nj} = -\omega\, \frac{t_j\,\langle n\,a\rangle [n\,j]}{\langle \lambda(\sigma_n)\,a\rangle} \prod_{l\in\tilde{h}\backslash\{a,y\}} (\omega - (j\,l))(j\,l). \tag{C.2.18}$$

Notably we do not get a soft singularity if n does not attach to an external line. If it does attach to an external line, the particle j is specified *a posteriori* as corresponding to this line[2].

Evaluating the integral (C.2.16) in the limit $\langle n\,j\rangle \to 0$, we see that all dependence on $F(\omega, \sigma_j)$ drops out, and we can set $\omega = 0$ and $\sigma_n = \sigma_j$ in the remaining integrand. The 'soft factor' multiplying the lower-point tree subamplitude with leg n removed is thus

$$\frac{\langle \lambda(\sigma_j)\,a\rangle}{t_j\,\langle j\,a\rangle\,\langle n\,j\rangle\,[n\,j]} = \frac{1}{2\,k_n \cdot k_j} \,, \tag{C.2.19}$$

on the support of the holomorphic delta-functions for particle j. Summing over all the possible trees, we recover the full soft structure of BAS theory at tree-level (cf., [369]).

[2]If one does not choose j wisely in this way, we can not neglect roots where $F(\omega, \sigma_j)$ becomes small. This is because $1/\phi_{ni}$ for $i \neq j$ has a non-zero pole in ω that is independent of square brackets and can therefore coincide with a root of $F(\omega, \sigma_j)$.

Anti-holomorphic soft limit: The analysis for the anti-holomorphic soft limit, where $\tilde{\kappa}_n^{\dot\alpha} \to 0$ whilst keeping κ_n^α constant, is much easier as the only dependence on $[n\,j]$ is trivial in ϕ_{nj}, and is subleading in the twistor wavefunction. The only singularity is again when n attaches directly to the external leg j, appearing through

$$\frac{1}{\phi_{nj}} = -\frac{1}{t_n t_j\,[n\,j]\,(n\,j)} \prod_{l\in\tilde{h}\setminus\{a,y\}} \frac{1}{(n\,l)\,(j\,l)}. \tag{C.2.20}$$

The remaining n-dependence in ϕ_E^{-1} is subleading in the anti-holomorphic soft limit and therefore reduces to the product over edges in the reduced tree with leg n removed. We are thus interested in the singularity structure of

$$\int dt_n\, t_n\, D\sigma_n \prod_{l\in\tilde{h}\setminus\{a,y\}} (n\,l)^2\, \frac{1}{\phi_{nj}}\, \bar\delta^2(\kappa_n - t_n\lambda(\sigma_n))\, e^{it_n\,[\mu(\sigma_n)\,n]}\,. \tag{C.2.21}$$

Performing the integral over t_n against one of the holomorphic delta functions and keeping track of the remaining n-dependence gives

$$-\int \frac{D\sigma_n}{\langle\lambda(\sigma_n)\,a\rangle}\frac{\langle a\,b\rangle}{t_j\,[n\,j]\,\langle n\,a\rangle\,(n\,j)} \prod_{l\in\tilde{h}\setminus\{a,y\}} \frac{(n\,l)}{(i\,l)}\,\bar\delta\!\left(\frac{\langle n\,b\rangle}{\langle n\,a\rangle} - \frac{\langle\lambda(\sigma_n)\,b\rangle}{\langle\lambda(\sigma_n)\,a\rangle}\right) e^{it_n\,[\mu(\sigma_n)\,n]}, \tag{C.2.22}$$

$$t_n \to \frac{\langle n\,a\rangle}{\langle\lambda(\sigma_n)\,a\rangle}\,, \tag{C.2.23}$$

as in (C.2.12), but without the extra factors coming from ϕ_E^{-1} which have already simplified in this anti-holomorphic soft limit.

The exponential terms are subleading because of their dependence on $\tilde{\kappa}_n$. The remaining holomorphic delta function cannot be further massaged in this case as there are no small parameters in which to expand, but its argument can still be localized on the support of the holomorphic delta functions corresponding to particle j (which we have not written explicitly). This can be used to set

$$t_j = \frac{\langle j\,a\rangle}{\langle\lambda(\sigma_j)\,b\rangle}\,, \tag{C.2.24}$$

for $\{a^\alpha, b^\alpha\}$ again an arbitrary spinor dyad. Collecting factors and applying the Schouten identity to the argument of the holomorphic delta function gives

$$-\frac{1}{[n\,j]}\int D\sigma_n\, \frac{\langle\lambda(\sigma_j)\,a\rangle}{\langle j\,a\rangle\,(n\,j)} \prod_{l\in\tilde{h}\setminus\{a,y\}} \frac{(n\,l)}{(j\,l)}\,\bar\delta(\langle\lambda(\sigma_n)\,n\rangle)\,. \tag{C.2.25}$$

Recalling the definition (5.2.18)

$$\bar\delta(\langle\lambda(\sigma_n)\,n\rangle) \coloneqq \frac{1}{2\pi i}\,\bar\partial\left(\frac{1}{\langle\lambda(\sigma_n)\,n\rangle}\right), \tag{C.2.26}$$

we see that an integration-by-parts can be performed in (C.2.25).

In particular, note that given meromorphic functions f and g in z, it follows that

$$\int_{\mathbb{C}^*} dz\, f(z)\, \bar{\delta}(g(z)) = \frac{1}{2\pi i}\int_{\mathbb{C}^*} dz \wedge d\bar{z}\, \frac{\partial}{\partial \bar{z}}\left(\frac{f(z)}{g(z)}\right) - \int_{\mathbb{C}^*} dz\, \frac{1}{g(z)}\, \bar{\delta}\left(\frac{1}{f(z)}\right). \tag{C.2.27}$$

The first term on the left-hand side is a boundary term, equal to the pole at $z = \infty$ of $dz\, f(z)/g(z)$. If no such pole exists, then

$$\int_{\mathbb{C}^*} dz\, f(z)\, \bar{\delta}(g(z)) = -\int_{\mathbb{C}^*} dz\, \frac{1}{g(z)}\, \bar{\delta}\left(\frac{1}{f(z)}\right). \tag{C.2.28}$$

A rational differential form does not have a pole at infinity if it behaves at least as $O(z^{-2})$ as $z \to \infty$. This is true for our integrand

$$-\frac{\langle \lambda(\sigma_j)\, a\rangle}{\langle j\, a\rangle (n\, j)} \prod_{l \in \tilde{\mathbf{h}} \setminus \{a,y\}} \frac{(n\, l)}{(j\, l)}\; \frac{1}{\langle \lambda(\sigma_n)\, n\rangle}\,, \tag{C.2.29}$$

with σ_n playing the role of z, due to $\lambda(\sigma_n)$ being a polynomial of degree $d \geqslant 1$ in σ_n, and $\tilde{\mathbf{h}}$ being a set of $d+1$ elements.

So finally, (C.2.25) is turned into

$$\frac{1}{[n\, j]} \int D\sigma_n\, \frac{\langle \lambda(\sigma_j)\, a\rangle}{\langle j\, a\rangle\, \langle \lambda(\sigma_n)\, n\rangle} \prod_{l \in \tilde{\mathbf{h}} \setminus \{a,y\}} \frac{(n\, l)}{(j\, l)}\; \bar{\delta}((n\, j))\,. \tag{C.2.30}$$

The new holomorphic delta function, which emerges after the integration-by-parts, enables us to set $\sigma_n = \sigma_j$, and then on the support of the holomorphic delta functions for particle j we can replace $\lambda(\sigma_j)$ with κ_j since the whole expression has balanced homogeneity. This leaves the final soft factor

$$\frac{1}{[n\, j]\, \langle n\, j\rangle} = \frac{1}{2\, p_n \cdot p_j}\,, \tag{C.2.31}$$

where j is the external leg that n attaches to directly in the selected tree subamplitude. Repeating this for all trees contributing to the formula, we recover the full soft structure of BAS from the anti-holomorphic soft limit.

The same analysis can be adapted to the case of multi-particle soft limits, verifying that the amplitude has no unphysical poles. This is in contrast to the scalar blocks constructed in [311], for which it is easy to see that there are spurious soft singularities that are additionally dependent on the 'parity' of the soft limit taken.

C.3 Non-trivial backgrounds in twistor theory

Here we consider generalisations of the previous section to non-trivial backgrounds. In particular we consider three examples: space-times with a negative cosmological constant, self-dual radiative gauge fields, and asymptotically flat self-dual radiative space times. These reviews are primarily adapted from the papers [232, 233, 312], which we encourage the reader to refer to throughout.

C.3.1 AdS$_4$

Twistors naturally encode spacetime in a conformally invariant way. The breaking of this conformal invariance is done by introducing an additional structure known as the *infinity twistor* [26, 330] I_{AB}. The conformally flat spacetime metric is

$$ds^2 = \frac{dX_{AB}\, dX^{AB}}{(I_{CD}\, X^{CD})^2} \tag{C.3.32}$$

where X^{AB} is a skew bi-twistor corresponding to a point x in spacetime. Let Z_1, Z_2 be two points in $\mathbb{PT}$ which lie on the line X corresponding to x via the incidence relations. Then

$$X^{AB} = Z_1^{[A} Z_2^{B]} = \langle \lambda_1\, \lambda_2 \rangle \begin{pmatrix} \frac{1}{2}\epsilon^{\dot\alpha\dot\beta} x^2 & x^{\dot\alpha}_{\beta} \\ -x^{\dot\beta}_{\alpha} & \epsilon_{\alpha\beta} \end{pmatrix}. \tag{C.3.33}$$

AdS$_4$ corresponds to the choice of infinity twistor and its dual

$$I^{AB} = \begin{pmatrix} \epsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & \Lambda\, \epsilon_{\alpha\beta} \end{pmatrix}, \qquad I_{AB} = \begin{pmatrix} \Lambda\, \epsilon_{\dot\alpha\dot\beta} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}, \tag{C.3.34}$$

where $\Lambda < 0$ is the cosmological constant, of mass dimension $+2$. The metric (C.3.32) is therefore

$$ds^2 = \frac{dx_{\alpha\dot\alpha} dx^{\alpha\dot\alpha}}{(1+\Lambda x^2)^2}, \tag{C.3.35}$$

written in an affine Minkowski space patch. These coordinates have the advantage of admitting a smooth flat space $\Lambda \to 0$ limit. The boundary is given in these coordinates by the hypersurface $x^2 = -1/\Lambda$. Finally, the twistor space of AdS$_4$ is

$$\mathbb{PT}_\Lambda = \{Z \in \mathbb{P}^3 | I_{AB} Z^B \neq 0\}, \tag{C.3.36}$$

where the excluded subset $\{Z \in \mathbb{P}^3 | I_{AB} Z^B = 0\}$ represents the boundary. The infinity twistor also shows its marks in the contraction of twistor variables

$$\langle A\, B \rangle \coloneqq I_{IJ} A^I B^J, \qquad [C\, D] \coloneqq I^{IJ} C_I D_J. \tag{C.3.37}$$

Zero-rest-mass fields of helicity h in AdS$_4$ can still be represented as cohomology classes $H^{0,1}(\mathbb{PT}_\Lambda, \mathcal{O}(2h-2))$ in twistor space using the Penrose transform. However, the correct choice of physical states relevant for explicit calculations in AdS$_4$ is subtle. Therefore we will remain agnostic about the precise form of the external states.

The proposed formula for gravitational scattering in AdS$_4$ [312] looks superficially identical to the Cachazo-Skinner formula (5.2.33):

$$\mathcal{M}^{\Lambda}_{n,d} = \int d\mu_d \, |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}_\Lambda) \det{}'\left(\mathbb{H}^\vee_\Lambda\right) \prod_{i\in\mathbf{h}} h_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j(Z(\sigma_j))\,, \tag{C.3.38}$$

with the main distinction being in the entries of the matrices $\mathbb{H}_\Lambda$ and $\mathbb{H}^\vee_\Lambda$, which are where the formula is explicitly sensitive to the cosmological constant. The entries of the matrices are explicitly dependent on the square and angle brackets (C.3.37):

$$\begin{aligned} \mathbb{H}_{\Lambda\, ij} &= \frac{\sqrt{D\sigma_i\, D\sigma_j}}{(i\, j)} \left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_j)}\right], \qquad i, j \in \mathbf{h}\,, i \neq j\,, \\ \mathbb{H}_{\Lambda\, ii} &= -D\sigma_i \sum_{\substack{j\in\mathbf{h}\\ j\neq i}} \frac{1}{(i\, j)} \left[\frac{\partial}{\partial Z(\sigma_i)}, \frac{\partial}{\partial Z(\sigma_j)}\right] \prod_{l\in\tilde{\mathbf{h}}} \frac{(j\, l)}{(i\, l)}\,, \quad i \in \mathbf{h}\,. \end{aligned} \tag{C.3.39}$$

a $(n-d-1)\times(n-d-1)$ matrix as before, and

$$\begin{aligned} \mathbb{H}^\vee_{\Lambda\, ij} &= \frac{\left\langle Z(\sigma_i),\, Z(\sigma_j)\right\rangle}{(i\, j)}\,, \qquad i, j \in \tilde{\mathbf{h}}\,, i \neq j \\ \mathbb{H}^\vee_{\Lambda\, ii} &= -\frac{\langle Z(\sigma_i)\, dZ(\sigma_i)\rangle}{D\sigma_i}\,, \qquad i \in \tilde{\mathbf{h}}\,, \end{aligned} \tag{C.3.40}$$

a $(d+1)\times(d+1)$ matrix as before. The reduced determinants appearing in (6.5.152) are defined in the same way as before, by (5.2.30), (5.2.32).

As in Minkowski space, the reduced determinant $\det{}'(\mathbb{H}^\vee_\Lambda)$ can still be interpreted as a resultant, but now for the full map $Z : \mathbb{P}^1 \to \mathbb{PT}_\Lambda$, ensuring that the entire expression only has support for holomorphic maps which land inside the twistor space $\mathbb{PT}_\Lambda$. Since all of the ingredients in the formula (6.5.152) have a smooth flat space limit, in the sense that when $\Lambda \to 0$, the formula reduces to the original Cachazo-Skinner formula (5.2.33)[3].

As Yang-Mills theory is classically conformally invariant, the function form of the RSVW formula should be un-modified on AdS_4, with the only imprint of the conformal structure coming from boundary conditions in the moduli integrals associated with the new hypersurface at infinity. Therefore the formula for gluon scattering in AdS_4 can be similarly proposed to be

$$\mathcal{A}^{\Lambda}_{n,d}[\rho] = \int d\mu_d \, |\tilde{\mathbf{g}}|^4 \, \mathrm{PT}[\rho] \prod_{i\in\mathbf{h}} a_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{h}}} \tilde{a}_j(Z(\sigma_j))\,, \tag{C.3.41}$$

where $a_i \in H^{0,1}(\mathbb{PT}_\Lambda, \mathcal{O}(0))$ and $\tilde{a}_i \in H^{0,1}(\mathbb{PT}_\Lambda, \mathcal{O}(-4))$ as before.

Of note here is that the precise object that these formulae describe — and indeed what the form of the external state representatives should be — has not been established. For example, they could be missing boundary terms such as those that contribute to AdS correlators with all external particles of the same helicity [331].

[3] Observe that (6.5.152) is polynomial in Λ, so it can also be viewed as a formula for graviton scattering in dS_4 by simply continuing to $\Lambda > 0$.

C.3.2 Self-dual radiative gauge fields

There exists a construction in twistor space that describes self-dual gauge fields known as the *Ward correspondence* [25]:

Theorem 3 (Ward [25]). *There is a one-to-one correspondence between:*

- *self-dual gauge fields on* $\mathbb{M}$*, and*
- *holomorphic vector bundles* $E \to \mathbb{PT}$*, with* $E|_X$ *topologically trivial for every twistor line* $X \cong \mathbb{P}^1$ *corresponding to a point* $x \in \mathbb{M}$.

In this subsection, we will use this correspondence to describe self-dual *radiative* gauge fields in twistor space. This will then build up to presenting the version of the RSVW formula on these backgrounds, which will be used in the next chapter.

Self-dual radiative gauge fields are self-dual, source-free, asymptotically flat at past and future conformal null infinity $\mathscr{I}^- \cup \mathscr{I}^+ = \mathscr{I}$ and determined by the free characteristic data at either $\mathscr{I}^+$ or $\mathscr{I}^-$. More concretely, we will consider complex four-dimensional Minkowski space, $\mathbb{M}$, with metric

$$ds^2 = dx^{\alpha\dot{\alpha}} dx_{\alpha\dot{\alpha}}. \tag{C.3.42}$$

A 2-form F decomposes into self-dual (SD) and anti-self-dual parts in four dimensions:

$$F = F^+ + F^-, \qquad *F^\pm = \pm i F^\pm \tag{C.3.43}$$

where $*$ is the Hodge star operator of the four-dimensional Minkowski metric. A 2-form F is *SD* if $F^- = 0$. A generic F can be further represented in the 2-spinor formalism as

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta} \tilde{F}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} F_{\alpha\beta}, \tag{C.3.44}$$

where the SD field strength is encoded by $\tilde{F}_{\dot{\alpha}\dot{\beta}} = \tilde{F}_{(\dot{\alpha}\dot{\beta})}$ and the ASD field strength is encoded by $F_{\alpha\beta} = F_{(\alpha\beta)}$. In this formalism, the self-duality condition is $F_{\alpha\beta} = 0$.

To describe the conformal null infinity $\mathscr{I}$ of Minkowski space, and thus the radiative data, we conformally compactify the space. This is usually done in terms of the non-homogeneous retarded coordinates $(u, r, \zeta, \bar{\zeta})$ in which the metric (C.3.42) becomes (see e.g. [370])

$$ds^2 = -du^2 - 2du\, dr + r^2 \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \tag{C.3.45}$$

Applying the inversion $R = r^{-1}$ and conformally rescaling the metric we have

$$d\hat{s}^2 := R^2\, ds^2 = R^2\, du^2 - 2du\, dR - \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \tag{C.3.46}$$

This gives the degenerate metric on $\mathscr{I}^+$ when we take $R \to 0$

$$ds^2_{\mathscr{I}^+} = 0 \times du^2 - \frac{4d\zeta d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2}. \tag{C.3.47}$$

The celestial sphere $\mathbb{P}^1$ is described in this construction by the non-homogeneous complex coordinates $(\zeta, \bar{\zeta})$. Here it will be useful for us to replace these with *homogeneous* coordinates [371, 372]. We can instead describe Minkowski space-time using the coordinates $(u, r, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ where $\lambda_\alpha = (\lambda_0, \lambda_1)$ are holomorphic homogeneous coordinates subject to the equivalence relation

$$(u, r, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}) \sim (|b|^2 u, |b|^{-2} r, b\lambda_\alpha, \bar{b}\,\bar{\lambda}_{\dot{\alpha}}), \qquad \forall b \in \mathbb{C}^*. \tag{C.3.48}$$

In terms of these coordinates, the conformal equivalence class of degenerate metrics on $\mathscr{I}^+$ (conformally related to (C.3.47)) is

$$ds^2_{\mathscr{I}^+} = 0 \times du^2 + D\lambda\, D\bar{\lambda}, \qquad \text{where} \quad D\lambda := \langle \lambda\, d\lambda \rangle := \lambda^\alpha d\lambda_\alpha. \tag{C.3.49}$$

Future null infinity $\mathscr{I}^+ \cong \mathbb{R} \times S^2$ is therefore described by the projective Bondi coordinates $(u, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$. Due to the projective weights of $(\lambda, \bar{\lambda})$, this gives rise to the line bundles $\mathcal{O}(p, q) \to \mathscr{I}^+$ whose sections are homogeneous functions $f(u, \lambda, \bar{\lambda})$ obeying $f(|b|^2 u, b\lambda, \bar{b}\bar{\lambda}) = b^p \bar{b}^q f(u, \lambda, \bar{\lambda})$ for any $b \in \mathbb{C}^*$.

An identical analysis can be applied for $\mathscr{I}^-$ using advanced coordinates $(v, r, \lambda, \bar{\lambda})$.

We will consider a gauge field A in four-dimensional Minkowski spacetime, valued in the Lie algebra $\mathfrak{g}$ of a compact gauge group. Using retarded Bondi coordinates (C.3.45), the fall-off conditions of a gauge field in temporal gauge $A_u = 0$ mean that restricted to $\mathscr{I}^+$ we have

$$A|_{\mathscr{I}^+} = \mathcal{A}^0(u, \lambda, \bar{\lambda})\, D\lambda + \tilde{\mathcal{A}}^0(u, \lambda, \bar{\lambda})\, D\bar{\lambda}. \tag{C.3.50}$$

This is pulled back to the projective coordinates $(u, \lambda, \bar{\lambda})$, and the weights imply that $\mathcal{A}^0$ takes values in $\mathcal{O}(-2, 0) \otimes \mathfrak{g}$ an $\bar{\mathcal{A}}^0$ in $\mathcal{O}(0, -2) \otimes \mathfrak{g}$. Taking the *leading* parts of the associated 2-form field strength F at $\mathscr{I}^+$ we have the SD and ASD parts

$$F^+|_{\mathscr{I}^+} = \partial_u \tilde{\mathcal{A}}^0 du \wedge D\bar{\lambda}, \qquad F^-|_{\mathscr{I}^+} = \partial_u \mathcal{A}^0 du \wedge D\lambda. \tag{C.3.51}$$

Restricting now to self-dual fields, we see that the leading part of $F^+|_{\mathscr{I}^+}$ is determined by $\partial_u \tilde{\mathcal{A}}^0$. In fact this determines the radiative data for the fully non-linear gauge fields provided we assume appropriate regularity conditions.

Therefore we define a *self-dual radiative gaugle field* as a complex radiative gauge field that is asymptotically flat almost everywhere and determined by its radiative data $\tilde{\mathcal{A}}^0$, with $\mathcal{A} = 0$.

The condition of being asymptotically flat almost everywhere arises because it is natural to include plane waves (reviewed in Chapter 6.1.1) in this definition. These will have a singularity at one of the points of the celestial sphere corresponding to the null vector n_μ.

We can now apply Theorem 3 to self-dual radiative spacetimes. We express the complex structure on the Ward bundle $E \to \mathbb{PT}$ in terms of a $\bar{\partial}$-operators $\bar{D}$ that takes sections of E to $(0, 1)$ forms and satisfies $\bar{D}^2 = 0$. This is simply given in terms of the quantities defined above, and coordinates $Z^A = (\mu^{\dot\alpha}, \lambda_\alpha)$ on twistor space as

$$\bar{D} = \bar\partial + a, \qquad a := \tilde{\mathcal{A}}^0(\mu^{\dot\alpha}\bar\lambda_{\dot\alpha}, \lambda, \bar\lambda)\, D\bar\lambda, \tag{C.3.52}$$

with a valued in $\mathrm{End}\,E$.

The bundle E is holomorphically trivial. This means that restricting to a line $X \subset \mathbb{PT}$ corresponding to a spacetime point x, there exists a frame $H(x, \lambda, \bar\lambda) : E|_X \to \mathbb{C}^r$ so that $\bar{D}|_X H = 0$. In general it's difficult to find such an H, but when $\tilde{\mathcal{A}}^0$ takes values in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. It can be shown that

$$a|_X = \bar\partial|_X g(x, \lambda), \tag{C.3.53}$$

for some Cartan-valued $g(x, \lambda)$ in $\mathcal{O}(0, 0)$. The 'frame' H is therefore given by

$$H(x, \lambda) = \exp\big(-g(x, \lambda)\big), \tag{C.3.54}$$

using the exponential map of the gauge group. The spacetime connection associated with this Cartan valued radiative data is $D_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} - i\, A_{\alpha\dot\alpha}$ where $A_{\alpha\dot\alpha}$ is a function on $\mathbb{M}$ taking values in the adjoint of the gauge group and the derivative $\partial_{\alpha\dot\alpha}$ is with respect to spacetime points $x^{\alpha\dot\alpha}$. In terms of $g(x, \lambda)$, it can be shown to satisfy

$$\lambda^\alpha A_{\alpha\dot\alpha} = -i\lambda^\alpha \partial_{\alpha\dot\alpha} g(x, \lambda) \tag{C.3.55}$$

which can be solved explicity by

$$A_{\alpha\dot\alpha}(x) = \frac{o_\alpha}{2\pi} \int_X \frac{D\lambda}{\langle o\, \lambda\rangle} \frac{\partial a}{\partial \mu^{\dot\alpha}}\Big|_X \tag{C.3.56}$$

where o_α is a reference spinor that amounts to an arbitrary gauge choice.

Having reviewed the twistor theoretic construction of Cartan-valued self-dual radiative gauge field backgrounds, we can now write down what the scattering amplitudes of gluons in such a Cartan-valued background. The MHV amplitude can be derived using the MHV generating functional for gauge theory, as outlined in [232]. For negative helicity gluons r, s rest positive we have

$$\mathcal{M}_{n,1}[\rho] = g^{n-2} \langle r\, s\rangle^4 \prod_{i=1}^{n} \frac{1}{\langle \rho(i)\, \rho(i+1)\rangle} \int_{\mathbb{M}} d^4x \, \exp\left[\sum_{i=1}^{n} (i\, k_i \cdot x + e_i g(x, \kappa_i))\right] \tag{C.3.57}$$

where e_i is the charge of gluon i with respect to the colour vector of the Cartan-valued background, and the particles have external momenta $k_{\alpha\dot\alpha} = \kappa_\alpha \tilde\kappa_{\dot\alpha}$. As was reviewed earlier, MHV amplitudes involve an integral over maps $Z : \mathbb{P}^1 \to \mathbb{PT}$ of degree one. The moduli space of these maps is $\mathbb{M}$ according to the incidence relations, and this gives rise to the spacetime integral in (C.3.57).

For higher degrees, these formulae are conjectural, but rest on several non-trivial properties such as the correct flat space and perturbative limits. Recall from the flat space section that at N^{d-1}MHV, amplitude formulae for gluons or gravitons are supported on curves of degree d from $\mathbb{P}^1$ to $\mathbb{PT}$. Whereas before we trivialised the Ward bundle $E \to \mathbb{PT}$ on a twistor line corresponding to each spacetime point x, here we restrict to a curve $Z(\sigma)$ of degree d, identified with it's moduli U in the usual way

$$Z^I(\sigma) = U^I_{a_1\cdots a_d}\sigma^{a_1}\cdots\sigma^{a_d} =: U^I_{a(d)}\sigma^{a(d)}. \tag{C.3.58}$$

On restriction to such a curve, the bundle will (generically) be holomorphically trivial with a trivialisation $H(U,\sigma)$ defined by $\bar{D}|_{Z(\sigma)}H = 0$. Just as before (assuming the background is Cartan-valued), we can find a $g(U,\sigma)$ valued in $\mathfrak{g}$ so that $H(U,\sigma) = \exp[-g(U,\sigma)]$.

This naturally leads us to the generalisation of the RSVW formula in self-dual radiative gauge field backgorunds as [232]

$$\mathcal{A}_{n,d}[\rho] = \int d\mu_d\, |\tilde{\mathbf{g}}|^4 \mathrm{PT}_n[\rho] \prod_{i\in\mathbf{g}} a_i\, e^{e_i\, g(U,\sigma_i)} \prod_{j\in\tilde{\mathbf{g}}} b_j e^{e_j\, g(U,\sigma_j)}. \tag{C.3.59}$$

The functions $g(U,\sigma)$ now contain dependence on all the map moduli in the integral. Therefore in general this formula will have $4d$ residual integrals not fixed by δ-functions. However, this is still much less than the number of integrations expected for scattering amplitudes derived using, for example, the perturbiner method.

It should be emphasised that whilst the MHV amplitude (C.3.57) has been proven at tree level, (C.3.59) is a conjecture, albeit a well-motivated one. Proofs of formulae such as this usually proceed via analyticity and associated recursion relations. How these methods could generalise to background formulae is not yet known.

C.3.3 Self-dual radiative spacetimes

Self-dual complexified four-dimensioned spacetimes can be described using twistor theory through Penrose's non-linear graviton construction [26]:

Theorem 4 (Penrose [26])**.** *There is a one-to-one correspondence between*

- *suitably convex regions of Ricci-flat, self dual 4-manifolds* $(\mathcal{M}, g_{ab})$*, and*
- *3-dimensional complex manifolds* $\mathbb{P}\mathcal{T}$ *that are complex deformations of a region in* $\mathbb{PT}$*, contain a holomorphic* $\mathbb{CP}^1$ *with normal bundle* $\mathcal{O}(1)\oplus\mathcal{O}(1)$*, have a holomorphic fibration*

  $$\pi : \mathbb{P}\mathcal{T} \to \mathbb{P}^1 \tag{C.3.60}$$

  and have a holomorphic Poisson structure I *up the fibres of* π *with values in the pullback of* $\mathcal{O}(-2)$ *from* $\mathbb{P}^1$.

In this subsection, we will make use of this correspondence to find formulae for scattering amplitudes in self-dual *radiative* spacetimes. The final result will be a version of the Cachazo-Skinner formula on these backgrounds.

Self-dual radiative spacetimes are self-dual, source-free, almost everywhere asymptotically flat at past and future conformal null infinity $\mathscr{I}^- \cup \mathscr{I}^+ = \mathscr{I}$ and determined by the free characteristic data at either $\mathscr{I}^+$ or $\mathscr{I}^-$. Self-duality of a metric is encoded in its Weyl tensor C_{abcd}, the traceless component of the Riemann tensor given by

$$C_{abcd} = R_{abcd} + \frac{1}{2}(R_{ad}g_{bc} - R_{ac}g_{bd} + R_{bc}g_{ad} - R_{bd}g_{ac}) + \frac{1}{6}R(g_{ab}g_{cd} - g_{ad}g_{bd}). \quad \text{(C.3.61)}$$

Using the 2-spinor formalism, the Weyl tensor decomposes into its SD and ASD parts [373]

$$C_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} = \tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \Psi_{\alpha\beta\gamma\delta}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} \quad \text{(C.3.62)}$$

with $\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ corresponding the the SD part C^+_{abcd}, and $\Psi_{\alpha\beta\gamma\delta}$ corresponding to the ASD part C^-_{abcd}. A self dual spacetimes therefore corresponds to one where $\Psi_{\alpha\beta\gamma\delta} = 0$.

Using the same retarded coordinates $(u, r, \lambda, \bar{\lambda})$ as in the previous subsection, an asymptotically flat metric in Bondi-Sachs gauge [374, 375] admits a large-r expansion

$$ds^2 = 2\,du\,dr - r^2\left(D\lambda\,D\bar{\lambda} - \frac{\boldsymbol{\sigma}^0}{r}D\lambda^2 - \frac{\bar{\boldsymbol{\sigma}}^0}{r}D\bar{\lambda}^2\right) + \left(\frac{1}{\langle\lambda\,\hat{\lambda}\rangle} - 2\frac{m_B}{r}\right)du^2 \\ - du(\eth\bar{\boldsymbol{\sigma}}^0\,D\bar{\lambda} + \bar{\eth}\boldsymbol{\sigma}^0\,D\lambda) + O(r^{-1}). \quad \text{(C.3.63)}$$

Here $D\lambda \coloneqq \langle\lambda\,d\lambda\rangle, D\bar{\lambda} \coloneqq [\bar{\lambda}\,d\bar{\lambda}]$ and $\eth, \bar{\eth}$ are the spin-weighted covariant derivatives on the sphere. All other quantities are functions of $(u, \lambda, \bar{\lambda})$ with projective weights. The Bondi mass aspect m_B is valued in $\mathcal{O}(-3, -3)$, whereas the asymptotic shear optical scalars $\boldsymbol{\sigma}^0, \bar{\boldsymbol{\sigma}}^0$ are valued in $\mathcal{O}(-3, 1)$ and $\mathcal{O}(1, -3)$ respectively.

Again rescaling with the conformal factor $R = r^{-1}$ we have the leading behaviour

$$d\hat{s}^2 \coloneqq R^2 ds^2 = -2\,du\,dR - D\lambda\,D\bar{\lambda} + R\,\boldsymbol{\sigma}^0 D\lambda^2 + R\,\bar{\boldsymbol{\sigma}}^0 D\bar{\lambda}^2 + O(R^2). \quad \text{(C.3.64)}$$

It can be shown that taking $\boldsymbol{\sigma}^0 = 0$ corresponds to an asymptotically self-dual spacetime. A *self-dual radiative space-time* is then a complex radiative spacetime determined by its raditive data $\bar{\boldsymbol{\sigma}}^0$, with $\boldsymbol{\sigma}^0 = 0$. In particular this means that $m_B \to 0$ as $u \to +\infty$.

A useful quantity in the next discussions will be the *news function*

$$N(u, \lambda, \bar{\lambda}) \coloneqq -\partial_u \bar{\boldsymbol{\sigma}}(u, \lambda, \bar{\lambda}) \quad \text{(C.3.65)}$$

on $\mathscr{I}^+$. This encodes the energy-momentum radiated through $\mathscr{I}^+$ via the Bondi mass-loss theorem [375].

In contrast to the gauge theory case, we will immediately consider scattering at generic N^{d-1}MHV degree. The twistorial description of self-dual radiative spacetimes, following Penrose's non-linear graviton construction in Theorem 4, involves a deformation of the complex structure on $\mathbb{PT}$ in terms of the Dolbeault operator

$$\bar{\nabla} = \bar{\partial} + \bar{\boldsymbol{\sigma}}\,D\bar{\lambda}\bar{\lambda}^{\dot{\alpha}}\,\frac{\partial}{\partial\mu^{\dot{\alpha}}}. \quad \text{(C.3.66)}$$

Additionally, we will use the holomorphic frame of the self-dual spinor bundle on the SD radiative spacetime $H^{\dot{\alpha}\dot{\beta}}(x,\lambda)$. This is a holomorphic trivialisation of the SD spin connection on the manifold $\mathcal{M}$, and given by

$$\bar{\partial}|_X H^{\dot{\alpha}}{}_{\dot{\gamma}}(x,\lambda) = \bar{\lambda}^{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}}\, H^{\dot{\beta}}{}_{\dot{\gamma}}(x,\lambda)\, N|_X\, D\bar{\lambda}, \tag{C.3.67}$$

where this is evaluated on the twistor line (of degree one) X in $\mathbb{PT}$ corresponding to each space-time point x.

At higher degrees, we generalise this notion to higher degree curves $Z(\sigma)$, parametrised by a coordinate on $\mathbb{P}^1$. Parametrising these curves by a collection of moduli $U_{\mathbf{a}_d}$ as before, the frame satisfies

$$\bar{\partial} H^{\dot{\alpha}}{}_{\dot{\gamma}}(U,\sigma) = \bar{\lambda}^{\dot{\alpha}}(\sigma)\bar{\lambda}_{\dot{\beta}}(\sigma)\, H^{\dot{\beta}}{}_{\dot{\gamma}}(U,\sigma)\, N(\sigma)\, D\bar{\lambda}(\sigma). \tag{C.3.68}$$

The N^{d-1}MHV scattering amplitude of gravitons in a self-dual radiative spacetime is conjectured to be given by [233]

$$\begin{aligned}\mathcal{M}_{n,d} = \sum_{t=0}^{n-d-3} \sum_{p_1,\dots,p_t} \int d\mu_d\, |\tilde{\mathbf{h}}|^8 \det{}'(\mathbb{H}^\vee) \prod_{i\in\mathbf{h}} h_i(Z(\sigma_i)) \prod_{j\in\tilde{\mathbf{h}}} \tilde{h}_j(Z(\sigma_j)) \\ \times \left(\prod_{m=1}^{t} D\sigma_m \wedge D\bar{\lambda}(\sigma_m)\, \frac{N^{(p_m-2)}(\sigma_m)}{p_m!} \frac{\partial^{p_m}}{\partial\varepsilon_m^{p_m}}\right) \det{}'(\mathcal{H})\Bigg|_{\varepsilon=0}.\end{aligned} \tag{C.3.69}$$

Here, the sum over $t = 0, \dots, n-d-3$ indexes the number of tail contributions to the amplitude, where external gravitons scatter off the background curvature leading to graviton-background and background-background interactions. Each index p_m, for $m = 1, \dots, t$ is summed over all values greater than two, and encodes how many times a given background insertion contributes to the amplitude. These background insertions appear through

$$N^{(k)}(\sigma_m) := -\frac{\partial^{k+1}}{\partial u^{k+1}} \tilde{\boldsymbol{\sigma}}^0(u,\lambda(\sigma_m),\bar{\lambda}(\sigma_m))\Big|_{u=[\mu(\sigma_m)\,\bar{\lambda}(\sigma_m)]}, \tag{C.3.70}$$

the k-times differentiated news function of the SD radiative background.

The matrix $\mathbb{H}^\vee$ is unchanged from Minkowski space due to the self-duality of the background. However, the matrix $\mathbb{H}$ in the Cachazo-Skinner formula is replaced by a $(n+t-d-1)\times(n+t-d-1)$ matrix $\mathcal{H}$, with entries dependent on the background insertions. These are given by

$$\mathcal{H}_{ij} = \frac{[K_i\, K_j]}{(i\,j)}, \qquad i \neq j \tag{C.3.71}$$

$$\mathcal{H}_{ii} = -\sum_{j\in\mathbf{h}\cup\mathbf{t}\setminus\{i\}} \frac{[K_i\, K_j]}{(i\,j)} \prod_{l\in\tilde{\mathbf{h}}} \frac{(l\,j)}{(l\,i)}, \tag{C.3.72}$$

where the indices i, j take values in the set $\mathbf{h}\cup\mathbf{t}$ and $\mathbf{t}$ is the set of background insertions. The 'dressed momentum' is defined dependendent on whether it corresponds to a particle or a background insertion as

$$K_i^{\dot\alpha} = \begin{cases} i\, s_i\, H^{\dot\beta\dot\alpha}(U,\sigma_i)\,\tilde\kappa_{i\,\dot\beta} & i \in \mathbf{h}, \\ i\, \epsilon_i\, H^{\dot\beta\dot\alpha}(U,\sigma_i)\,\bar\lambda_{\dot\beta}(\sigma_i) & i \in \mathbf{t}. \end{cases} \tag{C.3.73}$$

The reduced determinant $\det'$ is then defined just as before by removing a row and column and dividing by the appropriate Vandermonde determinant.

Whilst this formula has not been proven for generic degrees (as there are no recursive arguments based on analyticity in curved spacetimes), at $d = 1$, it has been proven using the MHV generating functional for gravity. It also passes non-trivial checks such as flat space and perturbative limits.

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